Functional Ito calculus and Functional Kolmogorov equations

Rama CONT

Scuola Normale Superiore (Pisa, April-May 2013)

Opening Lecture: Monday April 8, 10:00-13:00
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Abstract:
The *Functional Ito Calculus* [1, 2, 3, 6] is a non-anticipative calculus which Ito's stochastic calculus to path-dependent functionals of stochastic processes [1, 2, 3]. These lectures provide an introduction to this theory, its links with Malliavin calculus and a glimpse of applications in stochastic analysis, mathematical finance and stochastic control theory.

A key ingredient of the approach is the functional Ito formula [1, 6], which we will derive using two different approaches: a pathwise, analytical approach [3] and a probabilistic construction [2].

The functional Ito fomula is then used to obtain explicit *martingale representation formulas* [2, 5] for functionals of a square-integrable martingale. By contrast with the Clark-Ocone formula, which is based on Malliavin calculus, these formulas are based on non-anticipative quantities which may be computed pathwise using simple numerical schemes. In mathematical finance, these formulas lead to a universal formula for hedging strategies for path-dependent derivatives.

One application of the functional Ito formula is to obtain *martingale representation formulas* [2, 5] for functionals of a square-integrable martingale. By contrast with the Clark-Ocone formula, which is based on Malliavin calculus, these formulas are based on non-anticipative quantities which may be computed pathwise using simple numerical schemes. In mathematical finance, these formulas allow explicit computations of hedging strategies for path-dependent derivatives.

The functional Ito calculus also leads to a new class of (functional) partial differential equations on path space, which share many properties with parabolic PDEs on finite dimensional spaces. We show in particular that a large class of martingales may be characterized as solutions to *functional Kolmogorov equations*. We study existence, and uniqueness of solutions and comparison principle of such equations and show that they lead to Feynman-Kac formulas for path-dependent functionals of a square-integrable martingale. These functional PDEs exhibit a natural link with Backward stochastic differential equations.

OUTLINE

1. Motivation.
2. Functional representation of stochastic processes
   (a) The space of stopped paths.
   (b) Adapted processes as non-anticipative functionals.
3. Pathwise calculus for non-anticipative functionals
   (a) Horizontal and vertical derivatives of a non-anticipative functional
   (b) Paths with finite quadratic variation. Föllmer's pathwise Ito formula.
   (c) A pathwise change of variable formula for non-anticipative functionals.
4. Functional Ito calculus
   (a) Functional change of variable formula for semimartingales
   (b) Vertical derivative of a non-anticipative process
   (c) A martingale representation formula
5. Extension to square integrable functionals of a semimartingale.
   (a) Vertical derivative of a square integrable martingale
   (b) A general martingale representation formula.
   (c) Application in finance: pricing and hedging of path-dependent options.
   (d) Relation with Malliavin calculus. Lifting theorem.
6. Extensions
   (a) Functionals of quadratic variation.
   (b) Locally regular functionals and functionals involving exit times.
7. Functional Kolmogorov equations and functional PDEs.
   (a) Functional Kolmogorov equations for martingales.
   (b) Harmonic functionals.
   (c) Classical solutions: definition, comparison principle, uniqueness.
   (d) Feynman-Kac formula for path-dependent functionals.

8. Functional calulus for discontinuous processes.
   (a) Martingale representation formula for Poisson random measures.
   (b) Functional PIDEs.
   (c) Harmonic functionals of a Lévy process.
9. Relation with Forward-Backward stochastic differential equations (FBSDEs)
10. Applications to stochastic control
   (a) Stochastic control problems and the martingale optimality principle.
   (b) A verification theorem for non-Markovian stochastic control problems.
   (c) Relation with Backward SDEs.

References

[1] R Cont and D Fournié (2010) A functional extension of the Ito formula, *Comptes Rendus de l'Académie des Sciences*, Volume 348, Issues 1-2, January 2010, Pages 57-61.

[2] R Cont and D Fournié (2013) Functional Ito calculus and stochastic integral representation of martingales, *Annals of Probability*, Vol 41, No 1, 109–133.

[3] R Cont and D Fournié (2010) Change of variable formulas for non-anticipative functionals on path space, *Journal of Functional Analysis*, Volume 259, No 4, Pages 1043-1072.

[4] R Cont and D Fournié (2010) Functional Kolmogorov equations, Working Paper.

[5] R Cont (2011) Martingale representation formulas for Poisson random measures, Working Paper.

[6] B Dupire (2009) Functional Ito calculus, `www.ssrn.com`.

[7] H. Föllmer (1981) Calcul d'Itô sans probabilités, *Séminaire de Probabilités* Vol. XV, Springer, Berlin, pp. 143–150.

[8] D Nualart (2009) *Malliavin Calculus and Its Applications* (CBMS Regional Conference Series in Mathematics), American Mathematical Society.