Divergence-measure fields: generalizations of the Gauss-Green formulas

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Classical Gauss-Green formula

Theorem

Let $E \subset \Omega$ be an open regular set; that is, $\mathrm{int}(\bar{E}) = E$ and ∂E is a C^1 $(n-1)$-manifold in Ω. Then $\forall \phi \in C^1_c(\Omega; \mathbb{R}^n)$

$$\int_E \mathrm{div}\phi \, dx = -\int_{\partial E} \phi \cdot \nu_E \, d\mathscr{H}^{n-1},$$

where ν_E is the interior unit normal to ∂E.

BV THEORY

- $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is a *function of bounded variation in* Ω, $u \in BV(\Omega)$, if $u \in L^1(\Omega)$ and the distributional gradient Du is a finite Radon measure; that is, a vector valued Borel measure with finite total variation on Ω.
- A set E *of (locally) finite perimeter in* Ω is a set whose characteristic function χ_E is a (locally) BV function in Ω. By the polar decomposition of Radon measures, $D\chi_E = \nu_E |D\chi_E|$, for some Borel function ν_E with norm 1 $|D\chi_E|$-a.e.
- Relevant subsets of the topological boundary of E:
  - *the reduced boundary*, (De Giorgi) $\partial^* E := \{x \in \Omega : \exists \lim_{r\to 0} \frac{D\chi_E(B(x,r))}{|D\chi_E|(B(x,r))} = \nu_E(x) \in \mathbb{S}^{n-1}\}$, on which the unit vector ν_E is well defined and called *measure theoretic interior unit normal*, since we have the blow-up property $(E - x)/r \to \{(y - x) \cdot \nu_E \geq 0\} =: H^+_{\nu_E}(x)$ in measure as $r \to 0$ for any $x \in \partial^* E$;
  - *the measure theoretic boundary*, (Federer) $\partial^m E := \Omega \setminus (E^0 \cup E^1)$, where $E^d := \{x \in \mathbb{R}^n : \lim_{r\to 0} \frac{|E \cap B(x,r)|}{|B(x,r)|} = d\}$, which satisfies $\partial^m E \supset \partial^* E$ and $\mathscr{H}^{n-1}(\partial^m E \setminus \partial^* E) = 0$. Hence, we can integrate on $\partial^m E$ or $\partial^* E$ with respect to $\mathscr{H}^{n-1}$ indifferently.
- $|D\chi_E| = \mathscr{H}^{n-1} \llcorner \partial^* E$ (De Giorgi's theorem).

Gauss-Green formula for sets of finite perimeter

We just need to apply the definition of distributional derivative

$$\int_\Omega \chi_E \mathrm{div}\phi \, dx = -\int_\Omega \phi \cdot dD\chi_E = -\int_\Omega \phi \cdot \nu_E d|D\chi_E|$$

and then De Giorgi's theorem.

Theorem (De Giorgi and Federer)

Let $E \subset \Omega$ be a set of locally finite perimeter. Then $\forall \phi \in C^1_c(\Omega; \mathbb{R}^n)$

$$\int_E \mathrm{div}\phi \, dx = -\int_{\partial^* E} \phi \cdot \nu_E \, d\mathscr{H}^{n-1}.$$

- Aim: to weaken the regularity hypotheses on the vector fields.
- Strategy: to characterize the divergence in a weak sense (as a Radon measure) and the trace as an approximate limit or the density of a Radon measure.

Fine properties of BV functions

Important properties of BV functions:

- if $u \in BV(\Omega)$, then $|Du| \ll \mathscr{H}^{n-1}$;
- precise representative: any BV function u admits a representative u^* well defined $\mathscr{H}^{n-1}$-a.e. which satisfies $u^*(x) = \lim_{\varepsilon \to 0}(u \star \rho_\varepsilon)(x)$ $\mathscr{H}^{n-1}$-a.e. for any mollification of u. In particular, if E is a set of finite perimeter,

$$\chi_E^* = \chi_{E^1} + \frac{1}{2}\chi_{\partial^* E};$$

- if $u \in BV(\Omega)$ and $\mathrm{supp}(u) \Subset \Omega$, then $Du(\Omega) = 0$;
- Leibniz rule: if $u, v \in BV(\Omega) \cap L^\infty(\Omega)$, then $uv \in BV(\Omega) \cap L^\infty(\Omega)$ and

$$D(uv) = u^* Dv + v^* Du.$$

Gauss-Green formula for BV vector fields

Theorem (Vol'pert)

Let $u \in BV(\Omega;\mathbb{R}^n) \cap L^\infty(\Omega;\mathbb{R}^n)$ and $E \Subset \Omega$ be a set of finite perimeter, then

$$\int_{E^1} d\,\mathrm{div}(u) = \mathrm{div}u(E^1) = -\int_{\partial^* E} u_{\nu_E} \cdot \nu_E \, d\mathcal{H}^{n-1},$$

$$\int_{E^1 \cup \partial^* E} d\,\mathrm{div}(u) = \mathrm{div}u(E^1 \cup \partial^* E) = -\int_{\partial^* E} u_{-\nu_E} \cdot \nu_E \, d\mathcal{H}^{n-1},$$

where E^1 is the measure theoretic interior of E and $u_{\pm\nu_E}$ are respectively the interior and the exterior trace; that is, the approximate limits of u in $\mathscr{H}^{n-1}$-a.e. $x \in \partial^* E$ restricted to $H^{\pm}_{\nu_E}(x) := \{y \in \mathbb{R}^n : (y-x)\cdot(\pm\nu_E(x)) \geq 0\}$.

The boundedness assumption on u can be removed, if we assume $u_{\pm\nu_E} \in L^1(\partial^* E, \mathscr{H}^{n-1})$, as shown by Maz'ya and Ambrosio-Fusco-Pallara.

Divergence-measure fields: definition

Definition

- A vector field $F \in L^p(\Omega;\mathbb{R}^n), 1 \leq p \leq \infty$ is said to be a *divergence-measure field*, and we write $F \in \mathcal{DM}^p(\Omega)$, if $\mathrm{div} F$ is a finite Radon measure on Ω.
- A vector field F is a *locally divergence-measure field*, and we write $F \in \mathcal{DM}^p_{\mathrm{loc}}(\Omega)$, if $F \in \mathcal{DM}^p(W)$ for any open set $W \Subset \Omega$.

Brief history

Anzellotti (1983) was the first to study divergence-measure fields, even though he considered the special case $p = \infty$. Then, these new function spaces were introduced in the early 2000s by many authors for different purposes.

1. Chen and Frid were interested in the applications to the theory of systems of conservation laws with the Lax entropy condition and obtained a Gauss-Green formula for divergence-measure fields on open bounded sets with Lipschitz deformable boundary. Later, Chen, Torres and Ziemer extended this result to sets of finite perimeter in the case $p = \infty$.
2. Degiovanni, Marzocchi, Musesti, Šilhavý and Schuricht wanted to prove the existence of a normal trace under weak regularity hypotheses, in order to achieve a representation formula for Cauchy fluxes, contact interactions and forces in the context of continuum mechanics.
3. Ambrosio, Crippa and Maniglia studied a class of these vector fields induced by functions of bounded deformation, with the aim of extending theDiPerna-Lions theory of the transport equation.

A few recent applications

1. Phuc-Torres studied the existence of solutions to

$$\mathrm{div} F = \mu,$$

finding sufficient and necessary condition for a nonnegative measure μ on $\mathbb{R}^n$ in the case $F \in L^p(\mathbb{R}^n; \mathbb{R}^n)$ and F continuous, and for a signed Radon measure in the case $p = \infty$; moreover, this problem is also related to the characterization of the dual of the space BV.

2. Frid unified the theory of Chen-Frid and Šilhavý for extended divergence-measure fields and showed well-posedness of entropy solutions to conservation laws with suitable boundary conditions.

3. Schuricht, Kawohl, Scheven, Schmidt and many others rediscovered the techniques of Anzellotti, and applied the theory of divergence-measure fields to the study of 1-Laplace and minimal surface type equations, looking for super and subsolutions and dual formulations.

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Comparison with $BV(\Omega;\mathbb{R}^n)$ and absolute continuity

- $BV(\Omega;\mathbb{R}^n) \cap L^p(\Omega;\mathbb{R}^n) \subset \mathcal{DM}^p(\Omega)$. Indeed if $F = (F_1, ..., F_n) \in L^p(\Omega;\mathbb{R}^n)$ with $F_j \in BV(\Omega)$ for $j = 1, ...n$, then it is clear that $D_i F_j$ are finite Radon measures for each i, j and so $\mathrm{div} F = \sum_{j=1}^n D_j F_j$ is also a finite Radon measure.
- The condition $\mathrm{div} F = \mu$, with μ Radon measure, allows for cancellations; hence, for $n \geq 2$, the inclusion is strict. For example (Chen-Frid, 1999),

$$F(x, y) = \sin\left(\frac{1}{x - y}\right)(1, 1)$$

satisfies

$$F \in \mathcal{DM}^\infty(\mathbb{R}^2) \setminus BV_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2).$$

Absolute continuity and Leibniz rule

- (Šilahvý, 2005, Chen-Torres-Ziemer, 2009, and Phuc-Torres, 2008)
  If $n \geq 2$ and $F \in \mathcal{DM}^p_{\mathrm{loc}}(\Omega)$ for $\frac{n}{n-1} \leq p \leq \infty$, then we have $|\mathrm{div} F| \ll \mathscr{H}^{n-q}$, where $q := \frac{p}{p-1}$ is the conjugate exponent of p.
  This result is sharp: if $1 \leq p < \frac{n}{n-1}$, then for any arbitrary signed Radon measure μ with compact support inside Ω there exists $F \in \mathcal{DM}^p_{\mathrm{loc}}(\Omega)$ such that $\mathrm{div} F = \mu$. On the other hand, if $\frac{n}{n-1} \leq p \leq \infty$, then for any $s > n - q$ there exists a field $F \in \mathcal{DM}^p_{\mathrm{loc}}(\Omega)$ such that $|\mathrm{div} F|$ is not $\mathscr{H}^s$ absolutely continuous.
  Therefore, if $F \in \mathcal{DM}^\infty(\Omega)$, then $|\mathrm{div} F| \ll \mathscr{H}^{n-1}$.
- (Chen-Frid, 1999) If $g \in BV(\Omega) \cap L^\infty(\Omega)$ and $F \in \mathcal{DM}^\infty(\Omega)$, we have $gF \in \mathcal{DM}^\infty(\Omega)$ and

$$\mathrm{div}(gF) = g^* \mathrm{div} F + \overline{F \cdot Dg},$$

  where g^* is the precise representative of g and $\overline{F \cdot Dg}$ is the weak-star limit of $F \cdot \nabla(g * \rho_\delta)$ as $\delta \to 0$, which satisfies $|\overline{F \cdot Dg}| \ll |Dg|$. Hence, it is in particular possible to use this formula in the case $g = \chi_E$ with $E \Subset \Omega$ of finite perimeter.

Anzellotti's first investigations

Anzellotti (1983) was the first to define the space $\mathcal{DM}^\infty(\Omega)$, which he denoted by $X_\mu(\Omega)$. He considered the *pairing* between F and $u \in W^{1,1}(\Omega) \cap L^\infty(\Omega) \cap C^0(\Omega)$ (which we now call *normal trace functional* on the Lipschitz boundary $\partial\Omega$):

$$\langle F, u\rangle_{\partial\Omega} := \int_\Omega u \, d\mathrm{div} F + \int_\Omega F \cdot \nabla u \, dx$$

Theorem (Anzellotti, 1983)

Let Ω be a bounded open set with Lipschitz boundary, $F \in \mathcal{DM}^\infty(\Omega)$ and $u \in BV(\Omega) \cap L^\infty(\Omega) \cap C^0(\Omega)$. Then $\langle F, \cdot\rangle_{\partial\Omega}$ is a Radon measure on $\partial\Omega$, satisfying $\langle F, \cdot\rangle_{\partial\Omega} = [F \cdot \nu_\Omega]\mathscr{H}^{n-1} \llcorner \partial\Omega$. In addition, there exists a suitable Radon measure (F, Du) such that the following Gauss-Green formula holds:

$$\int_\Omega u \, d\mathrm{div} F + \int_\Omega d(F, Du) = -\int_{\partial\Omega} u \, [F \cdot \nu_\Omega] \, d\mathscr{H}^{n-1},$$

with $(F, Du) = F \cdot \nabla u \, dx$ if $u \in W^{1,1}(\Omega)$.

Definition

Let Ω be an open set in $\mathbb{R}^n$. We say that $\partial\Omega$ is a *deformable Lipschitz boundary* if Ω has Lipschitz boundary and there exists a Lipschitz deformation of the boundary; that is, a map $\Psi : \partial\Omega \times [0,1] \to \overline{\Omega}$ such that Ψ is a bi-Lipschitz homeomorphism onto its image and $\Psi(\cdot, 0) = \mathrm{Id}$ on $\partial\Omega$. We define $\partial\Omega_s := \Psi(\partial\Omega \times \{s\}), s \in [0,1]$ and we set Ω_s to be the open subset of Ω whose boundary is $\partial\Omega_s$.
The Lipschitz deformation is *regular* if

$$\lim_{\tau \to 0^+} J^{\partial\Omega}\Psi_\tau = 1 \quad \text{in} \quad L^1(\partial\Omega; \mathscr{H}^{n-1}),$$

where $\Psi_\tau(x) = \Psi(x, \tau)$.

Chen-Frid: The Gauss-Green formula on Lipschitz deformable open sets

Theorem (Chen-Frid, 1999, 2003)

Let $F \in \mathcal{DM}^p(\Omega)$ and Ω be a bounded open set with deformable Lipschitz boundary with deformation Ψ. Then, for any $\phi \in \mathrm{Lip}(\Omega) \cap L^\infty(\Omega)$, we have

$$\int_\Omega \phi \, d\mathrm{div} F + \int_\Omega F \cdot \nabla \phi \, dx = -\operatorname*{ess\,lim}_{s \to 0} \int_{\partial \Omega_s} \phi \, F \cdot \nu_{\Omega_s} \, d\mathscr{H}^{n-1}$$
$$= -\operatorname*{ess\,lim}_{s \to 0} \int_{\partial \Omega} (\phi \, F \cdot \nu_{\Omega_s}) \circ \Psi_s \, J^{\partial \Omega} \Psi_s \, d\mathscr{H}^{n-1}.$$

If $p = \infty$ and the deformation is regular, then the functional normal trace is represented by a function $\mathcal{F}_i \cdot \nu_\Omega \in L^\infty(\partial\Omega; \mathscr{H}^{n-1})$ such that $\|\mathcal{F}_i \cdot \nu_\Omega\|_{L^\infty(\partial\Omega; d\mathscr{H}^{n-1})} \leq \|F\|_{L^\infty(\Omega; \mathbb{R}^n)}$ and, for any $\phi \in \mathrm{Lip}(\partial\Omega)$,

$$\operatorname*{ess\,lim}_{s \to 0} \int_{\partial \Omega} (F \cdot \nu_{\Omega_s}) \circ \Psi_s \, \phi \, d\mathscr{H}^{n-1} = \int_{\partial \Omega} (\mathcal{F}_i \cdot \nu_\Omega) \, \phi \, d\mathscr{H}^{n-1}. \tag{1}$$

Šilhavý: classification of the normal trace functionals

Theorem (Šilhavý, 2005)

Let $F \in \mathcal{DM}^1_{\mathrm{loc}}(\mathbb{R}^n)$ and E be a set of finite perimeter, then there exists a linear functional $N^E(F,\cdot) : \mathrm{Lip}_c(\partial^* E) \to \mathbb{R}$ such that

$$N^E(F, \phi|_{\partial^* E}) = \int_{E^1} \phi \, d\mathrm{div} F + \int_E F \cdot \nabla \phi \, dx, \tag{2}$$

for any $\phi \in \mathrm{Lip}_c(\mathbb{R}^n)$.
If F is weakly dominated on $\partial^* E$; that is,
$\liminf_{r \to 0} \int_{\partial^* E} \int_{B(x,r)} |F(y) \cdot \nu_E(x)| \, dy \, d\mathscr{H}^{n-1}(x) < \infty$, *then $N^E(F,\cdot)$ is a measure supported on $\overline{\partial^* E}$.*
If $F \in \mathcal{DM}^\infty(\mathbb{R}^n)$, then $N^E(F, \phi|_{\partial^* E}) = -\int_{\partial^* E} \phi \, (\mathcal{F}_i \cdot \nu_E) \, d\mathscr{H}^{n-1}$ for some function $(\mathcal{F}_i \cdot \nu_E) \in L^\infty(\partial^* E; \mathscr{H}^{n-1})$. Also, for $\mathscr{H}^{n-1}$-a.e. $x \in \partial^* E$, we have

$$(\mathcal{F}_i \cdot \nu_E)(x) = \lim_{r \to 0} \frac{n}{\omega_{n-1} r^n} \int_{B(x,r) \cap H^+_{\nu_E}(x)} F(y) \cdot \frac{y - x}{|y - x|} \, d\mathscr{H}^{n-1}.$$

Degiovanni-Marzocchi-Musesti and Schuricht: families of admissible sets of finite perimeter

For the purpose of applications to the foundations of continuum mechanics, some more conditions are imposed on the admissible sets of finite perimeter: given $F \in \mathcal{DM}^1_{\mathrm{loc}}(\Omega)$, we consider sets of finite perimeter E such that

$$|\mathrm{div} F|(\partial^* E) = 0 \quad \text{and} \quad \int_{\partial^* E} h \, d\mathscr{H}^{n-1} < \infty. \tag{3}$$

where $h \in L^1_{\mathrm{loc}}(\Omega)$ is a nonnegative function such that one can extract a subsequence $\{F_k\}_{k\in\mathbb{N}}$ of the canonical mollification $F_k := F * \rho_{\varepsilon_k}$ of $F \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ satisfying [1]

$$F_k \to F \quad \text{in } L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^n) \tag{4}$$

$$F_k(x) \to F(x) \quad \text{for each } x \in \Omega \text{ such that } h(x) < +\infty \tag{5}$$

$$|F_k(x)| \leq h(x) \quad \text{for each } x \in \Omega \text{ and } k \in \mathbb{N}. \tag{6}$$

[1] Here and below we will still denote by F the particular representative which is the limit of the sequence F_k in the sense (5).

The Gauss-Green formula in $\mathcal{DM}^1_{\mathrm{loc}}$ for an admissible set of finite perimeter

Theorem (Degiovanni-Marzocchi-Musesti, 1999, Schuricht, 2007)

Given $F \in \mathcal{DM}^1_{\mathrm{loc}}(\Omega)$, E a set of finite perimeter admissible for F and $\phi \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ such that $\chi_E \phi$ has compact support in Ω, we have

$$\int_{E^1} \phi \, d\mathrm{div} F = -\int_{\partial^* E} \phi F \cdot \nu_E \, d\mathscr{H}^{n-1} - \int_E F \cdot \nabla \phi \, dx.$$

As a consequence, Schuricht (2007) proved the following Leibniz formula for χ_E and the particular representation of F described in (5):

$$\mathrm{div}(\chi_E F) = g_E \mathrm{div} F + F \cdot \nu_E \mathscr{H}^{n-1} \llcorner \partial^* E,$$

where $g_E \in L^\infty(\Omega; |\mathrm{div} F|)$ satisfies $0 \le g_E \le 1$ and $g_E(x) = \chi^*_E(x)$ at each x for which the Lebesgue density exists.
In addition, C.-Payne (2017) showed that, if $F \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega)$ and E is admissible, by passing to the limit in the boundary terms thanks to (5) and (6) we obtain $\mathcal{F}_i \cdot \nu_E = \mathcal{F}_e \cdot \nu_E = F \cdot \nu_E$ $\mathscr{H}^{n-1}$-a.e. on $\partial^* E$.

Smooth approximation of sets of finite perimeter

For any bounded set E of finite perimeter in $\mathbb{R}^n$, we consider the mollification of its characteristic function $u_k := \chi_E * \rho_{\varepsilon_k}$ and, for any $t \in (0,1)$, we define

$$A_{k;t} := \{u_k > t\}.$$

By Sard's theorem, $\partial A_{k;t}$ is smooth for $\mathscr{L}^1$-a.e. t and for any k; and clearly

$$|E \Delta A_{k;t}| \to 0$$

as $k \to +\infty$. It is also well known that

$$\mathscr{H}^{n-1}(\partial A_{k;t}) \to \mathscr{H}^{n-1}(\partial^* E)$$

for $\mathscr{L}^1$-a.e. $t \in (0,1)$ (Ambrosio-Fusco-Pallara, 2000, Maggi, 2012).

One-sided interior and exterior approximation of sets of finite perimeter

Theorem (Chen-Torres-Ziemer, 2009, C.-Torres, 2017)

Let μ be a Radon measure such that $\mu << \mathscr{H}^{n-1}$. Then:
(a) $|\mu|(E^1 \Delta A_{k;t}) \to 0$, *for* $\frac{1}{2} < t < 1$;
(b) $|\mu|((E^1 \cup \partial^* E)\Delta A_{k;t}) \to 0$, *for* $0 < t < \frac{1}{2}$.
In addition, there exists a sequence ε_k converging to 0 such that

$$\lim_{k\to+\infty} \mathscr{H}^{n-1}(\partial A_{k;t} \setminus E^1) = 0 \tag{7}$$

for $\mathscr{L}^1$-a.e. $t \in (\frac{1}{2}, 1)$, and

$$\lim_{k\to+\infty} \mathscr{H}^{n-1}(\partial A_{k;t} \setminus E^0) = 0 \tag{8}$$

for $\mathscr{L}^1$-a.e. $t \in (0, \frac{1}{2})$.

Sketch of the proof

Points **(a)** and **(b)** follow from the fact that $u_k(x) \to \chi_E^*(x)$ $\mathscr{H}^{n-1}$-a.e., and so $u_k \to \chi_E^*$ in $L^1(\mathbb{R}^n, |\mu|)$. Then, for any $\varepsilon > 0$, there exists k large enough such that

$$\varepsilon > \|u_k - \chi_E^*\|_{L^1(\mathbb{R}^n,|\mu|)} \geq \min\{t - 1/2, 1 - t\}|\mu|(E^1 \Delta A_{k;t}),$$

for any $t \in (1/2, 1)$. For $0 < t < 1/2$, we argue similarly.
As for the second part, we use the coarea formula and the properties of convolution to show that

$$\int_t^1 \mathscr{H}^{n-1}(u_k^{-1}(s) \setminus E^1)\, ds = \int_{A_{k;t}\setminus E^1} |\nabla u_k|\, dx \leq \int_{\partial^* E} (\rho_{\varepsilon_k} * \chi_{A_{k;t}\setminus E})\, d\mathscr{H}^{n-1},$$

and then, by a blow-up procedure, we prove that
$\lim_{k\to+\infty} u_k(x + \varepsilon_k z) =: v(x, z) \leq \frac{1}{2}$ for any $x \in \partial^* E$ and $z \in H^-_{\nu_E}(x)$, and

$$(\rho_{\varepsilon_k} * \chi_{A_{k;t}\setminus E})(x) \to \int_{B(0,1)} \rho(z)\, \chi_{\{v(x,z)>t\}}(z)\, \chi_{H^-_{\nu_E}(x)}(z)\, dz.$$

Since $t > 1/2$, we can conclude. The case $0 < t < 1/2$ is treated analogously.

Gauss-Green formula for $\mathcal{DM}^\infty_{\mathrm{loc}}$ fields on bounded sets of finite perimeter

Theorem (Chen-Torres-Ziemer, 2009)

Let $F \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega)$ and $E \Subset \Omega$ be a set of finite perimeter.
Then there exists interior and exterior normal traces $(\mathcal{F}_i \cdot \nu_E)$, $(\mathcal{F}_e \cdot \nu_E)$ of F such that:

1. $\displaystyle\int_{E^1} d\,\mathrm{div} F = -\int_{\partial^* E} (\mathcal{F}_i \cdot \nu_E)(x)\, d\mathscr{H}^{n-1}(x)$;
2. $(2F \cdot \nabla u_k)\chi_E \overset{*}{\rightharpoonup} (\mathcal{F}_i \cdot \nu_E)\mathscr{H}^{n-1} \llcorner \partial^* E$ *in* $\mathcal{M}(\Omega)$;
3. $\|\mathcal{F}_i \cdot \nu_E\|_{L^\infty(\partial^* E; \mathscr{H}^{n-1})} \leq \|F\|_{L^\infty(E)}$.
4. $\displaystyle\int_{E^1 \cup \partial^* E} d\,\mathrm{div} F = -\int_{\partial^* E} (\mathcal{F}_e \cdot \nu_E)(x)\, d\mathscr{H}^{n-1}(x)$;
5. $(2F \cdot \nabla u_k)\chi_{\Omega \setminus E} \overset{*}{\rightharpoonup} (\mathcal{F}_e \cdot \nu_E)\mathscr{H}^{n-1} \llcorner \partial^* E$ *in* $\mathcal{M}(\Omega)$;
6. $\|\mathcal{F}_e \cdot \nu_E\|_{L^\infty(\partial^* E; \mathscr{H}^{n-1})} \leq \|F\|_{L^\infty(\Omega \setminus E)}$.

Sketch of the proof (1)

The proof relies on the smooth interior (resp. exterior) approximation of E and on the following lemma.

Lemma (Chen-Torres-Ziemer, 2009)

Let $F \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega)$, F_ϵ be a mollification of F and $E \Subset \Omega$ be a set of finite perimeter. Then, if in addition we assume that

1. $F_\epsilon \to F$ $\mathscr{H}^{n-1}$*-a.e. on* $\partial^* E$,
2. $|\mathrm{div} F|(\partial^* E) = 0$,

then

$$\mathrm{div} F(E^1 \cup \partial^* E) = \mathrm{div} F(E^1) = -\int_{\partial^* E} F \cdot \nu_E \, d\mathscr{H}^{n-1}.$$

For $\mathscr{L}^1$-a.e. $t \in (0,1)$, we have that $\partial A_{k;t}$ is smooth, $|\mathrm{div} F|(\partial A_{k;t}) = 0$ and $F_\varepsilon \to F$ $\mathscr{H}^{n-1}$-a.e. (this is a consequence of the co-area formula).
Hence, for such t we define the Radon measure

$$\sigma_{k;t}(B) := \int_{B \cap \partial A_{k;t}} F \cdot \nu_{A_{k;t}} \, d\mathscr{H}^{n-1}.$$

Sketch of the proof (2)

By the lemma, we have $\mathrm{div} F(A_{k;t}) = -\sigma_{k;t}(\Omega)$ for any k and $\mathscr{L}^1$-a.e. t.
We know that, since $|\mathrm{div} F| \ll \mathscr{H}^{n-1}$, then $\mathrm{div} F(A_{k;t}) \to \mathrm{div} F(E^1)$ for $\mathscr{L}^1$-a.e. $t \in (\frac{1}{2}, 1)$; and $\mathrm{div} F(A_{k;t}) \to \mathrm{div} F(E^1 \cup \partial^* E)$ for $\mathscr{L}^1$-a.e. $t \in (0, \frac{1}{2})$.
Hence, the weak limit of $\sigma_{k;t}$, which exists for $\mathscr{L}^1$-a.e. t, up to subsequences, must be independent of $t \in (1/2, 1)$ or $(0, 1/2)$; thus there exists two signed measures σ_i, σ_e such that $\sigma_{k;t} \overset{*}{\rightharpoonup} \sigma_i$ for $\mathscr{L}^1$-a.e. $1/2 < t < 1$, and $\sigma_{k;t} \overset{*}{\rightharpoonup} \sigma_e$, for $\mathscr{L}^1$-a.e. $0 < t < 1/2$.
Then, one shows that $|\sigma_i| \ll |D\chi_E|$ and $\lim_{k\to+\infty} |\sigma_{k;t}|(E^0 \cup \partial^* E) = 0$ for $\mathscr{L}^1$-a.e. $1/2 < t < 1$, and analogously $|\sigma_e| \ll |D\chi_E|$ and $\lim_{k\to+\infty} |\sigma_{k;t}|(\Omega \setminus E^0) = 0$ for $\mathscr{L}^1$-a.e. $0 < t < 1/2$. All in all, we have

$$\mathrm{div} F(E^1) = -\sigma_i(\partial^* E),$$
$$\mathrm{div} F(E^1 \cup \partial^* E) = -\sigma_e(\partial^* E),$$

and Radon-Nikodym theorem allows us to conclude. The estimates follows from Lebesgue's differentiation theorem. As for the limits, they follow from the identity

$$\int_E F \cdot \nabla u_k \, dx = \int_0^1 \int_{E \cap \partial A_{k;t}} F \cdot \nu_{A_{k;t}} \, d\mathscr{H}^{n-1} \, dt.$$

Jump component of the divergence

- We have the following representation formula for the jump component of the divergence of F; that is, for any set of finite perimeter $E \Subset \Omega$ we have

$$\chi_{\partial^* E} \mathrm{div} F = (\mathcal{F}_i \cdot \nu_E - \mathcal{F}_e \cdot \nu_E)\, \mathscr{H}^{n-1} \llcorner \partial^* E$$

in the sense of Radon measures on Ω. Hence, we obtain also

$$|\mathrm{div} F|(\partial^* E) = \int_{\partial^* E} |\mathcal{F}_i \cdot \nu_E - \mathcal{F}_e \cdot \nu_E|\, d\mathscr{H}^{n-1}$$

and, for any Borel set $B \subset \partial^* E$,

$$\mathrm{div} F(B) = \int_B (\mathcal{F}_i \cdot \nu_E - \mathcal{F}_e \cdot \nu_E)\, d\mathscr{H}^{n-1}.$$

- If F is continuous, interior and exterior normal traces coincide on $\partial^* E$ as functions in $L^\infty(\partial^* E; \mathscr{H}^{n-1})$, and they admit a representative which is the classical scalar product $F \cdot \nu_E$. Therefore, the divergence of a continuous vector field does not have jump component ($|\mathrm{div} F|(\partial^* E) = 0$).

Integration by parts formulas

Theorem (Chen-Torres-Ziemer, 2009, C.-Payne, 2017)

Let $F \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega)$ and let $E \subset \Omega$ be a set of locally finite perimeter. Then, there are well defined interior and exterior normal traces of F on $\partial^* E$ satisfying $(\mathcal{F}_i \cdot \nu_E), (\mathcal{F}_e \cdot \nu_E) \in L^\infty_{\mathrm{loc}}(\partial^* E; \mathscr{H}^{n-1})$ such that for any $\phi \in \mathrm{Lip}_c(\Omega)$ we have

$$\int_{E^1} \phi \, d\mathrm{div} F = -\int_{\partial^* E} \phi(\mathcal{F}_i \cdot \nu_E) \, d\mathscr{H}^{n-1} - \int_E F \cdot \nabla \phi \, dx \tag{9}$$

and

$$\int_{E^1 \cup \partial^* E} \phi \, d\mathrm{div} F = -\int_{\partial^* E} \phi(\mathcal{F}_e \cdot \nu_E) \, d\mathscr{H}^{n-1} - \int_E F \cdot \nabla \phi \, dx. \tag{10}$$

In addition, for any compact K and open set U such that $K \subset U \subset\subset \Omega$ one has the estimates

$$||\mathcal{F}_i \cdot \nu_E||_{L^\infty((\partial^* E) \cap K; \mathscr{H}^{n-1})} \leq ||F||_{L^\infty(E \cap U; \mathbb{R}^n)} \tag{11}$$

and

$$||\mathcal{F}_e \cdot \nu_E||_{L^\infty((\partial^* E) \cap K; \mathscr{H}^{n-1})} \leq ||F||_{L^\infty((\Omega \setminus E) \cap U; \mathbb{R}^n)}. \tag{12}$$

Ambrosio-Crippa-Maniglia: foliation results

In general, if $F \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega)$, $\mathcal{F}_i \cdot \nu_E \neq F \cdot \nu_E$ and $\mathcal{F}_e \cdot \nu_E \neq F \cdot \nu_E$ $\mathscr{H}^{n-1}$-a.e., for a set of finite perimeter $E \subset \Omega$.
However, roughly speaking, the normal traces coincide with the classical one on almost every surface. Let $I \subset \mathbb{R}$ be an open interval and let $\{\Sigma_t\}_{t \in I}$ be a family of oriented hypersurfaces in Ω such that there exists $\Omega' \Subset \Omega$, $\Phi \in C^1(\overline{\Omega'})$ and a family of open set $\Omega_t \Subset \Omega'$, $t \in I$, with $\Phi(\Omega') = I, \{\Phi = t\} = \Sigma_t = \partial\Omega_t$ for any $t \in I, |\nabla\Phi| > 0$ in Ω' and Σ_t is oriented by $\nabla\Phi/|\nabla\Phi|$. Then, we have

$$\mathcal{F}_i \cdot \nu_{\Omega_t} = \mathcal{F}_e \cdot \nu_{\Omega_t} = F \cdot \nu_{\Omega_t} \quad \mathscr{H}^{n-1}\text{-a.e. on } \Sigma_t, \text{ for } \mathscr{L}^1\text{-a.e. } t \in I.$$

Normal traces and orientation of the reduced boundary

By generalizing an argument developed by Ambrosio-Crippa-Maniglia in the case of open bounded sets with C^1 boundary, we find that the normal traces are determined by $\partial^* E$ and its orientation.

Proposition (C.-Payne)

Let $F \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega)$ and let E_1, E_2 be sets of locally finite perimeter in Ω such that $\mathscr{H}^{n-1}(\partial^* E_1 \cap \partial^* E_2) \neq 0$. Then one has

$$\mathcal{F}_i \cdot \nu_{E_1} = \mathcal{F}_i \cdot \nu_{E_2} \quad \text{and} \quad \mathcal{F}_e \cdot \nu_{E_1} = \mathcal{F}_e \cdot \nu_{E_2} \tag{13}$$

for $\mathscr{H}^{n-1}$-a.e. $x \in \{y \in \partial^* E_1 \cap \partial^* E_2 : \nu_{E_1}(y) = \nu_{E_2}(y)\}$ and

$$\mathcal{F}_i \cdot \nu_{E_1} = -\mathcal{F}_e \cdot \nu_{E_2} \quad \text{and} \quad \mathcal{F}_e \cdot \nu_{E_1} = -\mathcal{F}_i \cdot \nu_{E_2} \tag{14}$$

for $\mathscr{H}^{n-1}$-a.e. $x \in \{y \in \partial^* E_1 \cap \partial^* E_2 : \nu_{E_1}(y) = -\nu_{E_2}(y)\}$.

In the case $E_1 = E$ and $E_2 = \Omega \setminus E$, we obtain $\mathcal{F}_i \cdot \nu_E = -\mathcal{F}_e \cdot \nu_{\Omega \setminus E}$ and $\mathcal{F}_e \cdot \nu_E = -\mathcal{F}_i \cdot \nu_{\Omega \setminus E}$.

Gluing and extension theorems

As a consequence of the Gauss-Green formula we obtain gluing and extension theorems.

Theorem (Chen-Torres, 2005, C.-Payne, 2017)

Let $W \Subset E^\circ \subset E \Subset U \subset \Omega$, where Ω, U and W are open sets and E is a set of finite perimeter in Ω. Let $F_1 \in \mathcal{DM}^\infty(U)$ and $F_2 \in \mathcal{DM}^\infty(\Omega \setminus \overline{W})$. Then

$$F(x) = \begin{cases} F_1(x) & \text{if } x \in E \\ F_2(x) & \text{if } x \in \Omega \setminus E \end{cases}$$

belongs to $\mathcal{DM}^\infty(\Omega)$, and

$$\mathrm{div} F = \chi_{E^1} \mathrm{div} F_1 + \chi_{E^0} \mathrm{div} F_2 + (\mathcal{F}_{i,1} \cdot \nu_E - \mathcal{F}_{e,2} \cdot \nu_E) \mathscr{H}^{n-1} \llcorner \partial^* E \tag{15}$$

where $\mathcal{F}_{i,1} \cdot \nu_E$ is the interior normal trace of F_1 over $\partial^* E$ and $\mathcal{F}_{e,2} \cdot \nu_E$ is the exterior normal trace of F_2 over $\partial^* E$, which in particular implies the following representation for the jump component:

$$\chi_{\partial^* E} \mathrm{div} F = (\mathcal{F}_{i,1} \cdot \nu_E - \mathcal{F}_{e,2} \cdot \nu_E) \mathscr{H}^{n-1} \llcorner \partial^* E. \tag{16}$$

Gluing and extensions theorems

Theorem (Chen-Torres-Ziemer, 2009, C.-Payne, 2017)

Let $U \Subset \Omega$ be open sets with $\mathscr{H}^{n-1}(\partial U) < \infty$, $F_1 \in \mathcal{DM}^\infty(U)$ and $F_2 \in \mathcal{DM}^\infty(\Omega \setminus \overline{U})$. Then, if we set

$$F(x) = \begin{cases} F_1(x) & \text{if } x \in U \\ F_2(x) & \text{if } x \in \Omega \setminus \overline{U} \end{cases},$$

we have $F \in \mathcal{DM}^\infty(\Omega)$ and we obtain

$$\operatorname{div} F = \chi_{U^1} \operatorname{div} \hat{F}_1 + \chi_{U^0} \operatorname{div} \hat{F}_2 + ((\hat{\mathcal{F}}_{1,i} \cdot \nu_U) - (\hat{\mathcal{F}}_{2,e} \cdot \nu_U)) \mathscr{H}^{n-1} \llcorner \partial^* U, \tag{17}$$

where $(\hat{\mathcal{F}}_{1,i} \cdot \nu_U)$ is the interior normal trace on $\partial^* U$ of the zero extension $\hat{F}_1$ and $(\hat{\mathcal{F}}_{2,e} \cdot \nu_U)$ is the exterior normal trace in $\partial^* U$ of $\hat{F}_2$. In particular,

$$\chi_{\partial^* U} \operatorname{div} F = ((\hat{\mathcal{F}}_{1,i} \cdot \nu_U) - (\hat{\mathcal{F}}_{2,e} \cdot \nu_U)) \mathscr{H}^{n-1} \llcorner \partial^* U. \tag{18}$$

The Gauss-Green formula up to the boundary of the definition domain

Corollary (C.-Payne, 2017)

If Ω is a bounded open set with Lipschitz boundary and $F \in \mathcal{DM}^\infty(\Omega)$, then its zero extension $\hat{F}$ to $\mathbb{R}^n$ belongs to $\mathcal{DM}^\infty(\mathbb{R}^n)$ and there exists the interior normal trace $(\hat{\mathcal{F}}_i \cdot \nu_\Omega)$ on $\partial\Omega$, while the exterior normal trace is zero. In addition, for any $\phi \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n)$, we have

$$\int_\Omega \phi \, d\mathrm{div} F = -\int_{\partial\Omega} \phi(\hat{\mathcal{F}}_i \cdot \nu_\Omega) \, d\mathscr{H}^{n-1} - \int_\Omega F \cdot \nabla\phi \, dx.$$

Šilhavý: general open sets

Theorem (Šilhavý, 2009)

Let $F \in \mathcal{DM}^p(\Omega)$, $1 \le p \le \infty$, $m \in \mathrm{Lip}_c(\overline{\Omega})$, $m > 0$ *in* Ω, *and, for each* $\varepsilon > 0$, $L_\varepsilon := \{x \in \Omega : m(x) < \varepsilon\}$. *Then, for any* $\phi \in \mathrm{Lip}(\Omega) \cap L^\infty(\Omega)$, *we have*

$$N^\Omega(F, \phi) = -\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{L_\varepsilon} \phi\, F \cdot \nabla m \, dx.$$

In addition, if $\liminf_{\varepsilon \to 0} \varepsilon^{-1} \int_{L_\varepsilon} |F \cdot \nabla m| \, dx < \infty$, *then* $N^\Omega(F, \phi)$ *is a measure on* $\partial\Omega$.

A typical example is $m(x) = \mathrm{dist}(x, \partial\Omega)$, for which $L_\varepsilon = \Omega \setminus \overline{\Omega_\varepsilon}$, where $\Omega_\varepsilon := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \varepsilon\}$.

Example of a normal trace which is not a measure

Let $\Omega = B(0,1) \cap \{x_2 < 0\}$ in $\mathbb{R}^2$ and

$$F(x_1, x_2) = \frac{1}{|x|^\alpha}(x_2, -x_1), \quad \alpha \in [1,3).$$

Then, we clearly have $\mathrm{div} F = 0$ and $F \in \mathcal{DM}^p(\Omega)$ for any $1 \le p < 2/(\alpha - 1)$ if $\alpha > 1$, and $1 \le p \le \infty$ if $\alpha = 1$. If $\phi \in \mathrm{Lip}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ is such that $\phi \equiv 0$ in a neighborhood of $\{x_2 < 0\} \cap \partial B(0,1)$, we can just take $m(x) = -x_2$ near $\{x_2 = 0\}$, so that we have

$$N^U(F, \phi) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^\varepsilon \int_{-1}^1 \phi(t,s) \frac{t}{(t^2 + s^2)^{\frac{\alpha}{2}}} \, dt \, ds.$$

This implies

$$N^U(F, \phi) = \int_{-1}^1 \phi(t,0) \, \mathrm{sgn}(t) |t|^{1-\alpha} \, dt \quad \text{if} \quad 1 \le \alpha < 2,$$

$$N^U(F, \phi) = \mathrm{p.v.} \int_{-1}^1 \phi(t,0) \, \mathrm{sgn}(t) |t|^{1-\alpha} \, dt \quad \text{if} \quad 2 \le \alpha < 3,$$

which is not a measure, being a principal value.

Approximation of compact sets

Given $F \in \mathcal{DM}^1_{\mathrm{loc}}(\Omega)$ and $K \subset \Omega$ compact, by choosing the Lipschitz test functions

$$\varphi^\varepsilon_K(x) := \begin{cases} 1 & \text{if } x \in K \\ 1 - \frac{1}{\varepsilon}\mathrm{dist}(x,K) & \text{if } x \in K_\varepsilon \setminus K \\ 0 & \text{if } x \notin K_\varepsilon \end{cases},$$

Schuricht (2007) showed that

$$\mathrm{div} F(K) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{K_\varepsilon \setminus K} F \cdot \nu^d_K \, dx, \tag{19}$$

where $K_\varepsilon = \{x \in \Omega : \mathrm{dist}(x,K) \leq \varepsilon\}$ and $\nu^d_K(x) = \nabla \mathrm{dist}(x,K)$ is a unit vector for $\mathscr{L}^n$-a.e. $x \in \Omega \setminus K$.

Smooth approximations of the normal trace

For any open set U, we define the *signed distance* from ∂U to be $d(x) = \mathrm{dist}(x, \mathbb{R}^n \setminus U) - \mathrm{dist}(x, U)$, and, for any $\varepsilon > 0$,

$$U^\varepsilon := \{x \in \mathbb{R}^n : d(x) > \varepsilon\}, \quad U_\varepsilon := \{x \in \mathbb{R}^n : d(x) > -\varepsilon\}.$$

Theorem (Chen-C.-Torres, 2017)

Let $U \Subset \Omega$ be an open set and $F \in \mathcal{DM}^p(\Omega)$. Then, for any $\phi \in C^0(\Omega) \cap L^\infty(\Omega)$ with $\nabla\phi \in L^{p'}(\Omega; \mathbb{R}^n)$, there exists a set $\mathcal{N}$ with $\mathcal{L}^1(\mathcal{N}) = 0$ such that, for every sequence $\{\varepsilon_k\}$ satisfying $\varepsilon_k \to 0$ and $\varepsilon_k \notin \mathcal{N}$, we have

$$\int_U \phi \, d\mathrm{div} F + \int_U F \cdot \nabla\phi \, dx = -\lim_{k\to+\infty} \int_{\partial^* U^{\varepsilon_k}} \phi F \cdot \nu_{U^{\varepsilon_k}} \, d\mathscr{H}^{n-1}, \tag{20}$$

$$\int_{\overline{U}} \phi \, d\mathrm{div} F + \int_{\overline{U}} F \cdot \nabla\phi \, dx = -\lim_{k\to+\infty} \int_{\partial^* U_{\varepsilon_k}} \phi F \cdot \nu_{U_{\varepsilon_k}} \, d\mathscr{H}^{n-1}, \tag{21}$$

In particular, if U has C^0 boundary, then one can obtain an analogous result using the regularized distance ρ (Lieberman, 1985,Ball-Zarnescu, 2017) and the smooth sets

$$U^{\varepsilon,\rho} := \{x \in \mathbb{R}^n : \rho(x) > \varepsilon\}, \quad U_{\varepsilon,\rho} := \{x \in \mathbb{R}^n : \rho(x) > -\varepsilon\}.$$

Lipschitz implies Lipschitz deformable

Theorem (Nečas, 1962, Verchota, 1982)

Let U be a bounded Lipschitz domain. Then the following statements hold:

1. *there exists a sequence of open sets U_k satisfying ∂U_k is of class C^∞, $U_k \Subset U_{k+1} \Subset U$ and $\bigcup_k U_k = U$;*
2. *there exists a covering of ∂U by coordinate cylinders such that, for any coordinate pair (Z, φ), with $\varphi \in \mathrm{Lip}_c(\mathbb{R}^{n-1})$, $Z^* \cap \partial U_k$ is the graph of a function $\varphi_k \in C_c^\infty(\mathbb{R}^{n-1})$ satisfying $\varphi_k \to \varphi$ uniformly, $\|\nabla \varphi_k\|_{L^\infty(\mathbb{R}^{n-1};\mathbb{R}^{n-1})} \leq \|\nabla \varphi\|_{L^\infty(\mathbb{R}^{n-1};\mathbb{R}^{n-1})}$, $\nabla \varphi_k \to \nabla \varphi$ $\mathscr{L}^{n-1}$-a.e. and in $L^q(\mathbb{R}^{n-1};\mathbb{R}^{n-1})$ for any $1 \leq q < \infty$;*
3. *there exists a sequence of Lipschitz diffeomorphisms $f_k : \mathbb{R}^n \to \mathbb{R}^n$ such that $f_k(\partial U) = \partial U_k$, the Lipschitz constants are uniformly bounded in k, $f_k \to \mathrm{Id}$ uniformly on ∂U and $J^{\partial\Omega} f_k \to 1$ in $L^q(\partial U; \mathscr{H}^{n-1})$, for any $1 \leq q < \infty$.*

Therefore, there exists a regular Lipschitz deformation for U: it is enough to set

$$\Psi(x,t) := (k+1-k(k+1)t) f_{k+1}(x) + (k(k+1)t - k) f_k(x) \quad \text{if } t \in \left(\frac{1}{k+1}, \frac{1}{k}\right].$$

Thank you for your attention!