Overview on Convenient Calculus and Differential Geometry in Infinite dimensions, with Applications to Diffeomorphism Groups and Shape Spaces, or Geodesic evolution equations on shape spaces and diffeormorphism groups

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Peter W. Michor
University of Vienna, Austria

ERC School on Geometric Evolution Problems
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Based on collaborations with: M. Bauer, M. Bruveris, P. Harms, D. Mumford

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A short introduction to convenient calculus in infinite dimensions.

Traditional differential calculus works well for finite dimensional vector spaces and for Banach spaces.
Beyond Banach spaces, the main difficulty is that composition of linear mappings stops to be jointly continuous at the level of Banach spaces, for any compatible topology.
For more general locally convex spaces we sketch here the convenient approach as explained in [Frölicher-Kriegl 1988] and [Kriegl-Michor 1997].

The c^∞-topology

Let E be a locally convex vector space. A curve $c : \mathbb{R} \to E$ is called *smooth* or C^∞ if all derivatives exist and are continuous. Let $C^\infty(\mathbb{R}, E)$ be the space of smooth functions. It can be shown that the set $C^\infty(\mathbb{R}, E)$ does not depend on the locally convex topology of E, only on its associated bornology (system of bounded sets).
The final topologies with respect to the following sets of mappings into E coincide:

1. $C^\infty(\mathbb{R}, E)$.
2. The set of all Lipschitz curves (so that $\{\frac{c(t)-c(s)}{t-s} : t \neq s, |t|, |s| \leq C\}$ is bounded in E, for each C).
3. The set of injections $E_B \to E$ where B runs through all bounded absolutely convex subsets in E, and where E_B is the linear span of B equipped with the Minkowski functional $\|x\|_B := \inf\{\lambda > 0 : x \in \lambda B\}$.
4. The set of all Mackey-convergent sequences $x_n \to x$ (there exists a sequence $0 < \lambda_n \nearrow \infty$ with $\lambda_n(x_n - x)$ bounded).

The c^∞-topology. II

This topology is called the c^∞-topology on E and we write $c^\infty E$ for the resulting topological space.

In general (on the space $\mathcal{D}$ of test functions for example) it is finer than the given locally convex topology, it is not a vector space topology, since scalar multiplication is no longer jointly continuous. The finest among all locally convex topologies on E which are coarser than $c^\infty E$ is the bornologification of the given locally convex topology. If E is a Fréchet space, then $c^\infty E = E$.

Convenient vector spaces

A locally convex vector space E is said to be a *convenient vector space* if one of the following holds (called c^∞-completeness):

1. For any $c \in C^\infty(\mathbb{R}, E)$ the (Riemann-) integral $\int_0^1 c(t)dt$ exists in E.
2. Any Lipschitz curve in E is locally Riemann integrable.
3. A curve $c : \mathbb{R} \to E$ is C^∞ if and only if $\lambda \circ c$ is C^∞ for all $\lambda \in E^*$, where E^* is the dual of all cont. lin. funct. on E.
   - Equiv., for all $\lambda \in E'$, the dual of all bounded lin. functionals.
   - Equiv., for all $\lambda \in \mathcal{V}$, where $\mathcal{V}$ is a subset of E' which recognizes bounded subsets in E.

   We call this *scalarwise* C^∞.
4. Any Mackey-Cauchy-sequence (i. e. $t_{nm}(x_n - x_m) \to 0$ for some $t_{nm} \to \infty$ in $\mathbb{R}$) converges in E. This is visibly a mild completeness requirement.

Convenient vector spaces. II

5. If B is bounded closed absolutely convex, then E_B is a Banach space.
6. If $f : \mathbb{R} \to E$ is scalarwise Lip^k, then f is Lip^k, for $k > 1$.
7. If $f : \mathbb{R} \to E$ is scalarwise C^∞ then f is differentiable at 0.

Here a mapping $f : \mathbb{R} \to E$ is called Lip^k if all derivatives up to order k exist and are Lipschitz, locally on $\mathbb{R}$. That f is scalarwise C^∞ means $\lambda \circ f$ is C^∞ for all continuous (equiv., bounded) linear functionals on E.

Smooth mappings

Let E*, and* F *be convenient vector spaces, and let* $U \subset E$ *be* c^∞*-open. A mapping* $f : U \to F$ *is called smooth or* C^∞*, if* $f \circ c \in C^\infty(\mathbb{R}, F)$ *for all* $c \in C^\infty(\mathbb{R}, U)$*.*
If E is a Fréchet space, then this notion coincides with all other reasonable notions of C^∞-mappings. Beyond Fréchet mappings, as a rule, there are more smooth mappings in the convenient setting than in other settings, e.g., C_c^∞.

Main properties of smooth calculus

1. For maps on Fréchet spaces this coincides with all other reasonable definitions. On $\mathbb{R}^2$ this is non-trivial [Boman,1967].
2. Multilinear mappings are smooth iff they are bounded.
3. If $E \supseteq U \xrightarrow{f} F$ is smooth then the derivative $df : U \times E \to F$ is smooth, and also $df : U \to L(E, F)$ is smooth where $L(E, F)$ denotes the space of all bounded linear mappings with the topology of uniform convergence on bounded subsets.
4. The chain rule holds.
5. The space $C^\infty(U, F)$ is again a convenient vector space where the structure is given by the obvious injection

$$C^\infty(U, F) \xrightarrow{C^\infty(c,\ell)} \prod_{c \in C^\infty(\mathbb{R}, U), \ell \in F^*} C^\infty(\mathbb{R}, \mathbb{R}), \quad f \mapsto (\ell \circ f \circ c)_{c,\ell},$$

where $C^\infty(\mathbb{R}, \mathbb{R})$ carries the topology of compact convergence in each derivative separately.

Main properties of smooth calculus, II

6. The exponential law holds: For c^∞-open $V \subset F$,

   $$C^\infty(U, C^\infty(V, G)) \cong C^\infty(U \times V, G)$$

   is a linear diffeomorphism of convenient vector spaces.
   Note that this is the main assumption of variational calculus. Here it is a theorem.
7. A linear mapping $f : E \to C^\infty(V, G)$ is smooth (by (2) equivalent to bounded) if and only if $E \xrightarrow{f} C^\infty(V, G) \xrightarrow{\mathrm{ev}_v} G$ is smooth for each $v \in V$.
   (*Smooth uniform boundedness theorem*, see [KM97], theorem 5.26).

Main properties of smooth calculus, III

8. The following canonical mappings are smooth.

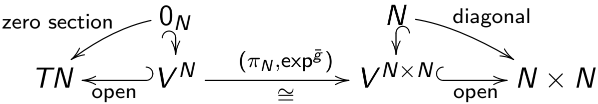
$$\begin{aligned}
&\mathrm{ev} : C^\infty(E,F)\times E \to F, \quad \mathrm{ev}(f,x) = f(x)\\
&\mathrm{ins} : E \to C^\infty(F, E\times F), \quad \mathrm{ins}(x)(y) = (x,y)\\
&(\quad)^\wedge : C^\infty(E, C^\infty(F,G)) \to C^\infty(E\times F, G)\\
&(\quad)^\vee : C^\infty(E\times F, G) \to C^\infty(E, C^\infty(F,G))\\
&\mathrm{comp} : C^\infty(F,G)\times C^\infty(E,F) \to C^\infty(E,G)\\
&C^\infty(\quad,\quad) : C^\infty(F,F_1)\times C^\infty(E_1,E) \to\\
&\qquad\qquad\qquad \to C^\infty(C^\infty(E,F), C^\infty(E_1,F_1))\\
&\qquad (f,g) \mapsto (h \mapsto f\circ h\circ g)\\
&\prod : \prod C^\infty(E_i,F_i) \to C^\infty(\prod E_i, \prod F_i)
\end{aligned}$$

This ends our review of the standard results of convenient calculus. Convenient calculus (having properties 6 and 7) exists also for:

- Real analytic mappings [Kriegl,M,1990]
- Holomorphic mappings [Kriegl,Nel,1985] (notion of [Fantappié, 1930-33])
- Many classes of Denjoy Carleman ultradifferentible functions, both of Beurling type and of Roumieu-type [Kriegl,M,Rainer, 2009, 2011, 2013]
- With some adaptations, Lip^k [Frölicher-Kriegl, 1988].
- With more adaptations, even $C^{k,\alpha}$ (k-th derivative Hölder-contin. with index α) [Faure,Frölicher 1989], [Faure, These Geneve, 1991]

Manifolds of mappings (with compact source) and diffeomorphism groups as convenient manifolds.

Let M be a compact (for simplicity's sake) fin. dim. manifold and N a manifold. We use an auxiliary Riemann metric $\bar{g}$ on N. Then



$C^\infty(M,N)$, the space of smooth mappings $M \to N$, has the following manifold structure. Chart, centered at $f \in C^\infty(M,N)$, is:

$$C^\infty(M,N) \supset U_f = \{g : (f,g)(M) \subset V^{N\times N}\} \xrightarrow{u_f} \tilde{U}_f \subset \Gamma(f^* TN)$$

$$u_f(g) = (\pi_N, \exp^{\bar{g}})^{-1} \circ (f,g), \quad u_f(g)(x) = (\exp^{\bar{g}}_{f(x)})^{-1}(g(x))$$

$$(u_f)^{-1}(s) = \exp^{\bar{g}}_f \circ s, \qquad (u_f)^{-1}(s)(x) = \exp^{\bar{g}}_{f(x)}(s(x))$$

Manifolds of mappings II

Lemma: $C^\infty(\mathbb{R}, \Gamma(M; f^*TN)) = \Gamma(\mathbb{R} \times M; {\mathrm{pr}_2}^* f^*TN)$
By Cartesian Closedness (after trivializing the bundle f^*TN).
Lemma: Chart changes are smooth (C^∞)
$\tilde{U}_{f_1} \ni s \mapsto (\pi_N, \exp^{\bar{g}}) \circ s \mapsto (\pi_N, \exp^{\bar{g}})^{-1} \circ (f_2, \exp^{\bar{g}}_{f_1} \circ s)$
since they map smooth curves to smooth curves.
Lemma: $C^\infty(\mathbb{R}, C^\infty(M, N)) \cong C^\infty(\mathbb{R} \times M, N)$.
By the first lemma.
Lemma: Composition $C^\infty(P, M) \times C^\infty(M, N) \to C^\infty(P, N)$, $(f, g) \mapsto g \circ f$, is smooth, since it maps smooth curves to smooth curves
Corollary (of the chart structure):
$TC^\infty(M, N) = C^\infty(M, TN) \xrightarrow{C^\infty(M, \pi_N)} C^\infty(M, N)$.

Regular Lie groups

We consider a smooth Lie group G with Lie algebra $\mathfrak{g} = T_e G$ modelled on convenient vector spaces. The notion of a regular Lie group is originally due to Omori et al. for Fréchet Lie groups, was weakened and made more transparent by Milnor, and then carried over to convenient Lie groups; see [KM97], 38.4.
A Lie group G is called *regular* if the following holds:

- For each smooth curve $X \in C^\infty(\mathbb{R}, \mathfrak{g})$ there exists a curve $g \in C^\infty(\mathbb{R}, G)$ whose right logarithmic derivative is X, i.e.,

$$\begin{cases} g(0) & = e \\ \partial_t g(t) & = T_e(\mu^{g(t)})X(t) = X(t).g(t) \end{cases}$$

  The curve g is uniquely determined by its initial value $g(0)$, if it exists.
- Put $\operatorname{evol}^r_G(X) = g(1)$ where g is the unique solution required above. Then $\operatorname{evol}^r_G : C^\infty(\mathbb{R}, \mathfrak{g}) \to G$ is required to be C^∞ also. We have $\operatorname{Evol}^X_t := g(t) = \operatorname{evol}_G(tX)$.

Diffeomorphism group of compact M

Theorem: *For each compact manifold M, the diffeomorphism group is a regular Lie group.*

Proof: $\mathrm{Diff}(M) \xrightarrow{open} C^\infty(M,M)$. Composition is smooth by restriction. Inversion is smooth: If $t \mapsto f(t, \quad)$ is a smooth curve in $\mathrm{Diff}(M)$, then $f(t, \quad)^{-1}$ satisfies the implicit equation $f(t, f(t, \quad)^{-1}(x)) = x$, so by the finite dimensional implicit function theorem, $(t,x) \mapsto f(t, \quad)^{-1}(x)$ is smooth. So inversion maps smooth curves to smooth curves, and is smooth.

Let $X(t,x)$ be a time dependent vector field on M (in $C^\infty(\mathbb{R}, \mathfrak{X}(M))$). Then $\mathrm{Fl}_s^{\partial_t \times X}(t,x) = (t+s, \mathrm{Evol}^X(t,x))$ satisfies the ODE $\quad \partial_t \mathrm{Evol}(t,x) = X(t, \mathrm{Evol}(t,x))$. If $X(s,t,x) \in C^\infty(\mathbb{R}^2, \mathfrak{X}(M))$ is a smooth curve of smooth curves in $\mathfrak{X}(M)$, then obviously the solution of the ODE depends smoothly also on the further variable s, thus evol maps smooth curves of time dependant vector fields to smooth curves of diffeomorphism. QED.

The principal bundle of embeddings

For finite dimensional manifolds M, N with M compact, $\mathrm{Emb}(M,N)$, the space of embeddings of M into N, is open in $C^\infty(M,N)$, so it is a smooth manifold. $\mathrm{Diff}(M)$ acts freely and smoothly from the right on $\mathrm{Emb}(M,N)$.

Theorem: $\mathrm{Emb}(M,N) \to \mathrm{Emb}(M,N)/\mathrm{Diff}(M)$ *is a principal fiber bundle with structure group* $\mathrm{Diff}(M)$.

Proof: Auxiliary Riem. metric $\bar g$ on N. Given $f \in \mathrm{Emb}(M,N)$, view $f(M)$ as submanifold of N. $TN|_{f(M)} = \mathrm{Nor}(f(M)) \oplus Tf(M)$.

$\mathrm{Nor}(f(M)) : \xrightarrow[\cong]{\exp^{\bar g}} W_{f(M)} \xrightarrow{p_{f(M)}} f(M)$ tubular nbhd of $f(M)$.

If $g : M \to N$ is C^1-near to f, then
$\varphi(g) := f^{-1} \circ p_{f(M)} \circ g \in \mathrm{Diff}(M)$ and
$g \circ \varphi(g)^{-1} \in \Gamma(f^* W_{f(M)}) \subset \Gamma(f^* \mathrm{Nor}(f(M)))$.
This is the required local splitting. QED

The orbifold bundle of immersions

$\mathrm{Imm}(M, N)$, the space of immersions $M \to N$, is open in $C^\infty(M, N)$, and is thus a smooth manifold. The regular Lie group $\mathrm{Diff}(M)$ acts smoothly from the right, but no longer freely.

Theorem: [Cervera,Mascaro,M,1991] *For an immersion $f : M \to N$, the isotropy group $\mathrm{Diff}(M)_f = \{\varphi \in \mathrm{Diff}(M) : f \circ \phi = f\}$ is always a finite group, acting freely on M; so $M \xrightarrow{p} M/\mathrm{Diff}(M)_f$ is a convering of manifold and f factors to $f = \bar{f} \circ p$.*

Thus $\mathrm{Imm}(M, N) \to \mathrm{Imm}(M, N)/Diff(M)$ is a projection onto an honest infinite dimensional orbifold.

A Zoo of diffeomorphism groups on $\mathbb{R}^n$

Theorem. *The following groups of diffeomorphisms on $\mathbb{R}^n$ are regular Lie groups:*

- $\mathrm{Diff}_{\mathcal{B}}(\mathbb{R}^n)$, *the group of all diffeomorphisms which differ from the identity by a function which is bounded together with all derivatives separately.*
- $\mathrm{Diff}_{H^\infty}(\mathbb{R}^n)$, *the group of all diffeomorphisms which differ from the identity by a function in the intersection H^∞ of all Sobolev spaces H^k for $k \in \mathbb{N}_{\geq 0}$.*
- $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n)$, *the group of all diffeomorphisms which fall rapidly to the identity.*
- $\mathrm{Diff}_c(\mathbb{R}^n)$ *of all diffeomorphisms which differ from the identity only on a compact subset. (well known since 1980)*

[M,Mumford,2013], partly [B.Walter,2012]; for Denjoy-Carleman ultradifferentiable diffeomorphisms [Kriegl, M, Rainer 2014].
In particular, $\mathrm{Diff}_{H^\infty}(\mathbb{R}^n)$ is essential if one wants to prove that the geodesic equation of a right Riemannian invariant metric is well-posed with the use of Sobolov space techniques.

We need more on convenient calculus.
[FK88], theorem 4.1.19.

Theorem. *Let $c : \mathbb{R} \to E$ be a curve in a convenient vector space E. Let $\mathcal{V} \subset E'$ be a subset of bounded linear functionals such that the bornology of E has a basis of $\sigma(E, \mathcal{V})$-closed sets. Then the following are equivalent:*

1. *c is smooth*
2. *There exist locally bounded curves $c^k : \mathbb{R} \to E$ such that $\ell \circ c$ is smooth $\mathbb{R} \to \mathbb{R}$ with $(\ell \circ c)^{(k)} = \ell \circ c^k$, for each $\ell \in \mathcal{V}$.*

If E is reflexive, then for any point separating subset $\mathcal{V} \subset E'$ the bornology of E has a basis of $\sigma(E, \mathcal{V})$-closed subsets, by [FK88], 4.1.23.

Faá di Bruno formula.
Let $g \in C^\infty(\mathbb{R}^n, \mathbb{R}^k)$ and let $f \in C^\infty(\mathbb{R}^k, \mathbb{R}^l)$. Then the p-th deivative of $f \circ g$ looks as follows where sym_p denotes symmetrization of a p-linear mapping:

$$\frac{d^p(f \circ g)(x)}{p!} = \\ = \text{sym}_p \Big(\sum_{j=1}^{p} \sum_{\substack{\alpha \in \mathbb{N}^j_{>0} \\ \alpha_1 + \dots + \alpha_j = p}} \frac{d^j f(g(x))}{j!} \Big(\frac{d^{\alpha_1} g(x)}{\alpha_1 !}, \dots, \frac{d^{\alpha_j} g(x)}{\alpha_j !} \Big) \Big)$$

The one dimensional version is due to [Faá di Bruno 1855], the only beatified mathematician.

Groups of smooth diffeomorphisms in the zoo

If we consider the group of all orientation preserving diffeomorphisms $\mathrm{Diff}(\mathbb{R}^n)$ of $\mathbb{R}^n$, it is not an open subset of $C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ with the compact C^∞-topology. So it is not a smooth manifold in the usual sense, but we may consider it as a Lie group in the cartesian closed category of Frölicher spaces, see [KM97], section 23, with the structure induced by the injection $f \mapsto (f, f^{-1}) \in C^\infty(\mathbb{R}^n, \mathbb{R}^n) \times C^\infty(\mathbb{R}^n, \mathbb{R}^)$.
We shall now describe regular Lie groups in $\mathrm{Diff}(\mathbb{R}^n)$ which are given by diffeomorphisms of the form $f = \mathrm{Id}_\mathbb{R} + g$ where g is in some specific convenient vector spaces of bounded functions in $C^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Now we discuss these spaces on $\mathbb{R}^n$, we describe the smooth curves in them, and we describe the corresponding groups.

The group $\mathrm{Diff}_{\mathcal{B}}(\mathbb{R}^n)$ in the zoo

The space $\mathcal{B}(\mathbb{R}^n)$ (called $\mathcal{D}_{L^\infty}(\mathbb{R}^n)$ by [L.Schwartz 1966]) consists of all smooth functions which have all derivatives (separately) bounded. It is a Fréchet space. By [Vogt 1983], the space $\mathcal{B}(\mathbb{R}^n)$ is linearly isomorphic to $\ell^\infty \hat{\otimes} \mathfrak{s}$ for any completed tensor-product between the projective one and the injective one, where $\mathfrak{s}$ is the nuclear Fréchet space of rapidly decreasing real sequences. Thus $\mathcal{B}(\mathbb{R}^n)$ is not reflexive, not nuclear, not smoothly paracompact.
The space $C^\infty(\mathbb{R}, \mathcal{B}(\mathbb{R}^n))$ of smooth curves in $\mathcal{B}(\mathbb{R}^n)$ consists of all functions $c \in C^\infty(\mathbb{R}^{n+1}, \mathbb{R})$ satisfying the following property:

- *For all $k \in \mathbb{N}_{\geq 0}$, $\alpha \in \mathbb{N}^n_{\geq 0}$ and each $t \in \mathbb{R}$ the expression $\partial_t^k \partial_x^\alpha c(t, x)$ is uniformly bounded in $x \in \mathbb{R}^n$, locally in t.*

To see this use thm FK for the set $\{\mathrm{ev}_x : x \in \mathbb{R}\}$ of point evaluations in $\mathcal{B}(\mathbb{R}^n)$. Here $\partial_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x^\alpha}$ and $c^k(t) = \partial_t^k f(t, \quad)$.
$\mathrm{Diff}^+_{\mathcal{B}}(\mathbb{R}^n) = \big\{ f = \mathrm{Id} + g : g \in \mathcal{B}(\mathbb{R}^n)^n, \det(\mathbb{I}_n + dg) \geq \varepsilon > 0 \big\}$ *denotes the corresponding group*, see below.

The group $\mathrm{Diff}_{H^\infty}(\mathbb{R}^n)$ in the zoo

The space $H^\infty(\mathbb{R}^n) = \bigcap_{k\geq 1} H^k(\mathbb{R}^n)$ is the intersection of all Sobolev spaces which is a reflexive Fréchet space. It is called $\mathcal{D}_{L^2}(\mathbb{R}^n)$ in [L.Schwartz 1966]. By [Vogt 1983], the space $H^\infty(\mathbb{R}^n)$ is linearly isomorphic to $\ell^2 \hat{\otimes}\, \mathfrak{s}$. Thus it is not nuclear, not Schwartz, not Montel, but still smoothly paracompact.
The space $C^\infty(\mathbb{R}, H^\infty(\mathbb{R}^n))$ of smooth curves in $H^\infty(\mathbb{R}^n)$ consists of all functions $c \in C^\infty(\mathbb{R}^{n+1}, \mathbb{R})$ satisfying the following property:

- *For all $k \in \mathbb{N}_{\geq 0}$, $\alpha \in \mathbb{N}^n_{\geq 0}$ the expression $\|\partial_t^k \partial_x^\alpha f(t, \quad)\|_{L^2(\mathbb{R}^n)}$ is locally bounded near each $t \in \mathbb{R}$.*

The proof is literally the same as for $\mathcal{B}(\mathbb{R}^n)$, noting that the point evaluations are continuous on each Sobolev space H^k with $k > \frac{n}{2}$. $\mathrm{Diff}^+_{H^\infty}(\mathbb{R}) = \{f = \mathrm{Id} + g : g \in H^\infty(\mathbb{R}), \det(\mathbb{I}_n + dg) > 0\}$ *denotes the correponding group.*

The group $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n)$ in the zoo

The algebra $\mathcal{S}(R^n)$ of rapidly decreasing functions is a reflexive nuclear Fréchet space.

The space $C^\infty(\mathbb{R}, \mathcal{S}(\mathbb{R}^n))$ of smooth curves in $\mathcal{S}(\mathbb{R}^n)$ consists of all functions $c \in C^\infty(\mathbb{R}^{n+1}, \mathbb{R})$ satisfying the following property:

- *For all $k, m \in \mathbb{N}_{\geq 0}$ and $\alpha \in \mathbb{N}^n_{\geq 0}$, the expression $(1 + |x|^2)^m \partial_t^k \partial_x^\alpha c(t, x)$ is uniformly bounded in $x \in \mathbb{R}^n$, locally uniformly bounded in $t \in \mathbb{R}$.*

$\mathrm{Diff}^+_{\mathcal{S}}(\mathbb{R}^n) = \left\{ f = \mathrm{Id} + g : g \in \mathcal{S}(\mathbb{R}^n)^n, \det(\mathbb{I}_n + dg) > 0 \right\}$ *is the correponding group.*

The group $\mathrm{Diff}_c(\mathbb{R}^n)$ in the zoo

The algebra $C_c^\infty(\mathbb{R}^n)$ of all smooth functions with compact support is a nuclear (LF)-space.

The space $C^\infty(\mathbb{R}, C_c^\infty(\mathbb{R}^n))$ of smooth curves in $C_c^\infty(\mathbb{R}^n)$ consists of all functions $f \in C^\infty(\mathbb{R}^{n+1}, \mathbb{R})$ satisfying the following property:

- *For each compact interval $[a, b]$ in $\mathbb{R}$ there exists a compact subset $K \subset \mathbb{R}^n$ such that $f(t, x) = 0$ for $(t, x) \in [a, b] \times (\mathbb{R}^n \setminus K)$.*

$\mathrm{Diff}_c(\mathbb{R}^n) = \{f = \mathrm{Id} + g : g \in C_c^\infty(\mathbb{R}^n)^n, \det(\mathbb{I}_n + dg) > 0\}$ *is the correponding group.*

Ideal properties of function spaces in the zoo

The function spaces are boundedly mapped into each other as follows:

$$C_c^\infty(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n) \longrightarrow H^\infty(\mathbb{R}^n) \longrightarrow \mathcal{B}(\mathbb{R}^n)$$

and each space is a bounded locally convex algebra and a bounded $\mathcal{B}(\mathbb{R}^n)$-module. Thus each space is an ideal in each larger space.

Theorem. *The sets of diffeomorphisms* $\mathrm{Diff}_c(\mathbb{R}^n)$, $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n)$, $\mathrm{Diff}_{H^\infty}(\mathbb{R}^n)$, *and* $\mathrm{Diff}_{\mathcal{B}}(\mathbb{R}^n)$ *are all smooth regular Lie groups. We have the following smooth injective group homomorphisms*

$$\mathrm{Diff}_c(\mathbb{R}^n) \longrightarrow \mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n) \longrightarrow \mathrm{Diff}_{H^\infty}(\mathbb{R}^n) \longrightarrow \mathrm{Diff}_{\mathcal{B}}(\mathbb{R}^n) \ .$$

Each group is a normal subgroup in any other in which it is contained, in particular in $\mathrm{Diff}_{\mathcal{B}}(\mathbb{R}^n)$.

Corollary. $\mathrm{Diff}_{\mathcal{B}}(\mathbb{R}^n)$ *acts on* Γ_c, $\Gamma_{\mathcal{S}}$ *and* Γ_{H^∞} *of any tensorbundle over* $\mathbb{R}^n$ *by pullback. The infinitesimal action of the Lie algebra* $\mathfrak{X}_{\mathcal{B}}(\mathbb{R}^n)$ *on these spaces by the Lie derivative thus maps each of these spaces into itself. A fortiori,* $\mathrm{Diff}_{H^\infty}(\mathbb{R}^n)$ *acts on* $\Gamma_{\mathcal{S}}$ *of any tensor bundle by pullback.*

Proof of the main zoo theorem

Let $\mathcal{A}$ denote any of $\mathcal{B}$, H^∞, $\mathcal{S}$, or c, and let $\mathcal{A}(\mathbb{R}^n)$ denote the corresponding function space. Let $f(x) = x + g(x)$ for $g \in \mathcal{A}(\mathbb{R}^n)^n$ with $\det(\mathbb{I}_n + dg) > 0$ and for $x \in \mathbb{R}^n$.

Each such f is a diffeomorphism. By the inverse function theorem f is a locally a diffeomorphism everywhere. Thus the image of f is open in $\mathbb{R}^n$. We claim that it is also closed. So let $x_i \in \mathbb{R}^n$ with $f(x_i) = x_i + g(x_i) \to y_0$ in $\mathbb{R}^n$. Then $f(x_i)$ is a bounded sequence. Since $g \in \mathcal{A}(\mathbb{R}^n) \subset \mathcal{B}(\mathbb{R}^n)$, the x_i also form a bounded sequence, thus contain a convergent subsequence. Without loss let $x_i \to x_0$ in $\mathbb{R}^n$. Then $f(x_i) \to f(x_0) = y_0$. Thus f is surjective. This also shows that f is a proper mapping (i.e., compact sets have compact inverse images under f). A proper surjective submersion is the projection of a smooth fiber bundle. In our case here f has discrete fibers, so f is a covering mapping and a diffeomorphism since $\mathbb{R}^n$ is simply connected.

$\mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^n)_0$ is closed under composition.

$$((\mathrm{Id}+f)\circ(\mathrm{Id}+g))(x) = x + g(x) + f(x+g(x))$$

We have to check that $x \mapsto f(x+g(x))$ is in $\mathcal{A}(\mathbb{R}^n)$ if $f, g \in \mathcal{A}(\mathbb{R}^n)^n$. For $\mathcal{A} = \mathcal{B}$ this follows by the Faà di Bruno formula. For $\mathcal{A} = \mathcal{S}$ or $\mathcal{S}_1$ we need furthermore: $(\partial_x^\alpha f)(x+g(x)) = O\Big(\frac{1}{(1+|x+g(x)|^2)^k}\Big) = O\Big(\frac{1}{(1+|x|^2)^k}\Big)$ which holds since $\frac{1+|x|^2}{1+|x+g(x)|^2}$ is globally bounded.
For $\mathcal{A} = H^\infty$ we also need that

$$\int_{\mathbb{R}^n} |(\partial_x^\alpha f)(x+g(x))|^2\,dx = \tag{3}$$
$$\int_{\mathbb{R}^n} |(\partial^\alpha f)(y)|^2 \frac{dy}{|\det(\mathbb{I}_n + dg)((\mathrm{Id}+g)^{-1}(y))|} \le C(g)\int_{\mathbb{R}^n} |(\partial^\alpha f)(y)|^2 dy;$$

this holds since the denominator is globally bounded away from 0 since g and dg vanish at ∞ by the lemma of Riemann-Lebesque. The case $\mathcal{A}(\mathbb{R}^n) = C_c^\infty(\mathbb{R}^n)$ or $C_{c,1}^\infty(\mathbb{R}^n)$ is easy and well known.

Multiplication is smooth on $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^n)$.

Suppose that the curves $t \mapsto \mathrm{Id} + f(t, \quad)$ and $t \mapsto \mathrm{Id} + g(t, \quad)$ are in $C^\infty(\mathbb{R}, \mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^n))$ which means that the functions $f, g \in C^\infty(\mathbb{R}^{n+1}, \mathbb{R}^n)$ satisfy condition $\mathcal{A}$.
We have to check that $f(t, x + g(t, x))$ also satisfies condition $\mathcal{A}$.
For this we reread the proof that composition preserves $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^n)$ and pay attention to the further parameter t.

The inverse $(\mathrm{Id}+g)^{-1}$ is again an element in $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^n)$

For $g \in \mathcal{A}(\mathbb{R}^n)^n$ we write $(\mathrm{Id}+g)^{-1} = \mathrm{Id}+f$.
We have to check that $f \in \mathcal{A}(\mathbb{R}^n)^n$.

$$(\mathrm{Id}+f) \circ (\mathrm{Id}+g) = \mathrm{Id} \implies x + g(x) + f(x+g(x)) = x$$
$$\implies x \mapsto f(x+g(x)) = -g(x) \text{ is in } \mathcal{A}(\mathbb{R}^n)^n.$$

First the case $\mathcal{A} = \mathcal{B}$. We know already that $\mathrm{Id}+g$ is a diffeomorphism. By definition, we have $\det(\mathbb{I}_n + dg(x)) \geq \varepsilon > 0$ for some ε. This implies that

$$\|(\mathbb{I}_n + dg(x))^{-1}\|_{L(\mathbb{R}^n,\mathbb{R}^n)} \quad \text{is globally bounded,}$$

using that $\|A^{-1}\| \leq \frac{\|A\|^{n-1}}{|\det(A)|}$ for any linear $A : \mathbb{R}^n \to \mathbb{R}^n$. Moreover,

$$(\mathbb{I}_n+df(x+g(x)))(\mathbb{I}_n+dg(x)) = \mathbb{I}_n \implies \det(\mathbb{I}_n+df(x+g(x))) =$$
$$= \det(\mathbb{I}_n + dg(x))^{-1} \geq \|\mathbb{I}_n + dg(x)\|^{-n} \geq \eta > 0 \text{ for all } x.$$

For higher derivatives we write the Faa di Bruno formula as:

$$\frac{d^p(f\circ(\mathrm{Id}+g))(x)}{p!} = \mathrm{sym}_p\Big(\sum_{j=1}^{p}\sum_{\substack{\alpha\in\mathbb{N}^j_{>0}\\ \alpha_1+\dots+\alpha_j=p}} \frac{d^j f(x+g(x))}{j!}\Big(\frac{d^{\alpha_1}(\mathrm{Id}+g)(x)}{\alpha_1!},\dots,\frac{d^{\alpha_j}(\mathrm{Id}+g)(x)}{\alpha_j!}\Big)\Big)$$

$$= \frac{d^p f(x+g(x))}{p!}\Big(\mathrm{Id}+dg(x),\dots,\mathrm{Id}+dg(x)\Big) \qquad \text{(top extra)}$$

$$+\,\mathrm{sym}_{p-1}\Big(\sum_{j=1}^{p-1}\sum_{\substack{\alpha\in\mathbb{N}^j_{>0}\\ \alpha_1+\dots+\alpha_j=p\\ (h_{\alpha_1},\dots,h_{\alpha_j})}} \frac{d^j f(x+g(x))}{j!}\Big(\frac{d^{\alpha_1}h_{\alpha_1}(x)}{\alpha_1!},\dots,\frac{d^{\alpha_j}h_{\alpha_j}(x)}{\alpha_j!}\Big)\Big)$$

where $h_{\alpha_i}(x)$ is $g(x)$ for $\alpha_i > 1$ (there is always such an i), and where $h_{\alpha_i}(x) = x$ or $g(x)$ if $\alpha_i = 1$.

Now we argue as follows:
The left hand side is globally bounded. We already know that $\mathrm{Id} + dg(x) : \mathbb{R}^n \to \mathbb{R}^n$ is invertible with $\|(\mathbb{I}_n + dg(x))^{-1}\|_{L(\mathbb{R}^n,\mathbb{R}^n)}$ globally bounded.
Thus we can conclude by induction on p that $d^p f(x + g(x))$ is bounded uniformly in x, thus also uniformy in $y = x + g(x) \in \mathbb{R}$.
For general $\mathcal{A}$ we note that the left hand side is in $\mathcal{A}$. Since we already know that $f \in \mathcal{B}$, and since $\mathcal{A}$ is a $\mathcal{B}$-module, the last term is in $\mathcal{A}$. Thus also the first term is in $\mathcal{A}$, and any summand there containing just one $dg(x)$ is in $\mathcal{A}$, so the unique summand $d^p f(x, g(x))$ is also in $\mathcal{A}$. Thus inversion maps $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$ into itself.

Inversion is smooth on $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^n)$.

We retrace the proof that inversion preserves $\mathrm{Diff}_{\mathcal{A}}$ assuming that $g(t, x)$ satisfies condition $\mathcal{A}$.
We see again that $f(t, x + g(t, x)) = -g(t, x)$ satisfies condition $\mathcal{A}$ as a function of t, x, and we claim that f then does the same.
We reread the proof paying attention to the parameter t and see that condition $\mathcal{A}$ is satisfied.

$\mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^n)$ is a regular Lie group

So let $t \mapsto X(t, \quad)$ be a smooth curve in the Lie algebra $\mathfrak{X}_{\mathcal{A}}(\mathbb{R}^n) = \mathcal{A}(\mathbb{R}^n)^n$, i.e., X satisfies condition $\mathcal{A}$.
The evolution of this time dependent vector field is the function given by the ODE

$$\mathrm{Evol}(X)(t,x) = x + f(t,x),$$
$$\begin{cases} \partial_t(x + f(t,x)) = f_t(t,x) = X(t, x + f(t,x)), \\ f(0,x) = 0. \end{cases} \tag{7}$$

We have to show
first that $f(t, \quad) \in \mathcal{A}(\mathbb{R}^n)^n$ for each $t \in \mathbb{R}$,
second that it is smooth in t with values in $\mathcal{A}(\mathbb{R}^n)^n$, and
third that $X \mapsto f$ is also smooth.

For $0 \leq t \leq C$ we consider

$$|f(t,x)| \leq \int_0^t |f_t(s,x)| ds = \int_0^t |X(s, x + f(s,x))| \, ds. \qquad (8)$$

Since $\mathcal{A} \subseteq \mathcal{B}$, the vector field $X(t,y)$ is uniformly bounded in $y \in \mathbb{R}^n$, locally in t. So the same is true for $f(t,x)$ by (7).

Next consider

$$
\begin{aligned}
\partial_t d_x f(t,x) &= d_x(X(t, x_f(t,x))) \qquad (9)\\
&= (d_x X)(t, x + f(t,x)) + (d_x X)(t, x + f(t,x)).d_x f(t,x)\\
\|d_x f(t,x)\| &\le \int_0^t \|(d_x X)(s, x + f(s,x))\| ds\\
&\quad + \int_0^t \|(d_x X)(s, x + f(s,x))\|.\|d_x f(s,x)\| ds\\
&=: \alpha(t,x) + \int_0^t \beta(s,x).\|d_x f(s,x)\| ds
\end{aligned}
$$

By the Bellman-Grönwall inequality,

$$\|d_x f(t,x)\| \le \alpha(t,x) + \int_0^t \alpha(s,x).\beta(s,x).e^{\int_s^t \beta(\sigma,x)\, d\sigma}\, ds,$$

which is globally bounded in x, locally in t.

For higher derivatives in x (where $p > 1$) we use Faá di Bruno as

$$\partial_t d_x^p f(t,x) = d_x^p(X(t,x+f(t,x))) = \mathrm{sym}_p\Big(\sum_{j=1}^{p} \sum_{\substack{\alpha\in\mathbb{N}_{>0}^j\\ \alpha_1+\cdots+\alpha_j=p}}$$

$$\frac{(d_x^j X)(t,x+f(t,x))}{j!}\Big(\frac{d_x^{\alpha_1}(x+f(t,x))}{\alpha_1!},\dots,\frac{d_x^{\alpha_j}(x+f(t,x))}{\alpha_j!}\Big)\Big)$$

$$= (d_x X)(t,x+f(t,x))\big(d_x^p f(t,x)\big) + \qquad (\textcolor{red}{\text{bottom extra}})$$

$$+\,\mathrm{sym}_p\Big(\sum_{j=2}^{p} \sum_{\substack{\alpha\in\mathbb{N}_{>0}^j\\ \alpha_1+\cdots+\alpha_j=p}}$$

$$\frac{(d_x^j X)(t,x+f(t,x))}{j!}\Big(\frac{d_x^{\alpha_1}(x+f(t,x))}{\alpha_1!},\dots,\frac{d_x^{\alpha_j}(x+f(t,x))}{\alpha_j!}\Big)\Big)$$

We can assume recursively that $d_x^j f(t,x)$ is globally bounded in x, locally in t, for $j < p$. Then we have reproduced the situation of (9) (with values in the space of symmetric p-linear mappings $(\mathbb{R}^n)^p \to \mathbb{R}^n$) and we can repeat the argument above involving the Bellman-Grönwall inequality to conclude that $d_x^p f(t,x)$ is globally bounded in x, locally in t.
To conclude the same for $\partial_t^m d_x^p f(t,x)$ we just repeat the last arguments for $\partial_t^m f(t,x)$. So we have now proved that $f \in C^\infty(\mathbb{R}, \mathfrak{X}_{\mathcal{B}}(\mathbb{R}^n))$.
To prove that $C^\infty(\mathbb{R}, \mathfrak{X}_{\mathcal{B}}(\mathbb{R}^n)) \ni X \mapsto \mathrm{Evol}(X)(1, \quad) \in \mathrm{Diff}_{\mathcal{B}}(\mathbb{R}^n)$ is smooth, we consider a smooth curve X in $C^\infty(\mathbb{R}, \mathfrak{X}_{\mathcal{B}}(\mathbb{R}^n))$; thus $X(t_1, t_2, x)$ is smooth on $\mathbb{R}^2 \times \mathbb{R}^n$, globally bounded in x in each derivative separately, locally in $t = (t_1, t_2)$ in each derivative. Or, we assume that t is 2-dimensional in the argument above. But then it suffices to show that $(t_1, t_2) \mapsto X(t_1, t_2, \quad) \in \mathfrak{X}_{\mathcal{B}}(\mathbb{R}^n)$ is smooth along smooth curves in $\mathbb{R}^2$, and we are again in the situation we have just treated.
Thus $\mathrm{Diff}_{\mathcal{B}}(\mathbb{R}^n)$ is a regular Lie group.

If $\mathcal{A} = \mathcal{S}$, we already know that $f(s,x)$ is globally bounded in x, locally in t. Thus may insert $X(s, x + f(s,x)) = O(\frac{1}{(1+|x+f(s,x)|^2)^k}) = O(\frac{1}{(1+|x|^2)^k})$ into (8) and can conclude that $f(t,x) = O(\frac{1}{(1+|x|^2)^k})$ globally in x, locally in t, for each k.
Using this argument, we can repeat the proof for the case $\mathcal{A} = \mathcal{B}$ from above.
Thus $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n)$ is a regular Lie group.

If $\mathcal{A} = H^\infty$ we first consider the differential of (8),

$$\|d_x f(s,x)\| = \left\| \int_0^t d_x(X(s, \quad))(x + f(s,x)).(\mathbb{I}_n + df(s, \quad)(x))\, ds \right\|$$

$$\leq \int_0^t \left\| d_x(X(s, \quad))(x + f(s,x)) \right\| .C, ds \tag{10}$$

since $d_x f(s,x)$ is globally bounded in x, locally in s, by the case $\mathcal{A} = \mathcal{B}$. The same holds for $f(s,x)$. Moreover, $X(s, \quad)$ vanishes near infinity by the lemma of Riemann-Lebesque, so that the same holds for $f(s, \quad)$ by (10).

Now we consider

$$\int_{\mathbb{R}^n} \|(d_x^p f)(t,x)\|^2\, dx = \int_{\mathbb{R}^n} \Big\| \int_0^t d_x^p\big(X(s, \mathrm{Id} + f(s, \quad))\big)(x)\, ds \Big\|^2 dx.$$

We apply Faá di Bruno in the form (top extra) to the integrand, remember that we already know that each $d^{\alpha_i}(\mathrm{Id} + f(s, \quad))(x)$ is globally bounded, locally in s, thus the last term is

$$\leq \int_{\mathbb{R}^n} \Big(\int_0^t \sum_{j=1}^p \|(d_x^j X)(s, x + f(s,x))\|.C_j\, ds \Big)^2 dx$$

$$= \int_{\mathbb{R}^n} \Big(\int_0^t \sum_{j=1}^p \|(d_x^j X)(s, y)\|.C_j\, ds \Big)^2 \frac{dy}{|\det(\mathbb{I}_n + df(s, \quad))((\mathbb{I}_n + f(s, \quad))^{-1}(y)|}$$

which is finite since $X(s, \quad) \in H^\infty$ and since the determinand in the denominator is bounded away from zero – we just checked that $d_x f(s, \quad)$ vanishes at infinity. We repeat this for $\partial_t^m d_x^p f(t,x)$. This shows that $\mathrm{Evol}(X)(t, \quad) \in \mathrm{Diff}_{H^\infty}(\mathbb{R}^n)$ for each t. Choosing t two-dimensional (as in the case $\mathcal{A} = \mathcal{B}$) we can conclude that $\mathrm{Diff}_{H^\infty}(\mathbb{R}^n)$ is a regular Lie group.

$\mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n)$ is a normal subgroup of $\mathrm{Diff}_{\mathcal{B}}(\mathbb{R}^n)$.

So let $g \in \mathcal{B}(\mathbb{R}^n)^n$ with $\det(\mathbb{I}_n + dg(x)) \geq \varepsilon > 0$ for all x, and $s \in \mathcal{S}(\mathbb{R}^n)^n$ with $\det(\mathbb{I}_n + ds(x)) > 0$ for all x. We consider

$$\begin{aligned}
&(\mathrm{Id} + g)^{-1}(x) = x + f(x) \quad \text{for } f \in \mathcal{B}(\mathbb{R}^n)^n \\
&\qquad\qquad \iff f(x + g(x)) = -g(x) \\
&\big((\mathrm{Id} + g)^{-1} \circ (\mathrm{Id} + s) \circ (\mathrm{Id} + g)\big)(x) = \big((\mathrm{Id} + f) \circ (\mathrm{Id} + s) \circ (\mathrm{Id} + g)\big)(x) = \\
&\qquad = x + g(x) + s(x + g(x)) + f\big(x + g(x) + s(x + g(x))\big) \\
&\qquad = x + s(x + g(x)) - f(x + g(x)) + f\big(x + g(x) + s(x + g(x))\big).
\end{aligned}$$

Since $g(x)$ is globally bounded we get $s(x + g(x)) = O((1 + |x + g(x)|^{-k}) = O((1 + |x|)^{-k})$ for each k. For $d_x^p(s \circ (\mathrm{Id} + g))(x)$ this follows from Faá di Bruno in the form of (top extra).

Moreover we have

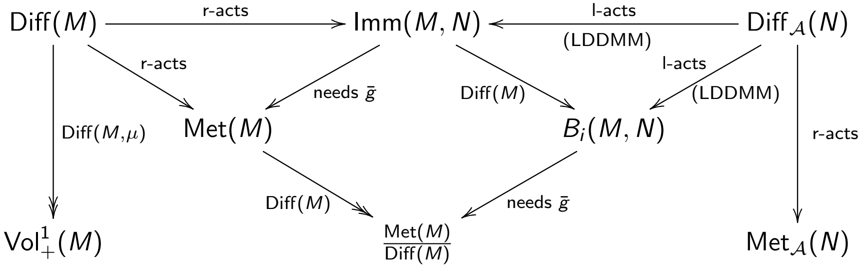
$$f\bigl(x+g(x)+s(x+g(x))\bigr)-f(x+g(x))=$$
$$=\int_0^1 df\bigl(x+g(x)+ts(x+g(x))\bigr)(s(x+g(x)))\,dt$$

which is in $\mathcal{S}(\mathbb{R}^n)^n$ as a function of x since df is in $\mathcal{B}$ and $s(x+g(x))$ is in $\mathcal{S}$.

$\mathrm{Diff}_{H^\infty}(\mathbb{R}^n)$ is a normal subgroup of $\mathrm{Diff}_{\mathcal{B}}(\mathbb{R}^n)$.

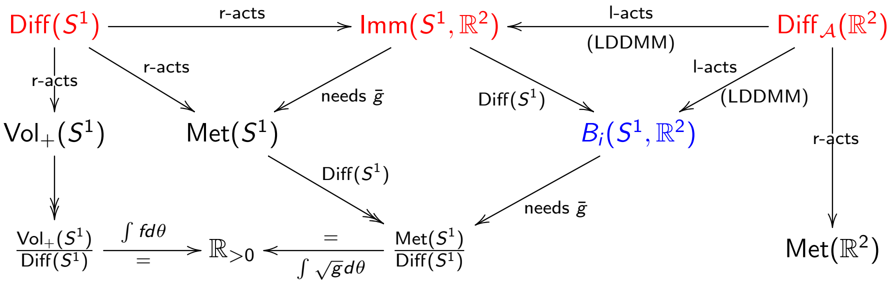
We redo the last proof under the assumption that $s \in H^\infty(\mathbb{R}^n)^n$.
By the argument in (3) we see that $s(x + g(x))$ is in H^∞ as a function of x.
The rest is as above.
This finishes the proof of the main theorem.

A diagram of actions of diffeomorphism groups.



M compact , N pssibly non-compact manifold

$\mathrm{Met}(N) = \Gamma(S^2_+ T^* N)$	space of all Riemann metrics on N
$\bar{g}$	one Riemann metric on N
$\mathrm{Diff}(M)$	Lie group of all diffeos on compact mf M
$\mathrm{Diff}_{\mathcal{A}}(N),\ \mathcal{A} \in \{H^\infty, \mathcal{S}, c\}$	Lie group of diffeos of decay $\mathcal{A}$ to Id_N
$\mathrm{Imm}(M, N)$	mf of all immersions $M \to N$
$B_i(M, N) = \mathrm{Imm}/\mathrm{Diff}(M)$	shape space
$\mathrm{Vol}^1_+(M) \subset \Gamma(\mathrm{vol}(M))$	space of positive smooth probability densities



$\text{Diff}(S^1)$	Lie group of all diffeos on compact mf S^1
$\text{Diff}_{\mathcal{A}}(\mathbb{R}^2),\ \mathcal{A} \in \{\mathcal{B}, H^\infty, \mathcal{S}, c\}$	Lie group of diffeos of decay $\mathcal{A}$ to $\text{Id}_{\mathbb{R}^2}$
$\text{Imm}(S^1, \mathbb{R}^2)$	mf of all immersions $S^1 \to \mathbb{R}^2$
$B_i(S^1, \mathbb{R}^2) = \text{Imm}/\text{Diff}(S^1)$	shape space
$\text{Vol}_+(S^1) = \left\{ f\, d\theta : f \in C^\infty(S^1, \mathbb{R}_{>0}) \right\}$	space of positive smooth probability densities
$\text{Met}(S^1) = \left\{ g\, d\theta^2 : g \in C^\infty(S^1, \mathbb{R}_{>0}) \right\}$	space of metrics on S^1

The manifold of immersions

Let M be either S^1 or $[0, 2\pi]$.

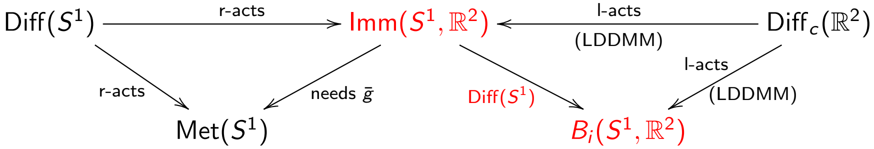
$$\mathrm{Imm}(M, \mathbb{R}^2) := \{c \in C^\infty(M, \mathbb{R}^2) : c'(\theta) \neq 0\} \subset C^\infty(M, \mathbb{R}^2).$$

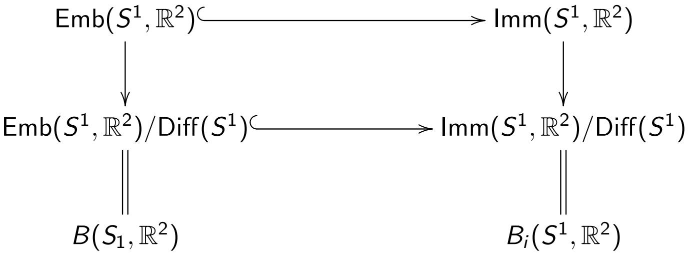
The tangent space of $\mathrm{Imm}(M, \mathbb{R}^2)$ at a curve c is the set of all vector fields along c:

$$T_c \,\mathrm{Imm}(M, \mathbb{R}^2) = \left\{ h : \begin{array}{ccc} & & T\mathbb{R}^2 \\ & \overset{h}{\nearrow} & \downarrow \pi \\ M & \xrightarrow{\;c\;} & \mathbb{R}^2 \end{array} \right\} \cong \{h \in C^\infty(M, \mathbb{R}^2)\}$$

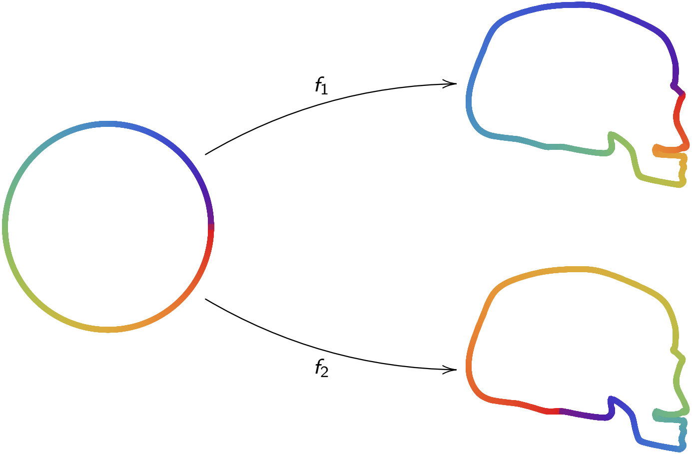
Some Notation:

$$v(\theta) = \frac{c'(\theta)}{|c'(\theta)|}, \quad n(\theta) = iv(\theta), \quad ds = |c'(\theta)| d\theta, \quad D_s = \frac{1}{|c'(\theta)|}\partial_\theta$$


$\mathrm{Diff}(S^1)$
r-acts
$\mathrm{Imm}(S^1, \mathbb{R}^2)$
l-acts
(LDDMM)
$\mathrm{Diff}_c(\mathbb{R}^2)$
r-acts
needs $\bar{g}$
$\mathrm{Diff}(S^1)$
l-acts
(LDDMM)
$\mathrm{Met}(S^1)$
$B_i(S^1, \mathbb{R}^2)$


$\mathrm{Emb}(S^1, \mathbb{R}^2)$
$\mathrm{Imm}(S^1, \mathbb{R}^2)$
$\mathrm{Emb}(S^1, \mathbb{R}^2)/\mathrm{Diff}(S^1)$
$\mathrm{Imm}(S^1, \mathbb{R}^2)/\mathrm{Diff}(S^1)$
$B(S_1, \mathbb{R}^2)$
$B_i(S^1, \mathbb{R}^2)$

Different parameterizations



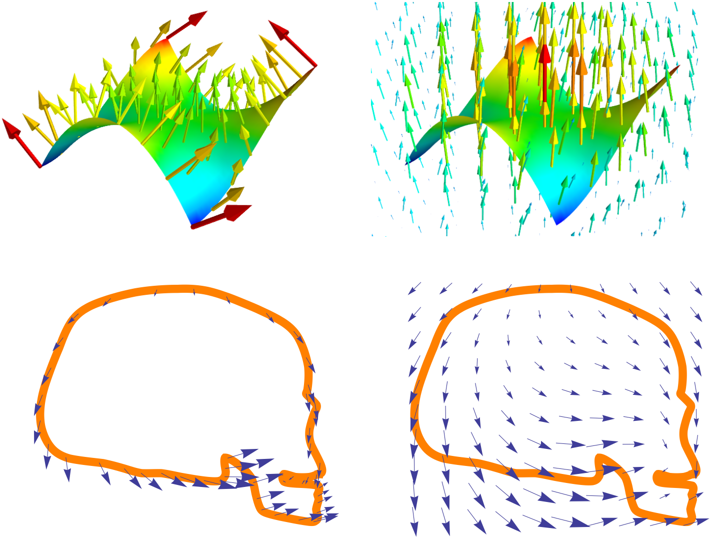
$$f_1, f_2 : S^1 \to \mathbb{R}^2, \qquad f_1 = f_2 \circ \varphi, \qquad \varphi \in \mathrm{Diff}(S^1)$$

Inducing a metric on shape space

$$\begin{array}{c} \operatorname{Imm}(M,N) \\ \big\downarrow \pi \\ B_i := \operatorname{Imm}(M,N)/\operatorname{Diff}(M) \end{array}$$

Every $\operatorname{Diff}(M)$-invariant metric "above" induces a unique metric "below" such that π is a Riemannian submersion.

Inner versus Outer



The vertical and horizontal bundle

- $T\,\mathrm{Imm} = \mathrm{Vert} \bigoplus \mathrm{Hor}$.
- The vertical bundle is

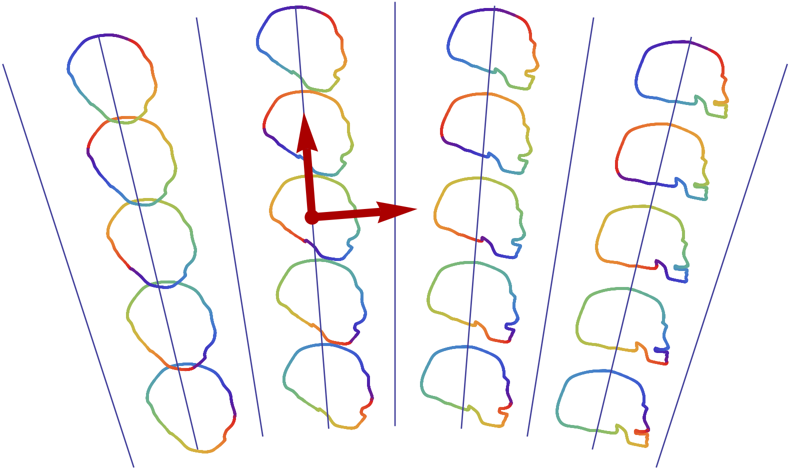
$$\mathrm{Vert} := \ker T\pi \subset T\,\mathrm{Imm}\,.$$

- The horizontal bundle is

$$\mathrm{Hor} := (\ker T\pi)^{\perp,G} \subset T\,\mathrm{Imm}\,.$$

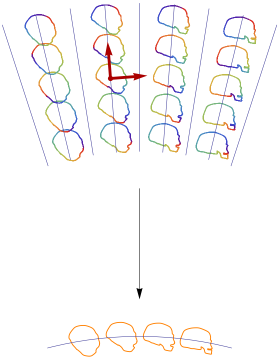
It might not be a complement - sometimes one has to go to the completion of $(T_f\,\mathrm{Imm}, G_f)$ in order to get a complement.

The vertical and horizontal bundle



Definition of a Riemannian metric

$$\mathrm{Imm}(M,N) \xrightarrow{\ \pi\ } B_i(M,N)$$



1. Define a $\mathrm{Diff}(M)$-invariant metric G on Imm.
2. If the horizontal space is a complement, then $T\pi$ restricted to the horizontal space yields an isomorphism

   $$(\ker T_f\pi)^{\perp,G} \cong T_{\pi(f)}B_i.$$

   Otherwise one has to induce the quotient metric, or use the completion.
3. This gives a metric on B_i such that $\pi : \mathrm{Imm} \to B_i$ is a *Riemannian submersion*.

Riemannian submersions

$$\begin{array}{c}\operatorname{Imm}(M,N)\\ \Big\downarrow{\scriptstyle\pi}\\ B_i := \operatorname{Imm}(M,N)/\operatorname{Diff}(M)\end{array}$$

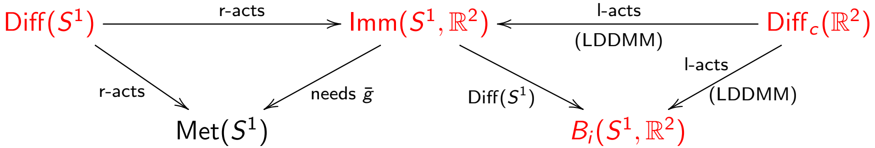
- Horizontal geodesics on $\operatorname{Imm}(M,N)$ project down to geodesics in shape space.
- O'Neill's formula connects sectional curvature on $\operatorname{Imm}(M,N)$ and on B_i.

[Micheli, M, Mumford, Izvestija 2013]

L^2 metric

$$G_c^0(h,k) = \int_M \langle h(\theta), k(\theta) \rangle ds.$$

Problem: The induced geodesic distance vanishes.



Movies about vanishing: $\text{Diff}(S^1)$ $\text{Imm}(S^1, \mathbb{R}^2)$

[MichorMumford2005a,2005b], [BauerBruverisHarmsMichor2011,2012]

The simplest (L^2-) metric on $\mathrm{Imm}(S^1, \mathbb{R}^2)$

$$G^0_c(h,k) = \int_{S^1} \langle h, k\rangle ds = \int_{S^1} \langle h,k\rangle |c_\theta|\, d\theta$$

Geodesic equation

$$\boxed{c_{tt} = -\frac{1}{2|c_\theta|}\partial_\theta\Big(\frac{|c_t|^2\, c_\theta}{|c_\theta|}\Big) - \frac{1}{|c_\theta|^2}\langle c_{t\theta}, c_\theta\rangle c_t.}$$

A relative of Burger's equation.
Conserved momenta for G^0 along any geodesic $t \mapsto c(\quad, t)$:

$\langle v, c_t\rangle |c_\theta|^2 \;\in \mathfrak{X}(S^1)$	reparam. mom.
$\int_{S^1} c_t ds \;\in \mathbb{R}^2$	linear moment.
$\int_{S^1} \langle Jc, c_t\rangle ds \in \mathbb{R}$	angular moment.

Horizontal Geodesics for G^0 on $B_i(S^1, \mathbb{R}^2)$

$\langle c_t, c_\theta \rangle = 0$ and $c_t = an = aJ\frac{c_\theta}{|c_\theta|}$ for $a \in C^\infty(S^1, \mathbb{R})$. We use functions a, $s = |c_\theta|$, and κ, only holonomic derivatives:

$$s_t = -a\kappa s, \quad a_t = \tfrac{1}{2}\kappa a^2,$$
$$\kappa_t = a\kappa^2 + \frac{1}{s}\left(\frac{a_\theta}{s}\right)_\theta = a\kappa^2 + \frac{a_{\theta\theta}}{s^2} - \frac{a_\theta s_\theta}{s^3}.$$

We may assume $s|_{t=0} \equiv 1$. Let $v(\theta) = a(0, \theta)$, the initial value for a. Then
$\frac{s_t}{s} = -a\kappa = -2\frac{a_t}{a}$, so $\log(sa^2)_t = 0$, thus
$s(t, \theta)a(t, \theta)^2 = s(0, \theta)a(0, \theta)^2 = v(\theta)^2$,
a conserved quantity along the geodesic. We substitute $s = \frac{v^2}{a^2}$ and $\kappa = 2\frac{a_t}{a^2}$ to get

$$
\begin{aligned}
&a_{tt} - 4\frac{a_t^2}{a} - \frac{a^6 a_{\theta\theta}}{2v^4} + \frac{a^6 a_\theta v_\theta}{v^5} - \frac{a^5 a_\theta^2}{v^4} = 0,\\
&a(0,\theta) = v(\theta),
\end{aligned}
$$

a nonlinear hyperbolic second order equation. Note that wherever $v = 0$ then also $a = 0$ for all t. So substitute $a = vb$. The outcome is

$$
\boxed{
\begin{aligned}
&(b^{-3})_{tt} = -\frac{v^2}{2}(b^3)_{\theta\theta} - 2vv_\theta(b^3)_\theta - \frac{3vv_{\theta\theta}}{2}b^3,\\
&b(0,\theta) = 1.
\end{aligned}
}
$$

This is the codimension 1 version where
Burgers' equation is the codimension 0 version.

Weak Riem. metrics on $\mathrm{Emb}(M,N) \subset \mathrm{Imm}(M,N)$.

Metrics on the space of immersions of the form:

$$G^P_f(h,k) = \int_M \bar g(P^f h, k)\,\mathrm{vol}(f^*\bar g)$$

where $\bar g$ is some fixed metric on N, $g = f^*\bar g$ is the induced metric on M, $h,k \in \Gamma(f^*TN)$ are tangent vectors at f to $\mathrm{Imm}(M,N)$, and P^f is a positive, selfadjoint, bijective (scalar) pseudo differential operator of order $2p$ depending smoothly on f. Good example: $P^f = 1 + A(\Delta^g)^p$, where Δ^g is the Bochner-Laplacian on M induced by the metric $g = f^*\bar g$. Also P has to be $\mathrm{Diff}(M)$-invariant: $\varphi^* \circ P_f = P_{f\circ\varphi} \circ \varphi^*$.

Elastic metrics on plane curves

Here $M = S^1$ or $[0, 1\pi]$, $N = \mathbb{R}^2$. The elastic metrics on $\operatorname{Imm}(M, \mathbb{R}^2)$ is

$$G_c^{a,b}(h,k) = \int_0^{2\pi} a^2 \langle D_s h, n\rangle \langle D_s k, n\rangle + b^2 \langle D_s h, v\rangle \langle D_s k, v\rangle \, ds,$$

with

$$\begin{aligned} P_c^{a,b}(h) = &- a^2 \langle D_s^2 h, n\rangle n - b^2 \langle D_s^2 h, v\rangle v \\ &+ (a^2 - b^2)\kappa\big(\langle D_s h, v\rangle n + \langle D_s h, n\rangle v\big) \\ &+ (\delta_{2\pi} - \delta_0)\big(a^2 \langle n, D_s h\rangle n + b^2 \langle v, D_s h\rangle v\big). \end{aligned}$$

Sobolev type metrics

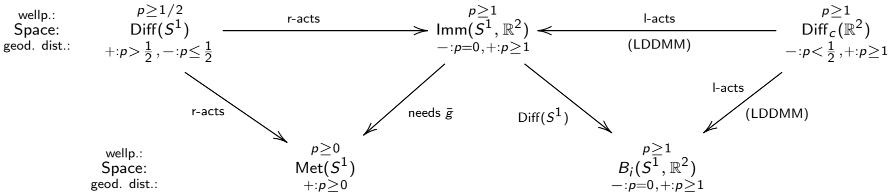
Advantages of Sobolev type metrics:

1. Positive geodesic distance
2. Geodesic equations are well posed
3. Spaces are geodesically complete for $p > \frac{\dim(M)}{2} + 1$.

[Bruveris, M, Mumford, 2013] for plane curves. A remark in [Ebin, Marsden, 1970], and [Bruveris, Meyer, 2014] for diffeomorphism groups.

Problems:

1. Analytic solutions to the geodesic equation?
2. Curvature of shape space with respect to these metrics?
3. Numerics are in general computational expensive



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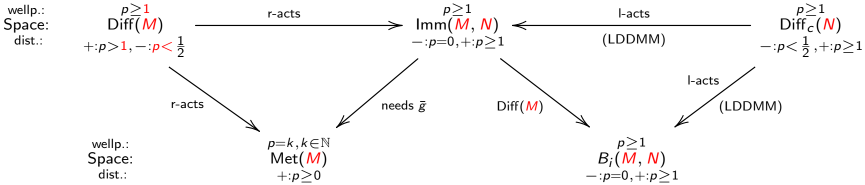


Geodesic equation.

The geodesic equation for a Sobolev-type metric G^P on immersions is given by

$$\begin{aligned}\nabla_{\partial_t} f_t =& \frac{1}{2} P^{-1}\Big(\operatorname{Adj}(\nabla P)(f_t, f_t)^\perp - 2.Tf.\bar{g}(Pf_t, \nabla f_t)^\sharp \\ & \qquad - \bar{g}(Pf_t, f_t).\operatorname{Tr}^g(S)\Big) \\ & - P^{-1}\Big((\nabla_{f_t} P)f_t + \operatorname{Tr}^g\big(\bar{g}(\nabla f_t, Tf)\big)Pf_t\Big).\end{aligned}$$

The geodesic equation written in terms of the momentum for a Sobolev-type metric G^P on Imm is given by:

$$\begin{cases} p = Pf_t \otimes \operatorname{vol}(f^*\bar{g}) \\ \nabla_{\partial_t} p = \frac{1}{2}\big(\operatorname{Adj}(\nabla P)(f_t, f_t)^\perp - 2Tf.\bar{g}(Pf_t, \nabla f_t)^\sharp \\ \qquad - \bar{g}(Pf_t, f_t)\operatorname{Tr}^{f^*\bar{g}}(S)\big) \otimes \operatorname{vol}(f^*\bar{g}) \end{cases}$$

Wellposedness

Assumption 1: $P, \nabla P$ and $\operatorname{Adj}(\nabla P)^\perp$ are smooth sections of the bundles

$$\begin{array}{ccc} L(T\mathrm{Imm}; T\mathrm{Imm}) & L^2(T\mathrm{Imm}; T\mathrm{Imm}) & L^2(T\mathrm{Imm}; T\mathrm{Imm}) \\ \downarrow & \downarrow & \downarrow \\ \mathrm{Imm} & \mathrm{Imm} & \mathrm{Imm}, \end{array}$$

respectively. Viewed locally in trivializ. of these bundles, $P_f h, \quad (\nabla P)_f(h,k), \quad \big(\operatorname{Adj}(\nabla P)_f(h,k)\big)^\perp$ are pseudo-differential operators of order $2p$ in h, k separately. As mappings in f they are non-linear, and we assume they are a composition of operators of the following type:

(a) Local operators of order $l \le 2p$, i.e., nonlinear differential operators $A(f)(x) = A(x, \hat\nabla^l f(x), \hat\nabla^{l-1} f(x), \dots, \hat\nabla f(x), f(x))$

(b) Linear pseudo-differential operators of degrees l_i,

such that the total (top) order of the composition is $\le 2p$.

Assumption 2: For each $f \in \mathrm{Imm}(M,N)$, the operator P_f is an elliptic pseudo-differential operator of order $2p$ for $p > 0$ which is positive and symmetric with respect to the H^0-metric on Imm, i.e.

$$\int_M \bar g(P_f h, k)\,\mathrm{vol}(g) = \int_M \bar g(h, P_f k)\,\mathrm{vol}(g) \qquad \text{for } h,k \in T_f\mathrm{Imm}.$$

Theorem [Bauer, Harms, M, 2011] *Let* $p \geq 1$ *and* $k > \dim(M)/2 + 1$, *and let* P *satisfy the assumptions.*
Then the geodesic equation has unique local solutions in the Sobolev manifold Imm^{k+2p} *of* H^{k+2p}*-immersions. The solutions depend smoothly on* t *and on the initial conditions* $f(0, \,.\,)$ *and* $f_t(0, \,.\,)$*. The domain of existence (in* t*) is uniform in* k *and thus this also holds in* $\mathrm{Imm}(M, N)$*.*
Moreover, in each Sobolev completion Imm^{k+2p}*, the Riemannian exponential mapping* $\exp^P$ *exists and is smooth on a neighborhood of the zero section in the tangent bundle, and* $(\pi, \exp^P)$ *is a diffeomorphism from a (smaller) neigbourhood of the zero section to a neighborhood of the diagonal in* $\mathrm{Imm}^{k+2p} \times \mathrm{Imm}^{k+2p}$*. All these neighborhoods are uniform in* $k > \dim(M)/2 + 1$ *and can be chosen* H^{k_0+2p}*-open, for* $k_0 > \dim(M)/2 + 1$*. Thus both properties of the exponential mapping continue to hold in* $\mathrm{Imm}(M, N)$*.*

Sobolev completions of $\Gamma(E)$, where $E \to M$ is a VB

Fix (background) Riemannian metric $\hat{g}$ on M and its covariant derivative ∇^M. Equip E with a (background) fiber Riemannian metric $\hat{g}^E$ and a compatible covariant derivative $\hat{\nabla}^E$. Then *Sobolev space* $H^k(E)$ is the completion of $\Gamma(E)$ for the Sobolev norm

$$\|h\|_k^2 = \sum_{j=0}^{k} \int_M (\hat{g}^E \otimes \hat{g}_j^0)\big((\hat{\nabla}^E)^j h, (\hat{\nabla}^E)^j h\big)\,\mathrm{vol}(\hat{g}).$$

This Sobolev space is independ of choices of $\hat{g}$, ∇^M, $\hat{g}^E$ and $\hat{\nabla}^E$ since M is compact: The resulting norms are equivalent.

Sobolev lemma: *If $k > \dim(M)/2$ then the identy on $\Gamma(E)$ extends to a injective bounded linear map $H^{k+p}(E) \to C^p(E)$ where $C^p(E)$ carries the supremum norm of all derivatives up to order p.*

Module property of Sobolev spaces: *If $k > \dim(M)/2$ then pointwise evaluation $H^k(L(E,E)) \times H^k(E) \to H^k(E)$ is bounded bilinear. Likewise all other pointwise contraction operations are multilinear bounded operations.*

Proof of well-posedness

By assumption 1 the mapping $P_f h$ is of order $2p$ in f and in h where f is the footpoint of h. Therefore $f \mapsto P_f$ extends to a smooth section of the smooth Sobolev bundle

$$L\big(T\mathrm{Imm}^{k+2p}; T\mathrm{Imm}^k \mid \mathrm{Imm}^{k+2p}\big) \to \mathrm{Imm}^{k+2p},$$

where $T\mathrm{Imm}^k \mid \mathrm{Imm}^{k+2p}$ denotes the space of all H^k tangent vectors with foot point a H^{k+2p} immersion, i.e., the restriction of the bundle $T\mathrm{Imm}^k \to \mathrm{Imm}^k$ to $\mathrm{Imm}^{k+2p} \subset \mathrm{Imm}^k$.
This means that P_f is a bounded linear operator

$$P_f \in L\big(H^{k+2p}(f^*TN), H^k(f^*TN)\big) \quad \text{for} \quad f \in \mathrm{Imm}^{k+2p}.$$

It is injective since it is positive. As an elliptic operator, it is an unbounded operator on the Hilbert completion of $T_f\mathrm{Imm}$ with respect to the H^0-metric, and a Fredholm operator $H^{k+2p} \to H^k$ for each k. It is selfadjoint elliptic, so the index $=0$. Since it is injective, it is thus also surjective.

By the implicit function theorem on Banach spaces, $f \mapsto P_f^{-1}$ is then a smooth section of the smooth Sobolev bundle

$$L\big(T\mathrm{Imm}^k \mid \mathrm{Imm}^{k+2p}; T\mathrm{Imm}^{k+2p}\big) \to \mathrm{Imm}^{k+2p}$$

As an inverse of an elliptic pseudodifferential operator, P_f^{-1} is also an elliptic pseudo-differential operator of order $-2p$.

By assumption 1 again, $(\nabla P)_f(m,h)$ and $\big(\mathrm{Adj}(\nabla P)_f(m,h)\big)^\perp$ are of order $2p$ in f, m, h (locally). Therefore $f \mapsto P_f$ and $f \mapsto \mathrm{Adj}(\nabla P)^\perp$ extend to smooth sections of the Sobolev bundle

$$L^2\big(T\mathrm{Imm}^{k+2p}; T\mathrm{Imm}^k \mid \mathrm{Imm}^{k+2p}\big) \to \mathrm{Imm}^{k+2p}$$

Using the module property of Sobolev spaces, one obtains that the "Christoffel symbols"

$$\begin{aligned}\Gamma_f(h,h) = \frac{1}{2}P^{-1}\Big(&\mathrm{Adj}(\nabla P)(h,h)^\perp - 2.Tf.\overline{g}(Ph,\nabla h)^\sharp\\ &- \overline{g}(Ph,h).\mathrm{Tr}^g(S) - (\nabla_h P)h - \mathrm{Tr}^g\big(\overline{g}(\nabla h, Tf)\big)Ph\Big)\end{aligned}$$

extend to a smooth (C^∞) section of the smooth Sobolev bundle

$$L^2_{\mathrm{sym}}\big(T\mathrm{Imm}^{k+2p}; T\mathrm{Imm}^{k+2p}\big) \to \mathrm{Imm}^{k+2p}$$

Thus $h \mapsto \Gamma_f(h,h)$ is a smooth quadratic mapping $T\mathrm{Imm} \to T\mathrm{Imm}$ which extends to smooth quadratic mappings $T\mathrm{Imm}^{k+2p} \to T\mathrm{Imm}^{k+2p}$ for each $k \geq \frac{\dim(2)}{2} + 1$. The geodesic equation $\boxed{\nabla^{\overline{g}}_{\partial_t} f_t = \Gamma_f(f_t, f_t)}$ can be reformulated using the linear connection $C^g : TN \times_N TN \to TTN$ (horizontal lift mapping) of $\nabla^{\overline{g}}$:

$$\partial_t f_t = C\Big(\frac{1}{2} H_f(f_t, f_t) - K_f(f_t, f_t), f_t\Big).$$

The right-hand side is a smooth vector field on $T\mathrm{Imm}^{k+2p}$, the geodesic spray. Note that the restriction to $T\mathrm{Imm}^{k+1+2p}$ of the geodesic spray on $T\mathrm{Imm}^{k+2p}$ equals the geodesic spray there. By the theory of smooth ODE's on Banach spaces, the flow of this vector field exists in $T\mathrm{Imm}^{k+2p}$ and is smooth in time and in the initial condition, for all $k \geq \frac{\dim(2)}{2} + 1$.
It remains to show that the domain of existence is independent of k. I omit this. QED

Sobolev metrics of order ≥ 2 on $\mathrm{Imm}(S^1, \mathbb{R}^2)$ are complete

Theorem. [Bruveris, M, Mumford, 2014] *Let $n \geq 2$ and the metric G on $\mathrm{Imm}(S^1, \mathbb{R}^2)$ be given by*

$$G_c(h,k) = \int_{S^1} \sum_{j=0}^{n} a_j \langle D_s^j h, D_s^j k \rangle \, \mathrm{d}s \,,$$

with $a_j \geq 0$ and $a_0, a_n \neq 0$. Given initial conditions $(c_0, u_0) \in T\,\mathrm{Imm}(S^1, \mathbb{R}^2)$ the solution of the geodesic equation

$$\partial_t \left(\sum_{j=0}^{n} (-1)^j \, |c'| \, D_s^{2j} c_t \right) = -\frac{a_0}{2} \, |c'| \, D_s \left(\langle c_t, c_t \rangle v \right)$$
$$+ \sum_{k=1}^{n} \sum_{j=1}^{2k-1} (-1)^{k+j} \frac{a_k}{2} \, |c'| \, D_s \left(\langle D_s^{2k-j} c_t, D_s^j c_t \rangle v \right) \,.$$

for the metric G with initial values (c_0, u_0) exists for all time.

Recall: $ds = |c'| d\theta$ is arc-length measure, $D_s = \frac{1}{|c'|}\partial_\theta$ is the derivative with respect to arc-length, $v = c'/|c'|$ is the unit length tangent vector to c and $\langle \; , \; \rangle$ is the Euclidean inner product on $\mathbb{R}^2$.

Thus if G is a Sobolev-type metric of order at least 2, so that

$$\int_{S^1} (|h|^2 + |D_s^2 h|^2) ds \leq C\, G_c(h, h),$$

then the Riemannian manifold $(\operatorname{Imm}(S^1, \mathbb{R}^2), G)$ is geodesically complete. If the Sobolev-type metric is invariant under the reparameterization group $\operatorname{Diff}(S^1)$, also the induced metric on shape space $\operatorname{Imm}(S^1, \mathbb{R}^2)/\operatorname{Diff}(S^1)$ is geodesically complete.

The proof of this theorem is surprisingly difficult.

The elastic metric

$$G_c^{a,b}(h,k) = \int_0^{2\pi} a^2 \langle D_s h, n\rangle \langle D_s k, n\rangle + b^2 \langle D_s h, v\rangle \langle D_s k, v\rangle \, ds,$$

$$\boxed{\begin{aligned} c_t &= u \in C^\infty(\mathbb{R}_{>0} \times M, \mathbb{R}^2) \\ P(u_t) &= P(\frac{1}{2} H_c(u,u) - K_c(u,u)) \\ &= \frac{1}{2}(\delta_0 - \delta_{2\pi})\big(\langle D_s u, D_s u\rangle v + \tfrac{3}{4}\langle v, D_s u\rangle^2 v \\ &\qquad - 2\langle D_s u, v\rangle D_s u - \tfrac{3}{2}\langle n, D_s u\rangle^2 v\big) \\ &\qquad + D_s\big(\langle D_s u, D_s u\rangle v + \tfrac{3}{4}\langle v, D_s u\rangle^2 v \\ &\qquad - 2\langle D_s u, v\rangle D_s u - \tfrac{3}{2}\langle n, D_s u\rangle^2 v\big) \end{aligned}}$$

Note: Only a metric on Imm/transl.

Representation of the elastic metrics

Aim: Represent the class of elastic metrics as the pullback metric of a flat metric on $C^\infty(M, \mathbb{R}^2)$, i.e.: find a map

$$R : \mathrm{Imm}(M, \mathbb{R}^2) \mapsto C^\infty(M, \mathbb{R}^n)$$

such that

$$G_c^{a,b}(h, k) = R^*\langle h, k\rangle_{L^2} = \langle T_c R.h, T_c R.k\rangle_{L^2}.$$

[YounesMichorShahMumford2008] [SrivastavaKlassenJoshiJermyn2011]

The R transform on open curves

Theorem

The metric $G^{a,b}$ is the pullback of the flat L^2 metric via the transform R:

$$R^{a,b}: \operatorname{Imm}([0,2\pi],\mathbb{R}^2) \to C^\infty([0,2\pi],\mathbb{R}^3)$$
$$R^{a,b}(c) = |c'|^{1/2}\left(a\begin{pmatrix} v \\ 0 \end{pmatrix} + \sqrt{4b^2-a^2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \quad .$$

The metric $G^{a,b}$ is flat on open curves, geodesics are the preimages under the R-transform of geodesics on the flat space $\operatorname{im} R$ *and the geodesic distance between $c, \overline{c} \in \operatorname{Imm}([0,2\pi],\mathbb{R}^2)/\operatorname{trans}$ is given by the integral over the pointwise distance in the image* $\operatorname{Im}(R)$*.*
The curvature on $B([0,2\pi],\mathbb{R}^2)$ is non-negative.

[BauerBruverisMarslandMichor2014]

The R transform on open curves II

Image of R is characterized by the condition:

$$(4b^2 - a^2)(R_1^2(c) + R_2^2(c)) = a^2 R_3^2(c)$$

Define the flat cone

$$C^{a,b} = \{q \in \mathbb{R}^3 : (4b^2 - a^2)(q_1^2 + q_2^2) = a^2 q_3^2, q_3 > 0\}.$$

Then $\operatorname{Im} R = C^\infty(S^1, C^{a,b})$. The inverse of R is given by:

$$\begin{aligned} &R^{-1} : \operatorname{im} R \to \operatorname{Imm}([0, 2\pi], \mathbb{R}^2)/\operatorname{trans} \\ &R^{-1}(q)(\theta) = p_0 + \frac{1}{2ab} \int_0^\theta |q(\theta)| \begin{pmatrix} q_1(\theta) \\ q_2(\theta) \end{pmatrix} d\theta\,. \end{aligned}$$

The R transform on closed curves I

Characterize image using the inverse:

$$R^{-1}(q)(\theta) = p_0 + \frac{1}{2ab} \int_0^\theta |q(\theta)| \begin{pmatrix} q_1(\theta) \\ q_2(\theta) \end{pmatrix} d\theta \,.$$

$R^{-1}(q)(\theta)$ is closed iff

$$F(q) = \int_0^{2\pi} |q(\theta)| \begin{pmatrix} q_1(\theta) \\ q_2(\theta) \end{pmatrix} d\theta = 0$$

A basis of the orthogonal complement $\left(T_q \mathscr{C}^{a,b}\right)^\perp$ is given by the two gradients $\operatorname{grad}^{L^2} F_i(q)$

The R transform on closed curves II

Theorem

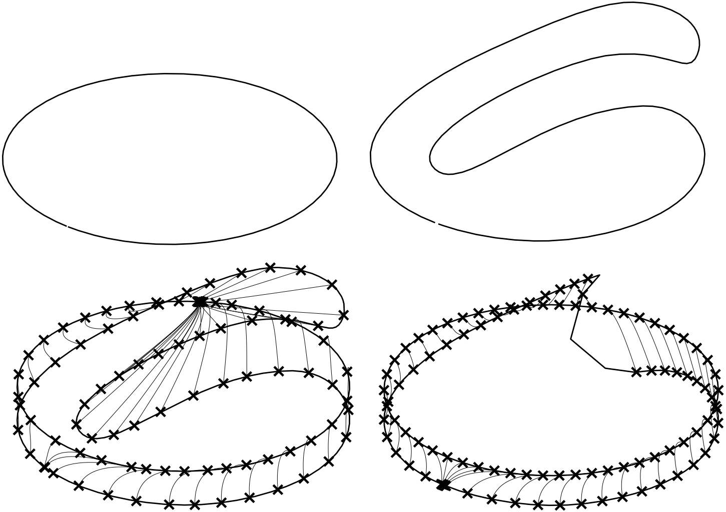
The image $\mathscr{C}^{a,b}$ of the manifold of closed curves under the R-transform is a codimension 2 submanifold of the flat space $\mathrm{Im}(R)_{open}$. A basis of the orthogonal complement $\left(T_q\mathscr{C}^{a,b}\right)^{\perp}$ is given by the two vectors

$$U_1(q) = \frac{1}{\sqrt{q_1^2+q_2^2}}\begin{pmatrix} 2q_1^2+q_2^2 \\ q_1 q_2 \\ 0 \end{pmatrix} + \frac{2}{a}\sqrt{4b^2-a^2}\begin{pmatrix} 0 \\ 0 \\ q_1 \end{pmatrix},$$

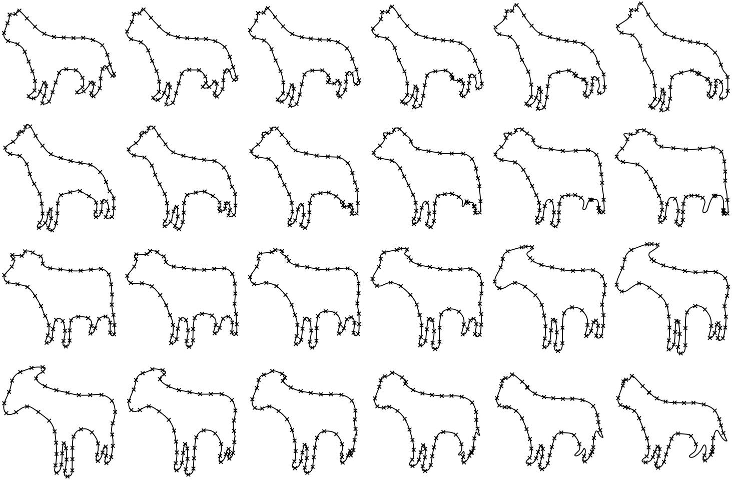
$$U_2(q) = \frac{1}{\sqrt{q_1^2+q_2^2}}\begin{pmatrix} q_1 q_2 \\ q_1^2+2q_2^2 \\ 0 \end{pmatrix} + \frac{2}{a}\sqrt{4b^2-a^2}\begin{pmatrix} 0 \\ 0 \\ q_2 \end{pmatrix}.$$

[BauerBruverisMarslandMichor2012]

compress and stretch

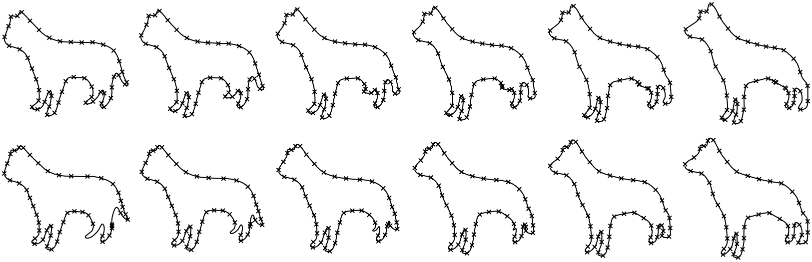


A geodesic Rectangle

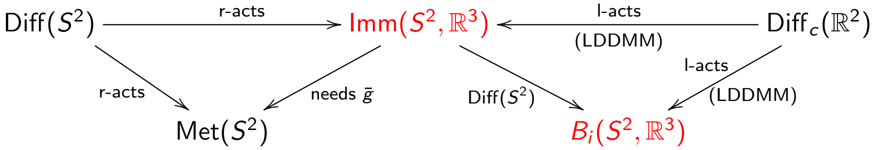


Non-symmetric distances

		$I_1 \to I_2$			$I_2 \to I_1$			
I_1	I_2	# iterations	# points	distance	# iterations	# points	distance	%diff
cat	cow	28	456	7.339	33	462	8.729	15.9
cat	dog	36	475	8.027	102	455	10.060	20.2
cat	donkey	73	476	12.620	102	482	12.010	4.8
cow	donkey	32	452	7.959	26	511	7.915	0.6
dog	donkey	15	457	8.299	10	476	8.901	6.8
shark	airplane	63	491	13.741	40	487	13.453	2.1



An example of a metric space with strongly negatively curved regions



$$G_f^{\Phi}(h,k) = \int_M \Phi(f)\overline{g}(h,k)\,\mathrm{vol}(g)$$

[BauerHarmsMichor2012]

Non-vanishing geodesic distance

The pathlength metric on shape space induced by G^{Φ} separates points if one of the following holds:

- $\Phi \geq C_1 + C_2 \|\mathrm{Tr}^g(S)\|^2$ with $C_1, C_2 > 0$ or
- $\Phi \geq C_3 \,\mathrm{Vol}$

This leads us to consider $\Phi = \Phi(\mathrm{Vol}, \|\mathrm{Tr}^g(S)\|^2)$.
Special cases:

- G^A-metric: $\Phi = 1 + A\|\mathrm{Tr}^g(S)\|^2$
- Conformal metrics: $\Phi = \Phi(\mathrm{Vol})$

Geodesic equation on shape space $B_i(M,\mathbb{R}^n)$, with $\Phi = \Phi(\mathrm{Vol}, \mathrm{Tr}(L))$

$$
\begin{aligned}
f_t &= a.\nu, \\
a_t &= \frac{1}{\Phi}\Big[\frac{\Phi}{2}a^2\,\mathrm{Tr}(L) - \frac{1}{2}\,\mathrm{Tr}(L)\int_M (\partial_1\Phi)a^2\,\mathrm{vol}(g) - \frac{1}{2}a^2\Delta(\partial_2\Phi) \\
&\quad + 2ag^{-1}(d(\partial_2\Phi), da) + (\partial_2\Phi)\|da\|^2_{g^{-1}} \\
&\quad + (\partial_1\Phi)a\int_M \mathrm{Tr}(L).a\,\mathrm{vol}(g) - \frac{1}{2}(\partial_2\Phi)\,\mathrm{Tr}(L^2)a^2\Big]
\end{aligned}
$$

Sectional curvature on B_i

Chart for B_i centered at $\pi(f_0)$ so that $\pi(f_0) = 0$ in this chart:

$$a \in C^\infty(M) \longleftrightarrow \pi(f_0 + a.\nu^{f_0}).$$

For a linear 2-dim. subspace $P \subset T_{\pi(f_0)}B_i$ spanned by a_1, a_1, the sectional curvature is defined as:

$$k(P) = -\frac{G^\Phi_{\pi(f_0)}\big(\mathcal{R}_{\pi(f_0)}(a_1, a_2)a_1, a_2\big)}{\|a_1\|^2\|a_2\|^2 - G^\Phi_{\pi(f_0)}(a_1, a_2)^2}\ , \text{ where}$$

$$\begin{aligned}
R_0(a_1, a_2, a_1, a_2) = G^\Phi_0(R_0(a_1, a_2)a_1, a_2) = \\
\frac{1}{2}d^2 G^\Phi_0(a_1, a_1)(a_2, a_2) + \frac{1}{2}d^2 G^\Phi_0(a_2, a_2)(a_1, a_1) \\
- d^2 G^\Phi_0(a_1, a_2)(a_1, a_2) \\
+ G^\Phi_0(\Gamma_0(a_1, a_1), \Gamma_0(a_2, a_2)) - G^\Phi_0(\Gamma_0(a_1, a_2), \Gamma_0(a_1, a_2)).
\end{aligned}$$

Sectional curvature on B_i for $\Phi = \mathrm{Vol}$

$$k(P) = -\frac{\mathcal{R}_0(a_1, a_2, a_1, a_2)}{\|a_1\|^2\|a_2\|^2 - G^{\Phi}_{\pi(f_0)}(a_1, a_2)^2}\ , \text{ where}$$

$$\begin{aligned}
R_0(a_1, a_2, a_1, a_2) = & -\frac{1}{2}\,\mathrm{Vol}\int_M \|a_1 da_2 - a_2 da_1\|^2_{g^{-1}}\,\mathrm{vol}(g) \\
& + \frac{1}{4\,\mathrm{Vol}}\overline{\mathrm{Tr}(L)^2}\Big(\overline{a_1^2}.\overline{a_2^2} - \overline{a_1.a_2}^2\Big) \\
& + \frac{1}{4}\Big(\overline{a_1^2}.\overline{\mathrm{Tr}(L)^2 a_2^2} - 2\overline{a_1.a_2}.\overline{\mathrm{Tr}(L)^2 a_1.a_2} + \overline{a_2^2}.\overline{\mathrm{Tr}(L)^2 a_1^2}\Big) \\
& - \frac{3}{4\,\mathrm{Vol}}\Big(\overline{a_1^2}.\overline{\mathrm{Tr}(L)a_2}^2 - 2\overline{a_1.a_2}.\overline{\mathrm{Tr}(L)a_1}.\overline{\mathrm{Tr}(L)a_2} + \overline{a_2^2}.\overline{\mathrm{Tr}(L)a_1}^2\Big) \\
& + \frac{1}{2}\Big(\overline{a_1^2}.\overline{\mathrm{Tr}^g((da_2)^2)} - 2\overline{a_1.a_2}.\overline{\mathrm{Tr}^g(da_1.da_2)} + \overline{a_2^2}\overline{\mathrm{Tr}^g((da_1)^2)}\Big) \\
& - \frac{1}{2}\Big(\overline{a_1^2}.\overline{a_2^2.\,\mathrm{Tr}(L^2)} - 2.\overline{a_1.a_2}.\overline{a_1.a_2.\,\mathrm{Tr}(L^2)} + \overline{a_2^2}.\overline{a_1^2.\,\mathrm{Tr}(L^2)}\Big).
\end{aligned}$$

Sectional curvature on B_i for $\Phi = 1 + A\,\mathrm{Tr}(L)^2$

$$k(P) = -\frac{\mathcal{R}_0(a_1, a_2, a_1, a_2)}{\|a_1\|^2\|a_2\|^2 - G^{\Phi}_{\pi(f_0)}(a_1, a_2)^2}\ , \text{ where}$$

$$R_0(a_1, a_2, a_1, a_2) = \int_M A(a_1\Delta a_2 - a_2\Delta a_1)^2 \,\mathrm{vol}(g)$$

$$+ \int_M 2A\,\mathrm{Tr}(L) g_2^0\big((a_1 da_2 - a_2 da_1) \otimes (a_1 da_2 - a_2 da_1), s\big)\,\mathrm{vol}(g)$$

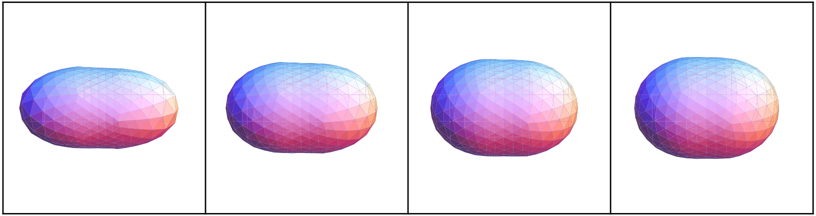
$$+ \int_M \frac{1}{1 + A\,\mathrm{Tr}(L)^2}\Big[- 4A^2 g^{-1}\big(d\,\mathrm{Tr}(L), a_1 da_2 - a_2 da_1\big)^2$$

$$- \Big(\frac{1}{2}\big(1 + A\,\mathrm{Tr}(L)^2\big)^2 + 2A^2\,\mathrm{Tr}(L)\Delta(\mathrm{Tr}(L)) + 2A^2\,\mathrm{Tr}(L^2)\,\mathrm{Tr}(L)^2\Big)\cdot$$

$$\cdot \|a_1 da_2 - a_2 da_1\|^2_{g^{-1}} + (2A^2\,\mathrm{Tr}(L)^2)\|da_1 \wedge da_2\|^2_{g_0^2}$$

$$+ (8A^2\,\mathrm{Tr}(L)) g_2^0\big(d\,\mathrm{Tr}(L) \otimes (a_1 da_2 - a_2 da_1), da_1 \wedge da_2\big)\Big]\,\mathrm{vol}(g)$$

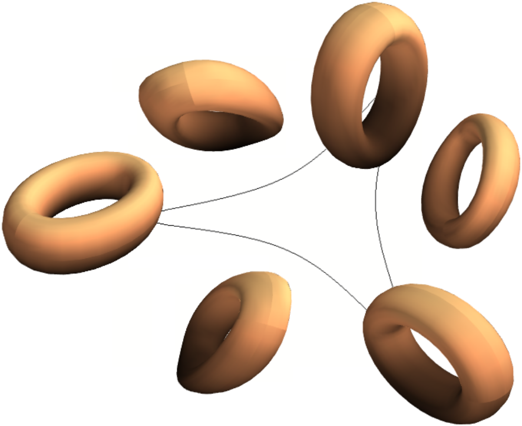
Negative Curvature: A toy example



Movies: Ex1: $\Phi = 1 + .4\,\mathrm{Tr}(L)^2$ Ex2: $\Phi = e^{Vol}$ Ex3: $\Phi = e^{Vol}$

Another toy example

$G_f^{\Phi}(h,k) = \int_{\mathbb{T}^2} \overline{g}((1+\Delta)h,k)\,\mathrm{vol}(g)$ on $\mathrm{Imm}(\mathbb{T}^2,\mathbb{R}^3)$:



[BauerBruveris2011]

Right invariant Riemannian geometries on Diffeomorphism groups.

For $M = N$ the space $\text{Emb}(M, M)$ equals the *diffeomorphism group of* M. An operator $P \in \Gamma\big(L(T\,\text{Emb}; T\,\text{Emb})\big)$ that is invariant under reparametrizations induces a right-invariant Riemannian metric on this space. Thus one gets the geodesic equation for right-invariant Sobolev metrics on diffeomorphism groups and well-posedness of this equation. The geodesic equation on $\text{Diff}(M)$ in terms of the momentum p is given by

$$\begin{cases} p = Pf_t \otimes \text{vol}(g), \\ \nabla_{\partial_t} p = -Tf.\bar{g}(Pf_t, \nabla f_t)^\sharp \otimes \text{vol}(g). \end{cases}$$

Note that this equation is not right-trivialized, in contrast to the equation given in [Arnold 1966]. The special case of theorem now reads as follows:

Theorem. [Bauer, Harms, M, 2011] *Let* $p \geq 1$ *and* $k > \frac{\dim(M)}{2} + 1$ *and let* P *satisfy the assumptions.*
The initial value problem for the geodesic equation has unique local solutions in the Sobolev manifold Diff^{k+2p} *of* H^{k+2p}*-diffeomorphisms. The solutions depend smoothly on* t *and on the initial conditions* $f(0, \,.\,)$ *and* $f_t(0, \,.\,)$*. The domain of existence (in* t*) is uniform in* k *and thus this also holds in* $\mathrm{Diff}(M)$*.*
Moreover, in each Sobolev completion Diff^{k+2p}*, the Riemannian exponential mapping* $\exp^P$ *exists and is smooth on a neighborhood of the zero section in the tangent bundle, and* $(\pi, \exp^P)$ *is a diffeomorphism from a (smaller) neigbourhood of the zero section to a neighborhood of the diagonal in* $\mathrm{Diff}^{k+2p} \times \mathrm{Diff}^{k+2p}$*. All these neighborhoods are uniform in* $k > \dim(M)/2 + 1$ *and can be chosen* H^{k_0+2p}*-open, for* $k_0 > \dim(M)/2 + 1$*. Thus both properties of the exponential mapping continue to hold in* $\mathrm{Diff}(M)$*.*

Arnold's formula for geodesics on Lie groups: Notation

Let G be a regular convenient Lie group, with Lie algebra $\mathfrak{g}$. Let $\mu : G \times G \to G$ be the group multiplication, μ_x the left translation and μ^y the right translation, $\mu_x(y) = \mu^y(x) = xy = \mu(x, y)$.

Let $L, R : \mathfrak{g} \to \mathfrak{X}(G)$ be the left- and right-invariant vector field mappings, given by $L_X(g) = T_e(\mu_g).X$ and $R_X = T_e(\mu^g).X$, resp. They are related by $L_X(g) = R_{\mathrm{Ad}(g)X}(g)$. Their flows are given by

$$\mathrm{Fl}_t^{L_X}(g) = g.\exp(tX) = \mu^{\exp(tX)}(g),$$
$$\mathrm{Fl}_t^{R_X}(g) = \exp(tX).g = \mu_{\exp(tX)}(g).$$

The right Maurer–Cartan form $\kappa = \kappa^r \in \Omega^1(G, \mathfrak{g})$ is given by $\kappa_x(\xi) := T_x(\mu^{x^{-1}}) \cdot \xi$.
The left Maurer–Cartan form $\kappa^l \in \Omega^1(G, \mathfrak{g})$ is given by $\kappa_x(\xi) := T_x(\mu_{x^{-1}}) \cdot \xi$.

κ^r satisfies the left Maurer-Cartan equation $d\kappa - \frac{1}{2}[\kappa,\kappa]^\wedge_{\mathfrak{g}} = 0$, where $[\quad,\quad]_\wedge$ denotes the wedge product of $\mathfrak{g}$-valued forms on G induced by the Lie bracket. Note that $\frac{1}{2}[\kappa,\kappa]_\wedge(\xi,\eta) = [\kappa(\xi),\kappa(\eta)]$. *$\kappa^l$ satisfies the right Maurer-Cartan equation $d\kappa + \frac{1}{2}[\kappa,\kappa]^\wedge_{\mathfrak{g}} = 0$.*

Proof: Evaluate $d\kappa^r$ on right invariant vector fields R_X, R_Y for $X, Y \in \mathfrak{g}$.

$$\begin{aligned}(d\kappa^r)(R_X,R_Y) &= R_X(\kappa^r(R_Y)) - R_Y(\kappa^r(R_X)) - \kappa^r([R_X,R_Y]) \\ &= R_X(Y) - R_Y(X) + [X,Y] = 0 - 0 + [\kappa^r(R_X),\kappa^r(R_Y)].\end{aligned}$$

The (exterior) derivative of the function $\mathrm{Ad} : G \to GL(\mathfrak{g})$ can be expressed by

$$d\,\mathrm{Ad} = \mathrm{Ad}\,.(\mathrm{ad}\circ\kappa^l) = (\mathrm{ad}\circ\kappa^r).\,\mathrm{Ad}$$

since we have
$d\,\mathrm{Ad}(T\mu_g.X) = \frac{d}{dt}|_0\,\mathrm{Ad}(g.\exp(tX)) = \mathrm{Ad}(g).\,\mathrm{ad}(\kappa^l(T\mu_g.X))$.

Geodesics of a Right-Invariant Metric on a Lie Group

Let $\gamma = \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be a positive-definite bounded (weak) inner product. Then

$$\gamma_x(\xi, \eta) = \gamma\big(T(\mu^{x^{-1}}) \cdot \xi,\ T(\mu^{x^{-1}}) \cdot \eta\big) = \gamma\big(\kappa(\xi),\ \kappa(\eta)\big)$$

is a right-invariant (weak) Riemannian metric on G. Denote by $\check{\gamma} : \mathfrak{g} \to \mathfrak{g}^*$ the mapping induced by γ, and by $\langle \alpha, X \rangle_{\mathfrak{g}}$ the duality evaluation between $\alpha \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$.
Let $g : [a, b] \to G$ be a smooth curve. The velocity field of g, viewed in the right trivializations, coincides with the right logarithmic derivative

$$\delta^r(g) = T(\mu^{g^{-1}}) \cdot \partial_t g = \kappa(\partial_t g) = (g^*\kappa)(\partial_t).$$

The energy of the curve $g(t)$ is given by

$$E(g) = \frac{1}{2} \int_a^b \gamma_g(g', g')dt = \frac{1}{2} \int_a^b \gamma\big((g^*\kappa)(\partial_t), (g^*\kappa)(\partial_t)\big)\, dt.$$

For a variation $g(s,t)$ with fixed endpoints we then use that

$$d(g^*\kappa)(\partial_t,\partial_s) = \partial_t(g^*\kappa(\partial_s)) - \partial_s(g^*\kappa(\partial_t)) - 0,$$

partial integration and the left Maurer–Cartan equation to obtain

$$\begin{aligned}
\partial_s E(g) &= \frac{1}{2}\int_a^b 2\gamma\big(\partial_s(g^*\kappa)(\partial_t),\, (g^*\kappa)(\partial_t)\big)\, dt \\
&= \int_a^b \gamma\big(\partial_t(g^*\kappa)(\partial_s) - d(g^*\kappa)(\partial_t,\partial_s),\, (g^*\kappa)(\partial_t)\big)\, dt \\
&= -\int_a^b \gamma\big((g^*\kappa)(\partial_s),\, \partial_t(g^*\kappa)(\partial_t)\big)\, dt \\
&\qquad - \int_a^b \gamma\big([(g^*\kappa)(\partial_t),(g^*\kappa)(\partial_s)],\, (g^*\kappa)(\partial_t)\big)\, dt \\
&= -\int_a^b \big\langle \check{\gamma}(\partial_t(g^*\kappa)(\partial_t)),\, (g^*\kappa)(\partial_s)\big\rangle_{\mathfrak{g}}\, dt \\
&\qquad - \int_a^b \big\langle \check{\gamma}((g^*\kappa)(\partial_t)),\, \mathrm{ad}_{(g^*\kappa)(\partial_t)}(g^*\kappa)(\partial_s)\big\rangle_{\mathfrak{g}}\, dt \\
&= -\int_a^b \big\langle \check{\gamma}(\partial_t(g^*\kappa)(\partial_t)) + (\mathrm{ad}_{(g^*\kappa)(\partial_t)})^*\check{\gamma}((g^*\kappa)(\partial_t)),\, (g^*\kappa)(\partial_s)\big\rangle_{\mathfrak{g}}\, dt.
\end{aligned}$$

Thus the curve $g(0,t)$ is critical for the energy if and only if

$$\check{\gamma}(\partial_t(g^*\kappa)(\partial_t)) + (\mathrm{ad}_{(g^*\kappa)(\partial_t)})^* \check{\gamma}((g^*\kappa)(\partial_t)) = 0.$$

In terms of the right logarithmic derivative $u : [a,b] \to \mathfrak{g}$ of $g : [a,b] \to G$, given by $u(t) := g^*\kappa(\partial_t) = T_{g(t)}(\mu^{g(t)^{-1}}) \cdot g'(t)$, the geodesic equation has the expression

$$\boxed{\partial_t u = -\,\check{\gamma}^{-1}\,\mathrm{ad}(u)^*\,\check{\gamma}(u)} \qquad (1)$$

Thus the geodesic equation exists in general if and only if $\mathrm{ad}(X)^*\check{\gamma}(X)$ is in the image of $\check{\gamma} : \mathfrak{g} \to \mathfrak{g}^*$, i.e.

$$\mathrm{ad}(X)^*\check{\gamma}(X) \in \check{\gamma}(\mathfrak{g}) \qquad (2)$$

for every $X \in \mathfrak{X}$. Condition (2) then leads to the existence of the Christoffel symbols. [Arnold 1966] has the more restrictive condition $\mathrm{ad}(X)^*\check{\gamma}(Y) \in \check{\gamma} \in \mathfrak{g}$. The geodesic equation for the *momentum* $p := \gamma(u)$:

$$p_t = -\,\mathrm{ad}(\check{\gamma}^{-1}(p))^* p.$$

Soon we shall encounter situations where only the more general condition is satisfied, but where the usual transpose $\mathrm{ad}^\top(X)$ of $\mathrm{ad}(X)$,

$$\mathrm{ad}^\top(X) := \check{\gamma}^{-1} \circ \mathrm{ad}^*_X \circ \check{\gamma}$$

does not exist for all X.

Groups related to $\mathrm{Diff}_c(\mathbb{R})$

The reflexive nuclear (LF) space $C_c^\infty(\mathbb{R})$ of smooth functions with compact support leads to the well-known regular Lie group $\mathrm{Diff}_c(\mathbb{R})$.
Define $C_{c,2}^\infty(\mathbb{R}) = \{f : f' \in C_c^\infty(\mathbb{R})\}$ to be the space of antiderivatives of smooth functions with compact support. It is a reflexive nuclear (LF) space. We also define the space $C_{c,1}^\infty(\mathbb{R}) = \left\{ f \in C_{c,2}^\infty(\mathbb{R}) \ : \ f(-\infty) = 0 \right\}$ of antiderivatives of the form $x \mapsto \int_{-\infty}^x g\, dy$ with $g \in C_c^\infty(\mathbb{R})$.
$\mathrm{Diff}_{c,2}(\mathbb{R}) = \left\{ \varphi = \mathrm{Id} + f : f \in C_{c,2}^\infty(\mathbb{R}), f' > -1 \right\}$ *is the corresponding group.*
Define the two functionals $\mathrm{Shift}_\ell, \mathrm{Shift}_r : \mathrm{Diff}_{c,2}(\mathbb{R}) \to \mathbb{R}$ by

$$\mathrm{Shift}_\ell(\varphi) = \mathrm{ev}_{-\infty}(f) = \lim_{x \to -\infty} f(x), \quad \mathrm{Shift}_r(\varphi) = \mathrm{ev}_{\infty}(f) = \lim_{x \to \infty} f(x)$$

for $\varphi(x) = x + f(x)$.

Then the short exact sequence of smooth homomorphisms of Lie groups

$$\mathrm{Diff}_c(\mathbb{R}) \rightarrowtail \mathrm{Diff}_{c,2}(\mathbb{R}) \xrightarrow{(\mathrm{Shift}_\ell,\mathrm{Shift}_r)} (\mathbb{R}^2,+)$$

describes a semidirect product, where a smooth homomorphic section $s:\mathbb{R}^2\to\mathrm{Diff}_{c,2}(\mathbb{R})$ is given by the composition of flows $s(a,b)=\mathrm{Fl}^{X_\ell}_a\circ\mathrm{Fl}^{X_r}_b$ for the vectorfields $X_\ell=f_\ell\partial_x$, $X_r=f_r\partial_x$ with $[X_\ell,X_r]=0$ where $f_\ell,f_r\in C^\infty(\mathbb{R},[0,1])$ satisfy

$$f_\ell(x)=\begin{cases}1 & \text{for } x\le -1\\ 0 & \text{for } x\ge 0,\end{cases}\qquad f_r(x)=\begin{cases}0 & \text{for } x\le 0\\ 1 & \text{for } x\ge 1.\end{cases}\tag{3}$$

The normal subgroup
$\mathrm{Diff}_{c,1}(\mathbb{R})=\ker(\mathrm{Shift}_\ell)=\{\varphi=\mathrm{Id}+f: f\in C^\infty_{c,1}(\mathbb{R}), f'>-1\}$ of diffeomorphisms which have no shift at $-\infty$ will play an important role later on.

Some diffeomorphism groups on $\mathbb{R}$

We have the following smooth injective group homomorphisms:

$$\begin{array}{ccccccc}
\mathrm{Diff}_c(\mathbb{R}) & \longrightarrow & \mathrm{Diff}_{\mathcal{S}}(\mathbb{R}) & \longrightarrow & \mathrm{Diff}_{W^{\infty,1}}(\mathbb{R}) & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{Diff}_{c,1}(\mathbb{R}) & \longrightarrow & \mathrm{Diff}_{\mathcal{S}_1}(\mathbb{R}) & \longrightarrow & \mathrm{Diff}_{W_1^{\infty,1}}(\mathbb{R}) & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{Diff}_{c,2}(\mathbb{R}) & \longrightarrow & \mathrm{Diff}_{\mathcal{S}_2}(\mathbb{R}) & \longrightarrow & \mathrm{Diff}_{W_2^{\infty,1}}(\mathbb{R}) & \longrightarrow & \mathrm{Diff}_{\mathcal{B}}(\mathbb{R})
\end{array}$$

Each group is a normal subgroup in any other in which it is contained, in particular in $\mathrm{Diff}_{\mathcal{B}}(\mathbb{R})$.
For $\mathcal{S}$ and $W^{\infty,1}$ this works the same as for C_c^∞. For $H^\infty = W^{\infty,2}$ it is surprisingly more subtle.

Solving the Hunter-Saxton equation: The setting

We will denote by $\mathcal{A}(\mathbb{R})$ any of the spaces $C_c^\infty(\mathbb{R})$, $\mathcal{S}(\mathbb{R})$ or $W^{\infty,1}(\mathbb{R})$ and by $\operatorname{Diff}_{\mathcal{A}}(\mathbb{R})$ the corresponding groups $\operatorname{Diff}_c(\mathbb{R})$, $\operatorname{Diff}_{\mathcal{S}}(\mathbb{R})$ or $\operatorname{Diff}_{W^{\infty,1}}(\mathbb{R})$.
Similarly $\mathcal{A}_1(\mathbb{R})$ will denote any of the spaces $C_{c,1}^\infty(\mathbb{R})$, $\mathcal{S}_1(\mathbb{R})$ or $W_1^{\infty,1}(\mathbb{R})$ and $\operatorname{Diff}_{\mathcal{A}_1}(\mathbb{R})$ the corresponding groups $\operatorname{Diff}_{c,1}(\mathbb{R})$, $\operatorname{Diff}_{\mathcal{S}_1}(\mathbb{R})$ or $\operatorname{Diff}_{W_1^{\infty,1}}(\mathbb{R})$.
The $\dot{H}^1$-metric. For $\operatorname{Diff}_{\mathcal{A}}(\mathbb{R})$ and $\operatorname{Diff}_{\mathcal{A}_1}(\mathbb{R})$ the homogeneous H^1-metric is given by

$$G_\varphi(X \circ \varphi, Y \circ \varphi) = G_{\mathrm{Id}}(X, Y) = \int_{\mathbb{R}} X'(x) Y'(x)\, dx\ ,$$

where X, Y are elements of the Lie algebra $\mathcal{A}(\mathbb{R})$ or $\mathcal{A}_1(\mathbb{R})$. We shall also use the notation

$$\langle \cdot, \cdot \rangle_{\dot{H}^1} := G(\cdot, \cdot)\ .$$

Theorem

On $\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ *the geodesic equation is the Hunter-Saxton equation*

$$\boxed{(\varphi_t)\circ\varphi^{-1} = u \qquad u_t = -uu_x + \frac{1}{2}\int_{-\infty}^{x}(u_x(z))^2\,dz\ ,}$$

and the induced geodesic distance is positive.
On the other hand the geodesic equation does not exist on the subgroups $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$*, since the adjoint* $\mathrm{ad}(X)^*\check{G}_{\mathrm{Id}}(X)$ *does not lie in* $\check{G}_{\mathrm{Id}}(\mathcal{A}(\mathbb{R}))$ *for all* $X \in \mathcal{A}(\mathbb{R})$*.*
One obtains the classical form of the Hunter-Saxton equation by differentiating:

$$u_{tx} = -uu_{xx} - \frac{1}{2}u_x^2\ ,$$

Note that $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$ is a natural example of a non-robust Riemannian manifold.

Proof

Note that $\check{G}_{\mathrm{Id}} : \mathcal{A}_1(\mathbb{R}) \to \mathcal{A}_1(\mathbb{R})^*$ is given by $\check{G}_{\mathrm{Id}}(X) = -X''$ if we use the L^2-pairing $X \mapsto (Y \mapsto \int XYdx)$ to embed functions into the space of distributions. We now compute the adjoint of $\mathrm{ad}(X)$:

$$\begin{aligned}\left\langle \mathrm{ad}(X)^* \check{G}_{\mathrm{Id}}(Y), Z \right\rangle &= \check{G}_{\mathrm{Id}}(Y, \mathrm{ad}(X)Z) = G_{\mathrm{Id}}(Y, -[X, Z]) \\ &= \int_{\mathbb{R}} Y'(x)\big(X'(x)Z(x) - X(x)Z'(x)\big)'\,dx \\ &= \int_{\mathbb{R}} Z(x)\big(X''(x)Y'(x) - (X(x)Y'(x))''\big)\,dx\,.\end{aligned}$$

Therefore the adjoint as an element of $\mathcal{A}_1^*$ is given by

$$\mathrm{ad}(X)^* \check{G}_{\mathrm{Id}}(Y) = X''Y' - (XY')''\,.$$

For $X = Y$ we can rewrite this as

$$\begin{aligned}\mathrm{ad}(X)^* \check{G}_{\mathrm{Id}}(X) &= \tfrac{1}{2}\big((X'^2)' - (X^2)'''\big) = \frac{1}{2}\Big(\int_{-\infty}^{x} X'(y)^2\,dy - (X^2)'\Big)'' \\ &= \frac{1}{2}\check{G}_{\mathrm{Id}}\Big(-\int_{-\infty}^{x} X'(y)^2\,dy + (X^2)'\Big)\,.\end{aligned}$$

If $X \in \mathcal{A}_1(\mathbb{R})$ then the function $-\frac{1}{2}\int_{-\infty}^{x} X'(y)^2\,dy + \frac{1}{2}(X^2)'$ is again an element of $\mathcal{A}_1(\mathbb{R})$. This follows immediately from the definition of $\mathcal{A}_1(\mathbb{R})$. Therefore the geodesic equation exists on $\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ and is as given.
However if $X \in \mathcal{A}(\mathbb{R})$, a neccessary condition for $\int_{-\infty}^{x}(X'(y))^2 dy \in \mathcal{A}(\mathbb{R})$ would be $\int_{-\infty}^{\infty} X'(y)^2 dy = 0$, which would imply $X' = 0$. Thus the geodesic equation does not exist on $\mathcal{A}(\mathbb{R})$.
The positivity of geodesic distance will follow from the explicit formula for geodesic distance below. QED.

Theorem.

[BBM2014] [A version for $Diff(S^1)$ is by J.Lenells 2007,08,11]
We define the R-map by:

$$R : \begin{cases} \mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R}) \to \mathcal{A}\big(\mathbb{R}, \mathbb{R}_{>-2}\big) \subset \mathcal{A}(\mathbb{R}, \mathbb{R}) \\ \varphi \mapsto 2\left((\varphi')^{1/2} - 1\right) . \end{cases}$$

The R-map is invertible with inverse

$$R^{-1} : \begin{cases} \mathcal{A}\big(\mathbb{R}, \mathbb{R}_{>-2}\big) \to \mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R}) \\ \gamma \mapsto x + \frac{1}{4}\int_{-\infty}^{x} \gamma^2 + 4\gamma \, dx . \end{cases}$$

The pull-back of the flat L^2-metric via R is the $\dot{H}^1$-metric on $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$, *i.e.,*

$$R^*\langle \cdot, \cdot \rangle_{L^2} = \langle \cdot, \cdot \rangle_{\dot{H}^1} .$$

Thus the space $\left(\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R}), \dot{H}^1\right)$ *is a flat space in the sense of Riemannian geometry.*
Here $\langle \cdot, \cdot \rangle_{L^2}$ denotes the L^2-inner product on $\mathcal{A}(\mathbb{R})$ with constant volume dx.

Proof

To compute the pullback of the L^2-metric via the R-map we first need to calculate its tangent mapping. For this let $h = X \circ \varphi \in T_\varphi \mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ and let $t \mapsto \psi(t)$ be a smooth curve in $\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ with $\psi(0) = \mathrm{Id}$ and $\partial_t|_0 \psi(t) = X$. We have:

$$\begin{aligned}
T_\varphi R.h &= \partial_t|_0 R(\psi(t)\circ\varphi) = \partial_t|_0 2\Big(((\psi(t)\circ\varphi)_x)^{1/2} - 1\Big) \\
&= \partial_t|_0 2((\psi(t)_x \circ \varphi)\,\varphi_x)^{1/2} \\
&= 2(\varphi_x)^{1/2}\partial_t|_0((\psi(t)_x)^{1/2}\circ\varphi) = (\varphi_x)^{1/2}\Big(\frac{\psi_{tx}(0)}{(\psi(0)_x)^{-1/2}}\circ\varphi\Big) \\
&= (\varphi_x)^{1/2}(X'\circ\varphi) = (\varphi')^{1/2}(X'\circ\varphi)\,.
\end{aligned}$$

Using this formula we have for $h = X_1 \circ \varphi, k = X_2 \circ \varphi$:

$$R^*\langle h, k\rangle_{L^2} = \langle T_\varphi R.h, T_\varphi R.k\rangle_{L^2} = \int_{\mathbb{R}} X_1'(x) X_2'(x)\,dx = \langle h, k\rangle_{\dot{H}^1} \;\mathit{QED}$$

Corollary

Given $\varphi_0, \varphi_1 \in \mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ *the geodesic* $\varphi(t,x)$ *connecting them is given by*

$$\varphi(t,x) = R^{-1}\Big((1-t)R(\varphi_0) + tR(\varphi_1)\Big)(x)$$

and their geodesic distance is

$$d(\varphi_0,\varphi_1)^2 = 4\int_{\mathbb{R}} \left((\varphi_1')^{1/2} - (\varphi_0')^{1/2}\right)^2 dx\ .$$

But this construction shows much more: For $\mathcal{S}_1$, C_1^∞, and even for many kinds of Denjoy-Carleman ultradifferentiable model spaces (not explained here). This shows that Sobolev space methods for treating nonlinear PDEs is not the only method.

Corollary: *The metric space* $\left(\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R}), \dot{H}^1\right)$ *is path-connected and geodesically convex but not geodesically complete. In particular, for every* $\varphi_0 \in \mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ *and* $h \in T_{\varphi_0}\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$, $h \neq 0$ *there exists a time* $T \in \mathbb{R}$ *such that* $\varphi(t,\cdot)$ *is a geodesic for* $|t| < |T|$ *starting at* φ_0 *with* $\varphi_t(0) = h$, *but* $\varphi_x(T,x) = 0$ *for some* $x \in \mathbb{R}$.

Theorem: *The square root representation on the diffeomorphism group* $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$ *is a bijective mapping, given by:*

$$R : \begin{cases} \mathrm{Diff}_{\mathcal{A}}(\mathbb{R}) \to \left(\mathrm{Im}(R), \|\cdot\|_{L^2}\right) \subset \left(\mathcal{A}\big(\mathbb{R}, \mathbb{R}_{>-2}\big), \|\cdot\|_{L^2}\right) \\ \qquad\qquad \varphi \mapsto 2\left((\varphi')^{1/2} - 1\right). \end{cases}$$

The pull-back of the restriction of the flat L^2*-metric to* $\mathrm{Im}(R)$ *via* R *is again the homogeneous Sobolev metric of order one. The image of the* R*-map is the splitting submanifold of* $\mathcal{A}(\mathbb{R}, \mathbb{R}_{>-2})$ *given by:*

$$\mathrm{Im}(R) = \left\{\gamma \in \mathcal{A}(\mathbb{R}, \mathbb{R}_{>-2}) : F(\gamma) := \int_{\mathbb{R}} \gamma(\gamma + 4)\, dx = 0\right\}.$$

On the space $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$ the geodesic equation does not exist. Still:
Corollary: *The geodesic distance* $d^{\mathcal{A}}$ *on* $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$ *coincides with the restriction of* $d^{\mathcal{A}_1}$ *to* $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$*, i.e., for* $\varphi_0, \varphi_1 \in \mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$ *we have*

$$d^{\mathcal{A}}(\varphi_0, \varphi_1) = d^{\mathcal{A}_1}(\varphi_0, \varphi_1)\ .$$

Continuing Geodesics Beyond the Group, or How Solutions of the Hunter–Saxton Equation Blow Up

Consider a straight line $\gamma(t) = \gamma_0 + t\gamma_1$ in $\mathcal{A}(\mathbb{R}, \mathbb{R})$. Then $\gamma(t) \in \mathcal{A}(\mathbb{R}, \mathbb{R}_{>-2})$ precisely for t in an open interval (t_0, t_1) which is finite at least on one side, say, at $t_1 < \infty$. Note that

$$\varphi(t)(x) := R^{-1}(\gamma(t))(x) = x + \frac{1}{4}\int_{-\infty}^{x} \gamma^2(t)(u) + 4\gamma(t)(u)\, du$$

makes sense for all t, that $\varphi(t) : \mathbb{R} \to \mathbb{R}$ is smooth and that $\varphi(t)'(x) \geq 0$ for all x and t; thus, $\varphi(t)$ is monotone non-decreasing. Moreover, $\varphi(t)$ is proper and surjective since $\gamma(t)$ vanishes at $-\infty$ and ∞. Let

$$\mathrm{Mon}_{\mathcal{A}_1}(\mathbb{R}) := \left\{ \mathrm{Id} + f : f \in \mathcal{A}_1(\mathbb{R}, \mathbb{R}), f' \geq -1 \right\}$$

be the monoid (under composition) of all such functions.

For $\gamma \in \mathcal{A}(\mathbb{R},\mathbb{R})$ let $x(\gamma) := \min\{x \in \mathbb{R} \cup \{\infty\} : \gamma(x) = -2\}$. Then for the line $\gamma(t)$ from above we see that $x(\gamma(t)) < \infty$ for all $t > t_1$. Thus, if the 'geodesic' $\varphi(t)$ leaves the diffeomorphism group at t_1, it never comes back but stays inside $\mathrm{Mon}_{\mathcal{A}_1}(\mathbb{R}) \setminus \mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ for the rest of its life. In this sense, $\mathrm{Mon}_{\mathcal{A}_1}(\mathbb{R})$ is a *geodesic completion* of $\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$, and $\mathrm{Mon}_{\mathcal{A}_1}(\mathbb{R}) \setminus \mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ is the *boundary*.

What happens to the corresponding solution $u(t,x) = \varphi_t(t, \varphi(t)^{-1}(x))$ of the HS equation? In certain points it has infinite derivative, it may be multivalued, or its graph can contain whole vertical intervals. If we replace an element $\varphi \in \mathrm{Mon}_{\mathcal{A}_1}(\mathbb{R})$ by its graph $\{(x, \varphi(x)) : x \in \mathbb{R}\} \subset \mathbb{R}$ we get a smooth 'monotone' submanifold, a smooth monotone relation. The inverse φ^{-1} is then also a smooth monotone relation. Then $t \mapsto \{(x, u(t,x)) : x \in \mathbb{R}\}$ is a (smooth) curve of relations. Checking that it satisfies the HS equation is an exercise left for the interested reader. What we have described here is the *flow completion* of the HS equation in the spirit of [KhesinMichor2004].

Soliton-Like Solutions of the Hunter Saxton equation

For a right-invariant metric G on a diffeomorphism group one can ask whether (generalized) solutions $u(t) = \varphi_t(t) \circ \varphi(t)^{-1}$ exist such that the momenta $\check{G}(u(t)) =: p(t)$ are distributions with finite support. Here the geodesic $\varphi(t)$ may exist only in some suitable Sobolev completion of the diffeomorphism group. By the general theory, the momentum $\mathrm{Ad}(\varphi(t))^* p(t) = \varphi(t)^* p(t) = p(0)$ is constant. In other words,

$$p(t) = (\varphi(t)^{-1})^* p(0) = \varphi(t)_* p(0),$$

i.e., the momentum is carried forward by the flow and remains in the space of distributions with finite support. The infinitesimal version (take ∂_t of the last expression) is

$$p_t(t) = -\mathcal{L}_{u(t)} p(t) = -\operatorname{ad}_{u(t)}{}^* \, p(t).$$

The space of N-solitons of order 0 consists of momenta of the form $p_{y,a} = \sum_{i=1}^{N} a_i \delta_{y_i}$ with $(y, a) \in \mathbb{R}^{2N}$. Consider an initial soliton $p_0 = \check{G}(u_0) = -u_0'' = \sum_{i=1}^{N} a_i \, \delta_{y_i}$ with $y_1 < y_2 < \cdots < y_N$. Let H be the Heaviside function

$$H(x) = \begin{cases} 0, & x < 0, \\ \frac{1}{2}, & x = 0, \\ 1, & x > 0, \end{cases}$$

and $D(x) = 0$ for $x \leq 0$ and $D(x) = x$ for $x > 0$. We will see later why the choice $H(0) = \frac{1}{2}$ is the most natural one; note that the behavior is called the Gibbs phenomenon. With these functions we can write

$$u_0''(x) = -\sum_{i=1}^{N} a_i \delta_{y_i}(x)$$

$$u_0'(x) = -\sum_{i=1}^{N} a_i H(x - y_i)$$

$$u_0(x) = -\sum_{i=1}^{N} a_i D(x - y_i).$$

We will assume henceforth that $\sum_{i=1}^{N} a_i = 0$. Then $u_0(x)$ is constant for $x > y_N$ and thus $u_0 \in H_1^1(\mathbb{R})$; with a slight abuse of notation we assume that $H_1^1(\mathbb{R})$ is defined similarly to $H_1^\infty(\mathbb{R})$. Defining $S_i = \sum_{j=1}^{i} a_j$ we can write

$$u_0'(x) = -\sum_{i=1}^{N} S_i \left(H(x - y_i) - H(x - y_{i+1})\right).$$

This formula will be useful because
$\operatorname{supp}(H(. - y_i) - H(. - y_{i+1})) = [y_i, y_{i+1}]$.
The evolution of the geodesic $u(t)$ with initial value $u(0) = u_0$ can be described by a system of ordinary differential equations (ODEs) for the variables (y, a).
Theorem *The map $(y, a) \mapsto \sum_{i=1}^{N} a_i \delta_{y_i}$ is a Poisson map between the canonical symplectic structure on $\mathbb{R}^{2N}$ and the Lie–Poisson structure on the dual $T^*_{\mathrm{Id}} \operatorname{Diff}_{\mathcal{A}}(\mathbb{R})$ of the Lie algebra.*

In particular, this means that the ODEs for (y, a) are Hamilton's equations for the pullback Hamiltonian

$$E(y, a) = \frac{1}{2} G_{\mathrm{Id}}(u_{(y,a)}, u_{(y,a)}),$$

with $u_{(y,a)} = \check{G}^{-1}(\sum_{i=1}^{N} a_i \delta_{y_i}) = -\sum_{i=1}^{N} a_i D(. - y_i)$. We can obtain the more explicit expression

$$E(y, a) = \frac{1}{2} \int_{\mathbb{R}} \left(u_{(y,a)}(x)' \right)^2 \, dx = \frac{1}{2} \int_{\mathbb{R}} \left(\sum_{i=1}^{N} S_i \mathbb{1}_{[y_i, y_{i+1}]} \right)^2 \, dx$$
$$= \frac{1}{2} \sum_{i=1}^{N} S_i^2 (y_{i+1} - y_i).$$

Hamilton's equations $\dot{y}_i = \partial E / \partial a_i$, $\dot{a}_i = -\partial E / \partial y_i$ are in this case

$$\dot{y}_i(t) = \sum_{j=i}^{N-1} S_i(t)(y_{i+1}(t) - y_i(t)),$$
$$\dot{a}_i(t) = \frac{1}{2} \left(S_i(t)^2 - S_{i-1}(t)^2 \right).$$

Using the R-map we can find explicit solutions for these equations as follows. Let us write $a_i(0) = a_i$ and $y_i(0) = y_i$. The geodesic with initial velocity u_0 is given by

$$\varphi(t,x) = x + \frac{1}{4}\int_{-\infty}^{x} t^2(u_0'(y))^2 + 4tu_0'(y)\,dy$$

$$u(t,x) = u_0(\varphi^{-1}(t,x)) + \frac{t}{2}\int_{-\infty}^{\varphi^{-1}(t,x)} u_0'(y)^2\,dy.$$

First note that

$$\varphi'(t,x) = \left(1 + \frac{t}{2}u_0'(x)\right)^2$$

$$u'(t,z) = \frac{u_0'\left(\varphi^{-1}(t,z)\right)}{1 + \frac{t}{2}u_0'\left(\varphi^{-1}(t,z)\right)}.$$

Using the identity $H(\varphi^{-1}(t,z)-y_i) = H(z-\varphi(t,y_i))$ we obtain

$$u_0'\left(\varphi^{-1}(t,z)\right) = -\sum_{i=1}^{N} a_i H\left(z-\varphi(t,y_i)\right),$$

and thus

$$\left(u_0'\left(\varphi^{-1}(t,z)\right)\right)' = -\sum_{i=1}^{N} a_i \delta_{\varphi(t,y_i)}(z).$$

Combining these we obtain

$$\begin{aligned} u''(t,z) &= \frac{1}{\left(1+\frac{t}{2}u_0'\left(\varphi^{-1}(t,z)\right)\right)^2}\left(-\sum_{i=1}^{N} a_i \delta_{\varphi(t,y_i)}(z)\right) \\ &= \sum_{i=1}^{N} \frac{-a_i}{\left(1+\frac{t}{2}u_0'(y_i)\right)^2}\delta_{\varphi(t,y_i)}(z). \end{aligned}$$

From here we can read off the solution of Hamilton's equations

$$\begin{aligned} y_i(t) &= \varphi(t,y_i) \\ a_i(t) &= -a_i\left(1+\tfrac{t}{2}u_0'(y_i)\right)^{-2}. \end{aligned}$$

When trying to evaluate $u_0'(y_i)$,

$$u_0'(y_i) = a_i H(0) - S_i,$$

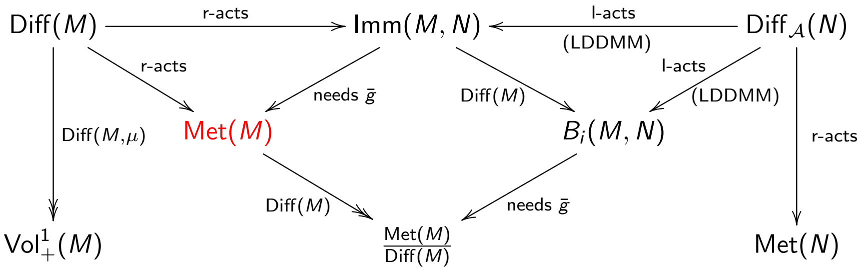
we see that u_0' is discontinuous at y_i and it is here that we seem to have the freedom to choose the value $H(0)$. However, it turns out that we observe the Gibbs phenomenon, i.e., only the choice $H(0) = \frac{1}{2}$ leads to solutions of Hamilton's equations. Also, the regularized theory of multiplications of distributions (Colombeau, Kunzinger et.al.) leads to this choice. Thus we obtain

$$y_i(t) = y_i + \sum_{j=1}^{i-1} \left(\frac{t^2}{4} S_j^2 - t S_j \right) (y_{j+1} - y_j)$$

$$a_i(t) = \frac{-a_i}{\left(1 + \frac{t}{2}\left(\frac{a_i}{2} - S_i\right)\right)^2} = -\left(\frac{S_i}{1 - \frac{t}{2} S_i} - \frac{S_{i-1}}{1 - \frac{t}{2} S_{i-1}} \right).$$

It can be checked by direct computation that these functions indeed solve Hamilton's equations.

Riemannian geometries on spaces of Riemannian metrics and pulling them back to diffeomorphism groups.



Weak Riemann metrics on $\mathrm{Met}(M)$

All of them are $\mathrm{Diff}(M)$-invariant; natural, tautological.

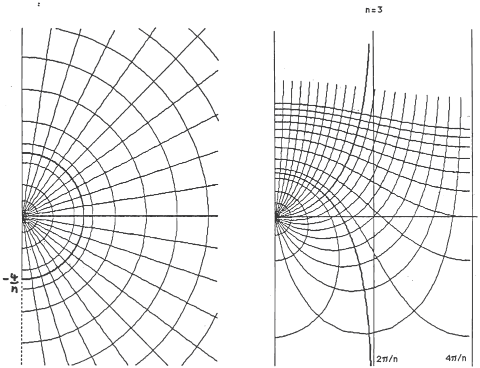
$$
\begin{aligned}
G_g(h,k) &= \int_M g_2^0(h,k)\,\mathrm{vol}(g) = \int \mathrm{Tr}(g^{-1}hg^{-1}k)\,\mathrm{vol}(g), \quad L^2\text{-metr.}\\
\text{or } &= \Phi(\mathrm{Vol(g)})\int_M g_2^0(h,k)\,\mathrm{vol}(g) \qquad \text{conformal}\\
\text{or } &= \int_M \Phi(\mathrm{Scal}^g).g_2^0(h,k)\,\mathrm{vol}(g) \qquad \text{curvature modified}\\
\text{or } &= \int_M g_2^0((1+\Delta^g)^p h,k)\,\mathrm{vol}(g) \qquad \text{Sobolev order } p\\
\text{or } &= \int_M \Big(g_2^0(h,k) + g_3^0(\nabla^g h, \nabla^g k) + \dots\\
&\qquad + g_p^0((\nabla^g)^p h, (\nabla^g)^p k)\Big)\,\mathrm{vol}(g)
\end{aligned}
$$

where Φ is a suitable real-valued function, $\mathrm{Vol} = \int_M \mathrm{vol}(g)$ is the total volume of (M,g), Scal is the scalar curvature of (M,g), and where g_2^0 is the induced metric on $\binom{0}{2}$-tensors.

The L^2-metric on the space of all Riemann metrics

[Ebin 1970]. Geodesics and curvature [Freed Groisser 1989]. [Gil-Medrano Michor 1991] for non-compact M. [Clarke 2009] showed that geodesic distance for the L^2-metric is positive, and he determined the metric completion of $\mathrm{Met}(M)$.
The geodesic equation is completely decoupled from space, it is an ODE:

$$g_{tt} = g_t g^{-1} g_t + \tfrac{1}{4}\,\mathrm{Tr}(g^{-1} g_t g^{-1} g_t)\, g - \tfrac{1}{2}\,\mathrm{Tr}(g^{-1} g_t)\, g_t$$



$$\exp_0(A) = \frac{2}{n}\log\left(\left(1+\frac{1}{4}\operatorname{Tr}(A)\right)^2 + \frac{n}{16}\operatorname{Tr}(A_0^2)\right) Id$$
$$+ \frac{4}{\sqrt{n\operatorname{Tr}(A_0^2)}}\arctan\left(\frac{\sqrt{n\operatorname{Tr}(A_0^2)}}{4+\operatorname{Tr}(A)}\right) A_0.$$

Back to the the general metric on $\mathrm{Met}(M)$.

We describe all these metrics uniformly as

$$\begin{aligned} G_g^P(h,k) &= \int_M g_2^0(P_g h, k)\,\mathrm{vol}(g) \\ &= \int_M \mathrm{Tr}(g^{-1}.P_g(h).g^{-1}.k)\,\mathrm{vol}(g), \end{aligned}$$

where

$$P_g : \Gamma(S^2T^*M) \to \Gamma(S^2T^*M)$$

is a positive, symmetric, bijective pseudo-differential operator of order $2p, p \geq 0$, depending smoothly on the metric g, and also $\mathrm{Diff}(M)$-equivariantly:

$$\varphi^* \circ P_g = P_{\varphi^* g} \circ \varphi^*$$

The geodesic equation in this notation:

$$g_{tt} = P^{-1}\Big[(D_{(g,.)}Pg_t)^*(g_t) + \frac{1}{4}.g.\operatorname{Tr}(g^{-1}.Pg_t.g^{-1}.g_t) + \frac{1}{2}g_t.g^{-1}.Pg_t + \frac{1}{2}Pg_t.g^{-1}.g_t - (D_{(g,g_t)}P)g_t - \frac{1}{2}\operatorname{Tr}(g^{-1}.g_t).Pg_t\Big]$$

We can rewrite this equation to get it in a slightly more compact form:

$$\begin{aligned}(Pg_t)_t &= (D_{(g,g_t)}P)g_t + Pg_{tt} \\ &= (D_{(g,.)}Pg_t)^*(g_t) + \frac{1}{4}.g.\operatorname{Tr}(g^{-1}.Pg_t.g^{-1}.g_t) \\ &\quad + \frac{1}{2}g_t.g^{-1}.Pg_t + \frac{1}{2}Pg_t.g^{-1}.g_t - \frac{1}{2}\operatorname{Tr}(g^{-1}.g_t).Pg_t\end{aligned}$$

Well posedness of geodesic equation.

Assumptions *Let* $P_g(h)$, $P_g^{-1}(k)$ *and* $(D_{(g,.)}Ph)^*(m)$ *be linear pseudo-differential operators of order* $2p$ *in* m, h *and of order* $-2p$ *in* k *for some* $p \geq 0$.
As mappings in the foot point g, *we assume that all mappings are non-linear, and that they are a composition of operators of the following type:*
(a) Non-linear differential operators of order $l \leq 2p$, *i.e.,*

$$A(g)(x) = A\big(x, g(x), (\hat{\nabla} g)(x), \ldots, (\hat{\nabla}^l g)(x)\big),$$

(b) Linear pseudo-differential operators of order $\leq 2p$,
such that the total (top) order of the composition is $\leq 2p$.
Since $h \mapsto P_g h$ induces a weak inner product, it is a symmetric and injective pseudodifferential operator. We *assume* that it is elliptic and selfadjoint. Then it is Fredholm and has vanishing index. Thus it is invertible and $g \mapsto P_g^{-1}$ is smooth
$H^k(S^2_+ T^*M) \to L(H^k(S^2 T^*M), H^{k+2p}(S^2 T^*M))$ by the implicit function theorem on Banach spaces.

Theorem. [Bauer, Harms, M. 2011] *Let the assumptions above hold. Then for* $k > \frac{\dim(M)}{2}$*, the initial value problem for the geodesic equation has unique local solutions in the Sobolev manifold* $\mathrm{Met}^{k+2p}(M)$ *of* H^{k+2p}*-metrics. The solutions depend* C^∞ *on* t *and on the initial conditions* $g(0,\ .\ ) \in \mathrm{Met}^{k+2p}(M)$ *and* $g_t(0,\ .\ ) \in H^{k+2p}(S^2 T^* M)$*. The domain of existence (in* t*) is uniform in* k *and thus this also holds in* $\mathrm{Met}(M)$*.*
Moreover, in each Sobolev completion $\mathrm{Met}^{k+2p}(M)$*, the Riemannian exponential mapping* $\exp^P$ *exists and is smooth on a neighborhood of the zero section in the tangent bundle, and* $(\pi, \exp^P)$ *is a diffeomorphism from a (smaller) neighborhood of the zero section to a neighborhood of the diagonal in* $\mathrm{Met}^{k+2p}(M) \times \mathrm{Met}^{k+2p}(M)$*. All these neighborhoods are uniform in* $k > \frac{\dim(M)}{2}$ *and can be chosen* H^{k_0+2p}*-open, where* $k_0 > \frac{\dim(M)}{2}$*. Thus all properties of the exponential mapping continue to hold in* $\mathrm{Met}(M)$*.*

Conserved Quantities on Met(M).

Right action of Diff(M) on Met(M) given by

$$(g, \phi) \mapsto \phi^* g.$$

Fundamental vector field (infinitesimal action):

$$\zeta_X(g) = \mathcal{L}_X g = -2\,\mathrm{Sym}\,\nabla(g(X)).$$

If metric G^P is invariant, we have the following conserved quantities

$$\begin{aligned}
\text{const} &= G^P(g_t, \zeta_X(g)) \\
&= -2 \int_M g_1^0\big(\nabla^*\,\mathrm{Sym}\,Pg_t, g(X)\big)\,\mathrm{vol}(g) \\
&= -2 \int_M g\big(g^{-1}\nabla^* Pg_t, X\big)\,\mathrm{vol}(g)
\end{aligned}$$

Since this holds for all vector fields X,

$$\boxed{(\nabla^* Pg_t)\,\mathrm{vol}(g) \in \Gamma(T^*M \otimes_M \mathrm{vol}(M)) \text{ is const. in } t.}$$

For which metric is the Ricci flow a gradient flow

$\mathrm{Met}(M)$ is convex and open subset, thus contractible. A necessary and sufficient condition for Ricci curvature to be a gradient vector field with respect to the G^P-metric is that the following exterior derivative vanishes:

$$\big(dG^P(\mathrm{Ric},\cdot)\big)(h,k) = hG^P(\mathrm{Ric},k) - kG^P(\mathrm{Ric},h) - G^P(\mathrm{Ric},[h,k]) = 0.$$

It suffices to look at constant vector fields h, k, in which case $[h,k] = 0$. We have

$$\begin{aligned}
&hG^P(\mathrm{Ric},k) - kG^P(\mathrm{Ric},h)\\
&= \int \Big(- \mathrm{Tr}\left(g^{-1}hg^{-1}(P\,\mathrm{Ric})g^{-1}k\right) + \mathrm{Tr}\left(g^{-1}kg^{-1}(P\,\mathrm{Ric})g^{-1}h\right)\\
&+ \mathrm{Tr}\left(g^{-1}D_{g,h}(P\,\mathrm{Ric})g^{-1}k\right) - \mathrm{Tr}\left(g^{-1}D_{g,k}(P\,\mathrm{Ric})g^{-1}h\right)\\
&- \mathrm{Tr}\left(g^{-1}(P\,\mathrm{Ric})g^{-1}hg^{-1}k\right) + \mathrm{Tr}\left(g^{-1}(P\,\mathrm{Ric})g^{-1}kg^{-1}h\right)\\
&+ \frac{1}{2}\mathrm{Tr}\left(g^{-1}(P\,\mathrm{Ric})g^{-1}k\right)\mathrm{Tr}(g^{-1}h) - \frac{1}{2}\mathrm{Tr}\left(g^{-1}(P\,\mathrm{Ric})g^{-1}h\right)\mathrm{Tr}(g^{-1}k)\Big)
\end{aligned}$$

Some terms in this formula cancel out because for symmetric A, B, C one has

$$\mathrm{Tr}(ABC) = \mathrm{Tr}((ABC)^\top) = \mathrm{Tr}(C^\top B^\top A^\top) = \mathrm{Tr}(A^\top C^\top B^\top) = \mathrm{Tr}(ACB).$$

Therefore

$$\begin{aligned}
&hG^P(\mathrm{Ric}, k) - kG^P(\mathrm{Ric}, h) \\
&= \int \Big(\mathrm{Tr}\left(g^{-1} D_{g,h}(P\,\mathrm{Ric}) g^{-1} k\right) - \mathrm{Tr}\left(g^{-1} D_{g,k}(P\,\mathrm{Ric}) g^{-1} h\right) \\
&\qquad + \frac{1}{2}\mathrm{Tr}\left(g^{-1}(P\,\mathrm{Ric}) g^{-1} k\right) \mathrm{Tr}(g^{-1} h) \\
&\qquad - \frac{1}{2}\mathrm{Tr}\left(g^{-1}(P\,\mathrm{Ric}) g^{-1} h\right) \mathrm{Tr}(g^{-1} k)\Big) \mathrm{vol}(g).
\end{aligned}$$

We write $D_{g,h}(P\,\mathrm{Ric}) = Q(h)$ for some differential operator Q mapping symmetric two-tensors to themselves and Q^* for the adjoint of Q with respect to $\int_M g_2^0(h,k)\,\mathrm{vol}(g)$.

$$hG^P(\mathrm{Ric},k) - kG^P(\mathrm{Ric},h)$$
$$= \int \Big(g_2^0(Q(h),k) - g_2^0(Q(k),h)$$
$$+ \frac{1}{2} g_2^0(P\,\mathrm{Ric},k)\,\mathrm{Tr}(g^{-1}h) - \frac{1}{2} g_2^0(P\,\mathrm{Ric},h)\,\mathrm{Tr}(g^{-1}k)\Big)\,\mathrm{vol}(g)$$
$$= \int g_2^0\Big(Q(h) - Q^*(h) + \frac{1}{2}(P\,\mathrm{Ric}).\,\mathrm{Tr}(g^{-1}h) - \frac{1}{2} g.g_2^0(P\,\mathrm{Ric},h), k\Big)\,\mathrm{vol}(g).$$

We have proved:

Lemma. *The Ricci vector field* Ric *is a gradient field for the* G^P*-metric if and only if the equation*

$$2\big(Q(h) - Q^*(h)\big) + (P\,\mathrm{Ric}).\,\mathrm{Tr}(g^{-1}h) - g.g_2^0(P\,\mathrm{Ric},h) = 0,$$
$$\text{with } Q(h) = Q_g(h) = D_{g,h}(P_g\,\mathrm{Ric}_g),$$

is satisfied for all $g \in \mathrm{Met}(M)$ *and and all symmetric* $\binom{0}{2}$*-tensors* h.

None of the specific metrics mentioned here satisfies the Lemma in general dimension. Note that the Lemma is trivially satisfied in dimension $\dim(M) = 1$. In dimension 2 the equation $\mathrm{Ric}_g = \frac{1}{2}\,\mathrm{Scal}_g$ holds and the operator $P_g h = 2\,\mathrm{Scal}_g^{-1}\,h$ satisfies equation (1) on the open subset $\{g : \mathrm{Scal}_g \neq 0\}$. Generally, equation (1) is satisfied if $P_g\,\mathrm{Ric}_g = g$, but this cannot hold on the space of all metrics if $\dim(M) > 2$.

On $\mathbb{R}^n$: $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^n)$ acts on $\mathrm{Met}_{\mathcal{A}}(\mathbb{R}^n)$

For $\mathcal{A} = C_c^\infty, \mathcal{S}, H^\infty$, we consider here the right action

$$r : \mathrm{Met}_{\mathcal{A}}(\mathbb{R}^n) \times \mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^n) \to \mathrm{Met}_{\mathcal{A}}(\mathbb{R}^n)$$
$$\mathrm{Met}_{\mathcal{A}}(\mathbb{R}^n) := \{g \in \mathrm{Met}(\mathbb{R}^n) : g - \mathrm{can} \in \Gamma_{\mathcal{A}}(S^2 T^*\mathbb{R}^n)\}$$

which is given by $r(g,\varphi) = \varphi^* g$, together with its partial mappings $r(g,\varphi) = r^\varphi(g) = r_g(\varphi) = \mathrm{Pull}^g(\varphi)$.

Lemma. *For* $g \in \mathrm{Met}_{\mathcal{A}}(\mathbb{R}^n)$ *the isometry group* $\mathrm{Isom}(g)$ *has trivial intersection with* $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^n)$*.*

Proof. The Killing equation is an elliptic equation whose coefficients are bounded away from 0. Thus each Killing vector field grows linearly and cannot lie in $\mathfrak{X}_{\mathcal{A}}(\mathbb{R}^n)$.
Alternatively, since g falls towards the standard metric $\bar{g}$, each isometry of g fall towards an isometry of $\bar{g}$, i.e., towards an element of $O(n)$. But $O(n) \cap \mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^n) = \{\mathrm{Id}\}$. □

Theorem. *If* $n \geq 2$*, the image of* $\mathrm{Pull}^{\bar{g}}$*, i.e., the* $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^n)$*-orbit through* $\bar{g}$*, is the set* $\mathrm{Met}^{\mathrm{flat}}_{\mathcal{A}}(\mathbb{R}^n)$ *of all flat metrics in* $\mathrm{Met}_{\mathcal{A}}(\mathbb{R}^n)$*.*
Proof. Curvature $R^{\varphi^* \bar{g}} = \varphi^* R^{\bar{g}} = \varphi^* 0 = 0$, so the orbit consists of flat metrics. To see the converse, let $g \in \mathrm{Met}^{\mathrm{flat}}_{\mathcal{A}}(\mathbb{R}^n)$ be a flat metric. Considering g as a symmetric positive matrix, let $s := \sqrt{g}$. We search for an orthogonal matrix valued function $u \in C^\infty(\mathbb{R}^n, SO(n))$ such that $u.s = d\varphi$ for a diffeomorphism φ. Let $\sigma_i := \sum_j s_{ij} dx^j$ be the rows of s. Then for the metric we have $g = \sum_i \sigma_i \otimes \sigma_i$, thus the column vector $\sigma = (\sigma_1, \dots, \sigma_n)^t$ of 1-forms is a global orthonormal coframe. We want $u.\sigma = d\varphi$, so the 2-form $d(u.\sigma)$ should vanish. But

$$0 = d(u.\sigma) = du \wedge \sigma + u.d\sigma \iff 0 = u^{-1}.du \wedge \sigma + d\sigma$$

This means that the $\mathfrak{o}(n)$-valued 1-form $\omega := u^{-1}.du$ is the connection 1-form for the Levi-Civita connection of the metric g. Since g is flat, the curvature 2-form $\Omega = d\omega + \omega \wedge \omega$ vanishes.

We consider now the trivial principal bundle $\mathrm{pr}_1 : \mathbb{R}^n \times SO(n) \to \mathbb{R}^n$ and the principal connection form $\mathrm{pr}_1^* \omega$ on it which is flat, so the horizontal distribution is integrable. Let $L(u_0) \subset \mathbb{R}^n \times SO(n)$ be the horizontal leaf through the point $(0, u_0) \in \mathbb{R}^n \times SO(n)$, then the restriction $\mathrm{pr}_1 : L(u_0) \to \mathbb{R}^n$ is a covering map and thus a diffeomorphism whose inverse furnishes us the required $u \in C^\infty(\mathbb{R}^n, SO(n))$ which is unique up to right multiplication by $u_0 \in SO(n)$. The function $u : \mathbb{R}^n \to SO(n)$ is also called the Cartan development fo ω.

Thus $u.\sigma = d\varphi$ for for a column vector $\varphi = (\varphi^1, \dots, \varphi^n)$ of functions which defines a smooth map $\varphi : \mathbb{R}^n \to \mathbb{R}^n$. Since $d\varphi = u.\sigma$ is everywhere invertible, φ is locally a diffeomorphism. Since g falls to $\bar{g} = \mathbb{I}_n$ as a function in $\mathcal{A}$, the same is true for σ and thus also for $u.\sigma$ since u is bounded. So $d(\varphi - \mathrm{Id}_{\mathbb{R}^n})$ is asymptotically 0, thus $\varphi - \mathrm{Id}_{\mathbb{R}^n}$ is asymptotically a constant matrix A; here we need $n \geq 2$. Replacing φ by $\varphi - A$ we see that φ then falls asymptotically towards $\mathrm{Id}_{\mathbb{R}^n}$. Thus φ is a proper mapping and thus has closed image, which is also open since φ is still a local diffeomorphism. Thus $\varphi \in \mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^n)$.
Finally, $d\varphi^t.d\varphi = (u.\sigma)^t.u.\sigma = \sigma^t.u^t.u.\sigma = \sigma^t.\sigma = g$. Note that φ is unique in $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^n)$. This is also clear from the fact, that the fiber of $\mathrm{Pull}^{\bar{g}}$ over $\varphi^*\bar{g}$ consists of all isometries of $\varphi^*\bar{g}$ which is the group $\varphi^{-1} \circ (\mathbb{R}^n \ltimes O(n)) \circ \varphi \subset \mathrm{Diff}(\mathbb{R}^n)$ whose intersection with $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^n)$ is trivial. □

The pullback of the Ebin metric to $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n)$

The pullback of the Ebin metric to the diffeomorphism group is a right invariant metric G given by

$$G_{\mathrm{Id}}(X, Y) = 4 \int_{\mathbb{R}^n} \mathrm{Tr}\left((\mathrm{Sym}\, dX).(\mathrm{Sym}\, dY)\right) dx = \int_{\mathbb{R}^n} \langle X, PY \rangle dx$$

Using the inertia operator P we can write the metric as $\int_{\mathbb{R}^n} \langle X, PY \rangle dx$, with

$$P = -2(\mathrm{grad}\,\mathrm{div} + \Delta)\,.$$

The pullback of the general metric to $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n)$

We consider now a weak Riemannian metric on $\mathrm{Met}_{\mathcal{A}}(\mathbb{R}^n)$ in its general form

$$G_g^P(h,k) = \int_M g_2^0(P_g h, k)\,\mathrm{vol}(g) = \int_M \mathrm{Tr}(g^{-1}.P_g(h).g^{-1}.k)\,\mathrm{vol}(g),$$

where $P_g : \Gamma(S^2T^*M) \to \Gamma(S^2T^*M)$ is as described above. *If the operator P is equivariant for the action of $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^n)$ on $\mathrm{Met}_{\mathcal{A}}(\mathbb{R}^n)$, then the induced pullback metric $(\mathrm{Pull}^{\bar{g}})^*G^P$ on $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^n)$ is right invariant:*

$$G_{\mathrm{Id}}(X,Y) = -4\int_{\mathbb{R}^n} \partial_j(P_{\bar{g}}\,\mathrm{Sym}\,dX)^i_j.Y^i\,dx \tag{4}$$

Thus we we get the following formula for the corresponding inertia operator $(\tilde{P}X)^i = \sum_j \partial_j(P_{\bar{g}}\,\mathrm{Sym}\,dX)^i_j$. Note that the pullback metric $(\mathrm{Pull}^{\bar{g}})^*G^P$ on $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^n)$ is always of one order higher then the metric G^P on $\mathrm{Met}_{\mathcal{A}}(\mathbb{R}^n)$.

The Sobolev metric of order p.

The Sobolev metric G^P

$$G^P_g(h,k) = \int_{\mathbb{R}^n} \mathrm{Tr}(g^{-1}.((1+\Delta)^p h).g^{-1}.k)\,\mathrm{vol}(g)\,.$$

The pullback of the Sobolev metric G^P to the diffeomorphism group is a right invariant metric G given by

$$G_{\mathrm{Id}}(X,Y) = -2\int_{\mathbb{R}^n} \Big\langle (\mathrm{grad}\,\mathrm{div} + \Delta)(1-\Delta)^p X, Y \Big\rangle dx\,.$$

Thus the inertia operator is given by

$$\tilde{P} = -2(1-\Delta)^p(\Delta + \mathrm{grad}\,\mathrm{div}) = -2(1-\Delta)^p(\Delta + \mathrm{grad}\,\mathrm{div})\,.$$

It is a linear isomorphism $H^s(\mathbb{R}^n)^n \to H^{s-2p-2}(\mathbb{R}^n)^n$ for every s.

Approximating the Euler equation of fluid mechanics on $\mathrm{Diff}_{H^\infty}(\mathbb{R}^n)$

On the Lie algebra of VF $\mathfrak{X}_{H^\infty}(\mathbb{R}^n) = H^\infty(\mathbb{R}^n)^n$ we consider a weak inner product of the form $\|v\|_L^2 = \int_{\mathbb{R}^n} \langle Lv, v\rangle\, dx$ where L is a positive L^2-symmetric (pseudo-) differential operator (inertia operator). Leads to a right invariant metric on $\mathrm{Diff}_{H^\infty}(\mathbb{R}^n)$ whose geodesic equation is

$$\partial_t \varphi = u \circ \varphi, \quad \partial_t u = -\operatorname{ad}_u^\top u, \text{ where}$$
$$\int \langle L(\operatorname{ad}_u^\top u), v\rangle\, dx = \int \langle L(u), -[u,v]\rangle\, dx$$

Condsider the *momentum* $m = L(v)$ of a vector field, so that $\langle v, w\rangle_L = \int \langle m, w\rangle dx$. Then the geodesic equation is of the form:

$$\partial_t m = -(v\cdot\nabla)m - \operatorname{div}(v)m - m\cdot(Dv)^t$$
$$\partial_t m_i = -\sum_j (v_j \partial_{x_j} m_i + \partial_{x_j} v_j \cdot m_i + m_j \partial_{x_i} v_j)$$
$$v = K * m, \qquad \text{where } K \text{ is the matrix-valued Green function of } L.$$

Suppose, the time dependent vector field v integrates to a flow φ via

$$\partial_t \varphi(x,t) = v(\varphi(x,t),t)$$

and we describe the momentum by a *measure-valued 1-form*

$$\widetilde{m} = \sum_i m_i dx_i \otimes (dx_1 \wedge \cdots \wedge dx_n)$$

so that $\|v\|_L^2 = \int (v, \widetilde{m})$ makes intrinsic sense. Then the geodesic equation is equivalent to: $\widetilde{m}$ is *invariant* under the flow φ, that is,

$$\widetilde{m}(\cdot,t) = \varphi(\cdot,t)_* \widetilde{m}(\cdot,0),$$

whose infinitesimal version is the following, using the Lie derivative:

$$\partial_t \widetilde{m}(\cdot,t) = -\mathcal{L}_{v(\cdot,t)} \widetilde{m}(\cdot,t).$$

Because of this invariance, if a geodesic begins with momentum of compact support, its momentum will always have compact support; and if it begins with momentum which, along with all its derivatives, has 'rapid' decay at infinity, that is it is in $O(\|x\|^{-n})$ for every n, this too will persist, because $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n) \subset \mathrm{Diff}_{H^\infty}(\mathbb{R}^n)$ is a normal subgroup.
Moreover this invariance gives us a Lagrangian form of EPDiff:

$$\begin{aligned}\partial_t \varphi(x,t) = \int K^{\varphi(\cdot,t)}(x,y)(\varphi(y,t)_* \widetilde{m}(y,0)) \\ = K^{\varphi(\cdot,t)} * (\varphi(\cdot,t)_* \widetilde{m}(\cdot,0)) \\ \text{where } K^{\varphi}(x,y) = K(\varphi(x),\varphi(y))\end{aligned}$$

Aim of this talk: Solutions of Euler's equation are limits of solutions of equations in the EPDiff class with the operator:

$$L_{\varepsilon,\eta} = (I - \tfrac{\eta^2}{p}\triangle)^p \circ (I - \tfrac{1}{\varepsilon^2}\nabla \circ \mathrm{div}), \qquad \text{for any } \varepsilon > 0, \eta \geq 0.$$

All solutions of Euler's equation are limits of solutions of these much more regular EPDiff equations and *give a bound on their rate of convergence*. In fact, so long as $p > n/2 + 1$, these EPDiff equations have a well-posed initial value problem with unique solutions for all time. Moreover, although $L_{0,\eta}$ does not make sense, the analog of its Green's function $K_{0,\eta}$ does make sense as do the geodesic equations in momentum form. These are, in fact, geodesic equations on the group of volume preserving diffeomorphisms SDiff and become Euler's equation for $\eta = 0$. An important point is that so long as $\eta > 0$, the equations have *soliton* solutions (called *vortons*) in which the momentum is a sum of delta functions.

Relation to Euler's equ. Oseledetz 1988

We use the kernel

$$K_{ij}(x) = \delta_{ij}\delta_0(x) + \partial_{x_i}\partial_{x_j} H$$

where H is the Green's function of $-\Delta$. But K now has a rather substantial pole at the origin. If $V_n = \mathrm{Vol}(S^{n-1})$,

$$H(x) = \begin{cases} \frac{1}{(n-2)V_n}(1/|x|^{n-2}) & \text{if } n > 2, \\ \frac{1}{V_2}\log(1/|x|) & \text{if } n = 2 \end{cases}$$

so that, *as a function*

$$(M_0)_{ij}(x) := \partial_{x_i}\partial_{x_j} H(x) = \frac{1}{V_n} \cdot \frac{n x_i x_j - \delta_{ij}|x|^2}{|x|^{n+2}}, \quad \text{if } x \neq 0.$$

Convolution with any $(M_0)_{ij}$ is still a Calderon-Zygmund singular integral operator defined by the limit as $\varepsilon \to 0$ of its value outside an ε-ball, so it is reasonably well behaved. *As a distribution* there is another term:

$$\partial_{x_i}\partial_{x_j} H \overset{\text{distribution}}{=} (M_0)_{ij} - \frac{1}{n}\delta_{ij}\delta_0$$

$$P_{\mathrm{div}=0} : m \mapsto v = \left(m + \partial^2(H)_{\mathrm{distr}}\right) = \left(\tfrac{n-1}{n} \cdot m + M_0 * m\right)$$

is the *orthogonal projection* of the space of vector fields m onto the subspace of divergence free vector fields v, orthogonal in each Sobolev space H^p, $p \in \mathbb{Z}_{\geq 0}$. (Hodge alias Helmholtz projection). The matrix $M_0(x)$ has $\mathbb{R}x$ as an eigenspace with eigenvalue $(n-1)/V_n|x|^n$ and $\mathbb{R}x^\perp$ as an eigenspace with eigenvalue $-1/V_n|x|^n$. Let $P_{\mathbb{R}x}$ and $P_{\mathbb{R}x^\perp}$ be the orthonormal projections onto the eigenspaces, then

$$\begin{aligned} &P_{\mathrm{div}=0}(m)(x) = \tfrac{n-1}{n} \cdot m(x) + \\ &+ \frac{1}{V_n} \cdot \lim_{\varepsilon \to 0} \int_{|y| \geq \varepsilon} \frac{1}{|y|^n} \left((n-1) P_{\mathbb{R}y}(m(x-y)) - P_{\mathbb{R}y^\perp}(m(x-y)\right) dy. \end{aligned}$$

With this K, EPDiff in the variables (v, m) is the Euler equation in v with pressure a function of (v, m). *Oseledets's form for Euler*:

$$\boxed{\begin{aligned} v &= P_{\text{div}=0}(m) \\ \partial_t m &= -(v \cdot \nabla)m - m \cdot (Dv)^t \end{aligned}}$$

Let $\widetilde{m} = \sum_i m_i dx_i$ be the 1-form associated to m. Since $\text{div}\, v = 0$, we can use $\widetilde{m}$ instead of $\sum_i m_i dx_i \otimes dx_1 \wedge \ldots dx_n$. Integrated form:

$$\boxed{\begin{aligned} \partial_t \varphi &= P_{\text{div}=0}(m) \circ \varphi \\ \widetilde{m}(\cdot, t) &= \varphi(\cdot, t)_* \widetilde{m}(\cdot, 0) \end{aligned}}$$

This uses the variables v, m instead of v and pressure.
Advantage: m, like vorticity, is constant when transported by the flow. m determines the vorticity the 2-form $\omega = d(\sum_i v_i dx_i)$, because v and m differ by a gradient, so $\omega = d\widetilde{m}$ also. Thus: vorticity is constant along flows follows from the same fact for momentum 1-form $\widetilde{m}$.

However, these equ. are not part of the true EPDiff framework because the operator $K = P_{\text{div}=0}$ is not invertible and there is no corresponding differential operator L.
In fact, v does not determine m as we have rewritten Euler's equation using extra non-unique variables m, albeit ones which obey a conservation law so they may be viewed simply as extra parameters.

Approximationg Euler by EPDiff

Replace the Green's function H of $-\Delta$ by the Green's function H_ε of the positive $\varepsilon^2 I - \triangle$ for $\varepsilon > 0$ (whose dimension is length^{-1}). The Green's function is be given explicitly using the 'K' Bessel function via the formula

$$H_\varepsilon(x) = c_n \varepsilon^{n-2} |\varepsilon x|^{1-n/2} K_{n/2-1}(|\varepsilon x|)$$

for a suitable constant c_n independent of ε. Then we get the modified kernel

$$(K_\varepsilon)_{ij} = \delta_{ij}\delta_0 + (\partial_{x_i}\partial_{x_j} H_\varepsilon)_{\mathrm{distr}}$$

This has exactly the same highest order pole at the origin as K did and the second derivative is again a Calderon-Zygmund singular integral operator minus the same delta function. The main difference is that this kernel has exponential decay at infinity, not polynomial decay. By weakening the requirement that the velocity be divergence free, the resulting integro-differential equation behaves much more locally, more like a hyperbolic equation rather than a parabolic one.

The corresponding inverse is the differential operator

$$L_\varepsilon = I - \tfrac{1}{\varepsilon^2}\nabla \circ \mathrm{div}$$
$$v = K_\varepsilon * m, \quad m = L_\varepsilon(v)$$
$$\|v\|^2_{L_\varepsilon} = \int \langle v, v\rangle + \mathrm{div}(v).\,\mathrm{div}(v)dx$$

Geodesic equation:

$$\partial_t(v_i) = (K_\varepsilon)_{ij} * \partial_t(m_j)$$
$$= -(K_\varepsilon)_{ij} * (v_k v_{j,k}) - v_i\,\mathrm{div}(v) - \tfrac{1}{2}(K_\varepsilon)_{ij} * \left(|v(x)|^2 + (\tfrac{\mathrm{div}(v)}{\varepsilon})^2\right)_{,j}$$

Curiously though, the parameter ε can be scaled away. That is, if $v(x,t), m(x,t)$ is a solution of EPDiff for the kernel K_1, then $v(\varepsilon x, \varepsilon t), m(\varepsilon x, \varepsilon t)$ is a solution of EPDiff for K_ε.

Regularizing more

Compose L_ε with a scaled version of the standard regularizing kernel $(I - \triangle)^p$ to get

$$\begin{aligned} L_{\varepsilon,\eta} &= (I - \tfrac{\eta^2}{p}\triangle)^p \circ (I - \tfrac{1}{\varepsilon^2}\nabla \circ \mathrm{div}) \\ K_{\varepsilon,\eta} &:= {L_{\varepsilon,\eta}}^{-1} = G_\eta^{(p)} * K_\varepsilon \end{aligned}$$

where $G_\eta^{(p)}$ is the Green's function of $(I - \frac{\eta^2}{p}\triangle)^p$ and is again given explicitly by a 'K'-Bessel function $d_{p,n}\eta^{-n}|x|^{p-n/2}K_{p-n/2}(|x|/\eta)$. For $p \gg 0$, the kernel converges to a Gaussian with variance depending only on η, namely $(2\sqrt{\pi}\eta)^{-n}e^{-|x|^2/4\eta^2}$. This follows because the Fourier transform takes $G_\eta^{(p)}$ to $\left(1 + \frac{\eta^2|\xi|^2}{p}\right)^{-p}$, whose limit, as $p \to \infty$, is $e^{-\eta^2|\xi|^2}$. These approximately Gaussian kernels lie in C^q if $q \leq p - (n+1)/2$.
So long as the kernel is in C^1, it is known that EPDiff has solutions for all time, as noted first by A.Trouve and L.Younes.

Theorem

Let $F(x) = f(|x|)$ be any integrable C^2 radial function on $\mathbb{R}^n$. Assume $n \geq 3$. Define:

$$H_F(x) = \int_{\mathbb{R}^n} \min\left(\tfrac{1}{|x|^{n-2}}, \tfrac{1}{|y|^{n-2}}\right) F(y)dy$$
$$= \frac{1}{|x|^{n-2}} \int_{|y|\leq|x|} F(y)dy + \int_{|y|\geq|x|} \frac{F(y)}{|y|^{n-2}} dy$$

Then H_F is the convolution of F with $\frac{1}{|x|^{n-2}}$, is in C^4 and:

$$\partial_i(H_F)(x) = -(n-2)\frac{x_i}{|x|^n} \int_{|y|\leq|x|} F(y)dy$$
$$\partial_i\partial_j(H_F)(x) = (n-2)\left(\frac{nx_ix_j - \delta_{ij}|x|^2}{|x|^{n+2}} \int_{|y|\leq|x|} F(y)dy - V_n \frac{x_ix_j}{|x|^2} F(x)\right)$$

If $n = 2$, the same holds if you replace $1/|x|^{n-2}$ by $\log(1/|x|)$ and omit the factors $(n-2)$ in the derivatives.

no L	$K_{0,0} = P_{\mathrm{div}=0} = \delta_{ij}\delta_0 + (\partial_i\partial_j H)_{\mathrm{distr}}$
no L	$K_{0,\eta} = G_\eta^{(p)} * P_{\mathrm{div}=0}$ – see above
$L_{\varepsilon,0} = I - \frac{1}{\varepsilon^2}\nabla \circ \mathrm{div}$	$K_{\varepsilon,0} = \delta_{ij}\delta_0 + \partial_i\partial_j H_\varepsilon$
$L_{\varepsilon,\eta} = \left(I - \frac{\eta^2}{p}\triangle\right)^p \circ \circ \left(I - \frac{1}{\varepsilon^2}\nabla \circ \mathrm{div}\right)$	$K_{\varepsilon,\eta} = \delta_{ij} G_\eta^{(p)} + \partial_i\partial_j (G_\eta^{(p)} * H_\varepsilon)$

Theorem: *Let $\varepsilon \geq 0, \eta > 0, p \geq (n+3)/2$ and $K = K_{\varepsilon,\eta}$ be the corresponding kernel. For any vector-valued distribution m_0 whose components are finite signed measures, consider the Lagrangian equation for a time varying C^1-diffeomorphism $\varphi(\cdot, t)$ with $\varphi(x,0) \equiv x$:*

$$\partial_t \varphi(x,t) = \int K(\varphi(x,t) - \varphi(y,t))(D\varphi(y,t))^{-1,\top} m_0(y) dy.$$

Here $D\varphi$ is the spatial derivative of φ. This equation has a unique solution for all time t.

Proof: The Eulerian velocity at φ is:

$$V_\varphi(x) = \int K(x - \varphi(y))(D\varphi(y))^{-1,\top} m_0(y)dy$$

and $W_\varphi(x) = V_\varphi(\varphi(x))$ is the velocity in 'material' coordinates. Note that because of our assumption on m_0, if φ is a C^1-diffeomorphism, then V_φ and W_φ are C^1 vector fields on $\mathbb{R}^n$; in fact, they are as differentiable as K is, for suitably decaying m. The equation can be viewed as a the flow equation for the vector field $\varphi \mapsto W_\varphi$ on the union of the open sets

$$U_c = \left\{\varphi \in C^1(\mathbb{R}^n)^n : \| \operatorname{Id} - \varphi\|_{C^1} < 1/c,\ \det(D\varphi) > c\right\},$$

where $c > 0$. The union of all U_c is the group $\operatorname{Diff}_{C_b^1}(\mathbb{R}^n)$ of all C^1-diffeomorphisms which, together with their inverses, differ from the identity by a function in $C^1(\mathbb{R}^n)^n$ with bounded C^1-norm. We claim this vector field is locally Lipschitz on each U_c:

$$\|W_{\varphi_1} - W_{\varphi_2}\|_{C^1} \le C.\|\varphi_1 - \varphi_2\|_{C^1}$$

where C depends only on c: Use that K is uniformly continuous and use $\|D\varphi^{-1}\| \le \|D\varphi\|^{n-1}/|\det(D\varphi)|$.

As a result we can integrate the vector field for short times in $\mathrm{Diff}_{C^1_b}(\mathbb{R}^n)$. But since $(D\varphi(y,t))^{-1,\top} m_0(y)$ is then again a signed finite $\mathbb{R}^n$-valued measure,

$$\int V_{\varphi(\cdot,t)}(x)(D\varphi(y,t))^{-1,\top} m_0(y)dx = \|V_{\varphi(\cdot,t)}\|_{L_{\varepsilon,\eta}}$$

is actually finite for each t. Using the fact that in EPDiff the $L_{\varepsilon,\eta}$-energy $\|V_{\varphi(\cdot,t)}\|_{L_{\varepsilon,\eta}}$ of the $L_{\varepsilon,\eta}$-geodesic is constant in t, we get a bound on the norm $\|V_{\varphi(\cdot,t)}\|_{H^p}$, depending of course on η but independent of t, hence a bound on $\|V_{\varphi(\cdot,t)}\|_{C^1}$. Thus $\|\varphi(\cdot,t)\|_{C^0}$ grows at most linearly in t. But $\partial_t D\varphi = DW_\varphi = DV_\varphi \cdot D\varphi$ which shows us that $D\varphi$ grows at most exponentially in t. Hence $\det D\varphi$ can shrink at worst exponentially towards zero, because $\partial_t \det(D\varphi) = \mathrm{Tr}(\mathrm{Adj}(D\varphi).\partial_t D\varphi)$. Thus for all finite t, the solution $\varphi(\cdot,t)$ stays in a bounded subset of our Banach space and the ODE can continue to be solved. QED.

Lemma: *If $\eta \geq 0$ and $\varepsilon > 0$ are bounded above, then the norm*

$$\|v\|^2_{k,\varepsilon,\eta} = \sum_{|\alpha|\leq k} \int \langle D^\alpha L_{\varepsilon,\eta} v, D^\alpha v \rangle dx$$

is bounded above and below by the metric, with constants independent of ε and η:

$$\|v\|^2_{H^k} + \frac{1}{\varepsilon^2}\|\mathrm{div}(v)\|^2_{H^k} + \sum_{k+1\leq|\alpha|\leq k+p} \eta^{2(|\alpha|-k)} \int |D^\alpha v|^2 + \frac{1}{\varepsilon^2}|D^\alpha \mathrm{div}(v)|^2$$

Main estimate: *Assume k is sufficiently large, for instance $k \geq (n+2p+4)$ works, then the velocity field of a solution satisfies:*

$$\boxed{\left|\partial_t \left(\|v\|^2_{k,\varepsilon,\eta}\right)\right| \leq C.\|v\|^3_{k,\varepsilon,\eta}}$$

where, so long ε and η are bounded above, the constant C is independent of ε and η.

Theorem: *Fix k, p, n with $p > n/2 + 1, k \geq n + 2p + 4$ and assume $(\varepsilon, \eta) \in [0, M]^2$ for some $M > 0$. Then there are constants t_0, C such that for all initial $v_0 \in H^{k+p+1}$, there is a unique solution $v_{\varepsilon,\eta}(x, t)$ of EPDiff (including the limiting Euler case) for $t \in [0, t_0]$. The solution $v_{\varepsilon,\eta}(\cdot, t) \in H^{k+p+1}$ depends continuously on $\varepsilon, \eta \in [0, M]^2$ and satisfies $\|v_{\varepsilon,\eta}(\cdot, t)\|_{k,\varepsilon,\eta} < C$ for all $t \in [0, t_0]$.*

Theorem: *Take any k and M and any smooth initial velocity $v(\cdot, 0)$. Then there are constants t_0, C such that Euler's equation and $(\varepsilon, 0)$-EPDiff have solutions v_0 and v_ε respectively for $t \in [0, t_0]$ and all $\varepsilon < M$ and these satisfy:*

$$\|v_0(\cdot, t) - v_\varepsilon(\cdot, t)\|_{H^k} \leq C\varepsilon.$$

Theorem: *Let $\varepsilon > 0$. Take any k and M and any smooth initial velocity $v(\cdot, 0)$. Then there are constants t_0, C such that $(\varepsilon, 0)$-EPDiff and (ε, η)-EPDiff have solutions v_0 and v_η respectively for $t \in [0, t_0]$ and all $\varepsilon, \eta < M$ and these satisfy:*

$$\|v_0(\cdot, t) - v_\eta(\cdot, t)\|_{H^k} \leq C\eta^2.$$

Vortons: Soliton-like solutions via landmark theory

We have a C^1 kernel, so we can consider solutions in which momentum m is supported in a finite set $\{P_1, \cdots, P_N\}$, so that the components of the momentum field are given by $m^i(x) = \sum_a m_{ai}\delta(x - P_a)$. The support is called the set of landmark points and in this case, EPDiff reduces to a set of Hamiltonian ODE's based on the kernel $K = K_{\varepsilon,\eta}, \varepsilon \geq 0, \eta > 0$:

$$\text{Energy } E = \sum_{a,b} m_{ai} K_{ij}(P_a - P_b) m_{jb}$$

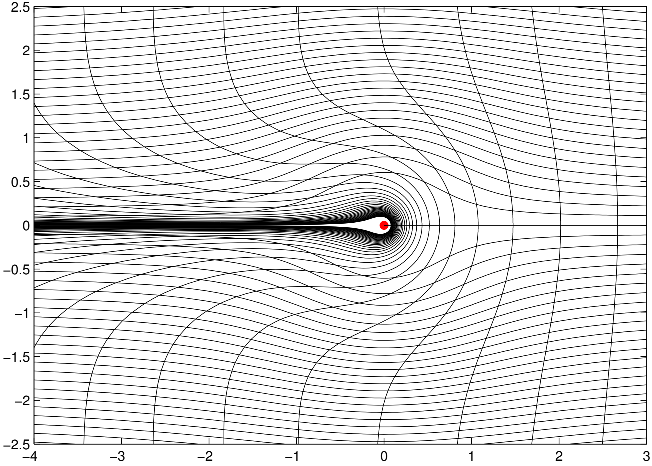
$$\frac{dP_{ai}}{dt} = \sum_{b,j} K_{ij}(P_a - P_b) m_{bj}$$

$$\frac{dm_{ai}}{dt} = -\sum_{b,j,k} \partial_{x_i} K_{jk}(P_a - P_b) m_{aj} m_{bk}$$

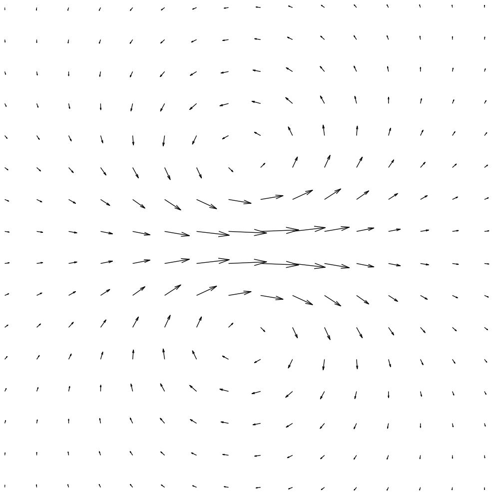
where a, b enumerate the points and i, j, k the dimensions in $\mathbb{R}^n$. These are essentially Roberts' equations from 1972.

One landmark point

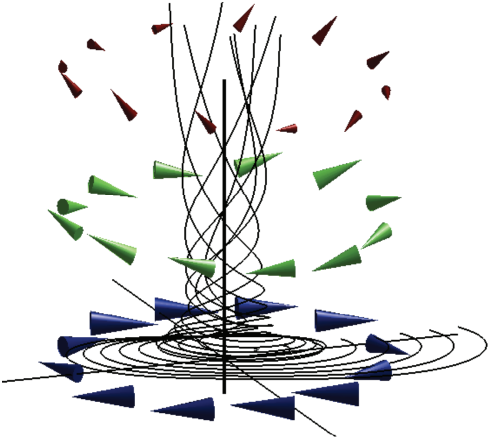
Its momentum must be constant hence so is its velocity. Therefore the momentum moves uniformly in a straight line ℓ from $-\infty$ to $+\infty$.



Momentum is tranformed to vortex-like velocity field by kernel $K_{0,\varepsilon}$

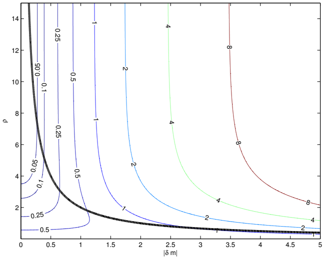


The dipole given by the kernel $K_{0,\eta}$ in dimension 2.



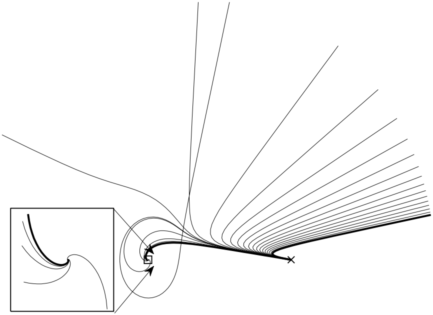
Streamlines and MatLab's ‘coneplot’ to visualize the vector field given by the x_1-derivative of the kernel $K_{0,1}$ times the vector $(1, 2, 0)$.

Two landmark points



Level sets of energy for the collision of two vortons with $\overline{m} = 0$, $\eta = 1$, $\omega = 1$. The coordinates are $\rho = |\delta P|$ and $|\delta m|$, and the state space is the double cover of the area above and right of the heavy black line, the two sheets being distinguished by the sign of $\langle \delta m, \delta P \rangle$. The heavy black line which is the curve $\rho \cdot |\delta m| = \omega$ where $\langle \delta m, \delta P \rangle = 0$. Each level set is a geodesic. If they hit the black line, they flip to the other sheet and retrace their path. Otherwise ρ goes to zero at one end of the geodesic.

Geodesics in the δP plane all starting at the point marked by an X but with $\overline{m} = m_1 + m_2$ =const. along the y-axis varying from 0 to 10. Here $\eta = 1$, the initial point is $(5, 0)$ and the initial momentum is $(-3, .5)$. Note how the two vortons repel each other on some geodesics and attract on others. A blow up shows the spiraling behavior as they collapse towards each other.



Robust Infinite Dimensional Riemannian manifolds, Sobolev Metrics on Diffeomorphism Groups, and the Derived Geometry of Shape Spaces

Based on: [Mario Micheli, Peter W. Michor, David Mumford: Sobolev Metrics on Diffeomorphism Groups and the Derived Geometry of Spaces of Submanifolds. Izvestiya: Mathematics 77:3 (2013), 541-570.]

Recall: Geodesics of a right invariant metric

Let $\gamma = \langle \ , \ \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be a positive definite bounded weak inner product; so $\check{\gamma} : \mathfrak{g} \to \mathfrak{g}^*$ is injective. Then

$$\gamma_x(\xi, \eta) = \langle T(\mu^{x^{-1}}) \cdot \xi, T(\mu^{x^{-1}}) \cdot \eta) \rangle = \langle \kappa(\xi), \kappa(\eta) \rangle$$

is a right invariant weak Riemannian metric on G.
Let $g : [a, b] \to G$ be a smooth curve. In terms of its right logarithmic derivative $u : [a, b] \to \mathfrak{g}$,
$u(t) := g^*\kappa(\partial_t) = T_{g(t)}(\mu^{g(t)^{-1}}) \cdot g'(t) = g'(t).g(t)^{-1}$, the geodesic equation is

$$\boxed{\check{\gamma}(u_t) = -\operatorname{ad}(u)^*\check{\gamma}(u)}$$

Condition for the existence of the geodesic equation:

$$X \mapsto \check{\gamma}^{-1}(\operatorname{ad}(X)^*\check{\gamma}(X))$$

is bounded quadratic $\overline{\mathfrak{g}} \to \overline{\mathfrak{g}}$.

We treated in detail a situation where $\mathrm{ad}(X)^*\check{\gamma}(X) \in \check{\gamma}(\overline{g})$ for all $X \in \overline{g}$, but the more general [Arnold 1966] condition $\mathrm{ad}(X)^*\check{\gamma}(Y) \in \check{\gamma}(\overline{g})$ for all $X, Y \in \overline{g}$ does not hold.
[M. Bauer, M. Bruveris, P. Michor: The homogeneous Sobolev metric of order one on diffeomorphism groups on the real line. Journal of Nonlinear Science. doi:10.1007/s00332-014-9204-y. arXiv:1209.2836]

Note also the geodesic equation for the *momentum* $\alpha = \check{\gamma}(u)$:

$$\boxed{\alpha_t = -\,\mathrm{ad}(\gamma^{-1}(\alpha))^*\,\alpha = -\,\mathrm{ad}(\alpha^\sharp)^*\alpha. = -\,\mathrm{ad}(u)^*\,\alpha.}$$

The covariant derivative for a right invariant metric

The right trivialization, or framing, $(\pi_G, \kappa) : TG \to G \times \mathfrak{g}$ induces the isomorphism $R : C^\infty(G, \mathfrak{g}) \to \mathfrak{X}(G)$, given by $R(X)(x) := R_X(x) := T_e(\mu^x) \cdot X(x)$, for $X \in C^\infty(G, \mathfrak{g})$ and $x \in G$. Here $\mathfrak{X}(G) := \Gamma(TG)$ denote the Lie algebra of all vector fields. For the Lie bracket and the Riemannian metric we have

$$[R_X, R_Y] = R(-[X, Y]_\mathfrak{g} + dY \cdot R_X - dX \cdot R_Y),$$
$$R^{-1}[R_X, R_Y] = -[X, Y]_\mathfrak{g} + R_X(Y) - R_Y(X),$$
$$\gamma_x(R_X(x), R_Y(x)) = \gamma(X(x), Y(x)),\ x \in G.$$

In the sequel we shall compute in $C^\infty(G, \mathfrak{g})$ instead of $\mathfrak{X}(G)$. In particular, we shall use the convention

$$\nabla_X Y := R^{-1}(\nabla_{R_X} R_Y) \quad \text{for } X, Y \in C^\infty(G, \mathfrak{g}).$$

to express the Levi-Civita covariant derivative.

Lemma

Assume that for all $\xi \in \mathfrak{g}$ *the element* $\mathrm{ad}(\xi)^*\check{\gamma}(\xi) \in \mathfrak{g}^*$ *is in the image of* $\check{\gamma} : \mathfrak{g} \to \mathfrak{g}^*$ *and that* $\xi \mapsto \check{\gamma}^{-1}\,\mathrm{ad}(\xi)^*\check{\gamma}(\xi)$ *is bounded quadratic (equivalently, smooth). Then the Levi-Civita covariant derivative of the metric* γ *exists and is given for any* $X, Y \in C^\infty(G, \mathfrak{g})$ *in terms of the isomorphism* R *by*

$$\nabla_X Y = dY.R_X + \rho(X)Y - \tfrac{1}{2}\,\mathrm{ad}(X)Y \ ,$$

where

$$\begin{aligned}\rho(\xi)\eta &= \tfrac{1}{4}\check{\gamma}^{-1}\big(\,\mathrm{ad}^*_{\xi+\eta}\,\check{\gamma}(\xi+\eta) - \mathrm{ad}^*_{\xi-\eta}\,\check{\gamma}(\xi-\eta)\big)\\ &= \tfrac{1}{2}\check{\gamma}^{-1}\big(\,\mathrm{ad}^*_{\xi}\,\check{\gamma}(\eta) + \mathrm{ad}^*_{\eta}\,\check{\gamma}(\xi)\big)\end{aligned}$$

is the polarized version. $\rho : \mathfrak{g} \to L(\mathfrak{g}, \mathfrak{g})$ *is bounded and we have* $\rho(\xi)\eta = \rho(\eta)\xi$*. We also have:*

$$\begin{aligned}\gamma\big(\rho(\xi)\eta, \zeta\big) &= \tfrac{1}{2}\gamma(\xi, \mathrm{ad}(\eta)\zeta) + \tfrac{1}{2}\gamma(\eta, \mathrm{ad}(\xi)\zeta) \ ,\\ \gamma(\rho(\xi)\eta, \zeta) &+ \gamma(\rho(\eta)\zeta, \xi) + \gamma(\rho(\zeta)\xi, \xi) = 0 \ .\end{aligned}$$

The curvature

For $X, Y \in C^\infty(G, \mathfrak{g})$ we have

$$[R_X, \mathrm{ad}(Y)] = \mathrm{ad}(R_X(Y)) \quad \text{and} \quad [R_X, \rho(Y)] = \rho(R_X(Y)).$$

The Riemannian curvature is then computed by

$$\begin{aligned}
\mathcal{R}(X, Y) &= [\nabla_X, \nabla_Y] - \nabla_{-[X,Y]_\mathfrak{g} + R_X(Y) - R_Y(X)} \\
&= [R_X + \rho_X - \tfrac{1}{2}\,\mathrm{ad}_X, R_Y + \rho_Y - \tfrac{1}{2}\,\mathrm{ad}_Y] \\
&\quad - R(-[X, Y]_\mathfrak{g} + R_X(Y) - R_Y(X)) \\
&\quad - \rho(-[X, Y]_\mathfrak{g} + R_X(Y) - R_Y(X)) \\
&\quad + \tfrac{1}{2}\,\mathrm{ad}(-[X, Y]_\mathfrak{g} + R_X(Y) - R_Y(X)) \\
&= [\rho_X, \rho_Y] + \rho_{[X,Y]_\mathfrak{g}} - \tfrac{1}{2}[\rho_X, \mathrm{ad}_Y] + \tfrac{1}{2}[\rho_Y, \mathrm{ad}_X] - \tfrac{1}{4}\,\mathrm{ad}_{[X,Y]_\mathfrak{g}}
\end{aligned}$$

which visibly is a tensor field.

Sectional Curvature. For a linear 2-dim. subspace $P \subseteq \mathfrak{g}$ spanned by $X, Y \in \mathfrak{g}$, the sectional curvature is defined as:

$$k(P) = -\frac{\gamma\big(\mathcal{R}(X,Y)X, Y\big)}{\|X\|_\gamma^2 \|Y\|_\gamma^2 - \gamma(X,Y)^2} \ .$$

For the numerator we get:

$$\begin{aligned}\gamma\big(\mathcal{R}(X,Y)X, Y\big) &= \gamma(\rho_X X, \rho_Y Y) - \|\rho_X Y\|_\gamma^2 + \tfrac{3}{4}\|[X,Y]\|_\gamma^2 \\ &\quad - \tfrac{1}{2}\gamma(X,[Y,[X,Y]]) + \tfrac{1}{2}\gamma(Y,[X,[X,Y]]). \\ &= \gamma(\rho_X X, \rho_Y Y) - \|\rho_X Y\|_\gamma^2 + \tfrac{3}{4}\|[X,Y]\|_\gamma^2 \\ &\quad - \gamma(\rho_X Y,[X,Y]]) + \gamma(Y,[X,[X,Y]]) \ .\end{aligned}$$

If the adjoint $\mathrm{ad}(X)^\top : \mathfrak{g} \to \mathfrak{g}$ exists, this is easily seen to coincide with Arnold's original formula [Arnold1966],

$$\begin{aligned}\gamma(\mathcal{R}(X,Y)X,Y) &= -\tfrac{1}{4}\|\,\mathrm{ad}(X)^\top Y + \mathrm{ad}(Y)^\top X\|_\gamma^2 \\ &\quad + \gamma(\mathrm{ad}(X)^\top X, \mathrm{ad}(Y)^\top Y) \\ &\quad + \tfrac{1}{2}\gamma(\mathrm{ad}(X)^\top Y - \mathrm{ad}(Y)^\top X, \mathrm{ad}(X)Y) + \tfrac{3}{4}\|[X,Y]\|_\gamma^2 \ .\end{aligned}$$

A covariant formula for curvature and its relations to O'Neill's curvature formulas.

Mario Micheli in his 2008 thesis derived the the coordinate version of the following formula for the sectional curvature expression, which is valid for **closed** 1-forms α, β on a Riemannian manifold (M, g), where we view $g : TM \to T^*M$ and so g^{-1} is the dual inner product on T^*M. Here $\alpha^\sharp = g^{-1}(\alpha)$.

$$
\begin{aligned}
& g\big(R(\alpha^\sharp, \beta^\sharp)\alpha^\sharp, \beta^\sharp\big) = \\
& -\tfrac{1}{2}\alpha^\sharp\alpha^\sharp(\|\beta\|^2_{g^{-1}}) - \tfrac{1}{2}\beta^\sharp\beta^\sharp(\|\alpha\|^2_{g^{-1}}) + \tfrac{1}{2}(\alpha^\sharp\beta^\sharp + \beta^\sharp\alpha^\sharp)g^{-1}(\alpha, \beta) \\
& \qquad\qquad \big(\text{last line } = -\alpha^\sharp\beta([\alpha^\sharp, \beta^\sharp]) + \beta^\sharp\alpha([\alpha^\sharp, \beta^\sharp]])\big) \\
& -\tfrac{1}{4}\|d(g^{-1}(\alpha, \beta))\|^2_{g^{-1}} + \tfrac{1}{4}g^{-1}\big(d(\|\alpha\|^2_{g^{-1}}), d(\|\beta\|^2_{g^{-1}})\big) \\
& +\tfrac{3}{4}\big\|[\alpha^\sharp, \beta^\sharp]\big\|^2_g
\end{aligned}
$$

Mario's formula in coordinates

Assume that $\alpha = \alpha_i dx^i, \beta = \beta_i dx^i$ where the coefficients α_i, β_i are *constants*, hence α, β are closed.
Then $\alpha^\sharp = g^{ij}\alpha_i\partial_j, \beta^\sharp = g^{ij}\beta_i\partial_j$ and we have:

$$\boxed{\begin{aligned}&4g\big(R(\alpha^\sharp, \beta^\sharp)\beta^\sharp, \alpha^\sharp\big)\\ &= (\alpha_i\beta_k - \alpha_k\beta_i)\cdot(\alpha_j\beta_l - \alpha_l\beta_j)\cdot\\ &\cdot\Big(2g^{is}(g^{jt}g^{kl}_{,t})_{,s} - \tfrac{1}{2}g^{ij}_{,s}g^{st}g^{kl}_{,t} - 3g^{is}g^{kp}_{,s}g_{pq}g^{jt}g^{lq}_{,t}\Big)\end{aligned}}$$

Covariant curvature and O'Neill's formula, finite dim.

Let $p : (E, g_E) \to (B, g_B)$ be a Riemannian submersion:
For $b \in B$ and $x \in E_b := p^{-1}(b)$ the g_E-orthogonal splitting

$$T_x E = T_x(E_{p(x)}) \oplus T_x(E_{p(x)})^{\perp, g_E} =: T_x(E_{p(x)}) \oplus \mathrm{Hor}_x(p).$$
$$T_x p : (\mathrm{Hor}_x(p), g_E) \to (T_b B, g_B)$$

is an isometry. A vector field $X \in \mathfrak{X}(E)$ is decomposed as $X = X^{\mathrm{hor}} + X^{\mathrm{ver}}$ into horizontal and vertical parts. Each vector field $\xi \in \mathfrak{X}(B)$ can be uniquely lifted to a smooth horizontal field $\xi^{\mathrm{hor}} \in \Gamma(\mathrm{Hor}(p)) \subset \mathfrak{X}(E)$.

O'Neill's formula says that for any two horizontal vector fields X, Y on E and any $x \in E$, the sectional curvatures of E and B are related by:

$$\boxed{\begin{aligned} & g^B_{p(x)}(R^B(p_*(X_x), p_*(Y_x))p_*(Y_x), p_*(X_x)) \\ & = g^E_x(R^E(X_x, Y_x)Y_x, X_x) + \tfrac{3}{4}\|[X,Y]^{ver}\|^2_x. \end{aligned}}$$

Comparing Mario's formula on E and B gives an immediate proof of this fact. Namely: If $\alpha \in \Omega^1(B)$, then the vector field $(p^*\alpha)^\sharp$ is horizontal and we have $Tp \circ (p^*\alpha)^\sharp = \alpha^\sharp \circ p$. Therefore $(p^*\alpha)^\sharp$ equals the horizontal lift $(\alpha^\sharp)^{\mathrm{hor}}$. For each $x \in E$ the mapping $(T_x p)^* : (T^*_{p(x)}B, g_B^{-1}) \to (T^*_x E, g_E^{-1})$ is an isometry. We also use:

$$\|[(p^*\alpha)^\sharp, (p^*\beta)^\sharp]^{\mathrm{hor}}\|^2_{g_E} = p^*\|[\alpha^\sharp, \beta^\sharp]\|^2_{g_B}$$

Requirements for infinite dimensional manifolds

Let (M, g) be a *weak Riemannian manifold*, modeled on convenient locally convex vector spaces: Weak: $g_x : T_x M \to T_x^* M$ is only injective. The image $g(TM) \subset T^* M$ is called the *smooth cotangent bundle* associated to g. Now $\Omega_g^1(M) := \Gamma(g(TM))$ and $\alpha^\sharp = g^{-1}\alpha \in \mathfrak{X}(M), X^\flat = gX$ are as above.
$d : \Omega_g^1(M) \to \Omega^2(M) = \Gamma(L^2_{\text{skew}}(TM; \mathbb{R}))$.
Existence of the Levi-Civita covariant derivative is equivalent to: The metric itself admits *symmetric* gradients with respect to itself. Locally: If M is c^∞-open in a convenient vector space V_M. Then:

$$\boxed{\begin{aligned} D_{x,X} g_x(X, Y) &= g_x(X, \operatorname{grad}_1 g(x)(X, Y)) \\ &= g_x(\operatorname{grad}_2 g(x)(X, X), Y) \end{aligned}}$$

where $\operatorname{grad}_1 g, \operatorname{sym} \operatorname{grad}_2 g : M \times V_M \times V_M \to V_M$ given by $(x, X) \mapsto \operatorname{grad}_{1,2} g(x)(X, X)$ are smooth and quadratic in $X \in V_M$. Then the rest of the derivation of Mario's formula goes through and the final formula for curvature holds in both the finite and infinite dimensional cases.

Robust weak Riemannian manifolds

Another problem: Some constructions lead to vector fields whose values do not lie in $T_x M$, but in the Hilbert space completion $\overline{T_x M}$ with respect to the inner product g_x. We need that $\bigcup_{x \in M} \overline{T_x M}$ forms a smooth vector bundle over M. In a coordinate chart on open $U \subset M$, $TM|_U$ is a trivial bundle $U \times V$ and all the inner products $g_x, x \in U$ define inner products on the same topological vector space V. They all should be bounded with respect to each other, so that the completion $\overline{V}$ of V with respect to g_x does not depend on x and $\bigcup_{x \in U} \overline{T_x M} \cong U \times \overline{V}$. This means that $\bigcup_{x \in M} \overline{T_x M}$ forms a smooth vector bundle over M with trivialisations the linear extensions of the trivialisations of the bundle $TM \to M$.

Definition A convenient weak Riemannian manifold (M, g) will be called a *robust* Riemannian manifold if

- The Levi-Civita convariant derivative of the metric g exists: The symmetric gradients gradients should exist.
- The completions $\overline{T_x M}$ form a vector bundle as above.

Covariant curvature and O'Neill's formula

Some subtle considerations lead to:

Theorem. *Let $p : (E, g_E) \to (B, g_B)$ be a Riemann submersion between infinite dimensional robust Riemann manifolds. Then for 1-forms $\alpha, \beta \in \Omega^1_{g_B}(B)$ O'Neill's formula holds in the form:*

$$\boxed{\begin{aligned} g_B\big(R^B(\alpha^\sharp, \beta^\sharp)\beta^\sharp, \alpha^\sharp\big) \\ &= g_E\big(R^E((p^*\alpha)^\sharp, (p^*\beta)^\sharp)(p^*\beta)^\sharp, (p^*\alpha)^\sharp\big) \\ &+ \tfrac{3}{4}\|[(p^*\alpha)^\sharp, (p^*\beta)^\sharp]^{ver}\|^2_{g_E} \end{aligned}}$$

Semilocal version of Mario's formula, force, and stress

Let (M, g) be a robust Riemannian manifold, $x \in M$, $\alpha, \beta \in g_x(T_x M)$. Assume we are given local smooth vector fields X_α and X_β such that:

1. $X_\alpha(x) = \alpha^\sharp(x), \quad X_\beta(x) = \beta^\sharp(x)$,
2. Then $\alpha^\sharp - X_\alpha$ is zero at x hence has a well defined derivative $D_x(\alpha^\sharp - X_\alpha)$ lying in $\mathrm{Hom}(T_x M, T_x M)$. For a vector field Y we have $D_x(\alpha^\sharp - X_\alpha).Y_x = [Y, \alpha^\sharp - X_\alpha](x) = \mathcal{L}_Y(\alpha^\sharp - X_\alpha)|_x$. The same holds for β.
3. $\mathcal{L}_{X_\alpha}(\alpha) = \mathcal{L}_{X_\alpha}(\beta) = \mathcal{L}_{X_\beta}(\alpha) = \mathcal{L}_{X_\beta}(\beta) = 0$,
4. $[X_\alpha, X_\beta] = 0$.

Locally constant 1-forms and vector fields will do. We then define:

$$\mathcal{F}(\alpha, \beta) := \tfrac{1}{2} d(g^{-1}(\alpha, \beta)), \qquad \text{a 1-form on } M \text{ called the } \textit{force},$$

$$\begin{aligned} \mathcal{D}(\alpha, \beta)(x) &:= D_x(\beta^\sharp - X_\beta).\alpha^\sharp(x) \\ &= d(\beta^\sharp - X_\beta).\alpha^\sharp(x), \quad \in T_x M \text{ called the } \textit{stress}. \end{aligned}$$

$$\Longrightarrow \mathcal{D}(\alpha, \beta)(x) - \mathcal{D}(\beta, \alpha)(x) = [\alpha^\sharp, \beta^\sharp](x)$$

Then in the notation above:

$$g\big(R(\alpha^\sharp,\beta^\sharp)\beta^\sharp,\alpha^\sharp\big)(x) = R_{11}+R_{12}+R_2+R_3$$

$$R_{11} = \tfrac{1}{2}\Big(\mathcal{L}^2_{X_\alpha}(g^{-1})(\beta,\beta) - 2\mathcal{L}_{X_\alpha}\mathcal{L}_{X_\beta}(g^{-1})(\alpha,\beta) + \mathcal{L}^2_{X_\beta}(g^{-1})(\alpha,\alpha)\Big)(x)$$

$$R_{12} = \langle\mathcal{F}(\alpha,\alpha),\mathcal{D}(\beta,\beta)\rangle + \langle\mathcal{F}(\beta,\beta),\mathcal{D}(\alpha,\alpha)\rangle - \langle\mathcal{F}(\alpha,\beta),\mathcal{D}(\alpha,\beta)+\mathcal{D}(\beta,\alpha)\rangle$$

$$R_2 = \Big(\|\mathcal{F}(\alpha,\beta)\|^2_{g^{-1}} - \big\langle\mathcal{F}(\alpha,\alpha)),\mathcal{F}(\beta,\beta)\big\rangle_{g^{-1}}\Big)(x)$$

$$R_3 = -\tfrac{3}{4}\|\mathcal{D}(\alpha,\beta)-\mathcal{D}(\beta,\alpha)\|^2_{g_x}$$

Diffeomorphism groups

Let N be a manifold. We consider the following regular Lie groups:

$\mathrm{Diff}(N)$, the group of all diffeomorphisms of N if N is compact.
$\mathrm{Diff}_c(N)$, the group of diffeomorphisms with compact support.

If (N, g) is a *Riemannian manifold of bounded geometry*, we also may consider:

$\mathrm{Diff}_{\mathcal{S}}(N)$, the group of all diffeos which fall rapidly to the identity.
$\mathrm{Diff}_{H^\infty}(N)$, the group of all diffeos which are modelled on the space $\Gamma_{H^\infty}(TM)$, the intersection of all Sobolev spaces of vector fields.

The Lie algebras are the spaces $\mathfrak{X}_{\mathcal{A}}(N)$ of vector fields, where $\mathcal{A} \in \{C_c^\infty, \mathcal{S}, H^\infty\}$, with the negative of the usual bracket as Lie bracket.

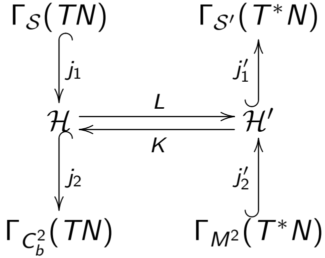
Riemann metrics on Diff(N).

The concept of robust Riemannian manifolds, and also the reproducing Hilbert space approach in Chapter 12 of [Younes 2010] leads to:
We construct a right invariant weak Riemannian metric by assuming that we have a Hilbert space $\mathcal{H}$ together with two bounded injective linear mappings

$$\mathfrak{X}_{\mathcal{S}}(N) = \Gamma_{\mathcal{S}}(TN) \xrightarrow{j_1} \mathcal{H} \xrightarrow{j_2} \Gamma_{C^2_b}(TN) \tag{1}$$

where $\Gamma_{C^2_b}(TN)$ is the Banach space of all C^2 vector fields X on N which are globally bounded together with $\nabla^g X$ and $\nabla^g \nabla^g X$ with respect to g, such that $j_2 \circ j_1 : \Gamma_{\mathcal{S}}(TN) \to \Gamma_{C^2_b}(TN)$ is the canonical embedding. We also assume that j_1 has dense image.

Dualizing the Banach spaces in equation (1) and using the canonical isomorphisms between $\mathcal{H}$ and its dual $\mathcal{H}'$ – which we call L and K, we get the diagram:



Here $\Gamma_{\mathcal{S}'}(T^*N)$, the space of *1-co-currents*, is the dual of the space of smooth vector fields $\Gamma_{\mathcal{S}}(TN) = \mathfrak{X}_{\mathcal{S}}(N)$. It contains the space $\Gamma_{\mathcal{S}}(T^*N \otimes \mathrm{vol}(N))$ of smooth measure valued cotangent vectors on N, and also the bigger subspace of second derivatives of finite measure valued 1-forms on N, written as $\Gamma_{M^2}(T^*N)$ which is part of the dual of $\Gamma_{C_b^2}(TN)$. In what follows, we will have many momentum variables with values in these spaces.

The restriction of L to $\mathfrak{X}_{\mathcal{S}}(N)$ via j_1 gives us a positive definite weak inner product on $\mathfrak{X}_{\mathcal{S}}(N)$ which may be defined by a distribution valued kernel – which we also write as L:

$$\langle \quad , \quad \rangle_L : \mathfrak{X}_{\mathcal{S}}(N) \times \mathfrak{X}_{\mathcal{S}}(N) \to \mathbb{R}, \quad \text{defined by}$$
$$\langle X, Y\rangle_L = \langle j_1 X, j_1 Y\rangle_{\mathcal{H}} = \iint_{N\times N} (X(y_1) \otimes Y(y_2), L(y_1, y_2)),$$
$$\text{where } L \in \Gamma_{\mathcal{S}'}(\mathrm{pr}_1^*(T^*N) \otimes \mathrm{pr}_2^*(T^*N))$$

Extending this weak inner product right invariantly over $\mathrm{Diff}_{\mathcal{S}}(N)$, we get a robust weak Riemannian manifold.
Given an operator L with appropriate properties we can reconstruct the Hilbert space $\mathcal{H}$ with the two bounded injective mappings j_1, j_2.

In the case (called the *standard case* below) that $N = \mathbb{R}^n$ and that

$$\langle X, Y\rangle_L = \int_{\mathbb{R}^n} \langle (1 - A\Delta)^l X, Y\rangle \, dx$$

we have

$$L(x,y) = \Big(\frac{1}{(2\pi)^n} \int_{\xi \in \mathbb{R}^n} e^{i\langle \xi, x-y\rangle} (1 + A|\xi|^2)^l d\xi \Big)$$
$$\sum_{i=1}^{n} (du^i|_x \otimes dx) \otimes (du^i|_y \otimes dy)$$

where $d\xi$, dx and dy denote Lebesque measure, and where (u^i) are linear coordinates on $\mathbb{R}^n$. Here $\mathcal{H}$ is the space of Sobolev H^l vector fields on N.

Construction of the reproducing kernel K

The inverse map K of L is given by a C^2 tensor, the reproducing kernel. Namely, $\Gamma_{M^2}(T^*N)$ contains the measures supported at one point x defined by an element $\alpha_x \in T_x^*N$. Then $j_2(K(j_2'(\alpha_x)))$ is given by a C^2 vector field K_{α_x} on N which satisfies:

$$\langle K_{\alpha_x}, X\rangle_{\mathcal{H}} = \alpha_x(j_2 X)(x) \quad \text{for all } X \in \mathcal{H}, \alpha_x \in T_x^*N. \qquad (*)$$

The map $\alpha_x \mapsto K_{\alpha_x}$ is weakly C_b^2, thus by [KM97, theorem 12.8] strongly Lip^1. Since $\mathrm{ev}_y \circ K : T_x^*N \ni \alpha_x \mapsto K_{\alpha_x}(y) \in T_yN$ is linear we get a corresponding element
$K(x,y) \in L(T_x^*N, T_yN) = T_xN \otimes T_yN$ with
$K(y,x)(\alpha_x) = K_{\alpha_x}(y)$.
Using (*) twice we have (omitting j_2)

$$\beta_y.K(y,x)(\alpha_x) = \langle K(\quad, x)(\alpha_x), K(\quad, y)(\beta_y)\rangle_{\mathcal{H}} = \alpha_x.K(x,y)(\beta_y)$$

so that:

- $K(x,y)^\top = K(y,x) : T_y^*N \to T_xN$,
- $K \in \Gamma_{C_b^2}(\mathrm{pr}_1{}^*\, TN \otimes \mathrm{pr}_2^*\, TN)$.

Moreover the operator K defined directly by integration

$$K : \Gamma_{M^2}(T^*N) \to \Gamma_{\mathcal{S}}(TN)$$
$$K(\alpha)(y_2) = \int_{y_1 \in N} (K(y_1, y_2), \alpha(y_1)).$$

is the same as the inverse K to L.
We will sometimes use the abbreviations $\langle\alpha|K|$, $|K|\beta\rangle$ and $\langle\alpha|K|\beta\rangle$ for the contraction of the vector values of K in its first and second variable against 1-forms α and β. Often these are measure valued 1-forms so after contracting, there remains a measure in that variable which can be integrated.

Thus the C^2 tensor K determines L and hence $\mathcal{H}$ and hence the whole metric on $\mathrm{Diff}_{\mathcal{S}}(N)$. It is tempting to start with the tensor K, assuming it is symmetric and positive definite in a suitable sense. But rather subtle conditions on K are required in order that its inverse L is defined on all infinitely differentiable vector fields. For example, if $N = \mathbb{R}$, the Gaussian kernel $K(x,y) = e^{-|x-y|^2}$ does not give such an L.

In the standard case we have

$$K(x,y) = K_l(x-y) \sum_{i=1}^{n} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial y^i},$$

$$K_l(x) = \frac{1}{(2\pi)^n} \int_{\xi \in \mathbb{R}^n} \frac{e^{i\langle \xi, x \rangle}}{(1 + A|\xi|^2)^l} d\xi$$

where K_l is given by a classical Bessel function of differentiability class C^{2l}.

The zero compressibility limit

The case originally studied by Arnold – the L^2 metric on volume preserving diffeomorphisms – is not included in the family of metrics described above. But they do include metrics which have this case as a limit. Taking $N = \mathbb{R}^n$ and starting with the standard Sobolev metric, we can add a divergence term with a coefficient B:

$$\langle X, Y \rangle_L = \int_{\mathbb{R}^n} \Big(\langle (1 - A\Delta)^l X, Y \rangle + B.\mathrm{div}(X)\mathrm{div}(Y) \Big) \, dx$$

Note that as B approaches ∞, the geodesics will tend to lie on the cosets with respect to the subgroup of volume preserving diffeomorphisms. And when, in addition, A approaches zero, we get the simple L^2 metric used by Arnold. This suggests that, as in the so-called 'zero-viscosity limit', we should be able to construct geodesics in Arnold's metric, i.e. solutions of Euler's equation, as limits of geodesics for this larger family of metrics on the full group.

The resulting kernels L and K are no longer diagonal. To L, we must add

$$B\sum_{i=1}^{n}\sum_{j=1}^{n}\Big(\frac{1}{(2\pi)^n}\int_{\xi\in\mathbb{R}^n} e^{i\langle \xi, x-y\rangle}\xi_i.\xi_j d\xi\Big)(du^i|_x\otimes dx)\otimes(du^j|_y\otimes dy).$$

It can be checked that the corresponding kernel K will have the form

$$K(x,y) = K_0(x-y)\sum_{i=1}^{n}\frac{\partial}{\partial x^i}\otimes\frac{\partial}{\partial y^i}+\sum_{i=1}^{n}\sum_{j=1}^{n}(K_B)_{,ij}(x-y)\frac{\partial}{\partial x^i}\otimes\frac{\partial}{\partial y^j}$$

where K_0 is the kernel as above for the standard norm of order l and K_B is a second radially symmetric kernel on $\mathbb{R}^n$ depending on B.

The geodesic equation on $\mathrm{Diff}_{\mathcal{S}}(N)$

According to [Arnold 1966], slightly generalized as explained above: Let $\varphi : [a,b] \to \mathrm{Diff}_{\mathcal{S}}(N)$ be a smooth curve. In terms of its right logarithmic derivative $u : [a,b] \to \mathfrak{X}_{\mathcal{S}}(N)$, $u(t) := \varphi^*\kappa(\partial_t) = T_{\varphi(t)}(\mu^{\varphi(t)^{-1}}) \cdot \varphi'(t) = \varphi'(t) \circ \varphi(t)^{-1}$, the geodesic equation is

$$\boxed{L(u_t) = -\operatorname{ad}(u)^* L(u)}$$

Condition for the existence of the geodesic equation:

$$X \mapsto K(\operatorname{ad}(X)^* L(X))$$

is bounded quadratic $\mathfrak{X}_{\mathcal{S}}(N) \to \mathfrak{X}_{\mathcal{S}}(N)$.

The Lie algebra of $\mathrm{Diff}_{\mathcal{S}}(N)$ is the space $\mathfrak{X}_{\mathcal{S}}(N)$ of all rapidly decreasing smooth vector fields with Lie bracket the negative of the usual Lie bracket $\operatorname{ad}_X Y = -[X, Y]$.

Using *Lie derivatives*, the computation of ad^*_X is especially simple. Namely, for any section ω of $T^*N \otimes \mathrm{vol}$ and vector fields $\xi, \eta \in \mathfrak{X}_{\mathcal{S}}(N)$, we have:

$$\int_N (\omega, [\xi, \eta]) = \int_N (\omega, \mathcal{L}_\xi(\eta)) = -\int_N (\mathcal{L}_\xi(\omega), \eta),$$

hence $\operatorname{ad}^*_\xi(\omega) = +\mathcal{L}_\xi(\omega)$.
Thus the Hamiltonian version of the geodesic equation on the smooth dual $L(\mathfrak{X}_{\mathcal{S}}(N)) \subset \Gamma_{C^2}(T^*N \otimes \mathrm{vol})$ becomes

$$\boxed{\partial_t \alpha = -\operatorname{ad}^*_{K(\alpha)} \alpha = -\mathcal{L}_{K(\alpha)} \alpha,}$$

or, keeping track of everything,

$$\boxed{\begin{aligned} \partial_t \varphi &= u \circ \varphi, \\ \partial_t \alpha &= -\mathcal{L}_u \alpha \\ u = K(\alpha) &= \alpha^\sharp, \quad \alpha = L(u) = u^\flat. \end{aligned}} \tag{1}$$

One can also derive the geodesic equation from the conserved momentum mapping $J : T\,\mathrm{Diff}_{\mathcal{S}}(N) \to \mathfrak{X}_{\mathcal{S}}(N)'$ given by $J(g, X) = L \circ \mathrm{Ad}(g)^{\top} X$ where $\mathrm{Ad}(g)X = Tg \circ X \circ g^{-1}$. This means that $\mathrm{Ad}(g(t))u(t)$ is conserved and $0 = \partial_t \mathrm{Ad}(g(t))u(t)$ leads quickly to the geodesic equation. It is remarkable that the momentum mapping exists if and only if $(\mathrm{Diff}_{\mathcal{S}}(N), \langle\ \ ,\ \ \rangle_L)$ is a robust weak Riemannian manifold.

Landmark space as homogeneus space of solitons

A *landmark* $q = (q_1, \dots, q_N)$ is an N-tuple of distinct points in $\mathbb{R}^n$; so $\mathrm{Land}^N \subset (\mathbb{R}^n)^N$ is open. Let $q^0 = (q_1^0, \dots, q_N^0)$ be a fixed standard template landmark. Then we have the the surjective mapping

$$\begin{aligned}
&\mathrm{ev}_{q^0} : \mathrm{Diff}(\mathbb{R}^n) \to \mathrm{Land}^N,\\
&\varphi \mapsto \mathrm{ev}_{q^0}(\varphi) = \varphi(q^0) = (\varphi(q_1^0), \dots, \varphi(q_N^0)).
\end{aligned}$$

The fiber of ev_{q^0} over a landmark $q = \varphi_0(q^0)$ is

$$\begin{aligned}
&\{\varphi \in \mathrm{Diff}(\mathbb{R}^n) : \varphi(q^0) = q\}\\
&= \varphi_0 \circ \{\varphi \in \mathrm{Diff}(\mathbb{R}^n) : \varphi(q^0) = q^0\}\\
&= \{\varphi \in \mathrm{Diff}(\mathbb{R}^n) : \varphi(q) = q\} \circ \varphi_0;
\end{aligned}$$

The tangent space to the fiber is

$$\{X \circ \varphi_0 : X \in \mathfrak{X}_{\mathcal{S}}(\mathbb{R}^n), X(q_i) = 0 \text{ for all } i\}.$$

A tangent vector $Y \circ \varphi_0 \in T_{\varphi_0} \operatorname{Diff}_{\mathcal{S}}(\mathbb{R}^n)$ is $G^L_{\varphi_0}$-perpendicular to the fiber over q if

$$\int_{\mathbb{R}^n} \langle LY, X \rangle \, dx = 0 \quad \forall X \text{ with } X(q) = 0.$$

If we require Y to be smooth then $Y = 0$. So we assume that $LY = \sum_i P_i . \delta_{q_i}$, a distributional vector field with support in q. Here $P_i \in T_{q_i}\mathbb{R}^n$. But then

$$Y(x) = L^{-1}\Big(\sum_i P_i . \delta_{q_i} \Big) = \int_{\mathbb{R}^n} K(x-y) \sum_i P_i . \delta_{q_i}(y) \, dy$$

$$= \sum_i K(x - q_i) . P_i$$

$$T_{\varphi_0}(\mathrm{ev}_{q^0}).(Y \circ \varphi_0) = Y(q_k)_k = \sum_i (K(q_k - q_i).P_i)_k$$

Now let us consider a tangent vector $P = (P_k) \in T_q \operatorname{Land}^N$. Its horizontal lift with footpoint φ_0 is $P^{\text{hor}} \circ \varphi_0$ where the vector field P^{hor} on $\mathbb{R}^n$ is given as follows: Let $K^{-1}(q)_{ki}$ be the inverse of the $(N \times N)$-matrix $K(q)_{ij} = K(q_i - q_j)$. Then

$$P^{\text{hor}}(x) = \sum_{i,j} K(x - q_i) K^{-1}(q)_{ij} P_j$$

$$L(P^{\text{hor}}(x)) = \sum_{i,j} \delta(x - q_i) K^{-1}(q)_{ij} P_j$$

Note that P^{hor} is a vector field of class H^{2l-1}.

The Riemannian metric on Land^N induced by the g^L-metric on $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n)$ is

$$
\begin{aligned}
g^L_q(P,Q) &= G^L_{\varphi_0}(P^{\mathrm{hor}}, Q^{\mathrm{hor}}) \\
&= \int_{\mathbb{R}^n} \langle L(P^{\mathrm{hor}}), Q^{\mathrm{hor}} \rangle \, dx \\
&= \int_{\mathbb{R}^n} \Big\langle \sum_{i,j} \delta(x-q_i) K^{-1}(q)_{ij} P_j, \\
&\qquad\qquad \sum_{k,l} K(x-q_k) K^{-1}(q)_{kl} Q_l \Big\rangle \, dx \\
&= \sum_{i,j,k,l} K^{-1}(q)_{ij} K(q_i - q_k) K^{-1}(q)_{kl} \langle P_j, Q_l \rangle \\
g^L_q(P,Q) &= \sum_{k,l} K^{-1}(q)_{kl} \langle P_k, Q_l \rangle. \qquad (1)
\end{aligned}
$$

The geodesic equation in vector form is:

$$\boxed{\begin{aligned}&\ddot{q}_n = \\ &-\frac{1}{2}\sum_{k,i,j,l} K^{-1}(q)_{ki}\,\mathrm{grad}\,K(q_i - q_j)(K(q)_{in} - K(q)_{jn}) \\ &\qquad\qquad K^{-1}(q)_{jl}\langle \dot{q}_k, \dot{q}_l\rangle \\ &+\sum_{k,i} K^{-1}(q)_{ki}\Big\langle\,\mathrm{grad}\,K(q_i - q_n), \dot{q}_i - \dot{q}_n\Big\rangle \dot{q}_k\end{aligned}}$$

The geodesic equation on $T^*\mathrm{Land}^N(\mathbb{R}^n)$

.

The cotangent bundle $T^*\mathrm{Land}^N(\mathbb{R}^n) = \mathrm{Land}^N(\mathbb{R}^n) \times ((\mathbb{R}^n)^N)^* \ni (q, \alpha)$. We shall treat $\mathbb{R}^n$ like scalars; $\langle \quad , \quad \rangle$ is always the standard inner product on $\mathbb{R}^n$. The metric looks like

$$(g^L)_q^{-1}(\alpha, \beta) = \sum_{i,j} K(q)_{ij} \langle \alpha_i, \beta_j \rangle,$$

$$K(q)_{ij} = K(q_i - q_j).$$

The energy function

$$E(q,\alpha) = \tfrac{1}{2}(g^L)_q^{-1}(\alpha,\alpha) = \tfrac{1}{2}\sum_{i,j} K(q)_{ij}\langle \alpha_i, \alpha_j\rangle$$

and its Hamiltonian vector field (using $\mathbb{R}^n$-valued derivatives to save notation)

$$\begin{aligned} H_E(q,\alpha) = \sum_{i,k=1}^{N} \Big(& K(q_k - q_i)\alpha_i \frac{\partial}{\partial q_k} \\ & + \operatorname{grad} K(q_i - q_k)\langle \alpha_i, \alpha_k\rangle \frac{\partial}{\partial \alpha_k}\Big). \end{aligned}$$

So the geodesic equation is the flow of this vector field:

$$\boxed{\begin{aligned} \dot{q}_k &= \sum_i K(q_i - q_k)\alpha_i \\ \dot{\alpha}_k &= -\sum_i \operatorname{grad} K(q_i - q_k)\langle \alpha_i, \alpha_k\rangle \end{aligned}}$$

Stress and Force

$$\alpha_k^\sharp = \sum_i K(q_k - q_i)\alpha_i, \quad \alpha^\sharp = \sum_{i,k} K(q_k - q_i)\langle \alpha_i, \tfrac{\partial}{\partial q^k}\rangle$$

$$\mathcal{D}(\alpha,\beta) := \sum_{i,j} dK(q_i - q_j)(\alpha_i^\sharp - \alpha_j^\sharp)\Big\langle \beta_j, \frac{\partial}{\partial q_i}\Big\rangle, \quad \text{the } \textit{stress.}$$

$$\mathcal{D}(\alpha,\beta) - \mathcal{D}(\beta,\alpha) = (D_{\alpha^\sharp}\beta^\sharp) - D_{\beta^\sharp}\alpha^\sharp = [\alpha^\sharp, \beta^\sharp], \quad \textit{Lie bracket.}$$

$$\mathcal{F}_i(\alpha,\beta) = \frac{1}{2}\sum_k \operatorname{grad} K(q_i - q_k)(\langle \alpha_i, \beta_k\rangle + \langle \beta_i, \alpha_k\rangle)$$

$$\mathcal{F}(\alpha,\beta) := \sum_i \langle \mathcal{F}_i(\alpha,\beta), dq_i\rangle = \frac{1}{2}\, d\, g^{-1}(\alpha,\beta) \quad \text{the } \textit{force}.$$

The geodesic equation on $T^*\,\mathrm{Land}^N(\mathbb{R}^n)$ then becomes

$$\boxed{\begin{aligned} \dot q &= \alpha^\sharp \\ \dot\alpha &= -\mathcal{F}(\alpha,\alpha) \end{aligned}}$$

Curvature via the cotangent bundle

From the semilocal version of Mario's formula for the sectional curvature expression for constant 1-forms α, β on landmark space, where $\alpha_k^\sharp = \sum_i K(q_k - q_i)\alpha_i$, we get directly:

$$
\boxed{
\begin{aligned}
& g^L\big(R(\alpha^\sharp, \beta^\sharp)\alpha^\sharp, \beta^\sharp\big) = \\
& = \big\langle \mathcal{D}(\alpha,\beta) + \mathcal{D}(\beta,\alpha), \mathcal{F}(\alpha,\beta)\big\rangle \\
& \quad - \big\langle \mathcal{D}(\alpha,\alpha), \mathcal{F}(\beta,\beta)\big\rangle - \big\langle \mathcal{D}(\beta,\beta), \mathcal{F}(\alpha,\alpha)\big\rangle \\
& \quad - \tfrac{1}{2}\sum_{i,j}\Big(d^2K(q_i - q_j)(\beta_i^\sharp - \beta_j^\sharp, \beta_i^\sharp - \beta_j^\sharp)\langle \alpha_i, \alpha_j\rangle \\
& \qquad\qquad - 2d^2K(q_i - q_j)(\beta_i^\sharp - \beta_j^\sharp, \alpha_i^\sharp - \alpha_j^\sharp)\langle \beta_i, \alpha_j\rangle \\
& \qquad\qquad + d^2K(q_i - q_j)(\alpha_i^\sharp - \alpha_j^\sharp, \alpha_i^\sharp - \alpha_j^\sharp)\langle \beta_i, \beta_j\rangle\Big) \\
& \quad - \|\mathcal{F}(\alpha,\beta)\|^2_{g^{-1}} + g^{-1}\big(\mathcal{F}(\alpha,\alpha), \mathcal{F}(\beta,\beta)\big). \\
& \quad + \tfrac{3}{4}\|[\alpha^\sharp, \beta^\sharp]\|^2_g
\end{aligned}
}
$$

Bundle of embeddings over the differentiable Chow variety.

Let M be a compact connected manifold with $\dim(M) < \dim(N)$. The smooth manifold $\mathrm{Emb}(M, N)$ of all embeddings $M \to N$ is the total space of a smooth principal bundle with structure group $\mathrm{Diff}(M)$ acting freely by composition from the right hand side.

The quotient manifold $B(M, N)$ can be viewed as the space of all submanifolds of N of diffeomorphism type M; we call it the *differentiable Chow manifold* or the *non-linear Grassmannian*.

$B(M, N)$ is a smooth manifold with charts centered at $F \in B(M, N)$ diffeomorphic to open subsets of the Frechet space of sections of the normal bundle $TF^{\perp,g} \subset TN|_F$.

Let $\ell : \mathrm{Diff}_{\mathcal{S}}(N) \times B(M, N) \to B(M, N)$ be the smooth left action. In the following we will consider just one open $\mathrm{Diff}_{\mathcal{S}}(N)$-orbit $\ell(\mathrm{Diff}_{\mathcal{S}}(N), F_0)$ in $B(M, N)$.

The induced Riemannian cometric on $T^*B(M,N)$

We follow the procedure used for $\mathrm{Diff}_{\mathcal{S}}(N)$. For any $F \subset N$, we decompose $\mathcal{H}$ into:

$$\mathcal{H}_F^{\text{vert}} = j_2^{-1}\big(\{X \in \Gamma_{C_b^2}(TN) : X(x) \in T_xF, \text{ for all } x \in F\}\big)$$

$$\mathcal{H}_F^{\text{hor}} = \text{perpendicular complement of } \mathcal{H}_F^{\text{vert}}$$

It is then easy to check that we get the diagram:

$$\begin{array}{ccccc}
\Gamma_{\mathcal{S}}(TN) & \overset{j_1}{\hookrightarrow} & \mathcal{H} & \overset{j_2}{\hookrightarrow} & \Gamma_{C_b^2}(TN) \\
\downarrow{\scriptstyle \text{res}} & & \downarrow & & \downarrow{\scriptstyle \text{res}} \\
\Gamma_{\mathcal{S}}(\mathrm{Nor}(F)) & \overset{j_1^f}{\hookrightarrow} & \mathcal{H}_F^{\text{hor}} & \overset{j_2^f}{\hookrightarrow} & \Gamma_{C_b^2}(\mathrm{Nor}(F)).
\end{array}$$

Here $\mathrm{Nor}(F) = TN|_F / TF$.

As this is an orthogonal decomposition, L and K take $\mathcal{H}_F^{\text{vert}}$ and $\mathcal{H}_F^{\text{hor}}$ into their own duals and, as before we get:

$$
\begin{array}{ccc}
\Gamma_{\mathcal{S}}(\mathrm{Nor}(F)) & & \Gamma_{\mathcal{S}'}(\mathrm{Nor}^*(F)) \\
\Big\downarrow{\scriptstyle j_1} & & \Big\uparrow{\scriptstyle j_1'} \\
\mathcal{H}_F^{\text{hor}} & \overset{L_F}{\underset{K_F}{\rightleftarrows}} & (\mathcal{H}_F^{\text{hor}})' \\
\Big\downarrow{\scriptstyle j_2} & & \Big\uparrow{\scriptstyle j_2'} \\
\Gamma_{C_b^2}(\mathrm{Nor}(F)) & & \Gamma_{M^2}(\mathrm{Nor}^*(F))
\end{array}
$$

K_F is just the restriction of K to this subspace of $\mathcal{H}'$ and is given by the kernel:

$$K_F(x_1, x_2) := \text{image of } K(x_1, x_2) \in \mathrm{Nor}_{x_1}(F) \otimes \mathrm{Nor}_{x_2}(F)), \quad x_1, x_2 \in F.$$

This is a C^2 section over $F \times F$ of $\mathrm{pr}_1^* \mathrm{Nor}(F) \otimes \mathrm{pr}_2^* \mathrm{Nor}(F)$.

We can identify $\mathcal{H}_F^{\text{hor}}$ as the closure of the image under K_F of measure valued 1-forms supported by F and with values in $\text{Nor}^*(F)$. A dense set of elements in $\mathcal{H}_F^{\text{hor}}$ is given by either taking the 1-forms with finite support or taking smooth 1-forms. In the smooth case, fix a volume form κ on M and a smooth covector $\xi \in \Gamma_{\mathcal{S}}(\text{Nor}^*(F))$. Then $\xi.\kappa$ defines a horizontal vector field h like this:

$$h(x_1) = \int_{x_2 \in F} \big| K_F(x_1, x_2) \big| \xi(x_2).\kappa(x_2) \big\rangle$$

The horizontal lift h^{hor} of any $h \in T_F B(M, N)$ is then:

$$h^{\text{hor}}(y_1) = K(L_F h)(y_1) = \int_{x_2 \in F} \big| K(y_1, x_2) \big| L_F h(x_2) \big\rangle, \quad y_1 \in N.$$

Note that all elements of the cotangent space $\alpha \in \Gamma_{\mathcal{S}'}(\text{Nor}^*(F))$ can be pushed up to N by $(j_F)_*$, where $j_F : F \hookrightarrow N$ is the inclusion, and this identifies $(j_F)_*\alpha$ with a 1-co-current on N.

Finally the induced homogeneous weak Riemannian metric on $B(M,N)$ is given like this:

$$\begin{aligned}\langle h,k\rangle_F &= \int_N (h^{\text{hor}}(y_1), L(k^{\text{hor}})(y_1)) = \int_{y_1\in N} (K(L_F h))(y_1), (L_F k)(y_1)) \\ &= \int_{(y_1,y_2)\in N\times N} (K(y_1,y_2), (L_F h)(y_1)\otimes (L_F k)(y_2)) \\ &= \int_{(x_1,x_1)\in F\times F} \Big\langle L_F h(x_1)\Big| K_F(x_1,x_2)\Big| L_F h(x_2)\Big\rangle\end{aligned}$$

With this metric, the projection from $\mathrm{Diff}_{\mathcal{S}}(N)$ to $B(M,N)$ is a submersion.

The inverse co-metric on the smooth cotangent bundle $\bigsqcup_{F\in B(M,N)} \Gamma(\mathrm{Nor}^*(F)\otimes \mathrm{vol}(F)) \subset T^*B(M,N)$ is much simpler and easier to handle:

$$\boxed{\langle\alpha,\beta\rangle_F = \iint_{F\times F} \big\langle \alpha(x_1)\big| K_F(x_1,x_2)\big|\beta(x_1)\big\rangle.}$$

It is simply the restriction to the co-metric on the Hilbert sub-bundle of $T^*\,\mathrm{Diff}_{\mathcal{S}}(N)$ defined by $\mathcal{H}'$ to the Hilbert sub-bundle of subspace $T^*B(M,N)$ defined by $\mathcal{H}'_F$.

Because they are related by a submersion, the geodesics on $B(M,N)$ are the horizontal geodesics on $\mathrm{Diff}_{\mathcal{S}}(N)$. We have two variables: a family $\{F_t\}$ of submanifolds in $B(M,N)$ and a time varying momentum $\alpha(t,\cdot) \in \mathrm{Nor}^*(F_t) \otimes \mathrm{vol}(F_t)$ which lifts to the horizontal 1-co-current $(j_{F_t})_*(\alpha(t,\cdot)$ on N. Then the horizontal geodesic on $\mathrm{Diff}_{\mathcal{S}}(N)$ is given by the same equations as before:

$$\boxed{\begin{aligned} \partial_t(F_t) &= \mathrm{res}_{\mathrm{Nor}(F_t)}(u(t,\cdot)) \\ u(t,x) &= \int_{(F_t)_y} \big|K(x,y)\big|\alpha(t,y)\big\rangle \in \mathfrak{X}_{\mathcal{S}}(N) \\ \partial_t\left((j_{F_t})_*(\alpha(t,\cdot)\right) &= -\mathcal{L}_{u(t,\cdot)}((j_{F_t})_*(\alpha(t,\cdot)). \end{aligned}}$$

This is a complete description for geodesics on $B(M,N)$ but it is not very clear how to compute the Lie derivative of $(j_{F_t})_*(\alpha(t,\cdot)$. One can unwind this Lie derivative via a torsion-free connection, but we turn to a different approach which will be essential for working out the curvature of $B(M,N)$.

Auxiliary tensors on $B(M, N)$

For $X \in \mathfrak{X}_{\mathcal{S}}(N)$ let B_X be the infinitesimal action on $B(M, N)$ given by $B_X(F) = T_{\mathrm{Id}}(\ell^F)X$ with its flow $\mathrm{Fl}_t^{B_X}(F) = \mathrm{Fl}_t^X(F)$. We have $[B_X, B_Y] = B_{[X,Y]}$.
$\{B_X(F) : X \in \mathfrak{X}_{\mathcal{S}}(N)\}$ equals the tangent space $T_F B(M, N)$.

Note that $B(M, N)$ is naturally submanifold of the vector space of m-currents on N:

$$B(M, N) \hookrightarrow \Gamma_{\mathcal{S}'}(\Lambda^m T^* N), \quad \text{via } F \mapsto \left(\omega \mapsto \int_F \omega\right).$$

Any $\alpha \in \Omega^m(N)$ is a linear coordinate on $\Gamma_{\mathcal{S}'}(TN)$ and this restricts to the function $B_\alpha \in C^\infty(B(M, N), \mathbb{R})$ given by $B_\alpha(F) = \int_F \alpha$.
If $\alpha = d\beta$ for $\beta \in \Omega^{m-1}(N)$ then

$$B_\alpha(F) = B_{d\beta}(F) = \int_F j_F^* d\beta = \int_F d j_F^* \beta = 0$$

by Stokes' theorem.

For $\alpha \in \Omega^m(N)$ and $X \in \mathfrak{X}_{\mathcal{S}}(N)$ we can evaluate the vector field B_X on the function B_α:

$$\begin{aligned} B_X(B_\alpha)(F) &= dB_\alpha(B_X)(F) = \partial_t|_0 B_\alpha(Fl_t^X(F)) \\ &= \int_F j_F^* \mathcal{L}_X \alpha = B_{\mathcal{L}_X(\alpha)}(F) \\ \text{as well as } &= \int_F j_F^*(i_X d\alpha + d i_X \alpha) = \int_F j_F^* i_X d\alpha = B_{i_X(d\alpha)}(F) \end{aligned}$$

If $X \in \mathfrak{X}_{\mathcal{S}}(N)$ is tangent to F along F then $B_X(B_\alpha)(F) = \int_F \mathcal{L}_{X|_F} j_F^* \alpha = 0$.

More generally, a *pm*-form α on N^k defines a function $B_\alpha^{(p)}$ on $B(M, N)$ by $B_\alpha^{(p)}(F) = \int_{F^p} \alpha$.

For $\alpha \in \Omega^{m+k}(N)$ we denote by B_α the k-form in $\Omega^k(B(M,N))$ given by the skew-symmetric multi-linear form:

$$(B_\alpha)_F(B_{X_1}(F), \ldots, B_{X_k}(F)) = \int_F {j_F}^*(i_{X_1 \wedge \cdots \wedge X_k} \alpha).$$

This is well defined: If one of the X_i is tangential to F at a point $x \in F$ then ${j_F}^*$ pulls back the resulting m-form to 0 at x.

Note that any smooth cotangent vector a to $F \in B(M,N)$ is equal to $B_\alpha(F)$ for some closed $(m+1)$-form α. Smooth cotangent vectors at F are elements of $\Gamma_{\mathcal{S}}(F, \mathrm{Nor}^*(F) \otimes \Lambda^m T^*(F))$.

Likewise, a $2m+k$ form $\alpha \in \Omega^{2m+k}(N^2)$ defines a k-form on $B(M,N)$ as follows: First, for $X \in \mathfrak{X}_{\mathcal{S}}(N)$ let $X^{(2)} \in \mathfrak{X}(N^2)$ be given by

$$X^{(2)}_{(n_1,n_2)} := (X_{n_1} \times 0_{n_2}) + (0_{n_1} \times X_{n_2})$$

Then we put

$$(B^{(2)}_\alpha)_F(B_{X_1}(F),\dots,B_{X_k}(F)) = \int_{F^2} {j_{F^2}}^*(i_{X_1^{(2)}\wedge\dots\wedge X_k^{(2)}}\alpha).$$

This is just B applied to the submanifold $F^2 \subset N^2$ and to the special vector fields $X^{(2)}$.

Using this for $p=2$, we find that for any two m-forms α,β on N, the inner product of B_α and B_β is given by:

$$g_B^{-1}(B_\alpha, B_\beta) = B^{(2)}_{\langle\alpha|K|\beta\rangle}.$$

We have

$$\begin{aligned} i_{B_X} B_\alpha &= B_{i_X\alpha} \\ dB_\alpha &= B_{d\alpha} \quad \text{for any } \alpha \in \Omega^{m+k}(N) \\ \mathcal{L}_{B_X} B_\alpha &= B_{\mathcal{L}_X\alpha} \end{aligned}$$

Force and Stress

Moving to curvature, fix F. Then we claim that for any two smooth co-vectors a, b at F, we can construct not only two closed $(m+1)$-forms α, β on N as above but also two commuting vector fields X_α, X_β on N in a neighborhood of F such that:

1. $B_\alpha(F) = a$ and $B_\beta(F) = b$,
2. $B_{X_\alpha}(F) = a^\sharp$ and $B_{X_\beta}(F) = b^\sharp$
3. $\mathcal{L}_{X_\alpha}(\alpha) = \mathcal{L}_{X_\alpha}(\beta) = \mathcal{L}_{X_\beta}(\alpha) = \mathcal{L}_{X_\beta}(\beta) = 0$
4. $[X_\alpha, X_\beta] = 0$

The force is

$$2\mathcal{F}(\alpha, \beta) = d(\langle B_\alpha, B_\beta \rangle) = d\left(B^{(2)}_{\langle \alpha | K | \beta \rangle}\right) = B^{(2)}_{d(\langle \alpha | K | \beta \rangle)}.$$

The stress $\mathcal{D} = \mathcal{D}_N$ on N can be computed as:

$$\mathcal{D}(\alpha, \beta, F)(x) = (\text{restr. to Nor}(F)) \left(- \int_{y \in F} \left| \mathcal{L}_{X^{(2)}_\alpha}(x, y) K(x, y) \right| \beta(y) \right\rangle \right)$$

The curvature

Finally, the semilocal Mario formula and some computations lead to:

$$\langle R_{B(M,N)}(B^\sharp_\alpha, B^\sharp_\beta)B^\sharp_\beta, B^\sharp_\alpha\rangle(F) = R_{11} + R_{12} + R_2 + R_3$$

$$R_{11} = \frac{1}{2}\iint_{F\times F}\Big(\big\langle\beta\big|\mathcal{L}_{X^{(2)}_\alpha}\mathcal{L}_{X^{(2)}_\alpha}K\big|\beta\big\rangle + \big\langle\alpha\big|\mathcal{L}_{X^{(2)}_\beta}\mathcal{L}_{X^{(2)}_\beta}K\big|\alpha\big\rangle - 2\big\langle\alpha\big|\mathcal{L}_{X^{(2)}_\alpha}\mathcal{L}_{X^{(2)}_\beta}K\big|\beta\big\rangle\Big)$$

$$R_{12} = \int_F\Big(\langle\mathcal{D}(\alpha,\alpha,F),\mathcal{F}(\beta,\beta,F)\rangle + \langle\mathcal{D}(\beta,\beta,F),\mathcal{F}(\alpha,\alpha,F)\rangle - \langle\mathcal{D}(\alpha,\beta,F)+\mathcal{D}(\beta,\alpha,F),\mathcal{F}(\alpha,\beta,F)\rangle\Big)$$

$$R_2 = \|\mathcal{F}(\alpha,\beta,F)\|^2_{K_F} - \big\langle\mathcal{F}(\alpha,\alpha,F)),\mathcal{F}(\beta,\beta,F)\big\rangle_{K_F}$$

$$R_3 = -\frac{3}{4}\|\mathcal{D}(\alpha,\beta,F) - \mathcal{D}(\beta,\alpha,F)\|^2_{L_F}$$

Thank you for listening