Metric measure spaces with Riemannian Ricci curvature bounded from below

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The course will provide an introduction to non-smooth analysis in a metric-measure space $(X, \mathsf{d}, \mathfrak{m})$ and its recent developments regarding first order calculus tools, Sobolev spaces, energy forms, heat flow, lower Ricci curvature bounds in terms of optimal transport and Bakry-Emery "Gamma-calculus".

After a quick review of the basic results in a smooth differentiable Riemannian setting, we will consider the case of a complete metric space (X, d) endowed with a Borel probability measure $\mathfrak{m}$ and we will try to present and discuss:

1. the construction of the Sobolev space $W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ and of the Cheeger energy obtained by relaxing the $L^2(X, \mathfrak{m})$ norm of the local slope of Lipschitz functions;
2. the equivalent characterization by weak upper gradients and derivatives along "good" collections of absolutely continuous curves, defined by modulus or test-plans;
3. the construction of a (possibly nonlinear) heat flow as the L^2-gradient flow of the Cheeger energy or the Wasserstein W_2-gradient flow of the relative entropy functional;
4. the role of the curvature-dimension condition $\mathrm{CD}(K, \infty)$ in terms of geodesic convexity of the entropy functional in the Wasserstein space;
5. the distinguished case when the Cheeger energy is quadratic: $\mathrm{RCD}(K, \infty)$ spaces and their characterization in terms of Evolution Variational Inequalities;
6. the point of view of Dirichlet forms: Γ-calculus, Bakry-Émery condition, intrinsic Biroli-Mosco distance;
7. the equivalence between the "metric-measure-transport" and the "energy-measure-semigroup" approaches.

Each topic will be supplemented with examples, applications, recent new results and research directions.

The course will be mainly based on [17, 30, 41]; an indicative (but largely incomplete) list of references follows.

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