Metric measure spaces with Riemannian Ricci curvature bounded from below
Lecture IV

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Outline

The abtract framework for Γ-calculus

- A (Polish) topological space $(\boldsymbol{X}, \tau)$
- A probability Borel measure $\mathfrak{m}$ with full support
- a **strongly local Dirichlet form** $\mathcal{E}$ in $L^2(\boldsymbol{X}, \mathfrak{m})$, i.e. a closed, symmetric, nonnegative bilinear form on $D(\mathcal{E}) \subset L^2(\boldsymbol{X}, \mathfrak{m})$ satisfying

$$\mathcal{E}(f_+, f_+) \le \mathcal{E}(f, f), \quad \mathcal{E}(f, h) = 0 \quad \text{if } f, h \in D(\mathcal{E}),\ fh = 0.$$

- $(\mathsf{P}_t)_{t \ge 0}$ is the **positivity and mass preserving Markov semigroup** in $L^2(\boldsymbol{X}, \mathfrak{m})$ (in fact in any $L^p(\boldsymbol{X}, \mathfrak{m})$) generated by $\mathcal{E}$
- $-\mathsf{L} : D(\mathsf{L}) \subset L^2(\boldsymbol{X}, \mathfrak{m})$ is the **selfadjoint accretive operator**

$$-\int \mathsf{L}u \, \varphi \, \mathrm{d}\mathfrak{m} = \mathcal{E}(u, \varphi), \quad -\int \mathsf{L}u \, u \, \mathrm{d}\mathfrak{m} = \mathcal{E}(u, u) \ge 0.$$

Bakry-Émery condition $\mathsf{BE}(K,\infty)$ in energy-measure spaces

$\mathsf{BE}(K,\infty)$: Weak form

for every $f \in L^2(\boldsymbol{X},\mathfrak{m})$, $h \in L^\infty(\boldsymbol{X},\mathfrak{m})$, $h \geq 0$, $t > 0$, the quantity

$$A_s[f,h] := \frac{1}{2}\int \big|\mathsf{P}_s f\big|^2 \mathsf{P}_{t-s} h \,\mathrm{d}\mathfrak{m}$$

satisfies

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}A_s[f,h] + 2K\frac{\mathrm{d}}{\mathrm{d}s}A_s[f,h] \geq 0 \quad \text{in } \mathscr{D}'(0,t)$$

Energy density: if $\mathsf{BE}(K,\infty)$ holds there exists a bilinear map $\Gamma : D(\mathcal{E}) \to L^1(\boldsymbol{X},\mathfrak{m})$ ($\Gamma(f)$ plays the role of $|\mathrm{D}f|_w^2$) such that

$$-\frac{1}{2}\mathcal{E}(f^2,h) + \mathcal{E}(f,fh) = \int \Gamma(f)\,h\,\mathrm{d}\mathfrak{m} \qquad \text{for every } f,h \in D(\mathcal{E}) \cap L^\infty$$

$$\mathcal{E}(f,h) = \int \Gamma(f,h)\,\mathrm{d}\mathfrak{m}.$$

Pointwise gradient commutation estimate: for every $f \in D(\mathcal{E})$

$$\Gamma\big(\mathsf{P}_t f\big) \leq \mathrm{e}^{-2Kt}\mathsf{P}_t\big(\Gamma(f)\big)$$

Strong form: Γ_2 tensor $\Gamma_2(f) = \frac{1}{2}\mathsf{L}\Gamma(f) - \Gamma(f,\mathsf{L}f) \geq K\Gamma(f)$ can be recovered in a measure-theoretic sense, useful for further applications.

BE $\Rightarrow$ RCD: program.

1. Starting from the Dirichlet form $\mathcal{E}$ study the property of the induced Biroli-Mosco distance $\mathsf{d}_{\mathcal{E}}$ and the corresponding Cheeger energy $\mathsf{Ch}_{\mathsf{d}_{\mathcal{E}}}$
2. By identifying $\mathcal{E}$ with $\mathsf{Ch}_{\mathsf{d}_{\mathcal{E}}}$ we prove that the Markov semigroup P_t as the L^2-gradient flow of the Dirichlet form $\mathcal{E}$ coincides with the Wasserstein gradient flow of the Entropy.
3. Prove the Wasserstein contraction property in order to extend P_t to a semigroup S_t defined on probability measures.
4. Prove that **S_t is a metric K-flow of the Entropy.**

Intrinsic distance

"1-Lipschitz" functions induced by Γ:

$$\mathbb{L} := \Big\{\psi \in D(\mathcal{E}) : \Gamma(\psi) \le 1 \ \mathfrak{m}\text{-a.e.}\Big\}$$

Assumption I Every function in $\mathbb{L}$ admits a continuous representative.

Biroli-Mosco distance [Feffereman-Sanchez Calle, Nagel-Stein-Wanger, Sturm]

$$\boxed{\mathsf{d}_{\mathcal{E}}(x,y) := \sup_{\psi\in\mathbb{L}} |\psi(x) - \psi(y)|}$$

- $\mathsf{d}_{\mathcal{E}}$ is always τ-lower semicontinuous
- $\mathsf{d}_{\mathcal{E}}$ is a distance (possibly assuming $+\infty$) whenever $\mathbb{L}$ separates the points of $\boldsymbol{X}$.

Assumption II $(\boldsymbol{X}, \mathsf{d}_{\mathcal{E}})$ is a complete and separable metric space.

Completeness is not an issue, since one can always take the abstract completion of $\boldsymbol{X}$. The crucial point here is **separability.**
By replacing τ with the topology induced by $\mathsf{d}_{\mathcal{E}}$, one can always assume that the topologies coincide. $\mathfrak{m}$-measurable sets are not affected.

"Singular" examples: extended distances and strict inequality

$\boldsymbol{X}$ = unit square of $\mathbb{R}^2$, $\mathfrak{m}$ the Lebesgue measure

$$\boxed{\mathcal{E}(f) = \int (\partial_x f)^2 \,\mathrm{d}x\mathrm{d}y}, \quad D(\mathcal{E}) = \Big\{ f \in L^2, \ \partial_x f \in L^2 \Big\}$$

$$\mathsf{d}_{\mathcal{E}}\big((x_1, y_1), (x_2, y_2)\big)^2 = \begin{cases} |x_1 - x_2|^2 & \text{if } y_1 = y_2, \\ +\infty & \text{otherwise.} \end{cases}$$

[Sturm] For every $\varepsilon > 0$ there exists a function $\mathsf{g} : \boldsymbol{X} \to [1/2, 1)$ such that $\mathscr{L}^2[\mathsf{g} > 1/2] < \varepsilon$ such that

$$\mathcal{E}(f) := \int \mathsf{g} |\mathrm{D} f|^2 \,\mathrm{d}x$$

produces $\mathsf{d}_{\mathcal{E}}(x, y) = |x - y|$ as for $\mathsf{g} \equiv 1$, so that

$$\mathsf{Ch}(f) = \frac{1}{2} \int |\mathrm{D} f|^2 \,\mathrm{d}x > \frac{1}{2} \mathcal{E}(f).$$

The identity $\mathsf{Ch} = \frac{1}{2}\mathcal{E}$ holds if g is a continuous function.

General properties

- [Sturm, Stollmann] $(\boldsymbol{X}, \mathsf{d}_{\mathcal{E}})$ is always **a length space** (i.e. $\mathsf{d}_{\mathcal{E}}(x_0, x_1)$ is the infimum of the length of the curves connecting x_0 to x_1).
- $\mathbb{L}$ is a convex subset of $D(\mathcal{E})$ which is **closed in $L^2(\boldsymbol{X}, \mathfrak{m})$.**
- Every function in $\mathbb{L}$ is $1-$lipschitz w.r.t. $\mathsf{d}_{\mathcal{E}}$
- **Every bounded 1-Lipschitz function w.r.t. $\mathsf{d}_{\mathcal{E}}$ belongs to $\mathbb{L}$.**

Proof

General properties of local Dirichlet form:

$$\Gamma(f\vee g)=\begin{cases}\Gamma(f) & \text{where } f\ge g,\\ \Gamma(g) & \text{where } f\le g,\end{cases}\qquad \Gamma(f\vee M)=\begin{cases}\Gamma(f) & \text{where } f< M,\\ 0 & \text{where } f\ge M,\end{cases}$$

$$\Gamma(\phi(f))=(\phi'(f))^2\Gamma(f)\quad \text{if } \phi\in \mathrm{Lip}(\mathbb{R}).$$

There exists a countable set $(\psi_n)_n\subset\mathbb{L}$ such that

$$\mathsf{d}_{\mathcal{E}}(x,y):=\sup_n|\psi_n(x)-\psi_n(y)|=\lim_{n\to\infty}\sup_{1\le m\le n}|\psi_m(x)-\psi_m(y)|$$

For every fixed $y\in\boldsymbol{X}$

$$x\mapsto \mathsf{d}_{n,k}(x)=\Big(\sup_{1\le m\le n}|\psi_m(x)-\psi_m(y)|\Big)\wedge k\quad \text{belongs to } \mathbb{L}$$

so that $x\mapsto\mathsf{d}_k(x,\bar x)=\mathsf{d}_{\mathcal{E}}(x,\bar x)\wedge k=\lim_{n\to\infty}\mathsf{d}_{n,k}(x)$ belongs to $\mathbb{L}$.
If now f is 1-Lipschitz and bounded (without restriction $0\le f\le k$, i.e. $f(x)-f(y)\le\mathsf{d}_k(x,y)$) we have $f\in\mathbb{L}$ since for a **countable dense** $(y_n)_n$

$$f(x)=\inf_y(f(y)+\mathsf{d}_k(x,y))=\inf_n f(y_n)+\mathsf{d}_k(x,y_n)$$

$$=\lim_{n\to\infty}\Big(\inf_{1\le j\le n}f(y_j)+\mathsf{d}_k(x,y_j)\Big)$$

Comparison with the Cheeger energy

Let Ch be the Cheeger energy induced by $\mathsf{d}_{\mathcal{E}}$ with minimal weak upper gradient $|\mathrm{D}\cdot|_w$. Then

$$D(\mathsf{Ch}) \subset D(\mathcal{E}), \quad \mathsf{Ch}(f) \ge \frac{1}{2}\mathcal{E}(f,f)$$

and for every $f \in D(\mathcal{E})$

$$|\mathrm{D}f|_w^2 \ge \Gamma(f) \quad \mathfrak{m}\text{-a.e.}$$

The Hopf-Lax semigroup

Assume that $\boldsymbol{X}$ is compact. Let $\phi \in \mathrm{Lip}(\boldsymbol{X})$ and

$$\boxed{\mathsf{Q}_t\phi(x) := \min_y \frac{1}{2t}\mathsf{d}^2(x,y) + \phi(y).}$$

Let

$$Y_t(x) := \operatorname*{argmin}_y \frac{1}{2t}\mathsf{d}^2(x,y) + \phi(y), \quad D_t^+(x) := \max_{y \in Y_t(x)} \mathsf{d}(x,y)$$

The map $t \mapsto \mathsf{Q}_t\phi$ is **Lipschitz** from $[0,\infty)$ to $\mathrm{C}(\boldsymbol{X})$ and $\mathsf{Q}_t\phi$ is **Lipschitz** for every $t \geq 0$.

$$|\mathrm{D}\phi|(x) \leq \frac{D_t^+(x)}{t}$$

For every $x \in \boldsymbol{X}$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{Q}_t\phi + \frac{1}{2}\Big(\frac{D_t^+(x)}{t}\Big)^2 = 0} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\mathsf{Q}_t\phi + \frac{1}{2}|\mathrm{D}\mathsf{Q}_t\phi|^2 \leq 0 \qquad \text{(HJ)}$$

for every $t > 0$ with at most countably many exceptions.
If $(\boldsymbol{X}, \mathsf{d})$ is a geodesic space, then equality holds in (HJ) for every $t > 0$ with at most countably many exceptions.

Proof (compact case)

- By definition of Cheeger energy, it is not restrictive to **assume f Lipschitz** and prove that

$$|\mathrm{D}f|^2 \geq \Gamma(f) \quad \mathfrak{m}\text{-a.e.}$$

- By truncation, we can also assume $0 \leq f \leq 1$ and $\mathsf{d}_{\mathcal{E}} \leq 1$.
- Proof for $\mathsf{Q}_t f$, via **approximated Hopf lax formula**

$$\mathsf{Q}_t f(x) = \inf_y \frac{1}{2t}\mathsf{d}^2_{\mathcal{E}}(x,y) + f(y) = \lim_{n\to\infty} f^n_t(x),$$

$$f^n_t(x) := \inf_{1\leq j\leq n} \frac{1}{2t}\mathsf{d}^2_{\mathcal{E}}(x,y_j) + f(y_j);$$

$y^n_t(x) :=$ any point where the inf is attained.

- $\Gamma\big(f^n_t\big)(x) \leq A^n_t(x) := \frac{1}{t^2}\mathsf{d}^2_{\mathcal{E}}(x, y^n_t(x))$
- $0 \leq \mathsf{Q}_t f(x) \leq f^n_t(x) \leq 1 + \frac{1}{2t}$,
  $f^n_t(x) = \frac{1}{2t}\mathsf{d}^2_{\mathcal{E}}(x, y^n_t(x)) + f(y^n_t(x)) \downarrow \mathsf{Q}_t f(x)$
- $\Gamma(\mathsf{Q}_t f) \leq A_t(x) := \limsup_{n\to\infty} A^n_t(x) \leq (D^+_t(x)/t)^2 \leq 2\,\mathrm{Lip}(f)$.
- $\partial_t \mathsf{Q}_t f(x) + \frac{1}{2}A_t(x) \leq 0, \quad |\mathrm{D}f|^2(x) \geq \limsup_{t\downarrow 0} \int_0^1 A_{tr}(x)\,\mathrm{d}r$.

The reverse inequality: upper semicontinuous envelope of the slope

Let $f \in \mathbb{L}$, $\zeta : \boldsymbol{X} \to \mathbb{R}$ upper semicontinuous.

$$\text{if } \Gamma(f) \le \zeta \quad \text{then} \quad |\mathrm{D}f| \le \zeta.$$

Since $\boldsymbol{X}$ is a length space, $|\mathrm{D}f|^* \le \zeta$.

Proof: fix $x_0 \in \boldsymbol{X}$, $\varepsilon > 0$, $Z_\varepsilon = \sup_{B_\varepsilon(x_0)} \zeta$ and consider the Lipschitz function

$$\psi(x) = \Big[|f(x) - f(x_0)| \vee Z_\varepsilon \mathsf{d}(x, x_0)\Big] \wedge \varepsilon Z_\varepsilon.$$

$\Gamma(\psi) \le Z_\varepsilon$ so that ψ is Z_ε-Lipschitz.

$\psi(x) \le Z_\varepsilon \mathsf{d}(x, x_0)$ and

$$|\mathrm{D}f|(x_0) \le \limsup_{x \to x_0} \frac{|f(x) - f(x_0)|}{\mathsf{d}_{\mathcal{E}}(x, x_0)} \le \limsup_{x \to x_0} \frac{\psi(x)}{\mathsf{d}_{\mathcal{E}}(x, x_0)} \le Z_\varepsilon.$$

We conclude by letting $\varepsilon \downarrow 0$, using the u.s.c. of ζ.

Upper regular Dirichlet forms

$\mathcal{E}$ is upper-regular if for every f in a dense subset of $D(\mathcal{E})$ there exists sequences $f_n \in D(\mathcal{E})$ and ζ_n u.s.c. and bounded such that

$$\Gamma(f_n) \le \zeta_n, \quad f_n \xrightarrow{L^2} f, \quad \liminf_{n\to\infty} \int \zeta_n \,\mathrm{d}\mathbf{m} \le \mathcal{E}(f)$$

If $\mathcal{E}$ is upper regular then

$$\boxed{\mathsf{Ch}(f) = \frac{1}{2}\mathcal{E}(f,f), \quad D(\mathcal{E}) = D(\mathsf{Ch}), \quad |\mathrm{D}f|_w^2 = \Gamma(f).}$$

Bakry-Emery yields upper regularity

If $\mathsf{BE}(K,\infty)$ holds then $\mathcal{E}$ **is upper regular** and $\mathsf{Ch} = \frac{1}{2}\mathcal{E}$. Moreover

$$|\mathrm{D}\mathsf{P}_t f|^2 \le \mathrm{e}^{-2Kt}\mathsf{P}_t\big(|\mathrm{D}f|_w^2\big) \quad \text{for every Lipschitz function } f.$$

Proof.
$L^\infty \rightsquigarrow$ Lip regularization: if $\zeta \in L^\infty(\boldsymbol{X},\mathfrak{m})$ then

$$C(t)\Gamma(\mathsf{P}_t\zeta) \le \|\zeta\|_{L^\infty}, \quad \mathsf{P}_t(\zeta) \in \mathrm{C}_b(\boldsymbol{X}).$$

Take now $f \in \mathbb{L}$ and $\zeta = \Gamma(f)$.

$$\Gamma(\mathsf{P}_t f) \le \mathrm{e}^{-2Kt}\mathsf{P}_t\zeta, \quad \mathsf{P}_t \to f, \quad \int \mathsf{P}_t\zeta \to \int \Gamma(f)\,\mathrm{d}\mathfrak{m}$$

as $t \downarrow 0$.

Bakry-Émery yields Wasserstein contraction

Define a semigroup S_t on (absolutely continuous w.r.t. $\mathfrak{m}$) probability meaures by the formula

$$\mathsf{S}_t(f\mathfrak{m}) := (\mathsf{P}_t f)\,\mathfrak{m} \quad \text{for every } f \in L^1_+(\boldsymbol{X}, \mathfrak{m}).$$

Contraction and fundamental solutions

If $\mathsf{BE}(K,\infty)$ holds then

$$W_2(\mathsf{S}_t\mu, \mathsf{S}_t\nu) \le \mathrm{e}^{-Kt} W_2(\mu,\nu)$$

S_t can be extended to $\mathscr{P}_2(\boldsymbol{X})$ by density.
Precise representative: for every bounded or nonnegative Borel function

$$\mathsf{P}_t f(x) = \int f\,\mathrm{d}(\mathsf{S}_t\delta_x)$$

Proof [Kuwada]

($K = 0$, $\boldsymbol{X}$-compact)

- Contraction in W_1 just by duality with Lipschitz function.
- Extension of S_t to arbitrary probability measure since W_1 provides a distance for the weak convergence in $\mathscr{P}(\boldsymbol{X})$.
- Kantorovich duality and Hopf-Lax

$$\boxed{\mathsf{P}_t\mathsf{Q}_1 f(x) - \mathsf{P}_t f(y) \leq \frac{1}{2}\mathsf{d}^2_{\mathcal{E}}(x, y).}$$

Commutation inequality between P and Q [Ledoux]

$$\boxed{\mathsf{P}_t\mathsf{Q}_1 f \leq \mathsf{Q}_1\mathsf{P}_t f}$$