Metric measure spaces with Riemannian Ricci curvature bounded from below
Lecture III

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Outline

1. $\mathsf{CD}(K,\infty)$ **and** $\mathsf{RCD}(K,\infty)$ **metric-measure spaces**
2. **A stronger notion of metric flows via Evolution Variational inequalities**
3. $\mathsf{RCD} \Rightarrow \mathsf{BE}$
4. **Stability under Sturm-Gromov-Hausdorff convergence and spectral convergence**

CD(K,∞) metric measure spaces.

$(\boldsymbol{X},\mathsf{d},\mathfrak{m})$: $(\boldsymbol{X},\mathsf{d})$ is a complete and separable metric space, $\mathfrak{m}$ is a Borel probability measure in $\mathscr{P}(\boldsymbol{X})$ with full support

CD(K,∞) spaces

For every $\mu_0,\mu_1\in\mathscr{P}(\boldsymbol{X})$ with finite entropy there exists $\mu_\vartheta\in\mathscr{P}(\boldsymbol{X})$ such that:

- **Geodesic interpolation in the transport metric**:

$$W_2(\mu_\vartheta,\mu_0)=\vartheta W_2(\mu_0,\mu_1),\quad W_2(\mu_\vartheta,\mu_1)=(1-\vartheta)W_2(\mu_0,\mu_1),$$

- ***K*-convexity of the Entropy:**

$$\mathrm{Ent}_{\mathfrak{m}}(\mu_\vartheta)\le(1-\vartheta)\mathrm{Ent}_{\mathfrak{m}}(\mu_0)+\vartheta\mathrm{Ent}_{\mathfrak{m}}(\mu_1)-\frac{K}{2}\vartheta(1-\vartheta)W_2^2(\mu_0,\mu_1).$$

Riemannian metric measure spaces: the $\mathsf{RCD}(K,\infty)$ condition

Even if $(\boldsymbol{X},\mathsf{d},\mathfrak{m})$ satisfies the $\mathsf{CD}(K,\infty)$ condition, **in general** the Cheeger energy is not a quadratic form and the heat semigroup is not linear.
If one hope to compare the LSV and the BE approaches, it is necessary to impose these properties: they lead to the definition of $\mathsf{RCD}(K,\infty)$ spaces:

$\mathsf{RCD}(K,\infty)$ spaces

A metric measure space satisfies the Riemannian $\mathsf{RCD}(K,\infty)$ condition if

- **$(\boldsymbol{X},\mathsf{d},\mathfrak{m})$ is a $\mathsf{CD}(K,\infty)$ space**
- **the Cheeger energy $\mathsf{Ch}(f)=\frac{1}{2}\int|\mathrm{D}f|_w^2\,\mathrm{d}\mathfrak{m}$ is quadratic** (equivalently P is a linear semigroup).

Theorem ($\mathsf{RCD}(K,\infty)$ spaces satisfies the Bakry-Émery $\mathsf{BE}(K,\infty)$ condition)

If $(\boldsymbol{X},\mathsf{d},\mathfrak{m})$ is a $\mathsf{RCD}(K,\infty)$ metric measure space then

- *the **Cheeger energy is a strongly local Dirichlet form***
- *$(\boldsymbol{X},\mathfrak{m},\mathsf{Ch})$ **satisfies the Bakry-Émery $\mathsf{BE}(K,\infty)$ condition.***
- *$|\mathrm{D}u|_w^2=\Gamma(u)$ and functions with $|\mathrm{D}u|_w\le L$ $\mathfrak{m}$-a.e. are L-Lipschitz.*

Description of the Heat flow: Cheeger-L^2 vs transport-entropy

$L^2(\boldsymbol{X}, \mathfrak{m})$ **framework**: $f_t = \mathsf{P}_t f$ are **functions**, the evolution is obtained by

"maximizing" the $L^2(\boldsymbol{X}, \mathfrak{m})$-dissipation rate of the Cheeger energy,

Along an arbitrary curve $h_t \in \mathrm{AC}^2$: $-\dfrac{\mathrm{d}}{\mathrm{d}t}\mathsf{Ch}(h_t) \le \|\dot h_t\|_2\,\|\Delta h_t\|_2$

Along the heat flow $f_t = \mathsf{P}_t f$: $-\dfrac{\mathrm{d}}{\mathrm{d}t}\mathsf{Ch}(f_t) = \|\dot f_t\|_2\,\|\Delta f_t\|_2 = \|\dot f_t\|_2^2 = \|\Delta f_t\|_2^2$

Dual point of view: f_t are **probability densities**, associated to the evolving measures $\mu_t = f_t\mathfrak{m}$.
The evolution is obtained by

"maximizing" the dissipation rate of the Entropy functional

$$-\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Ent}_{\mathfrak{m}}(\mu_t) = \sqrt{\mathsf{F}(f_t)}|\dot\mu_t| = \mathsf{F}(f_t) = |\dot\mu_t|^2, \qquad \mu_t = f_t\mathfrak{m}$$

with respect to the transport distance W.

Are there better characterizations, as for convex functionals in Hilbert spaces?

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|f_t - v\|_{L^2}^2 \le \mathsf{Ch}(v) - \mathsf{Ch}(f_t)$$

EVI: euristics in the case of convex functionals

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\mathsf{x}_t - y\|^2 \le \Phi(y) - \Phi(\mathsf{x}_t) \quad \text{for every } y \in H} \tag{EVI}$$

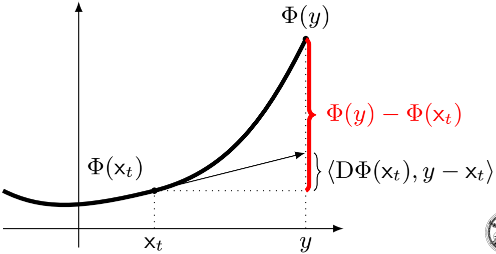
EVI is modeled on the variational characterization of **gradient flows of K-convex functionals Φ in a Hilbert space H**: in this case a curve $t \mapsto \mathsf{x}_t$ solves the differential equation

$$\dot{\mathsf{x}}_t = -\mathrm{D}\Phi(\mathsf{x}_t)$$

if and only if

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\mathsf{x}_t - y\|^2 \le \Phi(y) - \Phi(\mathsf{x}_t) \quad \text{for every } y \in H \tag{EVI}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\mathsf{x}_t - y\|^2 &= \langle \dot{\mathsf{x}}_t, \mathsf{x}_t - y\rangle \\ &= \langle \mathrm{D}\Phi(\mathsf{x}_t), y - \mathsf{x}_t\rangle \\ &\le \Phi(y) - \Phi(\mathsf{x}_t)\end{aligned}$$



Evolution variational inequality for the Entropy and metric K-flows

Let μ be a given initial measure in $\mathscr{P}_2(\boldsymbol{X})$ and $K \in \mathbb{R}$.

$\mathbf{EVI}_K(\mu)$ and Metric K-flows

A locally Lipschitz curve $\mu : (0,\infty) \to \mathscr{P}_2(\boldsymbol{X})$ is a solution of the **Evolution Variational Inequality** $\mathsf{EVI}_K(\mu)$ if for a.e. $t > 0$ and for every $\nu \in \mathscr{P}_2(\boldsymbol{X})$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(\mu_t,\nu)+\frac{K}{2}W_2^2(\mu_t,\nu) \le \mathrm{Ent}_{\mathfrak{m}}(\nu) - \mathrm{Ent}_{\mathfrak{m}}(\mu_t)} \tag{EVI}$$

and $\lim_{t\downarrow 0}\mu_t = \mu \quad$ in $\mathscr{P}_2(\boldsymbol{X})$.
$(\mathsf{S}_t)_{t\ge 0}$ is **a metric K-flow** in $D(\mathrm{Ent}_{\mathfrak{m}})$ if $\mu_t := \mathsf{S}_t(\mu)$ solves $\mathsf{EVI}_K(\mu)$ for every $\mu \in D(\mathrm{Ent}_{\mathfrak{m}})$.

Properties of solutions to EVI_K

Let μ, ν be solutions to EVI_K with initial data $\bar\mu, \bar\nu \in \mathscr{P}_2(X)$.

- **Uniqueness and K-contraction:**

$$\boxed{W_2(\mu_t, \nu_t) \le \mathrm{e}^{-Kt} W_2(\bar\mu, \bar\nu)}$$

- **Entropy dissipation:** The map $t \mapsto \mathrm{Ent}_{\mathfrak{m}}(\mu_t)$ is nonincreasing, locally semi-convex, and satisfies the **Entropy dissipation identity**

$$\boxed{-\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Ent}_{\mathfrak{m}}(\mu_t) = |\dot\mu_t|^2 = \mathsf{F}(\mu_t)} \qquad \text{(EDI)}$$

  In particular $\mu_t = (\mathsf{P}_t f)\mathfrak{m}$ whenever $\bar\mu = f\mathfrak{m}$.

- **Regularizing effect:** For $t > 0$ we have $\mu_t \in D(\mathsf{F}) \subset D(\mathrm{Ent}_{\mathfrak{m}})$. If e.g. $K \ge 0$, we have for every $\nu \in D(\mathsf{F})$

$$\mathrm{Ent}_{\mathfrak{m}}(\mu_t) \le \mathrm{Ent}_{\mathfrak{m}}(\nu) + \frac{1}{2t} W_2^2(\mu_t, \nu), \qquad \mathsf{F}(\mu_t) \le \mathsf{F}(\nu) + \frac{1}{t^2} W_2^2(\mu_t, \nu)$$

Metric K-flows, $\mathsf{RCD}(K,\infty)$-spaces and $\mathsf{BE}(K,\infty)$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(\mu_t,\nu)+\frac{K}{2}W_2^2(\mu_t,\nu)\le \mathrm{Ent}_{\mathfrak{m}}(\nu)-\mathrm{Ent}_{\mathfrak{m}}(\mu_t)} \qquad \text{(EVI)}$$

Theorem (Metric K-flow)

Let us suppose that the entropy functional admits a K-flow $(\mathsf{S}_t)_{t\ge 0}$ in $(\boldsymbol{X},\mathsf{d},\mathfrak{m})$. Then

- S_t ***coincides with Heat semigroup*** P_t *(equivalently defined as the Wasserstein gradient flow of the Entropy or the L^2-flow of the Cheeger energy).*
- *The* ***Entropy functional is K-convex****, i.e. $(\boldsymbol{X},\mathsf{d},\mathfrak{m})$ is a $\mathsf{CD}(K,\infty)$ space.*
- S_t ***is a linear semigroup and the Cheeger energy is quadratic.*** *In particular $(\boldsymbol{X},\mathsf{d},\mathfrak{m})$ is a Riemannian $\mathsf{RCD}(K,\infty)$ space.*
- S_t ***satisfies the Wasserstein contraction*** *estimate*

$$W_2(\mathsf{S}_t\mu,\mathsf{S}_t\nu)\le \mathrm{e}^{-Kt}W_2(\mu,\nu).$$

and [Kuwada] $\mathsf{BE}(K,\infty)$ *holds for the Heat semigroup* P*:*

$$\big|\mathrm{D}\mathsf{P}_t u\big|_w^2\le \mathrm{e}^{-2Kt}\mathsf{P}_t\big|\mathrm{D}u\big|_w^2$$

Contraction

$\mu_s = \mathsf{S}_s\mu,\ \nu_t = \mathsf{S}_t\nu$

$$\frac{\partial}{\partial s}\frac{1}{2}W_2^2(\mu_s,\nu_t) \le \mathrm{Ent}_{\mathfrak{m}}(\nu_t) - \mathrm{Ent}_{\mathfrak{m}}(\mu_s)$$
$$\frac{\partial}{\partial t}\frac{1}{2}W_2^2(\mu_s,\nu_t) \le \mathrm{Ent}_{\mathfrak{m}}(\mu_s) - \mathrm{Ent}_{\mathfrak{m}}(\nu_t)$$
$$\frac{\partial}{\partial s}\frac{1}{2}W_2^2(\mu_s,\nu_t) + \frac{\partial}{\partial t}\frac{1}{2}W_2^2(\mu_s,\nu_t) \le 0$$

"$s = t$"

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(\mu_t,\nu_t) \le 0$$

Convexity $(K = 0)$

Let μ_ϑ be a geodesic in $\mathscr{P}_2(\boldsymbol{X})$, $\mu_\vartheta(t) = \mathsf{S}_t \mu_\vartheta$.

$$\frac{1}{2}W_2^2(\mu_\vartheta(t), \mu_0) - \frac{1}{2}W_2^2(\mu_\vartheta, \mu_0) \le t\Big(\mathrm{Ent}_{\mathfrak{m}}(\mu_0) - \mathrm{Ent}_{\mathfrak{m}}(\mu_\vartheta(t))\Big) \quad \rightsquigarrow \times(1-\vartheta)$$
$$\frac{1}{2}W_2^2(\mu_\vartheta(t), \mu_1) - \frac{1}{2}W_2^2(\mu_\vartheta, \mu_1) \le t\Big(\mathrm{Ent}_{\mathfrak{m}}(\mu_1) - \mathrm{Ent}_{\mathfrak{m}}(\mu_\vartheta(t))\Big) \quad \rightsquigarrow \times\vartheta$$

$$(1-\vartheta)\mathrm{Ent}_{\mathfrak{m}}(\mu_0) + \vartheta \mathrm{Ent}_{\mathfrak{m}}(\mu_1) - \mathrm{Ent}_{\mathfrak{m}}(\mu_\vartheta(t)) \ge 0$$

since along the geodesic

$$(1-\vartheta)W_2^2(\mu_\vartheta, \mu_0) + \vartheta W_2^2(\mu_\vartheta, \mu_1) = \vartheta(1-\vartheta)W_2^2(\mu_0, \mu_1)$$

and the triangle inequality yields

$$(1-\vartheta)W_2^2(\mu_\vartheta(t), \mu_0) + \vartheta W_2^2(\mu_\vartheta(t), \mu_1) \ge \vartheta(1-\vartheta)W_2^2(\mu_0, \mu_1)$$

Linearity

and set

$$G(\mu,\nu) := \mathrm{Ent}_{\mathfrak{m}}(\nu) - \mathrm{Ent}_{\mathfrak{m}}(\mu).$$

Let μ_t^1, μ_t^2 be two gradient flows; we know that for arbitrary ν^1, ν^2

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(\mu_t^1,\nu^1) \le G(\mu_t^1,\nu^1), \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(\mu_t^2,\nu^2) \le G(\mu_t^2,\nu^2).$$

Setting

$$\mu_t := \alpha\mu_t^1 + \beta\mu_t^2, \quad \alpha,\beta \ge 0,\ \alpha+\beta = 1,$$

we want to prove that

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(\mu_t,\nu) \le \mathrm{Ent}_{\mathfrak{m}}(\nu) - \mathrm{Ent}_{\mathfrak{m}}(\mu_t) = G(\mu_t,\nu)} \quad \forall\,\nu \in D(\mathrm{Ent}_{\mathfrak{m}})$$

Idea: fix a time t and split the test measure ν as $\nu = \alpha\nu^1 + \beta\nu^2$ (depending on t) so that at that time

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(\mu_t,\nu) \boxed{\le} \alpha\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(\mu_t^1,\nu^1) + \beta\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(\mu_t^2,\nu^2) \quad \text{[\textbf{Subadditivity}]}$$

$$G(\mu,\nu) \boxed{\ge} \alpha G(\mu^1,\nu^1) + \beta G(\mu^2,\nu^2) \quad \text{[\textbf{Superadditivity}]}$$

The choice of ν^1, ν^2

Fix t and let $\boldsymbol{\sigma}$ be an **optimal coupling** between μ_t and ν and let

$$\theta^1 := \alpha \frac{\mathrm{d}\mu_t^1}{\mathrm{d}\mu_t}, \quad \mu_t^1 = \theta^1 \mu_t; \qquad \theta^2 := \beta \frac{\mathrm{d}\mu_t^2}{\mathrm{d}\mu_t}, \quad \mu_t^2 = \theta^2 \mu_t; \qquad \theta^1(x)+\theta^2(x) \equiv 1.$$

We set $\pi^x(x,y) := x$, $\pi^y(x,y) := y$ and

$$\boldsymbol{\sigma}^1 := \theta^1(x)\boldsymbol{\sigma} = (\theta^1 \circ \pi^x)\boldsymbol{\sigma}, \quad \boldsymbol{\sigma}^2 := \theta^2(x)\boldsymbol{\sigma} = (\theta^2 \circ \pi^x)\boldsymbol{\sigma}; \qquad \boldsymbol{\sigma}^1 + \boldsymbol{\sigma}^2 = \boldsymbol{\sigma}$$

$\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2$ are still **optimal couplings,** since the optimality property depends only on the **support** of a coupling.

Correspondingly we set $\boxed{\nu^1 := \pi^y_\sharp \boldsymbol{\sigma}^1, \quad \nu^2 := \pi^y_\sharp \boldsymbol{\sigma}^2}$

Fundamental solution and Lipschitz estimates

$\eta_{t,x} = \mathsf{S}_t\delta_x \ll \mathfrak{m}$ if $t > 0$, by the regularization estimates.

$$\boxed{\mathsf{P}_t f(x) = \int f \,\mathrm{d}\eta_{t,x}}$$

If $f \in \mathrm{Lip}(\boldsymbol{X})$ then $\mathsf{P}_t f \in \mathrm{Lip}(\boldsymbol{X})$ and $\mathrm{Lip}(\mathsf{P}_t f) \le L = \mathrm{Lip}(f)$.

$$\begin{aligned}\mathsf{P}_t f(x) - \mathsf{P}_t f(y) &= \int f(z)\,\mathrm{d}\eta_{t,x}(z) - \int f(w)\,\mathrm{d}\eta_{t,y}(w)\\ &= \int \Big(f(z) - f(w)\Big)\,\mathrm{d}\boldsymbol{\mu}_{t,x,y}(v,w) \le L\int \mathsf{d}(z,w)\,\mathrm{d}\boldsymbol{\mu}_{t,x,y}\\ &= LW_2(\eta_{t,x},\eta_{t,y}) \le LW_2(\delta_x,\delta_y) = L\mathsf{d}(x,y).\end{aligned}$$

Bakry-Émery estimate

Take two points x_0, x_1 and a geodesic y connecting them. For every time $t > 0$ the curve $\vartheta \mapsto \eta_{t,\mathsf{y}(\vartheta)} = \mathsf{S}_t \delta_{\mathsf{y}(\vartheta)}$ is Lipschitz in $\mathscr{P}_2(\boldsymbol{X})$ by the contraction property. We lift it to a dynamic plan $\boldsymbol{\pi}$
If f is a Lipschitz function and $r = \mathsf{d}(x_0, x_1)$ we get

$$
\begin{aligned}
\mathsf{P}_t f(x_0) - \mathsf{P}_t f(x_1) &= \int f(z)\,\mathrm{d}\eta_{t,x_0}(z) - \int f(w)\,\mathrm{d}\eta_{t,x_1}(w) \\
&= \int\int_{\partial\mathsf{x}} f\,\mathrm{d}\boldsymbol{\pi} \le \int\int_{\mathsf{x}} |\mathrm{D}f|\,\mathrm{d}\boldsymbol{\pi} = \int_0^1\int |\mathrm{D}f|(\mathsf{x}(\vartheta))|\dot{\mathsf{x}}|(\vartheta)\,\mathrm{d}\boldsymbol{\pi}(\mathsf{x})\,\mathrm{d}\vartheta \le \\
&\le \int_0^1 \Big(\int |\mathrm{D}f|^2\,\mathrm{d}\eta_{t,\mathsf{y}(\vartheta)}\Big)^{1/2}\Big(\int |\dot{\mathsf{x}}|^2\,\mathrm{d}\boldsymbol{\pi}\Big)^{1/2}\mathrm{d}\vartheta \\
&\le \int_0^1 \Big(\mathsf{P}_t\big(|\mathrm{D}f|^2\big)(\mathsf{y}(\vartheta))\Big)^{1/2}|\dot{\mathsf{y}}|(\vartheta)\,\mathrm{d}\vartheta \le \mathsf{d}(x_0,x_1)\sup_{B_r(x_0)}\Big(\mathsf{P}_t\big(|\mathrm{D}f|^2\big)\Big)^{1/2}
\end{aligned}
$$

Dividing by $\mathsf{d}(x_0, x_1)$ and passing to the limit as $x_1 \to x_0$

$$|\mathrm{D}\mathsf{P}_t f|^2(x_0) \le \mathsf{P}_t\big(|\mathrm{D}f|^2\big)(x_0).$$

By approximation, whenever $f \in W^{1,2}(\boldsymbol{X}, \mathsf{d}, \mathfrak{m})$

$$|\mathrm{D}\mathsf{P}_t f|^2(x_0) \le \mathsf{P}_t\big(|\mathrm{D}f|_w^2\big)(x_0).$$

RCD = BE: exhibit a K-flow!

In order to prove the implication $\mathsf{RCD}(\mathsf{K},\infty) \;\Rightarrow\; \mathsf{BE}(K,\infty)$:

1. start from the Heat semigroup P_t as the L^2-gradient flow of the Cheeger energy
2. it coincides with the Wasserstein gradient flow of the entropy in the Entropy-dissipation sense.
3. Assuming moreover that P_t is linear (i.e. the Cheeger energy is quadratic) prove that it induces **a metric K-flow of the Entropy.**

Basic ingredients: Given $\mu_t = f_t\mathfrak{m}$ with $f_t = \mathsf{P}_t f$ starting from f with bounded density, and ν_ϑ a geodesic connecting μ_t to ν calculate

$$\text{the derivative } W' = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(\mu_t,\nu) \quad \text{at } t>0$$

$$\text{the right derivative } E' = \frac{\mathrm{d}}{\mathrm{d}\vartheta+}\mathrm{Ent}_{\mathfrak{m}}(\nu_\vartheta) \quad \text{at } \vartheta=0$$

Prove that $W' \le E'$. By convexity ($K=0$), $E' \le \mathrm{Ent}_{\mathfrak{m}}(\nu) - \mathrm{Ent}_{\mathfrak{m}}(\mu)$.

Dual Kantorovich characterization of the Wasserstein distance

Dual characterization:

$$\frac{1}{2}W_2^2(\mu,\nu) = \boxed{\sup\Big\{\int \mathsf{Q}_1\phi\,\mathrm{d}\mu - \int \phi\,\mathrm{d}\nu : \phi \in \mathrm{Lip}_b(\boldsymbol{X})\Big\}}$$

where

$$\boxed{\mathsf{Q}_t\phi(x) := \inf_y \frac{1}{2t}\mathsf{d}^2(x,y) + \phi(y).}$$

If $\boldsymbol{X}$ is compact, there exists a couple $\phi, \psi = \mathsf{Q}_1\phi$ in $\mathrm{Lip}(\boldsymbol{X})$ of optimal Kantorovich potentials satisfying

$$\psi(x) - \phi(y) \le \frac{1}{2}\mathsf{d}^2(x,y) \quad \forall x, y$$

$$\psi(x) - \phi(y) = \frac{1}{2}\mathsf{d}^2(x,y) \quad \text{if } x, y \in \operatorname{supp}\boldsymbol{\mu}, \quad \boldsymbol{\mu} \in \mathrm{Opt}(\mu,\nu).$$

Derivative of the Wasserstein distance along the heat flow

Let $\mu_t = f_t \mathfrak{m} \in \mathscr{P}_2(\boldsymbol{X})$, $f_t = \mathsf{P}_t f$, $f \in L^\infty(\boldsymbol{X}, \mathfrak{m})$. Let $\nu \in \mathscr{P}_2(\boldsymbol{X})$ and ψ_t a Kantorovich potential for the couple μ_t, ν at the time t.

For a.e. $t > 0$

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} W_2^2(\mu_t, \nu) = \int \psi_t \, \Delta f_t \, \mathrm{d}\mathfrak{m}.$$

Derivative of the entropy along a geodesic

Assume that $(\boldsymbol{X}, \mathsf{d}, \mathfrak{m})$ is a $\mathsf{CD}(K,\infty)$ space.

Let $\mu = f\mathfrak{m}, \nu \in \mathscr{P}_2(\boldsymbol{X})$ with bounded densities and $f \ge c > 0$ $\mathfrak{m}$-a.e., and let $(\nu_\vartheta)_\vartheta$ be a geodesic connecting μ to ν with **uniformly bounded densities [Rajala, Sturm]** along which $\mathrm{Ent}_\mathfrak{m}$ is convex.

Let ψ be the associated Kantorovich potential.

$$\frac{\mathrm{d}}{\mathrm{d}\vartheta+}\mathrm{Ent}_\mathfrak{m}(\mu_\vartheta) \ge \lim_{\varepsilon\downarrow 0} \frac{\mathsf{Ch}(\psi) - \mathsf{Ch}(\psi + \varepsilon f)}{\varepsilon}$$

If Ch is quadratic, $\mathsf{Ch}(f) = \frac{1}{2}\mathcal{E}(f,f)$ for a symmetric bilinear form $\mathcal{E}$,

$$\mathsf{Ch}(\psi) - \mathsf{Ch}(\psi + \varepsilon f) = -\varepsilon\mathcal{E}(\psi, f) - \frac{1}{2}\varepsilon^2\mathcal{E}(f,f)$$

$$\lim_{\varepsilon\downarrow 0} \frac{\mathsf{Ch}(\psi) - \mathsf{Ch}(\psi + \varepsilon f)}{\varepsilon} = -\mathcal{E}(\psi, f) = \int \psi\Delta f \,\mathrm{d}\mathfrak{m}$$

$$\mathrm{Ent}_\mathfrak{m}(\nu) - \mathrm{Ent}_\mathfrak{m}(\mu_t) \overset{K=0}{\ge} \frac{\mathrm{d}}{\mathrm{d}\vartheta+}\mathrm{Ent}_\mathfrak{m}(\mu_\vartheta) \ge \lim_{\varepsilon\downarrow 0} \frac{\mathsf{Ch}(\psi_t) - \mathsf{Ch}(\psi_t + \varepsilon f)}{\varepsilon}$$

$$\overset{\mathsf{Ch}\text{ quadratic}}{=} \int \psi_t\Delta f \,\mathrm{d}\mathfrak{m} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(\mu_t, \nu)$$

Characterization of metric-measure spaces

What is sufficient to characterize a metric measure space?

$(\boldsymbol{X}_1, \mathsf{d}_1, \mathfrak{m}_1) \sim (\boldsymbol{X}_2, \mathsf{d}_2, \mathfrak{m}_2)$ if there exists a **measure preserving isometry** $\mathsf{i} : \mathrm{supp}(\mathfrak{m}_1) \subset \boldsymbol{X}_1 \to \boldsymbol{X}_2$, i.e.

$$\mathsf{d}_2(\mathsf{i}(x), \mathsf{i}(y)) = \mathsf{d}_1(x, y), \quad \mathsf{i}_\sharp(\mathfrak{m}_1) = \mathfrak{m}_2 \quad \text{for every } x, y \in \boldsymbol{X}_1,\ A \subset \boldsymbol{X}_1$$

Consider independent and identically distributed $\boldsymbol{X}$-random variables $X_1, X_2, \cdots, X_N$ with law $\mathfrak{m}$ and consider metric-measure functionals

$$\Phi[\boldsymbol{X}, \mathsf{d}, \mathfrak{m}] = \mathbb{E}\Big[\Phi\big(\mathsf{d}(X_i, X_j)\big)_{i,j=1}^N\Big] = \int \Phi\big(\mathsf{d}(x_i, x_j)_{i,j=1}^N\big)\, \mathrm{d}\mathfrak{m}^{\otimes N}(x_1, x_2, \cdots, x_N)$$

where $\Phi : \mathbb{R}^{N\times N} \to \mathbb{R}$ continuous and bounded.

Theorem (Gromov reconstruction)

$(\boldsymbol{X}_1, \mathsf{d}_1, \mathfrak{m}_1) \sim (\boldsymbol{X}_2, \mathsf{d}_2, \mathfrak{m}_2)$ *if and only if* $\Phi[\boldsymbol{X}_1, \mathsf{d}_1, \mathfrak{m}_1] = \Phi[\boldsymbol{X}_2, \mathsf{d}_2, \mathfrak{m}_2]$ *for every metric-measure functional.*

Sturm-Gromov-Hausdorff convergence of metric-measure spaces

We say that $(\boldsymbol{X}_n, \mathsf{d}_n, \mathfrak{m}_n)$ converge to $(\boldsymbol{X}_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty)$ if

$$\lim_{n\to\infty} \Phi[\boldsymbol{X}_n, \mathsf{d}_n, \mathfrak{m}_n] = \Phi[\boldsymbol{X}_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty]$$

for every metric-measure functional Φ.

Equivalently [Sturm]: there exists a complete and separable metric space $(\boldsymbol{Y}, \mathsf{d})$ and isometries $\mathsf{i}_n : (\boldsymbol{X}_n, \mathsf{d}_n) \to (\boldsymbol{Y}, \mathsf{d})$, $n \in \mathbb{N} \cup \{\infty\}$, such that

$$(\mathsf{i}_n)_\sharp \mathfrak{m}_n \longrightarrow (\mathsf{i}_\infty)_\sharp \mathfrak{m}_\infty \quad \text{weakly in } \mathscr{P}(\boldsymbol{Y}).$$

Gromov's compactness theorem:

The class of Riemannian manifolds $(M, \boldsymbol{g})$ with

$$\mathsf{dim}(M) \le N, \quad \mathsf{diam}(M) \le D, \quad \mathsf{Ric}(M) \ge K$$

is pre-compact in the *SGH* topology.

The $\mathsf{CD}(K, \infty)$ condition is stable under Gromov-weak convergence. In particular, Gromov-weak limits of Riemannian manifolds with Ricci curvature (uniformly) bounded from below is a $\mathsf{CD}(K, \infty)$ space.

Stability of RCD, convergence of the metric flow and of the spectrum

Let $(\boldsymbol{X}^n, \mathsf{d}^n, \mathfrak{m}^n)$ be $\mathsf{RCD}(K,\infty)$ spaces SGH-converging to $(\boldsymbol{X}^\infty, \mathsf{d}^\infty, \mathfrak{m}^\infty)$.

Theorem (Stability of the RCD condition)

$(\boldsymbol{X}^\infty, \mathsf{d}^\infty, \mathfrak{m}^\infty)$ *is* $\mathsf{RCD}(K,\infty)$

Theorem (Convergence of the metric flow)

If S^n_t *be the* ***metric flow in*** $(\boldsymbol{X}^n, \mathsf{d}^n, \mathfrak{m}^n)$.
If μ^n ***"converges" to*** μ^∞***, then*** $\mathsf{S}^n_t\mu^n$ ***converges to*** $\mathsf{S}^\infty_t\mu^\infty$ ***for every*** $t > 0$.

Let us assume $K > 0$ and let $\lambda_1(\Delta_n) \le \lambda_2(\Delta_n) \le \cdots \le \lambda_k(\Delta_n) < \cdots$ be the (ordered) eigenvalues of the Laplace operator $-\Delta_n$ on $(\boldsymbol{X}_n, \mathsf{d}_n, \mathfrak{m}_n)$.

Theorem (Convergence of the spectrum)

$$\lim_{n\to\infty} \lambda_k(\Delta_n) = \lambda_k(\Delta_\infty).$$