Metric measure spaces with Riemannian Ricci curvature bounded from below
Lecture I

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Outline

1. **Smooth setting: energy forms, diffusion semgroups**
2. **Bochner identity and the Bakry-Émery approach to lower curvature bounds**
3. **Ricci curvature and optimal transport**

Smooth setting

$(\mathbb{M}, \mathsf{g})$ smooth complete Riemannian manifold of dimension n.

In a **local chart** $U \subset \mathbb{M}$, $x : U \to \Omega \subset \mathbb{R}^n$ is a system of local coordinates: $x = (x^i)_{i=1,\ldots,n}$. $\partial_i = \frac{\partial}{\partial x_i}$.

Tangent vector: $V = \sum V^i \partial_i$.

$$|V|^2_{\mathsf{g}} = \sum \mathrm{g}_{ij} V^i V^j, \quad \mathsf{g} = \sum \mathrm{g}_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j; \quad \langle V, W \rangle_{\mathsf{g}} = \mathsf{g}(V, W) = \sum \mathrm{g}_{ij} V^i W^j.$$

Smooth curve: $\mathsf{x} : [a, b] \to \mathbb{M}$, $\mathsf{x}(t) = (\mathsf{x}^i(t))$,

$$V^i := \dot{\mathsf{x}}^i, \qquad \boxed{|\dot{\mathsf{x}}|_{\mathsf{g}} = \sqrt{\sum g_{ij} \dot{\mathsf{x}}^i \dot{\mathsf{x}}^j}} \qquad \mathrm{Length}[\mathsf{x}] = \int_a^b |\dot{\mathsf{x}}|_{\mathsf{g}} \, \mathrm{d}t$$

Cotangent vector - differential form: $\omega = \sum \omega_i \mathrm{d}x^i$, with dual norm g^*

$$|\omega|^2_{\mathsf{g}} = \sum \mathrm{g}^{ij} \omega_i \omega_j, \quad \sum \mathrm{g}^{ij} \mathrm{g}_{jk} = \delta^i_k; \quad \langle \omega, \eta \rangle_{\mathsf{g}} = \mathsf{g}^*(\omega, \eta) = \sum \mathrm{g}^{ij} \omega_i \eta_j.$$

Differential of a function $f : \mathbb{M} \to \mathbb{R}$: $\mathrm{D}f = \mathrm{d}f = \sum \partial_i f \, \mathrm{d}x^i$

$$\boxed{|\mathrm{D}f|^2_{\mathsf{g}} = \sum \mathrm{g}^{ij} \partial_i f \partial_j f}$$

Volume measure:

$$\mathrm{Vol}_{\mathsf{g}} = \mathrm{e}^{-G} \mathscr{L}^n, \quad G := -\frac{1}{2} \log(\det \mathsf{g}) = \frac{1}{2} \log(\det \mathsf{g}^*).$$

Energy forms and differential operators

$\mathfrak{m} = \mathrm{e}^{-V}\mathrm{Vol}_{\mathrm{g}} = \mathrm{e}^{-(V+G)}\mathscr{L}^n$ is a **reference Borel measure,** $V : \mathbb{M} \to \mathbb{R}$ smooth.

Energy form:

$$\mathcal{E}(f,h) := \int_{\mathbb{M}} \langle \mathrm{D}f, \mathrm{D}h\rangle_{\mathrm{g}}\,\mathrm{d}\mathfrak{m},$$

$$\mathcal{E}(f) = \mathcal{E}(f,f) = \int_{\mathbb{M}} |\mathrm{D}f|^2_{\mathrm{g}}\,\mathrm{d}\mathfrak{m} = \int_{\mathbb{M}} \sum \mathrm{g}^{ij}\partial_i f\,\partial_j f\ \mathrm{e}^{-V}\,\mathrm{dVol}_{\mathrm{g}}$$

Sobolev space $D(\mathcal{E}) = W^{1,2}(\mathbb{M},\mathbf{g},\mathfrak{m})$: the completion of the space of smooth functions in $L^2(\mathbb{M},\mathfrak{m})$ with $\mathcal{E} < \infty$ endowed with the scalar product

$$\mathcal{E}_1(f,h) := \int f\,h\,\mathrm{d}\mathfrak{m} + \mathcal{E}(f,h).$$

L is the associated second order **drift-diffusion differential operator:**

$$f \in D(\mathsf{L}) \quad \Leftrightarrow \quad \mathsf{L}f \in L^2(\mathbb{M},\mathfrak{m}) : \quad \boxed{\mathcal{E}(f,h) = -\int \mathsf{L}f\,h\,\mathrm{d}\mathfrak{m} \quad \forall h \in D(\mathcal{E})}$$

$$\begin{aligned}\mathsf{L}f &= \mathrm{e}^{V+G}\sum \partial_i\big(\mathrm{e}^{-(V+G)}\mathrm{g}^{ij}\partial_j f\big)\\ &= \sum \partial_i\big(\mathrm{g}^{ij}\partial_j f\big) - \sum \mathrm{g}^{ij}\partial_i\big(V+G\big)\,\partial_j f\end{aligned}$$

Examples in $\mathbb{R}^n$

Euclidean Dirichlet energy:

$\mathbb{M} = \mathbb{R}^n$, $|V|_{\mathrm{g}} = |V|$, $\mathfrak{m} = \mathrm{Vol}_{\mathrm{g}} = \mathscr{L}^n$.

$$\mathcal{E}(f) = \int |\mathrm{D}f|^2 \,\mathrm{d}x, \qquad \mathsf{L}f = \Delta f = \sum \partial_i^2 f.$$

Weighted energy and drift-diffusion:

$\mathbb{M} = \mathbb{R}^n$, $|V|_{\mathrm{g}} = |V|$, $\mathfrak{m} = \mathrm{e}^{-V}\mathscr{L}^n$.

$$\mathcal{E}(f) = \int |\mathrm{D}f|^2 \,\mathrm{e}^{-V}\,\mathrm{d}x, \qquad \mathsf{L}f = \Delta f - \langle \mathrm{D}V, \mathrm{D}f\rangle$$

Gaussian and Ornstein-Uhelenbeck operator:

$V(x) := \frac{1}{2}|x|^2 - \frac{n}{2}\log(2\pi)$

$$\mathcal{E}(f) = \frac{1}{(2\pi)^{n/2}} \int |\mathrm{D}f|^2 \,\mathrm{e}^{-\frac{1}{2}|x|^2}\,\mathrm{d}x, \qquad \mathsf{L}f = \Delta f - \langle x, \mathrm{D}f\rangle$$

Elliptic operator in divergence form:

$\mathbb{M} = \mathbb{R}^n$, $|V|_{\mathrm{g}} = \mathrm{g}_{ij}Z^iZ^j$, $\mathfrak{m} = \mathscr{L}^n$.

$$\mathcal{E}(f) = \int \mathrm{g}^{ij}\partial_i f\,\partial_j f\,\mathrm{d}x, \qquad \mathsf{L}f = \sum \partial_i\big(\mathrm{g}^{ij}\partial_j f\big).$$

Examples

Laplace-Beltrami: $\mathfrak{m} = \mathrm{Vol}_{\mathrm{g}}$

$$\mathcal{E}(f) = \int \mathrm{g}^{ij}\partial_i f\,\partial_j f\,\mathrm{e}^{-G}\,\mathrm{d}x,$$

$$\mathsf{L}f = \Delta_{\mathrm{g}} f = \mathrm{e}^{G}\sum \partial_i\big(\mathrm{e}^{-G}\mathrm{g}^{ij}\partial_j f\big) = \sum \partial_i\big(\mathrm{g}^{ij}\partial_j f\big) - \langle \mathrm{D}G, \mathrm{D}f\rangle_{\mathrm{g}}$$

Conformal geometry: $\mathsf{g} = \mathrm{g}\,\mathsf{Id}$, $|Z|^2_{\mathrm{g}} = \mathrm{g}|Z|^2$, $\quad \mathfrak{m} = \mathrm{Vol}_{\mathrm{g}} = \mathrm{g}^{n/2}\mathscr{L}^n$.

$$\mathcal{E}(f) = \int \mathrm{g}^{n/2-1}|\mathrm{D}f|^2\,\mathrm{d}x,$$

$$\mathsf{L}f = \frac{1}{\mathrm{g}^{n/2}}\sum \partial_i\big(\mathrm{g}^{n/2-1}\partial_i f\big) = \frac{1}{\mathrm{g}}\Big(\Delta f + (n/2-1)\langle \mathrm{D}\log \mathrm{g}, \mathrm{D}f\rangle\Big)$$

In particular, when $n = 1$ $Lf = \frac{1}{\mathrm{g}^2}\Big(g\,f'' - \frac{1}{2}g' f'\Big)$.

When $n = 2$ $\boxed{Lf = \frac{1}{\mathrm{g}}\Delta f}$.

Weighted geometry: $\mathfrak{m} = \mathrm{e}^{-V}\,\mathrm{Vol}_{\mathrm{g}}$

$$\mathcal{E}(f) = \int |\mathrm{D}f|^2_{\mathrm{g}}\,\mathrm{e}^{-V}\,\mathrm{dVol}_{\mathrm{g}}, \qquad \mathsf{L}f = \Delta_{\mathrm{g}} f - \langle \mathrm{D}V, \mathrm{D}f\rangle_{\mathrm{g}}$$

Diffusion semigroup

Diffusion semigroup in $L^2(\mathbb{M}, \mathfrak{m})$ generated by $\mathcal{E}$: $(\mathsf{P}_t)_{t\geq 0}$.
For every $f \in L^2(\mathbb{M}, \mathfrak{m})$ $f_t = \mathsf{P}_t f \in D(\mathsf{L})$, $t > 0$, is the unique solution of

$$\frac{\partial}{\partial t} f_t = \mathsf{L} f_t, \quad \lim_{t\downarrow 0} f_t = f \quad \text{in } L^2(\mathbb{M}, \mathfrak{m}).$$

Variational formulation:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int f_t h \,\mathrm{d}\mathfrak{m} + \mathcal{E}(f_t, h) = 0 \quad \forall h \in D(\mathcal{E}).$$

$(\mathsf{P}_t)_{t\geq 0}$ is

- **symmetric:** $\int \mathsf{P}_t f \, h \,\mathrm{d}\mathfrak{m} = \int f \, \mathsf{P}_t h \,\mathrm{d}\mathfrak{m}.$
- **contractive** in every L^p, $1 \leq p \leq \infty$: $\|\mathsf{P}_t f\|_{L^p} \leq \|f\|_{L^p}$
- **analytic** in L^p, $1 < p < \infty$: $\|\mathsf{L}\mathsf{P}_t f\|_{L^p} \leq C t^{-1} \|f\|_{L^p}$
- **order preserving:** $f \leq h \ \Rightarrow\ \mathsf{P}_t f \leq \mathsf{P}_t h$. In particular $f \geq 0 \ \Rightarrow\ \mathsf{P}_t f \geq 0$
- **mass preserving,** if $\mathfrak{m}(B_r(\bar{x})) \leq A \mathrm{e}^{B r^2}$: $\mathsf{P}_t c \equiv c$, $\int \mathsf{P}_t f \,\mathrm{d}\mathfrak{m} = \int f \,\mathrm{d}\mathfrak{m}$

Γ tensor, $\mathsf{L}f^2$, commutation and Lebnitz rule

Γ-tensor $$\boxed{\Gamma(f,h) := \frac{1}{2}\Big(\mathsf{L}(fh) - f\,\mathsf{L}h - h\,\mathsf{L}f\Big)}$$

Leibnitz rule yields $\Gamma(f,g) = \langle \mathrm{D}f, \mathrm{D}h\rangle_{\mathsf{g}}$.

Choosing $f = h$

$$\Gamma(f) = \boxed{\Gamma(f,f) = \frac{1}{2}\mathsf{L}f^2 - f\,\mathsf{L}f}, \qquad \boldsymbol{\Gamma(f) = |\mathrm{D}f|^2_{\mathsf{g}}}.$$

From the Energy form $\mathcal{E}$ it is possibile to recover the energy density:

$$\int |\mathrm{D}f|^2_{\mathsf{g}}\,h\,\mathrm{d}\mathbf{m} = \int \Gamma(f)\,h\,\mathrm{d}\mathbf{m} = \int \Big(\frac{1}{2}\mathsf{L}f^2 - f\,\mathsf{L}f\Big)h\,\mathrm{d}\mathbf{m}$$
$$= -\frac{1}{2}\mathcal{E}(f^2,h) + \mathcal{E}(f,fh)$$

Computation of $\mathsf{L}|\mathrm{D}f|^2_{\mathsf{g}}$

In $\mathbb{R}^n$

$$\boxed{\frac{1}{2}\Delta|\mathrm{D}f|^2 = \langle \mathrm{D}f, \mathrm{D}\Delta f\rangle + |\mathrm{D}^2 f|^2} \qquad |\mathrm{D}^2 f|^2 = \sum \big(\partial^2_{ij} f\big)^2.$$

Drift part $Z(f) = \langle \mathrm{D}V, \mathrm{D}f\rangle$:

$$\frac{1}{2}Z\big(|\mathrm{D}f|^2\big) = \langle \mathrm{D}f, \mathrm{D}Z(f)\rangle - \boldsymbol{\mathrm{D}^2 V(\mathrm{D}f, \mathrm{D}f)}$$

$$\mathsf{L}f = \Delta f - Z(f), \qquad \frac{1}{2}\mathsf{L}|\mathrm{D}f|^2 = \langle \mathrm{D}f, \mathrm{D}\mathsf{L}f\rangle + |\mathrm{D}^2 f|^2 + \boldsymbol{\mathrm{D}^2 V(\mathrm{D}f, \mathrm{D}f)}$$

$$\boldsymbol{\Gamma_2(f)} := \boxed{\frac{1}{2}\mathsf{L}\Gamma(f) - \Gamma(f, \mathsf{L}f)} = \frac{1}{2}\mathsf{L}|\mathrm{D}f|^2 - \langle \mathrm{D}f, \mathrm{D}\mathsf{L}f\rangle$$

$$\Gamma_2(f) = |\mathrm{D}^2 f|^2 + \boldsymbol{\mathrm{D}^2 V(\mathrm{D}f, \mathrm{D}f)}$$

Gaussian: $\mathfrak{m} = \frac{1}{(2\pi\lambda)^{n/2}} \mathrm{e}^{\frac{1}{2\lambda}|x|^2} \mathscr{L}^n.$ $\qquad \Gamma_2(f) = |\mathrm{D}^2 f|^2 + \boldsymbol{\frac{1}{\lambda}|\mathrm{D}f|^2}$

Computation of $\mathsf{L}|\mathrm{D}f|_{\mathrm{g}}^2$: Laplacian and covariant derivative

Riemannian connection ∇; $\nabla_i = \nabla_{\partial_i}$.

$$\nabla_Z \langle X, Y\rangle_{\mathrm{g}} = \langle \nabla_Z X, Y\rangle_{\mathrm{g}} + \langle X, \nabla_Z Y\rangle_{\mathrm{g}}, \quad \nabla_X Y - \nabla_Y X = [X, Y]$$

$$\nabla_i \partial_j = \gamma_{ij}^k \partial_k, \quad (\nabla_i Z)^k = \partial_i Z^k + \sum_j \boldsymbol{\gamma_{ij}^k Z^j}, \quad (\nabla_i \omega)_j = \partial_i \omega_j - \sum_k \boldsymbol{\gamma_{ij}^k \omega_k}$$

Hessian:

$$\mathrm{D}^2 f = \nabla \mathrm{d} f = \sum \mathrm{H}_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j, \quad \mathrm{H}_{ij} = \partial_{ij}^2 f - \sum_k \boldsymbol{\gamma_{ij}^k\, \partial_k f}$$

Laplacian:

$$\Delta_{\mathrm{g}} f = \mathrm{trace}(\mathrm{D}^2 f) = \sum_{ij} \mathrm{g}^{ij} \mathrm{H}_{ij} = \sum_{ij} \mathrm{g}^{ij} \Big(\partial_{ij}^2 f - \sum_k \boldsymbol{\gamma_{ij}^k \partial_k f} \Big)$$

The "variational" and the "covariant" representation of Δ_{g} coincide!

Ricci curvature and the Bochner's formula

Second order derivative: $\nabla^2_{X,Y} := \nabla_X \nabla_Y - \nabla_{\nabla_X Y};$

$$\nabla^2_{ij} = \nabla_i \nabla_j - \gamma^k_{ij} \nabla_k$$

Riemann curvature tensor:

$$\mathrm{Rm}(X,Y) = \nabla^2_{X,Y} - \nabla^2_{Y,X}, \qquad \mathrm{Rm}(X,Y;Z,W) := \langle \nabla^2_{X,Y} Z - \nabla^2_{Y,X} Z, W \rangle_{\mathrm{g}}$$

Ricci curvature tensor:

$$\mathbf{Ric}(X,Y) := \sum_i \mathrm{Rm}(X, E_i; Y, E_i), \quad E_i \quad \text{orthonormal frame}$$

Bochner identity

$$\boxed{\frac{1}{2}\Delta_{\mathrm{g}} |\mathrm{D} f|^2_{\mathrm{g}} = \langle \mathrm{D} f, \mathrm{D}\Delta_{\mathrm{g}} f \rangle_{\mathrm{g}} + |\mathrm{D}^2 f|^2_{\mathrm{g}} + \mathbf{Ric}(\mathrm{D} f, \mathrm{D} f)}$$

$$Z = \nabla V, \qquad \frac{1}{2} Z\big(|\mathrm{D} f|^2_{\mathrm{g}}\big) = \langle \mathrm{D} f, \mathrm{D} Z(f) \rangle_{\mathrm{g}} - \mathbf{D}^2 V(\mathrm{D} f, \mathrm{D} f).$$

$$\mathsf{L} f = \Delta_{\mathrm{g}} f - Z(f), \qquad \boxed{\frac{1}{2}\mathsf{L} |\mathrm{D} f|^2_{\mathrm{g}} = \langle \mathrm{D} f, \mathrm{D}\mathsf{L} f \rangle_{\mathrm{g}} + |\mathrm{D}^2 f|^2_{\mathrm{g}} + \mathbf{Ric}_{\mathsf{L}}(\mathrm{D} f, \mathrm{D} f)}$$

$$\mathbf{Ric}_{\mathsf{L}} = \mathbf{Ric} + \mathbf{D}^2 V$$

Examples

Sphere: $\mathbb{S}^n \subset \mathbb{R}^{n+1}$; local chart $\boldsymbol{x} = (x^1, \cdots, x^n, y)$, $|x| < 1$; $y := \sqrt{1-|x|^2}$

$$\mathrm{g}_{ij}(x) = \delta_{ij} + \frac{x^i x^j}{1-|x|^2}, \quad \mathrm{g}^{ij}(x) = \delta^{ij} - x^i x^j, \quad |\mathrm{D}f|_{\mathrm{g}}^2 = \sum (\delta^{ij} - x^i x^j)\partial_i f \, \partial_j f$$

$$G = \frac{1}{2}\log\Big(1-|x|^2\Big), \quad \mathrm{Vol}_{\mathrm{g}} = \frac{1}{\sqrt{1-|x|^2}}\mathscr{L}^n.$$

$$\boxed{\Delta_{\mathbb{S}^n} f = \sum (\delta^{ij} - x^i x^j)\partial^2_{ij} f - n \sum x^i \partial_i f}$$

$$\mathbf{Ric(D}f, \mathbf{D}f) = \mathbf{Ric_L(D}f, \mathbf{D}f) = (n-1)|\mathbf{D}f|^2_{\mathrm{g}}$$

Hyperbolic space: $\mathbb{H}^n = \{(x^1, x^2, \cdots, x^{n-1}, x^n) \in \mathbb{R}^n : x^n > 0\}$, $z = x^n$;

$$\mathrm{g}_{ij}(x) = \frac{1}{z^2}\delta_{ij}, \quad \mathrm{g}^{ij}(x) = z^2 \delta^{ij}, \quad |\mathrm{D}f|^2_{\mathrm{g}} = z^2 |\mathrm{D}f|^2$$

$$G = n \log z, \quad \mathrm{Vol}_{\mathrm{g}} = z^{-n}\mathscr{L}^n$$

$$\boxed{\Delta_{\mathbb{H}^n} f = z^2 \Delta f - (n-2)\partial_z f}$$

$$\mathbf{Ric(D}f, \mathbf{D}f) = \mathbf{Ric_L(D}f, \mathbf{D}f) = -(n-1)|\mathbf{D}f|^2_{\mathrm{g}}$$

Bakry-Émery Γ_2 conditions

$$\boldsymbol{\Gamma_2(f)} := \frac{1}{2}\mathsf{L}\Gamma(f) - \Gamma(f, \mathsf{L}f)$$
$$= \frac{1}{2}\mathsf{L}|\mathrm{D}f|_{\mathsf{g}}^2 - \langle \mathrm{D}f, \mathrm{D}\mathsf{L}f\rangle_{\mathsf{g}}$$

$$\boldsymbol{\Gamma_2(f)} = |\mathrm{D}^2 f|_{\mathsf{g}}^2 + \mathbf{Ric}_{\mathsf{L}}(\mathrm{D}f, \mathrm{D}f)$$

Bakry-Émery condition $\mathsf{BE}(K,N)$**,** $K \in \mathbb{R}$**,** $N \geq n$**:**

- $\Gamma_2(f) \geq K\,\Gamma(f) + \frac{1}{N}\big(\mathsf{L}f\big)^2$
- $\mathbf{Ric}_{\mathsf{L}}(\mathrm{D}f, \mathrm{D}f) \geq K|\mathrm{D}f|_{\mathsf{g}}^2 + \frac{1}{N-n}\langle V, \mathrm{D}f\rangle_{\mathsf{g}}^2$

When $N = \infty$

$$\boxed{\Gamma_2(f) \geq K\,\Gamma(f)} \qquad \Longleftrightarrow \qquad \boxed{\mathbf{Ric}_{\mathsf{L}}(\mathrm{D}f, \mathrm{D}f) \geq K|\mathrm{D}f|_{\mathsf{g}}^2}$$

Pointwise gradient bounds for the diffusion semigroup

Fix $t > 0$, f smooth and define for $0 < s < t$

$$A_s(f) := \frac{1}{2}\mathsf{P}_{t-s}\big(\mathsf{P}_s f\big)^2, \quad B_s(f) := \frac{1}{2}\mathsf{P}_{t-s}|\mathrm{D}\mathsf{P}_s f|^2_{\mathsf{g}} = \frac{1}{2}\mathsf{P}_{t-s}\Gamma(\mathsf{P}_s f).$$

Setting $f_s := \mathsf{P}_s f$

$$\frac{\mathrm{d}}{\mathrm{d}s}A_s(f) = -\mathsf{P}_{t-s}\Big(\frac{1}{2}\mathbf{L}f_s^2 - f_s\mathbf{L}f_s\Big) = -\mathsf{P}_{t-s}\Gamma(f_s) = -2B_s(f)$$

$$\frac{\mathrm{d}}{\mathrm{d}s}B_s(f) = -\mathsf{P}_{t-s}\Big(\frac{1}{2}\mathbf{L}\Gamma(f_s) - \Gamma(f_s, \mathbf{L}f_s)\Big) = -\mathsf{P}_{t-s}\Gamma_2(f_s)$$

If $\mathsf{BE}(K,\infty)$ holds, i.e. $\boxed{\Gamma_2 \geq K\Gamma}$

$$\frac{\mathrm{d}}{\mathrm{d}s}B_s(f) \leq -K\mathsf{P}_{t-s}\Gamma(f_s) = -2KB_s(f)$$

$$\boldsymbol{B' + 2KB \leq 0, \quad A'' - 2KA' \geq 0}$$

$$2B_t(f) = |\mathrm{D}\mathsf{P}_t f|^2_{\mathsf{g}} \leq 2\mathrm{e}^{-2Kt}B_0(f) = \mathrm{e}^{-2Kt}\mathsf{P}_t|\mathrm{D}f|^2_{\mathsf{g}}$$

$$\boxed{\Gamma(\mathsf{P}_t f) \leq \mathrm{e}^{-2Kt}\mathsf{P}_t\Gamma(f)}$$

Lipschitz regularization

$$B_t \le \mathrm{e}^{2K(t-s)} B_s, \quad A'_s = -2B_s \le -2\mathrm{e}^{2K(t-s)} B_t$$

Integrating w.r.t. s

$$A_t - A_0 \le -2B_t \mathrm{I}_{2K}(t), \quad \mathrm{I}_{2K}(t) := \int_0^t \mathrm{e}^{2Kr}\,\mathrm{d}r = \begin{cases} t & \text{if } K = 0 \\ (2K)^{-1}\big(\mathrm{e}^{2Kt} - 1\big) & \text{if } K \ne 0 \end{cases}$$

$$f_t^2 + 2\mathrm{I}_{2K}(t)\Gamma(f_t) \le f^2$$

$$\boxed{\sqrt{2\mathrm{I}_{2K}(t)}\,\mathrm{Lip}(f) \le \|f\|_\infty}$$

The abtract framework for Γ-calculus

- A (Polish) topological space $(\boldsymbol{X}, \tau)$
- A probability Borel measure $\mathfrak{m}$
- a **strongly local Dirichlet form** $\mathcal{E}$ in $L^2(\boldsymbol{X}, \mathfrak{m})$, i.e. a closed, symmetric, nonnegative bilinear form on $D(\mathcal{E}) \subset L^2(\boldsymbol{X}, \mathfrak{m})$ satisfying

$$\mathcal{E}(f_+, f_+) \le \mathcal{E}(f, f), \quad \mathcal{E}(f, h) = 0 \quad \text{if } f, h \in D(\mathcal{E}),\ fh = 0.$$

- $(\mathsf{P}_t)_{t\ge 0}$ is the **positivity and mass preserving Markov semigroup** in $L^2(\boldsymbol{X}, \mathfrak{m})$ (in fact in any $L^p(\boldsymbol{X}, \mathfrak{m})$) generated by $\mathcal{E}$
- $-\mathsf{L} : D(\mathsf{L}) \subset L^2(\boldsymbol{X}, \mathfrak{m})$ is the **selfadjoint accretive operator**

$$-\int \mathsf{L}u\, \varphi \,\mathrm{d}\mathfrak{m} = \mathcal{E}(u, \varphi), \quad -\int \mathsf{L}u\, u \,\mathrm{d}\mathfrak{m} = \mathcal{E}(u, u) \ge 0.$$

- **Energy density**: there exists a bilinear map $\Gamma : D(\mathcal{E}) \to L^1(\boldsymbol{X}, \mathfrak{m})$:

$$-\frac{1}{2}\mathcal{E}(f^2, h) + \mathcal{E}(f, fh) = \int \Gamma(f)\, h \,\mathrm{d}\mathfrak{m} \qquad \text{for every } f, h \in D(\mathcal{E}) \cap L^\infty$$

$$\mathcal{E}(f, h) = \int \Gamma(f, h) \,\mathrm{d}\mathfrak{m}.$$

$\Gamma(f)$ plays the role of $|\mathrm{D}f|^2_{\mathsf{g}}$, $\Gamma(f, h)$ corresponds to $\langle \mathrm{D}f, \mathrm{D}h\rangle_{\mathsf{g}}$.

Bakry-Émery condition $\mathsf{BE}(K,\infty)$ in energy-measure spaces

- **Strong form:** Γ_2 tensor

$$\Gamma_2(f) = \frac{1}{2}\mathsf{L}\Gamma(f) - \Gamma(f,\mathsf{L}f) \geq K\Gamma(f)$$

- **Pointwise gradient commutation estimate:**

$$\Gamma\big(\mathsf{P}_t f\big) \leq \mathrm{e}^{-2Kt}\mathsf{P}_t\big(\Gamma(f)\big)$$

- **Weak form:** the quantity $A_s[f,h] := \frac{1}{2}\int \big|\mathsf{P}_s f\big|^2 \mathsf{P}_{t-s}h\,\mathrm{d}\mathfrak{m}$ satisfies

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}A_s[f,h] + 2K\frac{\mathrm{d}}{\mathrm{d}s}A_s[f,h] \geq 0 \quad \text{in } \mathscr{D}'(0,t)$$

for every $f \in L^2(\boldsymbol{X},\mathfrak{m}),\ h \in L^\infty(\boldsymbol{X},\mathfrak{m}),\ \varphi \geq 0$

Applications (Bakry, Ledoux, Lott, Gentil, Qian, Wang, Wei, ...): volume and geometric comparison in weighted Riemannian manifold, **Log-Sobolev** and **spectral-gap** inequalities, **hypercontractivity** of the Markov semigroup, **Levy-Gromov** isoperimetric inequality in infinite dimension, **Li-Yau and Harnack** inequalities (for the finite dimensional version $\mathsf{BE}(K,N)$), ...

Riemannian distance, minimal geodesics and exponential map

$$\mathsf{d}_{\mathrm{g}}(x_0, x_1) = \min\Big\{\mathrm{Length}[\mathsf{x}] : \mathsf{x} \text{ smooth curve joining } x_0 \text{ to } x_1\Big\}$$

$(\mathbb{M}, \mathsf{d}_{\mathrm{g}})$ is a **complete metric space.**

$$\mathsf{d}^2_{\mathrm{g}}(x_0, x_1) = \min\Big\{\int_0^1 |\dot{\mathsf{x}}|^2\,\mathrm{d}r : \mathsf{x} : [0,1] \to \mathbb{M},\ \mathsf{x}(i) = x_i,\ i = 0,1\Big\}$$

$\mathsf{x} : [0,1] \to \mathbb{M}$ is a **minimal, constant speed geodesic** if

$$\mathsf{d}_{\mathrm{g}}(\mathsf{x}(s), \mathsf{x}(t)) = |t - s|\,\mathsf{d}_{\mathrm{g}}(\mathsf{x}(0), \mathsf{x}(1)).$$

In local coordinates

$$\ddot{\mathsf{x}}^k(t) = -\gamma^k_{ij}(\mathsf{x}(t))\dot{\mathsf{x}}^i(t)\mathsf{x}^j(t). \qquad (\star)$$

Exponential map: if Z is a vector field and $x \in \mathbb{M}$, $\exp_x(tZ)$ is the value $\mathsf{x}(t)$ of the solution of $(\star)$ with initial conditions

$$\mathsf{x}^k(0) = x^k_0, \quad \dot{\mathsf{x}}^k(0) = Z^k(x).$$

If Z is smooth, $\mathsf{T}_t(x) := \exp_x(tZ(x))$ is smooth flow in $\mathbb{M}$, with $\mathsf{T}_0(x) = x$.

Push forward of measures and Ricci curvature

X, Y are separable metric space, $\mathsf{T} : X \to Y$ is a Borel map, $\mu \in \mathscr{P}(X)$ is a Borel probability measure.

$$\boxed{\nu = \mathsf{T}_\sharp \mu \in \mathscr{P}(Y), \quad \nu(B) = \mu(\mathsf{T}^{-1}(B))} \quad \forall B \in \mathscr{B}(Y),\ \text{Borel.}$$

$$\int_Y f \,\mathrm{d}\nu(y) = \int_X f(\mathsf{T}(x)) \,\mathrm{d}\mu(x) \quad f \text{ bounded or nonnegative, Borel.}$$

$X = Y = \mathbb{M}$, T_t smooth with invertible differential $\mathrm{d}\mathsf{T}_t$, $\mu = \varrho \mathfrak{m}$, $\mu_t = \varrho_t \mathfrak{m} = (\mathsf{T}_t)_\sharp \mu$.

$$\varrho_t(\mathsf{T}_t(x))\mathrm{e}^{-V(\mathsf{T}_t(x))} \det(\mathrm{d}\mathsf{T}_t(x)) = \varrho(x)\mathrm{e}^{-V(x)}$$

$$J_t(x) := \log\Big(\varrho_t(\mathsf{T}_t x)/\varrho(x)\Big) = V(\mathsf{T}_t(x)) - V(x) - \log\big(\det \mathrm{d}\mathsf{T}_t(x)\big).$$

When $\mathsf{T}_t(x) := \exp_x(t\nabla\Psi(x))$ then

$$\boxed{\ddot{J}_t(x) \geq \frac{1}{N}(\dot{J}_t(x))^2 + \mathbf{Ric}_{\mathsf{L}}(\dot{\mathsf{T}}_t(x), \dot{\mathsf{T}}_t(x))}$$

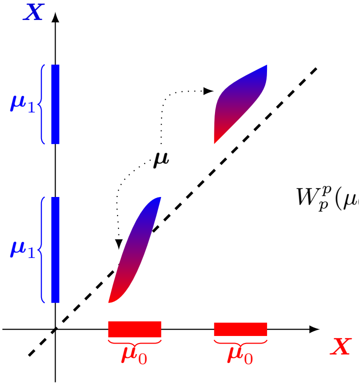
Couplings and Wasserstein distance

$(\boldsymbol{X},\mathsf{d})$ is a metric space, $\mu_0,\mu_1 \in \mathscr{P}(\boldsymbol{X})$, $\pi^i : \boldsymbol{X}\times\boldsymbol{X} \to \boldsymbol{X}$ are the projections $\pi^i(x_0,x_1) = x_i$.

Coupling $\boldsymbol{\mu} \in \mathscr{P}(\boldsymbol{X}\times\boldsymbol{X})$ between μ_0,μ_1: $(\pi^i)_\sharp\boldsymbol{\mu} = \mu_i$, i.e. $\mu_0(A) = \boldsymbol{\mu}(A\times\boldsymbol{X}),\ \mu_1(B) = \boldsymbol{\mu}(\boldsymbol{X}\times B)$.

$\mathscr{P}_p(\boldsymbol{X})$: space of Borel probability measures with **finite p-moment:**

$$\int \mathsf{d}^p(x,\bar{x})\,\mathrm{d}\mu(x) < \infty \quad \text{for some } \bar{x}\in\boldsymbol{X}.$$



$$\mu_0,\mu_1 \in \mathscr{P}_p(\boldsymbol{X}),$$

$$W_p^p(\mu_0,\mu_1) := \min\Big\{ \int \mathsf{d}^p(x_0,x_1)\,\mathrm{d}\boldsymbol{\mu}(x_0,x_1) : \boldsymbol{\mu} \text{ coupling for } \mu_0,\mu_1 \Big\}$$

Metric properties of $\mathscr{P}_p(\boldsymbol{X})$.

- **Optimal coupling:** $\boldsymbol{\mu} \in \mathrm{Opt}(\mu_0, \mu_1)$ such that

$$\int \mathsf{d}^p(x_0, x_1)\,\mathrm{d}\boldsymbol{\mu} = W_p^p(\mu_0, \mu_1)$$

- $(\mathscr{P}_p(\boldsymbol{X}), W_p)$ is a **metric space.**
- If $(\boldsymbol{X}, \mathsf{d})$ is **complete (resp. separable, compact, length, geodesic)**, then $(\mathscr{P}_p(\boldsymbol{X}), W_p)$ is complete (resp. separable, compact, length, geodesic).

  A metric space is *geodesic* if every couple of points can be connected by a geodesic.
- $W_p(\mu_n, \mu) \to 0$ iff

$$\int f\,\mathrm{d}\mu_n \to \int f\,\mathrm{d}\mu \quad \text{for every } f \in \mathrm{C}(\boldsymbol{X}),\ |f(x)| \le A + B\,\mathsf{d}^p(x, \bar{x}).$$

- If d **is bounded** then $\mathscr{P}_p(\boldsymbol{X}) = \mathscr{P}(\boldsymbol{X})$ and the topology induced by W_p coincides with the usual weak topology in $\mathscr{P}(\boldsymbol{X})$, in duality with $\mathrm{C}_b(\boldsymbol{X})$

Dynamic properties of $\mathscr{P}_p(\boldsymbol{X})$

Path space: $C([0,1];\boldsymbol{X})$. $\mathrm{Geo}(\boldsymbol{X})$ subset of all minimal geodesics.

Evaluation map $\mathsf{e}_t : C([0,1];\boldsymbol{X}) \to \boldsymbol{X}$, $\mathsf{e}_t(\mathsf{x}) := \mathsf{x}(t)$.

Dynamic plans: probability measures $\boldsymbol{\pi}$ on $C([0,1];\boldsymbol{X})$. $\boldsymbol{\pi}$ is a geodesic plan if it is concentrated on $\mathrm{Geo}(\boldsymbol{X})$, i.e. $\boldsymbol{\pi}(\mathrm{Geo}(\boldsymbol{X})) = 1$.

- If $\boldsymbol{\pi}$ is a dynamic plan, $\mu_t = (\mathsf{e}_t)_\sharp\boldsymbol{\pi}$ is a continuous curve in $\mathscr{P}(\boldsymbol{X})$.
- If $\boldsymbol{X}$ is **geodesic** and $\boldsymbol{\mu} \in \mathrm{Opt}(\mu_0, \mu_1)$ then there exists $\boldsymbol{\pi} \in \mathscr{P}(\mathrm{Geo}(\boldsymbol{X}))$ such that $\boldsymbol{\mu} = (\mathsf{e}_0, \mathsf{e}_1)_\sharp\boldsymbol{\pi}$.

  In this case $\boldsymbol{\pi} \in \mathrm{GeoOpt}(\mu_0, \mu_1)$ and $\mu_t = (\mathsf{e}_t)_\sharp\boldsymbol{\pi}$ is a minimal, constant speed, geodesic in $\mathscr{P}_p(\boldsymbol{X})$.
- **Geodesic parametrization:** conversely, if $t \mapsto \mu_t$ is a geodesic in $\mathscr{P}_p(\boldsymbol{X})$ there exists an optimal geodesic plan $\boldsymbol{\pi} \in \mathrm{GeoOpt}(\mu_0, \mu_1)$ such that $\mu_t = (\mathsf{e}_t)_\sharp\boldsymbol{\pi}$ for every $t \in [0,1]$.

Optimal transport in Riemannian manifold

Suppose $\boldsymbol{X} = \mathbb{M}$, $\mathfrak{m} = \mathrm{e}^{-V}\mathrm{Vol}_{\mathrm{g}}$ and $\mu_i = \varrho_i \mathfrak{m} \in \mathscr{P}_2(\boldsymbol{X})$.

- There exists a unique geodesic $(\mu_t)_{t\in[0,1]}$ connecting μ_0 to μ_1 and a unique geodesic optimal plan $\boldsymbol{\pi} \in \mathrm{GeoOpt}(\mu_0, \mu_1)$, $\mu_t = (\mathsf{e}_t)_\sharp \boldsymbol{\pi}$.
- $\exists\ \mathsf{T}_t(x) = \exp(tZ)$ such that $\mathsf{x}(t) = \mathsf{T}_t(\mathsf{x}(0))$ for $\boldsymbol{\pi}$-a.e. x,

$$\mu_t = (\mathsf{T}_t)_\sharp \mu_0, \quad W_2^2(\mu_s, \mu_t) = \int \mathsf{d}^2(\mathsf{T}_s(x), \mathsf{T}_t(x))\,\mathrm{d}\mu_0(x).$$

- $\mu_t = \varrho_t \mathfrak{m}$ and $J_t(x) := \log\Big(\varrho_t(\mathsf{T}_t(x))/\varrho_0(x)\Big)$ satisfies

$$\ddot{J}_t(x) \geq \frac{1}{N}(\dot{J}_t(x))^2 + \mathbf{Ric}_{\mathsf{L}}(\dot{\mathsf{T}}_t(x), \dot{\mathsf{T}}_t(x))$$

- If $\mathbf{Ric}_{\mathsf{L}} \geq K\mathsf{g}$ the Relative entropy functional

$$\mathrm{Ent}_{\mathfrak{m}}(\mu_t) := \int \varrho_t \log \varrho_t \,\mathrm{d}\mathfrak{m} \qquad \text{satisfies}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathrm{Ent}_{\mathfrak{m}}(\mu_t) \geq K W_2^2(\mu_0, \mu_1).$$

[Brenier, McCann, Otto-Villani, Cordero Erausquin-McCann-Schmuckenschlager, Von Renesse-Sturm...]

Second derivative of the entropy

$$E(t) := \mathrm{Ent}_{\mathfrak{m}}(\mu_t) = \int \varrho_t \log \varrho_t \,\mathrm{d}\mathfrak{m} = \int \log \varrho_t \,\mathrm{d}\mu_t =$$
$$= \int \log(\varrho_t(\mathsf{T}_t(x)))\,\mathrm{d}\mu_0 = \int J_t(x)\,\mathrm{d}\mu_0 + E(0).$$

$$\ddot{E}(t) \geq \int \ddot{J}_t(x)\,\mathrm{d}\mu_0 \geq K \int |\dot{\mathsf{T}}_t(x)|^2_{\mathsf{g}}\,\mathrm{d}\mu_0 = K \int \mathsf{d}^2_g(\mathsf{T}_1(x), x)\,\mathrm{d}\mu_0$$
$$\geq K\, W_2^2(\mu_1, \mu_0)$$

$$\boxed{E(t) - \frac{1}{2} K W_2^2(\mu_1,\mu_0) t^2 \quad \text{is convex}},$$

$$\boxed{E(t) \leq (1-t)E(0) + tE(1) - \frac{K}{2} t(1-t) W_2^2(\mu_1,\mu_0)}$$

Metric measure spaces satisfying a lower Ricci curvature bound: the approach by Lott, Sturm, Villani.

The basic object is a **metric measure space:**

$$(\boldsymbol{X}, \mathsf{d}, \mathfrak{m}) : \quad \begin{array}{l} (\boldsymbol{X}, \mathsf{d}) \text{ is a complete and separable metric space,} \\ \mathfrak{m} \text{ is a Borel probability measure in } \mathscr{P}(\boldsymbol{X}) \end{array}$$

$\mathsf{CD}(K, \infty)$ spaces

$(\boldsymbol{X}, \mathsf{d}, \mathfrak{m})$ satisfies the lower Ricci curvature bound $\mathsf{CD}(K, \infty)$ according to Lott-Sturm-Villani if for every $\mu_0, \mu_1 \in \mathscr{P}(\boldsymbol{X})$ with finite entropy there exists $\mu_\vartheta \in \mathscr{P}(\boldsymbol{X})$ such that:

- **Geodesic interpolation in the transport metric**:

$$W_2(\mu_\vartheta, \mu_0) = \vartheta W_2(\mu_0, \mu_1), \quad W_2(\mu_\vartheta, \mu_1) = (1-\vartheta) W_2(\mu_0, \mu_1),$$

- ***K*-convexity of the Entropy:**

$$\mathrm{Ent}_{\mathfrak{m}}(\mu_\vartheta) \le (1-\vartheta)\mathrm{Ent}_{\mathfrak{m}}(\mu_0) + \vartheta \mathrm{Ent}_{\mathfrak{m}}(\mu_1) - \frac{K}{2}\vartheta(1-\vartheta) W_2^2(\mu_0, \mu_1).$$

Intrinsic metric approach, Bonnet-Myers diameter comparison, Bishop-Gromov volume comparison, stability w.r.t. Sturm-Gromov-Hausdorff convergence (Cheeger-Colding: limits of Riemannian manifold), nonsmooth calculus and

Main problem: how to connect BE to LSV?

- **Bakry-Émery:** gradient commutation along the **Heat flow**

$$\Gamma\big(\mathsf{P}_t u\big) \le \mathrm{e}^{-2Kt}\mathsf{P}_t\Gamma(u).$$

- **Lott-Sturm-Villani:** K-convexity of the entropy along **"geodesic interpolation" of measures.**

$$\mathrm{Ent}_{\mathfrak{m}}(\mu_\vartheta) \le (1-\vartheta)\mathrm{Ent}_{\mathfrak{m}}(\mu_0) + \vartheta\mathrm{Ent}_{\mathfrak{m}}(\mu_1) - \frac{K}{2}\vartheta(1-\vartheta)W_2^2(\mu_0,\mu_1).$$