Quantization for the prescribed Q-curvature equation on open domains

Luca Martinazzi*

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Abstract

We discuss compactness, blow-up and quantization phenomena for the prescribed Q-curvature equation $(-\Delta)^m u_k = V_k e^{2mu_k}$ on open domains of $\mathbb{R}^{2m}$. Under natural integral assumptions we show that when blow-up occurs, up to a subsequence

$$\lim_{k\to\infty} \int_{\Omega_0} V_k e^{2mu_k} dx = L\Lambda_1,$$

where $\Omega_0 \subset\subset \Omega$ is open and contains the blow-up points, $L \in \mathbb{N}$ and $\Lambda_1 := (2m-1)!\,\mathrm{vol}(S^{2m})$ is the total Q-curvature of the round sphere S^{2m}. Moreover, under suitable assumptions, the blow-up points are isolated. We do not assume that V is positive.

1 Introduction

Let $\Omega \subset \mathbb{R}^{2m}$ be a connected open set and consider a sequence (u_k) of solutions to the equation

$$(-\Delta)^m u_k = V_k e^{2mu_k} \quad \text{in } \Omega, \tag{1}$$

where

$$V_k \to V_0 \quad \text{in } C^0_{\mathrm{loc}}(\Omega), \tag{2}$$

and, for some $\Lambda > 0$,

$$\int_\Omega e^{2mu_k} dx \le \Lambda. \tag{3}$$

Equation (1) arises in conformal geometry, as it is the higher-dimensional generalization of the Gauss equation for the prescribed Gaussian curvature. In fact, if u_k satisfies (1), then the conformal metric

$$g_k := e^{2u_k}|dx|^2$$

has Q-curvature V_k (here $|dx|^2$ denotes the Euclidean metric). For the definition of Q-curvature and for more details about the geometric meaning of (1) we refer to the introduction in [Mar1].

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An important example of solutions to (1)-(3) can be constructed as follows. It is well known that the Q-curvature of the round sphere S^{2m} is $(2m-1)!$. Then, if $\pi : S^{2m} \to \mathbb{R}^{2m}$ is the stereographic projection, the metric $g_1 := (\pi^{-1})^* g_{S^{2m}}$ also has Q-curvature $(2m-1)!$. Since $g_1 = e^{2\eta_0}|dx|^2$, with $\eta_0(x) = \log \frac{2}{1+|x|^2}$, it follows that

$$\begin{aligned} &(-\Delta)^m \eta_0 = (2m-1)! e^{2m\eta_0}, \\ &(2m-1)! \int_{\mathbb{R}^{2m}} e^{2m\eta_0} dx = (2m-1)! \operatorname{vol}(S^{2m}) =: \Lambda_1. \end{aligned} \tag{4}$$

The purpose of this paper is to study the compactness properties of (1), and show analogies and differences with previous results in this direction. We start by considering the following model case. The sequence of functions $u_k(x) := \log \frac{2k}{1+k^2|x|^2}$ satifies (1) on $\Omega = \mathbb{R}^{2m}$ with $V_k \equiv (2m-1)!$ and $\int_{\mathbb{R}^{2m}} e^{2mu_k} dx = \operatorname{vol}(S^{2m})$ for every k. On the other hand (u_k) is not precompact, as $u_k(0) \to \infty$ and $u_k \to -\infty$ locally uniformly on $\mathbb{R}^{2m} \setminus \{0\}$ so that

$$V_k e^{2mu_k} dx \rightharpoonup \Lambda_1 \delta_0$$

in the sense of measures as $k \to \infty$.

For $m = 1$, Brezis and Merle in their seminal work [BM] proved that a sequence (u_k) of solutions to (1)-(3) is either bounded in $C^{1,\alpha}_{\mathrm{loc}}(\Omega)$, or $u_k \to -\infty$ uniformly locally in $\Omega \backslash S$, where $S = \{x^{(1)}, \ldots, x^{(I)}\}$ is a finite set. In particular one has

$$V_k dx \rightharpoonup \sum_{i=1}^{I} \alpha_i \delta_{x^{(i)}}$$

in the sense of measures. Brezis and Merle also conjectured that, at least for $V_0 > 0$, in the latter case one has $\alpha_i = 4\pi L_i$ for some positive integers L_i. This was shown to be true by Li and Shafrir [LS]. Notice that $4\pi = \Lambda_1$ for $m = 1$.

For $m \geq 2$ things are more complex. In [CC] Chang and Chen proved that for every $\alpha \in (0, \Lambda_1)$ there exists a solution v to $(-\Delta)^m v = (2m-1)! e^{2mv}$ on $\mathbb{R}^{2m}$ and with $(2m-1)! \int_{\mathbb{R}^{2m}} e^{2mv} dx = \alpha$. Then, setting

$$u_k(x) = v(kx) + \log k,$$

we find a non-compact sequence of solutions to (1), (2), (3) with $V_k \equiv (2m-1)!$ and

$$\int_{\mathbb{R}^{2m}} V_k e^{2mu_k} dx \to \alpha \notin \Lambda_1 \mathbb{N}.$$

Moreover for $m = 2$ Adimurthi, Robert and Struwe [ARS] gave examples of sequences (u_k) with $u_k \to \infty$ on a hyperplane. These facts suggest that in order to obtain a situation similar to the results of Brezis-Merle (finiteness of the blow-up set) and of Li-Shafrir (quantization of the total Q-curvature), we should make further assumption. In this setting this was first done by Robert for $m = 2$, and Theorem 1 below is a generalization of Robert's result to the case when m is arbitrary.

Theorem 1 *Let* $(u_k) \subset C^{2m}_{\mathrm{loc}}(\Omega)$ *be solutions to* (1), (2) *and* (3), *and assume that there is a ball* $B_\rho(\xi) \subset \Omega$ *such that*

$$\|\Delta u_k\|_{L^1(B_\rho(\xi))} \leq C. \tag{5}$$

Then there is a finite (possibly empty) set $S = \{x^{(1)}, \ldots, x^{(I)}\}$ *such that one of the following is true:*

(i) up to a subsequence $u_k \to u_0$ *in* $C^{2m-1}_{\mathrm{loc}}(\Omega \setminus S)$ *for some* $u_0 \in C^{2m}(\Omega \setminus S)$ *solving* $(-\Delta)^m u_0 = V_0 e^{2mu_0}$*, or*

(ii) up to a subsequence $u_k \to -\infty$ *locally uniformly in* $\Omega \setminus S$.

If $S \neq \emptyset$ *and* $V(x^{(i)}) > 0$ *for some* $1 \leq i \leq I$*, then case* (ii) *occurs.*

Moreover, if we also assume that

$$\|(\Delta u_k)^-\|_{L^1(\Omega)} \leq C, \quad \text{with } (\Delta u_k)^- := \min\{\Delta u_k, 0\}, \tag{6}$$

we have in case (i) *that* $S = \emptyset$ *and in case* (ii) *that* $V_0(x^{(i)}) > 0$ *for* $1 \leq i \leq I$ *and*

$$V_k e^{2mu_k} dx \rightharpoonup \sum_{i=1}^{I} \alpha_i \delta_{x^{(i)}} \tag{7}$$

in the sense of measures in Ω*, where* $\alpha_i = L_i \Lambda_1$ *for some positive* $L_i \in \mathbb{N}$*. In particular, in case* (ii) *for any open set* $\Omega_0 \subset\subset \Omega$ *with* $S \subset \Omega_0$ *we have*

$$\int_{\Omega_0} V_k e^{2mu_k} dx \to L\Lambda_1 \tag{8}$$

for some $L \in \mathbb{N}$ *(*$L = 0$ *if* $S = \emptyset$*).*

Notice that the hypothesis (5) and (6) are natural, since for $m = 1$ they already follow from (1), (2) and (3), and the counterexample quoted above show that they are necessary to some extent (see the first open problem in the last section). Moreover, contrary to [Rob2] and [LS], we do not assume that $V_0 > 0$. In fact, as already discussed in [Mar3], if V_0 has changing sign, one can show using the results of [Mar2] that, if (6) holds, blow-up happens only at points where $V_0 > 0$. We also point out that when $m = 2$, F. Robert [Rob3] proved a version of Theorem 1 where the assumptions (3), (5) and (6) are replaced by $\|\Delta u_k\|_{L^1(\Omega)} \leq C$. This does not seem possible for $m > 2$ without further assumptions of $\Delta^j u_k$ for $2 \leq j \leq m-1$.

A different approach to compactness can be given by working on a closed Riemannian manifold instead of an open set, see Druet-Robert [DR], Malchiodi [Mal], Martinazzi [Mar3] and Ndiaye [Ndi], or by assuming Ω bounded and imposing a Dirichlet or a Navier boundary condition, see Wei [Wei], Robert-Wei [RW] and Martinazzi-Petrache [MP]. In this case the quantization is even stronger, as one shows that $\alpha_i = \Lambda_1$ in (7) and $L = I$ in (8). It turns out that the ideas of [DR] and [Mar3] can be applied in the present context of an open domain if we assume an a-priori L^1-bound on ∇u_k in place of the bound on Δu_k:

Theorem 2 *Let* $(u_k) \subset C^{2m}_{\mathrm{loc}}(\Omega)$ *be solutions to* (1) *and* (3)*, where*

$$V_k \to V_0 \quad \text{in } C^1_{\mathrm{loc}}(\Omega). \tag{9}$$

Assume further that there is a ball $B_\rho(\xi) \subset \Omega$ *such that*

$$\|\nabla u_k\|_{L^1(B_\rho(\xi))} \le C. \tag{10}$$

Then there is a finite (possibly empty) set $S = \{x^{(1)}, \dots, x^{(I)}\}$ *such that one of the following is true:*

(i) up to a subsequence $u_k \to u_0$ *in* $C^{2m-1}_{\mathrm{loc}}(\Omega \setminus S)$ *for some* $u_0 \in C^{2m}(\Omega \setminus S)$ *solving* $(-\Delta)^m u_0 = V_0 e^{2mu_0}$*, or*

(ii) up to a subsequence $u_k \to -\infty$ *locally uniformly in* $\Omega \setminus S$*.*

If $S \neq \emptyset$ *and* $V(x^{(i)}) > 0$ *for some* $1 \le i \le I$*, then case* (ii) *occurs.*

Moreover, if we also assume that

$$\|\nabla u_k\|_{L^1(\Omega)} \le C, \tag{11}$$

we have that in case (i) $S = \emptyset$ *and in case* (ii) $V_0(x^{(i)}) > 0$ *for* $1 \le i \le I$ *and*

$$V_k e^{2mu_k} dx \rightharpoonup \sum_{i=1}^{I} \Lambda_1 \delta_{x^{(i)}} \tag{12}$$

in the sense of measures. In particular, for any open set $\Omega_0 \subset\subset \Omega$ *with* $S \subset \Omega_0$ *we have*

$$\int_{\Omega_0} V_k e^{2mu_k} dx \to I\Lambda_1. \tag{13}$$

The difference between Theorem 1 and Theorem 2 is that under the hypothesis of Theorem 2 one can prove uniform bounds for $\nabla^\ell u_k$, $1 \le \ell \le 2m-2$ (Propositions 12 and 13), which in turn allow us to apply a clever technique of Druet and Robert [DR] to rule out the occurrence of multiple blow-up points. In Theorem 1 one can only prove bounds for $\nabla^{\ell-2}\Delta u_k$, $2 \le \ell \le 2m-1$ (Propositions 5 and 7 below). This is not just a technical issue, as the result of Theorem 2 is stronger than that of Theorem 1. Indeed X. Chen [Che] showed that already for $m = 1$, under the assumptions of Theorem 1, there exist sequences with multiple blow-up points.

The paper is organized as follows. In Section 2 we prove Theorem 1, in section 3, we prove Theorem 2 and in the last section we collect some open problems. The letter C always denotes a generic large constant which can change from line to line, and even within the same line.

I am grateful to F. Robert for suggesting me to work on this problems.

2 Proof of Theorem 1

In the proof of Theorem 1 we use the strategy of extracting blow-up profiles (Proposition 6 below), in the spirit of Struwe [Str1], [Str2] and of Brézis-Coron

[BC1], [BC2]. We classify such profiles thanks to the results of [Mar1] and [Mar2], and finally we use Harnack-type estimates inspired from [Rob2]. Since Propositions 4 and 5 below don't work for $m = 1$, in this section we shall assume that $m > 1$. For the case $m = 1$ we refer to [LS], noticing that their assumption $V_k \geq 0$ can be easily dropped (particularly in their Lemma 1), since there are no solutions to the equation

$$-\Delta u = Ve^{2u} \text{ in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} e^{2u} dx < \infty, \quad V \equiv const < 0,$$

see Theorem 1 in [Mar2].

Proposition 3 *Let* (u_k) *be a sequence of solutions to* (1)-(3) *satisfying* (5) *for some ball* $B_\rho(\xi) \subset \Omega$ *and set*

$$S := \left\{ y \in \Omega : \lim_{r \to 0^+} \liminf_{k \to \infty} \int_{B_r(y)} |V_k| e^{2mu_k} dy \geq \frac{\Lambda_1}{2} \right\}. \tag{14}$$

Then S *is finite (possibly empty) and up to selecting a subsequence one of the following is true:*

(i) $u_k \to u_0$ *in* $C^{2m-1}_{\mathrm{loc}}(\Omega \backslash S)$ *for some* $u_0 \in C^{2m}(\Omega \backslash S)$*;*

(ii) $u_k \to -\infty$ *locally uniformly in* $\Omega \backslash S$*.*

If $S \neq \emptyset$ *and* $V(x^{(i)}) > 0$ *for some* $1 \leq i \leq I$*, then case* (ii) *occurs.*

Proof. By Theorem 1 in [Mar3] (compare [ARS]) we have that S is finite and either

(a) $u_k \to u_0$ in $C^{2m-1}_{\mathrm{loc}}(\Omega \backslash S)$ for some $u_0 \in C^{2m}(\Omega \backslash S)$, or

(b) $u_k \to -\infty$ locally uniformly in $\Omega \backslash (S \cup \Gamma)$, where Γ is a closed set of Hausdorff dimension at most $2m-1$. Moreover there are numbers $\beta_k \to \infty$ such that

$$\frac{u_k}{\beta_k} \to \varphi \quad \text{in } C^{2m-1}_{\mathrm{loc}}(\Omega \backslash (S \cup \Gamma)), \tag{15}$$

where $\varphi \in C^\infty(\Omega \backslash S)$, $\Gamma = \{x \in \Omega \setminus S : \varphi(x) = 0\}$ and

$$\Delta^m \varphi \equiv 0, \quad \varphi \leq 0, \quad \varphi \not\equiv 0 \quad \text{in } \Omega \backslash S. \tag{16}$$

Clearly case (a) corresponds to case (i) in the proposition. We need to show that if (b) occurs, then $\Gamma = \emptyset$, so that $\varphi < 0$ on $\Omega \backslash S$ and case (ii) follows from (15). In order to show that $\Gamma = \emptyset$, observe that $\Delta \varphi \equiv 0$ in $\Omega \backslash S$. Otherwise, since $\Delta \varphi$ is analytic[1], we would have

$$\int_{B_\rho(\xi)} |\Delta \varphi| dx > 0,$$

where $B_\rho(\xi) \subset \Omega$ is as in (5). Then (15) would imply

$$\lim_{k \to \infty} \int_{B_\rho(\xi)} |\Delta u_k| dx = \lim_{k \to \infty} \beta_k \int_{B_\rho(\xi)} |\Delta \varphi| dx = +\infty,$$

[1] we have $\Delta^{m-1}(\Delta \varphi) = 0$, and polyharmonic functions are analytic.

contradicting (5). Therefore $\Delta\varphi \equiv 0$. Then the maximum principle and (16) imply that $\varphi < 0$ in $\Omega \backslash S$, i.e. $\Gamma = \emptyset$, as wished. Also the last claim follows from Theorem 1 in [Mar3]. $\square$

Proposition 3 completes the proof of the first part of Theorem 1. In the remaining part of this section we shall assume that (u_k) satisfies all the hypothesis of Theorem 1, including (6) in particular, and we shall prove the second part of Theorem 1. If $S = \emptyset$, it is clear that the proof of Theorem 1 is complete. Therefore we shall also assume that $S \neq \emptyset$, and we shall prove that consequently we are in case (ii) of Theorem 1.

Proposition 4 *For every open set $\Omega_0 \subset\subset \Omega\backslash S$ there is a constant $C(\Omega_0)$ independent of k such that*

$$\|\Delta u_k\|_{C^{2m-3}(\Omega_0)} \leq C(\Omega_0). \tag{17}$$

Proof. If case (i) of Proposition 3 occurs the proof of (17) is trivial, hence we shall assume that we are in case (ii). Up to restricting the ball $B_\rho(\xi)$ given in (5), we can assume that $B_{2\rho}(\xi) \cap S = \emptyset$, so that $u_k \leq C = C(\rho)$ on $B_\rho(\xi)$. Consequently $|\Delta^m u_k| \leq C$ on $B_\rho(\xi)$. This, (5) and elliptic estimates (see e.g. [Mar1], Lemma 20) imply that

$$\|\Delta u_k\|_{C^{2m-3}(B_{\rho/2}(\xi))} \leq C. \tag{18}$$

Elliptic estimates and (6) imply that either $\Delta u_k \to +\infty$ locally uniformly in $\Omega \setminus S$, or $(\Delta u_k)_{k\in\mathbb{N}}$ is uniformly bounded locally in $\Omega \setminus S$. In the first case (18) cannot hold, so we are in the second situation, and (17) follows at once from elliptic estimates, since $|\Delta^m u_k| \leq C(\Omega_0)$ on Ω_0. $\square$

Proposition 5 *For every open set $\Omega_0 \subset\subset \Omega$ there is a constant C independent of k such that*

$$\int_{B_r(x_0)} |\nabla^{\ell-2}\Delta u_k| dx \leq C r^{2m-\ell}, \tag{19}$$

for $2 \leq \ell \leq 2m-1$ and for every ball $B_r(x_0) \subset \Omega_0$.

Proof. Fix

$$\delta = \frac{1}{16}\min\left\{\min_{1\leq i\neq j\leq I} |x^{(i)} - x^{(j)}|, \mathrm{dist}(\partial\Omega, \partial\Omega_0)\right\}.$$

By a covering argument, it is enough to prove (19) for $0 < r \leq \delta$. Given $B_r(x_0) \subset \Omega_0$ with $r \leq \delta$, we can choose a ball $B_{4\delta}(\xi) \subset \Omega$ such that $B_r(x_0) \subset B_{2\delta}(\xi)$, $\mathrm{dist}(\partial B_{4\delta}(\xi), S) \geq 2\delta$. For $x \in B_{2\delta}(\xi)$, let $G_x(y)$ be the Green function for the operator Δ^{m-1} in $B_{4\delta}(\xi)$ with respect to the Navier boundary condition:

$$\Delta^{m-1}G_x = \delta_x \text{ in } B_{4\delta}(\xi), \quad G_x = \Delta G_x = \ldots = \Delta^{m-2}G_x = 0 \text{ on } \partial B_{4\delta}(\xi).$$

Then we can write

$$\begin{aligned}\Delta u_k(x) = &\int_{B_{4\delta}(\xi)} G_x(y)\Delta^{m-1}\Delta u_k(y)dy \\ &+ \sum_{j=0}^{m-2}\int_{\partial B_{4\delta}(\xi)} \frac{\partial}{\partial\nu}(\Delta^{m-j-2}G_x)\Delta^j(\Delta u_k)d\sigma.\end{aligned} \tag{20}$$

Differentiating and using the bound $|\nabla^{\ell-2}G_x(y)| \leq \frac{C}{|x-y|^\ell}$ (see [DAS]) and (17) on $\partial B_{4\delta}(\xi)$, we infer for $x \in B_{2\delta}(\xi)$

$$\begin{aligned}|\nabla^{\ell-2}\Delta u_k(x)| \leq & C\int_{B_{4\delta}(\xi)} \frac{|V_k(y)|e^{2mu_k(y)}}{|x-y|^\ell}dy \\ & + C\sum_{j=0}^{m-2} \sup_{\partial B_{4\delta}(\xi)} \Big(\Delta^{j+1}u_k\Big)\int_{\partial B_{4\delta}(\xi)} \frac{d\sigma(y)}{|x-y|^{\ell+2m-2j-3}} \\ \leq & C\int_{B_{4\delta}(\xi)} \frac{e^{2mu_k(y)}}{|x-y|^\ell}dy + C.\end{aligned} \tag{21}$$

Integrating on $B_r(x_0)$ and using Fubini's theorem, we finally get

$$\begin{aligned}\int_{B_r(x_0)} |\nabla^{\ell-2}\Delta u_k(x)|dx \leq & C\int_{B_r(x_0)}\int_{B_{4\delta}(\xi)} \frac{e^{2mu_k(y)}}{|x-y|^\ell}dydx + Cr^{2m} \\ \leq & C\int_{B_{4\delta}(\xi)} e^{2mu_k(y)}\bigg(\int_{B_r(x_0)} \frac{1}{|x-y|^\ell}dx\bigg)dy + Cr^{2m} \\ \leq & Cr^{2m-\ell}\int_{B_{4\delta}(\xi)} e^{2mu_k(y)}dy + Cr^{2m} \\ \leq & Cr^{2m-\ell} + Cr^{2m} \leq Cr^{2m-\ell},\end{aligned}$$

where in the last inequality we used that $r \leq \delta$. □

Proposition 6 *Let $\Omega_0 \subset\subset \Omega$ be an open set such that $S \subset \Omega_0$. Then up to a subsequence we have*

$$\lim_{k\to\infty} \sup_{\Omega_0} u_k = +\infty, \tag{22}$$

and case (ii) of Theorem 1 occurs. There exist $L \geq I$ converging sequences of points $x_{i,k} \to x^{(i)} \in \Omega$ such that $u_k(x_{i,k}) \to \infty$ as $k \to \infty$, $S = \{x^{(1)}, \ldots, x^{(L)}\}$, $V(x^{(i)}) > 0$ for $1 \leq i \leq L$, and there exist L sequences of positive numbers

$$\mu_{i,k} := 2\bigg(\frac{(2m-1)!}{V_0(x^{(i)})}\bigg)^{\frac{1}{2m}} e^{-u_k(x_{i,k})} \to 0 \tag{23}$$

such that the following holds:

(a) for $1 \leq i, j \leq L$, $i \neq j$

$$\lim_{k\to\infty} \frac{|x_{i,k} - x_{j,k}|}{\mu_{i,k}} = \infty;$$

(b) setting $\eta_{i,k} := u_k(x_{i,k} + \mu_{i,k}x) - u_k(x_{i,k}) + \log 2$, one has

$$\lim_{k\to\infty} \eta_{i,k}(x) = \eta_0(x) = \log\frac{2}{1+|x|^2} \quad \text{in } C^{2m-1}_{\mathrm{loc}}(\mathbb{R}^{2m}),$$

and

$$\lim_{R\to\infty}\lim_{k\to\infty}\int_{B_{R\mu_{i,k}}(x_{i,k})} V_k e^{2mu_k}dx = \Lambda_1; \tag{24}$$

(c) for every $\Omega_0 \subset\subset \Omega$ we have

$$\inf_{1\le i\le L} |x - x_{i,k}| e^{u_k(x)} \le C = C(\Omega_0). \tag{25}$$

Proof. Step 1. If $\sup_{\Omega_0} u_k \le C$, then by (14) we have $S = \emptyset$, contrary to the assumption we made after Proposition 3. Therefore we can assume that (22) holds.

Step 2. Since u_k is locally bounded in $\Omega \setminus S$ uniformly in k if case (i) of Theorem 1 holds, and $u_k \to -\infty$ uniformly locally in $\Omega \setminus S$, one can find $x_k \in \Omega_0$ such that

$$u_k(x_k) = \sup_{\Omega_0} u_k \to \infty, \quad \text{as } k \to \infty.$$

Moreover up to a subsequence $x_k \to x_0 \in S$. In particular $\mathrm{dist}(x_k, \partial\Omega_0) \ge \delta > 0$ for some $\delta > 0$. Setting $\sigma_k = e^{-u_k(x_k)}$, we define

$$z_k(y) = u_k(x_k + \sigma_k y) + \log(\sigma_k) \le 0 \quad \text{in } B_{\delta/\sigma_k}(0).$$

We claim that up to a subsequence $z_k \to z_0$ in $C^{2m-1,\alpha}_{\mathrm{loc}}(\mathbb{R}^{2m})$, where

$$(-\Delta)^m z_0 = V_0(x_0) e^{2mz_0}, \quad \int_{\mathbb{R}^{2m}} e^{2mz_0} dx < \infty. \tag{26}$$

This follows by elliptic estimates, using that $z_k \le 0$, $z_k(0) = 0$ and Proposition 5. With the same technique of the proof of Proposition 8 in [Mar3], step 3, one proves that $V_0(x_0) > 0$. Since we have found a point $x_0 \in S$ with $V_0(x_0) > 0$, Proposition 3 implies that we are in case (ii) of Theorem 1.

Step 3. Now we define $x_{1,k} := x_k \to x_0 =: x^{(1)}$. Also set $\mu_{1,k}$ and $\eta_{1,k}$ as in the statement of the proposition. Then, still following [Mar3], Proposition 8, we infer that $\eta_{1,k}(x) \to \log \frac{2}{1+|x|^2}$ in $C^{2m-1,\alpha}_{\mathrm{loc}}(\mathbb{R}^{2m})$.

Step 4. We now proceed by induction, as follows. Assume that we have already found L sequences $(x_{i,k})$ and $(\mu_{i,k})$, $1 \le i \le L$, such that (a) and (b) holds, we either have that also (c) holds, and we are done, or we construct a new sequence $x_{L+1,k} = x_k \to x_0 \in S$, and $\sigma_k = \sigma_{L+1,k} := e^{-u_k(x_k)}$ such that

$$\inf_{1\le i\le L} |x_k - x^{(i)}| e^{u_k(x_k)} = \max_{x\in\Omega_0} \inf_{1\le i\le L} |x - x^{(i)}| e^{u_k(x)}.$$

Then we define $z_k \to z_0$ as before, we prove that $V_0(x_0) > 0$, so that we can define $\mu_{L+1,k}$ and $\eta_{L+1,k}$ as in the statement of the proposition and $\eta_{L+1,k}(x) \to \log \frac{2}{1+|x|^2}$ in $C^{2m-1,\alpha}_{\mathrm{loc}}(\mathbb{R}^{2m})$. Moreover (a) holds with $L+1$ instead of L. Taking into account (a) and (b), we see that

$$\limsup_{k\to\infty} \int_{\Omega_0} V_k e^{2mu_k} dx \ge (L+1)\Lambda_1.$$

This, (2) and (3) imply that after a finite number of steps the procedure stops and (c) holds. The missing details are as in Step 1 of the proof of Theorem 1 in [DR]. $\square$

Remark. In general, as shown by X. Chen [Che], it is possible that $L > I$, hence $x^{(i)} = x^{(j)}$ for some $i \neq j$. In this case we will stick to the notation $S = \{x^{(i)}, \ldots, x^{(I)}\}$, i.e. $x^{(i)} \neq x^{(j)}$ for $i \neq j$, $1 \le i,j \le I$.

Proposition 7 *For $2 \leq \ell \leq 2m-1$ and $\Omega_0 \subset\subset \Omega$ we have*

$$\inf_{1\leq i\leq L} |x - x_{i,k}|^\ell |\nabla^{\ell-2}\Delta u_k(x)| \leq C = C(\Omega_0), \quad \textit{for } x \in \Omega_0. \tag{27}$$

Proof. Let us consider a ball $B_\delta(\xi)$ as in the proof of Proposition 5, so that we have

$$|\nabla^{\ell-2}\Delta u_k(x)| \leq C \int_{B_{4\delta}(\xi)} \frac{e^{2mu_k(y)}}{|x-y|^\ell} dy + C$$

for $x \in B_{2\delta}(\xi)$ which we now fix. Set for $1 \leq i \leq L$

$$\Omega_{i,k} := \Big\{ y \in B_{2\delta}(\xi) : \inf_{1\leq j\leq L} |y - x_{j,k}| = |y - x_{i,k}| \Big\},$$

and, assuming $x \neq x_{i,k}$ for $1 \leq i \leq L$ (otherwise (27) is trivial), set

$$\Omega_{i,k}^{(1)} := \Omega_{i,k} \cap B_{|x_{i,k}-x|/2}(x_{i,k}), \quad \Omega_{i,k}^{(2)} := \Omega_{i,k} \backslash B_{|x_{i,k}-x|/2}(x_{i,k}).$$

Observing that for $y \in \Omega_{i,k}^{(1)}$ we have $\frac{1}{|x-y|} \leq \frac{2}{|x-x_{i,k}|}$ and using (c) from Proposition 6, we infer

$$\begin{aligned} \int_{\Omega_{i,k}} \frac{e^{2mu_k}}{|x-y|^\ell} dy &\leq \frac{C}{|x-x_{i,k}|^\ell} \int_{\Omega_{i,k}^{(1)}} e^{2mu_k(y)} dy \\ &\quad + C \int_{\Omega_{i,k}^{(2)}} \frac{dy}{|x-y|^\ell |y-x_{i,k}|^{2m}}. \end{aligned}$$

The first integral on the right-hand side is bounded by $\frac{C}{|x-x_{i,k}|^\ell}$. As for the integral over $\Omega_{i,k}^{(2)}$, write $\Omega_{i,k}^{(2)} = \Omega_{i,k}^{(3)} \cup \Omega_{i,k}^{(4)}$, with

$$\Omega_{i,k}^{(3)} = \Omega_{i,k}^{(2)} \cap B_{2|x-x_{i,k}|}(x), \quad \Omega_{i,k}^{(4)} = \Omega_{i,k}^{(2)} \backslash B_{2|x-x_{i,k}|}(x).$$

We have

$$\begin{aligned} \int_{\Omega_{i,k}^{(3)}} \frac{dy}{|x-y|^\ell |y-x_{i,k}|^{2m}} &\leq \frac{C}{|x-x_{i,k}|^{2m}} \int_{\Omega_{i,k}^{(3)}} \frac{dy}{|x-y|^\ell} \\ &\leq \frac{C}{|x-x_{i,k}|^{2m}} \int_0^{2|x-x_{i,k}|} r^{2m-\ell-1} dr \\ &\leq \frac{C}{|x-x_{i,k}|^\ell}. \end{aligned}$$

Observing that

$$\frac{1}{C}|y-x_{i,k}| \leq |x-y| \leq C|y-x_{i,k}| \quad \text{on } \Omega_{i,k}^{(4)},$$

we estimate

$$\begin{aligned} \int_{\Omega_{i,k}^{(4)}} \frac{dy}{|x-y|^\ell |y-x_{i,k}|^{2m}} &\leq C \int_{\Omega_{i,k}^{(4)}} \frac{dy}{|x-y|^{2m+\ell}} \\ &\leq C \int_{2|x-x_{i,k}|}^\infty r^{-\ell-1} dr \\ &\leq \frac{C}{|x-x_{i,k}|^\ell}. \end{aligned}$$

Putting these inequalities together yields

$$|\nabla^{\ell-2}\Delta u_k(x)| \leq \frac{C}{\inf_{1\leq i\leq L}|x-x_{i,k}|^\ell} + C.$$

This gives (27) for $x \in B_{2\delta}(\xi)\setminus S$ and for $\mathrm{dist}(x,S)\leq 1$. For $\mathrm{dist}(x,S)\geq 1$, (27) follows from Proposition 4. By a simple covering argument, we conclude. $\square$

Analogous to Proposition 4.1 in [Rob2] we have the following result, which is the key step in showing that the contributions given by (24) for $1\leq i\leq L$ asymptotically exhaust the whole energy.

Proposition 8 *Consider $x_0 \in S$, $0<\delta<\frac{\mathrm{dist}(x_0,\partial\Omega)}{4}$, such that $V_k(x)\geq V_k(x_0)/2>0$ for $x\in B_{4\delta}(x_0)$ and k large enough. Up to relabelling assume that*

$$\lim_{k\to\infty} x_{i,k} = x_0, \quad \textit{for } 1\leq i\leq N,$$

for some positive integer $N\leq L$, and set $x_k := x_{1,k}$, $\mu_k := \mu_{1,k}$. Assume that for a sequence $0\leq \rho_k\to 0$ we have

$$\inf_{1\leq i\leq N}|x-x_{i,k}|e^{u_k(x)}\leq C, \quad \inf_{1\leq i\leq N}|x-x_{i,k}|^\ell|\nabla^{\ell-2}\Delta u_k(x)|\leq C \tag{28}$$

for $x\in B_{2\delta(x_k)}\setminus B_{\rho_k}(x_k)$ and $2\leq\ell\leq 2m-1$. Let $r_k>0$ be such that $r:=\lim_{k\to\infty} r_k\in[0,\delta]$, $\lim_{k\to\infty}\frac{\mu_k}{r_k}=\lim_{k\to\infty}\frac{\rho_k}{r_k}=0$ and set

$$J := \Big\{ i\in\{2,\dots,N\} : \limsup_{k\to\infty}\frac{|x_{i,k}-x_k|}{r_k}<\infty\Big\}.$$

Up to a subsequence, define $\tilde{x}_i := \lim_{k\to\infty}\frac{x_{i,k}-x_k}{r_k}$, for $i\in J$. Assume that $\tilde{x}_i\neq 0$ for $i\in J$ and let ν and R be such that

$$0<\nu<\frac{1}{10}\min\big\{\{|\tilde{x}_i| : i\in J\}\cup\{|\tilde{x}_i-\tilde{x}_j| : i,j\in J, \tilde{x}_i\neq\tilde{x}_j\}\big\}, \tag{29}$$

and

$$3\max\{|\tilde{x}_i| : i\in J\} < R < \frac{\delta}{2r}, \tag{30}$$

where $\frac{\delta}{2r}:=\infty$ if $r=0$. Then we have

$$\lim_{k\to\infty}\int_{\left(B_{Rr_k}(x_k)\setminus\bigcup_{i\in J}\overline{B}_{\nu r_k}(x_{i,k})\right)\setminus\overline{B}_{3\rho_k}(x_k)} e^{2mu_k}dx = 0, \quad \textit{if } \mu_k/\rho_k\to 0, \tag{31}$$

as $k\to\infty$, and

$$\lim_{\tilde{R}\to\infty}\lim_{k\to\infty}\int_{\left(B_{Rr_k}(x_k)\setminus\bigcup_{i\in J}\overline{B}_{\nu r_k}(x_{i,k})\right)\setminus\overline{B}_{\tilde{R}\mu_k}(x_k)} e^{2mu_k}dx = 0, \quad \textit{if } \rho_k\leq C\mu_k. \tag{32}$$

Remark. For a better understanding of the above proposition one can first consider the simplified case when $N=L=1$ (only one blow-up sequence), $r_k=\delta$, $\rho_k=0$, $R=\frac{1}{4}$ and $J=\emptyset$. Then (32) reduces to

$$\lim_{\tilde{R}\to\infty}\lim_{k\to\infty}\int_{B_{\delta/4}(x_k)\setminus\overline{B}_{\tilde{R}\mu_k}(x_k)} e^{2mu_k}dx = 0.$$

This and (24) imply (7) with $\alpha_1 = \Lambda_1$, hence the proof of Theorem 1 is complete in this special case.

In the general case we point out that the estimates in (28) are stronger than (25) and (27) in that the infimum is not taken over all $1 \le i \le L$, and weaker in that they need not hold in $B_{\rho_k}(x_k)$.

Proof. First observe that if $\rho_k \le C\mu_k$, upon redefining ρ_k larger, we see that (31) implies (32), hence we shall assume that $\lim_{k\to\infty} \mu_k/\rho_k = 0$.

Step 1. Set

$$\Omega_k := \Big(B_{3R}(0) \setminus \bigcup_{i\in J} \overline{B}_{\frac{\nu}{2}}(\tilde{x}_i)\Big) \setminus \overline{B}_{\frac{\rho_k}{r_k}}(0).$$

Then, as in [Rob2], we easily get that for $x \in \Omega_k$ and k large enough

$$\inf_{1\le i\le N} |x_k + r_k x - x_{i,k}| \ge C(\nu, R) r_k |x|, \tag{33}$$

and

$$x_k + r_k x \in B_{2\delta}(x_k) \setminus \overline{B}_{\rho_k}(x_k).$$

Set $\tilde{u}_k(x) := u_k(x_k + r_k x) + \log r_k$ for $x \in B_{3R}(0)$, satisfying

$$(-\Delta)^m \tilde{u}_k = \tilde{V}_k e^{2m\tilde{u}_k} \quad \text{in } B_{3R}(0)$$

for $\tilde{V}_k(x) := V_k(x_k + r_k x)$. According to (28) we have

$$|x| e^{\tilde{u}_k(x)} \le C, \quad |x|^{2\ell} |\Delta^\ell \tilde{u}_k(x)| \le C \quad \text{for } x \in \Omega_k,\ 1 \le \ell \le m-1. \tag{34}$$

Step 2. There are constants $C = C(\nu, R)$, $\beta = \beta(\nu, R) > 0$ such that

$$\sup_{\substack{|x|=r \\ x\notin \bigcup_{i\in J} \overline{B}_\nu(\tilde{x}_i)}} (\beta \tilde{u}_k(x)) \le \inf_{\substack{|x|=r \\ x\notin \bigcup_{i\in J} \overline{B}_\nu(\tilde{x}_i)}} \tilde{u}_k(x) + (1-\beta)\log r + C, \tag{35}$$

for all $r \in]3\rho_k/r_k, 2R]$. This follows exactly as in step 4.2 of [Rob2], using (34) and Harnack's inequality.

Step 3. We claim that there exists $\alpha > 0$ such that

$$\sup_{\substack{|x|=r \\ x\notin \bigcup_{i\in J} \overline{B}_\nu(\tilde{x}_i)}} \tilde{u}_k(x) \le -(1+\alpha)\log r - \alpha \log\frac{r_k}{\mu_k} + C \tag{36}$$

for all $r \in]3\rho_k/r_k, 2R]$. In order to prove this claim, fix $s_k \in]3\rho_k/r_k, 2R]$ and set

$$U_k(x) := \tilde{u}_k(s_k x) + \log s_k \quad \text{for } x \in B_{\frac{3R}{s_k}}(0).$$

Assume that $0 < s_k < 8\nu$, so that

$$B_1(0) \cap \Big(\bigcup_{i\in J} \overline{B}_{\frac{2\nu}{s_k}}(s_k^{-1}\tilde{x}_i)\Big) = \emptyset,$$

and let H be the Green's function of Δ^m on B_1 with Navier boundary condition, that is the only function satisfying

$$\Delta^m H = \delta_0 \text{ on } B_1, \qquad H = \Delta H = \cdots = \Delta^{m-1} H = 0 \text{ on } \partial B_1.$$

Then we have

$$U_k(0) = \int_{B_1} H(y)\Delta^m U_k(y)dy + \sum_{\ell=0}^{m-1} \int_{\partial B_1} \frac{\partial \Delta^{m-1-\ell} H(y)}{\partial n} \Delta^\ell U_k(y) d\sigma(y). \quad (37)$$

Using (29) and (30) we infer that $\partial B_1 \subset s_k^{-1}\Omega_k$. Moreover (34) yields

$$U_k(x) \le C, \quad |\Delta^\ell U_k(x)| \le C \quad \text{for } |x| = 1,\ 1 \le \ell \le m-1.$$

This implies

$$\left| \int_{\partial B_1} \frac{\partial \Delta^{m-1-\ell} H(y)}{\partial n} \Delta^\ell U_k(y) d\sigma(y) \right| \le C, \quad \text{for } 1 \le \ell \le m-1,$$

and

$$\int_{\partial B_1} \frac{\partial \Delta^{m-1} H(y)}{\partial n} U_k(y) d\sigma(y) \ge \inf_{\partial B_1} U_k,$$

where we used the identity $\int_{\partial B_1} \frac{\partial \Delta^{m-1} H(y)}{\partial n} d\sigma(y) = 1$. This in turn can be checked by testing (37) with $U_k \equiv 1$. Then, also observing that $(-1)^m H \ge 0$ and $(-\Delta)^m U_k \ge 0$, (37) gives

$$\begin{aligned} U_k(0) &\ge \int_{B_1} (-1)^m H(y)(-\Delta)^m U_k(y)dy + \inf_{\partial B_1} U_k - C \\ &\ge \int_{B_{\frac{\tilde{R}\mu_k}{s_k r_k}}} (-1)^m H(y)(-\Delta^m) U_k(y)dy + \inf_{\partial B_1} U_k - C, \end{aligned} \quad (38)$$

for any $\tilde{R} > 0$ and $k \ge k_0$ such that $B_{\frac{\tilde{R}\mu_k}{s_k r_k}} \subset B_{\frac{1}{2}}$. We have that

$$(-1)^m H(y) \ge \frac{2}{\Lambda_1} \log \frac{1}{|y|} - C, \quad (39)$$

which follows by elliptic estimates and the fact that $K(x) := \frac{2}{\Lambda_1} \log \frac{1}{|x|}$ satisfies $(-\Delta)^m K = \delta_0$ (see e.g. [Mar1, Proposition 22]), hence $\Delta^m((-1)^m K - H) = 0$. Plugging (39) into (38) we can further estimate

$$U_k(0) - \inf_{\partial B_1} U_k + C \ge \int_{B_{\frac{\tilde{R}\mu_k}{s_k r_k}}} \left(\frac{2}{\Lambda_1} \log \frac{1}{|y|} - C \right) (-\Delta)^m U_k(y)dy =: I. \quad (40)$$

Scaling back, recalling that $u_k(x_k) = -\log \mu_k + \frac{1}{2m} \log \frac{(2m-1)!}{V_0(x_0)}$, and performing the change of variable $y = \frac{\mu_k}{s_k r_k} z$, we obtain

$$\begin{aligned} I &= \int_{B_{\frac{\tilde{R}\mu_k}{s_k r_k}}} \left(\frac{2}{\Lambda_1} \log \frac{1}{|y|} - C \right) V_k(x_k + r_k s_k y) e^{2mU_k(y)} dy \\ &= \int_{B_{\tilde{R}}} \frac{2}{\Lambda_1} \left(\log \frac{1}{|z|} + \log \frac{s_k r_k}{\mu_k} - C \right) \frac{(2m-1)! V_k(x_k + \mu_k z)}{V_0(x_0)} e^{2m\eta_k} dz, \end{aligned}$$

with $\eta_k = \eta_{1,k}$ is as in Proposition 6, part b. Then Proposition 6 implies for $k \geq k_0(\tilde{R})$

$$I \geq (1+o(1))\frac{2}{\Lambda_1}\log\frac{s_k r_k}{\mu_k}\int_{B_{\tilde{R}}}(2m-1)!e^{2m\eta_0}dz,$$

with error $o(1) \to 0$ as $k \to \infty$. Then with (4) we get

$$I \geq (2+\theta_k(\tilde{R}))\log\frac{s_k r_k}{\mu_k}$$

for some function $\theta_k(\tilde{R})$ with $\lim_{\tilde{R}\to\infty}\lim_{k\to\infty}\theta_k(\tilde{R}) = 0$. Going back to (40) and observing that $U_k(0) = \log\frac{r_k s_k}{\mu_k} + C$, we conclude

$$(1+\theta_k(\tilde{R}))\log\frac{s_k r_k}{\mu_k} + \inf_{\partial B_1} U_k \leq C,$$

for $k \geq k_0(\tilde{R})$ large enough. Upon choosing $\tilde{R}$ large, we see that there exists $\theta > -1$ such that

$$(1+\theta)\log\frac{s_k r_k}{\mu_k} + \inf_{\partial B_1} U_k \leq C$$

for all k large enough. Combining this with (35) we obtain (36) with $\alpha := \frac{1+\theta}{\beta} > 0$, at least under the assumption that $r < 8\nu$. For $r \geq 8\nu$ (36) follows from the case $r = 7\nu$ and (35).

Step 4. We now complete the proof of (31). For $y \in B_R(0) \setminus \bigcup_{i=1}^{\ell} \overline{B}_{\nu/2}(\tilde{x}_i)$ we get from (36) (upon taking ν smaller)

$$\tilde{u}_k(y) \leq -(1+\alpha)\log|y| - \alpha\log\frac{r_k}{\mu_k} + C.$$

Finally, scaling back to u_k and observing that $\overline{B}_{\nu/2}(\tilde{x}_i) \subset \overline{B}_\nu\left(\frac{x_{i,k}-x_k}{r_k}\right)$ for k large enough, one gets

$$\begin{aligned}
&\int_{\left(B_{Rr_k}(x_k)\setminus\bigcup_{i\in J}\overline{B}_{\nu r_k}(x_{i,k})\right)\setminus\overline{B}_{3\rho_k}(x_k)} e^{2mu_k}dx \\
&\leq \int_{\left(B_R\setminus\bigcup_{i\in J}\overline{B}_{\frac{\nu}{2}}(\tilde{x}_i)\right)\setminus\overline{B}_{\frac{3\rho_k}{r_k}}} e^{2m\tilde{u}_k(y)}dy \\
&\leq \int_{\mathbb{R}^{2m}\setminus\overline{B}_{\frac{3\rho_k}{r_k}}} C\left(\frac{\mu_k}{r_k}\right)^{2m\alpha}\frac{1}{|y|^{2m(1+\alpha)}}dy \\
&\leq C\left(\frac{\mu_k}{\rho_k}\right)^{2m\alpha} \to 0, \quad \text{as } k\to\infty.
\end{aligned}$$

□

Finally we claim that for any $N > 0$ the following proposition holds.

Proposition 9 *Given a ball $B_{4\delta}(x_0) \subset \mathbb{R}^{2m}$, let $(u_k) \subset C^{2m}(B_{4\delta(x_0)})$ be a sequence of solutions to* (1), (2), (3) *with $\Omega = B_{4\delta}(x_0)$, $V_k \geq V_0(x_0)/2 > 0$. Let $x_{i,k}$ and $\mu_{i,k}$, $1 \leq i \leq L$ be as in Proposition 6, and assume that $1 \leq L \leq N$, and* $\lim_{k\to\infty} x_{i,k} = x_0$ *for $1 \leq i \leq L$. Then*

$$\lim_{k\to\infty}\int_{B_\delta(x_0)} V_k e^{2mu_k}dx = L\Lambda_1.$$

The proof of Proposition 9 follows from Proposition 8 and (24) by induction on N as in [Rob2], Proposition (H_N), with only minor and straightforward modifications.

Proof of Theorem 1. Fix $\Omega_0 \subset\subset \Omega$ open with $S \subset \Omega_0$ and choose $\delta > 0$ such that $B_{4\delta}(x^{(i)}) \subset \Omega_0$ for $1 \le i \le I$ and and $B_{4\delta}(x^{(i)}) \cap B_{4\delta}(x^{(j)}) = \emptyset$ for $1 \le i \ne j \le I$ (remember that $x^{(i)} \ne x^{(j)}$ for $1 \le i \ne j \le I$) and such that $V_k \ge V_k(x^{(i)})/2 > 0$ on $B_{4\delta}(x^{(i)})$ for k large enough and $1 \le i \le I$. We fix $i \in \{1, \dots, I\}$ and apply Proposition 9 to the function u_k restricted to $B_\delta(x^{(i)})$ together with the $N = L_i \ge 1$ blow-up sequences converging to $x^{(i)}$, hence getting

$$\lim_{k\to\infty} \int_{B_\delta(x^{(i)})} V_k e^{2mu_k} dx = L_i \Lambda_1.$$

Moreover, since $u_k \to -\infty$ uniformly locally in $\Omega \setminus S$, it follows that

$$\lim_{k\to\infty} \int_{\Omega_0 \setminus \bigcup_{i=1}^I B_\delta(x^{(i)})} V_k e^{2mu_k} dx = 0,$$

whence (7) and (8) follow at once. □

3 Proof of Theorem 2

Here the Harnack-type estimates of [Rob2] are replaced by a technique of [DR], reminiscent of the Pohozaev inequality. For this it is crucial to have the gradient estimates of Propositions 11 and 12 below, which correspond to (and in fact are stronger than) Propositions 4 and 5 of the previous section, and which also work in the case $m = 1$.

Proposition 10 *Let (u_k) be a sequence of solutions to* (1), (3) *and* (9) *satisfying* (10) *for some ball $B_\rho(\xi) \subset \Omega$, and let S be as in* (14). *Then S is finite (possibly empty) and one of the following is true:*

(i) $u_k \to u_0$ *in* $C^{2m-1}_{\mathrm{loc}}(\Omega\backslash S)$ *for some* $u_0 \in C^{2m}(\Omega\backslash S)$*;*

(ii) $u_k \to -\infty$ *locally uniformly in* $\Omega\backslash S$.

If $S \ne \emptyset$ and $V_0(x) > 0$ for some $x \in S$, then case (ii) occurs.

Proof. The proof is analogous to the proof of Proposition 3. Following that proof and its notation, it is enough to show that if case (b) occurs, then $\Gamma = \emptyset$. In order to show this, observe that $\nabla\varphi \equiv 0$ in $\Omega\backslash S$. Otherwise, since $\nabla\varphi$ is analytic, we would have

$$\int_{B_\rho(\xi)} |\nabla\varphi| dx > 0,$$

where $B_\rho(\xi) \subset \Omega$ is as in (10). Then (15) would imply

$$\lim_{k\to\infty} \int_{B_\rho(\xi)} |\nabla u_k| dx = \infty,$$

contradicting (10). Therefore $\varphi \equiv$ *const* and (16) implies that $\varphi < 0$ in $\Omega\backslash S$, i.e. $\Gamma = \emptyset$, as claimed. □

This completes the proof of the first part of Theorem 2 and, as we did in the last section, we shall now assume that (u_k) satisfies all the hypothesis of Theorem 2, including (11). As before, if $S = \emptyset$ the proof of Theorem 2 is complete, hence we shall also assume that $S \neq \emptyset$ and we shall prove that we are in case (ii) of the theorem.

Proposition 11 *For every open set $\Omega_0 \subset\subset \Omega \backslash S$ there is a constant $C = C(\Omega_0)$ such that*

$$\|u_k - \bar{u}_k\|_{C^{2m-1}(\Omega_0)} \leq C, \tag{41}$$

where $\bar{u}_k := ⨍_{\Omega_0} u_k dx$.

Proof. If case (i) of Proposition 10 occurs the proof is trivial, hence we shall assume that we are in case (ii). Consider an open set $\tilde{\Omega}_0 \subset\subset \Omega \setminus S$ with smooth boundary and with $\Omega_0 \subset\subset \tilde{\Omega}_0$. Write $u_k = w_k + h_k$ in $\tilde{\Omega}_0$, with $\Delta^m h_k = 0$ and $w_k = \Delta w_k = \ldots = \Delta^{m-1} w_k = 0$ on $\partial\tilde{\Omega}_0$. Since

$$|\Delta^m w_k| = |\Delta^m u_k| \leq C = C(\tilde{\Omega}_0) \quad \text{on } \tilde{\Omega}_0,$$

by elliptic estimates we have

$$\|w_k\|_{C^{2m-1}(\tilde{\Omega}_0)} \leq C.$$

This and (11) give $\|\nabla h_k\|_{L^1(\tilde{\Omega}_0)} \leq C$, hence, since $\Delta^m(\nabla h_k) = 0$, by elliptic estimates we infer

$$\|\nabla h_k\|_{C^\ell(\Omega_0)} \leq C = C(\ell, \Omega_0, \tilde{\Omega}_0)$$

for every $\ell \geq 0$, see e.g. Proposition 4 in [Mar1]. Therefore

$$\|\nabla u_k\|_{C^{2m-2}(\Omega_0)} \leq C = C(\Omega_0, \tilde{\Omega}_0),$$

and (41) follows at once. □

Proposition 12 *For every open set $\Omega_0 \subset\subset \Omega$ there is a constant C independent of k such that*

$$\int_{B_r(x_0)} |\nabla^\ell u_k| dx \leq C r^{2m-\ell}, \tag{42}$$

for $1 \leq \ell \leq 2m-1$ and for every ball $B_r(x_0) \subset \Omega_0$.

Proof. Going back to the proof of Proposition 5, we only need to replace (20) by

$$\begin{aligned} u_k(x) - \bar{u}_k = &\int_{B_{4\delta}(\xi)} G_x(y) \Delta^m u_k(y) dy \\ &+ \sum_{j=0}^{m-1} \int_{\partial B_{4\delta}(\xi)} \frac{\partial}{\partial\nu} (\Delta^{m-j-1} G_x) \Delta^j (u_k - \bar{u}_k) d\sigma, \end{aligned} \tag{43}$$

where now

$$\Delta^m G_x = \delta_x \text{ in } B_{4\delta}(\xi), \quad G_x = \Delta G_x = \ldots = \Delta^{m-1} G_x = 0 \text{ on } \partial B_{4\delta}(\xi),$$

and

$$\bar{u}_k := \fint_{B_{4\delta}(\xi)} u_k dx.$$

Differentiating and using $|\nabla^\ell G_x(y)| \leq \frac{C}{|x-y|^\ell}$ (see e.g. [DAS]) and (41) (with $\Omega_0 = B_{4\delta}(\xi)$) on $\partial B_{4\delta}(\xi)$, we infer for $x \in B_{2\delta}(\xi)$

$$|\nabla^\ell u_k(x)| \leq C \int_{B_{4\delta}(\xi)} \frac{e^{2mu_k(y)}}{|x-y|^\ell} dy + C.$$

Integrating on $B_r(x_0) \subset B_{2\delta}(\xi)$ and using Fubini's theorem as before, we finally get

$$\int_{B_r(x_0)} |\nabla^\ell u_k(x)| dx \leq C \int_{B_r(x_0)} \int_{B_{4\delta}(\xi)} \frac{e^{2mu_k(y)}}{|x-y|^\ell} dy dx + Cr^{2m} \leq Cr^{2m-\ell}.$$

□

Proposition 6 also holds with the same proof. Proposition 7 has the following analogue, which can be proved as above. Notice that at this point we are not yet excluding that $L > I$.

Proposition 13 *For* $1 \leq \ell \leq 2m-2$ *and* $\Omega_0 \subset\subset \Omega$ *we have*

$$\inf_{1 \leq i \leq L} |x - x_{i,k}|^\ell |\nabla^\ell u_k(x)| \leq C = C(\Omega_0), \quad \textit{for } x \in \Omega_0. \tag{44}$$

Taking into account Proposition 6 and Proposition 13, one can follow the proof of step 4 of Theorem 2 in [Mar3], in order to prove that the concentration points are isolated, i.e. $x^{(i)} \neq x^{(j)}$ for $i \neq j$, $I = L$, and that for $\delta > 0$ small enough

$$\lim_{R \to \infty} \lim_{k \to \infty} \int_{B_\delta(x_{i,k}) \setminus B_{R\mu_{i,k}}(x_{i,k})} V_k e^{2mu_k} dx = 0.$$

This and (24) complete the proof of Theorem 2.

4 A few open questions

1) Necessity of hypothesis (6) *and* (11). Is the assumption (6) (resp. (11)) necessary in order to have quantization in the second part of Theorem 1 (resp. Theorem 2), or is (5) (resp. (10)) enough?

For instance, is it possible to find a sequence (u_k) of solutions to

$$(-\Delta)^m u_k = e^{2mu_k} \quad \text{in } B_1(0)$$

with

$$\lim_{k \to \infty} \int_{B_1(0)} e^{2mu_k} dx = \alpha \in (0, \Lambda_1)$$

and

$$\int_{B_\rho(\xi)} |\Delta u_k| dx \leq C$$

for a ball $B_\rho(\xi) \subset B_1(0)$? To our knowledge, this is unknown even in the case when u_k is radially symmetric, see [Rob1].

2) If case (i) of Theorem 1 (or equivalently Theorem 2) occurs, is it possible to have $S \neq \emptyset$? If instead of (2) we only assume the bound $\|V_k\|_{L^\infty(\Omega)} \leq C$, the answer is negative, as shown for $m = 1$ by Shixiao Wang [Wan].

3) Boundedness from above. Given a solution u to

$$(-\Delta)^m u = V e^{2mu} \quad \text{in } \mathbb{R}^{2m},$$

with $V \in L^\infty(\mathbb{R}^{2m})$, $e^{2mu} \in L^1(\mathbb{R}^{2m})$, is it true that $\sup_{\mathbb{R}^{2m}} u < \infty$?

For $m = 1$ this was proven by Brézis and Merle, [BM, Theorem 2], but their simple technique, which rests on the mean-value theorem for harmonic functions, cannot be applied when $m > 1$. It is only known that when $V \equiv const \geq 0$ the answer is positive, see [Lin, Theorem 1], [Mar1, Theorem 1] and [Mar2, Theorem 3].

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