# Uniqueness of the Cheeger set of a convex body

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**Abstract**

We prove that if $C \subset \mathbb{R}^N$ is a an open bounded convex set, then there is only one Cheeger set inside $C$ and it is convex. A Cheeger set of $C$ is a set which minimizes the ratio perimeter over volume among all subsets of $C$.

*Key words:* Cheeger set, convex set, set of finite perimeter, mean curvature
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Dedicated to the memory of Thomas Lachand-Robert

## 1 Introduction

Given an nonempty open bounded subset $\Omega$ of $\mathbb{R}^N$, we call Cheeger constant of $\Omega$ the quantity

$$h_\Omega \ := \ \min_{F \subseteq \Omega} \frac{P(F)}{|F|} \,. \tag{1}$$

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Here $|F|$ denotes de $N$-dimensional volume of $F$ and $P(F)$ denotes the perimeter of $F$. The minimum in (1) is taken over all nonempty sets of finite perimeter contained in $\Omega$. Note that the minimum in (1) cannot be attained at a set $G$ whose distance from the boundary of $\Omega$ is positive, otherwise we could diminish the quotient $P(G)/|G|$ by rescaling $G$ with a factor larger than one. A Cheeger set of $\Omega$ is any set $G \subseteq \Omega$ which minimizes (1).

For any set $F$ of finite perimeter in $I\!R^N$, let us define

$$\lambda_F := \frac{P(F)}{|F|}.$$

Notice that for any Cheeger set $G$ of $\Omega$, $\lambda_G = h_G$. Observe also that $G$ is a Cheeger set of $\Omega$ if and only if $G$ minimizes

$$\min_{F\subseteq\Omega} P(F) - \lambda_G |F|. \tag{2}$$

We say that a set $\Omega \subset I\!R^N$ is calibrable if $\Omega$ minimizes the problem

$$\min_{F\subseteq\Omega} P(F) - \lambda_\Omega |F|, \tag{3}$$

or, equivalently, if $\Omega$ is Cheeger in itself. Notice that, if $G$ is a Cheeger set of $\Omega$, then $G$ is calibrable.

Finding the Cheeger sets of a given set $\Omega$ is, in general, a difficult task. This task is simplified if $\Omega$ is a convex set and $N = 2$. In that case, there is a unique Cheeger set of $\Omega$ and is given by $\Omega^R \oplus B(0,R)$ where $\Omega^R := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > R\}$ is such that $|\Omega^R| = \pi R^2$ [3,29] (we denote by $X \oplus Y$ the set $\{x + y : x \in X,\, y \in Y\}$, $X, Y \subset I\!R^2$). In particular, we observe that the Cheeger set of $\Omega$ is convex. Both features, uniqueness and convexity of the Cheeger set of $\Omega$ are due to the convexity of $\Omega$ (a counterexample is given in [29] when $\Omega$ is not convex) and our main purpose is to prove that the same results hold when $\Omega$ is a convex body in $I\!R^N$.

Recall that a convex body in $I\!R^N$ is a compact convex subset of $I\!R^N$. We say that a convex body is non-trivial if it has nonempty interior. The purpose of this paper is to prove the following Theorem:

**Theorem 1** *There is a unique Cheeger set inside any non-trivial convex body in $I\!R^N$. The Cheeger set is convex and of class $C^{1,1}$.*

A similar result was proved in [11] under the additional assumption that the convex body is uniformly convex and of class $C^2$. Our proof of Theorem 1 uses some technical results proved in [11]. We notice that, for convex bodies, the $C^{1,1}$ regularity of Cheeger sets is a consequence of the results in [20,21,37], the existence of convex Cheeger sets was shown in [27], and the main assertion of Theorem 1 concerns the uniqueness of Cheeger sets.

As a consequence of Theorem 1 we have the following result (proved in [17] when $N = 2$): if $\Omega \subseteq I\!\!R^N$ is a non-trivial convex body and it is calibrable, then it is strictly calibrable, that is, for any set $F \subset \Omega$ of finite perimeter such that $F \neq \Omega$, we have

$$0 = P(\Omega) - \lambda_\Omega|\Omega| < P(F) - \lambda_\Omega|F|. \tag{4}$$

As it was proved in [17], these inequalities imply that the capillary problem in absence of gravity (with vertical contact angle at the boundary)

$$\begin{aligned} &-\mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \lambda_\Omega \quad \text{in } \Omega \\ &-\frac{Du}{\sqrt{1+|Du|^2}} \cdot \nu^\Omega = 1 \qquad\quad \text{in } \partial\Omega \end{aligned} \tag{5}$$

has a solution $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$. Conversely, if (5) has a solution $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$, then $\Omega$ is strictly calibrable [17,28].

These results can be complemented with a characterization of calibrable non-trivial convex bodies (of class $C^{1,1}$) in terms of the mean curvature of its boundary. Assume that $\Omega$ is a non-trivial convex body in $I\!\!R^N$ whose boundary is of class $C^{1,1}$. Observe that in this case the mean curvature exists at almost any point (with respect to the $(N-1)$-Hausdorff measure $\mathcal{H}^{N-1}$) $x \in \partial\Omega$ ; let us denote it by $\mathbf{H}_\Omega(x)$. Then the set $\Omega$ is calibrable if and only if

$$(N-1)\mathrm{ess\,sup}_{x\in\partial\Omega}\mathbf{H}_\Omega(x) \le \lambda_\Omega. \tag{6}$$

When $N = 2$ this result has been proved by several authors [17,7,29,3,30], and extended to the general case $N \ge 2$ in [2]. As we observe in the present paper this result can be slightly strengthened to say that a non-trivial convex body $\Omega \subset I\!\!R^N$ is calibrable if and only if is of class $C^{1,1}$ and (6) holds. This represents an extension of Giusti's results [17] to non-trivial convex bodies in $\mathbb{R}^N$.

Let us finally comment on the role played by the Cheeger constant in other contexts. Given an open bounded set $\Omega \subseteq \mathbb{R}^N$ with Lipschitz boundary and $p \in (1, \infty)$, the Cheeger constant of $\Omega$ permits to give a lower bound on the first eigenvalue of the $p$-Laplacian on $\Omega$ with Dirichlet boundary conditions. Indeed, if we define

$$\lambda_p(\Omega) := \min_{0 \neq v \in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla v|^p \, dx}{\int_\Omega |v|^p \, dx}, \tag{7}$$

then

$$\lambda_p(\Omega) \geq \left( \frac{h_\Omega}{p} \right)^p . \tag{8}$$

This result was proved in [12] when $p = 2$ and extended to any $p \in (1, \infty)$ in [32]. When $p = 1$ the first eigenvalue of the 1-Laplacian is defined by

$$\lambda_1(\Omega) := \min_{0 \neq v \in BV(\Omega)} \frac{\int_\Omega |Dv| + \int_{\partial\Omega} |v| \, d\mathcal{H}^{N-1}}{\int_\Omega |v| \, dx}, \tag{9}$$

where $BV(\Omega)$ denotes the space of functions of bounded variation in $\Omega$. Then $\lambda_1(\Omega) = h_\Omega$ and both problems are equivalent in the following sense: A function $u \in BV(\Omega)$ is a minimum of (9) if and only if almost every level set is a Cheeger set (see [26,27]). These results have been extended in several directions, in particular, using weighted volume and perimeter [10,9] and for anisotropic versions of the perimeter [30]. Let us also mention that Cheeger sets are related to the global behavior of solutions of the time-dependent constant-mean-curvature equation under vanishing initial condition and Dirichlet boundary data [33,28]. Finally, let us mention that there is an interesting interpretation of the Cheeger constant in terms of the max flow min cut theorem [36,22].

Let us explain the plan of the paper. In Section 2 we reduce the proof of Theorem 1 to the case of non-trivial convex bodies of class $C^{1,1}$. For that we prove the existence of a maximal Cheeger set (which is of class $C^{1,1}$) inside any non-trivial convex body in $\mathbb{R}^N$. The rest of the paper is devoted to the proof of Theorem 1 for non-trivial convex bodies of class $C^{1,1}$. In Section 3 we recall some well-known basic linear algebra inequalities for positive definite matrices. They will be used in Section 4 to study the behavior of the mean curvature of the boundary of the set obtained by convex combination of two smooth strictly convex sets. In Section 5 we prove an auxiliary property, namely that the free boundary of an isoperimetric region inside a convex body of class $C^1$ is strictly convex. Finally, in Section 6 we prove the uniqueness of Cheeger sets inside non-trivial convex bodies of class $C^{1,1}$.

## 2 The maximal Cheeger set inside a non-trivial convex body

The purpose of this Section is to prove the existence of a maximal Cheeger set of class $C^{1,1}$ inside any non-trivial convex body in $\mathbb{R}^N$. Moreover, the maximal Cheeger set is convex. This reduces the proof of Theorem 1 to the class of calibrable sets of class $C^{1,1}$. Let us first recall the notion of function of bounded variation in $\mathbb{R}^N$ and the notion of perimeter.

A function $u \in L^1(\mathbb{R}^N)$ whose gradient $Du$ in the sense of distributions is a (vector valued) Radon measure with finite total variation in $\mathbb{R}^N$ is called a function of bounded variation. The class of such functions will be denoted by $BV(\mathbb{R}^N)$. The total variation of $Du$ on $\mathbb{R}^N$ turns out to be

$$\sup\left\{\int_{\mathbb{R}^N} u \ \mathrm{div} z \ dx : z \in C_0^\infty(\mathbb{R}^N;\mathbb{R}^N), |z(x)| \le 1 \ \forall x \in \mathbb{R}^N\right\}, \tag{10}$$

(where for a vector $v = (v_1, \dots, v_N) \in \mathbb{R}^N$ we set $|v|^2 := \sum_{i=1}^N v_i^2$) and will be denoted by $\int_{\mathbb{R}^N} |Du|$. The map $u \to \int_{\mathbb{R}^N} |Du|$ is $L^1_{\mathrm{loc}}(\mathbb{R}^N)$-lower semicontinuous.

A measurable set $E \subseteq \mathbb{R}^N$ is said to be of finite perimeter if (10) is finite when $u$ is substituted with the characteristic function $\chi_E$ of $E$. The perimeter of $E$ in $\mathbb{R}^N$ is defined as $P(E) := \int_{\mathbb{R}^N} |D\chi_E|$. We denote by $\mathcal{H}^{N-1}$ the $(N-1)$-dimensional Hausdorff measure in $\mathbb{R}^N$. For more information on functions of bounded variation we refer to [18].

Let us recall some results proved in [2].

**Lemma 2.1** *([2, Lemma 4]) Let $C$ be a bounded convex subset of $\mathbb{R}^N$. For any $\mu > 0$, the problem*

$$(P)_\mu : \quad \min_{F \subseteq C} P(F) - \mu|F|. \tag{11}$$

*has always a minimizer. The following properties hold:*

*(i)* *Let $C_\lambda, C_\mu$ be minimizers of $(P)_\lambda$, and $(P)_\mu$, respectively. If $\lambda < \mu$, then $C_\lambda \subseteq C_\mu$.*

*(ii)* *Let $\lambda_n \uparrow \lambda$. Then $C_\lambda^\cup := \bigcup_n C_{\lambda_n}$ is a minimizer of $(P)_\lambda$. Moreover $P(C_{\lambda_n}) \to P(C_\lambda^\cup)$. Similarly, if $\lambda_n \downarrow \lambda$, then $C_\lambda^\cap := \bigcap_n C_{\lambda_n}$ is a minimizer of $(P)_\lambda$, and $P(C_{\lambda_n}) \to P(C_\lambda^\cap)$.*

If $C \subseteq I\!R^N$ is be a non-trivial convex body of class $C^{1,1}$, the mean curvature exists $\mathcal{H}^{N-1}$-almost everywhere on $\partial C$ and we denote it by $\mathbf{H}_C$. Recall that, the set $C$ being convex, $\mathbf{H}_C$ is a nonnegative function. Moreover, if $C$ is of class $C^2$, then $\mathbf{H}_C$ is defined everywhere on $\partial C$.

**Theorem 2** *([2, Theorems 9 and 10]) Let $C \subseteq I\!R^N$ be a non-trivial convex body of class $C^{1,1}$. Then there is a convex calibrable set $K \subseteq C$ which is the maximal Cheeger set contained in $C$. Therefore $K$ minimizes*

$$\min_{F \subseteq C} P(F) - \lambda_K |F| \qquad \text{where } \lambda_K := \tfrac{P(K)}{|K|}. \tag{12}$$

*For any $\mu > \lambda_K$, there is a unique minimizer $C_\mu$ of $(P)_\mu$, the function $\mu \to C_\mu$ is increasing and continuous and $C_\mu \to K$ as $\mu \to \lambda_K+$. Moreover, we have $C_\mu = C$ if and only if $\mu \geq \max(\lambda_C, (N-1)\|\mathbf{H}_C\|_\infty)$.*

As a consequence of Theorem 1 we will be able to say that $K$ is the Cheeger set of $C$ and $\lambda_K = h_C$. Let us refine a result proved in [2].

**Proposition 2.2** *Let $C$ be a non-trivial convex body in $I\!R^N$. Let $u_\lambda \in BV(I\!R^N) \cap L^2(I\!R^N)$ be the (unique) solution of the variational problem*

$$(\mathrm{Q})_{\lambda,C}: \qquad \min_{u \in BV(I\!R^N)\cap L^2(I\!R^N)} \left\{ \int_{I\!R^N} |Du| + \frac{\lambda}{2} \int_{I\!R^N} (u - \chi_C)^2 \, dx \right\}. \tag{13}$$

*Then $0 \leq u_\lambda \leq 1$. Let $E_s := [u_\lambda \geq s]$, $s \in (0,1]$. Then $E_s \subseteq C$, and, for any $s \in (0,1]$, $E_s$ is a minimum of $(P)_\mu$ for $\mu = \lambda(1-s)$. Moreover, each level set $E_s$ is convex and the function $u_\lambda$ restricted to $[u_\lambda > 0]$ is concave.*

**Proof.** The facts that $0 \leq u_\lambda \leq 1$ and $E_s$ is a solution of (11) with $\mu = \lambda(1-s)$ were proved in [2, Proposition 4]. The rest of assertions were proved in [2, Theorem 5], assuming that $C$ is $C^{1,1}$ and $\lambda \geq 2N(N-1)\|\mathbf{H}_C\|_\infty$. Let us extend them to the case of a general convex set and any $\lambda > 0$. First we assume that $C$ is $C^{1,1}$ and $\lambda > 0$. We follow the construction in [2, Section

5.3]. Let $K$ be the calibrable set contained in $C$ given by Theorem 2. For each $\mu \in (0,\infty)$ let $C_\mu$ be the solution of $(P)_\mu$. We take $C_\mu = \emptyset$ for any $\mu < \lambda_K$, and, by Theorem 2, we have that $C_\mu = C$ for any $\mu \geq \max(\lambda_C, (N-1)\|\mathbf{H}_C\|_\infty)$. Following the approach in [2] (see also [6,19]), using the monotonicity of $C_\mu$ and $|C \setminus \cup\{C_\mu : \mu > 0\}| = 0$, we define

$$M_C(x) := \begin{cases} -\inf\{\mu : x \in C_\mu\} & \text{if } x \in C \\ \\ 0 & \text{if } I\!R^N \setminus C. \end{cases} \tag{14}$$

Observe that $M_C(x) = -\lambda_K$ for any $x \in K$. Then, working as in the proof of [2, Theorem 17], we have that $u_\lambda(x) = (1 + \lambda M_C(x))^+ \chi_C$ for any $\lambda > 0$. Moreover, we have that $u_\lambda$ is positive and concave in $C$ for any $\lambda \geq 2N(N-1)\|\mathbf{H}_C\|_\infty$ [2, Theorem 5]. This amounts to say that $M_C(x)$ is also a concave function in $C$. Now, this implies that for any $s \in (0,1]$ and any $\lambda > 0$ the level set $[u_\lambda \geq s]$ is convex and $u_\lambda$ restricted to $[u_\lambda > 0]$ is concave.

Assume that $C$ is any bounded convex set in $I\!R^N$ and $\lambda > 0$. Let $C_n$ be bounded convex subsets of $I\!R^N$ of class $C^{1,1}$ such that $C \subseteq C_n$ and $C_n \to C$ in the Hausdorff distance (such sets exist, see for instance, [34], pp. 158-160, [5, Proposition 1.9], or Lemma 4.3 below). Let $u_{n,\lambda}$, $u_\lambda$ be the solutions of $(\mathrm{Q})_{\lambda,C}$ and $(\mathrm{Q})_{\lambda,C_n}$, respectively. We know that $0 \leq u_\lambda \leq u_{n,\lambda} \leq 1$, $u_{n,\lambda} = 0$ outside $C_n$, $u_\lambda = 0$ outside $C$, and $u_{n,\lambda} \to u_\lambda$ in $L^2(I\!R^N)$. Since the level sets $[u_{n,\lambda} \geq s]$, $s \in (0,1]$, are convex and $u_{n,\lambda}$ restricted to $[u_{n,\lambda} > 0]$ is concave, we deduce that for almost any $s \in (0,1]$ the level sets $[u_\lambda \geq s]$ are convex and $u_\lambda$ restricted to $[u_\lambda > 0]$ is concave. Hence $u_\lambda$ is continuous in $[u_\lambda > 0]$ and the level sets $[u_\lambda \geq s]$ are convex for any $s \in (0,1]$. $\square$

**Remark 2.3** Notice that, by [2, Lemma 3], we have that $u_\lambda \neq \chi_C$ for any $\lambda > 0$ and $u_\lambda \to \chi_C$ in $L^2(I\!R^N)$ as $\lambda \to \infty$.

**Proposition 2.4** *Let $C \subseteq I\!R^N$ be a non-trivial convex body. For any $\mu > h_C$, there is a unique solution $C_\mu$ of $(P)_\mu$ and the set $K := \cap_{\mu > h_C} C_\mu$ is a solution of $(P)_{h_C}$. Moreover, the sets $C_\mu$, $K$ are convex and $K$ is the maximal Cheeger set of $C$. The function $\mu \in [h_C, \infty) \to C_\mu$ is increasing, continuous and $C_\mu \to C$ as $\mu \to \infty$.*

**Proof.** Notice that the isoperimetric inequality implies that any Cheeger set has positive measure and $h_C > 0$. Let $\mu > h_C$. Let $\lambda > 0$ and $s \in (0,1)$ be such that $\mu = \lambda(1-s)$. Using Remark 2.3, we observe that, by taking $\lambda > 0$

large enough (e.g., $\lambda = 2\mu$), we may assume that $s < \|u_\lambda\|_\infty$, where $u_\lambda$ is the solution of $(Q)_{\lambda,C}$. Then, by Proposition 2.2, $[u_\lambda \geq s]$ is a solution of $(P)_\mu$. Now, if $G$ is any other solution of $(P)_\mu$, then by Lemma 2.1.(i) we have

$$[u_\lambda > s] = \cup_{\epsilon>0}[u_\lambda \geq s+\epsilon] \subseteq G \subseteq \cap_{\epsilon>0}[u_\lambda \geq s-\epsilon] = [u_\lambda \geq s]. \tag{15}$$

Since $u_\lambda$ is concave in $[u_\lambda > 0]$ and $s < \|u_\lambda\|_\infty$, we have that $G = [u_\lambda > s] = [u_\lambda \geq s]$ modulo a null set. Thus, the solution of $(P)_\mu$ is unique and convex.

By Lemma 2.1.(ii), the set $K = \cap_{\mu > h_C} C_\mu$ is a convex solution of $(P)_{h_C}$. Notice that $P(K) - h_C|K| \leq P(K') - h_C|K'| = 0$. Hence $K$ is a Cheeger set. Notice that, by Lemma 2.1.(i), any Cheeger set is contained in $K$.

The construction of $K$ together with the concavity of $u_\lambda$ in $[u_\lambda > 0]$ prove that the map $\mu \in [h_C, \infty) \to C_\mu$ is continuous. By Remark 2.3, we know that $u_\lambda \to \chi_C$ as $\lambda \to \infty$, and this implies that $C_\mu \to C$ as $\mu \to \infty$. □

**Remark 2.5** Thanks to Proposition 2.4, we may repeat the construction of $M_C(x)$ in the proof of Proposition 2.2 to conclude that $u_\lambda(x) = (1 + \lambda M_C(x))^+ \chi_C$ is the solution of $(\mathrm{Q})_{\lambda,C}$ for any $\lambda > 0$. Moreover, the set $[u_\lambda = \|u_\lambda\|_\infty] = K$.

**Remark 2.6** As in [2, Theorem 11], we can prove that for any $V \in [|K|, |C|]$ there is a unique solution of the isoperimetric problem with fixed volume

$$\min_{F \subseteq C, |F| = V} P(F). \tag{16}$$

Moreover, this solution is convex.

**Proposition 2.7** *Using the notation of Proposition 2.4, the maximal Cheeger set $K$ is $C^{1,1}$.*

**Proof.** Since $K$ is a solution of $(P)_{h_C}$, classical computations (see, for instance, [37]) prove that $0 \leq \mathbf{H}_K \leq h_C$. Since $K$ is convex, it follows that $K$ is $C^{1,1}$ (see, for instance, [5, Proposition 1.3] for a more general statement). □

**Remark 2.8** As we proved in [2], as a consequence of Theorem 2, if $C \subseteq I\!R^N$ is non-trivial convex body of class $C^{1,1}$, then $C$ is calibrable if and only if $(N-1)\operatorname{ess\,sup}_{x\in\partial C}\mathbf{H}_C(x) \leq \lambda_C$. Notice that Proposition 2.7 implies that if $C$ is non-trivial convex body in $I\!R^N$, then $C$ is calibrable if and only if $C$ is of class $C^{1,1}$ and $(N-1)\operatorname{ess\,sup}_{x\in\partial C}\mathbf{H}_C(x) \leq \lambda_C$.

## 3 Some linear algebra inequalities

We begin with some classical inequalities inside the cone of symmetric positive definite matrices. Though they can be found in the appendix of [4], we slightly strengthen them and we include its proof here for the sake of completeness.

Let us denote by $S_N^{++}(\mathbb{R})$ the set of real symmetric positive definite matrices of size $N \times N$.

**Proposition 3.1** *The map $A \mapsto A^{-1}$ is strictly convex in $S_N^{++}(\mathbb{R})$, i.e. $\forall A, B \in S_N^{++}(\mathbb{R}), A \neq B, \forall \lambda \in (0,1)$, we have*

$$(\lambda A + (1-\lambda)B)^{-1} - \lambda A^{-1} - (1-\lambda)B^{-1} \in S_N^{++}(\mathbb{R}). \tag{17}$$

**Proof.** From a classical result on the simultaneous diagonalization of two quadratic forms [16], we know that there exists an invertible matrix $P$ and a diagonal matrix $D = \mathrm{diag}(d_i)_{i\in\{1,\ldots,N\}}$ such that $A = {}^tPP$ and $B = {}^tPDP$, where ${}^tP$ denotes the tranpose of $P$. Using this, we can write

$$\begin{aligned} &(\lambda A + (1-\lambda)B)^{-1} - \lambda A^{-1} - (1-\lambda)B^{-1} \\ =& P^{-1}\left((\lambda I_N + (1-\lambda)D)^{-1} - \lambda I_N - (1-\lambda)D^{-1}\right)({}^tP)^{-1} \end{aligned}$$

where $I_N$ denotes the $N \times N$ identity matrix. Now, the result follows by observing that, since $x \mapsto \frac{1}{x}$ is strictly convex for $x > 0$, each diagonal element of $(\lambda I_N + (1-\lambda)D)^{-1} - \lambda I_N - (1-\lambda)D^{-1}$ is positive. □

Since $\mathrm{Tr}(A) > 0$ for any $A \in S_N^{++}(\mathbb{R})$, we get the following useful consequence.

**Corollary 3.2** *$A \mapsto \mathrm{Tr}(A^{-1})$ is strictly convex in $S_N^{++}(\mathbb{R})$.*

**Proposition 3.3** *Let $A$ and $B \in \mathcal{S}_N^{++}(\mathbb{R})$. Then*

$$\frac{1}{\mathrm{Tr}((A+B)^{-1})} \geq \frac{1}{\mathrm{Tr}(A^{-1})} + \frac{1}{\mathrm{Tr}(B^{-1})}. \tag{18}$$

*Moreover, the equality holds if and only if $A$ and $B$ are homothetic, i.e., if it exists $\lambda > 0$ with $A = \lambda B$.*

**Proof.** Observe that we can rewrite the inequality (18) as

$$\mathrm{Tr}((A+B)^{-1})\mathrm{Tr}(A^{-1}+B^{-1}) - \mathrm{Tr}(A^{-1})\mathrm{Tr}(B^{-1}) \leq 0. \tag{19}$$

Let $P$ and $D$ be as in the proof of Proposition 3.1. We may write

$$\begin{aligned}\mathrm{Tr}((A+B)^{-1}) &= \mathrm{Tr}(({}^tP(I_N+D)P)^{-1}) = \mathrm{Tr}((I_N+D)^{-1}({}^tP)^{-1}P^{-1}),\\ \mathrm{Tr}(A^{-1}) &= \mathrm{Tr}(P^{-1}({}^tP)^{-1}) = \mathrm{Tr}(({}^tP)^{-1}P^{-1}),\\ \mathrm{Tr}(B^{-1}) &= \mathrm{Tr}(P^{-1}D^{-1}({}^tP)^{-1}) = \mathrm{Tr}(D^{-1}({}^tP)^{-1}P^{-1}).\end{aligned}$$

Let us write $\widetilde{C} = (c_{ij})_{i,j=1}^N := ({}^tP)^{-1}P^{-1} \in \mathcal{S}_N^{++}(\mathbb{R})$. Using the above identities, proving (19) is equivalent to prove that

$$\mathrm{Tr}((I_n+D)^{-1}\widetilde{C})\mathrm{Tr}(C+D^{-1}\widetilde{C}) - \mathrm{Tr}(\widetilde{C})\mathrm{Tr}(D^{-1}\widetilde{C}) \leq 0. \tag{20}$$

Since $c_{ii} > 0$ for all $i = 1, \dots, N$, the result follows from the following elementary computations

$$\begin{aligned}&\mathrm{Tr}((I_N+D)^{-1}\widetilde{C})\mathrm{Tr}(\widetilde{C}+D^{-1}\widetilde{C}) - \mathrm{Tr}(\widetilde{C})\mathrm{Tr}(D^{-1}\widetilde{C})\\ &= \sum_{i=1}^N \frac{c_{ii}}{1+d_i}\sum_{j=1}^N c_{jj}(1+\frac{1}{d_j}) - \sum_{i=1}^N c_{ii}\sum_{j=1}^N \frac{c_{jj}}{d_j}\\ &= \sum_{i=1}^N\sum_{j=1}^N c_{ii}c_{jj}\frac{d_j-d_i}{d_j(1+d_i)} = \sum_{1\leq i<j\leq N} c_{ii}c_{jj}\left(\frac{(d_i-d_j)(d_j(1+d_i)-d_i(1+d_j))}{d_id_j(1+d_i)(1+d_j)}\right)\\ &= -\sum_{1\leq i<j\leq N} c_{ii}c_{jj}\frac{(d_i-d_j)^2}{d_id_j(1+d_i)(1+d_j)} \leq 0\,.\end{aligned}$$

Observe that the last inequality becomes an equality if and only if $D = d_1 I_N$, that is, when $A$ and $B$ are homothetic. □

## 4 Some convexity properties of the mean curvature

In this section, we apply the inequalities proved in last Section to study the behavior of the mean curvature of the boundary of the convex combination of two smooth convex or strictly convex sets.

We denote by $X \oplus Y$ the Minkowski's addition of two convex sets $X, Y \subseteq \mathbb{R}^N$, i.e., $X \oplus Y := \{x + y : x \in X,\ y \in Y\}$.

In this Section $K$ and $L$ will be two non-empty open bounded convex sets in $\mathbb{R}^N$. For all $t \in [0,1]$, let

$$K_t := (1-t)K \oplus tL = \{(1-t)x + ty : (x,y) \in K \times L\}.$$

Notice that $K_t$ is also an open bounded convex set.

**Lemma 4.1** *Assume that $\nu \in S^{N-1}$ is a normal to $\partial K$ at $x$ and to $\partial L$ at $y$, and let $x_t = (1-t)x + ty$. Then $x_t \in \partial K_t$ and $\nu$ is normal to $\partial K_t$ at $x_t$.*

**Proof.** Recall that $\nu$ is normal to $\partial K_t$ at $x_t$ if $K_t \subset \Pi^-_{x_t,\nu} := \{z \in I\!R^N : \langle z, \nu\rangle < \langle x_t, \nu\rangle\}$ with $x_t \in \overline{K_t}$. Observe that, since $x \in \overline{K}$ and $y \in \overline{K}$, by continuity of the addition we have $x_t \in \overline{K_t}$. Now, as $\nu$ is normal to $\partial K$ at $x$ and to $\partial L$ at $y$, we have that $K \subset \Pi^-_{x,\nu}$ and $L \subset \Pi^-_{y,\nu}$, where $\Pi^-_{x,\nu}$, $\Pi^-_{y,\nu}$ denote the corresponding half-spaces. It follows that $K_t = (1-t)K \oplus tL \subset (1-t)\Pi^-_{x,\nu} \oplus t\Pi^-_{y,\nu} = \Pi^-_{x_t,\nu}$. $\square$

When $K$ is of class $C^1$, we denote by $\nu^K(x)$ the outer unit normal to $x \in \partial K$, so that $\nu^K : \partial K \to S^{N-1}$ is the spherical image map. Assume that $K$ is a convex body of class $C^1$. Let $\Omega$ be an open set of $\partial K$ with the relative topology. We say that $\Omega$ is $C^k_+$ $(2 \leq k \leq \infty)$ if $\Omega$ is a manifold of class $C^k$ and $\nu^K|_\Omega$ is a diffeomorphism (onto $\nu^K(\Omega)$) [34, Section 2.5]. The last assertion is equivalent to the assumption that the principal curvatures at points $x \in \Omega$ (which are the eigenvalues of the differential of $\nu^K$ at $x$) are all positive [34, Section 2.5]. We say that $K$ is of class $C^k_+$ if $\partial K$ is of class $C^k_+$. We say that $K$ is $C^k_+$ near $x \in \partial K$ if there is a neighborhood of $\Omega_x \subseteq \partial K$ of $x$ which is $C^k_+$. Instead of saying that $\Omega$ is $C^k_+$, some authors say that $\Omega$ of class $C^k$ and strictly convex [24,13]. When convenient, we also use this terminology here.

The following result is an application of the linear algebra inequalities of the previous section.

**Theorem 3** *Suppose that $K$ and $L$ are $C^2_+$ near $x$ and $y$, respectively, and $\nu \in S^{N-1}$ is normal to $\partial K$ at $x$ and to $\partial L$ at $y$. Let $x_t = (1-t)x + ty$. Then $K_t$ is $C^2_+$ near $x_t$ and the functions $t \in [0,1] \to \mathbf{H}_{K_t}(x_t) \in (0,\infty)$ and $t \in [0,1] \to \frac{1}{\mathbf{H}_{K_t}(x_t)}$ are convex and concave in $t$, respectively.*

**Proof.** Recall that the support function of a convex body $B \subset I\!R^N$ is defined by $h_B(u) = \sup_{x\in B}\langle x, u\rangle$, $\forall u \in I\!R^N$. It is a sublinear function in $u$ and is additive with respect to the Minkowski sum (in particular, we have $h_{K_t} = (1-t)h_K + th_L$) [34]. It is also well-known that if the convex body $B$ is smooth, the eigenvalues of its Hessian matrix at $\nu^B(x)$ are 0 (with eigenvector $\nu^B(x)$) and the principal radii of curvature $r_1, \ldots, r_{N-1}$ of $\partial B$ at $x$ [34, Corollary 2.5.2, p. 109].

First, we observe that our assumptions imply that $K_t$ remains $C^2_+$ near $x_t$

because this property is equivalent to have a $C^2$ support function with bounded positive radii of curvature locally around $x_t$.

Let $\nu = \nu^K(x)$ and let $(e_1, ..., e_{N-1}, \nu)$ be an orthonormal basis of $\mathbb{R}^N$. Let $A, B$ be the Hessian matrices of $h_K$ and $h_L$ restricted to $\nu^\perp$, i.e.,

$$A = \left(\frac{\partial^2 h_K(\nu)}{\partial x_i \partial x_j}\right)_{1\le i,j\le N-1} \text{ and } B = \left(\frac{\partial^2 h_L(\nu)}{\partial x_i \partial x_j}\right)_{1\le i,j\le N-1}.$$

Then $A, B \in \mathcal{S}^{++}_{N-1}(\mathbb{R})$ because all radii of curvature are positive. The mean-curvature $\mathbf{H}_{K_t}(x_t)$ is given by

$$\mathbf{H}_{K_t}(x_t) = \frac{\text{Tr}(((1-t)A + tB)^{-1})}{N-1}.$$

Now, Corollary 3.2 shows that $t \mapsto \mathbf{H}_{K_t}(x_t)$ is convex, with strict convexity if $A \neq B$, and Proposition 3.3 shows that

$$\frac{1}{\mathbf{H}_{K_t}(x_t)} \ge \frac{1-t}{\mathbf{H}_K(x)} + \frac{t}{\mathbf{H}_L(y)}. \tag{21}$$

This proves the concavity of the function $t \mapsto \mathbf{H}_{K_t}(x_t)^{-1}$. $\square$

**Corollary 4.2** *Let $K, L$ be two nonempty open bounded convex sets in $\mathbb{R}^N$ of class $C^{1,1}$. Then $K_t$ is $C^{1,1}$ and, if $H(t) = \operatorname{ess\,sup}_{x\in\partial K_t} \mathbf{H}_{K_t}(x)$, then the functions $t \in [0,1] \mapsto H(t)$ and $t \in [0,1] \mapsto \frac{1}{H(t)}$ are convex and concave, respectively.*

**Proof.** If $K$ and $L$ are $C^2_+$, this is a straightforward consequence of the previous theorem as the supremum of convex functions is convex, and the infimum of concave functions is also concave.

The general case is a consequence of the previous case and the following convergence and approximation result concerning $C^{1,1}$ convex sets.

**Lemma 4.3** $(i)$ *Convergence: If $(K_n)_{n\in\mathbb{N}}$ a sequence of $C^{1,1}$ convex bodies in $\mathbb{R}^N$ with $\operatorname{ess\,sup}_{x\in\partial K_n} \mathbf{H}_{K_n}(x) \le H$ for all $n \in \mathbb{N}$, and $K_n \to K$ in the Hausdorff sense, then $K$ is $C^{1,1}$ and $\operatorname{ess\,sup}_{x\in\partial K} \mathbf{H}_K(x) \le H$.*

$(ii)$ *Approximation: Let $K$ be a $C^{1,1}$ convex body in $\mathbb{R}^N$ with $\operatorname{ess\,sup}_{x\in\partial K} \mathbf{H}_K(x) \le H$. Then there exists a sequence $K_n \in C^2_+$ with $K_n \to K$ in the Hausdorff sense and $\max_{x\in\partial K_n} \mathbf{H}_{K_n}(x) \le H_n$ with $H_n \to H$.*

**Proof.** $(i)$ It is a straightforward application of Blaschke's Rolling Theorem [34, Theorem 3.2.9] extended in [5, Corollary 1.13] to the case of $C^{1,1}$ convex sets (see also [8] where such an extension is derived in the general context of smooth anisotropic norms).

Observe that at almost any point (with respect to $\mathcal{H}^{N-1}$) of $\partial K_n$, the principal curvatures are bounded by $(N-1)H$, because $\operatorname{ess\,sup}_{x\in\partial K_n} \mathbf{H}_{K_n}(x) \leq H$. Using [5, Corollary 1.13], we deduce that a ball $B(r)$ of radius $r = \frac{1}{(N-1)H} > 0$ "rolls freely" inside $K_n$, i.e., there exists a convex body $K'_n$ such that $K_n = K'_n \oplus B(r)$. In particular, we have $h_{K_n} = h_{K'_n} + h_{B(r)}$.

Notice that $h_{K'_n} = h_{K_n} - h_{B(r)}$ are sublinear convex functions uniformly convergent to $h_K - h_{B(r)}$, because $h_{K_n}$ converges uniformly to $h_K$. We deduce that $h_K - h_{B(r)}$ is a sublinear convex function, so there exists a convex body $K'$ in $I\!R^N$ such that $h_{K'} = h_K - h_{B(r)}$ [34]. We deduce that $K = K' \oplus B(r)$ and, therefore, $K$ is is of class $C^{1,1}$.

The fact that the mean curvature remains bounded above by $H$ is a consequence of the well-known property that the curvature measures of $K_n$ weakly converge to the curvature measures of $K$ [34].

$(ii)$ We approximate $K$ by $K(t)$ where $K(t)$ is the motion by mean curvature of $K$ at time $t > 0$. By the results in [14,15], for any initial convex set, and in particular for $K$, there is a generalized motion by mean curvature $K(t)$ such that $K(t) \to K$ as $t \to 0^+$ in the Hausdorff sense. Moreover, there exists $T > 0$ such that $K(t)$ is smooth ($C^\infty$) for any $t \in (0, T]$ and satisfies

$$X_t = -\mathbf{H}_{K(t)} \nu^{K(t)} ,$$

where $X$ is a parameterization of $K(t)$ and $\nu^{K(t)}$ is the outer unit normal to $K(t)$.

Now, the results in [8] for smooth anisotropies prove that if $K$ is $C^{1,1}$, then the flow $t \in [0, T] \to K(t)$ can be approximated using the Almgren-Taylor-Wang iterative scheme [1,8]: for any $h > 0$ and any $i \in I\!N \cup \{0\}$ we define

$$K_h^{i+1} := \operatorname{argmin}_F \left\{ P(F) + \frac{1}{h} \int_{F \Delta K_h^i} d(x, \partial K_h^i)\, dx \right\},$$

where the minimum is taken over all subsets $F$ of $I\!R^N$ with finite perimeter, $d(x, \partial K_h^i)$ denotes the distance from $x$ to $\partial K_h^i$, and $K_h^0 = K$. Moreover, there is some $T > 0$ such that we have a uniform bound for the mean curvature of $K_h^i$ as long as $ih \leq T$ [8, Theorem 8]. Then we have that there exists the limit $\lim_{h\to 0+} K_h^{[t/h]}$ (where $[t/h]$ denotes the integer part of $t/h$) and coincides with $K(t)$ for any $t \in [0,T]$, hence $K(t)$ is also $C^{1,1}$ and there is a uniform bound for the mean curvature of $K(t)$ in $[0,T]$ [8, Theorem 1]. If we define $\mathcal{M}_{h,i} := \|\mathbf{H}_{K_h^i}\|_{L^\infty(\partial K_h^i, \mathcal{H}^{N-1})}$, then by equation (59) in [8] we have that

$$\mathcal{M}_{h,i+1} \leq \mathcal{M}_{h,i}(1+Qh) \tag{22}$$

for some constant $Q > 0$ as long as $ih \leq T$. Using the above convergence results and passing to the limit in (22) we obtain

$$\|\mathbf{H}_{K(t)}\|_\infty \leq \|\mathbf{H}_K\|_\infty e^{Qt}.$$

Then

$$\|\mathbf{H}_K\|_\infty \leq \liminf_{t\to 0+} \|\mathbf{H}_{K(t)}\|_\infty \leq \limsup_{t\to 0+} \|\mathbf{H}_{K(t)}\|_\infty \leq \lim_{t\to 0+} \|\mathbf{H}_K\|_\infty e^{Qt} = \|\mathbf{H}_K\|_\infty.$$

Finally, we observe that, by the results in [24,13], the sets $K(t)$ are $C^2_+$. □

**Remark 4.4** We have derived the approximation Lemma 4.3.$(ii)$ as a consequence of the estimates in [8] though it could also be derived with some additional work from the estimates in [24].

In the statement of next theorem we use the notation of Theorem 3.

**Proposition 4.5** *Let $\Omega \subset \partial K$, $\Omega' \subset \partial L$ be open, connected, $C^2_+$, and suppose that $\Omega' = \nu_L^{-1} \circ \nu_K(\Omega)$. Assume that*

*(i) $\mathbf{H}_K(x) = \mathbf{H}_L(\nu_L^{-1} \circ \nu_K(x))$, $\forall x \in \Omega$,*

*(ii) for any $x \in \Omega$, the function $t \in (0,1) \to \mathbf{H}_{K_t}(x_t)$ is not strictly convex, where $x_t = tx + (1-t)y$, $y = \nu_L^{-1} \circ \nu_K(x)$.*

*Then $\Omega'$ is a translate of $\Omega$, i.e. there exists $z \in I\!R^N$ with $\Omega' = z + \Omega$. The same result holds if, instead of $(ii)$, we assume that for any $x \in \Omega$ the function $t \in [0,1] \to 1/\mathbf{H}_{K_t}(x_t)$ is not strictly concave.*

**Proof.** Let $x \in \Omega$, $y = \nu_L^{-1} \circ \nu_K(x)$, and $x_t = tx + (1-t)y$, $t \in (0,1)$. We use the same notation as in the proof of Theorem 3. By Corollary 3.2, the equality $\mathbf{H}_{K_t}(x_t) = t\mathbf{H}_K(x) + (1-t)\mathbf{H}_L(y)$ only holds if $A = B$. On the other hand, by Proposition 3.3, the equality in (21) arises if and only if there is $\lambda > 0$ such that $A = \lambda B$. Since $\mathrm{Tr}(A^{-1}) = \mathbf{H}_K(x) = \mathbf{H}_L(y) = \mathrm{Tr}(B^{-1})$, the equality in (21) arises if and only if $A = B$.

Thus, we have that $d^2h_K(\nu) = d^2h_L(\nu)$, $\forall \nu \in \nu_K(\Omega) = \nu_L(\Omega')$. As $h_K$ and $h_L$ are positively homogeneous (of degree 1), this equation extends to a neighborhood $U \subset I\!R^N$ of $\nu_K(\Omega)$ which can be chosen connected because $\nu_K(\Omega)$ is connected. This shows that there exist $z \in I\!R^N$ and $\alpha \in I\!R$ such that

$$h_L(u) = h_K(u) + \langle z, u \rangle + \alpha, \quad \forall u \in U.$$

Since $h_K(0) = h_L(0) = 0$, we deduce that $\alpha = 0$. As the support function describes the convex set locally, we get that $\Omega' = z + \Omega$. □

## 5 Strict convexity of the free boundary of an isoperimetric region

In order to prove Proposition 5.2 we state without proof the following known result about convex sets.

**Lemma 5.1** *Let $K \subseteq I\!R^N$ be a convex set. Let $x, y \in \partial K$ and $\nu \in S^{N-1}$ be such that $\nu$ is normal to $\partial K$ at $x, y$. Then the segment $[x, y] \subseteq \partial K$ and $\nu$ is also normal to $\partial K$ at the points of $[x, y]$.*

**Proposition 5.2** *Let $K$ be a non-trivial convex body of class $C^1$, and $C \subset K$ be an isoperimetric region inside $K$. Assume that $C$ is convex. Then $\partial C \backslash \partial K$ is $C^\infty_+$.*

We say that $C \subset K$ is an isoperimetric region inside $K$ if $C$ minimizes the perimeter with a volume constraint among all sets contained in $K$ which satisfy the constraint.

**Proof.** As $C$ is an isoperimetric region inside $K$, we know that the set $\Sigma = \partial C \backslash \partial K$ satisfies [20,21,38,18]:

(1) There is a closed singular set $\Sigma_s \subset \Sigma$ of Hausdorff dimension less than or equal to $N-8$ such that $\Sigma_r = \Sigma \backslash \Sigma_s$ is a smooth embedded hypersurface;
(2) $\partial C$ is of class $C^1$ on a neighborhood of $\partial K \cap \partial C$;
(3) At every point $x \in \Sigma_s$, there is a tangent minimal cone $C_x$ different from a hyperplane. The square sum $|\sigma|^2 = k_1^2 + \ldots + k_{N-1}^2$ of the principal curvatures of $\Sigma$ tends to $\infty$ when we approach $x$ from $\Sigma_r$;
(4) $\Sigma_r$ has constant mean curvature with respect to the inner normal.

In our case, as $C$ is a convex set, the tangent minimal cone is included in a half-space, and the only kind of such a minimal cone is the hyperplane [35].

Hence $\Sigma_s = \emptyset$, and this implies that $\Sigma$ is a $C^\infty$ constant mean curvature surface.

In order to prove that $\Sigma$ is $C^\infty_+$, by [23, Theorem 3, p.297], we know that for constant mean curvature hypersurfaces with non-negative sectional curvatures, its Gaussian curvature $\mathbf{K}$ satisfies a strong minimum principle. When applied to $\Sigma_\epsilon := \{x \in \Sigma : \mathrm{dist}(x, \partial K) \geq \epsilon\}$, we have

$$\min_{x\in\Sigma_\epsilon} \mathbf{K}_C(x) = \min_{x\in\partial\Sigma_\epsilon} \mathbf{K}_C(x)\,,$$

where $\mathbf{K}_C$ (the Gaussian curvature of $\partial C$) has no interior minimum except if it is constant. So, if there exists $a \in \Sigma_\epsilon$ with $\mathbf{K}_C(a) = 0$, then $\mathbf{K}_C(a) = 0$ $\forall a \in \Sigma_\epsilon$, hence $\Sigma_\epsilon$ is part of a cylinder. Thus, either the statement of this theorem is true, or $\Sigma$ is part of a cylinder. The last possibility cannot happen. Indeed, let $L$ be a maximal segment contained in $\Sigma$. Notice that its extrema points, call them $x, y$, are in $\partial C \cap \partial K$. Since $\nu^K(x) = \nu^K(y)$, by Lemma 5.1, we deduce that $L \subset \partial K$. This is a contradiction, since $L \subset \partial C \setminus \partial K$. Hence, $\Sigma$ is $C^\infty_+$. $\square$

## 6 Uniqueness of the Cheeger set inside a $C^{1,1}$ convex body

In this section, we prove the following result, which (in view of Proposition 2.7) implies Theorem 1.

**Theorem 4** *Let $C$ a $C^{1,1}$ convex body in $\mathbb{R}^N$. Then we have a unique Cheeger set inside $C$.*

Let $C$ be a convex body in $\mathbb{R}^N$ of class $C^{1,1}$. By the results in [11] we know that there exist two convex sets $C_*$ and $C^*$ which are the minimal and maximal (with respect to inclusion) Cheeger sets of $C$. Both are solutions of $\min_{E\subseteq C} P(E) - h_C|E|$ [2,11]. Thus, we know that $(N-1)\mathbf{H}_{C_*}, (N-1)\mathbf{H}_{C^*} \leq h_C$ ($\mathcal{H}^{N-1}$ a.e. on its respective boundaries) with equality inside $C$ (see Proposition 2.7). Since they are convex, we have that they are of class $C^{1,1}$. The uniqueness of Cheeger sets inside $C$ is implied if we prove that $C_* = C^*$. This was done in [11] when $C$ is of class $C^2$ and strictly convex (we used the terminology uniformly convex there). We are going to remove both assumptions.

Thus, in the rest of this section, we suppose that $C^* \neq C_*$, and write $h_C = \frac{P(C_*)}{|C_*|} = \frac{P(C^*)}{|C^*|}$, the Cheeger constant.

**Proposition 6.1** *For any $t \in [0,1]$, $C_t := (1-t)C_* \oplus tC^*$ is a Cheeger set.*

**Proof.** As $C^*$ and $C_*$ are $C^{1,1}$ convex Cheeger sets of $C$ with

$$\operatorname*{ess\,sup}_{x\in\partial C^*} \mathbf{H}_{C^*}(x) \le \frac{h_C}{N-1} \quad \text{and} \quad \operatorname*{ess\,sup}_{x\in\partial C_*} \mathbf{H}_{C_*}(x) \le \frac{h_C}{N-1},$$

from Corollary 4.2, we obtain that $C_t$ is $C^{1,1}$ and

$$\operatorname*{ess\,sup}_{x\in\partial C_t} \mathbf{H}_{C_t}(x) \le \frac{h_C}{N-1}. \tag{23}$$

Observe that $h_C \le \frac{P(C_t)}{|C_t|}$, since $C_t \subset C^*$. Together with the inequality (23) and the characterization of calibrable sets proved in [2], this shows that $C_t$ is calibrable. In other words, $C_t$ minimizes

$$\min_{E\subseteq C_t} P(E) - \lambda_{C_t}|E| \qquad \text{where } \lambda_{C_t} = \tfrac{P(C_t)}{|C_t|}.$$

But $C_* \subset C_t$, and this implies that $\frac{P(C_t)}{|C_t|} \le \frac{P(C_*)}{|C_*|} = h_C$. We conclude that $C_t$ is a Cheeger set. $\Box$

**Proposition 6.2** *For any $t \in [0,1]$ the sets $C_*$ and $C_t$ are equivalent by telescoping, more precisely, $\exists \overline{z} \in I\!R^N$ such as $C_t$ is a translate of $C_* \oplus [0,t]\overline{z}$.*

**Proof.** In the context of this proof we assume that $C_*$ and $C^*$ are open sets. Since the result is obviously true for $t = 0$ (take $z = 0$) and follows for $t = 1$ by passing to the limit as $t \to 1-$, we may assume that $t \in (0,1)$.

*Step 1.* Let $\Omega$ be a connected component of $\partial C_* \setminus \partial C^*$ and let

$$\Omega_t := (\nu^{C_t})^{-1} \circ \nu^{C_*}(\Omega) \subset \partial C_t\,.$$

Then both $\Omega$ and $\Omega^t \subset \partial C_t \setminus \partial C^*$ are open, connected, and $C^2_+$. In particular, $\nu^{C_*}$ and $\nu^{C_t}$ are diffeomorphism from $\Omega$, resp. $\Omega_t$, onto $\nu^{C_*}(\Omega)$.

As $\partial C_* \setminus \partial C^*$ is an open set of $\partial C_*$, $\Omega$ is also an open set of $\partial C_*$. By Proposition 5.2 we know that $\Omega$ is $C^2_+$. This implies that $\nu^{C_*}|_\Omega$ is a diffeomorphism onto its image. Then, by definition of $\Omega_t$, we know that $\Omega_t$ is an open set. Let us prove that $\Omega_t \cap \partial C^* = \emptyset$. Indeed, if $p \in \Omega_t \cap \partial C^*$, then there is $\overline{x} \in \Omega$ such that $\nu^{C_*}(\overline{x}) = \nu^{C_t}(p) = \nu^{C^*}(p)$. Then, by Lemma 4.1, $p_t := (1-t)\overline{x} + tp \in \partial C_t$, $\nu^{C_t}(p_t) = \nu^{C_t}(p)$ and $p_t \neq p$, a contradiction. Since $\Omega_t \subset \partial C_t \setminus \partial C^*$ and $C_t$ is a Cheeger set, by Proposition 5.2, we know that $\Omega_t$ is $C^2_+$. Then $\nu^{C_t}$ is a diffeomorphism from $\Omega_t$ onto $\nu^{C_*}(\Omega)$. In particular, $\Omega_t$ is connected.

Before going into Step 2, observe that if $x \in \Omega$, $y \in \partial C^*$ are such that $\nu^{C_*}(x) = \nu^{C^*}(y)$ and $x_t := (1-t)x + ty$, then, by Lemma 4.1, $x_t \in \partial C_t \setminus \partial C^*$ and $\nu^{C_t}(x_t) = \nu^{C_*}(x)$. Thus $x_t \in \Omega_t$.

*Step 2.* Let us prove that there exists $z \in I\!R^N$, $z \neq 0$, such that $\Omega_t = tz + \Omega$ for all $t \in (0,1)$, and

$$\nu^{C_*}(\Omega) = S_z^+ \ , \quad \text{where } S_z^+ = \{u \in S^{N-1}, \langle u, z\rangle > 0\}. \tag{24}$$

Thus, we conclude that $\nu^{C_*}$ and $\nu^{C_t}$ are diffeomorphisms from $\Omega$ and $\Omega_t$, respectively, onto $S_z^+$.

To prove the first assertion, we observe that, by Proposition 6.1 and Step 1, $\Omega$ and $\Omega_t$ ($\forall t \in (0,1)$) satisfy $\mathbf{H}_{C_*}|_\Omega = \mathbf{H}_{C_t}|_{\Omega_t} = \frac{h_C}{N-1}$ together with the other assumptions of Proposition 4.5. Thus, for any $t \in (0,1)$ $\Omega_t$ is a translation of $\Omega$. By Step 1 and the observation previous to Step 2, we know that all $x_t \in \Omega_t$ with the same normal are collinear. This implies that there exists $z \in I\!R^N$, $z \neq 0$, with $\Omega_t = tz + \Omega$ where $z$ does not depend on $t \in (0,1)$.

To prove (24) we prove both that

$$\langle \nu^{C_*}(x), z\rangle > 0 \qquad \forall x \in \Omega, \tag{25}$$

and

$$\langle \nu^{C_*}(x), z\rangle = 0 \qquad \forall x \in \partial_{\partial C_*}\Omega. \tag{26}$$

To prove (25), observe that for any $x \in \Omega$, writing $x_t := x + tz \in \partial C_t$ and knowing that $C_t$ is $C^2_+$ near $x_t$, we obtain

$$\langle \nu^{C_*}(x), z\rangle = \langle \nu^{C_t}(x_t), z\rangle = \langle \nu^{C_t}(x_t), \frac{x_t - x}{t}\rangle > 0\,. \tag{27}$$

To prove (26), let $x \in \partial_{\partial C_*}\Omega$. By approximating $x$ by points inside $\Omega$ and using (25) we have that $\langle \nu^{C_*}(x), z\rangle \geq 0$. On the other hand, $x \in \partial C^*$ and, by letting $t \to 1-$ in $x_t = x + tz \in \partial C_t$, we also have that $x + z \in \partial C^*$. This implies that

$$\langle \nu^{C_*}(x), z\rangle = \langle \nu^{C^*}(x), x + z - x\rangle \leq 0.$$

Now we observe that (25) and (26) can be written respectively as $\nu^{C_*}(\Omega) \subseteq S_z^+$ and $\nu^{C_*}(\partial_{\partial C_*}\Omega) \subseteq S_z^0 := \{u \in S^{N-1} : u \perp z\}$. On one hand, we know that

$\nu^{C_*}(\Omega)$ is open in $S_z^+$. On the other hand, since $\overline{\nu^{C_*}(\Omega)} = \nu^{C_*}(\overline{\Omega})$, we also have that $\nu^{C_*}(\Omega)$ is closed in $S_z^+$. Indeed

$$\overline{\nu^{C_*}(\Omega)} \cap S_z^+ = \nu^{C_*}(\overline{\Omega}) \cap S_z^+ = \Big(\nu^{C_*}(\Omega) \cap S_z^+\Big) \cup \Big(\nu^{C_*}(\partial_{\partial C_*}\Omega) \cap S_z^+\Big) = \nu^{C_*}(\Omega) \cap S_z^+.$$

Since $\nu^{C_*}(\Omega)$ is nonempty, open and closed in $S_z^+$, we have (24).

*Step 3. Conclusion.* If $\Omega$ is the only connected component of $\partial C_* \backslash \partial C^*$, then the equality (24) implies that $C_t = C_* \oplus [0,1]tz$. In this case, we take $\overline{z} = z$. If $\Omega'$ is another connected component of $\partial C_* \backslash \partial C^*$, by applying Step 2 we know that there exists $z' \in I\!R^N$, $z' \neq 0$, such that $\nu^{C_*}$ and $\nu^{C_t}$ are diffeomorphisms from $\Omega'$ and $\Omega'_t := (\nu^{C_t})^{-1}(\nu^{C_*}(\Omega'))$, respectively, onto $S_{z'}^+$. Moreover $\Omega'_t = \Omega' + tz'$. Notice that, since $\Omega \cap \Omega' = \emptyset$ and $\nu^{C_*}$ is a diffeomorphism when restricted to $\Omega$ and $\Omega'$, we have $S_z^+ \cap S_{z'}^+ = \emptyset$. This implies that there exists $\alpha > 0$ with $z' = -\alpha z$, and we deduce that $C_t = C_* \oplus [0,t]z \oplus [0,t](-\alpha z) = C_* \oplus [0,1]t(1+\alpha)z - t\alpha z$. In this case, we take $\overline{z} = (1+\alpha)z$. $\square$

**Proposition 6.3** *For all $t \geq 0$, let $C^t := C_* \oplus [0,t]\overline{z}$, $\overline{z}$ being the vector found in Proposition 6.2, i.e., such that $C_t$ is a translate of $C^t$ for any $t \in [0,1]$. Then $C^t$ is $C^{1,1}$ and calibrable with $\frac{P(C^t)}{|C^t|} = h_C$.*

From now on, we denote by $\overline{z}$ the vector found in Proposition 6.2.

**Proof.** As $C^t = C_* \oplus [0,t]z$, we know that $P(C^t)$ and $|C^t|$ are two linear functions of $t$, i.e., there exists $\alpha, \beta > 0$ such that $P(C^t) = P(C_*) + \alpha t$ and $|C^t| = |C_*| + \beta t$ [34, Theorem 6.7.1, p.379]. As $P(C^t) = P(C_t)$ and $|C^t| = |C_t|$ for any $t \in [0,1]$, and $\frac{P(C_t)}{|C_t|} = h_C$, this equality extends to all $t \geq 0$, that is, $\frac{P(C^t)}{|C^t|} = h_C$ for all $t \geq 0$.

As $C_*$ is $C^{1,1}$ and $(N-1)\operatorname*{ess\,sup}_{x \in \partial C_*} \mathbf{H}_{C_*}(x) \leq h_C$, it is straightforward to show that $C^t$ is $C^{1,1}$ and

$$(N-1)\operatorname*{ess\,sup}_{x \in \partial C^t} \mathbf{H}_{C^t}(x) \leq h_C = \frac{P(C^t)}{|C^t|}.$$

Hence, using [2, Corollary 1], we have that $C^t$ is calibrable. $\square$

**Proposition 6.4** *Let $\Pi_{\overline{z}} = \{x \in I\!R^N : \langle x, \overline{z} \rangle = 0\}$. If $D$ is the projection of $C_*$ into $\Pi_{\overline{z}}$, then $\frac{P(D)}{|D|} = h_C$.*

Observe that $D$ is a convex body in $\Pi_{\overline{z}}$.

**Proof.** If $t' > t \geq 0$, then we have $(t'-t)P(D) = P(C^{t'}) - P(C^t) = h_C(|C^{t'}| - |C^t|) = h_C(t'-t)|D|$. $\square$

By translating the sets $C$ and $C^t$, and choosing $t$ large enough we may assume that $\Pi_{\overline{z}} \cap C^t = D$, and $C_* \subset C^t \cap \Pi_{\overline{z}}^-$, where $\Pi_{\overline{z}}^- = \{x \in I\!R^N : \langle x, \overline{z}\rangle < 0\}$. Let us consider the convex body $S := (C^t \cap \Pi_{\overline{z}}^-) \cup \mathrm{Sym}_{\Pi_{\overline{z}}}(C^t \cap \Pi_{\overline{z}}^-)$ which is symmetrical and $C^{1,1}$ (we denote by $\mathrm{Sym}_{\Pi_{\overline{z}}}(C^t \cap \Pi_{\overline{z}}^-)$ the symmetrization of $C^t \cap \Pi_{\overline{z}}^-$ with respect to $\Pi_{\overline{z}}$).

**Proposition 6.5** *$S$ is calibrable, with $\frac{P(S)}{|S|} = h_C$.*

**Proof.** Notice that choosing $t'$ big enough, we can translate $C^{t'}$ to have $S \subset C^{t'}$, and we get that $\frac{P(S)}{|S|} \geq h_C$. Since we have

$$(N-1)\operatorname*{ess\,sup}_{x \in \partial S} \mathbf{H}_S(x) \leq h_C \leq \frac{P(S)}{|S|},$$

by [2, Corollary 1] we obtain that $S$ is calibrable. Since we have chosen $t$ big enough to have $C_* \subset S$, then

$$\frac{P(S)}{|S|} \leq \frac{P(C^*)}{|C^*|} = h_C.$$

Thus $\frac{P(S)}{|S|} = h_C$. $\square$

**Proof of Theorem 4.** Suppose that $C_* \neq C^*$, and let $S$ be the set defined in the paragraph before Proposition 6.5. Observe that there is a function $u : D \to I\!R$ such that we can write $S$ as

$$S = \{x + tu(x)z_0, x \in D, t \in [-1,1]\}, \qquad \text{where } z_0 = \tfrac{\overline{z}}{\|\overline{z}\|}.$$

We have

$$\frac{|S|}{2} = \int_D u,$$

and

$$\frac{P(S)}{2} = \int_D \sqrt{1 + |Du|^2} + \int_{\partial D} u.$$

At the same time, $u$ is a solution of

$$-\mathrm{div}\Big(\frac{Du}{\sqrt{1+|Du|^2}}\Big) = h_C \tag{28}$$

and the graph of $u$ is a $C^{1,1}$ hypersurface above $D$ having zero contact angle with $\partial D \times I\!R$, i.e.

$$\frac{Du}{\sqrt{1+|Du|^2}} \cdot \nu^D = -1\,, \tag{29}$$

where $\nu^D$ denotes the outer unit normal to $\partial D$.

Then we compute

$$\begin{aligned}
\frac{P(S)}{2} = h_C \frac{|S|}{2} &= -\int_D \mathrm{div}\,\Big(\frac{Du}{\sqrt{1+|Du|^2}}\Big) u \\
&= \int_D \frac{|Du|^2}{\sqrt{1+|Du|^2}} - \int_{\partial D} \left(\frac{Du}{\sqrt{1+|Du|^2}} \cdot \nu^D\right) u \\
&= \int_D \frac{|Du|^2}{\sqrt{1+|Du|^2}} + \int_{\partial D} u \\
&< \int_D \sqrt{1+|Du|^2} + \int_{\partial D} u = \frac{P(S)}{2}\,,
\end{aligned}$$

and we obtain a contradiction. Our statement is proved. □

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