LOWER SEMICONTINUITY IN SBV FOR INTEGRALS WITH VARIABLE GROWTH

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Abstract. We prove a lower semicontinuity result for free discontinuity energies with a quasiconvex volume term having non standard growth and a surface term.

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1. Introduction

In the last years models involving bulk and interfacial energies have been used to describe phenomena in fracture mechanics, phase transitions, image segmentation and static theory of liquid crystals (see [8], [11], [12], [17], [24], [26], [33], [34]). The problem consists in finding minima of an energy functional of this kind

$$\mathcal{F}(w) := \int_\Omega f(x, w, \nabla w)\, dx + \int_{J_w} \varphi(x, w^+, w^-, \nu_w)\, d\mathcal{H}^{N-1}, \tag{1.1}$$

where Ω is an open set of $\mathbb{R}^N$, $\mathcal{H}^{N-1}$ denotes the Hausdorff measure of dimension $(N-1)$, w belongs to the space of special functions of bounded variation denoted by $\mathrm{SBV}(\Omega, \mathbb{R}^m)$, J_w denotes the set of approximate jump points of w, and the distributional derivative Dw is represented by $Dw = \nabla w \mathcal{L}^N + (w^+ - w^-) \otimes \nu_w \mathcal{H}^{N-1} \lfloor J_w$, with ν_w being the normal to J_w and w^+ and w^- the so called upper and lower approximate limit of w at the point $x \in \Omega$.

The existence of minima can be proved by using the direct methods in the calculus of variations. Under appropriate boundedness constraints, Ambrosio in [3] (see also [6]) proved a compactness theorem in SBV, which combined with lower semicontinuity results guarantees the existence of minima. In [4] the author studied the lower semicontinuity of the functional (1.1) when the function f is convex and satisfies p-growth condition, while the function φ fulfills suitable conditions, and in [5] he extended this result under quasiconvexity assumption on the function f (see also [20]). Earlier works (see [18] and [2]) have addressed other lower semicontinuity results under different assumptions on the bulk and on the surface energy.

In this paper we present a lower semicontinuity result for free discontinuity energies with a quasiconvex volume term having variable exponent growth and a quite general surface term.

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During the last decade, function spaces with variable exponent have attracted a lot of interest. In fact, apart from interesting theoretical considerations, this framework occurs in various variational problems from mathematical physics, in particular in electrorheological fluids (see [28] [29], [30]) and in the theory of homogenization (see [35]). More recently, Chen, Levine, and Rao in [9] proposed a variable exponent formulation for the problem of image restoration (see also [19]).

A survey of the history of Lebesgue and Sobolev variable exponent spaces with a comprehensive bibliography is provided in [14] and [31].

In this paper we consider a free discontinuity energy of the type

$$F(u) = \int_{\Omega} f(x, w, \nabla w) dx + \int_{\Omega \cap J_w} \gamma(|w^+ - w^-|) k(x, \nu_w)\, d\mathcal{H}^{N-1}, \tag{1.2}$$

where Ω is a bounded open set of $\mathbb{R}^N$, $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{Nm} \to \mathbb{R}$, $\gamma : [0, +\infty) \to [0, +\infty)$ and $k : \Omega \times \mathbb{R}^N \to [0, +\infty)$. We prove a lower semicontinuity result for the functional above with respect to the L^1-convergence under variable exponent growth assumption on f. In order to prove this l.s.c result it will be sufficient to concentrate on the bulk energy term since the l.s.c. of the surface term is essentially addressed in [2] (see Theorem 3.3). Thus the main result will be the following theorem.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $p : \Omega \to [1, +\infty)$ be a measurable bounded function such that $1 < p^- \le p(x) \le p^+ < +\infty$ in Ω; moreover p is (locally) log-Hölder continuous (see definition in Sect. 2.4 below).*
Let $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{Nm} \to [0, +\infty)$ be a Carathéodory function such that

$$-c + |z|^{p(x)} \le f(x, u, z) \le a(x) + \Psi(|u|)(1 + |z|^{p(x)})\ \forall (x, u, z) \in \Omega \times \mathbb{R}^m \times \mathbb{R}^{Nm} \tag{1.3}$$

for some $c > 0$, $a \in L^1(\Omega)$, and some continuous function $\Psi : [0, \infty) \to [0, \infty)$. Let us assume that for every $(x, u) \in \Omega \times \mathbb{R}^m$ the function $z \mapsto f(x, u, z)$ is quasiconvex. Then

$$\liminf_{n \to \infty} \int_{\Omega} f(x, w^n, \nabla w^n)\, dx \ge \int_{\Omega} f(x, w, \nabla w)\, dx$$

for every sequence $\{w^n\} \subset \mathrm{SBV}(\Omega, \mathbb{R}^m)$ converging in $L^1(\Omega, \mathbb{R}^m)$ to a function $w \in \mathrm{SBV}(\Omega, \mathbb{R}^m)$ and satisfying $\sup_{n \in \mathbb{N}} \mathcal{H}^{N-1}(J_{w^n}) < +\infty$.

This result extends a well known l.s.c. theorem due to Ambrosio (see [5] or Theorem 5.29 in [7]) in this new framework of special function of bounded variation with variable exponent.

Let us also observe that in the Sobolev setting with variable exponent, the problem was considered in [23].

The main ingredients in order to prove the result are essentially the following: blow up argument and Lipschitz truncation Lemma.

The blow up argument is useful in order to treat firstly the case where w is a linear function and the measure of the jump set of w^n goes to zero. In this case the idea is to replace w^n by equi-Lipschitz functions $w^{n,j}$ which agree with w^n on large sets. In the Sobolev setting this Lipschitz truncation Lemma is proved in [22] and [1], and then generalized by Ambrosio in [5] to the SBV context. More recently, Diening, Malek and Steinhauer in [15] extended the Lipschitz truncation method to Sobolev functions of variable exponent. Following the ideas of [15], and using the maximal function of the total variation of w^n as in [5], we are able to construct a suitable sequence $w^{n,j}$ of Lipschitz functions whose gradient, where w^n differs from $w^{n,j}$, in the $L^{p(\cdot)}$-norm can be so small as needed, and not only bounded as in [1] and in [5] (see subsection 2.4 for the definition of the Lebesgue space $L^{p(\cdot)}(\Omega)$). It is worth pointing out that this fact also leads to a simplification of the proof of Theorem 1 in [5], when $p(x)$ is constant.

2. Preliminaries

Throughout the paper $N, m > 1$ are fixed integers. Moreover, Ω will be an open subset of $\mathbb{R}^N$. The letter c will denote a strictly positive constant, whose value may change from line to line.

By $B_r(x)$ we denote the open ball in $\mathbb{R}^N$ with radius r and centered at x, and, more briefly, by B_1 the open unit ball centered at 0.
Let $\mathcal{L}^N$ denote the Lebesgue measure on $\mathbb{R}^N$ and $\mathcal{H}^{N-1}$ the Hausdorff measure of dimension $(N-1)$ on $\mathbb{R}^N$.

2.1. **BV functions.**

If $u \in L^1_{\rm loc}(\Omega;\mathbb{R}^m)$ and $x \in \Omega$, the *approximate limit of u at x* is defined as the unique value $\widetilde{u}(x) \in \mathbb{R}^m$ such that

$$\lim_{\rho\to 0^+} \frac{1}{\rho^N} \int_{B_\rho(x)} |u(y) - \widetilde{u}(x)|\, dx = 0\,.$$

The set of points in Ω where the approximate limit is not defined is called the *approximate singular set* of u and denoted by S_u, while $\Omega\setminus S_u$ consists of *approximate continuity points.* It simply follows by definitions that any Lebesgue point $x \in \Omega$ of u is an approximate continuity point and $\widetilde{u}(x) = u(x)$. Moreover it can be proved that S_u is a $\mathcal{L}^N$-negligible Borel set and $\widetilde{u} : \Omega \setminus S_u \to \mathbb{R}^m$ is a Borel function, coinciding $\mathcal{L}^N$-a.e. in $\Omega \setminus S_u$ with u.

Let $u \in L^1_{\rm loc}(\Omega;\mathbb{R}^m)$ and $x \in \Omega$. We say that x is an approximate jump point of u if there exist $a, b \in \mathbb{R}^m$ and $\nu \in \mathbb{S}^{N-1}$, such that $a \neq b$ and

$$\lim_{\rho\to 0^+} \fint_{B^+_\rho(x,\nu)} |u(y) - a|\, dy = 0 \qquad \text{and} \qquad \lim_{\rho\to 0^+} \fint_{B^-_\rho(x,\nu)} |u(y) - b|\, dy = 0$$

where $B^{\pm}_\rho(x,\nu) := \{y \in B_\rho(x) \ : \ \langle y - x, \nu\rangle \gtrless 0\}$. The triplet (a, b, ν) is uniquely determined by the previous formulas, up to a permutation of a, b and a change of sign of ν, and it is denoted by $(u^+(x), u^-(x), \nu_u(x))$. The Borel functions u^+ and u^- are called the *upper and lower approximate limit* of u at the point $x \in \Omega$. The set of approximate jump points of u is denoted by J_u. The set J_u is a Borel subset of S_u.
We recall that the space $\mathrm{BV}(\Omega;\mathbb{R}^m)$ of *functions of bounded variation* is defined as the set of all $u \in L^1(\Omega;\mathbb{R}^m)$ whose distributional gradient Du is a bounded Radon measure

on Ω with values in the space $M\!\!I^{m\times N}$ of $m \times N$ matrices. If $u \in \mathrm{BV}(\Omega;\mathbb{R}^m)$ then $\mathcal{H}^{N-1}(S_u \setminus J_u) = 0$.
We recall the usual decomposition

$$Du = \nabla u\,\mathcal{L}^N + D^s u,$$

where ∇u is the Radon-Nikodým derivative of Du with respect to the Lebesgue measure and $D^s u$ is the singular part of Du with respect to $\mathcal{L}^N$. We also split $D^s u$ into two parts: the *Cantor part* $D^c u$ and the *jump* part $D^j u = (u^+ - u^-) \otimes \nu_u\,\mathcal{H}^{N-1}\lfloor J_u$.
If $u \in \mathrm{BV}(\Omega,\mathbb{R}^m)$, then $\nabla u(x)$ is the *approximate differential* of u for almost every $x \in \Omega$, i.e.

$$\lim_{\rho\to 0^+} \fint_{B_\rho(x)} \frac{|u(y) - u(x) - \nabla u(x)(y-x)|}{|y-x|}\,dy = 0 \tag{2.1}$$

for a.e. $x \in \Omega$. An important consequence of (2.1) is the fact that $\nabla u(x) = 0$ for almost every x in the set $\{y \in \Omega : u(x) = 0\}$. In particular we have

$$u, v \in \mathrm{BV}(\Omega) \Longrightarrow \nabla u(x) = \nabla v(x) \text{ for a.e. } x \in \Omega \text{ s.t. } u(x) = v(x). \tag{2.2}$$

We recall that the space $\mathrm{SBV}(\Omega;\mathbb{R}^m)$ of *special functions of bounded variation* is defined as the set of all $u \in \mathrm{BV}(\Omega;\mathbb{R}^m)$ such that the Cantor part of derivative $D^c u$ is 0.
Let $p > 1$. The space $\mathrm{SBV}^p(\Omega;\mathbb{R}^m)$ is defined as the set of functions $u \in \mathrm{SBV}(\Omega;\mathbb{R}^m)$ with $\nabla u \in L^p(\Omega;\mathbb{R}^{Nm})$ and $\mathcal{H}^{N-1}(S_u) < \infty$.
For a general survey on the spaces of BV, SBV and SBV^p functions we refer for instance to [7].

2.2. **Quasiconvex functions.** Let $f : \mathbb{R}^{Nm} \to \mathbb{R}$ be a continuous function. We say that f is quasiconvex if

$$\int_{B_1} f(z + \nabla\varphi(x))\,dx \geq f(z) \quad \forall z \in \mathbb{R}^{Nm}, \forall \varphi \in C_0^1(B_1,\mathbb{R}^m).$$

This property, introduced by Morrey (see [25]), is necessary for the sequential lower semicontinuity of the functional

$$F(u) = \int_{B_1} f(\nabla w)\,dx$$

with respect to the weak* topology of $W^{1,\infty}(B_1)$. We have the following result (see Theorem II.1 in [1]).

Theorem 2.1. *Let $f(x,u,z) : B_1 \times \mathbb{R}^m \times \mathbb{R}^{Nm} \to [0,+\infty)$ be a Carathéodory function, quasiconvex in z for any $x \in B_1$ and any $u \in \mathbb{R}^m$, such that*

$$\sup\left\{f(x,u,z) : x \in B_1, u \in \mathbb{R}^m, z \in \mathbb{R}^{Nm}, |u| + |z| \leq \lambda\right\} < +\infty$$

for any $\lambda > 0$. Then,

$$\liminf_{n\to+\infty} \int_{B_1} f(x, w^n, \nabla w^n)\,dx \geq \int_{B_1} f(x, w, \nabla w)\,dx$$

for any bounded sequence $\{w^n\} \subset W^{1,\infty}(B_1, \mathbb{R}^m)$ *uniformly converging in* B_1 *to* $w \in W^{1,\infty}(B_1, \mathbb{R}^m)$.

2.3. **Lusin approximation in BV.** For every positive, finite Radon measure μ in $\mathbb{R}^N$ we introduce the *maximal function*

$$M(\mu)(x) := \sup_{\rho>0} \frac{\mu(B_\rho(x))}{\omega_N \rho^N}, \quad x \in \mathbb{R}^N.$$

Using the Besicovitch covering theorem (see e.g. [32]), it can be proved the existence of a constant ξ depending on N such that

$$\mathcal{L}^N \left(\left\{x \in \mathbb{R}^N : M(\mu)(x) > \lambda\right\}\right) \leq \frac{\xi}{\lambda}\mu(\mathbb{R}^N). \tag{2.3}$$

Now we show that a BV-function can be replaced by a Lipschitz function which agrees with it on larger and larger sets whose union is $\mathcal{L}^N$-almost all the domain. Our result slightly improves Theorem 5.34 in [7].

Theorem 2.2. *For every* $u \in \mathrm{BV}(\mathbb{R}^N, \mathbb{R}^m)$ *and* $\vartheta, \lambda > 0$ *there exists a Lipschitz function* $v : \mathbb{R}^N \to \mathbb{R}^m$ *such that* $\|v\|_\infty \leq \vartheta$, $\|\nabla v\|_\infty \leq c\lambda$ *and up to a null set*

$$\{v \neq u\} \subseteq \{M(|u|) > \vartheta\} \cup \{M(|Du|) > \lambda\}. \tag{2.4}$$

Moreover if $\Omega \subset \mathbb{R}^N$ *is a bounded open set with the property that there exists a constant* $A \geq 1$ *such that for all* $x \in \Omega$

$$\mathcal{L}^N(B_{2\mathrm{dist}(x,\Omega^c)}(x)) \leq A\mathcal{L}^N(B_{2\mathrm{dist}(x,\Omega^c)}(x) \cap \Omega^c), \tag{2.5}$$

and if u *has compact support in* Ω *then* $v \in W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^m)$, $v = 0$ *in* $\mathbb{R}^N \setminus \Omega$, $\|v\|_\infty \leq \vartheta$, $\|\nabla v\|_\infty \leq c\lambda$ *and up to a null set*

$$\{v \neq u\} \subseteq \Omega \cap (\{M(|u|) > \vartheta\} \cup \{M(|Du|) > \lambda\}). \tag{2.6}$$

Proof. Firstly, we note that it is sufficient to prove the theorem for a scalar valued BV-function. In fact, since for every $u = (u^1, \ldots, u^m) \in \mathrm{BV}(\Omega, \mathbb{R}^m)$ and for every ball $B_\rho(x)$

$$\max_{1\leq\alpha\leq m} |Du^\alpha|(B_\rho(x)) \leq |Du|(B_\rho(x))$$

we obtain that

$$\max_{1\leq\alpha\leq m} M(|Du^\alpha|)(x) \leq M(|Du|)(x)$$

and hence, for each α,

$$\{M(|Du^\alpha|) > \lambda\} \subseteq \{M(|Du|) > \lambda\}\ .$$

Analogously, we have

$$\{M(|u^\alpha|) > \vartheta\} \subseteq \{M(|u|) > \vartheta\}\ .$$

By Lemma 3.81 and Remark 3.82 in [7], u has an approximate limit $\widetilde{u}$ at every point x such that $M(|Du|)(x) < \infty$ and for every $\rho > 0$

$$\frac{1}{\omega_N \rho^N} \int\limits_{B_\rho(x)} \frac{|u(y) - \widetilde{u}(x)|}{|y - x|}\, dy \leq \int\limits_0^1 \frac{|Du|(B_{t\rho}(x))}{\omega_N (t\rho)^N}\, dt \leq M(|Du|)(x)\,. \tag{2.7}$$

This inequality shows that for any $\lambda \geq 0$ the restriction of $\widetilde{u}$ to $\{M(|Du|) \leq \lambda\}$ is a Lipschitz function. In fact, if $x, x' \in \{M(|Du|) \leq \lambda\}$ and $\rho = |x - x'|$, setting $\gamma := \mathcal{L}^N(B_\rho(x) \cap B_\rho(x'))/\rho^N$ (independent of ρ, x, x'), we get

$$
\begin{aligned}
|\widetilde{u}(x) - \widetilde{u}(x')| &= \frac{1}{\gamma\rho^N} \int_{B_\rho(x)\cap B_\rho(x')} |\widetilde{u}(x) - \widetilde{u}(x')| \, dy \\
&\leq \frac{1}{\gamma\rho^N} \int_{B_\rho(x)\cap B_\rho(x')} [|u(y) - \widetilde{u}(x)| + |u(y) - \widetilde{u}(x')|] \, dy \\
&\leq \frac{\rho\omega_N}{\gamma}[M(|Du|)(x) + M(|Du|)(x')] \leq \frac{2\lambda\omega_N}{\gamma}|x - x'|.
\end{aligned} \tag{2.8}
$$

Moreover the restriction of $\widetilde{u}$ to $\{M(|u|) \leq \vartheta\} \cap \{M(|Du|) \leq \lambda\}$ is bounded by ϑ. In fact, for every $x_0 \in \{M(|u|) \leq \vartheta\} \cap \{M(|Du|) \leq \lambda\}$ we have

$$
|\widetilde{u}(x_0)| \leq \fint_{B_\rho(x_0)} |\widetilde{u}(x_0) - u(x)| \, dx + \fint_{B_\rho(x_0)} |u(x)| \, dx \leq \fint_{B_\rho(x_0)} |\widetilde{u}(x_0) - u(x)| \, dx + M(|u|)(x_0) \, .
$$

As $\rho \to 0$, since $x_0 \in \mathbb{R}^N \setminus S_u$ we have

$$
|\widetilde{u}(x_0)| \leq M(|u|)(x_0) \leq \vartheta \, .
$$

Letting $M_{\vartheta,\lambda} := \{M(|u|) \leq \vartheta\} \cap \{M(|Du|) \leq \lambda\}$ and letting v be any extension of $\widetilde{u}\big|_{M_{\vartheta,\lambda}}$ to $\mathbb{R}^N$ which is bounded (with the same bound ϑ) and it is Lipschitz continuous (with the same Lipschitz constant $c\lambda$) (see e.g. [16]), the conclusion of the first part of the theorem follows since, up to an $\mathcal{L}^N$-null set, the following inclusions hold

$$
\{v \neq u\} = \{v \neq \widetilde{u}\} \subseteq \mathbb{R}^N \setminus M_{\vartheta,\lambda} = \{M(|u|) > \vartheta\} \cup \{M(|Du|) > \lambda\} \, .
$$

For the second part of the theorem we have to proceed more carefully in order to obtain that the Lipschitz truncations vanish on the boundary.
Let $x \in M_{\vartheta,\lambda} \cap \Omega$ and set $r := 2\,\mathrm{dist}(x, \Omega^c)$. Then by assumption (2.5) and since u is zero on Ω^c we have

$$
\begin{aligned}
\fint_{B_r(x)} |u(y) - (u)_{B_r(x)}| \, dy &\geq \frac{1}{\mathcal{L}^N(B_r(x))} \int_{B_r(x)\cap\Omega^c} |u(y) - (u)_{B_r(x)}| \, dy \\
&= \frac{\mathcal{L}^N(B_r(x) \cap \Omega^c)}{\mathcal{L}^N(B_r(x))} |(u)_{B_r(x)}| \\
&\geq \frac{1}{A}|(u)_{B_r(x)}|.
\end{aligned} \tag{2.9}
$$

Let us now recall a variant of the Poincaré inequality (see Remark 3.50 in [7]):

$$
\fint_{B_r(x)} |u(y) - (u)_{B_r(x)}| \, dy \leq c\, r \frac{|Du|(B_r(x))}{\mathcal{L}^N(B_r(x))},
$$

which, together with (2.9), gives, for $x \in M_{\vartheta,\lambda} \cap \Omega$

$$|(u)_{B_r(x)}| \leq c\,A\,r\frac{|Du|(B_r(x))}{\mathcal{L}^N(B_r(x))} \leq c\,A\,r\,M(|Du|)(x) \leq c\,A\,r\lambda.$$

Consequently, using also (2.7), we obtain

$$|\widetilde{u}(x)| \leq rM(|Du|)(x) + |(u)_{B_r(x)}| \leq c\,A\,r\lambda. \tag{2.10}$$

It follows from (2.10) that for all $x \in M_{\vartheta,\lambda} \cap \Omega$ and all $x' \in \Omega^{\mathrm{c}}$ we have

$$|\widetilde{u}(x) - \widetilde{u}(x')| = |\widetilde{u}(x)| \leq c\,A\,\mathrm{dist}(x, \Omega^{\mathrm{c}})\lambda \leq c\,A\lambda|x - x'|, \tag{2.11}$$

since $\widetilde{u}$ is zero on Ω^{c}. Then (2.8) and (2.11) imply that

$$|\widetilde{u}(x) - \widetilde{u}(x')| \leq c\,A\lambda|x - x'| \quad \forall x, x' \in M_{\vartheta,\lambda} \cup \Omega^{\mathrm{c}}.$$

Hence $\widetilde{u}$ is Lipschitz continuous on $H_{\vartheta,\lambda} := M_{\vartheta,\lambda} \cup \Omega^{\mathrm{c}}$ with Lipschitz constant bounded by $c\lambda$. We also have that $\widetilde{u}$ is bounded by ϑ on $H_{\vartheta,\lambda}$. Therefore extending $\widetilde{u}\big|_{H_{\vartheta,\lambda}}$ to $\mathbb{R}^N$ with the same bound ϑ and with the same Lipschitz constant $c\lambda$ we proved also the second part of the theorem. □

Remark 2.3. *If $\Omega \subset \mathbb{R}^N$ is an open bounded set with Lipschitz boundary, then Ω satisfies assumption* (2.5).

2.4. **The Lebesgue space with variable exponent.** Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $p : \Omega \to [1, +\infty)$ be a measurable bounded function, called a variable exponent on Ω. We assume that there exist two numbers p^+, p^- such that $1 < p^- \leq p(x) \leq p^+ < +\infty$ for every $x \in \Omega$.

We define the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega; \mathbb{R}^m)$ (which we will denote by $L^{p(\cdot)}(\Omega)$ if $m = 1$) as the set of all measurable functions $f : \Omega \to \mathbb{R}^m$ for which the modular

$$\rho_{L^{p(\cdot)},\Omega}(f) = \int_\Omega |f(x)|^{p(x)}\,dx$$

is finite. We define the Luxemburg norm on this space by

$$\|f\|_{L^{p(\cdot)},\Omega} = \inf\left\{\lambda > 0 : \rho_{L^{p(\cdot)},\Omega}(f/\lambda) \leq 1\right\}$$

for every $f \in L^{p(\cdot)}(\Omega; \mathbb{R}^m)$. If there is no misunderstanding we will write $\rho_{L^{p(\cdot)}}(\cdot)$ and $\|\cdot\|_{L^{p(\cdot)}}$ for the modular and the norm. We remark that the set $L^{p(\cdot)}(\Omega; \mathbb{R}^m)$ with this norm is a Banach space. Let us now consider some simple relationships between norm and modular. First of all, it is very easy to check that:

$$\rho_{L^{p(\cdot)}}(f) \leq 1 \quad \text{if and only if} \quad \|f\|_{L^{p(\cdot)}} \leq 1. \tag{2.12}$$

This property can be generalized as follows.

Lemma 2.4. *Let* $f \in L^{p(\cdot)}(\Omega;\mathbb{R}^m)$, *then*

$$\text{if } \|f\|_{L^{p(\cdot)}} \le C, \text{ then } \int_\Omega |f(x)|^{p(x)}\,dx \le C^{\widehat{q}}, \tag{2.13}$$

where

$$\widehat{q} = \begin{cases} p^- & \text{if } \int_\Omega |f(x)|^{p(x)}\,dx \le 1, \\ p^+ & \text{otherwise.} \end{cases} \tag{2.14}$$

Moreover

$$\text{if } \int_\Omega |f(x)|^{p(x)}\,dx \le C, \text{ then } \|f\|_{L^{p(\cdot)}} \le \begin{cases} C^{\frac{1}{p^+}} & \text{if } C \le 1, \\ C & \text{otherwise.} \end{cases} \tag{2.15}$$

Proof. Let us observe that

$$\|f\|_{L^{p(\cdot)}} = \inf\left\{\lambda > 0 : \int_\Omega \frac{|f(x)|^{p(x)}}{\lambda^{p(x)}}\,dx \le 1\right\}$$

$$= \min\left\{\inf\left\{0<\lambda<1 : \int_\Omega \frac{|f(x)|^{p(x)}}{\lambda^{p(x)}}\,dx \le 1\right\}; \inf\left\{\lambda \ge 1 : \int_\Omega \frac{|f(x)|^{p(x)}}{\lambda^{p(x)}}\,dx \le 1\right\}\right\}.$$

Moreover, if $0 < \lambda < 1$ is involved in the first infimum, then $\lambda \ge \left(\int_\Omega |f(x)|^{p(x)}\,dx\right)^{\frac{1}{p^-}}$, while for the second one we have $\lambda \ge \left(\int_\Omega |f(x)|^{p(x)}\,dx\right)^{\frac{1}{p^+}}$. Then

$$\|f\|_{L^{p(\cdot)}} \ge \min\left\{\left(\int_\Omega |f(x)|^{p(x)}\,dx\right)^{\frac{1}{p^-}}; \left(\int_\Omega |f(x)|^{p(x)}\,dx\right)^{\frac{1}{p^+}}\right\},$$

which gives

$$\|f\|_{L^{p(\cdot)}} \ge \left(\int_\Omega |f(x)|^{p(x)}\,dx\right)^{\frac{1}{\widehat{q}}}, \tag{2.16}$$

with

$$\widehat{q} = \begin{cases} p^- & \text{if} \left(\int_\Omega |f(x)|^{p(x)}\,dx\right)^{\frac{1}{p^-}} \le \left(\int_\Omega |f(x)|^{p(x)}\,dx\right)^{\frac{1}{p^+}}, \\ p^+ & \text{otherwise}. \end{cases}$$

It is easy to check that this coincides with the number $\widehat{q}$ defined in (2.14).
By (2.16) we have (2.13), while for (2.15), let us note that in the first case, i.e. $\int_\Omega |f(x)|^{p(x)}\,dx \le C \le 1$, since $\frac{1}{C^{\frac{1}{p^+}}} \le \frac{1}{C^{\frac{1}{p(x)}}}$, we have

$$\int_\Omega \left|\frac{f(x)}{C^{\frac{1}{p^+}}}\right|^{p(x)} dx \le \int_\Omega \left|\frac{f(x)}{C^{\frac{1}{p(x)}}}\right|^{p(x)} dx = \frac{1}{C}\int_\Omega |f(x)|^{p(x)}\,dx \le 1.$$

Therefore, we obtain $\|f\|_{L^{p(\cdot)}} \le C^{\frac{1}{p^+}}$. In the second case, i.e. $\int_\Omega |f(x)|^{p(x)}\,dx \le C$ and $C > 1$, since $\left(\frac{1}{C}\right)^{p(x)} \le \frac{1}{C}$, we have

$$\int_\Omega \left|\frac{f(x)}{C}\right|^{p(x)} dx \le \frac{1}{C}\int_\Omega |f(x)|^{p(x)}\,dx \le 1\,.$$

Therefore, we obtain $\|f\|_{L^{p(\cdot)}} \le C$. □

We say that a variable exponent $p : \Omega \to [1, +\infty)$ is *(locally)* log-*Hölder continuous* if there exists a constant $c > 0$ such that

$$|p(x) - p(y)| \le \frac{c}{\log(\frac{1}{|x-y|})}$$

for every $x, y \in \Omega$ with $|x - y| < 1/2$.

We say that p is *globally* log-*Hölder continuous* if it is log-Hölder continuous and there exist constants $c > 0$ and $p_\infty \in [1, +\infty)$ such that for all points $x \in \Omega$ we have

$$|p(x) - p_\infty| \le \frac{c}{\log(e + |x|)}.$$

The following fact is proven in [13] and [10].

Proposition 2.5. *Let $p : \mathbb{R}^N \to [1, +\infty)$ be a variable exponent with $1 < p^- \le p(x) \le p^+ < +\infty$, for every $x \in \mathbb{R}^N$, which is globally* log-*Hölder continuous. Then the Hardy-Littlewood maximal operator M is bounded from $L^{p(\cdot)}(\mathbb{R}^N)$ to $L^{p(\cdot)}(\mathbb{R}^N)$.*

For other weaker results about the boundedness of the Hardy-Littlewood operator see [13], [21], [27].
The following fact (see Corollary 4.3 in [15]) will be used in Theorem 4.1 to apply the Lipschitz truncation Theorem 3.1 below.

Proposition 2.6. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary and let $p : \Omega \to [1, +\infty)$ be* log-*Hölder continuous with $1 < p^- \le p(x) \le p^+ < +\infty$ for every $x \in \Omega$. Then there exists an extension $\tilde{p} : \mathbb{R}^N \to [1, +\infty)$ with $1 < \tilde{p}^- \le \tilde{p}(x) \le \tilde{p}^+ < +\infty$ for $x \in \mathbb{R}^N$, such that the Hardy-Littlewood maximal operator M is bounded from $L^{\tilde{p}(\cdot)}(\mathbb{R}^N)$ to $L^{\tilde{p}(\cdot)}(\mathbb{R}^N)$.*

3. Lusin approximation in SBV with variable exponent

The space $\mathrm{SBV}^{p(\cdot)}(\Omega; \mathbb{R}^m)$ is defined as the set of functions $u \in \mathrm{SBV}(\Omega; \mathbb{R}^m)$ with $\nabla u \in L^{p(\cdot)}(\Omega; M\!\!\!I^{m\times N})$ and $\mathcal{H}^{N-1}(S_u) < \infty$.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set which satisfies assumption* (2.5)*, and let $p : \mathbb{R}^N \to [1, +\infty)$, with $1 < p^- \le p(x) \le p^+ < +\infty$ for every $x \in \mathbb{R}^N$, be such that the Hardy-Littlewood maximal operator is bounded from $L^{p(\cdot)}(\mathbb{R}^N)$ to $L^{p(\cdot)}(\mathbb{R}^N)$.*

Let $v^n \in \mathrm{SBV}^{p(\cdot)}(\mathbb{R}^N,\mathbb{R}^m)$ *be a sequence of functions with compact support in* Ω *such that* v^n *tends to* 0 *in* $L^1(\Omega,\mathbb{R}^m)$*, as* $n\to\infty$*. Moreover we assume that*

$$\sup_n \|v^n\|_\infty < +\infty\,; \tag{3.1}$$

$$\sup_n \int_\Omega |\nabla v^n|^{p(x)}\,dx < +\infty\,; \tag{3.2}$$

$$\gamma_n := \|v^n\|_{L^{p(\cdot)},\Omega} \to 0 \quad (n\to\infty)\,. \tag{3.3}$$

Let $\vartheta_n > 0$ *be such that*

$$\vartheta_n \to 0 \quad \textit{and} \quad \frac{\gamma_n}{\vartheta_n} \to 0 \quad (n\to\infty)\,.$$

Then there exist sequences $\mu_j, \lambda_{n,j} > 1$ *such that for every* $n, j \in \mathbb{N}$

$$\mu_j \le \lambda_{n,j} \le \mu_{j+1}, \tag{3.4}$$

and a sequence $v^{n,j} \in W^{1,\infty}(\mathbb{R}^N,\mathbb{R}^m)$*,* $v^{n,j} = 0$ *on* $\mathbb{R}^N\setminus\Omega$*, such that for every* $n,j\in\mathbb{N}$

$$\|v^{n,j}\|_\infty \le \vartheta_n\,; \tag{3.5}$$

$$\|\nabla v^{n,j}\|_\infty \le C\lambda_{n,j}\,. \tag{3.6}$$

Moreover, up to a null set,

$$\left\{v^{n,j} \ne v^n\right\} \subseteq \Omega \cap \left(\{M(|v^n|) > \vartheta_n\} \cup \{M(|Dv^n|) > 3K\lambda_{n,j}\}\right). \tag{3.7}$$

For every $j\in\mathbb{N}$ *and for* $n\to\infty$

$$\nabla v^{n,j} \rightharpoonup 0 \quad \textit{weakly}^* \textit{ in } L^\infty(\Omega,\mathbb{R}^{Nm}). \tag{3.8}$$

Finally, there exists a sequence $\varepsilon_j > 0$ *with* $\varepsilon_j \to 0$ *such that for every* $n,j\in\mathbb{N}$

$$\begin{aligned} &\|\nabla v^{n,j}\chi_{\{M(|v^n|)>\vartheta_n\}\cup\{M(|\nabla v^n|)>2K\lambda_{n,j}\}}\|_{L^{p(\cdot)},\Omega} \le \\ &C\|\lambda_{n,j}\chi_{\{M(|v^n|)>\vartheta_n\}\cup\{M(|\nabla v^n|)>2K\lambda_{n,j}\}}\|_{L^{p(\cdot)},\Omega} \le C\frac{\gamma_n}{\vartheta_n}\mu_{j+1} + \varepsilon_j. \end{aligned} \tag{3.9}$$

Proof. The proof follows the lines of the proof of Theorem 4.4 in [15]. Even if only few changes are significant we will write it for the sake of completeness.
By (3.3), (3.2), and by (2.15) of Lemma 2.4, we have that

$$\sup_n \|v^n\|_{L^{p(\cdot)},\mathbb{R}^N} + \sup_n \|\nabla v^n\|_{L^{p(\cdot)},\mathbb{R}^N} < +\infty,$$

which thanks to the boundedness of the Hardy-Littlewood maximal operator, implies

$$\|M(|v^n|)\|_{L^{p(\cdot)},\mathbb{R}^N} + \|M(|\nabla v^n|)\|_{L^{p(\cdot)},\mathbb{R}^N} \le K, \tag{3.10}$$

for every $n\in\mathbb{N}$. So, by (2.12) we get

$$\int_{\mathbb{R}^N} |M(|v^n|)/K|^{p(x)}\,dx + \int_{\mathbb{R}^N} |M(|\nabla v^n|)/K|^{p(x)}\,dx \le 1,$$

for every $n\in\mathbb{N}$.

Next, we note that for a function $g \in L^{p(x)}(\mathbb{R}^N)$ with $\|g\|_{L^{p(\cdot)},\mathbb{R}^N} \leq 1$ we have

$$\begin{aligned} 1 &\geq \int_{\mathbb{R}^N} |g(x)|^{p(x)}\,dx = \int_{\mathbb{R}^N}\int_0^\infty p(x)t^{p(x)-1}\chi_{\{|g|>t\}}\,dt\,dx \\ &\geq \int_{\mathbb{R}^N}\sum_{m\in\mathbb{Z}}\int_{2^m}^{2^{m+1}} p(x)t^{p(x)-1}\chi_{\{|g|>t\}}\,dt\,dx \\ &\geq \int_{\mathbb{R}^N}\sum_{m\in\mathbb{Z}}(2^m)^{p(x)}\chi_{\{|g|>2^{m+1}\}}\,dx \\ &\geq \int_{\mathbb{R}^N}\sum_{m\in\mathbb{N}}(2^m)^{p(x)}\chi_{\{|g|>2^{m+1}\}}\,dx \\ &\geq \sum_{j\in\mathbb{N}}\sum_{k=2^j}^{2^{j+1}-1}\int_{\mathbb{R}^N}(2^k)^{p(x)}\chi_{\{|g|>2^{k+1}\}}\,dx. \end{aligned} \tag{3.11}$$

The choice $g = \chi_\Omega M(|\nabla v^n|)/K$ yields

$$\sum_{j\in\mathbb{N}}\sum_{k=2^j}^{2^{j+1}-1}\int_\Omega (2^k)^{p(x)}\chi_{\{|M(|\nabla v^n|)/K|>2^{k+1}\}}\,dx \leq 1,$$

and thus, for all $j, n \in \mathbb{N}$,

$$\sum_{k=2^j}^{2^{j+1}-1}\int_\Omega (2^k)^{p(x)}\chi_{\{|M(|\nabla v^n|)/K|>2^{k+1}\}}\,dx \leq 1.$$

Since the sum contains 2^j addenda, there is at least one index $k_{n,j}$ such that

$$\int_\Omega (2^{k_{n,j}})^{p(x)}\chi_{\{|M(|\nabla v^n|)/K|>2\cdot 2^{k_{n,j}}\}}\,dx \leq 2^{-j}. \tag{3.12}$$

Letting $\varepsilon_j := 2^{-j/p^+}$, by (2.15) of Lemma 2.4, it follows from (3.12) that

$$\|2^{k_{n,j}}\chi_{\{|M(|\nabla v^n|)/K|>2\cdot 2^{k_{n,j}}\}}\|_{L^{p(\cdot)},\Omega} \leq \varepsilon_j. \tag{3.13}$$

Now we define $\lambda_{n,j} := 2^{k_{n,j}}$ and $\mu_j := 2^{2^j}$, then

$$\mu_j = 2^{2^j} \leq \lambda_{n,j} \leq 2^{2^{j+1}} = \mu_{j+1}, \tag{3.14}$$

and we conclude from (3.13) that

$$\|\lambda_{n,j}\chi_{\{M(|\nabla v^n|)>2K\lambda_{n,j}\}}\|_{L^{p(\cdot)},\Omega} \leq \varepsilon_j. \tag{3.15}$$

Next we observe that

$$\begin{aligned} &\|\lambda_{n,j}\chi_{\{M(|v^n|)>\vartheta_n\}\cup\{M(|\nabla v^n|)>2K\lambda_{n,j}\}}\|_{L^{p(\cdot)},\Omega} \\ \leq\; &\frac{\lambda_{n,j}}{\vartheta_n}\|\vartheta_n\chi_{\{M(|v^n|)>\vartheta_n\}}\|_{L^{p(\cdot)},\Omega} + \|\lambda_{n,j}\chi_{\{M(|\nabla v^n|)>2K\lambda_{n,j}\}}\|_{L^{p(\cdot)},\Omega} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{\lambda_{n,j}}{\vartheta_n}\|M(|v^n|)\|_{L^{p(\cdot)},\mathbb{R}^N}+\varepsilon_j \qquad (3.16)\\
&\leq c\frac{\lambda_{n,j}}{\vartheta_n}\|v^n\|_{L^{p(\cdot)},\Omega}+\varepsilon_j = c\frac{\gamma_n}{\vartheta_n}\lambda_{n,j}+\varepsilon_j\\
&\leq c\frac{\gamma_n}{\vartheta_n}\mu_{j+1}+\varepsilon_j.
\end{aligned}$$

Now we apply Theorem 2.2 (with ϑ_n and $3K\lambda_{n,j}$) and we find, for each $n, j \in \mathbb{N}$, a sequence $v^{n,j} \in W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^m)$, $v^{n,j} = 0$ in $\mathbb{R}^N \setminus \Omega$, such that

$$\|v^{n,j}\|_\infty \leq \vartheta_n, \quad \|\nabla v^{n,j}\|_\infty \leq 3cK\lambda_{n,j} := C\lambda_{n,j}, \tag{3.17}$$

and, up to a null set,

$$\{v^{n,j} \neq v^n\} \subseteq \Omega \cap (\{M(|v^n|) > \vartheta_n\} \cup \{M(|Dv^n|) > 3K\lambda_{n,j}\}) \tag{3.18}$$

holds. Finally (3.8) follows by (3.17) and (3.14), while, using (3.16) and (3.17) we obtain (3.9).

□

4. Lower semicontinuity results

In this section we establish the main result of the paper, i.e. a lower semicontinuity theorem for integral functionals defined in $\mathrm{SBV}(\Omega, \mathbb{R}^m)$, under a variable growth condition. More precisely, we consider energy functionals containing a volume term of quasiconvex type and a surface term, whose integrands admit a growth assumption with variable exponent and we prove their lower semicontinuity separately. The first result (Theorem 1.1 above) extends the Ambrosio lower semicontinuity theorem (see Theorem 1 in [5] or Theorem 5.29 in [7]) to the SBV framework with variable exponent. The second one (Theorem 4.3 below) is a result obtained in [2] for surface energies with integrands depending in a discontinuous way on the spatial variable.
As in [5], in order to obtain Theorem 1.1 we first prove the result in the particular case when Ω is the unit ball B_1, the limit function u is linear, $\mathcal{H}^{N-1}(J_{u_n})$ is infinitesimal and the integrand functions vary.

Theorem 4.1. *Let $p_n : B_1 \to (1, +\infty)$ be a sequence of* log-H*ölder continuous functions such that there exists a constant $c_1 > 0$ (independent of n) such that*

$$|p_n(x) - p_n(y)| \leq \frac{c_1}{\log(\frac{1}{|x-y|})}$$

for every $x, y \in B_1$ with $|x - y| < 1/2$. Moreover we assume that $1 < p^- \leq p_n(y) \leq p^+ < +\infty$ for every $y \in B_1$ and that there exists a function $p : B_1 \to (1, +\infty)$ such that $\lim_{n\to\infty} p_n(y) = p(y)$ uniformly in B_1.
Let $g_n : B_1 \times \mathbb{R}^m \times \mathbb{R}^{Nm} \to [0, +\infty)$ a sequence of Carathéodory functions such that
(4.1)
$$-c_2 + c_3|z|^{p_n(y)} \leq g_n(y, u, z) \leq a_n(y) + \Psi_n(|u|)(1 + |z|^{p_n(y)}) \;\; \forall (y, u, z) \in B_1 \times \mathbb{R}^m \times \mathbb{R}^{Nm},$$

for some positive constants c_2, c_3, some continuous functions $\Psi_n : [0,+\infty) \to [0,+\infty)$, with the sequence $\{\Psi_n\}$ convergent in $L^\infty_{\rm loc}([0,+\infty))$, and $a_n \in L^1(B_1)$, with the sequence $\{a_n\}$ convergent in $L^1(B_1)$. Let us assume that there exists a $\mathcal{L}^N$-null set $E \subset B_1$ and a Carathéodory function $g : B_1 \times \mathbb{R}^{Nm} \to [0,+\infty)$, quasiconvex (in the second variable), satisfying the condition

$$-c_2 + c_3|z|^{p(y)} \le g(y,z) \le c_4(1+|z|^{p(y)}) \quad \forall (y,z) \in B^1 \times \mathbb{R}^{Nm},$$

for $c_4 > 0$, such that $\lim_{n\to\infty} g_n(y,u,z) = g(y,z)$ locally uniformly in $\mathbb{R}^{m+Nm}$ for every $y \in B_1 \setminus E$. Then

$$\liminf_{n\to\infty} \int_{B_1} g_n(y, w^n, \nabla w^n)\, dy \ge \int_{B_1} g(y, \nabla w)\, dy \tag{4.2}$$

for every sequence $\{w^n\} \subset \mathrm{SBV}(B_1, \mathbb{R}^m)$ converging in $L^1(B_1,\mathbb{R}^m)$ to a linear function w and satisfying $\mathcal{H}^{N-1}(J_{w^n}) \to 0$, as $n \to \infty$.

Proof. First observe that the function p shares with p_n the same regularity.
Let $L = \nabla w$. Possibly replacing w^n by $w^n - w$ and $g_n(y,u,z)$ by $g_n(y,u,z+L)$ we can assume that $w = 0$, that is w^n converges to 0 in L^1. Without no loss of generality we can suppose that the liminf in (4.2) is a finite limit, so that, by (4.1), we easily get that

$$\limsup_{n\to\infty} \int_{B_1} |\nabla w^n|^{p_n(y)}\, dy < \infty. \tag{4.3}$$

This implies, among other things, that the sequence $\{|\nabla w^n|\}$ is bounded in $L^{p^-}(B_1, \mathbb{R}^{Nm})$. Recalling that p^- is greater than 1, the proof of STEP 2 in Proposition 5.37 of [7] can be carried out exactly in the same way. Hence we can assume the additional information that $\|w^n\|_\infty \le 3$. Moreover, by the uniform convergence of p_n to p, we can consider a number $0 < \eta < \eta_0$ and an index $n_\eta \in \mathbb{N}$, such that, for every $y \in B_1$ and for every $n \ge n_\eta$, $p(y) - \eta \le p_n(y)$, with η_0 such that $p^\ominus := p^- - \eta_0$ is still strictly greater than 1. Thus, for such η, defining $\bar p(y) := p(y) - \eta$, we derive from (4.3) the following estimate

$$\limsup_{n\to\infty} \int_{B_1} |\nabla w^n|^{\bar p(y)}\, dy < \infty. \tag{4.4}$$

Letting $\rho \in (0,1)$, the Theorem will be proved if we show that

$$\liminf_{n\to\infty} \int_{B_\rho} g_n(y, w^n, \nabla w^n)\, dy \ge \int_{B_\rho} g(y, 0)\, dy, \tag{4.5}$$

eventually letting ρ tend to 1. Moreover, possibly multiplying w^n by a cut off function $\zeta \in C^\infty(\mathbb{R}^N)$, with compact support in B_1 and identically equal to 1 in B_ρ, we can assume, without loss of generality, that $w^n \in \mathrm{SBV}^{p(x)}(\mathbb{R}^N, \mathbb{R}^m)$ have compact support in B_1. Notice that, by the L^∞ bound on w^n, the sequence $\{\nabla(w^n\zeta)\}$ still satisfies (4.4). Since $\bar p$ satisfies all the hypotheses of Proposition 2.6 we can extend it to all $\mathbb{R}^N$ (without renaming it) and apply Theorem 3.1 to w^n, obtaining a sequence $w^{n,j} \in W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^m)$, $w^{n,j} = 0$ in $\mathbb{R}^N \setminus B_1$, such that (3.5), (3.6), (3.7), (3.8) hold, and

$$\begin{aligned} &\|\nabla w^{n,j} \chi_{\{M(|w^n|)>\vartheta_n\}\cup\{M(|\nabla w^n|)>2K\lambda_{n,j}\}}\|_{L^{\bar{p}(\cdot)},B_1} \leq \\ &C\|\lambda_{n,j}\chi_{\{M(|w^n|)>\vartheta_n\}\cup\{M(|\nabla w^n|)>2K\lambda_{n,j}\}}\|_{L^{\bar{p}(\cdot)},B_1} \leq C\frac{\gamma_n}{\vartheta_n}\mu_{j+1} + \varepsilon_j, \end{aligned} \tag{4.6}$$

with $\mu_j, \lambda_{n,j} > 1$ such that $\mu_j \leq \lambda_{n,j} \leq \mu_{j+1}$. We have

$$\begin{aligned} &\int_{B_\rho} g_n(y, w^n, \nabla w^n)\, dy \geq \int_{B_\rho\cap\{w^n = w^{n,j}\}} g_n(y, w^{n,j}, \nabla w^{n,j})\, dy \\ &= \int_{B_\rho\setminus E} [g_n(y, w^{n,j}, \nabla w^{n,j}) - g(y, \nabla w^{n,j})]\, dy + \int_{B_\rho} g(y, \nabla w^{n,j})\, dy \\ &- \int_{B_\rho\cap\{w^n \neq w^{n,j}\}} g_n(y, w^{n,j}, \nabla w^{n,j})\, dy, \end{aligned} \tag{4.7}$$

obtaining, when n tends to ∞,

$$\begin{aligned} &\liminf_{n\to\infty} \int_{B_\rho} g_n(y, w^n, \nabla w^n)\, dy \geq \liminf_{n\to\infty} \int_{B_\rho} g(y, \nabla w^{n,j})\, dy \\ &- \limsup_{n\to\infty} \int_{B_\rho\cap\{w^n\neq w^{n,j}\}} g_n(y, w^{n,j}, \nabla w^{n,j})\, dy \geq \int_{B_\rho} g(y, 0)\, dy \\ &- \limsup_{n\to\infty} \int_{B_\rho\cap\{w^n\neq w^{n,j}\}} [a_n(y) + \Psi_n(|w^{n,j}|)(1 + |\lambda_{n,j}|^{p_n(y)})]\, dy, \end{aligned} \tag{4.8}$$

where we used the Lebesgue dominated convergence theorem, Theorem 2.1, and (4.1). Let us now focus our attention on the last term in (4.8). First we note that, since the sequence $\{w^{n,j}\}$ is bounded by a constant independent by n, j, the sequence $\{\Psi_n(|w^{n,j}|)\}$ is bounded by a constant c_4 independent by n, j. So, thanks to (3.7), we have

$$\begin{aligned} &\int_{B_\rho\cap\{w^n\neq w^{n,j}\}} [a_n(y) + \Psi_n(|w^{n,j}|)(1 + |\lambda_{n,j}|^{p_n(y)})]\, dy \\ &\leq \int_{B_1\cap(\{M(|w^n|)>\vartheta_n\}\cup\{M(|\nabla w^n|)>2K\lambda_{n,j}\})} [a_n(y) + c_4(1 + |\lambda_{n,j}|^{p_n(y)})]\, dy \\ &+ \int_{B_1\cap\{M(|D^s w^n|)>K\lambda_{n,j}\}} [a_n(y) + c_4(1 + |\lambda_{n,j}|^{p_n(y)})]\, dy \leq \omega(n) + I_1^{n,j} + I_2^{n,j} + I_3^{n,j}, \end{aligned} \tag{4.9}$$

where $\omega(n)$ is infinitesimal and

$$I_1^{n,j} = \int_{B_1\cap(\{M(|w^n|)>\vartheta_n\}\cup\{M(|\nabla w^n|)>2K\lambda_{n,j}\})} \bar{a}(y)\, dy + \int_{B_1\cap\{M(|D^j w^n|)>K\lambda_{n,j}\}} \bar{a}(y)\, dy,$$

with $\bar{a}$ the L^1-limit of a_n plus a constant,

$$I_2^{n,j} = \int\limits_{B_1\cap(\{M(|w^n|)>\vartheta_n\}\cup\{M(|\nabla w^n|)>2K\lambda_{n,j}\})} c|\lambda_{n,j}|^{p_n(y)}\,dy,$$

and

$$I_3^{n,j} = \int\limits_{B_1\cap\{M(|D^s w^n|)>K\lambda_{n,j}\}} c|\lambda_{n,j}|^{p_n(y)}\,dy.$$

Concerning $I_3^{n,j}$ we derive from (2.3) that

$$I_3^{n,j} \le c\frac{\xi}{K\lambda_{n,j}}\lambda_{n,j}^{p^+}|D^s w^n|(\mathbb{R}^N) \le c\,\xi\,\mu_{j+1}^{p^+-1}\,\mathcal{H}^{N-1}(J_{w^n}), \tag{4.10}$$

so that

$$\lim_{n\to\infty} I_3^{n,j} = 0. \tag{4.11}$$

Let us consider now $I_2^{n,j}$; we have

$$I_2^{n,j} \le |\lambda_{n,j}|^{\|p_n-p\|_\infty+\eta}\int\limits_{B_1} c|\lambda_{n,j}|^{\bar{p}(y)}\chi_{\{M(|w^n|)>\vartheta_n\}\cup\{M(|\nabla w^n|)>2K\lambda_{n,j}\}}\,dy,$$

which together with Lemma 2.4 (with $\widehat{q}=\widehat{q}_{n,j}\in\{p^+,p^\ominus\}$) and (4.6) gives

$$I_2^{n,j} \le c|\mu_{j+1}|^{\|p_n-p\|_\infty+\eta}\left[C\frac{\gamma_n}{\vartheta_n}\mu_{j+1}+\varepsilon_j\right]^{\widehat{q}_{n,j}},$$

and

$$\limsup_{j\to\infty}\lim_{\eta\to 0}\limsup_{n\to\infty} I_2^{n,j} = 0. \tag{4.12}$$

Finally, let us consider $I_1^{n,j}$. We already estimated the Lebesgue measure of $\{M(|D^s w^n|) > K\lambda_{n,j}\}$ in (4.10) obtaining that it goes to zero as n tends to infinity. On the other hand, by Chebyshev inequality

$$\mathcal{L}^N(B_1\cap\{M(|\nabla w^n|)>2K\lambda_{n,j}\}) \le \frac{c}{\lambda_{n,j}^{p^-}}\int\limits_{B_1\cap\{M(|\nabla w^n|)>2K\lambda_{n,j}\}} M^{p^-}(|\nabla w^n|)\,dy$$

$$\le \frac{c}{\mu_j^{p^-}}\int\limits_{B_1} M^{p^-}(|\nabla w^n|)\,dy \le \frac{c}{\mu_j^{p^-}}.$$

Similarly, using also Lemma 2.4 and the boundedness of the maximal function between $L^{\bar{p}(\cdot)}(\mathbb{R}^N)$ and $L^{\bar{p}(\cdot)}(\mathbb{R}^N)$ we have

$$\mathcal{L}^N(B_1\cap\{M(|w^n|)>\vartheta_n\}) \le \int\limits_{\mathbb{R}^N}\left(\frac{M(|w^n|)}{\vartheta_n}\right)^{\bar{p}(y)}\,dy$$

$$\le \left\|\frac{M(|w^n|)}{\vartheta_n}\right\|_{L^{\bar{p}(\cdot)},\mathbb{R}^N}^{\widehat{q}_n} \le c\left(\frac{\|w^n\|_{\bar{p}(\cdot)}}{\vartheta_n}\right)^{\widehat{q}_n} = c\left(\frac{\gamma_n}{\vartheta_n}\right)^{\widehat{q}_n},$$

with $\widehat{q}_n \in \{p^+, p^{\ominus}\}$. Thus we can infer that

$$\lim_{j\to\infty} \limsup_{n\to\infty} I_1^{n,j} = 0. \tag{4.13}$$

Finally, combining (4.11), (4.12), and (4.13) we obtain (4.5) and this concludes the proof of the theorem. □

Now a blow up argument, in conjunction with covering theorems and with the approximate differentiability of BV functions, allows us to reduce the general problem to the special case of Theorem 4.1.

Proof of Theorem 1.1. The proof is very similar to that of Theorem 5.29 in [7] (see also Theorem 4.3 in [5]). We will consider only the key points of the proof referring for the details to [7] or [5].
The aim is to show that the Radon measure μ, that is the weak* limit of (a subsequence of) $f(x, w^n, \nabla w^n)\mathcal{L}^N$, will be greater than $f(x, w, \nabla w)\mathcal{L}^N$. This is obtained proving that

$$\limsup_{\rho\to 0} \frac{\mu(B_\rho(x_0))}{\mathcal{L}^N(B_\rho(x_0))} \geq f(x_0, w(x_0), \nabla w(x_0)),$$

for a.e. $x_0 \in \Omega$. Choosing suitable points x_0, Theorem 4.1 is applied to the sequences $g_i(y, u, z) := f(x_0 + \rho_i y, w(x_0) + \rho_i u, z)$, $u^i(y) := [w^{n_i}(x_0 + \rho_i y) - w(x_0)]/\rho_i$, where $\rho_i \to 0$ as $i \to \infty$ and $u^i \to \nabla w(x_0)\, y$ in $L^1(B_1, \mathbb{R}^m)$, while $\mathcal{H}^{N-1}(J_{u^i}) \to 0$. On the other hand the functions g_i satisfy all the conditions of Theorem 4.1, besides the fact that

$$\int_{B_1} g_i(y, u^i, \nabla u^i)\, dy \leq \frac{\mu(B_{\rho_i}(x_0))}{\rho_i^N} + \rho_i. \tag{4.14}$$

Thus

$$\liminf_{i\to\infty} \int_{B_1} g_i(y, u^i, \nabla u^i)\, dy \geq \int_{B_1} f(x_0, w(x_0), \nabla w(x_0))\, dy = \mathcal{L}^N(B_1) f(x_0, w(x_0), \nabla w(x_0)),$$

and this inequality, in conjunction with (4.14) gives the desired result. □

Remark 4.2. *The continuous function* $\Psi : [0, \infty) \to [0, \infty)$ *in the growth hypothesis* (1.3) *of Theorem 1.1 can be replaced by an increasing function* $\Psi : [0, \infty) \to [0, \infty)$. *In this case* (4.1) *can be written for a constant sequence of increasing functions* $\Psi_n = \Psi : [0, \infty) \to [0, \infty)$ *and the proofs of both theorems do not require any remarkable change.*

Finally, in the next theorem we deal with the surface energy.

Theorem 4.3. *Let* $p : \Omega \to [1, +\infty)$ *be a log-Hölder continuous function, such that* $1 < p^- \leq p(x) \leq p^+ < +\infty$ *for every* $x \in \Omega$.
Let $k : \Omega \times \mathbb{R}^N \to [0, +\infty)$ *be a locally bounded Borel function satisfying*

(4.15) $k(\cdot, \xi)$ *is* C_1*-quasi lower semicontinuous for every* $\xi \in \mathbb{R}^N$;

(4.16) $k(x, \cdot)$ *is convex and positively* 1*-homogeneous in* $\mathbb{R}^N$ *for every* $x \in \Omega$;

(4.17) $k(x, \xi) = k(x, -\xi)$ *for every* $(x, \xi) \in \Omega \times \mathbb{R}^N$;

(4.18) $\quad k(x,\xi)>0 \quad$ *for every* $(x,\xi)\in(\Omega\setminus N_0)\times(\mathbb{R}^N\setminus\{0\})$*, where* $\mathcal{H}^{N-1}(N_0)=0$.

Let $\gamma:[0,+\infty)\to[0,+\infty)$ *be a locally bounded, lower semicontinuous, increasing and subadditive function such that* $\gamma(0)=0$*. Then, for every* $(w^n)\subset\mathrm{SBV}(\Omega,\mathbb{R}^{\mathrm{m}})$ *and* $w\in\mathrm{SBV}(\Omega,\mathbb{R}^{\mathrm{m}})$ *such that* $w^n(x)\to w(x)$ *for almost every* $x\in\Omega$ *and*

$$\sup_{n\in\mathbb{N}}\left[\|w^n\|_\infty+\|\nabla w^n\|_{L^{p(\cdot)}}+\mathcal{H}^{N-1}(J_{w^n})\right]<+\infty\,,$$

we have

(4.19)
$$\int_{\Omega\cap J_w}\gamma(|w^+-w^-|)k(x,\nu_w)\,d\mathcal{H}^{N-1}\le\liminf_{n\to+\infty}\int_{\Omega\cap J_{w^n}}\gamma(|(w^n)^+-(w^n)^-|)k(x,\nu_{w^n})\,d\mathcal{H}^{N-1}\,.$$

Proof. It is sufficient to observe that the equiboundedness (with respect to n) of $\{\nabla w^n\}$ in $L^{p(\cdot)}$ implies the same boundedness also in L^{p^-}. Then we can apply Theorem 3.3 in [2] with $p=p^->1$. □

References

[1] E. Acerbi, N. Fusco: Semicontinuity problems in the Calculus of Variations. *Arch. Rat. Mech. Anal.*, **86** (1984), 125–145.

[2] M. Amar, V. De Cicco, N. Fusco: Lower semicontinuity results for free discontinuity energies. *M3AS*, to appear.

[3] L. Ambrosio: A compactness theorem for a special class of function of bounded variation. *Boll. Un. Mat. Ital. B(7)* **3** (1989), 857–881.

[4] L. Ambrosio: Existence theory for a new class of variational problems. *Arch. Rat. Mech. Anal.* **111** (1990), 291–322.

[5] L. Ambrosio: On the lower semicontinuity of quasiconvex integrals in SBV. *Nonlinear. Anal.* **23** (1994), 405–425.

[6] L. Ambrosio: A new proof of the SBV compactness theorem. *Calc. Var. Partial Differential Equations* **3** (1995), 127–137.

[7] L. Ambrosio, N. Fusco, D. Pallara: *Functions of bounded variation and free discontinuity problems.* Oxford University Press, New York, 2000.

[8] A. Blake, A. Zisserman: *Visual Reconstruction*, M.I.T. Press, 1987.

[9] Y. Chen, S. Levine, M. Rao: Variable exponent, linear growth functionals in image restoration. *SIAM J. Appl. Math.* **66** (4) (2006), 1383–1406.

[10] D. Cruz-Uribe, A. Fiorenza, C. J. Neugebauer: The maximal function on variable L^p spaces. *Ann. Acad. Sci. Fenn. Math.* **28** (2003), 223–238.

[11] E. De Giorgi, L. Ambrosio: New functionals in the calculus of variations. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* (8) **82** (2) (1988), 199–210.

[12] E. De Giorgi, M. Carriero, A. Leaci: Existence theorem for a minimum problem with free discontinuity set. *Arch. Rat. Mech. Anal.* **108** (1989), 195–218.

[13] L. Diening: Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$. *Math. Inequal. Appl.* **7** (2) (2004), 245–254.

[14] L. Diening, P. Hästö, A. Nekvinda: Open problems in variable exponent Lebesgue and Sobolev spaces. *FSDONA04 Proceedings* (Drabek and Rakosnik eds.; Milovy, Czech republic 2004), 38–58.

[15] L. Diening, J. Malek, M. Steinhauer: On Lipschitz truncations of Sobolev functions (with variable exponent) and their selected applications. *ESAIM Control Optim. Calc. Var.,* **14** (2) (2008), 211–232.

[16] L.C. Evans, R. F. Gariepy: *Measure theory and fine properties of functions*. CRC Press, Boca Raton, FL, 1992.

[17] I. Fonseca, G. Francfort: Relaxation in BV versus quasiconvexification in $W^{1,p}$; a model for the interaction between fracture and damage. *Calc. Var. Partial Differential Equations* **3** (1995), 407–446.

[18] N. Fusco, C. Leone, R. March, A. Verde: A lower semicontinuity result for polyconvex functionals in SBV. *Proc. Roy. Soc. Edinburgh Sect. A* **136** (2) (2006), 321–336.

[19] P. Harjulehto, P. Hästö, V. Latvala: Minimizers of the variable exponent, non-uniformly convex Dirichlet energy. *J. Math. Pures Appl.* (9) **89** (2) (2008), 174–197.

[20] J. Kristensen: Lower semicontinuity in spaces of weakly differentiable functions. *Math. Ann.* **313** (1999), 653–710.

[21] A. Lerner: Some remarks on the Hardy-Littlewood maximal function on variable L^p spaces. *Math. Z.* **251** (2005), 509–521.

[22] F. C. Liu: A Lusin type property of Sobolevfunctions. *Indiana Univ. Math. J.* **26** (1977), 645–651.

[23] G. Mingione, D. Mucci: Integral functionals and the gap problem: sharp bounds for relaxation and energy concentration. *SIAM J. Math. Anal.* **36** (5) (2005), 1540–1579.

[24] J. M. Morel, S. Solimini: *Variational Methods in Image Segmentation*. Birkhäuser, Boston, 1995.

[25] C. B. Morrey: *Multiple integrals in the Calculus of Variations.* Springer, Berlin, (1966).

[26] D. Mumford, J. Shah: Optimal approximations by piecewise smooth functions and associated variational problems. *Comm. Pure Appl. Math.* **42** (5) (1989), 577–685.

[27] A. Nekvinda: Hardy-Littlewood maximal operator on $L^{p(x)}(\mathbb{R})$. *Math. Inequal. Appl.*, **7** (2004), 255–265.

[28] K. R. Rajagopal, M. Ružička: On the modeling of electrorheological materials. *Mech. Research Comm.* **23** (1996), 401–407.

[29] K. R. Rajagopal, M. Ružička: Mathematical modeling of electrorheological materials. *Cont. Mech. and Thermodyn.* **13** (2001), 59–78.

[30] M. Ružička: *Electrorheological Fluids: Modeling and Mathematical Theory.* Lecture Notes in Math., vol. **1748**, Springer-Verlag, Berlin, 2000.

[31] S. Samko: On a progress in the theory of Lebesgue spaces with variable exponent: maximal and singular operators. *Integral Transforms Spec. Funct.* **16** (5–6) (2005), 461–482.

[32] E. M. Stein: *Singular integrals and the differentiability properties of functions.* Princeton University Press, 1970.

[33] E. G. Virga: *Variational theories for liquid crystals.* Applied Mathematics and Mathematical Computation. **8** Chapman & Hall, London, (1994).

[34] E. G. Virga: Drops of nematic liquid crystals. *Arch. Rational Mech. Anal.* **107** (4) (1989), 371–390.

[35] V. V. Zhikov: On Lavrentiev's phenomenon. *Russian J. Math. Phys.* **3** (1995), 249–269.

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