A SHARP ATTAINMENT RESULT FOR NONCONVEX VARIATIONAL PROBLEMS

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ABSTRACT. We consider the problem of minimizing autonomous, multiple integrals like

$$(\mathcal{P}) \qquad \min\left\{\int_\Omega f\,(u\,,\nabla u)\;dx:\;u\in u_0+W^{1,p}(\Omega)\right\}$$

where $f\colon\mathbb{R}\times\mathbb{R}^N\to[0\,,\infty)$ is a continuous, possibly nonconvex function of the gradient variable ∇u. Assuming that the bipolar function f^{**} of f is affine as a function of the gradient ∇u on each connected component of the sections of the detachment set $\mathcal{D}=\{f^{**}<f\}$, we prove attainment for $(\mathcal{P})$ under mild assumptions on f and f^{**}. We present examples that show that the hypotheses on f and f^{**} considered here for attainment are essentially sharp.

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1. Introduction

Consider the following variational problem

$$(\mathcal{P}) \qquad \min\left\{I(u)=\int_{\Omega} f\left(u(x)\,,\nabla u(x)\right)\,dx:\ u\in u_0+W_0^{1,p}(\Omega)\right\}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ with $N\geq 2$, $1<p<+\infty$, $f\colon \mathbb{R}\times\mathbb{R}^N\to[0\,,+\infty)$ is a continuous function, possibly nonconvex with respect to its last argument and $u_0\in W^{1,p}(\Omega)$.
The lack of convexity of $f(u\,,\nabla u)$ with respect to the gradient variable ∇u prevents from establishing the existence of minimizers of $(\mathcal{P})$ via the Direct Method of the Calculus of Variations, nevertheless many nonconvex minimum problems actually have solutions. Therefore, the question of establishing which conditions on f, other than convexity, ensures the existence of solutions to $(\mathcal{P})$ has been receiving increasing attention in recent years.
As regards variational problems for multiple integrals, the most widely investigated cases concern Lagrangean functions featuring the following structure: either $f(u\,,\nabla u)=g(u)+h(\nabla u)$ or $f(u\,,\nabla u)=g(u)h(\nabla u)$ with continuous functions g and nonconvex h. For these models, a well developed theory regarding attainment versus non attainment phenomena has been set up in recent years, see [9], [10], [19], [40], [37], [4] and [6]: whenever g and h satisfy mild regularity and growth assumptions, minimizers do exist for a wide class of continuous functions g, provided the convex envelope h^{**} of h is affine on each connected component of the detachment set $\{h^{**}<h\}$. Otherwise minimizers do not likely exist even if $f(u\,,\nabla u)=h(\nabla u)$ and the boundary datum is affine, see [26] and the sharp results of [9] and [19]. The conditions on g that yield attainment are roughly speaking the following: (a) every point of $\mathbb{R}$ lies between two intervals where g is monotone, i.e. g does not oscillate too fast; and (b) g has no strict, local minima.
Bolza's type examples highlight the sharpness of these assumptions for both type of models: as regards the case of sum-like integrals for instance, the following variational problem:

$$\min\left\{\int_B \left[(|\nabla u(x)|-1)^2+(u(x))^2\right]\,dx:\ u\in W_0^{1,2}(B)\right\}$$

has no solutions for any open ball B in $\mathbb{R}^N$. Here, $g(u)=u^2$ and only condition (b) is violated. Yet, the same pathological situation may occur when (b) holds and (a) fails. Indeed, the very same argument can be applied to the same Dirichlet problem with $g(u)=\max\{0\,,u\sin(1/u)\}$.
Similar examples can be produced also for the product case, see for instance [20] and [6]. Among the many related papers, besides those mentioned above, we refer to [18], [33], [23], [1], [25], [34], [12], [11], [3], [39], [38], [8] and [2] for sum-like integrals and to [27], [21] and [5] for product-like ones respectively, though all these latter papers deal with one dimensional problems only. Passing on, we mention also that there is a broad literature concerning the special case of one dimensional nonconvex problems and we refer to [27] and the references therein. See also the references in [7] for later years.
As regards nonconvex Lagrangean functions of general form, the available results concern only the nonautonomous case $f(x\,,\nabla u)$ without dependence on u, see for instance [28], [29] and the recent result [17], or the one dimensional, autonomous case $f(u\,,u')$, see [35], [7] and the references therein. In this latter paper [7], it is shown that again the two assumptions on the function g considered above, suitably rewritten for f, ensure the existence of solutions to the minimization problem. The aim of this paper is to show that the same result can be recovered also for multiple integrals, provided we assume in addition that $\xi\to f^{**}(\eta\,,\xi)$ is affine on the connected components of the sections of the detachment set $\mathcal{D}=\{f^{**}<f\}$, i.e. where f and f^{**} are different. In particular, we wish to emphasize that we do not require neither any particular smoothness assumption besides continuity of f and f^{**} nor any qualified convexity hypothesis at infinity and outside $\mathcal{D}$. Indeed, we

associate with the convex envelope f^{**} of f with respect to ξ a function $q\colon \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ whose value at the point $(\eta\,,\xi)$ is, roughly speaking, the value at the origin of the supporting affine function to the graph of $\xi \to f(\eta\,,\xi)$ through the point $(\eta\,,\xi)$. When f^{**} is smooth, q is given by

$$f^{**}(\eta\,,\xi) - \langle \nabla_\xi f^{**}(\eta\,,\xi)\,,\xi\rangle$$

where ∇_ξ is the gradient with respect to ξ. Note also that q reduces to f^{**} itself for $\xi = 0$.
The hypotheses (a) and (b) considered before for sum-like and product-like integrals then turn into the following assumptions on the function q on the detachment set $\mathcal{D}$:

- (a') if $f^{**}(\eta_0\,,\xi_0) < f(\eta_0\,,\xi_0)$, there exists $\delta = \delta(\eta_0\,,\xi_0) > 0$ such that $\eta \to q(\eta\,,\xi_0)$ is monotone on both intervals $[\eta_0 - \delta\,,\delta]$ and $[\eta_0\,,\eta_0 + \delta]$;
- (b') the function $\eta \to f^{**}(\eta\,,0)$ has no strict, local minima on the section of $\mathcal{D}$ corresponding to $\xi = 0$.

We refer to the following Section 2 for the definition of q and the exact statement of the attainment result for $(\mathcal{P})$. Here, we wish to point out only that, when $f(\eta\,,0) = f^{**}(\eta\,,0)$ for every η, then the only condition we have to require is (a') which is a very weak requirement on the behaviour of f. Moreover, when $f(\eta\,,\xi) = g(\eta) + h(\xi)$ for instance and h^{**} is smooth, it turns out that q is given by $h^{**}(\xi) - \langle \nabla h^{**}(\xi)\,,\xi\rangle + g(\eta)$ and we thus recover the previous assumptions (a) and (b) of [4]. A similar remark holds for the product-like case as well.
Finally, we briefly outline the idea of the proof. As a common feature of all nonconvex minimization problems, we consider the relaxed problem of $(\mathcal{P})$, i.e.

$$(\mathcal{P}^*) \qquad \min\left\{\int_\Omega f^{**}\left(u(x)\,,\nabla u(x)\right)\,dx:\ u\in u_0 + W_0^{1,p}(\Omega)\right\}.$$

By the Direct Method of the Calculus of Variations, this problem has a solution, say v, and we consider the measurable set of those points x of Ω such that the vector $(v(x)\,,\nabla v(x))$ falls in the detachment set $\mathcal{D}$. Exploiting the main assumption that $\xi \to f^{**}(\eta\,,\xi)$ is affine on the connected components of the sections of the detachment set $\mathcal{D}$ together with the hypotheses (a') and (b'), we show that we can locally modify v around almost every such point x so as to find new solutions to $(\mathcal{P}^*)$ which have the further property that they stay, together with their gradients, on the boundary of $\mathcal{D}$, i.e. where f and f^{**} coincide, on a small neighbourhood of the point x. This is the main technical part of the proof and the construction of these new solutions is accomplished by convex integration of partial differential relations. Then, by a covering argument, we glue these new, locally modified solutions so as to find a further new solution u to $(\mathcal{P}^*)$ satisfying $f^{**}(u\,,\nabla u) = f(u\,,\nabla u)$ almost everywhere on Ω, thus proving attainment for $(\mathcal{P})$.

2. Notations and statement of the main result

We begin by recalling some elementary definitions, notations and results, mostly from convex analysis and measure theory.
We denote the norm of a vector ξ in $\mathbb{R}^N$ by $|\xi|$ and the scalar product of ξ and ζ by $\langle\xi\,,\zeta\rangle$. We also denote the standard basis of $\mathbb{R}^N$ by$\{e_1\,,\ldots,e_N\}$, the open ball of radius $\rho > 0$ in $\mathbb{R}^N$ centered at x_0 by $B_\rho(x_0)$ and the closed segment in $\mathbb{R}^N$ whose endpoints are the vectors ξ_1 and ξ_2 by $[\xi_1\,,\xi_2]$. If A is a set in $\mathbb{R}^N$, we let $\mathrm{int}\,(A)$, $\overline{A}$ and ∂A be the interior, the closure and the boundary of A respectively. The *convex hull* $\mathrm{co}\,(A)$ of A is the intersection of all convex sets containing A itself. Moreover, if C is a convex subset of $\mathbb{R}^N$, we denote its *polar set* by C^0 and we recall that the dimension of C is the dimension of the smallest affine subspace containing it.
Throughout the paper, we shall consider points and subsets of $\mathbb{R}\times\mathbb{R}^N$. We shall write $(\eta\,,\xi)$ for such points and, whenever $\mathcal{E}$ is a subset of $\mathbb{R}\times\mathbb{R}^N$, we denote its sections with either η or ξ fixed

by

$$\mathcal{E}_\eta = \left\{\xi \in \mathbb{R}^N : (\eta, \xi) \in \mathcal{E}\right\} \qquad \text{and} \qquad \mathcal{E}^\xi = \left\{\eta \in \mathbb{R} : (\eta, \xi) \in \mathcal{E}\right\}$$

respectively and, for every point (η, ξ), the connected component of $\mathcal{E}_\eta$ containing ξ by $\mathcal{E}_\eta(\xi)$.
Now, let $g\colon \mathbb{R}^N \to [0, +\infty)$ be a lower semicontinuous function. We recall that g is said to be *subdifferentiable* at some point ξ if there exists a vector $d \in \mathbb{R}^N$ such that

$$g(\zeta) \geq g(\xi) + \langle d, \zeta - \xi \rangle, \qquad \zeta \in \mathbb{R}^N. \tag{2.1}$$

Every such d is a *subgradient* of g at ξ and the set of all such vectors d is the *subdifferential* $\partial g(\xi)$ of g at ξ. When g is also convex, $\partial g(\xi)$ is a nonempty, convex, compact set for every ξ in $\mathbb{R}^N$ and g turns out to be locally Lipschitz continuous on $\mathbb{R}^N$ with $\partial g(\xi) = \{\nabla g(\xi)\}$ for almost every $\xi \in \mathbb{R}^N$.
We recall also that, if $g\colon \mathbb{R}^N \to [0, +\infty)$ is lower semicontinuous, the *polar function* of g is the lower semicontinuous, convex function $g^*\colon \mathbb{R}^N \to (-\infty, +\infty]$ defined by

$$g^*(\zeta) = \sup\left\{\langle \xi, \zeta \rangle - g(\xi) : \xi \in \mathbb{R}^N\right\}, \qquad \zeta \in \mathbb{R}^N,$$

(see [16]) and that the *bipolar function* or *convex envelope* of g is the polar $g^{**}\colon \mathbb{R}^N \to [0, +\infty)$ of g^*. Thus, g^{**} is convex and

$$g^{**}(\xi) \leq g(\xi), \qquad \xi \in \mathbb{R}^N.$$

Moreover, we recall that, whenever $d \in \mathbb{R}^N$ is a subgradient of g^{**} at some point ξ, the values of $g^{**}(\xi)$ and $g^*(d)$ are related by

$$g^{**}(\xi) + g^*(d) = \langle d, \xi \rangle \tag{2.2}$$

(see [16]) because of the equality $g^{***} = g^*$. Hence, writing (2.1) with g^{**} instead of g, it follows that the value at the origin of the supporting affine function to the graph of g^{**} through the point $(\xi, g^{**}(\xi))$ with slope d is given by $-g^*(d)$.
As to measure theoretic notations and results, we denote the Lebesgue measure of a measurable subset E of $\mathbb{R}^N$ by $|E|$ and, as usual, we call *negligible* those sets E having null measure. We recall that a family $\mathcal{K}$ of compact sets containing a given point $x \in \mathbb{R}^N$ is said *to shrink nicely* at x if

$$\text{(a)}\ \inf\{|K| : K \in \mathcal{K}\} = 0; \quad \text{and} \quad \text{(b)}\ \sup\left\{\frac{|K|}{|B|} : K \subset B,\, B \text{ closed ball}\right\} \geq c, \qquad K \in \mathcal{K};$$

for some constant $c > 0$. We recall also that a *Vitali covering* of a measurable set E is a family of compact sets $\mathcal{K}$ containing, for a.e. $x \in E$, a sequence that shrinks nicely at x itself. Though we shall not need this in the sequel, we remark that, in the definitions above, the compact sets K associated with x need not be neither centered at x nor nested. Then, Vitali's covering theorem (see [36]) states that every such covering contains a (at most) countable subfamily of sets $\{K_n\}_n$ consisting of pairwise disjoint sets that cover E up to a negligible set, i.e. $|E \setminus (\cup_n K_n)| = 0$.
As regards functional theoretic notations, we let Ω be an open, bounded set in $\mathbb{R}^N$ and we use standard notations for Lebesgue and Sobolev spaces of functions on Ω and their norms. In particular, we let p^* be the Sobolev exponent relative to $1 \leq p \leq N$, i.e. $p^* = pN/(N-p)$ for $1 \leq p < N$ and $p^* = +\infty$ for $p = N$.
Now, we introduce the class of integral functionals that we are going to consider in the sequel. Let $f\colon \mathbb{R} \times \mathbb{R}^N \to [0, +\infty)$ be a continuous function. We consider the following integral functional

$$I(u) = \int_\Omega f(u(x), \nabla u(x))\, dx, \qquad u \in W^{1,p}(\Omega),$$

and the associated minimum problem

$$\min\left\{I(u) : u \in u_0 + W_0^{1,p}(\Omega)\right\} \tag{$\mathcal{P}$}$$

where $1 < p < +\infty$ and u_0 is in $W^{1,p}(\Omega)$. We denote the polar and the bipolar functions of f with respect to the second variable ξ by $f^*\colon \mathbb{R}\times\mathbb{R}^N \to (-\infty\,,+\infty]$ and $f^{**}\colon \mathbb{R}\times\mathbb{R}^N \to [0\,,+\infty)$ respectively and, for every $\eta \in \mathbb{R}$, we denote also the subdifferential of the function $\xi \to f^{**}(\eta\,,\xi)$ at the point $\xi \in \mathbb{R}^N$ by $\partial f^{**}(\eta\,,\xi)$. Then, f^{**} is Borel measurable and we consider the auxiliary functional

$$I^{**}(u) = \int_\Omega f^{**}(u(x)\,,\nabla u(x))\,dx, \qquad u \in W^{1,p}(\Omega),$$

and the associated minimum problem

$$(\mathcal{P}^{**}) \qquad \min\left\{I^{**}(u) :\ u \in u_0 + W_0^{1,p}(\Omega)\right\}.$$

This auxiliary functional I^{**} coincides with the relaxed functional of I with respect to the weak topology of $W^{1,p}(\Omega)$ (see Theorem 3.8, Chapter 10 in [16]) as soon as f satisfies suitable growth assumptions like those in the first formula of (H2$_p$) below. Even if this does not hold, it is plain that $I^{**} \le I$ on $W^{1,p}(\Omega)$ so that any solution u to $(\mathcal{P}^{**})$ satisfying $f^{**}(u\,,\nabla u) = f(u\,,\nabla u)$ almost everywhere on Ω is a solution to $(\mathcal{P})$ as well.

Next, we describe the assumptions that we are going to consider on the function f and its convex envelope f^{**}. As regards the regularity and the behaviour at infinity, we assume that

$$\text{(H1)} \qquad f \text{ and } f^{**} \text{ are continuous on } \mathbb{R}\times\mathbb{R}^N;$$

and that f satisfies the following growth assumption (H2$_p$) according to the value of $1 < p < +\infty$: if $1 < p \le N$, f is supposed to be bounded from above and below by

$$\text{(H2}_p\text{)} \qquad c_1|\xi|^p - c_2\,(1 + |\eta|^q) \le f(\eta\,,\xi) \le c_3|\xi|^p + c_2\,(1 + |\eta|^q)\,, \qquad (\eta\,,\xi) \in \mathbb{R}\times\mathbb{R}^N,$$

for some constants $c_3 \ge c_1 > 0$ and $c_2 \ge 0$ with $1 \le q < p^*$ whereas, if $N < p < \infty$, f is only supposed to be such that

$$\text{(H2}_p\text{)} \qquad c_1|\xi|^p - c_2\,(1 + |\eta|^q) \le f(\eta\,,\xi), \qquad (\eta\,,\xi) \in \mathbb{R}\times\mathbb{R}^N,$$

for some constants $c_1 > 0$, $c_2 \ge 0$ and $q \ge 1$. Clearly f^{**} satisfies the very same growth properties as f. The growth assumption from below ensures that every minimizing sequence for the problem $(\mathcal{P}^{**})$ has weakly convergent subsequences. This, and the weak lower semicontinuity of I^{**} on $u_0 + W_0^{1,p}(\Omega)$ (see Theorem 3.4, Chapter 3 in [13]), ensure the existence of solutions to $(\mathcal{P}^{**})$. As regards the regularity of these solutions, when $1 < p \le N$, minimizers are Hölder continuous (see [22]) and almost everywhere (classically) differentiable (see [4]). Moreover, the growth assumption from above can be somewhat relaxed when the boundary datum u_0 is (essentially) bounded, see the remark following the statement of Theorem 2.1. Finally, for $p > N$, the same smoothness properties are shared by all Sobolev functions.

As a consequence of (H1), the *detachment set* $\mathcal{D}$ defined by

$$\mathcal{D} = \left\{(\eta\,,\xi) \in \mathbb{R}\times\mathbb{R}^N :\ f^{**}(\eta\,,\xi) < f(\eta\,,\xi)\right\}$$

is open and, as we agreed upon at the beginning, we denote its sections with either η or ξ fixed by $\mathcal{D}_\eta$ and $\mathcal{D}^\xi$ respectively and the connected component of $\mathcal{D}_\eta$ containing ξ by $\mathcal{D}_\eta(\xi)$.

In the sequel, we also assume that f^{**} features the following qualitative behaviour on the sections of the detachment set $\mathcal{D}$:

(H3) for every $\eta \in \mathbb{R}$, the function $\xi \to f^{**}(\eta\,,\xi)$ is affine on each connected component of $\mathcal{D}_\eta$.

To be explicit, this means that, whenever $(\eta_0\,,\xi_0)$ is a point of $\mathcal{D}$, there exist $m_0 \in \mathbb{R}^N$ and $q_0 \in \mathbb{R}$ such that

$$f^{**}(\eta_0\,,\xi) = \langle m_0\,,\xi\rangle + q_0, \qquad \xi \in \mathcal{D}_{\eta_0}(\xi_0). \tag{2.3}$$

Now, we consider the function $q\colon \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ defined by

$$q(\eta\,,\xi) = \sup\,\{-f^*(\eta\,,d):\ d\in\partial f^{**}(\eta\,,\xi)\}\,,\qquad (\eta\,,\xi)\in\mathbb{R}\times\mathbb{R}^N. \tag{2.4}$$

It is well defined and real-valued because f^{**} is convex with respect to ξ and, as recalled at the beginning of this section, its value at the point $(\eta\,,\xi)$ yields the largest among the values at the origin of the supporting affine functions to the graph of $\xi\to f^{**}(\eta\,,\xi)$ through the point ξ. Moreover, if it happens that f^{**} has continuous partial derivatives with respect to ξ, then $q(\eta\,,\xi)$ reduces to

$$q(\eta\,,\xi) = f^{**}(\eta\,,\xi) - \langle\nabla_\xi f^{**}(\eta\,,\xi)\,,\xi\rangle\,,\qquad (\eta\,,\xi)\in\mathbb{R}\times\mathbb{R}^N, \tag{2.5}$$

because of the basic equality (2.2).
The properties of the detachment set $\mathcal{D}$ and of the restrictions of f^{**} and q to $\mathcal{D}$ itself will be investigated in the following Section 3.
Now, the preliminaries are over and we can state the attainment result for the nonconvex problem $(\mathcal{P})$.

Theorem 2.1. *Let $f\colon \mathbb{R}\times\mathbb{R}^N \to [0\,,+\infty)$ satisfy* (H1), (H2$_p$) *for some $1<p<+\infty$ and* (H3). *Let q be defined by* (2.4) *and assume also that the following properties hold:*

- (2.6) *for every $(\eta_0,\xi_0)\in\mathcal{D}$, there is $\delta=\delta(\eta_0,\xi_0)>0$ such that $[\eta_0-\delta\,,\eta_0+\delta]\subset\mathcal{D}^{\xi_0}$ and such that the restriction $\eta\in[\eta_0-\delta\,,\eta_0+\delta]\to q(\eta\,,\xi_0)$ is monotone on each interval $[\eta_0-\delta\,,\eta_0]$ and $[\eta_0\,,\eta_0+\delta]$;*
- (2.7) *if $\mathcal{D}^0\neq\emptyset$, the restriction $\eta\in\mathcal{D}^0\to q(\eta\,,0)$ has no strict, local minima on $\mathcal{D}^0$.*

Then, the nonconvex problem $(\mathcal{P})$ has a solution for every boundary datum $u_0\in W^{1,p}(\Omega)$.

It turns out that $q(\eta\,,0)=f^{**}(\eta\,,0)$ for every η in $\mathcal{D}^0$, see (d) of Proposition 3.1 below, so that (2.7) can be equivalently stated by requiring that the restriction $\eta\in\mathcal{D}^0\to f^{**}(\eta\,,0)$ of f^{**} on the nonempty section $\mathcal{D}^0$ has no strict, local minima. Note also for future purposes that the other hypothesis (2.6) implies that

- (2.8) *if $\mathcal{D}^\xi\neq\emptyset$, the restriction $\eta\in\mathcal{D}^\xi\to q(\eta\,,\xi)$ has only finitely many strict, local extrema in every compact subinterval of $\mathcal{D}^\xi$.*

As regards the hypotheses of this attainment result, we have already pointed out that the growth assumption (H2$_p$) is related only with the coercivity of I^{**} and the regularity of minimizers of $(\mathcal{P}^{**})$. Moreover, if $1<p\le N$ and the boundary datum u_0 is in $L^\infty(\Omega)\cap W^{1,p}(\Omega)$, it can be relaxed by requiring only that

$$c_1|\xi|^p - c_2\,(1+|\eta|^q)\le f(\eta\,,\xi)\le c(\eta)|\xi|^p + c_2\,(1+|\eta|^q)\,,\qquad (\eta\,,\xi)\in\mathbb{R}\times\mathbb{R}^N, \tag{H2$'_p$}$$

for some constants $c_1>0$, $c_2\ge 0$ and for some function $c\in\mathcal{C}(\mathbb{R})$ such that $c(\eta)\ge c_1$ for every $\eta\in\mathbb{R}$ with $1\le q<p^*$ as before. Then, every minimizer u of $(\mathcal{P}^{**})$ is again Hölder continuous and almost everywhere (classically) differentiable on Ω (see Theorem 2.1 of [6]) and the rest of the proof of Theorem 2.1 remains unchanged.
As to the main qualitative hypotheses (H3), (2.6) and (2.7), we have already remarked that they cannot be dropped in general without affecting attainment for $(\mathcal{P})$ as the examples mentioned in the Introduction show. Thus, all the assumptions of our attainment result but (H1) - which is a very weak smoothness hypothesis on f and f^{**} - are needed in the sense specified above and we wish to emphasize once again that the only truly demanding ones among them are (H3) and (2.7) - provided $\mathcal{D}^0$ is nonempty - which are well known necessary assumptions for nonconvex variational problems involving multiple integrals. By contrast, (2.6) is a very weak assumption on the behaviour of f^{**} and, whenever f^{**} is smooth enough so that q reduces to (2.5), its fulfillment just requires that q does not oscillate on finer and finer scales as a function of η. Once f^{**} is computed, this can be

easily tested by checking that the derivative of q with respect to η vanishes finitely many times only on each subinterval of $\mathcal{D}^\xi$.

3. Some technical results

This section contains the main technical steps towards the proof of our attainment result, Theorem 2.1. Indeed, the program outlined at the end of the Introduction calls first for studying the properties of the convex envelope f^{**} and of other f^{**}-related functions such as q, then for studying the properties of the detachment set $\mathcal{D}$, namely the local properties of its boundary, and at last for defining new solutions to $(\mathcal{P}^{**})$ which stay on the boundary of $\mathcal{D}$. As regards the properties of $\mathcal{D}$, we prove – see Propositions 3.2 and 3.3 below – that, though $\mathcal{D}$ need not to have bounded connected components, the connected components of the vertical sections $\mathcal{D}_\eta$ of $\mathcal{D}$ are locally uniformly bounded with respect to η and the "trace" of the boundary of a section $\mathcal{D}_\eta$ in every given direction of $\mathbb{R}^N$ is a locally bounded, lower semicontinuous function of η. Either properties follow from the fact that $\mathcal{D}$ is a sublevel set of the continuous function $f^{**} - f$, i.e. $\mathcal{D} = \{f^{**} - f < 0\}$ and that both f and f^{**} have superlinear growth as $|\xi| \to +\infty$. Then, we exploit these properties of $\mathcal{D}$ in the following Proposition 3.4 to define new, local solutions to $(\mathcal{P}^{**})$ which now stay on the boundary of $\mathcal{D}$. This is the main technical point of the paper and, following the works initiated by De Blasi and Pianigiani on the Baire category method in [15] and by Müller and Šverák in [30] on the convex integration of partial differential relations of Gromov ([24]), it will be accomplished by applying this method to find special families of solutions to autonomous, first order partial differential equations in implicit form like

$$H(w(x)\,,\nabla w(x)) = 0 \qquad \text{for a.e. } x \in \Omega, \tag{3.1}$$

where the Hamiltonian function H is given by $H = f^{**} - f$. As we said, this latter problem has been receiving much attention in recent years and we refer to [30], [15], [14], [31] and [32] for an extensive and systematic discussion of this kind of equations though all these papers mainly deal with the case of vector-valued solutions w. Here, we just remark that, by contrast, we deal with the much simpler case of real-valued solutions but we want to select, among all such solutions w to (3.1), those featuring some kind of order-related property with respect to the original solution v to $(\mathcal{P}^{**})$. We refer to Proposition 3.4 below for the exact statement of this property.
Now, we start by studying the properties of f^{**} and q on the detachment set $\mathcal{D}$.

Proposition 3.1. *Let $f\colon \mathbb{R}\times\mathbb{R}^N \to [0,+\infty)$ satisfy* (H1) *and* (H3). *Then,*

- (a) *there exists $d\colon \mathcal{D} \to \mathbb{R}^N$ such that $\partial f^{**}(\eta,\xi) = \{d(\eta,\xi)\}$ for every $(\eta,\xi) \in \mathcal{D}$;*
- (b) *for every $\eta \in \mathbb{R}$, the restrictions $\xi \in \mathcal{D}_\eta \to d(\eta,\xi)$ and $\xi \in \mathcal{D}_\eta \to q(\eta,\xi)$ are constant on each connected component of $\mathcal{D}_\eta$;*
- (c) *both $d\colon \mathcal{D} \to \mathbb{R}^N$ and $q\colon \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ are continuous on $\mathcal{D}$;*
- (d) *if $\mathcal{D}^0 \neq \emptyset$, then $q(\eta,0) = f^{**}(\eta,0)$ for every $\eta \in \mathcal{D}^0$.*

Note that, if $(\eta_0,\xi_0) \in \mathcal{D}$, then (2.3) turns into

$$f^{**}(\eta_0,\xi) = \langle d(\eta_0,\xi_0),\xi\rangle + q(\eta_0,\xi_0), \qquad \xi \in \mathcal{D}_{\eta_0}(\xi_0). \tag{3.2}$$

Proof of Proposition 3.1. For every nonempty section $\mathcal{D}_\eta$ the function $\xi \in \mathbb{R}^N \to f^{**}(\eta,\xi)$ is affine on each connected component of $\mathcal{D}_\eta$ because of (H3). Hence, it has derivatives with respect to the components of ξ at every point ξ in the open section $\mathcal{D}_\eta$ so that (a) obviously holds with $d(\eta,\xi)$ given by $\nabla_\xi f^{**}(\eta,\xi)$, i.e. the gradient of f^{**} with respect to ξ at the point (η,ξ). From (H3) again, this gradient is obviously constant on each connected component of $\mathcal{D}_\eta$ and this proves the part of (b) regarding d. The part of (b) regarding q follows from this and the very definition of q, see (2.4). Then, recall that

$$f^*(\eta, d(\eta\,,\xi)) = \langle d(\eta\,,\xi)\,,\xi\rangle - f^{**}(\eta\,,\xi), \qquad (\eta\,,\xi) \in \mathcal{D},$$

because of (2.2). By (a) and (2.4), this equality proves (d) and shows that, because of (H1) again, the continuity of q on the set $\mathcal{D}$ will follow from the corresponding property of d.
Thus, we have to prove that d is continuous on $\mathcal{D}$. To see this, let $\delta > 0$ be such that the cylinder $C = [\eta_0 - \delta\,,\eta_0 + \delta]\times B_{2\delta}(\xi_0)$ is contained in the open set $\mathcal{D}$ and let d^n be the n^{th} component of the vector-valued function d. From (a), (b) and (H3) we have

$$d^n(\eta,\xi) = \frac{f^{**}(\eta\,,\xi+\delta e_n) - f^{**}(\eta\,,\xi-\delta e_n)}{2\delta}, \qquad |\eta-\eta_0| \le \delta, \quad |\xi-\xi_0| < \delta.$$

Thus, d is continuous on $\mathcal{D}$ because of (H1). □

It is useful to introduce the set where f^{**} is affine as a function of ξ. To this purpose, let $\mathcal{A}$ be the subset of $\mathbb{R}\times\mathbb{R}^N$ defined by the following property: for every $\eta \in \mathbb{R}$, the section $\mathcal{A}_\eta$ of $\mathcal{A}$ is the union of all maximal, compact, N-dimensional convex sets $\{K_i\}$ such that f^{**} is affine on K_i as a function of ξ, i.e. $f^{**}(\eta\,,\xi) = \langle m_i\,,\xi\rangle + q_i$ for every $\xi \in K_i$ for some $m_i \in \mathbb{R}^N$ and $q_i \in \mathbb{R}$. Here, maximal means that there is no other compact, N-dimensional convex set containing K_i where the above formula for f^{**} holds. Each section $\mathcal{A}_\eta$ is the union of at most countably many sets K_i with this property and every two such sets have pairwise disjoint interiors. Again, when $(\eta\,,\xi)$ is a point of $\mathcal{A}$, the connected component of $\mathcal{A}_\eta$ containing ξ will be denoted by $\mathcal{A}_\eta(\xi)$.
Thus, the assumption (H3) can be stated equivalently by saying that $\mathcal{D}_{\eta_0}(\xi_0) \subset \mathcal{A}_{\eta_0}(\xi_0)$ for every point $(\eta_0\,,\xi_0)$ in $\mathcal{D}$. Moreover, the formula (3.2) holds on the larger set $\mathcal{A}_{\eta_0}(\xi_0)$, i.e.

$$f^{**}(\eta_0\,,\xi) = \langle d(\eta_0\,,\xi_0)\,,\xi\rangle + q(\eta_0\,,\xi_0), \qquad \xi \in \mathcal{A}_{\eta_0}(\xi_0), \tag{3.3}$$

and it is easy to check that the set $\mathcal{A}_{\eta_0}(\xi_0)$ is just the set of points ξ where the previous formula holds.
The properties of the sections of $\mathcal{A}$ that will be useful in the sequel are proved in the following proposition. In particular, we show below that, for every point $(\eta_0\,,\xi_0)$ in $\mathcal{D}$, the connected components $\mathcal{A}_\eta(\xi_0)$ of the vertical sections $\mathcal{A}_\eta$ are uniformly bounded provided η is sufficiently close to η_0.

Proposition 3.2. *Let $f\colon \mathbb{R}\times\mathbb{R}^N \to [0\,,+\infty)$ satisfy* (H1), (H2$_p$) *for some $1 < p < +\infty$ and* (H3). *Let also $(\eta_0\,,\xi_0) \in \mathcal{D}$ and $\delta = \delta(\eta_0\,,\xi_0) > 0$ be such that $(\eta\,,\xi_0) \in \mathcal{D}$ for every $\eta \in [\eta_0 - \delta\,,\eta_0 + \delta]$. Then,*

(a) *$\mathcal{A}_{\eta_0}(\xi_0)$ is a compact, convex subset of $\mathbb{R}^N$ and $\mathcal{D}_{\eta_0}(\xi_0) \subset \mathcal{A}_{\eta_0}(\xi_0)$;*
(b) *there is $R = R(\eta_0\,,\xi_0\,,\delta) > 0$ such that $\mathcal{A}_\eta(\xi_0) \subset B_R(\xi_0)$ for every $|\eta - \eta_0| \le \delta$.*

Proof. The set $\mathcal{A}_{\eta_0}(\xi_0)$ is closed and bounded by (H1) and (H2$_p$) respectively and it is also convex because it is a level set of a convex function. The inclusion of $\mathcal{D}_{\eta_0}(\xi_0)$ into $\mathcal{A}_{\eta_0}(\xi_0)$ is (H3). As to (b), assume by contradiction that there are numbers η_k at most δ far from η_0 and vectors ξ_k in $\mathcal{A}_{\eta_k}(\xi_0)$ such that $|\xi_k| \to +\infty$ and $\eta_k \to \eta_\infty$ for some point η_∞ at most δ far from η_0 again. Then, $d(\eta_k\,,\xi_0) \to d(\eta_\infty\,,\xi_0)$ and $q(\eta_k\,,\xi_0) \to q(\eta_\infty\,,\xi_0)$ because all points $(\eta_k\,,\xi_0)$ and $(\eta_\infty\,,\xi_0)$ are in $\mathcal{D}$ and both q and d are continuous on $\mathcal{D}$ itself. Hence, (3.2) yields that

$$\lim_{k\to+\infty} \frac{f^{**}(\eta_k\,,\xi_k)}{|\xi_k|^p} = \lim_{k\to+\infty} \frac{\langle d(\eta_k\,,\xi_0)\,,\xi_k\rangle + q(\eta_k\,,\xi_0)}{|\xi_k|^p} = 0$$

because $p > 1$ but (H2$_p$) gives $f^{**}(\eta_k\,,\xi_k) \ge c_1|\xi_k|^p$ for every k with $c_1 > 0$ and this completes the proof. □

Next, we describe the local behaviour of the "trace" in a given direction of the boundary of the detachment set $\mathcal{D}$ as a function of η.

Proposition 3.3. *Let $f\colon \mathbb{R}\times\mathbb{R}^N \to [0,+\infty)$ satisfy* (H1), (H2$_p$) *for some $1 < p < +\infty$ and* (H3). *Let also $(\eta_0, \xi_0) \in \mathcal{D}$ and $\delta = \delta(\eta_0, \xi_0) > 0$ be such that $(\eta, \xi_0) \in \mathcal{D}$ for every $\eta \in [\eta_0 - \delta, \eta_0 + \delta]$. Then, for every unit vector $\nu \in \mathbb{R}^N$, there exists a function $a_\nu\colon [\eta_0 - \delta, \eta_0 + \delta] \to (0,+\infty)$ such that*

(a) *a_ν is lower semicontinuous and bounded;*

(b) *$f^{**}(\eta, \xi_0 + t\nu) < f(\eta, \xi_0 + t\nu)$ for every $0 \le t < a_\nu(\eta)$ and $\eta \in [\eta_0 - \delta, \eta_0 + \delta]$;*

(c) *$f^{**}(\eta, \xi_0 + a_\nu(\eta)\nu) = f(\eta, \xi_0 + a_\nu(\eta)\nu)$ for every $\eta \in [\eta_0 - \delta, \eta_0 + \delta]$.*

Proof. Set $a_\nu(\eta) = \sup\{t \ge 0 : f^{**}(\eta, \xi_0 + s\nu) < f(\eta, \xi_0 + s\nu)$ for every $0 \le s \le t\}$ for every $\eta \in [\eta_0 - \delta, \eta_0 + \delta]$. The resulting function is well defined. Indeed, (H1) and the choice of δ imply that a_ν is positive on the interval $[\eta_0 - \delta, \eta_0 + \delta]$ and (b) of Proposition 3.2 shows that it is also bounded on the same interval. Then, the properties (b) and (c) follow from (H1) again and the very definition of a_ν so that we only have to prove that it is lower semicontinuous. Suppose not, i.e.

$$\liminf_{\eta\to\eta'} a_\nu(\eta) < t_0 < a_\nu(\eta')$$

at some point η' in $[\eta_0 - \delta, \eta_0 + \delta]$ and for some $t_0 > 0$. It would follow that $f^{**}(\eta', \xi_0 + t\nu) < f(\eta', \xi_0 + t\nu)$ for every $0 \le t \le t_0$. Hence, the compact segment of $\mathbb{R}\times\mathbb{R}^N$ whose endpoints are (η', ξ_0) and $(\eta', \xi_0 + t_0\nu)$ would be contained in the open set $\mathcal{D}$ whence the same would hold for all segments having endpoints (η, ξ_0) and $(\eta, \xi_0 + t_0\nu)$ for every η in $[\eta_0 - \delta, \eta_0 + \delta]$ within some sufficiently small $\sigma > 0$ from η'. Thus, $a_\nu(\eta)$ would be at least t_0 for every η in $[\eta_0 - \delta, \eta_0 + \delta]$ such that $|\eta - \eta'| \le \sigma$ and this would give a contradiction. □

Next, we show that we can modify every minimizer v of $(\mathcal{P}^{**})$ such that $(v(x_0), \nabla v(x_0))$ is in the detachment set $\mathcal{D}$ so as to find new functions which now stay on the boundary of $\mathcal{D}$, i.e. where f and f^{**} coincide, on neighbourhoods of x_0. As it was mentioned above, this is a special instance of results regarding the Dirichlet problem for equations like (3.1) but the emphasis here is not on the smoothness of the Hamiltonian function $H = f^{**} - f$ which can be somewhat relaxed – see Theorem 3.2 in [32] – but on the fact that we wish to control the values of the modified functions with respect to the original function v on neighbourhoods of x_0.

Proposition 3.4. *Let $f\colon \mathbb{R}\times\mathbb{R}^N \to [0,+\infty)$ satisfy* (H1), (H2$_p$) *for some $1 < p < +\infty$ and* (H3). *Let also $v \in W^{1,p}(\Omega)$ be a continuous, almost everywhere differentiable function on Ω such that*

(a) *v is differentiable at some point $x_0 \in \Omega$ with (classical) gradient $\xi_0 = \nabla v(x_0)$;*

(b) *$f^{**}(v(x_0), \nabla v(x_0)) < f(v(x_0), \nabla v(x_0))$.*

Then, there exist $\varepsilon_0 = \varepsilon_0(x_0) > 0$ and two families of compact subsets $\mathcal{K}^\pm_{x_0} = \{K^\pm_{x_0,\varepsilon} : 0 < \varepsilon \le \varepsilon_0\}$ of Ω such that

(3.4) *each set $K^\pm_{x_0,\varepsilon}$ is a neighbourhood of x_0 and each family $\mathcal{K}^\pm_{x_0}$ shrinks nicely at x_0;*

and two corresponding families of continuous functions $\mathcal{V}^\pm_{x_0} = \{v^\pm_{x_0,\varepsilon} : 0 < \varepsilon \le \varepsilon_0\}$ in $W^{1,p}(\Omega)$ such that the following properties hold for every $0 < \varepsilon \le \varepsilon_0$:

(3.5) $v^\pm_{x_0,\varepsilon} = v$ *on* $\Omega \setminus \operatorname{int}\left(K^\pm_{x_0,\varepsilon}\right)$;

(3.6+) $v(x) < v^+_{x_0,\varepsilon}(x) \le v(x) + \varepsilon$ *for every* $x \in \operatorname{int}\left(K^+_{x_0,\varepsilon}\right)$;

(3.6−) $v(x) - \varepsilon \le v^-_{x_0,\varepsilon}(x) < v(x)$ *for every* $x \in \operatorname{int}\left(K^-_{x_0,\varepsilon}\right)$;

(3.7) $f^{**}\left(v^\pm_{x_0,\varepsilon}(x)\,, \nabla v^\pm_{x_0,\varepsilon}(x)\right) = f\left(v^\pm_{x_0,\varepsilon}(x)\,, \nabla v^\pm_{x_0,\varepsilon}(x)\right)$ *for a.e.* $x \in K^\pm_{x_0,\varepsilon}$;

(3.8) $\nabla v^\pm_{x_0,\varepsilon}(x) \in \mathcal{A}_{v^\pm_{x_0,\varepsilon}(x)}(\xi_0)$ *for a.e.* $x \in K^\pm_{x_0,\varepsilon}$;

(3.9) $\displaystyle\int_{K^\pm_{x_0,\varepsilon}} \left\langle h\left(v^\pm_{x_0,\varepsilon}(x)\right), \nabla v^\pm_{x_0,\varepsilon}(x)\right\rangle dx = \int_{K^\pm_{x_0,\varepsilon}} \langle h(v(x))\,, \nabla v(x)\rangle\, dx$ *for every* $h \in \mathcal{C}(\mathbb{R}, \mathbb{R}^N)$.

The main tool of the proof – besides Gromov's method of convex integration as modified and described in [30] – is a rather elementary but powerful construction which enables to glue piecewise affine and differentiable functions, thus modifying the gradient of the latter. This kind of argument was introduced first by De Blasi and Pianigiani in case of Lipschitz continuous functions and improved versions of this result have been given also in [37], [40] and [6]. Here, we are going to state a slightly modified version of Lemma 3.1 of this latter paper. The only differences are: (i) stronger estimates than the original estimate (3.5) of [6], see (3.12+) and (3.12−) below; and (ii) an additional remark concerning the boundaries of the sets defined in the Lemma 3.5 below. Since the proof is essentially the same as that of Lemma 3.1 of [6], we just sketch it and we refer to the paper for the details.

Lemma 3.5. *Let* $\nu_0, \ldots, \nu_N$ *be* $N+1$ *unit vectors of* $\mathbb{R}^N$ *such that*

(a) $0 \in \operatorname{int}(\operatorname{co}\{\nu_i : i = 0, \ldots, N\})$;

and let $w \in W^{1,p}(\Omega)$, $1 < p < +\infty$, *be a continuous, almost everywhere differentiable function on* Ω *such that*

(b) w *is differentiable at some point* $x_0 \in \Omega$ *with (classical) gradient* $\xi_0 = \nabla w(x_0)$.

Then, for every $N+1$ *positive numbers* $\lambda_0, \ldots, \lambda_N$, *there exist* $\varepsilon_0 > 0$ *and two families of compact neighbourhoods* $\{A^\pm_{x_0,\varepsilon} : 0 < \varepsilon \le \varepsilon_0\}$ *of* x_0 *contained in* Ω *such that*

(3.10) $$B_{r_1\varepsilon}(x_0) \subset A^\pm_{x_0,\varepsilon} \subset B_{r_2\varepsilon}(x_0) \subset\subset \Omega, \qquad 0 < \varepsilon \le \varepsilon_0,$$

for some numbers $r_1 = r_1(x_0)$ *and* $r_2 = r_2(x_0)$ *with* $0 < r_1 < r_2$ *with the following additional property: there exist also two families of corresponding continuous, almost everywhere differentiable functions* $\{w^\pm_{x_0,\varepsilon} : 0 < \varepsilon \le \varepsilon_0\}$ *in* $W^{1,p}(\Omega)$ *such that the following properties hold for every* ε:

(3.11) $w^\pm_{x_0,\varepsilon} = w$ *on* $\Omega \setminus \operatorname{int}(A^\pm_{x_0,\varepsilon})$;

(3.12+) $w(x) < w^+_{x_0,\varepsilon}(x) < w(x) + 2\varepsilon$ *for every* $x \in \operatorname{int}(A^+_{x_0,\varepsilon})$;

(3.12−) $w(x) - 2\varepsilon < w^-_{x_0,\varepsilon}(x) < w(x)$ *for every* $x \in \operatorname{int}(A^-_{x_0,\varepsilon})$;

(3.13+) $\varepsilon \ge w^+_{x_0,\varepsilon}(x) - [w(x_0) + \langle \nabla w(x_0)\,, x - x_0\rangle] \ge \varepsilon/2$ *for every* $x \in B_{r_1\varepsilon}(x_0)$;

(3.13−) $-\varepsilon/2 \ge w^-_{x_0,\varepsilon}(x) - [w(x_0) + \langle \nabla w(x_0)\,, x - x_0\rangle] \ge -\varepsilon$ *for every* $x \in B_{r_1\varepsilon}(x_0)$;

(3.14) $\nabla w^\pm_{x_0,\varepsilon}(x) \in \{\xi_0 + \lambda_i \nu_i : i = 0, \ldots, N\}$ *for a.e.* $x \in A^\pm_{x_0,\varepsilon}$.

Moreover, $\left|\partial\left(A^\pm_{x_0,\varepsilon}\right)\right| = 0$ *for all but at most countably many* $0 < \varepsilon \le \varepsilon_0$.

Note that (3.10) shows that the compact sets $A^\pm_{x_0,\varepsilon}$ of the previous lemma rescale properly with ε and, in particular, that they shrink nicely to x_0 as $\varepsilon \to 0_+$.

Proof of Lemma 3.5. Set $C = -\operatorname{co}(\{\lambda_i \nu_i : i = 0, \ldots, N\})$. The polar set C^0 of C is a compact, convex neighbourhood of the origin so that there exist $0 < r_1 < r_2$ such that $B_{2r_1} \subset \pm C^\circ \subset B_{r_2/2}$.

Then, choose $\rho > 0$ such that the closed ball centered at x_0 with radius ρ is contained in Ω and set

$$\eta(\varepsilon) = \frac{1}{\varepsilon} \sup \{ |w(x) - [w(x_0) + \langle \xi_0 , x - x_0 \rangle]| : |x - x_0| < r_2 \varepsilon \}, \qquad 0 < \varepsilon \le \rho / r_2.$$

Obviously, (b) implies that $\eta(\varepsilon) \to 0_+$ as $\varepsilon \to 0$ and we choose $\varepsilon_0 \le \rho/r_2$ so that $\eta(\varepsilon) \le 1/4$ for every $0 < \varepsilon \le \varepsilon_0$. We refer to Lemma 3.1 in [6] for the details. Next, for $\varepsilon > 0$, consider the open sets $V_\varepsilon^\pm = x_0 \pm \operatorname{int}(\varepsilon C^0)$ and the piecewise affine functions $a_\varepsilon^\pm \in W^{1,\infty}(\mathbb{R}^N)$ defined by

$$\begin{cases} a_\varepsilon^+(x) = \varepsilon - \max \{ \langle \xi_0 - \lambda_i \nu_i , x - x_0 \rangle : i = 0, \dots, N \} \\ a_\varepsilon^-(x) = \max \{ \langle \lambda_i \nu_i - \xi_0 , x - x_0 \rangle : i = 0, \dots, N \} - \varepsilon \end{cases} \qquad x \in \mathbb{R}^N, \quad \epsilon > 0.$$

Now, $\nabla a_\varepsilon^\pm(x) \in \{ \lambda_i \nu_i - \xi_0 : i = 0, \dots, N \}$ for a.e. $x \in \mathbb{R}^N$ and every ε and, arguing again as in Lemma 3.1 of [6], it is easy to check that $w(x) - [(w(x_0) + \langle \xi_0 , x - x_0 \rangle) + a_\varepsilon^+(x)]$ is negative on $V_{\varepsilon/2}^+$ and positive on the boundary of $V_{2\varepsilon}^+$. Obviously, the reversed inequalities hold for the same function with a_ε^- instead of a_ε^+ with respect to the sets $V_{\varepsilon/2}^-$ and $V_{2\varepsilon}^-$.
Thus, for $0 < \varepsilon \le \varepsilon_0$, we consider the functions

$$w_{x_0,\varepsilon}^+(x) = \begin{cases} \max \{ w(x) , w(x_0) + \langle \xi_0 , x - x_0 \rangle + a_\varepsilon^+(x) \} & x \in V_{2\varepsilon}^+, \\ w(x) & x \in \Omega \setminus V_{2\varepsilon}^+, \end{cases}$$

$$w_{x_0,\varepsilon}^-(x) = \begin{cases} \min \{ w(x) , w(x_0) + \langle \xi_0 , x - x_0 \rangle + a_\varepsilon^-(x) \} & x \in V_{2\varepsilon}^-, \\ w(x) & x \in \Omega \setminus V_{2\varepsilon}^-. \end{cases}$$

They are both continuous, almost everywhere differentiable on Ω and in $W^{1,p}(\Omega)$. We define the corresponding sets $A_{x_0,\varepsilon}^\pm$ to be the closure of the open sets $\{ w_{x_0,\varepsilon}^+ > w \}$ and $\{ w_{x_0,\varepsilon}^- < w \}$ respectively. Then, it is clear that (3.10) holds since

$$B_{r_1 \varepsilon}(x_0) \subset V_{x_0,\varepsilon/2}^\pm \subset A_{x_0,\varepsilon}^\pm \subset V_{x_0,2\varepsilon}^\pm \subset B_{r_2 \varepsilon}(x_0), \qquad 0 < \varepsilon \le \varepsilon_0,$$

and we note that the gradient of $w_{x_0,\varepsilon}^\pm$ takes the values $\xi_0 + \lambda_i \nu_i$ only almost everywhere on $A_{x_0,\varepsilon}^\pm$ whence (3.14). Moreover, from (3.10) we have $|w(x) - [w(x_0) + \langle \xi_0 , x - x_0 \rangle]| \le \varepsilon \eta(\varepsilon) \le \varepsilon/4$ and $|a_{x_0,\varepsilon}^\pm(x)| \le \varepsilon$ for every $x \in \operatorname{int}(A_{x_0,\varepsilon}^\pm)$ so that (3.12+) and (3.12−) follow. As to the other properties stated in the thesis, they can be proved by the very same arguments of Lemma 3.1 of [6].
Finally, $w = w_{x_0,\varepsilon}^\pm$ on the boundary of $A_{x_0,\varepsilon}^\pm$. Thus, these boundaries have to be disjoint for different values of ε because the functions $a_{x_0,\varepsilon}^\pm$ never take the same value at the same point for different ε. As they are all contained in some bounded set, they must be negligible for all but countably many ε at most. □

Proof of Proposition 3.4. We prove the + case only and we omit the superscript +. Set $\eta_0 = v(x_0)$ and $\xi_0 = \nabla v(x_0)$ so that $(\eta_0 , \xi_0) \in \mathcal{D}$ by (b) and let $\delta = \delta(\eta_0 , \xi_0) > 0$ be such that all points (η , ξ_0) are in $\mathcal{D}$ too for every $\eta \in I = [\eta_0 - \delta , \eta_0 + \delta]$. Then, choose $N+1$ unit vectors $\nu_0, \dots, \nu_N$ in $\mathbb{R}^N$ such that

$$0 \in \operatorname{int} (\operatorname{co} \{ \nu_i : i = 0, \dots, N \}) \tag{3.15}$$

and let $a^i : I \to (0 , +\infty)$ be the positive, bounded and lower semicontinuous functions associated with the vectors ν_i by Proposition 3.3. In particular, we have that

$$f^{**} \left(\eta , \xi_0 + a^i(\eta) \nu_i \right) = f \left(\eta , \xi_0 + a^i(\eta) \nu_i \right), \qquad \eta \in I. \tag{3.16}$$

Set $\sigma = \min \{ a^i(\eta) : \eta \in I, \, i = 0, \dots, N \} > 0$ and consider the Moreau-Yosida approximations (up to the constant $\sigma/2k$) of the functions a^i, i.e.

$$a_k^i(\eta) = \min \left\{ a^i(\eta') + k|\eta' - \eta| : \eta' \in I \right\} - \frac{\sigma}{2k}, \qquad \eta \in I.$$

These approximating functions a_k^i have the following properties:

(3.17) $a_1^i(\eta) \geq \frac{\sigma}{2}$ and $a_{k+1}^i(\eta) - a_k^i(\eta) \geq \Delta_k$ for every $\eta \in I$ and $k \geq 1$;

where $\Delta_k = \dfrac{\sigma}{2k(k+1)}$ and

(3.18) every a_k^i is Lipschitz continuous on I with Lipschitz constant k;

(3.19) $a_k^i(\eta) \to a^i(\eta)$ for every $\eta \in I$.

Now, the preliminaries are over and we divide the rest of the proof into three steps.

Step 1. There exists $\varepsilon_0 > 0$ and a family of compact neighbourhoods $\{K_{x_0,\varepsilon} : 0 < \varepsilon \leq \varepsilon_0\}$ of x_0 such that

$$|\partial (K_{x_0,\varepsilon})| = 0, \qquad 0 < \varepsilon \leq \varepsilon_0, \tag{3.20}$$

with the following property: whatever positive sequence $\{\omega_k\}_k$ we choose, there exists a corresponding sequence of continuous, almost everywhere differentiable functions $\{v_{k,x_0,\varepsilon}\}_k$ in $W^{1,p}(\Omega)$ such that $v_{0,x_0,\varepsilon} = v$ and

(3.21a) $v_{k,x_0,\varepsilon} = v$ on $\Omega \setminus \text{int } (K_{x_0,\varepsilon})$;

(3.21b) $0 < v_{k,x_0,\varepsilon}(x) - v_{k-1,x_0,\varepsilon}(x) < \min\left\{\dfrac{\varepsilon}{2^k}, \omega_k, \dfrac{\Delta_k}{4(k+1)}\right\}$ for every $x \in \text{int } (K_{x_0,\varepsilon})$;

(3.21c) $|v_{k,x_0,\varepsilon}(x) - \eta_0| \leq \dfrac{\delta}{2} + \cdots + \dfrac{\delta}{2^k}$ for every $x \in K_{x_0,\varepsilon}$;

(3.21d) $\nabla v_{k,x_0,\varepsilon}(x) \in \displaystyle\bigcup_{0 \leq i \leq N} \left[\xi_0 + a_k^i(v_{k,x_0,\varepsilon}(x))\nu_i \, , \xi_0 + a_{k+1}^i(v_{k,x_0,\varepsilon}(x))\nu_i\right]$ for a.e. $x \in K_{x_0,\varepsilon}$.

Recall that $[\xi_1 \, , \xi_2]$ stands for the closed segment in $\mathbb{R}^N$ whose endpoints are the vectors ξ_1 and ξ_2.

Proof of Step 1. Let $\{\omega_k\}$ be a sequence of positive numbers to be chosen in the following Step2. We apply Lemma 3.5 with $w = v$, x_0, the unit vectors $\nu_0, \ldots, \nu_N$ and the positive numbers

$$\lambda_i = \frac{a_1^i(\eta_0) + a_2^i(\eta_0)}{2}, \qquad i = 0, \ldots, N.$$

We thus find the numbers $0 < r_1 < r_2$ and $\varepsilon_0 > 0$, a family $\{A_{x_0,\varepsilon} : 0 < \varepsilon \leq \varepsilon_0\}$ of compact neighbourhoods of x_0 satisfying (3.10) and a corresponding family of continuous, almost everywhere differentiable functions $\{w_{x_0,\varepsilon} : 0 < \varepsilon \leq \varepsilon_0\}$ in $W^{1,p}(\Omega)$ such that (3.11), (3.12+), (3.13+) and (3.14) hold. We recall that the sets $A_{x_0,\varepsilon}$ have negligible boundaries for all but possibly countably many ε. We assume that $\varepsilon_0 \leq \delta/2$ and also that it is small enough so that

$$|v(x) - \eta_0| < \min\left\{\frac{\delta}{4}, \frac{\Delta_1}{8}\right\} \qquad x \in B_{r_2\varepsilon_0}(x_0). \tag{3.22}$$

As $w_{x_0,\varepsilon}$ converges to v uniformly on Ω as $\varepsilon \to 0_+$ by (3.11) and (3.12+), for every $0 < \varepsilon \leq \varepsilon_0$ we find $0 < \varepsilon' \leq \varepsilon$ such that $\left|\partial \left(A_{x_0,\varepsilon'}\right)\right| = 0$ and

$$0 < w_{x_0,\varepsilon'}(x) - v(x) < \min\left\{\frac{\varepsilon}{2}, \omega_1, \frac{\Delta_1}{8}\right\}, \qquad x \in \text{int } \left(A_{x_0,\varepsilon'}\right). \tag{3.23}$$

Then, for every $0 < \varepsilon \leq \varepsilon_0$, we set $v_{1,x_0,\varepsilon} = w_{x_0,\varepsilon'}$ and $K_{x_0,\varepsilon} = A_{x_0,\varepsilon'}$. It follows that $v_{1,x_0,\varepsilon}$ is a continuous, almost everywhere differentiable function in $W^{1,p}(\Omega)$ and we emphasize that (3.10) of Lemma 3.5 implies that the sets $\{K_{x_0,\varepsilon} : 0 < \varepsilon \leq \varepsilon_0\}$ shrink nicely at x_0 and that our choice of ε'

implies also that (3.20) holds true. Moreover, (3.21a) and (3.21b) obviously hold by construction – recall that $v_{0,x_0,\varepsilon} = v$ – and (3.21c) follows from (3.22) and (3.23) since $\varepsilon \le \varepsilon_0 \le \delta/2$.
Now, we claim that (3.21d) holds with $k = 1$, i.e.

$$\nabla v_{1,x_0,\varepsilon}(x) \in \bigcup_{0\le i\le N} \left[\xi_0 + a_1^i(v_{1,x_0,\varepsilon}(x))\nu_i \,, \xi_0 + a_2^i(v_{1,x_0,\varepsilon}(x))\nu_i\right] \qquad \text{for a.e. } x \in K_{x_0,\varepsilon}. \tag{3.24}$$

Since the values of $\nabla v_{1,x_0,\varepsilon}$ are taken among the $N+1$ vectors $\xi_0 + \frac{a_1^i(\eta_0)+a_2^i(\eta_0)}{2}\nu_i$ almost everywhere on $K_{x_0,\varepsilon}$ by construction, we have to check that

$$a_1^i(v_{1,x_0,\varepsilon}(x)) \le \frac{a_1^i(\eta_0)+a_2^i(\eta_0)}{2} \le a_2^i(v_{1,x_0,\varepsilon}(x))$$

for every i and every $x \in K_{x_0,\varepsilon}$. We prove the second inequality only, the first one being similar. Indeed, (3.17) yields

$$\frac{a_1^i(\eta_0)+a_2^i(\eta_0)}{2} \le a_2^i(\eta_0) - \frac{\Delta_1}{2} = a_2^i(v_{1,x_0,\varepsilon}(x)) + \left[a_2^i(\eta_0) - a_2^i(v_{1,x_0,\varepsilon}(x))\right] - \frac{\Delta_1}{2}$$

and then (3.18), (3.22) and (3.23) yield

$$a_2^i(\eta_0) - a_2^i(v_{1,x_0,\varepsilon}(x)) \le 2\,(|\eta_0 - v(x)| + |v(x) - v_{1,x_0,\varepsilon}(x)|) \le 2\left(\frac{\Delta_1}{8} + \frac{\Delta_1}{8}\right) = \frac{\Delta_1}{2}$$

so that the inequality follows.
Now, we want to define the second function $v_{2,x_0,\varepsilon}$ of the sequence on the same set $K_{x_0,\varepsilon}$. On account of (3.20), it is enough we do this on the interior of $K_{x_0,\varepsilon}$ only. To this aim, fix $0<\varepsilon\le\varepsilon_0$, let $v_{1,x_0,\varepsilon}$ be differentiable at some point $y \in \text{int}\,(K_{x_0,\varepsilon})$ where (3.24) holds with $x = y$ and set $\eta_0' = v_{1,x_0,\varepsilon}(y)$ and $\xi_0' = \nabla v_{1,x_0,\varepsilon}(y)$. Thus, there exist an index $i_0 \in \{0,\dots,N\}$ and a positive number μ such that

$$\xi_0' = \xi_0 + \mu\nu_{i_0} \qquad \text{and} \qquad a_1^{i_0}(\eta_0') \le \mu \le a_2^{i_0}(\eta_0'). \tag{3.25}$$

Then, we want to apply Lemma 3.5 to the function $w = v_{1,x_0,\epsilon}$ at the point $x_0 = y$, this time with new unit vectors

$$\nu_{i_0}' = \nu_{i_0} \qquad \text{and} \qquad \nu_i' = \frac{\frac{a_2^i(\eta_0')+a_3^i(\eta_0')}{2}\nu_i - \mu\nu_{i_0}}{\left|\frac{a_2^i(\eta_0')+a_3^i(\eta_0')}{2}\nu_i - \mu\nu_{i_0}\right|} \qquad \text{for } i \ne i_0,$$

and new positive numbers

$$\lambda_{i_0}' = \frac{a_2^i(\eta_0')+a_3^i(\eta_0')}{2} - \mu \qquad \text{and} \qquad \lambda_i' = \left|\frac{a_2^i(\eta_0')+a_3^i(\eta_0')}{2}\nu_i - \mu\nu_{i_0}\right| \qquad \text{for } i \ne i_0.$$

Indeed, λ_{i_0}' is positive by (3.25) and the monotonicity of the sequence a_k^i with respect to k and the other λ_i' are positive too because of (3.15). This latter property of the original unit vectors ν_i implies that the same property is shared by the new ν_i'. Note also that this choice of the unit vectors ν_i' and the positive numbers λ_i' gives

$$\xi_0' + \lambda_i'\nu_i' = \xi_0 + \frac{a_2^i(\eta_0')+a_3^i(\eta_0')}{2}\nu_i, \qquad i = 0,\dots,N.$$

Therefore, Lemma 3.5 yields this time a new family $\{A_{y,\vartheta}:\ 0<\vartheta\le\vartheta_0\}$ of compact neighbourhoods of y contained in the interior of $K_{x_0,\varepsilon}$ such that

$$B_{s_1\vartheta}(y) \subset A_{y,\vartheta} \subset B_{s_2\vartheta}(y) \subset\subset \text{int}\,(K_{x_0,\varepsilon}) \subset\subset \Omega \tag{3.26}$$

for some numbers $0 < s_1 < s_2$, $\vartheta_0 = \vartheta_0(y\,,\varepsilon) > 0$ and also a new family of continuous, almost everywhere differentiable functions $\{w_{y,\vartheta} :\ 0 < \vartheta \le \vartheta_0\}$ in $W^{1,p}(\Omega)$ such that the following properties hold for every $0 < \vartheta \le \vartheta_0$:

$$w_{y,\vartheta} = v_{1,x_0,\varepsilon} \text{ on } \Omega \setminus \text{int } (A_{y,\vartheta}); \tag{3.27a}$$

$$v_{1,x_0,\varepsilon}(x) < w_{y,\vartheta}(x) < v_{1,x_0,\varepsilon}(x) + 2\vartheta \text{ for every } x \in \text{int } (A_{y,\vartheta}); \tag{3.27b}$$

$$\nabla w_{y,\vartheta}(x) \in \left\{ \xi_0 + \frac{a_2^i(\eta_0') + a_3^i(\eta_0')}{2}\nu_i :\ i = 0, \dots, N \right\} \text{ for a.e. } x \in A_{y,\vartheta}. \tag{3.27c}$$

Moreover, all sets $A_{y,\vartheta}$ but countably many at most have negligible boundaries and, recalling that $v_{1,x_0,\varepsilon}$ is continuous, we may assume that ϑ_0 is small enough so that

$$|v_{1,x_0,\varepsilon}(x) - \eta_0'| \le \frac{\Delta_2}{12}, \qquad x \in B_{s_2\vartheta_0}(y).$$

Now, arguing as above, for every $0 < \vartheta \le \vartheta_0$, we choose $0 < \vartheta' \le \vartheta$ such that $\left|\partial\left(A_{y,\vartheta'}\right)\right| = 0$ and

$$0 < w_{y,\vartheta'}(x) - v_{1,x_0,\varepsilon}(x) \le \min\left\{ \frac{\varepsilon}{4}\,, \omega_2\,, \frac{\Delta_2}{12} \right\}, \qquad x \in \text{int } \left(A_{y,\vartheta'}\right). \tag{3.28}$$

This inequality, together with (3.21c) with $k = 1$ and the assumption $\varepsilon \le \varepsilon_0 \le \delta/2$, shows that

$$|w_{y,\vartheta'}(x) - \eta_0| \le \frac{\delta}{2} + \frac{\delta}{4}, \qquad x \in A_{y,\vartheta'}. \tag{3.29}$$

Moreover, from (3.27c) and the inequalities

$$a_2^i(w_{y,\vartheta'}(x)) \le \frac{a_2^i(\eta_0') + a_3^i(\eta_0')}{2} \le a_3^i(w_{y,\vartheta'}(x))$$

which hold for every i and $x \in A_{y,\vartheta'}$ and can be proved by the very same argument we have used for the case $k = 1$, we conclude that

$$\nabla w_{y,\vartheta'}(x) \in \bigcup_{0 \le i \le N} \left[\xi_0 + a_2^i(w_{y,\vartheta'}(x))\nu_i\,, \xi_0 + a_3^i(w_{y,\vartheta'}(x))\nu_i \right], \qquad \text{for a.e. } x \in A_{y,\vartheta'}. \tag{3.30}$$

When $v_{2,x_0,\varepsilon}$ will be defined, see (3.31) below, (3.28), (3.29) and (3.30) will imply (3.21b), (3.21c) and (3.21d) with $k = 2$, respectively. In the sequel, to simplify the notation, we just write $w_{y,\vartheta}$ and $A_{y,\vartheta}$ instead of $w_{y,\vartheta'}$ and $A_{y,\vartheta'}$ respectively.

Now, we have to glue the functions $w_{y,\vartheta}$ we have defined so far. Indeed, we have associated with every $0 < \varepsilon \le \varepsilon_0$ and every point $y \in \text{int } (K_{x_0,\varepsilon})$, where $v_{1,x_0,\varepsilon}$ is differentiable and (3.24) holds, a family $\{A_{y,\vartheta} :\ 0 < \vartheta \le \vartheta_0\}$ of compact neighbourhoods of y contained in the interior of $K_{x_0,\varepsilon}$ which, for all such y, give a Vitali's covering of int $(K_{x_0,\varepsilon})$. Hence, by Vitali's covering theorem, we can choose countably many points y_j in int $(K_{x_0,\varepsilon})$ and positive numbers $0 < \vartheta_j \le \vartheta_0(y_j, \varepsilon)$ such that the corresponding compact sets $A_j = A_{y_j,\vartheta_j}$ are pairwise disjoint and cover int $(K_{x_0,\varepsilon})$ – and hence the whole set $K_{x_0,\varepsilon}$ by (3.20) – up to a null set. We set also $w_j = w_{y_j,\vartheta_j}$ for every j.

Now, we define $v_{2,x_0,\varepsilon}$, the second term of the sequence $\{v_{k,x_0,\varepsilon}\}_k$ we are looking for, by gluing together the functions w_j, i.e.

$$v_{2,x_0,\varepsilon}(x) = v_{1,x_0,\varepsilon}(x) + \sum_{j \ge 1} \left[w_j(x) - v_{1,x_0,\varepsilon}(x) \right], \qquad x \in \Omega. \tag{3.31}$$

The support of each summand is contained in A_j and these sets are pairwise disjoint. Thus, there is at most one nonzero term of the series for every $x \in \Omega$ and the resulting function $v_{2,x_0,\varepsilon}$ is almost everywhere differentiable on Ω because each summand $w_j - v_{1,x_0,\varepsilon}$ enjoys the same property and the boundary of each set A_j is negligible. Next, we claim that the series converges also uniformly on Ω and strongly in $W^{1,p}(\Omega)$. Indeed, $|A_j| \to 0$ unless there are only finitely many sets A_j and hence (3.26) and (3.27b) imply that $w_j - v_{1,x_0,\varepsilon}$ goes to zero uniformly on Ω as $j \to +\infty$. Since the sets A_j are disjoint, this shows that the series converges uniformly on Ω. As to the gradients, the very same property of the sets A_j implies that

$$\sum_{j\geq 1} \int_\Omega |\nabla w_j - \nabla v_{1,x_0,\varepsilon}|^p \, dx \leq 2^{p-1} \sum_{j\geq 1} \int_{A_j} |\nabla w_j|^p \, dx + 2^{p-1} \int_\Omega |\nabla v_{1,x_0,\varepsilon}|^p \, dx$$

and the right hand side is finite because the gradients ∇w_j are uniformly bounded on A_j by (a) of Proposition 3.3. Thus, $v_{2,x_0,\varepsilon}$ is in $W^{1,p}(\Omega)$.
Finally, $v_{2,x_0,\varepsilon}$ satisfies the required properties (3.21) with $k = 2$ because of the corresponding properties (3.26), (3.28), (3.29) and (3.30) of the functions w_j. We wish to remark in particular that (3.21d) holds true not only on the interior but on the whole compact set $K_{x_0,\varepsilon}$ by (3.20).
The other terms of the sequence $\{v_{k,x_0,\varepsilon}\}_k$ are defined recursively in the very same way we have got $v_{2,x_0,\varepsilon}$ from $v_{1,x_0,\varepsilon}$ and this completes the proof of Step 1.
Before going on with the second step, we wish to point out that, by construction, each function $v_{k,x_0,\varepsilon}$ actually depends only on the choice of the numbers $\omega_1, \dots, \omega_k$ and not on the remaining terms ω_h for $h \geq k+1$. This will be crucial in the following step where we will exploit the possibility of choosing the sequence $\{\omega_k\}_k$ so as to get strong convergence in $W^{1,p}(\Omega)$ of the sequence of approximating functions $\{v_{k,x_0,\varepsilon}\}_k$.

Step 2. For every $0 < \varepsilon \leq \varepsilon_0$, there exists a sequence $\{\omega_k\}_k$ with $\omega_1 = 1$ such that the corresponding sequence $\{v_{k,x_0,\varepsilon}\}_k$ of Step 1 converges uniformly on Ω and strongly in $W^{1,p}(\Omega)$ to a continuous function $v_{x_0,\varepsilon} \in W^{1,p}(\Omega)$ satisfying (3.5), (3.6+) and

$$|v_{x_0,\varepsilon}(x) - \eta_0| \leq \delta, \qquad x \in K_{x_0,\varepsilon}. \tag{3.32}$$

Proof of Step 2. Fix $0 < \varepsilon \leq \varepsilon_0$ and let $v_{1,x_0,\varepsilon}$ be the function defined in Step 1 with $\omega_1 = 1$. We extend $\nabla v_{1,x_0,\varepsilon}$ to $\mathbb{R}^N$ setting it equal to zero outside Ω so that $\nabla v_{1,x_0,\varepsilon}$ is in $L^p(\mathbb{R}^N, \mathbb{R}^N)$. Let $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ be the standard mollifying kernel and set as usual $\varphi_r(x) = r^{-N}\varphi(x/r)$ for every $x \in \mathbb{R}^N$ and $r > 0$. We choose $0 < r_1 \leq 1/2$ such that

$$\int_{\mathbb{R}^N} |\varphi_{r_1} * \nabla v_{1,x_0,\varepsilon}(x) - \nabla v_{1,x_0,\varepsilon}(x)|^p \, dx \leq 1.$$

Then, set $\omega_2 = r_1\omega_1$ and let $v_{2,x_0,\varepsilon}$ be as in Step 1 for this value of ω_2. Extend again $\nabla v_{2,x_0,\varepsilon}$ to $\mathbb{R}^N$ setting it equal to zero outside Ω and choose $0 < r_2 \leq 1/4$ such that

$$\int_{\mathbb{R}^N} |\varphi_{r_2} * \nabla v_{2,x_0,\varepsilon}(x) - \nabla v_{2,x_0,\varepsilon}(x)|^p \, dx \leq \frac{1}{2}.$$

Iterating this argument, for every $0 < \varepsilon \leq \varepsilon_0$, we find a sequence of continuous functions $\{v_{k,x_0,\varepsilon}\}_k$ in $W^{1,p}(\Omega)$ and a corresponding sequence of positive numbers $r_k = r_k(\varepsilon)$ with $r_k \leq 1/2^k$ such that

$$\int_{\mathbb{R}^N} |\varphi_{r_k} * \nabla v_{k,x_0,\varepsilon}(x) - \nabla v_{k,x_0,\varepsilon}(x)|^p \, dx \leq \frac{1}{k} \tag{3.33}$$

for every k and such that (3.21) hold. In particular, we have

$$0 \leq v_{k+1,x_0,\varepsilon}(x) - v_{k,x_0,\varepsilon}(x) \leq \omega_{k+1}, \qquad x \in \Omega,$$

by (3.21a) and (3.21b). Since $\omega_{k+1}/\omega_k = r_k < 1/2^k$, it follows that $\{v_{k,x_0,\varepsilon}\}_k$ is uniformly Cauchy on Ω and hence it converges uniformly on Ω to some continuous function $v_{x_0,\varepsilon}$. Moreover, the limit function $v_{x_0,\varepsilon}$ thus obtained satisfies (3.5), (3.6+) and (3.32) because of the corresponding properties (3.21a), (3.21b) and (3.21c) respectively of the approximating functions $v_{k,x_0,\varepsilon}$.

Then, we wish to prove that $v_{x_0,\varepsilon}$ is in the Sobolev space $W^{1,p}(\Omega)$ and that $v_{k,x_0,\varepsilon}$ converges strongly to $v_{x_0,\varepsilon}$ in $W^{1,p}(\Omega)$.
Indeed, (3.21a), (3.21d) and the uniform boundedness of the functions a^i_k show that the sequence $\{v_{k,x_0,\varepsilon}\}_k$ is bounded in $W^{1,p}(\Omega)$ and this implies that $v_{x_0,\varepsilon}$ is in $W^{1,p}(\Omega)$ as well. Next, we prove that $\nabla v_{k,x_0,\varepsilon}$ actually converges strongly to $\nabla v_{x_0,\varepsilon}$ in $L^p(\Omega, \mathbb{R}^N)$. Indeed, recalling that everything is set equal to zero outside Ω, we have

$$\|\nabla v_{k,x_0,\varepsilon} - \nabla v_{x_0,\varepsilon}\|_p \leq$$
$$\leq \|\nabla v_{k,x_0,\varepsilon} - \varphi_{r_k} * \nabla v_{k,x_0,\varepsilon}\|_p + \|\varphi_{r_k} * (\nabla v_{k,x_0,\varepsilon} - \nabla v_{x_0,\varepsilon})\|_p + \|\varphi_{r_k} * \nabla v_{x_0,\varepsilon} - \nabla v_{x_0,\varepsilon}\|_p$$

and, from (3.33) and standard properties of convolutions, we conclude that the first and the third terms at the right hand side go to zero as $k \to +\infty$. As to the second term, integrating by parts, we get

$$\|\varphi_{r_k} * (\nabla v_{k,x_0,\varepsilon} - \nabla v_{x_0,\varepsilon})\|_p = \|\nabla \varphi_{r_k} * (v_{k,x_0,\varepsilon} - v_{x_0,\varepsilon})\|_p \leq r_k^{-1} |\Omega|^{1/p} \|\nabla \varphi\|_1 \|v_{k,x_0,\varepsilon} - v_{x_0,\varepsilon}\|_\infty$$

and we are left to estimate $\|v_{k,x_0,\varepsilon} - v_{x_0,\varepsilon}\|_\infty$. Recalling (3.6+) and that $\omega_{k+1}/\omega_k = r_k < 1/2^k$ by definition, we obtain

$$\|v_{k,x_0,\varepsilon} - v_{x_0,\varepsilon}\|_\infty \leq \sum_{h\geq 1} \|v_{k+h,x_0,\varepsilon} - v_{k+h-1,x_0,\varepsilon}\|_\infty \leq \sum_{h \geq 1} \omega_{k+h} \leq 2\omega_{k+1} = 2\omega_k r_k \tag{3.34}$$

so that combining the three previous inequalities, we conclude that $\nabla v_{k,x_0,\varepsilon} \to \nabla v_{x_0,\varepsilon}$ strongly in $L^p(\Omega, \mathbb{R}^N)$.

Step 3. For every $0 < \varepsilon \leq \varepsilon_0$ and for the same sequence $\{\omega_k\}_k$ of Step 2, we have that

$$a^i_k(v_{k,x_0,\varepsilon}(x)) \to a^i(v_{x_0,\varepsilon}(x)), \qquad x \in \Omega, \quad i = 0, \dots, N. \tag{3.35}$$

Proof of Step 3. First, notice that (3.35) is well defined because of (3.21c) and (3.32). If x is not in the interior of $K_{x_0,\varepsilon}$, then $v_{k,x_0,\varepsilon}(x) = v_{x_0,\varepsilon}(x)$ and the conclusion follows. Therefore, suppose x is in the interior of $K_{x_0,\varepsilon}$. Then,

$$|a^i_k(v_{k,x_0,\varepsilon}(x)) - a^i(v_{x_0,\varepsilon}(x))| \leq |a^i_k(v_{k,x_0,\varepsilon}(x)) - a^i_k(v_{x_0,\varepsilon}(x))| + |a^i_k(v_{x_0,\varepsilon}(x)) - a^i(v_{x_0,\varepsilon}(x))|$$

and, from (3.19), we conclude that the last summand goes to zero as $k \to +\infty$. Moreover, (3.18) and (3.34) yield

$$|a^i_k(v_{k,x_0,\varepsilon}(x)) - a^i_k(v_{x_0,\varepsilon}(x))| \leq k\,|v_{k,x_0,\varepsilon}(x) - v_{x_0,\varepsilon}(x)| \leq 2k\omega_k r_k \leq \frac{k}{2^{k-1}}$$

because ω_k is decreasing with $\omega_1 = 1$ and $0 < r_k \leq 1/2^k$ and this yields the conclusion.

We can now summarize the results we have obtained so far and draw the conclusion. The functions $\mathcal{V}_{x_0} = \{v_{x_0,\varepsilon} : 0 < \varepsilon \leq \varepsilon_0\}$ we have defined in Steps 1 and 2 satisfy (3.5) and (3.6+) and the corresponding compact sets $\mathcal{K}_{x_0} = \{K_{x_0,\varepsilon} : 0 < \varepsilon \leq \varepsilon_0\}$ shrink nicely at x_0 as proved in Step 1. Therefore, we are only left to prove that $v_{x_0,\varepsilon}$ satisfies (3.7), (3.8) and (3.9). Indeed, up to a subsequence, $\nabla v_{k,x_0,\varepsilon}(x) \to \nabla v_{x_0,\varepsilon}(x)$ for a.e. x in Ω because $v_{k,x_0,\varepsilon} \to v_{x_0,\varepsilon}$ strongly in $W^{1,p}(\Omega)$. Thus, from (3.21d) and (3.35), we see that

$$\nabla v_{x_0,\varepsilon}(x) \in \bigcup_{0 \leq i \leq N} \left\{\xi_0 + a^i(v_{x_0,\varepsilon}(x))\nu_i\right\} \qquad \text{for a.e. } x \in K_{x_0,\varepsilon}.$$

From (3.32) and (3.16) we conclude that (3.7) and (3.8) hold true. Finally, (3.9) follows immediately from (3.5) as we note that $\langle h\,(v_{x_0,\varepsilon})\ , \nabla v_{x_0,\varepsilon}\rangle - \langle h(v)\,, \nabla v\rangle$ is the divergence of a function in $W^{1,p}_0(\Omega)$. □

4. Proof of the main result

In this final section, we put together and exploit the tools developed in the previous sections and prove our attainment result, Theorem 2.1. Before starting with the proof, we wish to point out a property of the set $\mathcal{M}$ of those points $(\eta, \xi) \in \mathcal{D}$ such that η is a strict, local extremum point of the restriction of q to the section $\mathcal{D}^\xi$, i.e.

$$\mathcal{M} = \Big\{(\eta, \xi) \in \mathcal{D} : \eta \text{ is a strict, local extremum point of } \eta' \in \mathcal{D}^\xi \to q(\eta', \xi)\Big\}. \tag{4.1}$$

We have the following result about this set $\mathcal{M}$.

Proposition 4.1. *Let* $f\colon \mathbb{R}\times\mathbb{R}^N \to [0, +\infty)$ *satisfy* (H1), (H2$_p$) *for some* $1 < p < +\infty$ *and* (H3). *Assume also that* (2.6) *holds. Then,*

(a) $\mathcal{M} = \bigcup_j (\{m_j\}\times D_j)$ *where* D_j *is a connected component of* $\mathcal{D}_{m_j}$*;*

(b) $\mathcal{D} \setminus \mathcal{M}$ *is open.*

Proof. Choose (η_0, ξ_0) in $\mathcal{D}$ and let $\delta > 0$ be such that the cylinder $C_\delta = [\eta_0 - \delta, \eta_0 + \delta]\times B_\delta(\xi_0)$ remains in $\mathcal{D}$ as well. Then, the projection of $C_\delta \cap \mathcal{M}$ on $\mathbb{R}$ must be finite because of (2.8) which follows from (2.6) and because of (b) of Proposition 3.1. As $\mathcal{D}$ is covered by countably many cylinders C_δ, we conclude that the projection of $\mathcal{M}$ on $\mathbb{R}$ is countable, say $\{m_j\}_j$. Now, each section $\mathcal{D}_{m_j}$ has at most countably many connected components. Let $D^i_{m_j}$ be those connected components of $\mathcal{D}_{m_j}$ containing points of $\mathcal{M}$. Recalling (b) of Proposition 3.1 again and possibly relabelling the sets $\{m_j\}\times D^i_{m_j}$, we get (a).
To prove (b), choose (η_0, ξ_0) in $\mathcal{D}\setminus\mathcal{M}$ and let the cylinder C_δ be contained in $\mathcal{D}$ as before. By (2.6) and (2.8) again, we can assume that $\delta > 0$ is small enough to have that no strict, local extrema of $\eta \in \mathcal{D}^{\xi_0} \to q(\eta, \xi_0)$ fall into $[\eta_0 - \delta, \eta_0 + \delta]$. Thus, $C_\delta \subset \mathcal{D} \setminus \mathcal{M}$ by (b) of Proposition 3.1 once more. □

Now, we can start with the proof of the attainment result for the nonconvex problem $(\mathcal{P})$.

Proof of Theorem 2.1. Let v be a solution to $(\mathcal{P}^{**})$ such that $I^{**}(v) < +\infty$. This is obviously true for $1 < p \le N$ because of (H2$_p$) but need not be true for other values of p and special choices of f and u_0. However, were I^{**} identically equal to $+\infty$ on the set of feasible functions, the thesis of the theorem would be trivially true. Moreover, v is Hölder continuous and almost everywhere differentiable on Ω, either because it is in $W^{1,p}(\Omega)$ and $p > N$ or because of Theorem 3.1 in [4] if $1 < p \le N$.
The proof will be obtained by finding a new solution u to $(\mathcal{P}^{**})$ such that

$$f(u(x), \nabla u(x)) = f^{**}(u(x), \nabla u(x)) \qquad \text{for a.e. } x \in \Omega. \tag{4.2}$$

To this aim, recalling that f and f^{**} are different on $\mathcal{D}$ only, we first modify v so as to find this new solution u to $(\mathcal{P}^{**})$ with the further property that the set

$$E_u = \{x \in \Omega : u \text{ is differentiable at } x \text{ and } (u(x), \nabla u(x)) \in \mathcal{D} \setminus \mathcal{M}\} \tag{4.3}$$

is negligible. Here $\mathcal{M}$ is the set of strict, local extrema of q defined by (4.1). Then, we show that the set

$$F_u = \{x \in \Omega : u \text{ is differentiable at } x \text{ and } (u(x), \nabla u(x)) \in \mathcal{M}\}$$

is negligible too so that (4.2) follows. We divide the proof into two claims.

Claim 1. There exists a solution u to $(\mathcal{P}^{**})$ such that the set E_u is negligible.

Proof of Claim 1. Let v be the solution to $(\mathcal{P}^{**})$ considered at the beginning and let E_v be the set defined by (4.3) with v instead of u. Assume that E_v has positive measure otherwise we set $u = v$ and the claim is proved. Let also x_0 be a density point of E_v and set $\eta_0 = v(x_0)$

and $\xi_0 = \nabla v(x_0)$. Thus, $(\eta_0\,,\xi_0)$ is in $\mathcal{D}\setminus\mathcal{M}$ by definition and, since this set is open by (b) of Proposition 4.1, there is $\delta > 0$ such that $(\eta\,,\xi_0)$ is in $\mathcal{D}\setminus\mathcal{M}$ too for every $\eta \in I$, $I = [\eta_0 - \delta\,,\eta_0 + \delta]$. Thus, recalling (3.3), we have

$$f^{**}(\eta\,,\xi) = \langle d(\eta)\,,\xi\rangle + q(\eta), \qquad \xi \in \mathcal{A}_\eta(\xi_0), \tag{4.4}$$

for every $\eta \in I$ where we have set $d(\eta) = d(\eta\,,\xi_0)$ and $q(\eta) = q(\eta\,,\xi_0)$ for every $\eta \in I$ because of (b) of Proposition 3.1. Moreover, q is monotone on the interval I by the choice of δ. Now, since v is differentiable at x_0 and $f^{**}(\eta_0\,,\xi_0) < f(\eta_0\,,\xi_0)$, we apply Proposition 3.4 and we thus find two families of compact neighbourhoods $\mathcal{K}^\pm_{x_0} = \{K^\pm_{x_0,\varepsilon} : 0 < \varepsilon \le \varepsilon_0\}$ of x_0 that shrink nicely at x_0 and two corresponding families of continuous functions $\mathcal{V}^\pm_{x_0} = \{v^\pm_{x_0,\varepsilon} : 0 < \varepsilon \le \varepsilon_0\}$ in $W^{1,p}(\Omega)$ featuring the properties stated in Proposition 3.4. Moreover, we choose $0 < \varepsilon_0 \le 1$ small enough to have

$$|v(x) - \eta_0| \le \delta \quad \text{and} \quad |v^\pm_{x_0,\varepsilon}(x) - \eta_0| \le \delta$$

for every $x \in K^\pm_{x_0,\varepsilon}$ and every $0 < \varepsilon \le \varepsilon_0$.

Every modified function $v^\pm_{x_0,\varepsilon}$ is feasible for $(\mathcal{P}^{**})$ by (3.5) and we compare the values of I^{**} at v and $v^\pm_{x_0,\varepsilon}$. By (3.5) it is enough to compare the integrals on the sets $K^\pm_{x_0,\varepsilon}$ only. Since $\nabla v^\pm_{x_0,\varepsilon} \in \mathcal{A}_{v^\pm_{x_0,\varepsilon}}(\xi_0)$ for a.e. $x \in K^\pm_{x_0,\varepsilon}$ by (3.8), it follows from (4.4) and (3.9) that

$$\begin{aligned}\int_{K^\pm_{x_0,\varepsilon}} f^{**}\left(v^\pm_{x_0,\varepsilon}(x)\,,\nabla v^\pm_{x_0,\varepsilon}(x)\right)\,dx &= \int_{K^\pm_{x_0,\varepsilon}} \left[\left\langle d\left(v^\pm_{x_0,\varepsilon}(x)\right)\,,\nabla v^\pm_{x_0,\varepsilon}(x)\right\rangle + q\left(v^\pm_{x_0,\varepsilon}(x)\right)\right]\,dx = \\ &= \int_{K^\pm_{x_0,\varepsilon}} \left[\langle d(v(x))\,,\nabla v(x)\rangle + q\left(v^\pm_{x_0,\varepsilon}(x)\right)\right]\,dx.\end{aligned}$$

Then, $f^{**}(\eta\,,\xi) \ge \langle d(\eta)\,,\xi\rangle + q(\eta)$ for every $\xi \in \mathbb{R}^N$ and $\eta \in I$. Thus,

$$\begin{aligned}&\int_{K^\pm_{x_0,\varepsilon}} f^{**}\left(v^\pm_{x_0,\varepsilon}(x)\,,\nabla v^\pm_{x_0,\varepsilon}(x)\right)\,dx \le \\ &\le \int_{K^\pm_{x_0,\varepsilon}} f^{**}\left(v(x)\,,\nabla v(x)\right)\,dx + \int_{K^\pm_{x_0,\varepsilon}} \left[q\left(v^\pm_{x_0,\varepsilon}(x)\right) - q(v(x))\right]\,dx.\end{aligned} \tag{4.5}$$

Since q is monotone on the interval I, we can choose either the $+$ or the $-$ modified function from $v^\pm_{x_0,\varepsilon}$ according to the monotonicity of q so that the last summand in (4.5) is nonpositive.

We have thus proved that for every density point x_0 of E_v there is $\sigma = \sigma(x_0)$ chosen between $+$ and $-$ such that the corresponding functions from $\mathcal{V}^\sigma_{x_0}$ are still solutions to $(\mathcal{P}^{**})$ with the further property that each function $v^\sigma_{x_0,\varepsilon}$ from $\mathcal{V}^\sigma_{x_0}$ satisfies

$$f^{**}\left(v^\sigma_{x_0,\varepsilon}(x)\,,\nabla v^\sigma_{x_0,\varepsilon}(x)\right) = f\left(v^\sigma_{x_0,\varepsilon}(x)\,,\nabla v^\sigma_{x_0,\varepsilon}(x)\right) \qquad \text{for a.e. } x \in K^\sigma_{x_0,\varepsilon} \tag{4.6}$$

because of (3.7). Now, recalling that the corresponding sets $\mathcal{K}^\sigma_{x_0}$ shrink nicely at x_0, we apply Vitali's covering theorem thus finding countably many density points x_h of E_v, numbers $\varepsilon_h > 0$ and symbols $\sigma_h = \sigma(x_h) \in \{+\,,-\}$ such that the sets $K_h = K^{\sigma_h}_{x_h,\varepsilon_h}$ are pairwise disjoint and cover E_v up to a negligible set, i.e.

$$|E_v \setminus (\cup_h K_h)| = 0. \tag{4.7}$$

Then, let $v_h = v^{\sigma_h}_{x_h,\varepsilon_h}$ be the corresponding new solutions to $(\mathcal{P}^{**})$ and notice that (4.6) turns into

$$f^{**}\left(v_h(x)\,,\nabla v_h(x)\right) = f\left(v_h(x)\,,\nabla v_h(x)\right) \qquad \text{for a.e. } x \in K_h. \tag{4.8}$$

Then, we set

$$u(x) = v(x) + \sum_h \left[v_h(x) - v(x)\right], \qquad x \in \Omega, \tag{4.9}$$

and we check that it is a solution to $(\mathcal{P}^{**})$ for which $|E_u| = 0$.

First, recall that, by (3.5), all functions $v_h - v$ have pairwise disjoint supports. Hence, at every point x, there is at most one nonvanishing summand of (4.9) so that u is pointwise defined. Then, arguing as in Step 1 of the proof of Proposition 3.4, we show that u is in $W^{1,p}(\Omega)$. Indeed, the very same property of the supports of the functions $v_h - v$ together with either (3.6+) or (3.6−) and the assumption $\varepsilon_0 \le 1$ yield that

$$\int_\Omega |u|^p\,dx \le 2^{p-1}\int_\Omega |v|^p\,dx + 2^{p-1}\sum_h \int_{K_h} |v_h - v|^p\,dx \le 2^{p-1}\left(\int_\Omega |v|^p\,dx + |\Omega|\right).$$

As to the gradients, note that each partial sum of the series defining u is itself a minimizer of I^{**} by construction so that

$$I^{**}\left(v + \sum_{1\le h\le k}(v_h - v)\right) = I^{**}(v) \tag{4.10}$$

holds for every k. Hence, a standard argument based on the growth assumption ($\mathrm{H2}_p$) and Sobolev-Poincare's inequality yields that

$$\sum_h \int_\Omega |\nabla v_h - \nabla v|^p\,dx = \sum_h \int_{K_h} |\nabla v_h - \nabla v|^p\,dx < +\infty.$$

This implies that u is in $W^{1,p}(\Omega)$ and that the series (4.9) converges strongly to it in $W^{1,p}(\Omega)$. Moreover, u is feasible for $(\mathcal{P}^{**})$ and the sequential weak lower semicontinuity of I^{**} together with (4.10) show that u is a solution to $(\mathcal{P}^{**})$ as well. In particular, as a minimizer of I^{**}, u must be Hölder continuous and almost everywhere differentiable on Ω. Finally, $u = u_h$ on each set K_h and $u = v$ off the union of the sets K_h and the same holds true almost everywhere for the gradients. Hence, (4.8) and (4.7) show that E_u is negligible and this completes the proof of the claim.

Claim 2. The set $F_u = \{x \in \Omega :\ u \text{ is differentiable at } x \text{ and } (u(x), \nabla u(x)) \in \mathcal{M}\}$ is negligible.

Proof of Claim 2. By (b) of Proposition 4.1, we have $F_u = \bigcup_j F_{u,j}$ where the measurable sets $F_{u,j}$ are defined by

$$F_{u,j} = \{x \in \Omega :\ u \text{ is differentiable at } x \text{ and } (u(x), \nabla u(x)) \in \{m_j\}\times D_j\}.$$

Assume by contradiction that $|F_{u,j}| > 0$ for some j and, to simplify the notations, write F, m and D for $F_{u,j}$, m_j and D_j. As F is a level set of u, $\nabla u = 0$ almost everywhere on F. Thus, D must contain the origin and (2.7) forces m to be a strict local maximum point of the section $\eta \in \mathcal{D}^0 \to q(\eta, 0)$. Recall also that $q(\eta, 0) = f^{**}(\eta, 0)$ for every $\eta \in \mathcal{D}^0$ and, to simplify the notations, write $q(\eta) = f^{**}(\eta, 0)$ and $d(\eta) = d(\eta, 0)$ for every $\eta \in \mathcal{D}^0$.
Now, choose a density point x_0 in F where $\nabla u(x_0) = 0$ and let $\delta > 0$ be such that the interval $I = [m - 2\delta, m + 2\delta]$ is contained in $\mathcal{D}^0$. In addition, recalling (2.6) and that m is a strict local maximum point of q, we can assume also that q is increasing on the interval $[m - 2\delta, m]$ and decreasing on $[m, m + 2\delta]$. Then, choose $N+1$ unit vectors $\nu_0, \dots, \nu_N$ of $\mathbb{R}^N$ such that

$$0 \in \operatorname{int}\,(\operatorname{co}\{\nu_i :\ i = 0, \dots, N\})$$

as in (a) of Lemma 3.5 and let $\lambda_0, \dots, \lambda_N$ be such that $0 < \lambda_i \le \min\{a_i(\eta) :\ \eta \in I\}$ where the bounded, positive and lower semicontinuous functions $a_i\colon I \to (0, +\infty)$ are those associated with $\xi_0 = 0$ and the vectors $\nu_0, \dots, \nu_N$ by Proposition 3.3. Thus, $\xi_0 + \lambda_i\nu_i \in \mathcal{A}_\eta(0)$ for every i and $\eta \in I$. Then, we apply Lemma 3.5 to the function u at the point x_0 with $\eta_0 = m$, $\xi_0 = 0$, the unit vectors ν_i and the positive numbers λ_i defined above and we thus find two families of compact sets $\{A^\pm_{x_0,\varepsilon} :\ 0 < \varepsilon \le \varepsilon_0\}$ satisfying (3.10) for some numbers $0 < r_1 < r_2$ and two corresponding families of continuous, almost everywhere differentiable functions $\{u^\pm_{x_0,\varepsilon} :\ 0 < \varepsilon \le \varepsilon_0\}$ in $W^{1,p}(\Omega)$ such that (3.11), (3.12+), (3.12−), (3.13+), (3.13−) and (3.14) hold.

We suppose $\varepsilon_0 \le \delta/2$ is small enough so as to have $|u(x) - m| \le \delta$ for every $x \in A^{\pm}_{x_0,\varepsilon}$ and ε and we recall that $|u^{\pm}_{x_0,\varepsilon}(x) - u(x)| \le 2\varepsilon \le 2\varepsilon_0 \le \delta$ by either (3.12+) or (3.12−). Thus, $|u^{\pm}_{x_0,\varepsilon} - m| \le 2\delta$ on $A^{\pm}_{x_0,\varepsilon}$ for every ε.
Now, we wish to compare the values of I^{**} at u and $u^{\pm}_{x_0,\varepsilon}$. As $u = u^{\pm}_{x_0,\varepsilon}$ outside $A^{\pm}_{x_0,\varepsilon}$ by (3.11), it is enough we compare the integrals

$$\int_{A^{\pm}_{x_0,\varepsilon}} f^{**}\left(u^{\pm}_{x_0,\varepsilon}(x)\,,\nabla u^{\pm}_{x_0,\varepsilon}(x)\right)\,dx \qquad \text{and} \qquad \int_{A^{\pm}_{x_0,\varepsilon}} f^{**}\left(u(x)\,,\nabla u(x)\right)\,dx.$$

Arguing as in the proof of Claim 1, we get

$$\int_{A^{\pm}_{x_0,\varepsilon}} f^{**}\left(u^{\pm}_{x_0,\varepsilon}(x)\,,\nabla u^{\pm}_{x_0,\varepsilon}(x)\right)\,dx \le$$
$$\le \int_{A^{\pm}_{x_0,\varepsilon}} f^{**}\left(u(x)\,,\nabla u(x)\right)\,dx + \int_{A^{\pm}_{x_0,\varepsilon}} \left[q\left(u^{\pm}_{x_0,\varepsilon}(x)\right) - q(u(x))\right]\,dx$$

and we claim that we can choose + or − and ε such that the second term at the right hand side of the previous formula is negative, thus contradicting the minimality of u. As the argument is the very same of Step 2 of the proof of Theorem 1.1 in [4], we just sketch the main points of the proof and we refer to this paper for the details.
Indeed, choose a sequence $\{\varepsilon_k\}_k$ in $(0\,,\varepsilon_0]$ that goes to zero and set

$$\eta_k = \frac{1}{\epsilon_k}\sup\left\{|u(x) - m| :\ |x - x_0| < 2r_2\varepsilon_k\right\} \qquad \text{for every } k$$

where r_2 is the positive number given by Lemma 3.5 and appearing in (3.10). Obviously, $\eta_k \to 0_+$ since u is differentiable at x_0 with $\nabla u(x_0) = 0$ by assumption and we can assume also that $0 < \eta_k\epsilon_k \le \delta$ for every k. Then, recalling that m is a strict, local maximum point of q and possibly extracting a subsequence that we relabel as $\{\epsilon_k\}_k$, we can assume in addition that the minimum between $q(m - \eta_k\epsilon_k)$ and $q(m + \eta_k\epsilon_k)$ is actually achieved for every k by terms that always have the same sign inside, say $q(m + \eta_k\epsilon_k)$, so that

$$0 < q(m) - q(m + \eta_k\epsilon_k) = \max\left\{q(m) - q(m - \eta_k\epsilon_k)\,, q(m) - q(m + \eta_k\epsilon_k)\right\} \tag{4.11}$$

holds for every k.
According to this assumption, we choose the + functions and, to simplify the notations, we set $u_k = u^{+}_{x_0,\epsilon_k}$ and $A_k = A^{+}_{x_0,\epsilon_k}$ for every k. Of course, should the minimum between $q(m - \eta_k\epsilon_k)$ and $q(m + \eta_k\epsilon_k)$ be achieved at $q(m - \eta_k\epsilon_k)$, we would choose the − functions.
Finally, set $B_{i,k} = B_{r_i\varepsilon_k}(x_0)$ for $i = 1$ and 2 and every k so that (3.10) of Lemma 3.5 turns into

$$B_{1,k} \subset A_k \subset B_{2,k}. \tag{4.12}$$

We prove the claim by setting

$$J^1_k = \frac{1}{|A_k|}\int_{A_k}\left[q(m) - q(u_k(x))\right]\,dx \qquad \text{and} \qquad J^2_k = \frac{1}{|A_k|}\int_{A_k}\left[q(m) - q(u(x))\right]\,dx$$

and showing that $J^1_k - J^2_k > 0$ for some k.
Indeed, note first that (3.13+) of Lemma 3.5 reduces to $\epsilon_k/2 \le u_k(x) - m \le \epsilon_k$ for every $x \in B_{1,k}$. Hence, recalling (4.12) and that q is decreasing on the interval $[m\,, m + 2\delta]$, we find that

$$J^1_k \ge \frac{1}{|B_{2,k}|}\int_{B_{1,k}}\left[q(m) - q(u_k(x))\right]\,dx \ge$$
$$\ge \frac{1}{|B_{2,k}|}\int_{B_{1,k}}\left[q(m) - q(m + \epsilon_k/2)\right]\,dx = \left(\frac{r_1}{r_2}\right)^N\left[q(m) - q(m + \epsilon_k/2)\right]$$

for every k. As to J_k^2, we have

$$J_k^2 = \frac{1}{|A_k|} \int_{A_k \setminus F} [q(m) - q(u(x))] \, dx$$

for every k and $|m - u(x)| \leq \eta_k \epsilon_k$ for every $x \in A_k$ by the very definition of η_k. Hence,

$$\begin{aligned} 0 \leq q(m) - q(u(x)) &\leq \max \{q(m) - q(m - \eta_k \epsilon_k), q(m) - q(m + \eta_k \epsilon_k)\} = \\ &= q(m) - q(m + \eta_k \epsilon_k) \end{aligned}$$

for every $x \in A_k$ and every k because of the behaviour of q around m and by (4.11) whence

$$0 \leq J_k^2 \leq \frac{|A_k \setminus F|}{|A_k|} [q(m) - q(m + \eta_k \epsilon_k)] \qquad \text{for every } k.$$

Since $\eta_k \to 0_+$, it follows that eventually $q(m) - q(m + \epsilon_k/2) \geq q(m) - q(m + \eta_k \epsilon_k) > 0$ since q is increasing on $[m, m + 2\delta]$. As x_0 is a density point of F, (4.12) shows that the ratio $|A_k \setminus F|/|A_k|$ goes to zero and the conclusion follows. □

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