Γ-limit of periodic obstacles

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Abstract

We compute the Γ-limit of a sequence of obstacle functionals in the case of periodic obstacles.

1 Introduction

In this paper we study the asymptotic behaviour, as $k \to \infty$, of the solutions to the obstacle problems

$$\min\{\int_\Omega |Du|^2 dx + \langle f, u\rangle : u \geq \psi_k \text{ q.e. in } B, \ \ u \in H_0^1(\Omega)\}, \tag{1}$$

where Ω is a bounded open set of $I\!R^n$, $n \geq 3$, $f \in H^{-1}(\Omega)$, $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, and B is a relatively compact subset of Ω with $|\partial B| = 0$. The obstacles ψ_k are periodic functions of period $\frac{1}{k}e_1, \dots, \frac{1}{k}e_n$, where $e_1, \dots, e_n$ is the canonical basis of $I\!R^n$, and the restriction of ψ_k to the periodicity cell $Q_{\frac{1}{k}} = [-\frac{1}{2k}, \frac{1}{2k}]^n$ is obtained by rescaling a fixed function $\psi: I\!R^n \to \overline{I\!R}$, which is assumed to be quasi upper semicontinuous (see Section 2), bounded from above, and such that $\psi = -\infty$ out of the closed ball of centre 0 and radius 1. More precisely, we suppose that

$$\psi_k(x) = \psi(k^{\frac{n}{n-2}}x) \quad \text{ for every } \ x \in Q_{\frac{1}{k}}\,.$$

It is well known (see, e.g., [3]) that the exponent $\frac{n}{n-2}$ in the scaling factor is the critical exponent for this kind of problems: it is the only exponent which leads to a limit problem where the obstacles are replaced by a finite penalization term. The purpose of this paper is to compute this term explicitly.

For every $t \in I\!R$ we define

$$g(t) = \min\{\int_{I\!R^n} |Dv|^2 dx \ : \ v - t \in \mathcal{D}^{1,2}(I\!R^n),\, v \geq \psi \text{ q.e. on } I\!R^n\}, \tag{2}$$

where

$$\mathcal{D}^{1,2}(I\!R^n) = \{u \in L^{2^*}(I\!R^n) \ : \ Dv \in L^2(I\!R^n)\},$$

with $\frac{1}{2^*} = \frac{1}{2} - \frac{1}{n}$. The main result of this paper is the following theorem.

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Theorem 1.1 *For every k, let u_k be the solution to problem (1), and let u be the solution to the problem*

$$\min\{\int_\Omega |Du|^2 dx + \langle f, u\rangle + \int_B g(u)\, dx \ : u \in H^1_0(\Omega)\}.$$

Then (u_k) converges to u weakly in $H^1_0(\Omega)$ and

$$\lim_{k\to\infty} \int_\Omega |Du_k|^2 dx = \int_\Omega |Du|^2 dx + \int_B g(u)\, dx.$$

The result is proved using Γ-convergence techniques (for the definition and properties of the Γ-convergence we refer to [5]). For each k we consider the obstacle functional

$$F_{\psi_k}(u, B) = \begin{cases} 0 & \text{if } u \geq \psi_k \text{ q.e. on } B, \\ +\infty & \text{otherwise.} \end{cases}$$

Then Theorem 1.1 is a direct consequence of the following result (see [5], Theorems 7.8 and 13.12). We recall that $B \subset\subset \Omega$ means that B is relatively compact in Ω.

Theorem 1.2 *For every set $B \subset\subset \Omega$, with $|\partial B| = 0$, the sequence*

$$\int_\Omega |Du|^2 dx + F_{\psi_k}(u, B) \tag{3}$$

Γ-converges to the functional

$$\int_\Omega |Du|^2 dx + \int_B g(u)\, dx$$

in the weak topology of $H^1_0(\Omega)$.

Using the results of [8] and [6], it easy to see that a subsequence of (3) Γ-converges, in the weak topology of $H^1_0(\Omega)$, to a functional of the form

$$\int_\Omega |Du|^2 dx + \int_B h(u)\, dx, \tag{4}$$

and that the subsequence and the function h do not depend on B. In our proof we will identify the function h, showing that $h = g$. Since the result does not depend on the choice of the subsequence, this will prove Theorem 1.2.

The paper is organized as follows:

- in Section 2 we prove Theorem 1.2;
- in Section 3 we discuss an example, which shows that the knowledge of the capacity of the set $\{\psi \geq t\}$ for every t is not enough to determine the function g in the Γ-limit.

2 Proof of the main result

For every set E of $\mathbb{R}^n$ the *capacity* of E, denoted by $\operatorname{cap}(E)$, is defined as the infimum of

$$\int_{\mathbb{R}^n} |Du|^2 dx$$

on the set of all functions $u \in \mathcal{D}^{1,2}(\mathbb{R}^n)$ such that $u \geq 1$ almost everywhere (shortly a.e.) in a neighbourhood of E. A property is said to hold *quasi everywhere* (shortly *q.e.*) on a set E if it holds for every point of E, with the exception of the points of a set of capacity zero.

A function $u\colon \Omega \to \overline{\mathbb{R}}$ is said to be *quasi-continuous* (resp. *quasi-upper semicontinuous*) if, for each $\varepsilon > 0$, there exists an open set Ω_ε, with $\operatorname{cap}(\Omega_\varepsilon) < \varepsilon$, such that the restriction of u to $\Omega \setminus \Omega_\varepsilon$ is continuous (resp. upper semicontinuous). Any function $u \in H^1(\Omega)$ has a quasi-continuous representative, which is uniquely defined quasi everywhere (see [9], Section 4.8, [12], Section 7.2.4, [10], Section 4, or [14], Chapter 3), and we shall always identify a function $u \in H^1(\Omega)$ with its quasi continuous representative. Similar results hold also for the space $\mathcal{D}^{1,2}(\mathbb{R}^n)$.

We say that a positive Radon measure μ belongs to $H^{-1}(\Omega)$ if there exists a constant c such that

$$\left| \int_\Omega u \, d\mu \right| \leq c \, \|u\|_{H^1(\Omega)},$$

for every function $u \in C^1_c(\Omega)$. It is well known that these measures vanish on any set of capacity zero. The following property (see [3], Lemma 2.8) will be used in the proof of our result.

Proposition 2.1 *Let Ω be a bounded open set of $\mathbb{R}^n$. Let (μ_k) be a sequence of positive measures which converges strongly in $H^{-1}(\Omega)$, and let (ν_k) be a sequence of measures such that $0 \leq \nu_k \leq \mu_k$. Then $\nu_k \in H^{-1}(\Omega)$, (ν_k) is bounded in $H^{-1}(\Omega)$, and any weakly convergent subsequence converges strongly in $H^{-1}(\Omega)$.*

The open ball in $\mathbb{R}^n$ of centre x and radius r is denoted by $B(x, r)$. When $x = 0$, we write simply $B(r)$. For every $t \in \mathbb{R}$ and for every $R > 1$ we define

$$g_R(t) = \min\{\int_{B(R)} |Dv|^2 dx \ : \ v - t \in H^1_0(B(R)), \ v \geq \psi \ \text{ q.e. on } \ B(R)\}. \tag{5}$$

Note that the minima are achieved in problems (2) and (5), which define $g(t)$ and $g_R(t)$, and the minimum points are unique by the strict convexity of the functionals. Arguing as in [7], Lemma 6.2, we obtain

$$g(t) = \lim_{R\to\infty} g_R(t). \tag{6}$$

Proof of Theorem 1.2. It is enough to show that the function h in (4) is equal to g. To simplify the exposition we assume whole sequence (3) Γ-converges to the functional

(4) and that the cube $Q = [-\frac{1}{2}, \frac{1}{2}]^n$ is contained in Ω. Since h does not depend on B, to determine h we choose $B = Q$ in (3) and (4). Let us fix $t \in I\!R$ and a function $u \in C_0^\infty(\Omega)$ such that $u = t$ in a neighbourhood of Q. We divide the proof into three steps.

Step 1. Let us prove that $h(t) \leq g(t)$. Let us fix a real number α, with $1 < \alpha < \frac{n}{n-2}$, and put $r_k = k^{-\frac{n}{n-2}}$ and $\rho_k = k^{-\alpha}$. For every $k > 2^{\frac{1}{\alpha-1}}$ let w_k be the solution to the following variational problem on the periodicity cell $Q_{\frac{1}{k}}$:

$$\min\{\int_{Q_{\frac{1}{k}}} |Dw|^2 dx \, : \, w \geq \psi_k \ \text{ q.e. on } \ Q_{\frac{1}{k}}, \, w = t \ \text{ q.e. on } \ Q_{\frac{1}{k}} \setminus B(\rho_k)\}. \tag{7}$$

We extend w_k by $\frac{1}{k}$-periodicity to $Q_{\sigma_k} =]-\sigma_k/2, \sigma_k/2[^n$, with $\sigma_k = 1$ for k odd, and $\sigma_k = 1 + \frac{1}{k}$ for k even. Then we have

$$k^n \int_{Q_{\frac{1}{k}}} |Dw_k|^2 dx \leq \int_{Q_{\sigma_k}} |Dw_k|^2 dx \leq (k+1)^n \int_{Q_{\frac{1}{k}}} |Dw_k|^2 dx. \tag{8}$$

By the change of variables $y = k^{\frac{n}{n-2}} x$ the ball $B(\rho_k)$ is transformed into the ball $B(R_k)$, with $R_k = k^{\frac{n}{n-2}-\alpha}$, and the obstacle ψ_k is transformed into ψ. Therefore

$$g_{R_k}(t) \leq \int_{Q_{\sigma_k}} |Dw_k|^2 dx \leq (1 + \tfrac{1}{k})^n g_{R_k}(t). \tag{9}$$

From (6), (8), and (9) we get

$$\int_{Q_{\sigma_k}} |Dw_k|^2 dx \to g(t) < +\infty. \tag{10}$$

Since, by the choice of α, the sequence (w_k) converges to t in measure, from (10) we obtain also

$$\int_{Q_{\sigma_k}} |w_k - t|^2 dx \to 0. \tag{11}$$

Moreover, by construction,

$$w_k \geq \psi_k \quad \text{q.e. on } \ Q_{\sigma_k}. \tag{12}$$

Let v_k be the function defined by

$$v_k = \begin{cases} w_k & \text{on } Q_{\sigma_k}, \\ u & \text{on } \Omega \setminus Q_{\sigma_k}. \end{cases} \tag{13}$$

Since $u = t$ in a neighbourhood of Q, the function v_k belongs to $H_0^1(\Omega)$ for k large enough. By (10) and (11) we have that $v_k \rightharpoonup u$ weakly in $H_0^1(\Omega)$. Since the functionals (3) Γ-converge to (4) for $B = Q$, by the definition of Γ-convergence and by (10) and (12) we have

$$\int_{\Omega \setminus Q} |Du|^2 dx \, + \, h(t) \leq \lim_{k \to \infty} \int_\Omega |Dv_k|^2 dx = \int_{\Omega \setminus Q} |Du|^2 dx \, + \, g(t), \tag{14}$$

hence $h(t) \leq g(t)$, which concludes the proof of Step 1.

Step 2. Let us prove that $h(t) \geq g(t)$. Since the functionals (3) Γ-converge to (4) for $B = Q$, by the definition of Γ-convergence there exists a sequence (u_k) in $H_0^1(\Omega)$ such that $u_k \rightharpoonup u$ weakly in $H_0^1(\Omega)$, $u_k \geq \psi_k$ q.e. on Q, and

$$\int_\Omega |Du_k|^2 dx \to \int_{\Omega \setminus Q} |Du|^2 dx \,+\, h(t).$$

Let us consider the function v_k defined in (13). By (14) it is enough to prove that

$$\liminf_{k\to\infty} \int_\Omega |Du_k|^2 dx \geq \liminf_{k\to\infty} \int_\Omega |Dv_k|^2 dx,$$

which is a consequence of

$$\liminf_{k\to\infty} \int_\Omega (|Du_k|^2 - |Dv_k|^2)\, dx \geq 0. \tag{15}$$

By convexity we have

$$\int_\Omega (|Du_k|^2 - |Dv_k|^2)\, dx \geq 2 \int_\Omega (Dv_k, Du_k - Dv_k)\, dx = -2\langle \Delta v_k, u_k - v_k \rangle. \tag{16}$$

Let E_k and F_k be the intersections with Q_{σ_k} of the $\frac{1}{k}$-periodic extension of the sets $\overline{B}(r_k)$ and $\overline{B}(\rho_k)$ respectively. Since w_k is the solution of the minimum problem (7), and $\psi_k = -\infty$ on $Q_{\frac{1}{k}} \setminus \overline{B}(r_k)$, we have $-\Delta v_k = 0$ on $Q_{\sigma_k} \setminus (E_k \cup \partial F_k)$. More precisely, there exist two non-negative measures μ_k and ν_k on Ω such that

$$-\Delta v_k = -\Delta w_k = \mu_k - \nu_k \text{ on } Q_{\sigma_k},$$

$$\mu_k \text{ is supported by } E_k,$$

$$\nu_k(B) = -\int_{B \cap \partial F_k} \frac{\partial w_k}{\partial \eta}\, d\mathcal{H}^{n-1} \tag{17}$$

for every Borel set $B \subset \Omega$, where η is the exterior unit normal to ∂F_k. Moreover (see [2]) $\mu_k \in H^{-1}(\Omega)$ and $w_k = \psi_k$ μ_k-a.e. on Ω. Hence

$$\int_{Q_{\sigma_k}} (u_k - w_k)\, d\mu_k = \int_{Q_{\sigma_k}} (u_k - \psi_k)\, d\mu_k \geq 0.$$

It follows that

$$-\langle \Delta v_k, u_k - v_k \rangle \geq -\int_{Q_{\sigma_k}} (u_k - v_k)\, d\nu_k - \int_{\Omega \setminus Q_{\sigma_k}} \Delta u (u_k - u)\, dx. \tag{18}$$

By (16), in order to prove (15) it remains to check that the right hand side of (18) tends to zero. We observe that $u_k - v_k \rightharpoonup 0$ in $H_0^1(\Omega)$. To conclude, it is sufficient to prove that (ν_k) converges strongly in $H^{-1}(\Omega)$. Using Proposition 2.1, we only have to show that there exists a sequence of measures (λ_k) such that

$$0 \le \nu_k \le \lambda_k, \ \ \lambda_k \in H^{-1}(\Omega), \ \ \lambda_k \to 0 \ \text{strongly in} \ H^{-1}(\Omega).$$

Step 3. Construction of the sequence (λ_k). Let z_k be the solution of

$$\begin{cases} -\Delta z_k = 0 & \text{on } B(\rho_k) \setminus \overline{B}(r_k), \\ z_k = M & \text{on } B(r_k), \\ z_k = t & \text{on } Q_{\frac{1}{k}} \setminus B(\rho_k), \end{cases}$$

where M is an upper bound of ψ with $M \ge t$. By the comparison principle we have $z_k \ge w_k$ on $B(\rho_k) \setminus \overline{B}(r_k)$, from which we obtain

$$-\frac{\partial z_k}{\partial \eta} \ge -\frac{\partial w_k}{\partial \eta} \ge 0 \quad \text{on} \quad \partial B(\rho_k), \tag{19}$$

where η is the exterior unit normal. For each Borel set $B \subset Q_{\frac{1}{k}}$ we define

$$\lambda_k(B) = -\int_{B \cap \partial B(\rho_k)} \frac{\partial z_k}{\partial \eta}\, d\mathcal{H}^{n-1},$$

and we extend the measure λ_k to Q_{σ_k} by $\frac{1}{k}$-periodicity. By (17) and (19) we obtain $\lambda_k \ge \nu_k$. Arguing as in [1], pag. 76, we find that (λ_k) converges strongly in $H^{-1}(\Omega)$, which concludes the proof.

3 An example

If ψ_E^c is equal to a (finite) constant c on a set $E \subset \overline{B}(0,1)$ and $\psi_E^c = -\infty$ out of E, then the corresponding function g_E^c, defined by (2), is given by

$$g_E^c(t) = ([c-t]^+)^2 \operatorname{cap}(E) = (c-t)^2 \operatorname{cap}(\{\psi_E^c \ge t\}),$$

where $[\cdot]^+$ denotes the positive part. Let E_k be the $\frac{1}{k}$-periodic extension of the set $k^{-\frac{n}{n-2}}E$. The results of Theorem 1.1 remain true, with minor modifications in the proof, if we take $f = 0$ and replace Ω by $\mathbb{R}^n$ and $H_0^1(\Omega)$ by $\mathcal{D}^{1,2}(\mathbb{R}^n)$. In particular, for $\psi = \psi_E^c$ and $c = 1$, we obtain

$$\lim_{k\to\infty} \operatorname{cap}(E_k \cap B) = \min\{\int_{\mathbb{R}^n} |Dv|^2 dx + \operatorname{cap}(E) \int_B (1-v)^2 dx \ : \ v \in \mathcal{D}^{1,2}(\mathbb{R}^n)\},$$

for every $B \subset\subset \mathbb{R}^n$ with $|\partial B| = 0$. This shows that, if $\psi^{(1)}$ and $\psi^{(2)}$ satisfy

$$\operatorname{cap}(\{\psi^{(1)} \ge t\}) = \operatorname{cap}(\{\psi^{(2)} \ge t\}) \ \text{for every} \ t \in \mathbb{R}, \tag{20}$$

then the corresponding periodic obstacles $\psi_k^{(1)}$ and $\psi_k^{(2)}$ satisfy

$$\lim_{k\to\infty} \mathrm{cap}(\{\psi_k^{(1)} \geq t\} \cap B) = \lim_{k\to\infty} \mathrm{cap}(\{\psi_k^{(2)} \geq t\} \cap B), \tag{21}$$

for every $t \in I\!R$ and for every $B \subset\subset I\!R^n$ with $|\partial B| = 0$.

Under a very weak boundedness assumption, it is proved in [4] that, if the obstacles φ_k and φ satisfy

$$\lim_{k\to\infty} \mathrm{cap}(\{\varphi_k \geq t\} \cap B) = \mathrm{cap}(\{\varphi \geq t\} \cap B),$$

for every $t \in I\!R$ and for every $B \subset\subset I\!R^n$ with $|\partial B| = 0$, then the solution of problem (1) associated with φ_k converges to the solution of the same obstacle problem associated with φ. In this case, the knowledge of the limit

$$\lim_{k\to\infty} \mathrm{cap}(\{\varphi_k \geq t\} \cap B)$$

is enough to determine the asymptotic problem. This theorem can not be applied in the case (21), since the limits in (21) can not be expressed as $\mathrm{cap}(\{\varphi \geq t\} \cap B)$, for a suitable function φ.

We construct now two functions $\psi^{(1)}$ and $\psi^{(2)}$ which satisfy (20), and hence the corresponding periodic obstacles $\psi_k^{(1)}$ and $\psi_k^{(2)}$ satisfy (21). We will show that, in this example, the limit problems are different, since they contain two different functions $g^{(1)}$ and $g^{(2)}$.

Let $\psi^{(1)}, \psi^{(2)}: I\!R^n \to \overline{I\!R}$ be defined by

$$\psi^{(1)} = \begin{cases} 1 & \text{on } \overline{B}(0, \frac{1}{2}), \\ 0 & \text{on } \overline{B}(0,1) \setminus \overline{B}(0,\frac{1}{2}), \\ -\infty & \text{otherwise}, \end{cases} \qquad \psi^{(2)} = \begin{cases} 1 & \text{on } \overline{B}(\xi, \frac{1}{2}), \\ 0 & \text{on } \overline{B}(0,1) \setminus \overline{B}(\xi,\frac{1}{2}), \\ -\infty & \text{otherwise}, \end{cases}$$

where $\xi \in I\!R^n$ and $|\xi| = \frac{1}{2}$. Let $g^{(1)}$ and $g^{(2)}$ be the functions defined by (2), corresponding to $\psi^{(1)}$ and $\psi^{(2)}$ respectively. Clearly, (20) is satisfied for every $t \in I\!R$. We will prove that

$$g^{(1)}(t) = g^{(2)}(t) \quad \text{for } t \geq t^*, \tag{22}$$

$$g^{(1)}(t) < g^{(2)}(t) \quad \text{for } t < t^*, \tag{23}$$

where $t^* = -1/(3^{n-2} - 1)$.

It is clear that $g^{(1)}(t) = g^{(2)}(t) = 0$ for $t \geq 1$. For every $t < 1$ let $u_t^{(1)}$ and $u_t^{(2)}$ be the solutions of problem (2) relative to $\psi^{(1)}$ and $\psi^{(2)}$ respectively. Let $z_t^{(1)}$ be the solution of the problem

$$\min\{\int_{I\!R^n} |Dz|^2 dx \ : \ z - t \in \mathcal{D}^{1,2}(I\!R^n),\ z \geq 1 \text{ on } \overline{B}(0, \tfrac{1}{2})\}, \tag{24}$$

and let $z_t^{(2)}(x) = z_t^{(1)}(x - \xi)$. A direct computation yields

$$z_t^{(1)}(x) = (1-t)2^{2-n}|x|^{2-n} + t.$$

Let $t_* = -1/(2^{n-2}-1)$. It easy to see that if $t_* \leq t < 1$ we have $z_t^{(1)} \geq 0$ on $\overline{B}(0,1)$, hence $u_t^{(1)} = z_t^{(1)}$, and consequently

$$g^{(1)}(t) = (1-t)^2 \operatorname{cap}(B(0, \tfrac{1}{2})) \quad \text{for } t_* \leq t < 1. \tag{25}$$

If $t < t_*$, by comparison, we have $\psi^{(1)} \leq u_t^{(1)} \leq z_{t_*}^{(1)}$. This implies $\{\psi^{(1)} \geq 0\} \subset \{u_t^{(1)} \geq 0\} \subset \{z_{t_*}^{(1)} \geq 0\}$, hence

$$\{u_t^{(1)} \geq 0\} = \overline{B}(0,1) \text{ for } t < t_*. \tag{26}$$

If $t^* \leq t < 1$, by an elementary computation we see that $z_t^{(2)} \geq 0$ on $\overline{B}(0,1)$, hence $u_t^{(2)} = z_t^{(2)}$, and consequently

$$g^{(2)}(t) = (1-t)^2 \operatorname{cap}(B(\xi, \tfrac{1}{2})) = (1-t)^2 \operatorname{cap}(B(0, \tfrac{1}{2})) \quad \text{for } t^* \leq t < 1. \tag{27}$$

As $t_* < t^*$, from (25) and (27) we obtain (22).

Now we consider $t < t^*$. By comparison, we have $u_t^{(2)} \geq z_t^{(2)}$, from which

$$\{u_t^{(2)} \geq 0\} \supset B(0,1) \cup \{z_t^{(2)} \geq 0\} = B(0,1) \cup B(\xi, R_t), \tag{28}$$

with $R_t = \frac{1}{2}(1-\frac{1}{t})^{\frac{1}{n-2}}$. Since $t < t^*$, we have $\frac{1}{2} < R_t < \frac{3}{2}$. As $|\xi| = \frac{1}{2}$, we get

$$|(B(0,1) \cup B(\xi, R_t))| > \max\{|B(0,1)|, |\{z_t^{(1)} \geq 0\}|\}. \tag{29}$$

Let $\overline{u}_t^{(2)}$ be the spherical rearrangement of $u_t^{(2)}$. From the fact that $u_t^{(2)} \geq \psi^{(2)}$, we deduce that $\overline{u}_t^{(2)} \geq \overline{\psi}^{(2)} = \psi^{(1)}$. Let us prove that $\overline{u}_t^{(2)} \neq u_t^{(1)}$. If $t_* \leq t < t^*$, we have $u_t^{(1)} = z_t^{(1)}$ and, by (28) and (29),

$$|\{\overline{u}_t^{(2)} \geq 0\}| = |\{u_t^{(2)} \geq 0\}| > |\{z_t^{(1)} \geq 0\}| = |\{u_t^{(1)} \geq 0\}|.$$

If $t < t_*$ we have

$$|\{\overline{u}_t^{(2)} \geq 0\}| = |\{u_t^{(2)} \geq 0\}| > |B(0,1)| = |\{u_t^{(1)} \geq 0\}|,$$

where the last equality follows from (26). Therefore $\overline{u}_t^{(2)} \neq u_t^{(1)}$ for $t < t^*$. By the Pólya-Szegö principle (see [13]) we obtain

$$\int_{\mathbb{R}^n} |Du_t^{(2)}|^2 dx \geq \int_{\mathbb{R}^n} |D\overline{u}_t^{(2)}|^2 dx > \int_{\mathbb{R}^n} |Du_t^{(1)}|^2 dx,$$

where the last inequality follows from the uniqueness of the solution to problem (2) for $\psi^{(1)}$. This inequality proves (23).

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