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Barriers for systems of ordinary differential equations: an application to the two-body problem

ABSTRACT. — We adapt the barriers approach, originally introduced by De Giorgi for geometric evolution problems, to systems of ordinary differential equations. In particular, we compute the regularized minimal closed barrier (RMCB) for the two-body problem. It turns out that, in general, the RMCB does not satisfy the semigroup property. Moreover, in the example of the two-body problem, the fattening phenomenon does not appear.

Barriere per sistemi di equazioni differenziali ordinarie: una applicazione al problema dei due corpi

RIASSUNTO. — Adattiamo il metodo delle barriere, originariamente introdotto da De Giorgi per trattare problemi geometrici di evoluzione, al caso dei sistemi di equazioni differenziali ordinarie. In particolare, calcoliamo la minima barriera regolarizzata chiusa (RMCB) per il problema dei due corpi. Ne risulta che RMCB non verifica necessariamente la proprietá di semigruppo. Inoltre, nel problema dei due corpi, il fenomeno del fattening non si presenta.

1. Introduction

The barriers method was defined by De Giorgi in [22,23] to provide a unique and global in time solution for a large class of geometric evolution problems, in particular in the context of mean curvature flow. Concerning motion by mean curvature of boundaries this method was studied in [9,10,12,18,14,17], where the problem of comparing the minimal barriers with the viscosity solutions is considered. In this one-codimensional geometric context, it turns out that the barriers approach is very close to the set-theoretic subsolutions method as defined by Ilmanen in [27,28]. One of the original motivations of De Giorgi [23] was to include in the theory the mean curvature flow of a manifold of arbitrary codimension; following his ideas, the applicability of the method has been pursued successfully in [3] (see also [16] and [34]) where, again, the comparison with the level-set method is considered. The original definition [23] of barrier was given in a very general setting; interestingly enough, it turns out that the concept of barrier can be applied to a very different setting with respect to the one discussed above, namely to systems of ordinary differential equations (see [11, Esempio 3.1] for a very preliminary discussion in this context). The aim of this paper is precisely to deepen the application of the barriers approach to systems of ODEs, in particular to the two-body problem. More precisely, we want to define a global solution to the two-body problem which

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coincides with the classical solution till the latter exists, and which is meaningful also after that a singularity occurs (i.e. after a collision).

The definition of global solutions to the two-body problem is a classical topic, and indeed various notions of regularization and different methods have been used to define trajectories beyond the collisions. We recall here: 1) the method of Sundman, [37], that consists in finding a real analytic branch of the solution that can be regarded as a continuation of the solution after the collision; 2) the method of Levi-Civita [29], that consists in changing variables in such a way that the differential equations defining the flow have no singular points (this second method has been reformulated later by Easton [24] using topological tools and we shall refer to this method as to Easton's regularization); 3) the approach of McGehee [31]. We also quote the recent method based on the smoothing of the potential and a perturbation of the initial conditions, as developed in [19].

The unique weak global solution that we propose in the present paper, starting in ξ at time t_0, is denoted by $\mathfrak{M}^*_c(\xi, \mathcal{F}, t_0)(t)$ and is called regularized minimal closed barrier (RMCB); it is slightly different from the one obtained applying directly the abstract definition of minimal barrier, since an operation of topological closure is required, see Section 3 for further details. Our main result is the computation of the RMCB for the two-body problem, with an initial datum ξ leading to collision. Even if this example is quite elementary, it suggests some observations.

The RMCB does not satisfy the semigroup property in time, that is, given two values of time t_1, t_2 with $t_0 \leq t_1 < t_2$, $\mathfrak{M}^*_c(\xi, \mathcal{F}, t_0)(t_2)$ is not necessarily equal to $\mathfrak{M}^*_c\left(\mathfrak{M}^*_c(\xi, \mathcal{F}, t_0)(t_1), \mathcal{F}, t_1\right)(t_2)$. In particular, at the collision time $t_{\rm coll}$ the RMCB turns out to be empty, while, for times slightly larger than $t_{\rm coll}$, it is a singleton; this shows that, in general, the RMCB can pass from the empty set to a non-empty set (at subsequent times). Despite the lack of the semigroup property, still the RMCB is unique and has reasonable coincidence properties with classical solutions, whenever they exist.

As we have seen from the above discussion, the properties of the minimal barriers may differ when applied to different contexts, such as mean curvature flow or systems of ODEs. However, there is a common phenomenon that can be isolated, called (at least in the literature of geometric evolutions) the *fattening phenomenon*, see [25.8,15]. In Definition 3.5 we explain what we mean by fattening in the context of the present paper. Generally speaking, fattening appears in presence of non-uniqueness; for systems of ODEs, fattening means that $\mathfrak{M}^*_c(\xi, \mathcal{F}, t_0)(t)$ is not anymore a singleton in the phase space, but a set with positive Hausdorff dimension. This phenomenon is strictly related to our notion of solution, similarly to what happens for viscosity solutions in mean curvature flow. The Hausdorff dimension of the solution set, at fixed time, should indicate how complicated is the singularity, namely how different orbits can escape out of the singularity, for times larger and close to the singularity time.

The content of the paper is the following. In Section 2 we introduce some notation. In Section 3, after recalling the original abstract definition of barrier, we apply it to our context and we give the definition of $\mathfrak{M}^*_c(\xi, \mathcal{F}, t_0)$. In Section 4 we discuss some examples; in particular, we describe $\mathfrak{M}^*_c(\xi, \mathcal{F}, t_0)$ for some singular systems of ODEs considered by McGehee in [30]. In Section 5 we study some general properties of RMCB which will be useful in the sequel. In Section 6 we define the RMCB for the N-body problem, in particular we focus our attention to the case $N = 2$. The main result of the paper is Theorem 6.1 where we compute the

RMCB for the two-body problem, in particular after the collision time. The two-body problem does not present the fattening phenomenon. Eventually, in Section 7 we point out some comments and open problems.

2. Notation.

We indicate by $\mathcal{P}(\mathbb{R}^n)$ the class of all the subsets of $\mathbb{R}^n$, $n \geq 1$. Given $E \in \mathcal{P}(\mathbb{R}^n)$, we denote by $\overline{E}$ the topological closure of E. Given $\rho > 0$, we also define

$$E_\rho^+ := \{x \in \mathbb{R}^n : \mathrm{dist}(x, E) < \rho\}.$$

In particular, if ξ is a point of $\mathbb{R}^n$, ξ_ρ^+ is the open ball centered at ξ with radius ρ.

The symbol $\mathcal{H}^\alpha$, with $\alpha \in [0, n]$, stands for the α-dimensional Hausdorff measure in $\mathbb{R}^n$.

We denote by I an open interval of $\mathbb{R}$, t_0 is a point of I and we set $\mathcal{T} := \sup I$ and $L := [t_0, \mathcal{T}[$.

Given a set-valued map $\phi : L \to \mathcal{P}(\mathbb{R}^n)$, we indicate by $\overline{\phi}$ the set-valued map $t \in L \to \overline{\phi(t)}$ for any $t \in L$. Moreover, given two set-valued maps $\phi, \psi : L \to \mathcal{P}(\mathbb{R}^n)$, by $\phi \subseteq \psi$ we mean that $\phi(t) \subseteq \psi(t)$ for any $t \in L$.

The notation $(\xi_k)_k$ denotes a sequence of points $\xi_k \in \mathbb{R}^n$.

We say that a function f belongs to $\mathcal{C}^1([a, b]; \mathbb{R}^n)$ if there exists an open set $U \subseteq \mathbb{R}$ containing $[a, b]$ and a function $\widetilde{f} \in \mathcal{C}^1(U; \mathbb{R}^n)$ such that $\widetilde{f} = f$ on $[a, b]$.

The singleton containing the point $\xi \in \mathbb{R}^n$ is denoted by $\{\xi\}$.

3. Barriers for systems of ODEs. Definitions of MCB and RMCB.

We recall the abstract definition of barrier given by De Giorgi in [23].

Definition 3.1. *Let $I \subseteq \mathbb{R}$ be an interval, $\mathcal{S}$ an arbitrary set and $r \subseteq \mathcal{S}^2$. Let us assume that*

$$\mathcal{S} = \bigcap \{E \subseteq \mathcal{S} : r \subseteq E^2\} \,. \tag{3.1}$$

Let $\mathcal{F}$ be a family of functions of one real variable with the following property:

for each $f \in \mathcal{F}$ there exist two real numbers a, b with $a < b$ and $f : [a, b] \to \mathcal{S}$.

We say that a function ϕ is a barrier with respect to the pair $(r, \mathcal{F})$ if there exists a convex set $J \subseteq I$ such that $\phi : J \to \mathcal{S}$ and if a, b, f satisfy

$$[a, b] \subseteq J \,, \quad f : [a, b] \to \mathcal{S} \,, \quad f \in \mathcal{F} \,, \quad (f(a), \phi(a)) \in r \,,$$

then

$$(f(b), \phi(b)) \in r \,.$$

In this case we shall write $\phi \in \mathcal{B}(r, \mathcal{F}, J)$.

The symbol r stands for the graph of a binary relation on S, and condition (3.1) essentially means that any element of S is either first or second element of a pair belonging to r. Barriers have been studied for geometric evolution problems; on the other hand, the study of minimal barriers for first and second order partial differential equations deserves further investigation.

To apply Definition 3.1 to systems of ODEs, we choose

$$\mathcal{S} := \mathcal{P}(\mathbb{R}^n), \qquad r := \{(A, B) : A \subseteq B \in \mathcal{P}(\mathbb{R}^n)\} \tag{3.2}$$

in Definition 3.1.

Since we always choose r as in (3.2), from now on we omit the dependence on r in the notation of barriers and minimal barriers.

It remains to choose the family $\mathcal{F}$ of "test trajectories". Let G be a function belonging to $\mathcal{C}^0(I \times A; \mathbb{R}^n)$, for some open set $A \subseteq \mathbb{R}^n$. We give the following definition.

Definition 3.2. *A triplet $(f; a, b)$, with $a, b \in I, a < b$ and $f : [a, b] \to \mathcal{P}(\mathbb{R}^n)$, belongs to $\mathcal{F}$ if*

$$\begin{cases} f \in \mathcal{C}^1([a, b]; \mathbb{R}^n), \\ \dfrac{d}{dt} f(t) = G(t, f(t)), \qquad t \in [a, b]\,. \end{cases} \tag{3.3}$$

If $(f; a, b)$ belongs to $\mathcal{F}$, with a little abuse of notation we shall sometimes write $f \in \mathcal{F}$. Furthermore we shall always identify $f \in \mathcal{F}$ with the map taking $t \in [a, b]$ into the singleton $\{f(t)\} \in \mathcal{P}(\mathbb{R}^n)$.

With our choice in (3.2) and recalling Definition 3.1, we say that ϕ is a barrier if there exists an interval $J \subseteq I$ such that $\phi : J \to \mathcal{P}(\mathbb{R}^n)$ and for all $(f; a, b) \in \mathcal{F}$, with $[a, b] \subseteq J$, we have

$$f(a) \in \phi(a) \quad \Longrightarrow \quad f(b) \in \phi(b).$$

To indicate that ϕ is a barrier on J we write $\phi \in \mathcal{B}(\mathcal{F}, J)$.

Definition 3.3. *Let $E \subseteq \mathbb{R}^n$. The minimal closed barrier (*MCB*) with origin in E at time t_0 with respect to the family $\mathcal{F}$ is defined as*

$$\mathfrak{M}_c(E, \mathcal{F}, t_0)(t) := \bigcap \Big\{ \overline{\phi(t)} : \phi \in \mathcal{B}(\mathcal{F}, L);\ \phi(t_0) \supseteq E \ \Big\} \qquad \forall t \in L. \tag{3.4}$$

Our definition of weak global solution of (3.3) starting from a set $E \in \mathcal{P}(\mathbb{R}^n)$, is the following.

Definition 3.4. *Let $E \subseteq \mathbb{R}^n$. The regularized minimal closed barrier (*RMCB*) with origin in E at time t_0 with respect to the family $\mathcal{F}$ is defined as*

$$\mathfrak{M}_c^*(E, \mathcal{F}, t_0)(t) := \bigcap_{\rho > 0} \mathfrak{M}_c(E_\rho^+, \mathcal{F}, t_0)(t) \qquad \forall t \in L. \tag{3.5}$$

In particular, the definition of weak global solution of (3.3) starting from a singleton $\{\xi\} \in \mathcal{P}(\mathbb{R}^n)$, leading to the notion of weak solution to the system in (3.3) coupled with the initial condition

$$f(t_0) = \xi, \tag{3.6}$$

becomes

$$\mathfrak{M}_c^*(\xi, \mathcal{F}, t_0)(t) := \bigcap_{\rho > 0} \mathfrak{M}_c(\xi_\rho^+, \mathcal{F}, t_0)(t) \qquad \forall t \in L\,. \tag{3.7}$$

Concerning Definition 3.4, some comments are in order.

Remark 3.1. The RMCB in (3.7) is constructed as follows: it takes into account all orbits starting at distance $\rho > 0$ from the point ξ, whose union is a sort of thin tube in space time (the section at time t of such a tube is, roughly speaking, the set $\mathfrak{M}_c(\xi_\rho^+, \mathcal{F}, t_0)(t)$ which, by construction, is a closed set). Then $\mathfrak{M}_c^*(\xi, \mathcal{F}, t_0)$ is defined as the intersection of all such tubes as the width ρ at time t_0 goes to zero. Notice that, if there is a global $\mathcal{C}^1$ solution of (3.3) starting initially at a point η belonging to ξ_ρ^+, then all its orbit is contained, by the definition of barrier, in the tube $\mathfrak{M}_c(\xi_\rho^+, \mathcal{F}, t_0)$. The definition is, therefore, based on the following idea: we extract the solution of (3.3), (3.6) by looking at ("most of") the smooth solutions of (3.3) with $f(t_0) = \eta \in \xi_\rho^+$, for small ρ. Since in principle also these solutions may be not globally defined, we are lead to use the definition of barrier. Observe also that the family of tests $\mathcal{F}$ is made by *local in time* smooth solutions of (3.3), being the initial time of $f \in \mathcal{F}$ not fixed a priori. Therefore, to construct RMCB we do not need necessarily smooth global in time test solutions starting close to ξ.

Remark 3.2. The RMCB in (3.5) is constructed using: (i) the intersection between sets; (ii) the distance function in $\mathbb{R}^n$; (iii) the topological closure in $\mathbb{R}^n$. It is therefore a concept which, in principle, could be defined in much more generality; for instance, the space $\mathbb{R}^n$ could be replaced by a manifold M with a distance d (of course, the closure becomes the closure in M and the distance function is d). Another possible generalization could be to replace $\mathbb{R}^n$ with an infinite dimensional space (say, a Hilbert or a Banach space); we shall not discuss whether this concept becomes interesting in this case.

Remark 3.3. The reason for which we consider the closure of the sets $\phi(t)$ in Definition 3.3 is apparent in the analysis of the examples in Section 4, see for instance example (B) (and figure (ii)). One useful consequence of the closure operation relies in Remark 5.4

Definition 3.5. *Let $\alpha > 0$, $t_1, t_2 \in L$ and $t_2 > t_1$. We say that $\mathfrak{M}_c^*(\xi, \mathcal{F}, t_0)$ develops α-dimensional fattening at time t_2 if $\mathfrak{M}_c^*(\xi, \mathcal{F}, t_0)(t)$ is a singleton for all $t \in [t_1, t_2[$ and for any $\epsilon > 0$ there exists $t \in]t_2, t_2 + \epsilon[\subset I$ such that*

$$\mathcal{H}^\alpha(\mathfrak{M}_c^*(\xi, \mathcal{F}, t_0)(t)) > 0 \ .$$

If $\mathfrak{M}_c^*(\xi, \mathcal{F}, t_0)$ develops fattening, then at some time larger than t_2, it is not a singleton anymore. Heuristically, this means that, in the vicinity of the position (if any) corresponding to the singularity time t_2, there is a large number of starting different trajectories of the system. This phenomenon can be considered a lack of uniqueness of the flow. Note that the above definition does not prevent the possibility that $\mathfrak{M}_c^*(\xi, \mathcal{F}, t_0)(t_2) = \emptyset$.

Remark 3.4. Observe that $\mathfrak{M}_c^*(E, \mathcal{F}, t_0)$ and $\mathfrak{M}_c(E, \mathcal{F}, t_0)$ are closed-valued maps, $\mathfrak{M}_c^*(E, \mathcal{F}, t_0) \supseteq \mathfrak{M}_c(E, \mathcal{F}, t_0)$ and

$$\mathfrak{M}_c(\emptyset, \mathcal{F}, t_0)(t) = \mathfrak{M}_c^*(\emptyset, \mathcal{F}, t_0)(t) = \emptyset \qquad \forall t \in L. \tag{3.8}$$

Eventually, observe that $\mathcal{F} \subseteq \mathcal{G}$ implies $\mathcal{B}(\mathcal{F}, L) \supseteq \mathcal{B}(\mathcal{G}, L)$, hence $\mathfrak{M}_c(E, \mathcal{F}, t_0) \subseteq \mathfrak{M}_c(E, \mathcal{G}, t_0)$ and $\mathfrak{M}_c^*(E, \mathcal{F}, t_0) \subseteq \mathfrak{M}_c^*(E, \mathcal{G}, t_0)$.

4. Examples

Let us apply the above definitions to Peano's example: let $n = 1$, $G(t, v) = G(v) := 3v^{2/3}$ for any $t \in \mathbb{R}$ and $v \in A := \mathbb{R}$, $\xi = 0 \in \mathbb{R}$ and $t_0 := 0$. It is not difficult to check that $\mathfrak{M}_c^*(0, \mathcal{F}, 0)(t) = [0, t^3] \subset \mathbb{R}$ for any $t \geq 0$. We are therefore in presence of one-dimensional fattening at time 0. Observe that a suitable modification of the function G into a new function of class $\mathcal{C}^1$ which grows quadratically for $v \geq 1$ can force $\mathfrak{M}_c^*(0, \mathcal{F}, 0)(t)$ to become unbounded in a finite time.

Let $t_0 := 0$, $n = 2$ and $A := \mathbb{R}^2 \setminus \{(0,0)\}$; we describe $\mathfrak{M}_c^*((\xi_1, \xi_2), \mathcal{F}, 0)$, with $(\xi_1, \xi_2) \in A$, for the following plane autonomous systems:

$$\text{(A)} \begin{cases} \dot{x} = (x^2 + y^2)^{-\frac{1}{3}} \\ \dot{y} = 0 \end{cases} \qquad \text{(B)} \begin{cases} \dot{x} = -x(x^2 + y^2)^{-1} \\ \dot{y} = y(x^2 + y^2)^{-1} \end{cases}$$

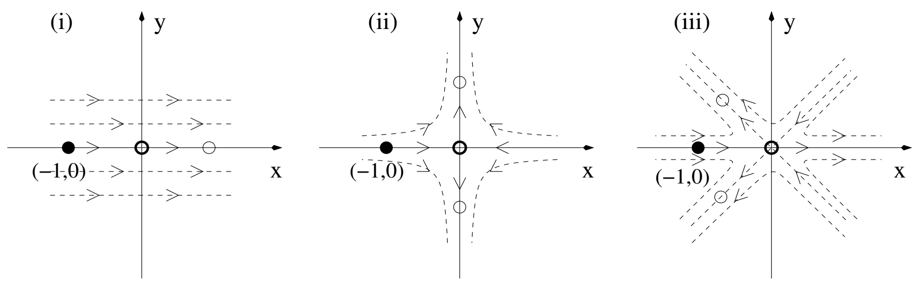
$$\text{(C)} \begin{cases} \dot{x} = (x^2 + y^2)^{-\frac{1}{2}} \\ \dot{y} = 0 \end{cases} \qquad \text{(D)} \begin{cases} \dot{x} = (x^2 - y^2)(x^2 + y^2)^{-\frac{4}{3}} \\ \dot{y} = -xy(x^2 + y^2)^{-\frac{4}{3}} \end{cases}$$

As observed in [30], we can apply both Easton's and Sundman's regularization to obtain a global solution to system (A) for any initial datum (ξ_1, ξ_2) (the two regularizations give rise to the same solution). Only Easton's regularization is appliable to system (C), only Sundman's regularization to system (D), neither Sundman's nor Easton's regularization to system (B).

The RMCB for these examples is drawn in the following figures for three different values of t: it is represented by a black small ball at the initial time and by thinner and thinner circles at subsequent times. We also draw the phase portrait of the equations.

In figure (i) we sketch the flow for both systems (A) and (C); in these cases the RMCB coincides with the classical solution obtained by Easton's regularization for any given initial conditions.

In figures (ii) and (iii) we sketch, respectively, the phase portraits and the RMCBs for systems (B) and (D). In these cases the RMCB coincides with the classical solution for all initial data not leading to the origin. For the remaining initial data (e.g. for $(\xi_1, \xi_2) = (-1, 0)$, as in the figure) the RMCB coincides with the origin at the singularity time t_{sing} (in particular, $\mathfrak{M}_c^*(0, \mathcal{F}, 0)(t_{\text{sing}})$ is not contained in the domain of G), and it splits into a pair of points with opposite y-coordinate for times larger than t_{sing}. To see that this is true, one can use, for instance, Remark 5.4 in Section 5 below.



5. General properties of the barriers MCB and RMCB

In this section we shall prove some general properties of barriers. We need the following lemma, which is standard in the theory of ODEs under our assumptions on G, see for instance [22,26].

Lemma 5.1. *Let $G \in \mathrm{Lip}_{\mathrm{loc}}(I \times A)$. The set S of all initial conditions $\xi \in A$ for which there exists a unique solution f_ξ of class $\mathcal{C}^1$ of the system in (3.3) in the common time interval $[a, b] \subset I$ with $f_\xi(a) = \xi$ is open.*

The following lemma asserts that the closure of a barrier is still a barrier. This property is in general not true for barriers to mean curvature flow.

Lemma 5.2. *Let $G \in \mathrm{Lip}_{\mathrm{loc}}(I \times A)$ and let $\phi \in \mathcal{B}(\mathcal{F}, L)$. Then $\overline{\phi} \in \mathcal{B}(\mathcal{F}, L)$.*

Proof. Let $(f; a, b) \in \mathcal{F}$ be such that $f(a) \in \overline{\phi(a)}$. We have to prove that $f(b) \subseteq \overline{\phi(b)}$. If $f(a) \in \phi(a)$ we have $f(b) \in \phi(b) \subseteq \overline{\phi(b)}$ and the assertion follows. On the other hand, if $f(a) \in \overline{\phi(a)} \setminus \phi(a)$, using also Lemma 5.1 we can choose a sequence $(p_k)_k$ of points of $\phi(a) \cap A$ converging to $f(a)$ as $k \to +\infty$, such that the solution f_k of the system in (3.3) with $f_k(a) = p_k$ is of class $\mathcal{C}^1$ and is defined in the time interval $[a, b]$ for any $k \in \mathbb{N}$. Clearly $f_k(b) \in \phi(b)$ for any $k \in \mathbb{N}$. By the continuous dependence on the initial data, we then have that the sequence $(f_k(b))_n$ converges to $f(b)$, and therefore $f(b) \in \overline{\phi(b)}$. It follows that $\overline{\phi} \in \mathcal{B}(\mathcal{F}, L)$. $\square$

Proposition 5.1. *Let $G \in \mathrm{Lip}_{\mathrm{loc}}(I \times A)$ and let $E \subseteq \mathbb{R}^n$. The* MCB *satisfies the following properties:*

(i) $\mathfrak{M}_c(E, \mathcal{F}, t_0) \in \mathcal{B}(\mathcal{F}, L)$;

(ii) $\mathfrak{M}_c(E, \mathcal{F}, t_0)(t) = \bigcap \left\{ \overline{\phi(t)} : \phi \in \mathcal{B}(\mathcal{F}, L), \phi(t_0) = \overline{E} \right\}$ *for any* $t \in L$;

(iii) $E_1 \subseteq E_2 \Rightarrow \mathfrak{M}_c(E_1, \mathcal{F}, t_0)(t) \subseteq \mathfrak{M}_c(E_2, \mathcal{F}, t_0)(t)$ *for any* $t \in L$;

(iv)

$$\mathfrak{M}_c(E, \mathcal{F}, t_0)(t_2) = \mathfrak{M}_c\Big(\mathfrak{M}_c(E, \mathcal{F}, t_0)(t_1), \mathcal{F}, t_1\Big)(t_2) \qquad t_0 \le t_1 \le t_2.$$

Proof. Property (i) follows from Lemma 5.2 and the fact that the intersections of barriers is a barrier. Property (ii) follows from the fact that $\mathfrak{M}_c(E, \mathcal{F}, t_0)(t_0) \supseteq \overline{E}$ and that the map $\psi : L \to \mathcal{P}(\mathbb{R}^n)$ defined as

$$\psi(t) := \begin{cases} \overline{E} & t = t_0 \\ \mathfrak{M}_c(E, \mathcal{F}, t_0)(t) & t > t_0, \ t \in L \end{cases}$$

belongs to $\mathcal{B}(\mathcal{F}, L)$. Property (iii) follows directly from the definitions. Property (iv) can be proved as follows (see Lemma 3.1 in [13]). Let $\phi : L \to \mathcal{P}(\mathbb{R}^n)$ be defined by

$$\phi(t) := \begin{cases} \mathfrak{M}_c(E, \mathcal{F}, t_0)(t) & t_0 \le t \le t_1 \\ \mathfrak{M}_c(\mathfrak{M}_c(E, \mathcal{F}, t_0)(t_1), \mathcal{F}, t_1)(t) & t \in \,]t_1, \mathcal{T}[\ ; \end{cases}$$

then $\phi(\bar{t}_0) = \overline{E}$ and $\phi \in \mathcal{B}(\mathcal{F}, L)$ by (i) and (ii) of Proposition 5.1. Hence

$$\mathfrak{M}_c(E, \mathcal{F}, t_0)(t_2) \subseteq \phi(t_2) = \mathfrak{M}_c(\mathfrak{M}_c(E, \mathcal{F}, t_0)(t_1), \mathcal{F}, t_1)(t_2)\,.$$

Conversely, since $\mathfrak{M}_c(E, \mathcal{F}, t_0)$ is a barrier on $[t_1, \mathcal{T}[$ which coincides with $\phi(t_1)$ at $t = t_1$, we have $\mathfrak{M}_c(E, \mathcal{F}, t_0)(t_2) \supseteq \mathfrak{M}_c(\phi(t_1), \mathcal{F}, t_1)(t_2) = \phi(t_2)$, and property (iv) is proved. $\square$

We call (ii) the inclusion property and (iv) the semigroup property for MCB.

Remark 5.1. In view of (ii) of Proposition 5.1, when considering $\mathfrak{M}_c(E, \mathcal{F}, t_0)$ we can always assume without loss of generality that E is closed.

Remark 5.2. Thanks to (3.8) and the semigroup property of MCB, it follows that, if for some $t \in L$ it happens that $\mathfrak{M}_c(E, \mathcal{F}, t_0)(t) = \emptyset$, then $\mathfrak{M}_c(E, \mathcal{F}, t_0)(\tau) = \emptyset$ for all $\tau \in L$. The interesting fact is that the same property for RMCB is not true in general.

Remark 5.3. Let $G \in \mathrm{Lip}_{\mathrm{loc}}(I \times A)$. Let $E \subseteq \mathbb{R}^n$ be a closed set. The RMCB satisfies the following properties:

(i) $\mathfrak{M}^*_c(E, \mathcal{F}, t_0) \in \mathcal{B}(\mathcal{F}, L)$;

(ii) $\mathfrak{M}^*_c(E, \mathcal{F}, t_0)(t) = \bigcap_{\rho>0} \bigcap \left\{ \overline{\phi(t)} : \phi \in \mathcal{B}(\mathcal{F}, L); \phi(t_0) = \overline{E^+_\rho} \right\}$ for any $t \in L$;

(iii) $E_1 \subseteq E_2 \Rightarrow \mathfrak{M}^*_c(E_1, \mathcal{F}, t_0)(t) \subseteq \mathfrak{M}^*_c(E_2, \mathcal{F}, t_0)(t)$ for any $t \in L$.

We shall see in Section 6.1 that $\mathfrak{M}^*_c(E, \mathcal{F}, t_0)$ does *not* verify the semigroup property.

We now prove two useful properties relatively to MCB and RMCB.

Lemma 5.3. *Let $G \in \mathrm{Lip}_{\mathrm{loc}}(I \times A)$ and let $E \subseteq \mathbb{R}^n$. Then*

$$\mathfrak{M}_c(E, \mathcal{F}, t_0)(t) = \overline{\bigcup_{z \in E} \mathfrak{M}_c(z, \mathcal{F}, t_0)(t)} \qquad \forall t \in L. \tag{5.2}$$

Proof. Set for notational simplicity

$$\Phi_E(t) := \bigcup_{z \in E} \mathfrak{M}_c(z, \mathcal{F}, t_0)(t) \qquad \forall t \in L.$$

For any $z \in E$ we have $\mathfrak{M}_c(z, \mathcal{F}, t_0) \subseteq \mathfrak{M}_c(E, \mathcal{F}, t_0)$ by the inclusion property; we deduce that $\Phi_E \subseteq \mathfrak{M}_c(E, \mathcal{F}, t_0)$, and therefore $\overline{\Phi_E} \subseteq \overline{\mathfrak{M}_c(E, \mathcal{F}, t_0)} = \mathfrak{M}_c(E, \mathcal{F}, t_0)$.

Conversely, we have $\Phi_E(t_0) = \bigcup_{z \in E} \mathfrak{M}_c(z, \mathcal{F}, t_0)(t_0) = E$. It is also easy to prove that $\Phi_E \in \mathcal{B}(\mathcal{F}, L)$. Therefore, by Lemma 5.2 we deduce that $\overline{\Phi_E} \in \mathcal{B}(\mathcal{F}, L)$, and we conclude that $\overline{\Phi_E} \supseteq \mathfrak{M}_c(E, \mathcal{F}, t_0)$. $\square$

Lemma 5.4. *Let $G \in \mathrm{Lip}_{\mathrm{loc}}(I \times A)$ and let $E \subseteq \mathbb{R}^n$. Then*

$$\mathfrak{M}^*_c(E, \mathcal{F}, t_0)(t) = \bigcap_{\rho>0} \overline{\bigcup_{\eta \in E^+_\rho} \mathfrak{M}_c(\eta, \mathcal{F}, t_0)(t)} = \bigcap_{\rho>0} \overline{\bigcup_{\eta \in E^+_\rho} \mathfrak{M}^*_c(\eta, \mathcal{F}, t_0)(t)} \qquad \forall t \in L. \tag{5.3}$$

Proof. Using (3.5), (5.2) and Remark 3.4 we have

$$\mathfrak{M}^*_c(E, \mathcal{F}, t_0) = \bigcap_{\rho>0} \mathfrak{M}_c(E^+_\rho, \mathcal{F}, t_0) = \bigcap_{\rho>0} \overline{\bigcup_{\eta \in E^+_\rho} \mathfrak{M}_c(\eta, \mathcal{F}, t_0)} \subseteq \bigcap_{\rho>0} \overline{\bigcup_{\eta \in E^+_\rho} \mathfrak{M}^*_c(\eta, \mathcal{F}, t_0)}$$

which in particular proves the first equality of (5.3).

Furthermore, by (5.2), given $\rho > 0$ we have

$$\bigcup_{\eta \in E^+_\rho} \mathfrak{M}^*_c(\eta, \mathcal{F}, t_0) = \bigcup_{\eta \in E^+_\rho} \bigcap_{r>0} \mathfrak{M}_c(\eta^+_r, \mathcal{F}, t_0) = \bigcup_{\eta \in E^+_\rho} \bigcap_{r>0} \overline{\bigcup_{z \in \eta^+_r} \mathfrak{M}_c(z, \mathcal{F}, t_0)}.$$

Now for any given $\eta \in E_\rho^+$, if $r > 0$ is small enough we have $\eta_r^+ \subset E_\rho^+$. Hence

$$\bigcap_{r>0} \overline{\bigcup_{z\in\eta_r^+} \mathfrak{M}_c(z,\mathcal{F},t_0)} \subseteq \overline{\bigcup_{\eta\in E_\rho^+} \mathfrak{M}_c(\eta,\mathcal{F},t_0)}.$$

Using again (5.2), it follows that

$$\overline{\bigcup_{\eta\in E_\rho^+} \overline{\bigcap_{r>0} \overline{\bigcup_{z\in\eta_r^+} \mathfrak{M}_c(z,\mathcal{F},t_0)}}} \subseteq \mathfrak{M}_c(E_\rho^+,\mathcal{F},t_0)\,.$$

Therefore

$$\mathfrak{M}_c(E_\rho^+,\mathcal{F},t_0) \supseteq \overline{\bigcup_{\eta\in E_\rho^+} \mathfrak{M}_c^*(\eta,\mathcal{F},t_0)}.$$

Taking the intersection with respect to $\rho > 0$, we deduce that

$$\mathfrak{M}_c^*(E,\mathcal{F},t_0) \supseteq \bigcap_{\rho>0} \overline{\bigcup_{\eta\in E_\rho^+} \mathfrak{M}_c^*(\eta,\mathcal{F},t_0)},$$

and this concludes the proof. $\square$

Remark 5.4. Assume that $E = \{\xi\}$, for $\xi \in \mathbb{R}^n$ and let $t \in L$. Then (5.3) is equivalent to

(5.4)
$$\mathfrak{M}_c^*(\xi,\mathcal{F},t_0)(t) = \{z \in \mathbb{R}^n : \exists(\xi_k)_k : \xi_k \to \xi,\ \exists w_k \in \mathfrak{M}_c^*(\xi_k,\mathcal{F},t_0)(t),\ w_k \to z\},$$

which, in turn, is equivalent to

(5.5)
$$\mathfrak{M}_c^*(\xi,\mathcal{F},t_0)(t) = \{z \in \mathbb{R}^n : \exists(\xi_k)_k : \xi_k \to \xi,\ \exists w_k \in \mathfrak{M}_c(\xi_k,\mathcal{F},t_0)(t),\ w_k \to z\}.$$

5.1. The RMCB of a single point.

The case $E = \{\xi\}$ where $\{\xi\}$ is a singleton is particularly interesting. If $\xi \in A$ we denote by u_ξ the unique maximal solution to the system in (3.3) with $u_\xi(t_0) = \xi$, on the maximal existence interval $[t_0, t_{\rm sing}[\subseteq L$.

The following propositions show in particular that the MCB and RMCB coincide with the classical solution till the latter exists.

Proposition 5.2. *Let $G \in \mathrm{Lip}_{\rm loc}(I \times A)$. Then*

$$\mathfrak{M}_c(\xi,\mathcal{F},t_0)(t) = \{u_\xi(t)\} \qquad \forall t \in [t_0, t_{\rm sing}[\tag{5.6}$$

and

$$\mathfrak{M}_c(\xi,\mathcal{F},t_0)(t) = \emptyset \qquad \forall t \in [t_{\rm sing}, \mathcal{T}[. \tag{5.7}$$

Proof. As $(u_\xi; t_0, t) \in \mathcal{F}$ for each $t \in]t_0, t_{\rm sing}[$ and $u_\xi(t_0) = \xi$ it follows

$$u_\xi(t) \in \mathfrak{M}_c(\xi,\mathcal{F},t_0)(t) \qquad \forall t \in [t_0, t_{\rm sing}[.$$

Conversely, set

$$\phi(t) := \begin{cases} \{u_\xi(t)\} & t_0 \le t < t_{\text{sing}} \\ \emptyset & t \in [t_{\text{sing}}, \mathcal{T}[. \end{cases}$$

To conclude the proof it is enough to show that $\phi \in \mathcal{B}(\mathcal{F}, L)$, since $\phi(t)$ is closed for any $t \in L$. Let $(f; a, b) \in \mathcal{F}$, with $t_0 \le a < b$ and $a < t_{\text{sing}}$, and let $f(a) \in \phi(a)$. It follows $f(a) = u_\xi(a)$, so that $f(t) = u_\xi(t)$ for all $t \in [a, b]$ by uniqueness. Furthermore $b < t_{\text{sing}}$ because, if not, u_ξ could be extended beyond t_{sing}. In particular $u_\xi(t) \in \phi(t)$ for any $t \in [a, b]$. We observe also that if $a \ge t_{\text{sing}}$ there is no function $f \in \mathcal{F}$, therefore in this case there is nothing to check for $f(b)$.

Proposition 5.3. *Let $G \in \mathrm{Lip}_{\mathrm{loc}}(I \times A)$. Then*

$$\mathfrak{M}_c^*(\xi, \mathcal{F}, t_0)(t) = \{u_\xi(t)\} \qquad t_0 \le t < t_{\text{sing}}. \tag{5.8}$$

Proof. Fix an arbitrary $t_1 < t_{\text{sing}}$. Using Lemma 5.4 and Proposition 5.2 we have

$$\mathfrak{M}_c^*(\xi, \mathcal{F}, t_0)(t_1) = \bigcap_{\rho > 0} \overline{\bigcup_{z \in \xi_\rho^+} u_z(t_1)} \ni u_\xi(t_1).$$

Assume by contradiction that there exists $w \in \mathbb{R}^n$, $w \neq u_\xi(t_1)$, with

$$w \in \mathfrak{M}_c^*(\xi, \mathcal{F}, t_0)(t_1). \tag{5.9}$$

By (5.9) it follows in particular that $w \in \overline{\bigcup_{z \in \xi_\sigma^+} u_z(t_1)}$ for any $\sigma > 0$. Choose $\eta > 0$ such that $|w - u_\xi(t_1)| > 2\eta$. By the continuous dependence on data we can select $\overline{\sigma} > 0$ such that $|u_z(t_1) - u_\xi(t_1)| < \eta$ for any $z \in \xi_{\overline{\sigma}}^+$, a contradiction. $\square$

Remark 5.5. No general assertion is stated on the behavior of $\mathfrak{M}_c^*(\xi, \mathcal{F}, t_0)$ after t_{sing}. In general, $\mathfrak{M}_c^*(\xi, \mathcal{F}, t_0)(t)$ may be *non-empty* for $t \ge t_{\text{sing}}$.

6. Minimal barriers for the N-body problem

Let $N > 1$. We consider the motion of N bodies with mass $m_1, \ldots, m_N$ that are subject only to their gravitational interaction. Let $n := 6N$ and $t_0 := 0$. The space $\mathbb{R}^{6N}$ is the phase space and $\mathcal{S} := \mathcal{P}(\mathbb{R}^{6N})$ is, as usual, the class of all subsets of $\mathbb{R}^{6N}$.

We use the notation $p := (q, v)$ with $q, v \in \mathbb{R}^{3N}$ and

$$\begin{cases} q = (q_1, \ldots, q_N) \\ v = (v_1, \ldots, v_N) \end{cases} \qquad \begin{cases} q_i = (q_{i,1}, q_{i,2}, q_{i,3}) \in \mathbb{R}^3 \\ v_i = (v_{i,1}, v_{i,2}, v_{i,3}) \in \mathbb{R}^3 \end{cases} \qquad i = 1, \ldots, N\,.$$

Let

$$\Delta := \bigcup_{1 \le i < j \le N} \left\{ (q, v) \in \mathbb{R}^{6N} : q_i = q_j \right\}\,.$$

We define $G(t, p) := G(p) = (G_1(p), \ldots, G_{6N}(p))$, with $t \in \mathbb{R}$ and $p \in \mathbb{R}^{6N} \setminus \Delta$, as

$$G_j(q, v) := \begin{cases} v_{r,s} \ \text{ with } r = \left[\dfrac{j-1}{3}\right] + 1;\ s = j - 3\left[\dfrac{j-1}{3}\right] & \text{if } 1 \le j \le 3N, \\ \displaystyle\sum_{h \neq j} \frac{m_h (q_{h,s} - q_{j,s})}{\|q_h - q_j\|^3} \ \text{ with } s = j - 3\left[\dfrac{j-1}{3}\right] & \text{if } 3N+1 \le j \le 6N\,, \end{cases}$$

where $[\cdot]$ denotes the integer part.

Definition 6.1. *A triplet $(f;a,b)$, with $a,b \in I$, $a<b$ and $f:[a,b] \to \mathcal{P}(\mathbb{R}^{6N})$, belongs to $\mathcal{F}$ if*

$$f(t) = \{(q_1(t),\dots,q_N(t),v_1(t),\dots,v_N(t))\} \qquad \forall t \in [a,b],$$

$q_i \in \mathcal{C}^1([a,b];\mathbb{R}^3)$ and $v_i \in \mathcal{C}^1([a,b];\mathbb{R}^3)$ for any $i=1,\dots,N$, and the system

$$f'(t) = G(f(t))$$

holds for any $t \in [a,b]$.

Let us select $\xi \in \mathbb{R}^{6N} \setminus \Delta$; we take

$$\mathfrak{M}_c^*(\xi,\mathcal{F},0)(t)$$

as generalized solution to the N-body problem with origin in ξ at time 0.

6.1. The RMCB for the two-body problem.

In this paragraph we let $N=2$. We shall study the properties of RMCB and MCB for the two-body problem, focusing our attention on initial data ξ leading to collisions. If ξ does not lead to a collision, then $\mathfrak{M}_c^*(\xi,\mathcal{F},0)$ coincides with the global classical solution by Proposition 5.3.

A useful property of the two-body problem is stated in the following remark, which follows from well-known results on the Kepler problem.

Remark 6.1. The problem is equivalent to a central motion of one particle on a plane, so that we can consider $\xi \in A := \big(\mathbb{R}^2 \setminus \{(0,0)\}\big) \times \mathbb{R}^2$. If ξ does not lead to a singularity (that is necessarily a collision), then there is an open neighborhood $U \subset A$ of ξ such that any $\eta \in U$ does not lead to a singularity. On the other hand, if ξ leads to a singularity and V is any neighborhood of ξ in A, then the set of all $\eta \in V$ not leading to a singularity is dense in V.

The main result of the paper is the following.

Theorem 6.1. *Let $N=2$. Let $\xi \in A$ be an initial condition leading to a collision at time $t_{\mathrm{coll}}>0$. Let u_ξ be the classical solution of the two-body problem on $[0,t_{\mathrm{coll}}[$ starting from ξ at time 0. Denote by T the length of the maximal interval (backward and forward in time) in which u_ξ is defined. Then*

$$\mathfrak{M}_c^*(\xi,\mathcal{F},0)(t) = \begin{cases} \{u_\xi(t)\} & 0 \le t < t_{\mathrm{coll}} \\ \emptyset & t = t_{\mathrm{coll}} + h\,T \\ \Big\{u_\xi\Big(t-(h+1)T\Big)\Big\} & t \in \,]t_{\mathrm{coll}}+hT, t_{\mathrm{coll}}+(h+1)T[\end{cases}$$

for any $h \in \mathbb{N}\cup\{0\}$. In particular $\mathfrak{M}_c^*(\xi,\mathcal{F},0)$ is T-periodic.

Proof. The assertion for times $t \in [0,t_{\mathrm{coll}}[$ is a consequence of Proposition 5.3. Let now $t \ge t_{\mathrm{coll}}$. Using (5.7) and Remark 6.1 it follows that

$$\mathfrak{M}_c^*(\xi,\mathcal{F},0)(t) = \bigcap_{\rho>0} \overline{\bigcup_{z\in\xi_\rho^+} \mathfrak{M}_c(z,\mathcal{F},0)(t)} = \bigcap_{\rho>0} \overline{\bigcup_{z\in W(\xi_\rho^+)} u_z(t)}$$

where $W(\xi_\rho^+)$ is the set of the initial conditions in ξ_ρ^+ *not* leading to a collision (hence corresponding to global solutions). Hence

$$(6.1) \qquad \mathfrak{M}_c^*(\xi, \mathcal{F}, 0)(t) = \{z \in \mathbb{R}^4 : \exists (\xi_k)_k : \xi_k \to \xi, \ \exists w_k \in u_{\xi_k}(t), \ w_k \to z\},$$

where each ξ_k does not lead to collision. Recall that $u_{\xi_k}(t)$ is a singleton. When $t = t_{\rm coll}$, by known properties of the solutions of the Kepler problem, if $\xi_k \to \xi$ then, writing $w_k = (w_k^q, w_k^v) \in \mathbb{R}^2 \times \mathbb{R}^2$, necessarily $|w_k^v| \to +\infty$. Hence (6.1) implies that $\mathfrak{M}_c^*(\xi, \mathcal{F}, 0)(t_{\rm coll})$ must be empty.

Assume now that $t > t_{\rm coll}$. Using the periodicity of u_{ξ_k}, from (6.1) we obtain

$$\mathfrak{M}_c^*(\xi, \mathcal{F}, 0)(t) = \{z \in \mathbb{R}^4 : z = \lim_{k \to +\infty} u_{\xi_k}(t - (h+1) T_{u_{\xi_k}}) \text{ for some } (\xi_k)_k : \xi_k \to \xi\},$$

where $T_{u_{\xi_k}}$ is the period of u_{ξ_k}.

By known properties of the solutions of the Kepler problem, if $\xi_k \to \xi$ then $T_{u_{\xi_k}} \to T$ and $\lim_{k \to +\infty} u_{\xi_k}(t - (h+1) T_{u_{\xi_k}}) = u_\xi(t - (h+1)T)$, and this concludes the proof. $\square$

Remark 6.2. The RMCB coincides (up to the discrete set of times when it is empty, i.e. the collision times) with the so-called reflection solution (or collision-ejection solution), that is when the velocity vector reverses direction at collision and the particles bounce off each other. The reflection solution has been obtained with several different methods, see for instance [1,29,35,31,32].

7. Final comments

We conclude the paper with some comments.

Remark 7.1. Nothing is said in the present paper about the properties (such as measurability, continuity and so on) of the map $t \to \mathfrak{M}_c^*(E, \mathcal{F}, t_0)(t) \in \mathcal{P}(\mathbb{R}^n)$.

Remark 7.2. It would be of some interest to see whether the barrier method can be applied to differential inclusions, and to compare it with the various notions of generalized solution present in the literature (see for instance [4,5,6,7,20,26,36]). We observe that a necessary requirement in order to implement the barrier method is to find a non-empty family $\mathcal{F}$ of "test" trajectories; the regularity of the elements f of $\mathcal{F}$ can be freely chosen for the particular problem at hand (such as absolutely continuity of f and so on).

Eventually, we point out an open problem. We have already seen that fattening does not appear after binary collision in the two-body problem. It would be interesting to know whether fattening happens after a multiple collision in the N-body problem (for instance at a triple collision for the three-body problem); in positive case, it would be interesting to estimate the Hausdorff dimension of RMCB (at fixed times).

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