Metrics in the space of curves

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Abstract

In this paper we study geometries on the manifold of curves.

We define a manifold M where objects $c \in M$ are curves, which we parameterize as $c : S^1 \to \mathbb{R}^n$ ($n \geq 2$, S^1 is the circle). Given a curve c, we define the tangent space T_cM of M at c including in it all deformations $h : S^1 \to \mathbb{R}^n$ of c.

We discuss Riemannian and Finsler metrics $F(c, h)$ on this manifold M, and in particular the case of the geometric H^0 metric $F(c, h) = \int |h|^2 ds$ of normal deformations h of c; we study the existence of minimal geodesics of H^0 under constraints; we moreover propose a conformal version of the H^0 metric.

1 Introduction

In this paper we study geometries on the manifold M of curves. This manifold contains curves c, which we parameterize as $c : S^1 \to \mathbb{R}^n$ (S^1 is the circle). Given a curve c, we define the tangent space T_cM of M at c including in it deformations $h : S^1 \to \mathbb{R}^n$, so that an infinitesimal deformation of the curve c in direction h will yield the curve $c(\theta) + \varepsilon h(\theta)$. This manifold M is the *Shape Space* that is studied in this paper.

We would like to define a *Riemannian metric* on the manifold M of curves: this means that, given two deformations $h, k \in T_cM$, we want to define a scalar product $\langle h, k\rangle_c$, possibly dependent on c. The Riemannian metric would then entail a *distance* $d(c_0, c_1)$ between the curves in M, defined as the infimum of the length $\mathrm{Len}(\gamma)$ of all smooth paths $\gamma : [0, 1] \to M$ connecting c_0 to c_1. We call *minimal geodesic* a path providing the minimum of $\mathrm{Len}(\gamma)$ in the class of γ with fixed endpoints.

A number of methods have been proposed in Shape Analysis to define distances between shapes, averages of shapes and optimal morphings between shapes; some of these approaches are reviewed in section §2. At the same time, there has been much previous work in Shape Optimization, for example Image Segmentation via Active Contours, 3D Stereo Reconstruction via Deformable Surfaces; in these later methods, many authors have defined Energy Functionals

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This work is dedicated to the memory of Anthony J. Yezzi, Sr., father of Anthony Yezzi, Jr., who passed away shortly after its initial completion. May he rest in peace and be remembered always as a loving husband, father, and grandfather who is and will continue to be dearly missed.

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on curves (or surfaces) and utilized the Calculus of Variations to derive curve evolutions to minimize the Energy Functionals; often referring to these evolutions as Gradient Flows. For example, the well known Geometric Heat Flow, popular for its smoothing effect on contours, is often referred as the *gradient flow for length.*

The reference to these flows as *gradient flows* implies a certain Riemannian metric on the space of curves; but this fact has been largely overlooked. We call this metric H^0 henceforth. If one wishes to have a consistent view of the geometry of the space of curves in both Shape Optimization and Shape Analysis, then one should use the H^0 metric when computing distances, averages and morphs between shapes.

In this paper we first introduce the metric H^0 in §2.1; we immediatly remark that, surprisingly, it does not yield a well define metric structure, since the associated distance is identically zero [(1)]. In §4 we analyse this metric; we show that the lower-semi-continuous relaxation of the associated energy functional is identically zero (see §3.1 and 4.15); but we prove in thm. 4.21 that, under additional constraints on the curvature of admissible curves, the metric H^0 admits minimum geodesics; we propose in §4.6 an example that justifies some of the hypotheses in 4.21. We can then define in §4.5 the *Shape Space* of curves with bounded curvature, where the metric H^0 entails a positive distance. These hypotheses on curvature, however, are not compatible with the classical definition of a Riemannian Geometry.

More recently, a Riemannian metric was proposed in [MM] for the space M of curves (see §2.1.3 here); this metric may fix the above problems; but it would significantly alter the nature of gradient flows used thus far in various Shape Optimization problems (assuming that one wishes to make those *gradient flows* consistent with this new metric). In this metric, distances measured between curves are defined using first and second order derivatives of the curves (and therefore the resulting optimality conditions involve up to fourth order derivatives); as a consequence, flows designed to converge towards these optimality conditions are necessarily fourth order, thereby precluding the use of *Level Set Methods* which have become popular in the field of Computer Vision and Shape Optimization.

We propose instead in §5 a class of *conformal metrics* that fix the above problems while minimally altering the earlier flows: in fact the new gradient flows will amount to a simple time reparameterization of the earlier flows. In addition the conformal metrics that we propose have some nice numerical and computational properties: distances measured between curves are defined using only first order derivatives (and therefore the resulting optimality conditions involve only second order derivatives); as a consequence, flows designed to converge towards these optimality conditions are second order, thereby allowing the use of Level Set methods: we indeed show such an implementation and a numerical example in §5.3. We also proposed in [YM04] a differential operator that is well adapted to the problem at hand: we review it here as well, in §5.1.1.

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(1) This striking fact was first described in [Mum]

1.1 Riemannian and Finsler geometries

Let M be a smooth connected differentiable manifold.(2) For any $c \in M$, let T_cM be *the tangent space at* c, that is the set of all vectors tangent to M at c; let TM be the tangent bundle, that is the bundle of all tangent spaces.

Definition 1.1 *Let X be a vector space; a norm $|\cdot|$ satisfies*

1. *$|v|$ is positive homogenous, i.e.*

$$\lambda|v| = |\lambda v| \quad \forall \lambda \geq 0 \quad ,$$

2. $$|v + w| \leq |v| + |w|$$

   (that, by (1), is equivalent to asking that $|v|$ be convex), and

3. *$|v| = 0$ only when $v = 0$.*

If the last condition is not satisfied, then $|\cdot|$ is a *seminorm*.

Definition 1.2 *We define a* Finsler metric *to be a function $F : TM \to \mathbb{R}^+$, such that*

- *F is lower semicontinuous and locally bounded from above, and,*
- *for all $c \in M$, $v \mapsto F(c, v)$ is a norm on T_cM.* (3)

  We will also consider it to be symmetric, that is, $F(c, -v) = F(c, v)$.

We will write $|v|_c$ for $F(c, v)$, or simply $|v|$ when the base point c can be easily inferred from the context.

We define the length of any locally Lipschitz path(4) $\gamma : [0, 1] \to M$ as

$$\operatorname{Len} \gamma = \int_0^1 |\dot{\gamma}(v)|_{\gamma(v)} \, dv$$

and the energy as

$$E(\gamma) = \int_0^1 |\dot{\gamma}(v)|^2_{\gamma(v)} \, dv$$

We define the *geodesic distance* $d(x, y)$ as the infimum

$$d(x, y) = \inf \operatorname{Len} \gamma \tag{1.3}$$

for all locally Lipschitz paths γ connecting x to y.

(2) If M is infinite dimensional, then we suppose that it is modeled on a Banach or Hilbert separable space (see [Lan99], ch.II).

(3) Sometimes F is called a “Minkowsky norm”

(4) As suggested in [MM], we want to avoid referring to γ as a *curve*, because confusion arises when we will introduce the manifold M of *closed curves*. So we will always talk of *paths* in the infinite dimensional manifold M. Note also that these *paths* are open-ended, while *curves* comprising M are closed.

The path connecting x to y that provides the $\min E(\gamma)$ in the class of all paths γ connecting x to y is the *(minimal length) geodesic connecting x to y.*[(5)]

Note that, in the classical books on Finsler Geometry (see for example [BCS]), M is finite dimensional, and $F(c,v)^2$ is considered to be smooth and *strongly convex* in the v variable (for $v \neq 0$) (see also 1.1.2); this hypothesis is not needed, though, for the theorems in §A.

1.1.1 Riemannian geometry

Definition 1.4 *Suppose that M is modeled on a Hilbert separable space H. A* Riemannian geometry *is defined by associating to M a* Riemannian metric *g; for any $c \in M$, $g(c)$ is a positive definite bilinear form on the tangent space T_cM of M at c: that is, if h, k are tangent to M at c, then $g(c)$ defines a scalar product $\langle h, k\rangle_c$. If the form g is positive semi-definite then the geometry is degenerate, and we will speak of a* pseudo-Riemannian metric*. If it is positive definite, then tangent space T_xM is isomorphic to H, by means of the metric g. See [Lan99], ch. VII.*

A Riemannian geometry is a special case of a Finsler geometry: we define the norm

$$|h|_c = \sqrt{\langle h, h\rangle} \quad ,$$

(pseudo-Riemannian geometries produce a seminorm $|h|_c$).

In the following we often drop the base point c from $\langle h, k\rangle_c$ and $|v|_c$.

If M is finite dimensional, then we can write

$$\langle h, k\rangle_c = h_i g^{i,j}(c) k_j$$

in a choice of local coordinates; then the matrix $g^{i,j}(c)$ is smooth and positive definite.

1.1.2 Geodesics and the exponential map

Suppose that the metric F is a *regular Finsler metric*, that is, F is of class C^2 and $F(c,\cdot)^2$ is strongly convex (for $v \neq 0$); such is the case when $F(x,v)^2 = |v|^2 = \langle v, v\rangle$ in a smooth Riemannian manifold.

Let

$$\ddot\gamma = \Gamma(\dot\gamma, \gamma)$$

be the Euler–Lagrange O.D.E. characterizing critical paths γ of

$$E(\gamma) = \int_0^1 |\dot\gamma(v)|^2 \, dv$$

Define the *exponential map* $\exp_c : T_pM \to M$ as $\exp_c(\eta) = \gamma(1)$ when γ is the *geodesic* curve solving

$$\left\{ \ddot\gamma(v) = \Gamma(\dot\gamma(v), \gamma(v)), \quad \gamma(0) = c, \quad \dot\gamma(0) = \eta \right. \tag{1.5}$$

Then we may state this extended version of the Hopf–Rinow theorem

(5) As explained in the proposition A.1 in the appendix A, we may equivalently define a minimal geodesic to be a minimum of $\min \mathrm{Len}(\gamma)$, that has been reparameterized to constant velocity

Theorem 1.6 (Hopf-Rinow) *Suppose that M is finite dimensional and connected, then these are equivalent:*

- *(M, d) is complete*
- *closed bounded sets are compact*
- *M is* geodesically complete *at a point c, that is, (1.5) can be solved for all $v \in \mathbb{R}$ and all η, that is, the map $\eta \mapsto \exp_c(\eta)$ is well defined*
- *for any c, the map $\eta \mapsto \exp_c(\eta)$ is well defined and surjective;*

and all those imply that $\forall x, y \in M$ there exist a minimal geodesic connecting x to y.

1.1.3 Submanifolds

The simplest examples of Riemannian manifolds are the submanifolds of a Hilbert space H. We think of the finite dimensional case, when $H = \mathbb{R}^n$, or the infinite dimensional case, when we assume that H is separable.

Proposition 1.7 *Define the distance $d(x, y) = \|x - y\|$ in H. Suppose that $M \subset H$ is a closed submanifold. We may view M as a metric space, (M, d): then it is complete.*

We may moreover induce a Riemannian structure on M using the scalar product of H: this in turn induces the geodesic distance d^g, as defined in (1.3). Then $d^g \geq d$, and (M, d^g) is complete as well. If M is of class C^2, moreover, then d and d^g are locally equivalent.[(6)]

It is not guaranteed that d and d^g are *globally* equivalent, as shown by this example

Example 1.8 [(7)] *Let $H = \mathbb{R}^2$ and $M = \{(s, \sin(s^2))\}$. Let $x_n = (\sqrt{\pi n}, 0) \in M$. Then $d(x_n, x_{n+1}) \to 0$ whereas $d^g(x_n, x_{n+1}) \geq 2$.*

In a certain sense, infinite dimensional Riemannian manifolds are simpler than their corresponding finite-dimensional: indeed, by [EE70],

Theorem 1.9 (Eells-Elworthy) *Any smooth differentiable manifold M modeled on an infinite dimensional Hilbert space H may be embedded as an open subset of a Hilbert space.*

With respect to geodesics, the matter is though much more complicated. Suppose M is infinite dimensional. In this case, if (M, d) is complete the equation (1.5) of *geodesic* curves can be solved for all $v \in \mathbb{R}$; but (unfortunately) many other important implications contained in the Hopf–Rinow theorem are false.

The most important example is due to Atkin [Atk75]:

Example 1.10 (Atkin) *There exists an infinite dimensional complete connected smooth Riemannian manifold M and $x, y \in M$ such that there is no geodesic connecting x to y.*

(6) Proof by standard arguments, see for example sec. VIII.§6 in [Lan99]

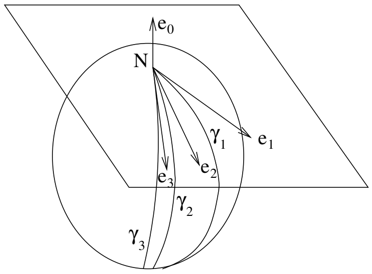
(7) We thank A.Abbondandolo for this remark.

We show a simpler example, of an infinite dimensional Riemannian manifold M such that the metric space (M, d) is complete, but there exist two points $x, y \in M$ that cannot be connected by a minimal geodesic.

Example 1.11 (Grossman) (8) *Let* $l^2(\mathbb{N})$ *be the Hilbert space of real sequences* $x = (x_0, x_1, \ldots)$ *with the scalar product*

$$\langle x, y \rangle = \sum_{i=0}^{\infty} x_i y_i$$

Let $e_i = (0, 0, \ldots, 0, 1, 0, \ldots)$ *where* 1 *is in the i-th position.*



We build an ellipsoid

$$M = \left\{ x \in l^2 \mid x_0^2 + \sum_{i=1} x_i^2/(1+1/i)^2 = 1 \right\}$$

in $l^2(\mathbb{N})$.
Since M *is closed, then it is complete (with the induced metric).*

Let $N = e_0 = (1, 0, 0, \ldots)$, $S = -e_0 = (-1, 0, 0, \ldots)$.

Let γ_i *be the geodesic starting from* $\gamma_i(0) = N$ *and with starting speed* $\dot\gamma_i(0) = e_i$*; then there exists a first* $\lambda_i > 0$ *such that* $\gamma_i(\lambda_i) = S$ *(moreover* $Len(\gamma_i) = \lambda_i$*).*

Then $Len(\gamma_i) \to \pi$*, but the sequence* γ_i *does not have a limit.*

Note that we may think of using weak convergence: but γ_i *weakly converges to the diameter; and* e_i *weakly converges to* 0.

See also [Eke78].

It is then, in general, quite difficult to prove that an infinite dimensional manifold admits minimal geodesics (even when it is known to be complete); a known result is

Theorem 1.12 (Cartan-Hadamard) *Suppose that* M *is connected, simply connected and has seminegative curvature; then these are equivalent:*

- (M, d) *is complete*
- *for a c, the map* $\eta \to \exp_c(\eta)$ *is well defined*

and then there exists an unique minimal geodesic connecting any two points. (9)

1.2 Geometries of curves

Now, suppose that $c(\theta)$ is an immersed curve $c : S_1 \to \mathbb{R}^n$, where S^1 is the circle; we want to define a geometry on M, the space of all such immersions c.

M is a manifold; the tangent space T_cM of M at c contains all the *deformations* $h \in T_cM$ of the curve c, that are all the vector fields along c. Then, an infinitesimal deformation of the curve c in "direction" h will yield (on first order) the curve $c(u) + \varepsilon h(u)$.

If $\gamma : [0, 1] \to M$ is a path connecting curves, then we may define a homotopy $C : S^1 \times [0, 1] \to \mathbb{R}^n$ associated to γ by $C(\theta, v) = \gamma(v)(\theta)$ (more is in §3.2.1).

(8) This example is also in a remark in sec. VIII.§6 in [Lan99]

(9) Corollary 3.9 and 3.11 in sec. IX.§3 in [Lan99]

1.2.1 Finsler geometry of curves

Any energy that we will study in this paper can be reconducted to this general form

$$E(\gamma) = \int_0^1 F\Big(\gamma(\cdot, v), \partial_v \gamma(\cdot, v)\Big)^2 \, dv \tag{1.13}$$

where $F(c,h)$ is defined when c is a curve, and $h \in T_c M$ is a deformation of c; note that F will be often a Minkowsky *seminorm* and not a *norm* [(10)] on the space M of immersions (see 1.18).

We look mainly for metrics in the space M that are independent on the parameterization of the curves c: to this end, [MM] define

$$B_i = B_i(S^1, \mathbb{R}^2) = \mathrm{Imm}(S^1, \mathbb{R}^2)/\mathrm{Diff}(S^1)$$

and

$$B_{i,f} = B_{i,f}(S^1, \mathbb{R}^2) = \mathrm{Imm}_f(S^1, \mathbb{R}^2)/\mathrm{Diff}(S^1)$$

that are the quotients of the spaces Imm_f of smooth immersion, and of the space $\mathrm{Imm}_f(S^1, \mathbb{R}^2)$ of smooth *free* immersion, with $\mathrm{Diff}(S^1)$ (the space of automorphisms of S^1). $B_{i,f}$ is a manifold, the base of a principal fiber bundle, as proved in §2.4.3 in [MM], while B_i is not. Any metric that does not depend on the parameterization of the curves c (as defined in eq.(1.15)) may be projected to B_i by means of the results in §2.5 in [MM] (the most important step appears also here as 3.10).

Remark 1.14 (extending M) *Imm*(S^1, M) *is on open subset of the Banach space* $C^1(S^1 \to \mathbb{R}^n)$*, where it is connected iff* $n \geq 3$*; whereas in the case* $n = 2$ *of planar curves, it is divided in connected components each containing curves with the same* winding number*.*

To define a Riemannian Geometry on $M =$ *Imm*(S^1, M)*, it may be convenient to view it as a subset of an Hilbert space such as* $H^1(S^1 \to \mathbb{R}^n)$*, and to complete it there.*

1.3 Abstract approach

As a first part of this paper, we want to cast the problem in an abstract setting. There are some general properties that we may ask of a metric defined as in sec. 1.2.1. We start with a fundamental property (that is a prerequisite to most of the others).

0. **[well-posedness and existence of minimal geodesics]** The Finsler metric F induces a well defined [(11)] geodesic distance d; (M, d) is complete (or, it may be completed inside the space of mappings $c : S^1 \to \mathbb{R}^n$); for any two curves in M, there exists a minimal geodesic connecting them.

Let C be a minimal geodesic connecting c_0 and c_1. We assume that C is a homotopy of c_0 to c_1.

[(10)] That is, it will fail to satisfy property 3 in definition 1.1

[(11)] That is, the distance between different points is positive, and d generates the same topology that the atlas of the manifold M induces

1. **[rescaling]** *if* $\lambda > 0$*, if we rescale* c_0, c_1 *to* $\lambda c_0, \lambda c_1$*, then we would like that* λC *be a minimal geodesic.*

   A sufficient condition is that $F(\lambda c, h) = \lambda^a F(c, h)$ for some $a \geq 0$, and all $\lambda \geq 0$. [12]

2. **[euclidean invariance]** *If we apply an euclidean transformation* A *to* c_0*,* c_1*, we would like that* AC *be a minimal geodesic connecting* Ac_0 *to* Ac_1

   If $F(Ac, Ah) = F(c, h)$ for all c, h, then the above is satisfied.

3. **[parameterization invariance]** there are two version of this: we define

   curve-wise parameterization invariance when the metric does not depend on the parameterization of the curve, that is

   $$F(\tilde{c}, \tilde{h}) = F(c, h) \tag{1.15}$$

   when $\tilde{c}(t) = c(\varphi(t))$ and $\tilde{h}(t) = h(\varphi(t))$ are reparameterizations of c, h

   homotopy-wise parameterization invariance *Define*

   $$\tilde{C}(\theta, v) = C(\varphi(\theta, v), v)$$

   where $\varphi : S^1 \times [0,1] \to S^1$*,* $\varphi \in C^1$*,* $\varphi(\cdot, v)$ *is a diffemorphism for all fixed* v*. We would like that, in this case,* $E(\tilde{C}) = E(C)$*.*
   If F can be written as

   $$F = F(c, \pi_N h) \tag{1.16}$$

   (that is, F depends only on the normal part of the deformation) and if it satisfies (1.15), then, by proposition 3.8, $E(\tilde{C}) = E(C)$.

   In both cases, the geometric structure we are building depends only on the embedding of the curves, and not on the parameterization.

The above properties defined in 1,2 are valid for all examples that we will show; property 3 is satisfied for some of them. Property 0 is possibly the most important argument.

Definition 1.17 *Any metric satisfying the above 1,2,3 is called a* **geometric metric**.

Note that

Remark 1.18 *If* F *satifies (1.16) then* $F(c, \cdot)$ *is necessarily a seminorm and not a norm*[13] *on the space* M*: we should then talk of a* pseudo-Riemannian *geometry of curves. The projection of* F *to the space* $B_{i,f}$ *may be nonetheless a norm.*

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[12] We could ask $F(\lambda c, h) = l(\lambda) F(c, h)$ for some $l : \mathbb{R}^+ \to \mathbb{R}^+$ monotone increasing, with $l(x) > 0$ for $x > 0$; but, if l is continuous, then $l(x) = x^a$ for some $a \geq 0$

[13] That is, it does not satisfy property 3 in the definition 1.1

These other properties would be very important in applications in Computer Vision. [14]

4. **[finite projection]** *There should be a finite dimensional approximation of our metric, for purposes numerical computation.*

   A sufficient condition is that the energy $E(C)$ should be well defined and continuous with respect to a norm of a Sobolov space $W^{k,p}(I)$ (with $k \in \mathbb{N}$ and $p \in [1, \infty)$): this would imply that we may approximate C by smooth functions C_h and $E(C_h) \to E(C)$.

5. **[embedding preserving]** *if c_0 and c_1 are embedded, we would like $C(\cdot, v)$ to be embedded at all v*

6. **[maximum principle]** In the following, suppose that curves are embedded in $\mathbb{R}^2$, and write $c \subset c'$ to mean that c' is contained in the bounded region of plane enclosed by c. [15]

   If $c_0 \subset c'_0$, $c_1 \subset c'_1$, then we would like that exists a minimal geodesic C' connecting c'_0 to c'_1 such that $C'(\cdot, v) \subset C(\cdot, v)$ for $v \in [0, 1]$.

   This is but another version of the Maximum Principle; it would imply that the minimal geodesic is unique. It is an important prerequisite if we want to implement numerical algorithms by using *Level Set Methods*.

7. **[convexity preserving]** *if c_0 and c_1 are convex, we would like $C(\cdot, v)$ to be convex at all v*

8. **[convex bounding]** *we would like that at all v, (the image of) the curve $C(\cdot, v)$ be contained in the convex envelope of the curves c_0, c_1*

9. **[translation]** *If $a \in \mathbb{R}^n$, if $c_1 = a + c_0$ is a translation of c_0, we would like that the uniform movement $C(\theta, v) = c_0(\theta) + va$ be a minimal geodesic from c_0 to c_1*

So we state the abstract problem: [16]

Problem 1.19 *Consider the space of curves M, and the family $\mathcal{G}$ of all Riemannian (or, regular Finsler) Geometries F on M.* [17]

Does there exist a metric $F \in \mathcal{G}$, satisfying the above properties 0,1,2,3?

Consider metrics $F \in \mathcal{G}$ that may be written in integral form

$$F(c, h) = \int_c f\big(c(s), d_s c(s), \ldots, d^j_{s^j} c(s),\, h(s), \ldots, d^i_{s^i} h(s)\big) ds$$

what is the relationship between the degrees i, j and the properties in this section?

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[14] Some of these properties are much trickier: we do not know sufficient conditions that imply them

[15] By Jordan's closed curve theorem, any embedded closed curve in the plane divides the plane in two regions, one bounded and one unbounded

[16] Solving this problem in abstract would be comparable to what Shannon did for communication theory, where in [Sha49] he asserted there would exist a code for communication on a noisy channel, without actually showing an efficient algorithm to compute it.

[17] $\mathcal{G}$ is non empty: see [Lan99]. By using 1.9, it would seem that there exists Riemannian metrics F on M such that (M, F) is geodesically complete; we did not carry on a detailed proof.

2 Examples, and different approaches and results

We now present some approaches and ideas that have been proposed to define a metric and a distance on the space of curves; we postpone exact definitions to section §3.2.

2.1 Riemannian geometries of curves

A Riemannian geometry is obtained by associating to T_cM the scalar product of an Hilbert space H of squared integrable functions. We actually have many choices for the definition of H.

2.1.1 Parametric (non-geometric) form of H^0

- We may define

$$H = H^0(S^1 \to \mathbb{R}^n) = L^2(S^1 \to \mathbb{R}^n)$$

endowed with the scalar product

$$\langle h, k \rangle = \int_{S_1} \langle h(\theta), k(\theta) \rangle d\theta \tag{2.1}$$

for all $h, k \in T_cM$; this is a common choice in analysis and geometry texts[(18)]; however, the resulting metric is not invariant with respect to parameterization of curves (see (3) on page 8) and is therefore not geometric.

Remark 2.2 *This is the most common choice in numerical applications: each curve is numerically represented by a finite number m of sample points; thereby discretizing the geometry of curves to the geometry of* $\mathbb{R}^{nm}$.

Therefore (1.13) takes the form

$$\int_0^1 \|\dot\gamma(v)\|^2_{H^0}\ dv = \int_0^1 \int_{S^1} |\partial_v \gamma(\theta, v)|^2\ d\theta dv \tag{2.3}$$

that is defined when $\gamma \in H^1([0, 1] \to M)$. The energy of a homotopy is then

$$E_s(C) = \int_I |\partial_v C|^2$$

Definition 2.4 *We define the space*

$$H^{1,0}\big(I \to \mathbb{R}^n\big) = H^1\Big([0, 1] \to H^0(S^1 \to \mathbb{R}^n)\Big)$$

and define the norm on the above space $H^{1,0}$ to be

$$\int_0^1 \int_{S^1} |\gamma(\theta, v)|^2 + |\partial_v \gamma(\theta, v)|^2\ d\theta dv \tag{2.4.$\star$}$$

(18) See ch 2.3 in [Kli82], or the long list of references at the end of II§1 in [Lan99]

then

Proposition 2.5 *$H^{1,0}$ is the space of all finite energy homotopies, and the norm (2.4.⋆) above is equivalent to the energy (2.3) on families γ with fixed end points (by prop. 3.14).*

$E_s(C)$ is strongly continuous in $H^{1,0}$ and is convex, and $E_s(C)$ is coercive (proof by using 3.14). So E_s has a very simple unique minimal geodesic, namely, the pointwise linear interpolation

$$C^*(\theta, v) = (1-v)c_0(\theta) + vc_1(\theta)$$

- As a second choice, we may define in H the scalar product

$$\langle h, k\rangle = \int_0^l \langle h(s), k(s)\rangle ds = \int_{S_1} \langle h(\theta), k(\theta)\rangle |\dot{c}(\theta)| d\theta \qquad (2.6)$$

  where l is the length of c, ds is the arc infinitesimal, and $\dot{c} = \partial_\theta c$.[(19)]

  This scalar product does not depend on the parameterization of the curve c; but the resulting metric is still not invariant with respect to reparameterization of homotopies (see 3 in 8).

  By projecting this metric onto $B_{i,f}$ and lifting it back to M (using 3.10), we then devise an appropriate geometric metric, as follows.

2.1.2 Geometric form of H^0

We then propose the scalar product

$$\langle h, k\rangle = \int_0^l \langle \pi_N h(s), \pi_N k(s)\rangle ds \qquad (2.7)$$

where π_N is the projection to the normal space N to the curve (see 3.2). From now on, when we speak of the H^0 metric, we will be implying this last definition.

Note that we may equivalently define the scalar product as in (2.6), and only accept in T_cM deformations that are orthogonal to the tangent of the curve. This would be akin to working in the quotient manifold $B_{i,f}$; or it may be viewed as a *sub-Riemannian geometry* on M itself.

This geometric metric generates the energy

$$E^N(C) \doteq \int_I |\pi_N \partial_v C|^2 |\dot{C}|\, d\theta\, dv \qquad (2.8)$$

which is invariant of reparameterizations of homotopies (see 3 in page 8). By proposition 3.10, the distance induced by this metric is equal to the distance induced by the previous one (2.6).

(19) There is an abuse of notation in (2.6); we would like, intuitively, to define h and k on the *immersed curve* c; to this end, we define $h(s)$ and $k(s)$ on the arc-parameterization $c(s) : [0, l] \to \mathbb{R}^n$ (with $l = \text{len}(c)$), and pull them back to $c(\theta) : S^1 \to \mathbb{R}^n$, where we write $h(\theta), k(\theta)$ instead of $h(s(\theta)), k(s(\theta))$

Unfortunately, it has been noted in [Mum] that the metric H^0 does not define a distance between curves, since

$$\inf E^N(C) = 0$$

(see C.1 and C.3). We will study this metric further in section 4.

There is a good reason to focus our attention on the properties of this metric (2.7) for curves. Namely, this is precisely the metric that is implicitly assumed in formulating *gradient flows* of contour based energy functionals in the vast literature on shape optimization. Consider for example the well known *geometric heat flow* ($\partial_t C = \partial_{ss} C$) [(20)] in which a curve evolves along the inward normal with speed equal to its signed curvature. This flow is widely considered to be the *gradient descent* of the Euclidean arclength functional. Its smoothing properties have led to its widespread use within the fields of computer vision and image processing. The only sense, however, in which this is truly a gradient flow is with respect to the H^0 metric as we see in the following calculation (where $L(t)$ denotes the time varying arclength of an evolving curve $C(u,t)$ with parameter $u \in [0,1]$).

$$\begin{aligned}
L(t) &= \int_C ds = \int_0^1 |C_u|\, du \\
L'(t) &= \int_0^1 \frac{C_{ut} \cdot C_u}{|C_u|}\, du = \int_0^1 C_{tu} \cdot C_s\, du = -\int_0^1 C_t \cdot C_{su}\, du = -\int_C C_t \cdot C_{ss}\, ds \\
&= -\Big\langle C_t, C_{ss} \Big\rangle_{H^0}
\end{aligned}$$

If we were to change the metric then the inner-product shown above would no longer correspond to the inner-product associated to the metric. As a consequence, the above flow could no longer be considered as the *gradient flow* with respect to the new metric. In other words, the gradient flow would be different. We will consider this consequence more at length in §5.

2.1.3 Michor-Mumford

To overcome the pathologies of the H^0 metric, Michor and Mumford [MM] propose the metric

$$G_c^A(h,k) = \int_0^1 (1 + A|\kappa_c|^2)\langle \pi_N h(u), \pi_N k(u)\rangle |\dot c(u)| du$$

on planar curves, where $\kappa_c(u)$ is the curvature of $c(u)$, and $A > 0$ is fixed. This may be generalized to the energy [(21)]

$$E^A(C) \doteq \int_0^1 \int_{S^1} |\pi_N \partial_v C|^2 (1 + A|H|^2)|\dot C|\; d\theta dv = E^N(C) + AJ(C) \tag{2.9}$$

on space homotopies $C(u,v) : [0,1]\times[0,1] \to \mathbb{R}^n$. This approach is discussed in §4.4.1.

[(20)] We will sometimes write C_v for $\partial_v C$ (and so on), to simplify the derivations

[(21)] $J(C)$ is defined in (4.2); H is the *mean curvature*, defined in 3.3. Note that both $E^N(C)$ and $J(C)$ are invariant with respect to reparameterization, in the sense defined in 3 on page 8.

2.1.4 Srivastava et al.

We consider here planar curves of length 2π and parameterized by arclength, using the notation $\xi(s) : S^1 \to \mathbb{R}^2$. Such curves are Lipschitz continuous.

If $\xi \in C^1$, then $|\dot\xi| = 1$, so we may lift the equality

$$\dot\xi(s) = \big(\cos(\theta(s)),\ \sin(\theta(s))\big) \tag{2.10}$$

to obtain a continuous function $\theta : \mathbb{R} \to \mathbb{R}$. This continuous lifting θ is unique up to addition of a constant $2\pi h$, $h \in \mathbb{Z}$; and $\theta(s + 2\pi) - \theta(s) = 2\pi i$, where $i \in \mathbb{Z}$ is the *winding number*, or *rotation index* of ξ. The addition of a generic constant to θ is equivalent to a rotation of ξ. We then understand that we may represent arc-parameterized curves $\xi(s)$, up to translation, scaling, and rotations, by considering a suitable class of liftings $\theta(s)$ for $s \in [0, 2\pi]$.

Two spaces are defined in [KSMJ03]; we present the case of *"Shape Representation using Direction Functions"*, where the space of (pre)-shapes is defined as the closed subset M of $L^2 = L^2([0, 2\pi])$,

$$M = \big\{\theta \in L^2([0, 2\pi]) \mid \phi(\theta) = (2\pi^2, 0, 0)\big\}$$

where $\phi : L^2 \to \mathbb{R}^3$ is defined by

$$\phi_1(\theta) = \int_0^{2\pi} \theta(s) ds \quad , \quad \phi_2(\theta) = \int_0^{2\pi} \cos\theta(s) ds \quad , \quad \phi_3(\theta) = \int_0^{2\pi} \sin\theta(s) ds$$

Define Z as the set of representations in M of flat curves; then Z is closed in L^2, and $M \setminus Z$ is a manifold: (22)

Proposition 2.11 *By the implicit function theorem, $M \setminus Z$ is a smooth immersed submanifold of codimension 3 in L^2.*

Note that $M \setminus Z$ contains the (representation of) all smooth immersed curves.

The manifold $M \setminus Z$ inherits a Riemannian structure, induced by the scalar product of L^2; geodesics may be prolonged smoothly as long as they do not meet Z. Even if M may not be a manifold at Z, we may define the geodesic distance $d^g(x, y)$ in M as the infimum of the length of Lipschitz paths $\gamma : [0, 1] \to L^2$ whose image is contained in M; (23) since $d^g(x, y) \geq \|x - y\|_{L^2}$, and M is closed in L^2, then the metric space (M, d^g) is complete.

We don't know if (M, d^g) admits minimal geodesics, or if it falls in the category of examples such as 1.11.

For any $\theta \in M$, it is possible to reconstruct the curve by integrating

$$\xi(s) = \int_0^s \cos(\theta(t)), \sin(\theta(t)))dt \tag{2.12}$$

This means that θ identifies an unique curve (of length 2π, and arc-parameterized) up to rotations and translations, and to the choice of the base point $\xi(0)$; for this last reason, M is called in [KSMJ03] a *preshape space*. The *shape space* S

(22) The details and the proof of 2.11 and 2.14 are in appendix §B

(23) It seems that M is Lipschitz-arc-connected, so $d^g(x, y) < \infty$; but we did not carry on a detailed proof

is obtained by quotienting M with the relation $\theta \sim \hat\theta$ iff $\theta(s) = \hat\theta(s-a)+b$, $a, b \in \mathbb{R}$. We do not discuss this quotient here.

We may represent any Lipschitz closed arc-parameterized curve ξ using a measurable $\theta \in M$: let $\mathrm{arc} : S^1 \to [0, 2\pi)$ be the inverse of $\theta \mapsto (\cos(\theta), \sin(\theta))$; arc() is a Borel function; then $((\mathrm{arc} \circ \dot\xi)(s) + a) \in M$, for an $a \in \mathbb{R}$. We remark that the measurable representation is never unique: for any measurable $A, B \subset [0, 2\pi]$ with $|A| = |B|$, $(\theta(s) + 2\pi(\mathbf{1}_A(s) - \mathbf{1}_B(s)))$ will as well represent ξ in M. This implies that the family A_ξ of θ that represent the same curve ξ is infinite. It may be then advisable to define a quotient distance as follows:

$$\hat d(\xi, \xi') \doteq \inf_{\theta \in A_\xi, \theta' \in A_{\xi'}} d(\theta, \theta') \tag{2.13}$$

where $d(\theta, \theta') = \|\theta - \theta'\|_{L^2}$, or alternatively $d = d^g$ is the geodesic distance on M.

If $\xi \in C^1$, we have an unique[(24)] continuous representation $\theta \in M$; but note that, even if $\xi, \xi' \in C^1$, the infimum (2.13) may not be given by the continuous representations θ, θ' of ξ, ξ'. Moreover there are curves ξ that do not admit a continuous representation θ. As a consequence, it will not be possible to define the rotation index of such curves ξ; indeed we prove this result:

Proposition 2.14 *For any $h \in \mathbb{Z}$, the set of closed smooth curves ξ with* rotation index *h, when represented in M using (2.10), is dense in $M \setminus Z$.*

2.1.5 Higher order Riemannian geometry

If we want an higher order model, we may define a metric mimicking the definition of the Hilbert space H^1, by defining

$$\langle h, k\rangle_{H^1} = \langle h, k\rangle_{H^0} + \langle \dot h, \dot k\rangle_{H^0} \tag{2.15}$$

We have again many different choices, since

- we may use in the RHS of (2.15) the parametric H^0 scalar product (2.1), in which case the scalar product $\langle h, k\rangle_{H^1}$ is the standard scalar product of $H^1(S^1 \to \mathbb{R}^n)$; then homotopies are in the space

  $$H^1\big([0,1] \to H^1(S^1 \to \mathbb{R}^n)\big)$$

  with norm

  $$\int_I |\gamma|^2 + \langle \partial_v \gamma \partial_v \gamma\rangle + \langle \partial_v \dot\gamma, \partial_v \dot\gamma\rangle$$

- we may use in the RHS of (2.15) the scalar product (2.6) or (2.7). Unfortunately none of these choices is invariant with respect to reparameterization of homotopies.

We don't know of many application of this idea; the only exception may be considered to be [You98].

[(24)] Indeed, the continuous lifting is unique up to addition of a constant to $\theta(s)$, which is equivalent to a rotation of ξ; and the constant is decided by $\phi_1(\theta) = 2\pi^2$

2.2 Finsler geometries of curves

To conclude, we present two examples of Finsler geometries of curves that have been used (sometimes covertly) in the literature.

2.2.1 L^∞ and Hausdorff metric

If we wish to define a norm on T_cM that is modeled on the norm of the Banach space $L^\infty(S^1 \to \mathbb{R}^n)$, we define

$$F^\infty(c,h) = \|\pi_N h\|_{L^\infty} = \operatorname{supess}_\theta |\pi_N h(\theta)|$$

This Finsler metric is geometric. The length of a homotopy is then

$$\operatorname{Len}(C) = \int_0^1 \operatorname{supess}_\theta |\pi_N \partial_v C(\theta,v)| dv$$

Hausdorff metric We recall the definition of the Hausdorff metric

$$d_H(A,B) \doteq \max\left\{\max_{x\in A} d(x,B), \max_{y\in B} d(A,y)\right\}$$

where $A,B \in \mathbb{R}^n$ are closed, and

$$d(x,A) \doteq \min_{y\in A} |x-y|$$

Let Ξ be the collection of all compact subsets of $\mathbb{R}^n$. We define the length of any continuous path $\xi : [0,1] \to \Xi$ by using the total variation, as follows

$$\operatorname{len}^H \gamma \doteq \sup_T \sum_{i=1}^j d_H\big(\xi(t_{i-1}), \xi(t_i)\big) \tag{2.16}$$

where the sup is carried out over all finite subsets $T = \{t_0, \cdots, t_j\}$ of $[0,1]$ and $t_0 \le \cdots \le t_j$.

The metric space (Ξ, d_H) is complete and path-metric,(25) and it is possible to connect any two $A,B \in \Xi$ by a minimal geodesic, of length $d_H(A,B)$.(26)

Let Ξ_c be the class of compact connected $A \subset \mathbb{R}^n$; Ξ_c is a closed subset of (Ξ, d_H); Ξ_C is Lipschitz-path-connected (27); for all above reasons, it is possible to connect any two $A,B \in \Xi_c$ by a minimal geodesic moving in Ξ_c; but note that Ξ_c is not geodesically convex in Ξ.(28) We don't know if (Ξ_c, d_H) is path-metric.

Projection of F^∞ into Hausdorff metric space When we associate to a continuous curve $c \in M$ its image $Im(c) \subset \mathbb{R}^n$, we are actually defining a natural projection

$$Im : M \to \Xi_c$$

(25)Path-metric: $d_H(A,B) = \inf \operatorname{len}^H \gamma$ where the infimum is computed in the class of Lip curves $\gamma : [0,1] \to \Xi$ connecting A to B

(26)To prove this, note that (Ξ, d_H) is locally compact and complete; and apply A.2

(27)That is, any $A,B \in \Xi_C$ can be connected by a Lipschitz arc $\gamma : [0,1] \to \Xi_c$

(28)There exist two points $A,B \in \Xi_c$ and a minimal geodesics ξ connecting A to B in the metric space (Ξ, d), such that the image of ξ is not contained inside Ξ_c

This projection transforms a path γ in M into a path $\xi : [0,1] \to \Xi_c$; if the homotopy $C(\theta, v) = \gamma(v)(\theta)$ is continuous, then ξ is continuous. Moreover the projection of all embedded curves c is dense in Ξ_c.

It is possible to prove that

$$\text{Len}(\gamma) = \text{len}^H(\xi)$$

for a large class of paths; and then the distance induced by the metric F^∞ coincides with $d_H(Im(c_0), Im(c_1))$.

This is quite useful, since in the metric space generated by the metric F^∞, it is possible to find two curves that cannot be connected by a geodesic (this is due to topological obstructions); whereas, the minimal geodesic will exist in the space (Ξ_c, d_H).

For this reason, [CFK03] proposed an approximation method to compute $\text{len}^H(\xi)$ by means of a family of energies defined using a smooth integrand (29), and successively to find approximation of geodesics.

Unfortunately, the geometry in (Ξ, d_H) is highly non regular: for example, it is possible to find two compact sets such that there are uncountably many minimal geodesics connecting them. (30)

2.2.2 L^1 and Plateau problem

If we wish to define a geometric norm on T_cM that is modeled on the norm of the Banach space $L^1(S^1 \to \mathbb{R}^n)$, we may define the metric

$$F^1(c,h) = \|\pi_N h\|_{L^1} = \int |\pi_N h(\theta)||\dot{c}(\theta)|d\theta$$

The length of a homotopy is then

$$\text{Len}(C) = \int_I |\pi_N \partial_v C(\theta, v)||\dot{C}(\theta, v)|d\theta dv$$

which coincides with

$$\text{Len}(C) = \int_I |\partial_v C(\theta, v) \times \partial_\theta C(\theta, v)|d\theta dv$$

This last is easily recognizable as the surface area of the homotopy (up to multiplicity); the problem of finding a minimal geodesic connecting c_0 and c_1 in the F^1 metric may be reconducted to the Plateau problem of finding a surface which is an immersion of $I = S^1 \times [0,1]$ and which has fixed borders to the curves c_0 and c_1. The Plateau problem is a wide and well studied subject upon which Fomenko expounds in the monograph [Fom90].

3 Basics

3.1 Relaxation of functionals

To prove that a Riemannian manifold admits minimal geodesics, we will study the energy $E(\gamma)$ by means of methods in Calculus of Variations; we review some basic ideas.

(29) The approximation is mainly based on the property $\|f\|_{L^p} \to_p \|f\|_{L^\infty}$, for any measurable function f defined on a bounded domain

(30) This is an unpublished result due to Alessandro Duci.

Let X be a topological space, endowed with a topology τ, and $\Omega \subset X$, and $f : \Omega \to \mathbb{R}$.

The function f is *lower semi continuous* if, for all $x \in X$,

$$\liminf_{y \to x} f(y) \geq f(x)$$

Note that, if the topology τ is not defined by a metric, then we can introduce a different condition: F is *sequentially lower semi continuous* if, for all $x \in X$, for all $x_n \to x$,

$$\liminf_n f(x_n) \geq f(x)$$

We define the *sequential relaxation* Γf *of* f *on* Ω to be the function

$$\Gamma f : \overline{\Omega} \to \mathbb{R}$$

that is the supremum of all $f' : \overline{\Omega} \to \mathbb{R}$ that are sequentially lower semicontinuous on $\overline{\Omega}$, and $f' \leq f$ in Ω. We have that, for all $x \in \overline{\Omega}$,

$$\Gamma_\Omega f(x) = \min_{(x_n), x_n \to x} \{\liminf_n f(x_n)\}$$

where the minimum is taken on all sequences x_n converging to x, with $x_n, x \in \Omega$.

f is sequentially lower semicontinuous, iff $\Gamma f|_\Omega = f$.

If X is a metric space, then "sequentially lower semicontinuous functions" are "lower semicontinuous functions", and viceversa; so we may drop the adjective "sequentially".

Consider again a function $f : X \to \mathbb{R}$: it is called *coercive in* X if $\forall M \in \mathbb{R}$,

$$\{x \in X : f(x) \leq M\}$$

is contained in a compact set.

Proposition 3.1 *If F is coercive in X and sequentally lower semi continuous then it admits a minimum on any closed set, that is, for all $C \subset X$ closed there exists $x \in C$ s.t.*

$$f(x) = \min_{y \in C} f(y)$$

This result is one of the pillars of the modern *Calculus of Variations*. We will see that unfortunately this result may not be applied directly to the problem at hand (see 4.1).

3.2 Curves and notations

Consider in the following a curve $c(\theta)$ defined as $c : S^1 \to \mathbb{R}^n$.

We write $\dot{c}$ for $\partial_\theta c$.

Definition 3.2 *Suppose that c is C^1; or suppose that $c \in W^{1,1}_{loc}$, that is, c admits a* weak derivative *$\dot{c}$. At all points where $\dot{c}(\theta) \neq 0$, we define the tangent vector*

$$T(\theta) = \frac{\dot{c}(\theta)}{|\dot{c}(\theta)|}$$

At the points where $\dot{c} = 0$ we define $T = 0$.

Let $v \in \mathbb{R}^n$*. We define the projection onto the normal space* $N = T^\perp$

$$\pi_N v = v - \langle v, T\rangle T$$

and on the tangent

$$\pi_T v = \langle v, T\rangle T$$

so $\pi_N v + \pi_T v = v$ *and* $|\pi_N v|^2 + |\pi_T v|^2 = |v|^2$ *(that implies* $|\pi_N v|^2 = |v|^2 - \langle v, T\rangle^2$*).*

If c admits the *weak derivative* $\partial_\theta c$ then T is measurable, so $T \in L^\infty$ and $\pi_T, \pi_N \in L^\infty(S^1 \to \mathbb{R}^{n\times n})$

A curve $c \in C^1(S^1 \to \mathbb{R}^n)$ is *immersed* when $|\dot c| > 0$ at all points. In this case, we can always define the arc parameter s so that

$$ds = |\dot c(\theta)| d\theta$$

and the derivation with respect to the arc parameter

$$\partial_s = \frac{1}{|\dot c|}\partial_\theta$$

We will also consider curves $c \in W^{1,1}(S^1 \to \mathbb{R}^n)$ such that $|\dot c| > 0$ at almost all points.

There are two different definitions of *curvature* of an immersed curve: *mean curvature* H and *signed curvature* κ, which is defined when c is valued in $\mathbb{R}^2$.

H and k are *extrinsic curvatures* [(31)]: they are properties of the *embedding* of c into $\mathbb{R}^n$.

Definition 3.3 (H) *If* c *is* C^2 *regular and immersed, we can define the mean curvature* H *of* c *as*

$$H = \partial_s T = \frac{1}{|\dot c|}\partial_\theta T$$

In general, we will say that a curve $c \in W^{1,1}_{loc}(S^1)$ *admits* mean curvature in the measure sense *if there exists a a vector valued measure* H *on* S^1 *such that*

$$\int_I T(s)\partial_s\phi(s)\ ds = -\int_I \phi(s)\, H(ds) \qquad \forall \phi \in C^\infty(S^1)$$

that is

$$\int_I T(\theta)\partial_\theta\phi(\theta)\, d\theta = -\int_I \phi(\theta)\, |\dot c| H(d\theta) \qquad \forall \phi \in C^\infty(S^1)$$

Note that the two definitions are related, since when $c \in C^2$*, the measure is* $H = \partial_s T \cdot ds$*. See also [Sim83], §7 and §16.*

We can then define the projection onto the curvature vector H by

$$\pi_H v = \frac{1}{|H|^2}\langle v, H\rangle H$$

---

(31) See also: Eric W. Weisstein. "Extrinsic Curvature." From MathWorld–A Wolfram Web Resource. `http://mathworld.wolfram.com/ExtrinsicCurvature.html`

Definition 3.4 (N) *When the curve c is in* $\mathbb{R}^2$*, and is immersed, we can define*[32] *a unit vector N such that* $N \perp T$ *and N is* $\pi/2$ *degree anticlockwise with respect to T. In this case for any vector* $V \in \mathbb{R}^2$*,* $\pi_N V = N\langle N, V\rangle$*, and,*

$$|V^2| = \pi_N V$$

Definition 3.5 (κ) *if c is in* $\mathbb{R}^2$ *then we can define a signed scalar curvature*

$$\kappa = \langle H, N\rangle$$

If H is a measure, then k is a real valued measure defined by

$$\kappa(A) = \sum_{i=1}^{n} \int_A N_i(\theta)\, dH_i(d\theta) \quad .$$

Note that $|H| = |\kappa|$.
When $c \in C^2(S^1 \to \mathbb{R}^2)$ *is immersed,*

$$\partial_s T = \kappa N \quad \text{and} \quad \partial_s N = -\kappa T \quad .$$

Remark 3.6 *When the curve c is in* $\mathbb{R}^2$*, and is immersed then*

$$\langle H, v\rangle = \kappa \pi_N v$$

whereas for immersed curves c in $\mathbb{R}^n$

$$|\langle H, v\rangle| \le |H||\pi_N v| \tag{3.6.⋆}$$

and we do not expect to have equality in general when $n \ge 3$*, since H is only a vector in the* $(n-1)$*-dimensional subspace* $N = T^\perp$.

3.2.1 Homotopies

Let $I = S^1 \times [0, 1]$. We define a homotopy to be a continuous function

$$C(\theta, v) : S^1 \times [0, 1] \to \mathbb{R}^n$$

This homotopy is a path connecting $c_0 = C(\cdot, 0)$ and $c_1 = C(\cdot, 1)$ in the space M of curves: indeed any path γ in M is associated to a homotopy C by $C(\theta, v) = \gamma(v)(\theta)$.

We extend the above definitions to homotopies C, isolating any curve c in the homotopy by defining $c(\theta) = C(\theta, v)$ for the corresponding fixed v; for example, when C admits the *weak derivative* $\partial_\theta C$ then

$$\pi_T, \pi_N \in L^\infty(S^1 \times [0, 1] \to \mathbb{R}^{n\times n})$$

Remark 3.7 *We extend the measure* $H(\cdot, v)$ *on* S^1*, (that is curvature of any single curve* $C(\cdot, v)$*) to a Borel measure* $\hat{H}$ *on I, by*

$$\hat{H}(A) = \int_0^1 H(A_v)dv \tag{3.7.⋆}$$

[32] There is a slight abuse of notation here, since in the definition $N = T^\perp$ in 3.2 we defined N to be a "vector space" and not a "vector"

where A_v is the section of A. H can be defined equivalently using the formula

$$\int_I T(\theta,v)\partial_\theta\phi(\theta,v) = -\int_I \phi(\theta,v)|\dot{C}(\theta,v)|H(d\theta,dv) \qquad \forall\phi\in C_c^\infty(I)$$

Moreover we define the length

$$\mathrm{len}(C)(v) = \int_{S^1} |\dot{C}(\theta,v)|\ d\theta$$

so that $\mathrm{len}(C) : [0,1] \to \mathbb{R}^+$ is a function of v.

3.3 Preliminary results

Definition 3.8 (Reparameterization) *We define the reparameterization of C to $\widetilde{C}$ as*

$$\widetilde{C}(\theta,v) = C(\varphi(\theta,v),v) \tag{3.8.$\star$}$$

where $\varphi : S^1\times[0,1]\to S^1$ is C^1 regular with $\partial_\theta\varphi\neq 0$. Then (by direct computation)

$$\partial_\theta\widetilde{C}(\theta,v) = \partial_\theta C(\tau,v)\partial_\theta\varphi(\theta,v) \tag{3.8.$\star\star$}$$

so that $T = \widetilde{T}\,sign(\partial_\theta\varphi)$; and

$$\pi_{\widetilde{N}}\partial_v\widetilde{C}(\theta,v) = \pi_N\partial_v C(\tau,v) \tag{3.8.$\diamond$}$$

whereas

$$\pi_{\widetilde{T}}\partial_v\widetilde{C}(\theta,v) = \pi_T\partial_v C(\tau,v) + \dot{C}(\tau,v)\partial_v\varphi(\theta,v) \tag{3.8.$\diamond\diamond$}$$

where $\tau \doteq \varphi(\theta,v)$.

We may choose to reparameterize using the arclength parameter as in the following proposition.

Proposition 3.9 (Arc parameter) *For any C^1 regular homotopy C such that all curves $\theta\mapsto C$ are immersed, there exists a φ as in (3.8.$\star$) above such that $|\partial_\theta\widetilde{C}|$ is constant in θ for any v (that is, there exists $l:[0,1]\to\mathbb{R}$ such that $|\partial_\theta\widetilde{C}(\theta,v)| = l(v)$)*

Proof. We just choose

$$\varphi(\theta,v) = \frac{2\pi}{\mathrm{len}\, C}\int_0^\theta |\dot{C}(t,v)|\ dt$$

□

On the other hand, we may reparameterize to eliminate $\pi_T\partial_v C$

Proposition 3.10 *For any C^2 regular homotopy C such that all curves $\theta\mapsto C$ are immersed there exists a φ as in (3.8.$\star$) above such that $\pi_{\widetilde{T}}\partial_v\widetilde{C} = 0$.*

Proof. Both $\pi_T \partial_v C$ and $\dot{C}\partial_v\varphi$ are parallel to T: so, if $\partial_v\varphi = -\langle\partial_v C, T\rangle/|\dot{C}|$, then $\pi_{\widetilde{T}}\partial_v\widetilde{C} = 0$

The O.D.E.

$$\begin{cases} \partial_v\varphi(\theta,v) = -\frac{\langle\partial_v C(\tau,v),T(\tau,v)\rangle}{|\dot{C}(\tau,v)|} \\ \tau\doteq\varphi(\theta,v) \\ \varphi(\theta,0)=\theta, \quad \theta\in S^1 \end{cases} \tag{3.10.$\star$}$$

can be solved for $v\in[0,1]$, since

$$M(\tau,v) = -\frac{\langle\partial_v C(\tau,v),T(\tau,v)\rangle}{|\dot{C}(\tau,v)|}$$

is periodic in τ and continuous, and then is bounded: that is,

$$\max_{S^1\times[0,1]} M < \infty$$

Defining $\psi = \partial_\theta\varphi = \dot{\varphi}$ we compute

$$\begin{aligned} \frac{d}{dv}\psi(\theta,v) = \frac{d}{d\theta}\frac{d}{dv}\varphi = -\frac{d}{d\theta}\frac{\langle\partial_v C(\tau,v),T(\tau,v)\rangle}{|\dot{C}(\tau,v)|} = \\ = -\psi(\theta,v)\frac{d}{d\tau}\left(\frac{\langle\partial_v C(\tau,v),T(\tau,v)\rangle}{|\dot{C}(\tau,v)|}\right) \end{aligned}$$

so ψ solves

$$\begin{cases} \partial_v\psi(\theta,v) = -\psi(\theta,v)\frac{d}{d\tau}\frac{\langle\partial_v C(\tau,v),T(\tau,v)\rangle}{|\dot{C}(\tau,v)|} \\ \psi(\theta,0)=1, \quad \theta\in S^1 \end{cases} \tag{3.10.$\star\star$}$$

and then $\partial_\theta\varphi > 0$ at all times v. □

Note that φ is not unique: we may change in (3.10.$\star$) and simply require that $\varphi(\cdot,0)$ be a diffeomorphism. The above result is stated in section 2.8 in [MM]; there $\widetilde{C}$ is called a *horizontal path*, since it is the canonical parallel lifting of a path in $B_{i,f}$ to a path in M.

For example, consider the unit circle translating with unit speed along the x-axis, giving rise to the following homotopy $C(\theta,v)$:

$$C(\theta,v) = (v+\cos\theta, \sin\theta)$$

While this is certainly the simplest way to parameterize the homotopy, it does not yield a motion purely in the normal direction at each point along each curve. However, the following reparameterization of the homotopy yields exactly the same family of translating circles (and therefore the same homotopy) but such that each point on each circle along the homotopy flows exclusively along the normal to the corresponding circle (i.e. in the radial direction).

$$\begin{aligned} \widetilde{C}(\theta,v) &= (\widetilde{x},\widetilde{y}) \\ \widetilde{x}(\theta,v) &= v + \frac{(1-e^{2v})+(1+e^{2v})\cos\theta}{(1+e^{2v})+(1-e^{2v})\cos\theta} \\ \widetilde{y}(\theta,v) &= \frac{2e^v\sin\theta}{(1+e^{2v})+(1-e^{2v})\cos\theta} \end{aligned}$$

In figure 1 we see a comparision between the the trajectories of various points (fixed values of θ) along the translating circle in the original homotopy C and its reparameterization $\widetilde{C}$.

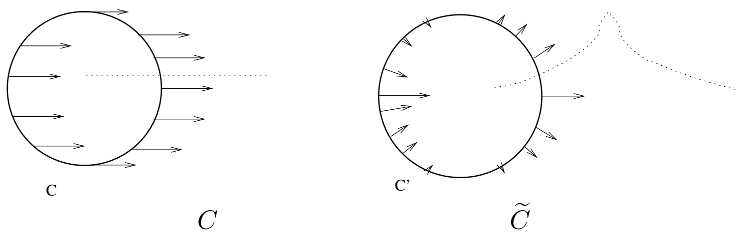


Figure 1: Reparameterization to $\pi_{\widetilde{T}}\partial_v\widetilde{C} = 0$ for the translation of the unit circle along the x-axis. The dotted line shows the trajectory of a point on the curve.

The above result is suprising, kind of "black magic": it seems to suggest that while modeling the motion of curves we can neglect the tangential part of the motion $\pi_T\partial_v C$. Unfortunately, however, this is not the case. We begin by providing a simple example.

Example 3.11 *The curve $C(\cdot, v)$ is translating as in figure 2, with unit speed:*

$$C(\theta, v) \doteq c_0(\theta) + ve_1$$

Figure 2: Stretching the parameterization

After the reparameterization 3.10, the points in the tracts BC and AD are motionless, whereas the parameterization in AB is stretching to produce new points for the curve, and in CD it is absorbing points: then, $\partial_\theta\widetilde{C} \to \infty$ if the curvature in AB goes to infinity.

We now consider the math in more detail.

Proposition 3.12 *Suppose that $n = 2$ for simplicity: define the curvature κ as per definition 3.5 on page 19.*

Suppose in particular that $|\dot{C}| = 1$ at all points: then

$$\partial_v|\dot{C}|^2 = 0 = 2\langle\partial_v\partial_\theta C, T\rangle$$

By deriving (3.8.$\star$) we obtain

$$\partial_\theta\widetilde{C}(\theta, v) = \partial_\theta C(\tau, v)\psi(\theta, v)$$

where $\psi = \partial_\theta\varphi$ solves (3.10.$\star\star$), that we can rewrite (using $|\dot{C}| = 1$) as

$$\partial_v\psi = -\psi\langle\partial_v C, N\rangle\kappa$$

(where C, κ, N, T are evaluated at $(\varphi(\theta, v), v)$).

Note that $|\langle \partial_v C, N \rangle| = |\pi_N \partial_v C|$. This implies that the parameterization of $\widetilde{C}$ will be highly affected at points where both κ and $\pi_N \partial_v C$ are big.

So the above teaches us that there are two different approches to the reparameterization: we may either use it to control $\dot{C}$ or $\pi_T \partial_v C$, but not both.

3.4 Homotopy classes

3.4.1 Class $\mathbb{C}$

Given an energy and two continuous curves c_0 and c_1, we will try to find a homotopy that minimizes this energy, searching in the class $\mathbb{C}$ so defined

Definition 3.13 (class $\mathbb{C}$) *Let $\mathbb{C}$ be the class of all homotopies $C : I \to \mathbb{R}^n$, continuous on I and locally Lipschitz in $S^1 \times (0, 1)$, such that $c_0 = C(\cdot, 0)$ and $c_1 = C(\cdot, 1)$.*

Such minimum will be a *minimal geodesic* connecting c_0 and c_1 in the space of curves. In this class $\mathbb{C}$ we can state

Proposition 3.14 (Poincarè inequality) *there are two constants $a', a'' > 0$ such that $\forall C \in \mathbb{C} \cap H^{1,0}$,*

$$\|C\|^2_{H^{1,0}(I)} \leq a' + a'' \int_I |\partial_v C|^2 \qquad (3.14.\star)$$

$a', a'' > 0$ depend on c_0 and c_1.

3.4.2 Class $\mathbb{F}$ of prescribed-parameter curves

In some of the following sections we will change our point of view with respect to the above assumption 3.13 by fixing a measurable positive function $l : [0, 1] \to \mathbb{R}^+$ and restricting our attention to a family of homotopies such that $|\partial_\theta C(\theta, v)| = l(v)$.

Definition 3.15 (class $\mathbb{F}$) *The class $\mathbb{F}$ contains all of $C \in \mathbb{C}$ such that*

$$|\partial_\theta C(\theta, v)| = l(v)$$

for all θ, v.

In particular, $\mathrm{len}(C) = 2\pi l(v)$.

Note that, by 4.3, if l is not Hölder continuous then the class $\mathbb{F}$ will be too poor to be useful (for example, it will not contain smooth functions). Unfortunately the class $\mathbb{F}$ is not closed with respect to the weak convergence in $W^{1,p}$

Proposition 3.16 *Suppose that l is bounded and $|\partial_\theta C_h(\theta, v)| = l(v)$ for all h, θ, v; assume that $1 \leq p < \infty$ and $\partial_\theta C_h \rightharpoonup V$ weakly in $L^p(I \to \mathbb{R}^n)$, or $p = \infty$ and $\partial_\theta C_n \rightharpoonup^* V$ weakly-* in $L^\infty(I \to \mathbb{R}^n)$.*

Then $|V(\theta, v)| \leq l(v)$.

Proof. Since $|\partial_\theta C_h| \leq \sup l$, we may always assume that $\partial_\theta C_h \rightharpoonup^* V$ weakly-* in $L^\infty(I \to \mathbb{R}^n)$; the result follows immediatly from theorem 1.1 in [Dac82] □

We conclude that

Theorem 3.17 *Let $1 \le p \le \infty$. The closure of $\mathbb{F} \cap W^{1,p}$ with respect to weak convergence $W^{1,p}$ is contained in the class $\overline{\mathbb{F}}$ of all $C \in W^{1,p}$ such that $C(\cdot, 0) = c_0$ and $C(\cdot, 1) = c_1$ are given and $|\partial_\theta C| \le l(v)$.*

Proof. Suppose $(C_h) \subset \mathbb{F}$ and $C_h \rightharpoonup C$: we prove that $C \in \overline{\mathbb{F}}$. (C_h) is bounded in $W^{1,p}$: by Rellich-Kondrachov theorem (see thm. IV.16 [Bre86]) (C_h) is precompact in L^p: up to a subsequence, we may assume that $C_h \to C$ in L^p and that $C_h \to C$ on almost all points: then $C(\cdot, 0) = c_0$ and $C(\cdot, 1) = c_1$. □

Remark 3.18 *We could just as well have defined a class of* homotopies with prescribed lengths*, where we fix a function $\hat{l}$ and include in the class all C such that* $\mathrm{len}(C) = \hat{l}(v)$ *for all v. However, if the energy E is geometric, then using the reparameterization 3.9, we can always replace any such C with a $\widetilde{C} \in \mathbb{F}$, and $E(C) = E(\widetilde{C})$.*

3.4.3 Factoring out reparameterizations

If we only consider curves c such that $|\dot{c}| \equiv 1$, then (as pointed in [MM]) the group of reparameterizations is $S^1 \ltimes \mathbb{Z}_2$, where

- the group S^1 is associated to the change of the initial point $c(0)$ for the parameterization, that is, the reparameterization $\tau \in S^1$ changes the curve c to $c(\theta + \tau)$,
- and $\mathbb{Z}_2$ means the operation of changing direction, that is, $c(\theta)$ becomes $c(-\theta)$

We extend the above to homotopies satisfying $|\partial_\theta C(\theta, v)| = l(v)$; the second reparameterization is not significant; the first one does not affect the normal velocity $\pi_N \partial_v C$; so the reparameterization in the class $\mathbb{F}$ does not affect energies that depend only on $\pi_N \partial_v C$ (as per eq. (1.16)).

4 Analysis of $E^N(C)$

In this section we will focus our attention on the energy

$$E^N(C) \doteq \int_I |\pi_N \partial_v C|^2 |\dot{C}| \, d\theta \, dv$$

which is associated with the geometric version of the H^0 metric. We will derive a result of existence of minimima of E^N in a class of homotopies such that the curvature is bounded.

Remark 4.1 (winding) *Unfortunately, any energy that is independent of parameterization cannot be coercive. It is then difficult to prove existence of geodesics by means of the standard procedure in the Calculus of Variations 3.1.*

Indeed, suppose that C is a homotopy, and define

$$C_k(\theta, v) = C(\theta + k2\pi v, v)$$

$\forall k \in \mathbb{Z}$; then

$$E^N(C_k) = E^N(C)$$

(As a special case, consider two curves $c_0 = c_1$, and $C_k(\theta, v) = c_0(\theta + k2\pi v)$; each C_k is a minimal geodesic connecting c_0, c_1, since $E^N(C_k) = 0$).

This proves that E^N is not coercive in $H^1(I)$, since $\int |\partial_v C_k|^2 \to \infty$ when $k \to \infty$.

4.1 Knowledge base

Some of the follwing results apply in the class $\mathbb{F}$ of homotopies with prescribed arc parameter $|\partial_\theta C(\theta, v)| = l(v)$ while others apply in the more general class $\mathbb{C}$.

Consider the integral

$$J(C) \doteq \int_I |H|^2 |\pi_N \partial_v C|^2 |\dot{C}| \, d\theta \, dv \tag{4.2}$$

where H is the curvature of C along θ. We now deduce the following result from [MM]

Proposition 4.3 *Suppose that the homotopy C is smooth and immersed: then by extending to $\mathbb{R}^n$ the computation in sec. 3.3 of [MM] we get*

$$\frac{d}{dv} \sqrt{\mathrm{len}(C)(v)} \leq \frac{1}{2} \sqrt{\int_{S^1} \langle H, \partial_v C \rangle^2 |\dot{C}| d\theta}$$

and then, for any $0 \leq v' < v'' \leq 1$,

$$\begin{aligned} \sqrt{\mathrm{len}(C)(v'')} - \sqrt{\mathrm{len}(C)(v')} &\leq \frac{1}{2} \int_{v'}^{v''} \left(\int_{S^1} \langle H, \partial_v C \rangle^2 |\dot{C}| d\theta \right)^{1/2} dv \leq \\ &\leq \frac{1}{2} \sqrt{v'' - v'} \left(\int_{v'}^{v''} \int_{S^1} \langle H, \partial_v C \rangle^2 |\dot{C}| d\theta \, dv \right)^{1/2} \leq \\ &\leq \frac{\sqrt{J(C)}}{2} \sqrt{v'' - v'} \end{aligned}$$

(by Cauchy-Schwarz and (3.6.$\star$)) and this implies that $\sqrt{\mathrm{len}(C)}$ is Hölder continuous when $J(C)$ is finite.

Remark 4.4 *Let $d(c_1, c_2)$ be the geodesic distance induce by the metric (2.9) defined in [MM]; the proposition above proves that, if $\mathrm{len}(c_1) \neq \mathrm{len}(c_2)$ then $d(c_1, c_2) > 0$.*

Proposition 4.5 (l.s.c. and polyconvex function) *Let $W, V \in \mathbb{R}^n$. Let $W \times V$ be the vector of all $n(n-1)/2$ determinants of all 2 by 2 minors of the matrix having W, V as columns.*

Consider a continuous function $f : \mathbb{R}^{n \times 2} \to \mathbb{R}$ such that

$$f(W, V) = g(W, V, W \times V)$$

where $g : \mathbb{R}^{n(n+3)/2} \to \mathbb{R}$ *is convex: then* f *is* polyconvex [33].

Let $p \geq 2$*, suppose* $f \geq 0$*: by theorem 4.1.5 and remark 4.1.6 in [But89] then*

$$\int_I f(\partial_\theta C, \partial_v C) d\theta dv$$

is $W^{1,p}$*-weakly-lower-semi-continuous.*[34] *This means that, if* $C_h \rightharpoonup C$ *weakly in* $W^{1,p}$*, that is*

$$C_h \to C, \quad \partial_v C_h \rightharpoonup \partial_v C \quad \partial_\theta C_h \rightharpoonup \partial_\theta C \tag{4.5.$\star$}$$

in L^p*,* [35] *then*

$$\liminf_h \int_I f(\partial_\theta C_h, \partial_v C_h) \geq \int_I f(\partial_\theta C, \partial_v C)$$

4.2 Compactness

We now list some simple lemmas.

Lemma 4.6 *let* $\widetilde{C}(\theta, v) = C(\theta + \varphi(v), v)$ *be a reparameterization; then*

$$\inf_\varphi \int |\pi_T \partial_v \widetilde{C}|^2 = \int_0^1 \int_{S^1} \left(\pi_T \partial_v C(\theta, v) - \fint_{S^1} \pi_T \partial_v C(s, v)\, ds \right)^2 d\theta\, dv \tag{4.6.$\star$}$$

Lemma 4.7 (Poincarè inequality in S^1) *If* $f : S^1 \to \mathbb{R}^n$ *is* C^1*, then*

$$\max |f| - \min |f| \leq \frac{1}{\pi} \int_{S^1} |df| \tag{4.7.$\star$}$$

so that

$$\sqrt{\int \left| f - \fint f \right|^2} \leq \frac{1}{2\pi} \int_{S^1} |df| \tag{4.7.$\star\star$}$$

For any $a \in \mathbb{R}, a \neq 0$*,*

$$\max |f| - \min |f| \leq \frac{2}{\pi} \int_{S^1} |df + a| \tag{4.7.$\diamond$}$$

so that

$$\sqrt{\int \left| f - \fint f \right|^2} \leq \frac{1}{\pi} \int_{S^1} |df + a| \tag{4.7.$\diamond\diamond$}$$

Lemma 4.8 *Suppose* $C \in C^2$*. If we derive* $\partial_v(|\dot{C}|^2)$ *we get*

$$\partial_v(|\dot{C}|^2) = 2\langle T, \partial_v \partial_\theta C \rangle |\dot{C}|$$

so,[36] *when* $|\dot{C}| \neq 0$*,*

[33] The more general definition and the properties may be found sec. 4.1 in [But89], or in 2.5 and 5.4 in [Dac82]

[34] Sequentially weakly-* if $p = \infty$

[35] We can write equivalently $C_h \to C$ or $C_h \rightharpoonup C$ in (4.5.$\star$), thanks to Rellich-Kondrachov theorem (see thm. IV.16 [Bre86])

[36] When $|\dot{C}| = 0$, then $T = 0$, by our definition, so the equation (4.8.$\star$) holds as well in the distributional sense

$$\partial_v(|\dot{C}|) = \langle T, \partial_v \partial_\theta C \rangle \tag{4.8.$\star$}$$

If the curves are in C^2 *and we derive* $\partial_\theta(\pi_T \partial_v C)$*, we get*

$$\partial_\theta(\pi_T \partial_v C) = \partial_\theta \langle T, \partial_v C \rangle = \langle H, \partial_v C \rangle |\dot{C}| + \langle T, \partial_\theta \partial_v C \rangle = \langle H, \partial_v C \rangle |\dot{C}| + \partial_v(|\dot{C}|) \tag{4.8.$\star\star$}$$

(where $\langle H, \partial_v C \rangle = \kappa \pi_N \partial_v C$ *for curves in the plane).*

Proposition 4.9 *Let* $M > 0$ *be a constant. Suppose that a* C^2 *homotopy* $C : I \to \mathbb{R}^n$ *satisfies* $|\dot{C}(\theta, v)| = l(v)$ *and*

$$\int_0^1 \left(\int_{S^1} \langle H, \partial_v C \rangle |\dot{C}| \, d\theta \right)^2 dv \le M \tag{4.9.$\star$}$$

where H *is the curvature of* C *along* θ*. Then there exists a suitable reparameterizations*

$$\widetilde{C}(\theta, v) = C(\theta + \varphi(v), v) \tag{4.9.$\star\star$}$$

such that

$$\int |\pi_T \partial_v \widetilde{C}|^2 \le 2M$$

Proof. Suppose that $l \in C^1$ (the general proof being obtained by an approximation argument). We summarize derivations in (4.8),

$$\partial_\theta(\pi_T \partial_v C) = \langle H, \partial_v C \rangle |\dot{C}| + \partial_v l$$

Applying eqns (4.6.$\star$) and (4.7.$\diamond\diamond$) (with $a = -\partial_v l$) we obtain

$$\inf_\varphi \int |\pi_T \partial_v \widetilde{C}|^2 = \int_0^1 \int_{S^1} \left| \pi_T \partial_v C(\theta, v) - \fint_{S^1} \pi_T \partial_v C(\tilde{\theta}, v) \, d\tilde{\theta} \right|^2 d\theta \, dv \le$$

$$\le \int_0^1 \left(\int_{S^1} |\partial_\theta(\pi_T \partial_v C) + a| \, d\theta \right)^2 \le \int_0^1 \left(\int_{S^1} |\langle H, \partial_v C \rangle |\dot{C}| + \partial_v l - \partial_v l| \, d\theta \right)^2 dv$$

□

Note that the reparametrization (4.9.$\star\star$) may be viewed as an *unwinding* when confronted with 4.1.

Remark 4.10 *1 Note that, if* $|\dot{C}(\theta, v)| = l(v)$ *then*

$$\int_0^1 \left(\int_{S^1} \langle H, \partial_v C \rangle |\dot{C}| \, d\theta \right)^2 dv \le \int_I \langle H, \partial_v C \rangle^2 l(v)^2 \le (\sup l) J(C)$$

So a bound on $J(C)$ *should provide compactness.*

4.3 Lower semicontinuity

Let $\alpha > 0, \beta > 0$, $V, W \in \mathbb{R}^n$, define

$$e(W,V) = |\pi_{W^\perp} V|^\alpha |W|^\beta$$

and define

$$E_{\alpha,\beta}(C) \doteq \int_I e(\partial_\theta C, \partial_v C)$$

Note that $E^N(C)$ is obtained by choosing $\alpha = 2, \beta = 1$. In general if $\beta = 1$ then $E_{\alpha,\beta}(C)$ is a geometric energy (see 1.17).

Let $W \times V$ be the vector of all $n(n-1)/2$ determinants of all 2 by 2 minors of the matrix having W, V as columns. The identity

$$|\pi_{W^\perp} V|^2 |W|^2 = |V|^2|W|^2 - \langle V, W\rangle^2 = |V \times W|^2$$

is easily checked [37]. Let

$$f(W,V) = |V \times W| \quad .$$

Note that f is a polyconvex function (see 4.5) and that

$$e(W,V) = |\pi_{W^\perp} V|^\alpha |W|^\beta = |W \times V|^\alpha |W|^{\beta-\alpha} \tag{4.11}$$

and

$$E_{\alpha,\beta}(C) = \int_I f(\partial_\theta C, \partial_v C)^\alpha |\partial_\theta C|^{\beta-\alpha}$$

We can provide this lower semicontinuity result in the class $\mathbb{F}$.

Proposition 4.12 *Let $p \geq 2$. Suppose $\alpha > \beta > 0$. Fix a continous function $l : [0,1] \to \mathbb{R}^+$, $l \geq 0$, and use it to build the class $\mathbb{F}$. Then $E_{\alpha,\beta}$ is $W^{1,p}$-weakly-lower-semi-continuous in the class $\mathbb{F}$.*

More precisely: let

$$C \in \mathbb{F}, \qquad (C_h)_h \subset \mathbb{F}$$

if $C_h \rightharpoonup C$ weakly in $W^{1,p}$, that is,

$$C_h \to C, \quad \partial_v C_h \rightharpoonup \partial_v C \quad \partial_\theta C_h \rightharpoonup \partial_\theta C$$

in L^p, and

$$l(v) = |\dot C| = |\dot C_h|$$

then

$$\liminf_h E_{\alpha,\beta}(C_h) \geq E_{\alpha,\beta}(C)$$

Proof. We prove the theorem in steps:

- Let $\lambda > 0$. Suppose $|\dot C| \equiv |\dot C_h| \equiv \lambda$; then

$$E_{\alpha,\beta}(C) = \lambda^{\beta-\alpha} \int f(\partial_\theta C, \partial_v C)^\alpha$$

  is l.s.c. (by 4.5).

[37] In $\mathbb{R}^3$ we have $\langle V, W\rangle = |V||W|\cos\alpha$ and $|V \times W| = |V||W|\sin\alpha$, where α is the angle between the two vectors

- for any homotopy C and any continuous $g : [0,1] \to \mathbb{R}^+$, define $e_g(C)(v) : [0,1] \to \mathbb{R}^+$,
$$e_g(C) \doteq \begin{cases} g(v)^{\beta-\alpha} \int_{S^1} f(\partial_\theta C, \partial_v C)^\alpha d\theta & \text{if } g(v) > 0 \\ 0 & \text{if } g(v) = 0 \end{cases}$$
If $C \in \mathbb{F}$ (that is $l(v) = |\dot{C}(\theta, v)|$) then
$$E_{\alpha,\beta}(C) = \int_0^1 e_l(C) dv$$
- Consider a piecewise function $g \geq 0$ defined
$$g = \sum_{i=1}^m g_i \chi_{[a_i, a_{i+1})} \tag{4.13}$$
and let
$$\hat{E}(C) \doteq \int_0^1 e_g(C) dv = \sum_{i=1}^m \int_{a_i}^{a_{i+1}} e_{g_i}(C) dv$$
then we apply the previous reasoning to all addends and conclude that
$$\liminf_h \hat{E}(C_h) \geq \hat{E}(C)$$
- Suppose l is continuous and $l \geq 0$. Let τ be the class of piecewise functions $g \geq 0$ defined as in (4.13), such that on any interval $[a_i, a_{i+1})$, either $g_i = 0$ or
$$g_i \geq \sup_{[a_i, a_{i+1})} l \quad .$$
Then for any such g and $C \in \mathbb{F}$,
$$\int_0^1 e_g(C) dv \leq E_{\alpha,\beta}(C) \quad .$$
- choose g in the class τ; then
$$\liminf_h E_{\alpha,\beta}(C_h) \geq \liminf_h \int_0^1 e_g(C_h) dv \geq \int_0^1 e_g(C) dv$$
- Fix C: it is possible to find a sequence $(g_j) \subset \tau$ such that
$$e_{g_j}(C)(v) \to_j e_l(C)(v)$$
for almost all points v, monotonically increasing: indeed, let
$$A_{j,i} = [i2^{-j}, (i+1)2^{-j}), \quad I_{j,i} = \inf_{v \in A_{j,i}} l(v), \quad S_{j,i} = \sup_{v \in A_{j,i}} l(v)$$
and
$$g_j(v) = \begin{cases} 0 & \text{if } v \in A_{j,i} \text{ and } I_{j,i} = 0 \\ S_{j,i} & \text{if } v \in A_{j,i} \text{ and } I_{j,i} > 0 \end{cases}$$
Then
$$E_{\alpha,\beta}(C) = \sup_j \int_0^1 e_{g_j}(C) dv$$

□

We would like to prove this more general statement

Conjecture 4.14 *choose Lipschitz homotopies*

$$C : I \to \mathbb{R}^n, \quad C_h : I \to \mathbb{R}^n$$

define

$$l(v) \doteq \operatorname{len} C, \quad l_h(v) \doteq \operatorname{len} C_h$$

If $l_h \to l$ uniformly and $C_h \rightharpoonup C$ weakly in $W^{1,p}$, then

$$\liminf_h E^N(C_h) \geq E^N(C)$$

Whereas we cannot generalize the theorem further: this is due to example 4.3.1.

4.3.1 Example

We want to show that E^N is not l.s.c. if we do not control the length.

Let $\alpha > \beta > 0$, define

$$e(W, V) = |\pi_{W^\perp} V|^\alpha |W|^\beta = |W \times V|^\alpha |W|^{\beta - \alpha}$$

and define

$$E_{\alpha,\beta}(C) \doteq \int_I e(\partial_\theta C, \partial_v C)$$

We will actually show that $E_{\alpha,\beta}$ is not l.s.c., and that

Proposition 4.15

$$\Gamma E_{\alpha,\beta}(C) = 0$$

where the relaxation is computed with respect to weak- L^∞ convergence of the derivatives.*

1. For simplicity, we temporarily drop the requirement that the curves $C(\cdot, v)$ be closed. We then redefine $I = [0,1] \times [0,1]$.

2. Let $\widetilde{C} : I \to I$ be a Lipshitz map such that

$$\widetilde{C}(\theta, 0) = (\theta, 0), \quad \widetilde{C}(\theta, 1) = (\theta, 1), \quad \widetilde{C}(0, v) = (0, v), \quad \widetilde{C}(1, v) = (1, v) \quad . \tag{4.16}$$

3. Let $h \geq 1$ be integers.

   We rescale $\tilde{C}$ and glue many copies of it to build C_h, as follows

$$C_h(\theta, v) \doteq \frac{1}{h} \tilde{C}\Big((h\theta) \operatorname{mod}(1), (hv) \operatorname{mod}(1)\Big) + b_h(\theta, v)$$

   where $b_h : I \to \mathbb{R}^2$ is the piecewise continuous function

$$b_h(\theta, v) \doteq \left(\frac{1}{h} \lfloor h\theta \rfloor, \frac{1}{h} \lfloor hv \rfloor \right)$$

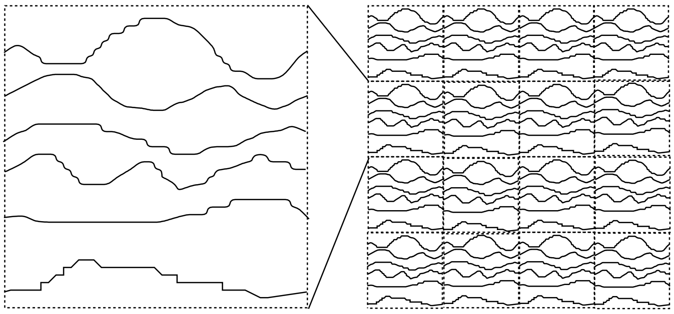


Figure 3: Tesselation of homotopy $\tilde{C}$ to form C_h

(in particular, $C_1 = \tilde{C}$). We represent this process in figure 3.
Then

$$\begin{aligned} \partial_\theta C_h(\theta, v) = &\quad \partial_\theta \tilde{C}\Big((h\theta) \bmod(1), (hv) \bmod(1)\Big) \\ \partial_v C_h(\theta, v) = &\quad \partial_v \tilde{C}\Big((h\theta) \bmod(1), (hv) \bmod(1)\Big) \end{aligned}$$

We may think of C_h as homotopies that connect the same two curves, namely,

$$C_h(\theta, 0) = (\theta, 0) = \tilde{C}(\theta, 0), \quad C_h(\theta, 1) = (\theta, 1) = \tilde{C}(\theta, 1) \quad ,$$

while extrema points move in a controlled way, namely

$$C_h(0, v) = (0, v) = \tilde{C}(0, v), \quad C_h(1, v) = (1, v) = \tilde{C}(1, v) \quad .$$

4. Let $C(\theta, v) \doteq (\theta, v)$ be the identity.
   The sequence C_h has the following properties
   (a) $C_h \to C$ in L^∞, and more precisely
   $$\sup_I |C_h - C| \leq \frac{2}{h}$$
   (b) $\partial_\theta C_h$ and $\partial_v C_h$ are bounded in L^∞, and more precisely,
   $$\sup_I |\partial_\theta C_h| \leq |\partial_\theta \tilde{C}|, \quad \sup_I |\partial_v C_h| \leq \sup_I |\partial_v \tilde{C}|$$
   and then all C_h are equi Lipschitz;
   (c)
   $$\partial_\theta C_h \rightharpoonup e_1 = (1, 0) = \partial_\theta C$$
   and
   $$\partial_v C_h \rightharpoonup e_2 = (0, 1) = \partial_v C$$
   weakly* in $L^\infty(I)$. (38)

(38) Proof by lemma 1.2 in [Dac82]

(d) let $\tilde{l}(v) \doteq \mathrm{len}(\tilde{C})$, $a \doteq \int \tilde{l}$ and

$$l_h(v) \doteq \mathrm{len}(C_h)(v) = \tilde{l}\big((hv) \bmod (1)\big)$$

then $l_h \rightharpoonup a$ weakly* in $L^\infty([0,1])$. [38]

(e) Suppose $\tilde{C}$ is piecewise smooth, so that the curvature H_h of C_h can be defined almost everywhere. By (4.16), $\tilde{l}(0) = \tilde{l}(1) = 1$. If $\tilde{l}(v) > 1$ at some points, then the sequence $l_h(v)$ is not equicontinuous: then, by proposition 4.3, the sequence of integrals

$$\int_0^1 \int_{S^1} \langle H_h, \partial_v C_h \rangle^2 |\dot{C}_h| d\theta \, dv$$

is unbounded in h. [39]

(f) $E_{\alpha,\beta}(C_h)$ is constant in h: indeed

$$\begin{aligned} \int_I e(\partial_\theta C_h, \partial_v C_h) &= h^2 \int_0^{1/h} \int_0^{1/h} e\big(\partial_\theta C_h, \partial_v C_h\big) d\theta \, dv = \\ &= h^2 \int_0^{1/h} \int_0^{1/h} e\big(\partial_\theta \tilde{C}(h\theta, hv), \partial_v \tilde{C}(h\theta, hv)\big) d\theta \, dv = \\ &= \int_I e\big(\partial_\theta \tilde{C}(\theta, v), \partial_v \tilde{C}(\theta, v)\big) d\theta \, dv \end{aligned}$$

that is

$$E_{\alpha,\beta}(C_h) = E_{\alpha,\beta}(\tilde{C}) \tag{4.17}$$

5. Let $j \geq 1$ be a fixed integer. Let $\tilde{C} : I \to \mathbb{R}^2$ be defined by

$$\gamma(v) = \begin{cases} v & \text{if } v \in [0, 1/2] \\ (1-v) & \text{if } v \in [1/2, 1] \end{cases} \tag{4.18}$$

and

$$\tilde{C}(\theta, v) = \big(\theta, v + \gamma(v) \sin(2\pi j \theta)\big) \tag{4.19}$$

(note that $\tilde{C}$ is a graph).

We have this further properties

- We know that $l_h \rightharpoonup a$ with $a = \int \mathrm{len}\,\tilde{C}$; but this limit a is strictly bigger than 1, whereas $\mathrm{len}\, C \equiv 1$
- For $\theta \in [0, 1/h]$ and $v \in [0, 1/h]$

$$\begin{aligned} \partial_v C_h(\theta, v) = \; & \partial_v \tilde{C}(h\theta, hv) &= (0,\ 1 + \gamma'(hv) \sin(2\pi j h\theta)) \\ \partial_\theta C_h(\theta, v) = \; & \partial_\theta \tilde{C}(h\theta, hv) &= (1,\ 2\pi j \gamma(hv) \cos(2\pi j h\theta)) \end{aligned}$$

Then

$$\sup_I |\partial_\theta C_h| \leq 1 + 2\pi j, \quad \sup_I |\partial_v C_h| \leq 2$$

[39] a fortiori, $J(C_h)$ is unbounded — $J(C)$ was defined in (4.2)

- We compute

$$E_{\alpha,\beta}(\tilde{C}) = \int_I e\big(\partial_\theta \tilde{C}(\theta,v), \partial_v \tilde{C}(\theta,v)\big) d\theta\, dv =$$

$$= \int_0^1 \int_0^1 |1+\gamma'(v)\sin(j\theta)|^\alpha |1+\gamma(v)^2 j^2 \cos(j\theta)^2|^{(\beta-\alpha)/2}\, d\theta\, dv$$

$$= 2\int_0^{1/2} \int_0^1 |1+\sin(j\theta)|^\alpha |1+v^2 j^2 \cos(j\theta)^2|^{(\beta-\alpha)/2}\, d\theta\, dv$$

then

$$\lim_{j\to\infty} E_{\alpha,\beta}(\tilde{C}) = 0 \tag{4.20}$$

6. Combining (4.17) and (4.20) we prove that $E_{\alpha,\beta}$ is not l.s.c. Indeed for j large
$$\lim_h E_{\alpha,\beta}(C_h) = E_{\alpha,\beta}(\tilde{C}) < E_{\alpha,\beta}(C) = 1$$
whereas $C_h \rightharpoonup C$.

7. to prove 4.15, consider any homotopy C; this may be approximated by a piecewise linear homotopy, which in turn may be approximated by many replicas of the above construction.

4.4 Existence of minimal geodesics

Theorem 4.21 *Let $M > 0$.*
Let $\mathcal{A}$ be the set of admissible curves $c : S^1 \to \mathbb{R}^n$, such that

- *$c : S^1 \to \mathbb{R}^n$ is Lipschitz, and c admits curvature H in the measure sense (see in 3.3), and moreover*
- *the total mass $|H|(S^1)$ of the curvature H of c is bounded uniformly*

$$|H|(S^1) \le M \quad . \tag{4.21.$\star$}$$

Let c_0, c_1 be curves in $\mathcal{A}$.
Fix a bounded continuous function $l : [0,1] \to \mathbb{R}^+$, with $\inf l > 0$.
Let $\mathcal{B}$ be the class of homotopies $C : I \to \mathbb{R}^n$ such that

- *$C \in H^1(I \to \mathbb{R}^n)$*
- *any given curve $\theta \mapsto C(\theta, v)$ is in $\mathcal{A}$*
- *the curvature can be extended to the homothopy (see 3.7), $\partial_v C$ is continuous, and*

$$\int_0^1 \left(\int_{S^1} \langle H, \partial_v C\rangle |\dot{C}|\, d\theta \right)^2 dv \le M \tag{(4.9.$\star$)}$$

- *$l(v) = \operatorname{len} C(v) = \int_{S^1} |\dot{C}|\, d\theta$*
- *$C(\theta, 0) = c_0(\theta)$ and $C(\theta, 1) = c_1$*

If $\mathcal{B}$ is non empty, then the functional E^N admits a minimizing homotopy C^*; this minimum C^* satisfies all the above requirements, but possibly for condition (4.9.$\star$).

Proof. The proof is divided in two important (and independent) steps

- Let C_h be a sequence such that

$$\lim E^N(C_h) = \inf_{C\in\mathcal{B}} E^N(C)$$

Up to reparameterization, assume

$$|\dot C_h(\theta, v)| = l(v) \quad .$$

By this bound, and the compactness result 4.9, we reparameterize any term C_h to $\widetilde C_h$ by

$$\widetilde C_h(\theta, v) = C_h(\theta + \varphi_h(v), v)$$

so that

$$\int |\pi_T \partial_v \widetilde C_h|^2 \le 2M \quad ;$$

moreover

$$\int |\pi_N \partial_v \widetilde C_h|^2 \le (\max \frac{1}{l}) \int |\pi_N \partial_v \widetilde C_h|^2 |\dot C| \le (\max \frac{1}{l})(1 + \inf E^N) \quad (4.22)$$

(definitively in h) and then

$$\int |\partial_v \widetilde C_h|^2 \le 2M + (\max \frac{1}{l})(1 + \inf E^N)$$

whereas

$$\int_I |\dot C_h|^2 = \int_0^1 l(v)^2 dv \quad :$$

then (by the Banach-Alaoglu-Bourbaki theorem [40]) up to a subsequence, $\widetilde C_h$ converges weakly in H^1 to a homotopy C^*.

- We want to prove that $|\dot C^*(\theta, v)| = l(v)$ for almost all θ, v. We know that $|\dot C^*(\theta, v)| \le l(v)$, by 3.16. Suppose on the opposite that $|\dot C^*(\theta, v)| < l(v)$: then there exists $\varepsilon > 0$ and an measurable subset $A \subset I$ with positive measure, such that $|\dot C^*(\theta, v)| < l(v) - \varepsilon$ for $(\theta, v) \in A$. Let $A_v = \{\theta : (\theta, v) \in A\}$ be the slice of A: then, by Fubini-Tonelli, there is a v such that the measure of A_v is positive. We fix that v. Suppose that H_h is the curvature of $C_h(\cdot, v)$: then

$$H_h = l(v) \partial_{\theta\theta} C_h$$

in the sense of measures. We know that H has bounded mass: so $\partial_{\theta\theta} C_h$ has: by Theorem 3.23 in [AFP00], $\partial_\theta C_h(\cdot, v)$ is compact in $L^1(S^1)$, so, (up to a subsequence), we would have that $\dot C_h(\cdot, v) \to \dot C^*(\cdot)$ *strongly* in $L^1(S^1)$: then $|\dot C^*(\theta, v)| = l(v)$ for that particular choice of v, achieving contradiction.

- By 4.12, $\liminf_h E^N(C_{k_h}) \ge E^N(C^*)$: then C^* is the minimum.

□

(40) See thm III.15 III.25 and cor III.26 in [Bre86]

Remark 4.23 *If we wish to extend the above theorem, we face some obstacles.*

- *If we do not enforce some bounds on curvature (as (4.21.$\star$) and (4.9.$\star$)), then the example in §4.6 shows that we cannot achieve compactness of a minimizing subsequence*
- *If we wish to remove the hypothesis "$\inf l > 0$",*[41] *we are faced with the following problem: if $l(v) = 0$, then the curve $C(\cdot, v)$ collapses to a point; consequently if $l(v) = 0$ on an interval $[a, b]$, then the homotopy collapses to path on that interval; moreover, the length and the energy of C restricted to $v \in [a, b]$ is necessarily 0, so that $E(C)$ provides no bound on the behaviour of C: we again lose compactness of a minimizing subsequence (indeed, the inequality (4.22) needs that $\inf_v \mathrm{len}(C)(v) > 0$, to be able to control $\int |\partial_v C|^2$).*

4.4.1 Michor-Mumford

Although we cannot currently prove a theorem of existence of minimal geodesics for the energy $E^A(C)$ (defined in eq.(2.9)), in this section we have though derived some insight; so we discuss the conjecture that is proposed in [MM]:

Conjecture 4.24 ([MM]) *Fix two curves c_0 and c_1. The energy $E^A(C)$ admits a minimum in the class $\mathbb{C}$ of homotopies connecting c_0 and c_1.*

May we improve the proof of theorem 4.21 to prove this conjecture? We discuss what is ok and what is wrong.

- We may want to use $J(C)$ to drop the requirement (4.21.$\star$) that is, in turn, used to have l.s.c. of the functional E^N; so we may think of proving this lemma

  > *"Suppose that we are given a sequence of smooth homotopies C_h, and $J(C_h) \le M$, and $C_h \rightharpoonup C$ weakly in $W^{1,p}$: then $\mathrm{len}(C_h) \to \mathrm{len}(C)$"*

  but this is wrong, as seen by this example

  Example 4.25 *Let $C_h : [0, 1] \times [0, 1] \to \mathbb{R}^2$ defined as*

  $$C_h(u, v) = \left(u,\ \frac{1}{h}\sin(2\pi h u)\right)$$

  and

  $$C(u, v) = (u, 0) \quad .$$

  These homotopies do not depend on v: then $J(C_h) = 0$. On the other hand, $C_h \rightharpoonup C$ but $\mathrm{len}(C_h)$ is constant and bigger than $1 = \mathrm{len}(C)$.

- We need way to be sure that curves in the homotopy do not collapse to points (as discussed in 4.23); so we may think of proving this lemma

  > *"Suppose that the homotopy C admits curvature, and $J(C) < \infty$: then $\inf_v \mathrm{len}(C)(v) > 0$"*

[41] note that the bound (4.21.$\star$) does not imply that $\inf l > 0$

but this is wrong, as seen by this example

Example 4.26 *Let*

$$c_1(\theta) = (\sin(\theta), \cos(\theta))$$

be the circle in $\mathbb{R}^2$*, and build the homotopy*

$$C(\theta, v) = v^4 c_1(\theta)$$

Then

$$J(C) = \int_0^1 \frac{1}{v^8}(4v^3)^2 2\pi v^4 dv = 32 \int_0^1 v^4 dv = 32\pi/5$$

- We need a semicontinuity result on $J(C)$. Indeed we cannot hope in any cancellation effect in the sum $E^N(C) + J(C)$ because the two energies scale in different ways: [42]

Remark 4.27 (rescaling) *Let* $\varepsilon > 0$ *and* $\tilde{C} = \varepsilon C$ *then* $E^N(\tilde{C}) = \varepsilon^3 E^N(C)$ *but* $J(\tilde{C}) = \varepsilon J(C)$

- On the bright side, we do not have a counterexample to show if $J(C)$ is not l.s.c.; actually, remark (4e) (on page 32) suggests that if $C_h \to C$ and the homotopies have a common border condition (such as (4.16)), and $J(C_h)$ is bounded, then $\liminf E^N(C_h) \geq E^N(C)$. Moreover, remark 4.4 shows that the energy cannot go to zero if the end curves have different length.
- As pointed in 4.10, the term $J(C)$ in the metric provides compactness in $H^1(I)$
- Moreover the example 4.6 shows that we do need to control the curvature of curves to be able to prove existence of minimal geodesics: this justifies the term $J(C)$ in the Michor-Mumford energy E^A (as well as the bound (4.9.$\star$) in theorem 4.21).

4.5 Space of Curves

By using the previous theorem 4.21, we immediatly obtain a metric on a Space of Shapes.

Fix $M > 0$. Let $\mathcal{S}$ be the space of closed immersed C^2 curves such that for any $c \in \mathcal{S}$, the curvature κ of c is bounded by M, as

$$|\kappa| \leq M \quad .$$

We may think of $\mathcal{S}$ as a "submanifold with border" in the manifold M of all closed immersed C^2 curves.

Then we can use the Riemannian metric

$$\langle h, k \rangle = \int_{S_1} \langle \pi_N h(\theta), \pi_N k(\theta) \rangle \; d\theta$$

to define a positive geodesic distance in $\mathcal{S}$: by the theorem in the previous section, this distance admits minimal geodesics:

[42] and this suggests that the energy $E^A(C)$ should not satisfy the rescaling property 1 on page 8

Proof. indeed, we may write

$$J(C) \leq K^2 E^N(C)$$

and then apply 4.3 to conclude that len(C) is always continuous; then we may use the remark 4.10. □

Unfortunately, since $\mathcal{S}$ has a border (given by the constraint $|\kappa| \leq M$) then the minimal geodesic will not, in general, satisfy the Euler-Lagrange ODE defined by E^N; moreover the formula of the metric is "geometric" (as defined in 1.17); but the minimal geodesics and the distance are not invariant with respect to rescaling, due to the bound $|\kappa| \leq M$.

4.6 The pulley

We show that there exists a sequence of Lipschitz functions $C_h : I \to \mathbb{R}^2$ such that $|\pi_N \partial_v C_h| \leq 1$, $|\partial_\theta C_h| = 1$, but $\int_I |\partial_v C_h| \to \infty$.

Consider the figure 4. The thick line $ABCDEF$ is the curve $\theta \mapsto C_5(\theta, 0)$. The thick arrows represent the normal part $\pi_N \partial_v C_5$ of the velocity, while the thin dashed arrows represent the tangent part $\pi_T \partial_v C_5$ of the velocity. The circles are just for fun, and represent the wheels of the pulley.

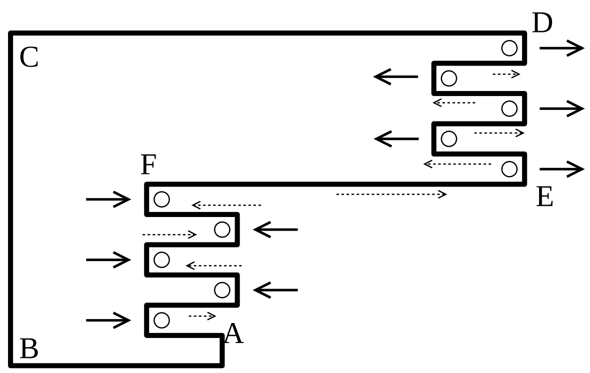


Figure 4: The pulley C_h, in case $h = 5$

The movement of $C_5(\theta, v)$, that is, its evolution in v, is so described: the part $ABCD$ is still, that is, it is constant in v; in the part DE (respectively FA) of the curve, vertical segments move apart (resp., together) as the thick arrows indicate, with horizontal velocity with norm $|\pi_N \partial_v C_5| = 1$; as the curve unravels, it is forced to move also parallel to itself.

The generic curve C_h has $2h$ wheels: h wheels in section DE, to pull apart, and h wheels in section AF, to pull together; the horizontal tracts in AF and DF are of lenght $\propto 1/h$, so the tract AF straightens up in a time $\Delta v \propto 1/h$: at that moment, the movement inverts: so while $v \in [0, 1]$, the cycle repeats for h times.

In this case, the tangent velocity in section DF is h times the normal velocity in sections AF and DE. Then, if we choose the normal velocity to have norm 1, the tangent velocity will explode when $h \to \infty$. This means that the family of homotopies C_h will not be compact in $H^1(I \to \mathbb{R}^1)$.

The first objection that comes to mind when reading the above is "*this example is not showing any problem with the curve itself, it is just giving problems with the parameterization of the curve*".

Indeed we may reparameterize the curves so that the tangent velocity will not explode when $h \to \infty$: by using 3.10, we obtain that $\pi_{\tilde{T}} \partial_v \tilde{C} = 0$.

Remembering remark 3.12, we understand that this is not going to help, though. So we point out this other problem.

Let $\lambda = \lambda(v, h)$ be the distance from feature point D to feature point E. Then $\partial_v \lambda \to \infty$ if $h \to \infty$.

5 Conformal metrics

We recall at this point that the well known *geometric heat flow* ($C_t = C_{ss}$)[(43)] is truly the *gradient descent* for the Euclidean arclength of a curve with respect to the H^0 metric. Unfortunately, given the pathologies encountered thus far with H^0, we see that this famous flow is not a gradient flow with respect to a well behaved Riemannian metric. If we propose a different metric, the new gradient descent flow for the Euclidean arclength of a curve will of course be entirely different. For example, the metric (2.9) proposed by [MM] yields the following gradient flow for arclength:

$$C_t = \frac{C_{ss}}{1 + A\, C_{ss} \cdot C_{ss}} = \frac{\kappa}{1 + A\kappa^2} N \tag{5.1}$$

Notice that the normal speed in (5.1) is *not monotic in the curvature*; and therefore the flow (5.1) will not share the nice properties of the *geometric heat flow* ($\partial_t C = \partial_{ss} C$). For example, embedded curves do not always remain embedded under this new flow ,as illustrated in Fig. 5.

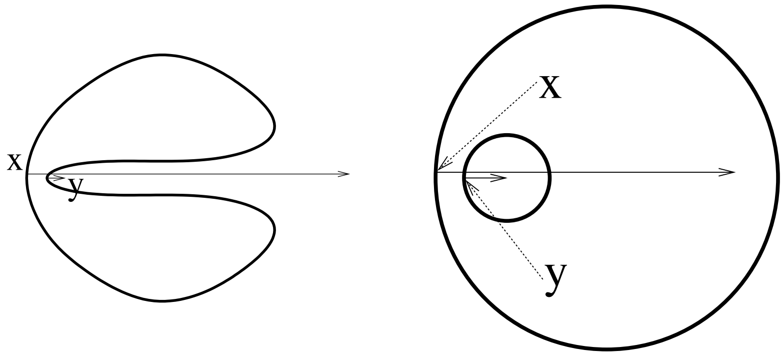


Figure 5: Intersections induced by flow (5.1): for big choices of A, the point x will travel faster than the point y (and in the same direction) and will eventually cross it.

Given the pathologies of H^0 we have no choice but to propose a new metric if we wish to construct a well behaved Riemannian geometry on the space of curves. However, we may seek a new metric whose gradient structure is as

(43) Recall that we write C_v for $\partial_v C$ (and so on), to simplify the derivations

similar as possible to that of the H^0 metric. In particular, for any functional $E : M \to \mathbb{R}$ we may ask that the gradient flow of E with respect to our new metric be related to that gradient flow of E with respect to H^0 by only a time reparameterization. In other words, if $C(t)$ represents a gradient flow trajectory according to H^0 and if $\hat{C}(t)$ represents the gradient flow trajectory according to our proposed new metric, then we wish that

$$\hat{C}(t) = C(f(t))$$

for some positive time reparameterization $f : \mathbb{R} \to \mathbb{R}$, $\dot{f} > 0$. The resulting gradient flows will then be related as follows.

$$\hat{C}_t = \dot{f}(t)\, C_t \tag{5.2}$$

The only class of new metrics that will satisfy (5.2) are *conformal* modifications of the original H^0 metric, which we will denote by H^0_ϕ. Such metrics are completely defined by combining the original H^0 metric with a positive *conformal factor* $\phi : M \to \mathbb{R}$ where $\phi(c) > 0$ may depend upon the curve c. The relationship between the inner products is given as follows.

$$\Big\langle h_1, h_2 \Big\rangle_{H^0_\phi} = \phi(c) \Big\langle h_1, h_2 \Big\rangle_{H^0} \tag{5.3}$$

Note that for any energy functional E of curves $C(t)$ we have the following equivalent expressions, where the first and last expressions are by definition of the gradient and the middle expression comes from the definition (5.3) of a conformal metric.

$$\frac{d}{dt}E(C(t)) = \left\langle \frac{\partial C}{\partial t}, \underbrace{\nabla^\phi E(C)}_{\substack{\text{Conformal}\\ \text{Gradient}}} \right\rangle_{H^0_\phi} = \phi \left\langle \frac{\partial C}{\partial t}, \underbrace{\nabla^\phi E(C)}_{\substack{\text{Conformal}\\ \text{Gradient}}} \right\rangle_{H^0} = \left\langle \frac{\partial C}{\partial t}, \underbrace{\nabla E(C)}_{\substack{\text{Original}\\ \text{Gradient}}} \right\rangle_{H^0} \tag{5.4}$$

We see from (5.4) that the *conformal gradient differs only in magnitude* from the original H^0 gradient

$$\nabla^\phi E = \frac{1}{\phi} \nabla E$$

and therefore the *conformal gradient flow differs only in speed* from the H^0 gradient flow.

$$\frac{\partial C}{\partial t} = -\nabla^\phi E(C) = -\frac{1}{\phi(C)} \nabla E(C)$$

As such and as we desired, the *solution differs only by a time reparameterization* f given by

$$\dot{f} = \frac{1}{\phi(C)}$$

The obvious question now is how to choose the conformal factor. Although we already know that the H^0 metric is not very useful, we may obtain a lot of insight into how to choose the conformal factor ϕ by observing the structure of the minimizing flow (which turns out to be unstable) for the H^0 energy in the space of homotopies. We will then try to choose the conformal factor in order to counteract the unstable elements of the H^0 flow.

5.1 The Unstable H^0 Flow

5.1.1 Geometric parameters s and v_*

We have denoted by $u \in [0,1]$ a parameter which traces out each curve in a parameterized homotopy $C(u,v)$ and we have denoted by $v \in [0,1]$ the parameter which moves us from curve to curve along the homotopy. Note that both of these parameters are arbitrary and not unique to the geometry of the curves comprising the homotopy. We now wish to construct more geometric parameters for the homotopy which will yield a more meaningful and intuitive expression for the minimizing flow we are about to derive. The most natural substitute for the curve parameter u is the arclength parmeter s. We must also address the parameter v, however. While v as a parameter ranging from 0 to 1 seems to have little to do with the arbitrary choice of the curve parameter u, the differential operator $\frac{\partial}{\partial v}$ depends heavily upon this prior choice. The desired effect of differentiating along the homotopy is mixed with the undesired effect of differentiating along the contour if flowing along corresponding values of u between curves in the homotopy requires some motion along the tangent direction. To see the dependence of $\frac{\partial}{\partial v}$ on u, note that $C(u,v)$ and $\hat{C}(u,v)$ where

$$\hat{C}(u,v) = C\left(u^{(1+v)}, v\right)$$

constitute the same homotopy geometrically, and yet $\frac{\partial C}{\partial v} \neq \frac{\partial \hat{C}}{\partial v}$.

We will therefore introduce the more geometric parameter v_* whose corresponding differential operator $\frac{\partial}{\partial v_*}$ yields the most efficient transport from one curve to another curve along the homotopy regardless of "correspondence" between values of the curve parameters. It is clear that such a transport must always move in the normal direction to the underlying curve since tangential motion along any curve does not contribute to movement along the homotopy. More preceisely, we define the parameteres s and v_* in terms of u and v as follows.

$$\frac{\partial}{\partial s} = \frac{1}{\|C_u\|}\frac{\partial}{\partial u} \qquad \text{and} \qquad \frac{\partial}{\partial v_*} = \frac{\partial}{\partial v} - \big(C_v \cdot C_s\big)\frac{\partial}{\partial s}$$

5.1.2 H^0 Minimizing Flow

Suppose we now consider a time varying family of homotopies $C(u,v,t) : [0,1] \times [0,1] \times (0,\infty) \to \mathbb{R}^n$ and compute the derivative of the H^0 energy along this family. Note that the H^0 energy, in terms of the new parameters s and v_* may be simply expressed as follows (since $\pi_N C_{v_*} = C_{v_*}$).

$$E(t) = \int_0^1 \int_0^L \left\|C_{v_*}\right\|^2 ds\, dv \tag{5.5}$$

In the appendix, we show that the derivative of E may be expressed as follows.

$$E'(t) = -2\int_0^1 \int_0^L C_t \cdot \Big(C_{v_* v_*} - (C_{v_* v_*} \cdot C_s)C_s - (C_{v_*} \cdot C_{ss})C_{v_*} + \frac{1}{2}\|C_{v_*}\|^2 C_{ss}\Big)\, ds\, dv \tag{5.6}$$

In the planar case, C_{v_*} and C_{ss} are linearly dependent (as both are orthogonal to C_s) which means that

$$(C_{v_*} \cdot C_{ss})C_{v_*} = (C_{v_*} \cdot C_{v_*})C_{ss} = \|C_{v_*}\|^2 C_{ss} \tag{5.7}$$

and therefore

$$E'(t) = -2\int_0^1 \int_0^L C_t \cdot \Big(\big(C_{v_* v_*} - (C_{v_* v_*} \cdot C_s)C_s\big) - \frac{1}{2}\|C_{v_*}\|^2 C_{ss} \Big)\, ds\, dv \quad (5.8)$$

by which we derive the minimization flow

$$C_t = C_{v_* v_*} - (C_{v_* v_*} \cdot C_s)C_s - \frac{1}{2}\|C_{v_*}\|^2 C_{ss}$$

which is geometrically equivalent to the following more simpler flow (by adding a tangential component):

$$C_t = C_{v_* v_*} - \frac{1}{2}\|C_{v_*}\|^2 C_{ss} \quad (5.9)$$

Note that the flow (5.9) consists of two orthogonal diffusion terms. The first term $C_{v_* v_*}$ is stable as it represents a forward diffusion along the homotopy, while the second term $-\|C_{v_*}\|^2 C_{ss}$ is an unstable backward diffusion term along each curve. Indeed, numerical experiments show a behaviour that parallels the phenomenon described in §4.3.1.

5.2 Conformal Versions of H^0

We now define the conformal H^0_ϕ energy (when the conformal factor ϕ is a function of the arclength L of each curve) as

$$E_\phi(t) = \int_0^1 \phi(L) \int_0^L \big\|C_{v_*}\big\|^2\, ds\, dv \quad (5.10)$$

Once again we compute (in the appendix) the derivative of this energy along a time varying family of homotopies $C(u, v, t)$.

$$E'_\phi(t) = -\int_0^1 \int_0^L C_t \cdot \Big(2\phi' L_{v_*} C_{v_*} + 2\phi C_{v_* v_*} - 2\phi (C_{v_* v_*} \cdot C_s) C_s$$
$$-2\phi (C_{v_*} \cdot C_{ss}) C_{v_*} + (\phi m + \phi' M) C_{ss} \Big)\, ds\, dv$$

where

$$m = \|C_{v_*}\|^2 \qquad \text{and} \qquad M = \int_0^L m\, ds = \int_0^L \|C_{v_*}\|^2\, ds.$$

As before, we now consider the planar case in which C_{v_*} and C_{ss} are linearly dependent and therefore $(C_{v_*} \cdot C_{ss})C_{v_*} = mC_{ss}$, yielding

$$E'(t) = -2\int_0^1 \int_0^L C_t \cdot \Big(\phi\big(C_{v_* v_*} - (C_{v_* v_*} \cdot C_s)C_s\big) + \phi' L_{v_*} C_{v_*} + \frac{1}{2}(\phi' M - \phi m) C_{ss} \Big)\, ds\, dv \quad (5.11)$$

from which we obtain the following minimizing flow

$$C_t = \phi\big(C_{v_* v_*} - (C_{v_* v_*} \cdot C_s)C_s\big) + \phi' L_{v_*} C_{v_*} + \frac{1}{2}(\phi' M - \phi m) C_{ss}$$

which is geometrically equivalent (by adding a tangential term) to

$$C_t = \phi C_{v_* v_*} + \phi' L_{v_*} C_{v_*} + \frac{1}{2}(\phi' M - \phi m) C_{ss} \quad (5.12)$$

5.2.1 Stable conformal factor

To stabilize the flow in last equation, we look for a ϕ such that

$$\phi' M - \phi m \geq 0 \qquad \text{for all } (s, v_*) \tag{5.13}$$

or (assuming $M \neq 0$)

$$\frac{\phi'}{\phi} = (\log \phi)' \geq \frac{m}{M} \qquad \text{for all } (s, v_*) \tag{5.14}$$

One way to satisfy this is to choose

$$(\log \phi)' = \max_{s, v_*} \frac{m}{M} \doteq \lambda \tag{5.15}$$

giving us

$$\phi = e^{\lambda L} \tag{5.16}$$

yielding the following flow of homotopies

$$C_t = e^{\lambda L} \Big(2 C_{v_* v_*} + 2\lambda L_{v_*} C_{v_*} + (\lambda M - m) C_{ss} \Big) \tag{5.17}$$

The choice of having $\phi = e^{\lambda L}$ agrees also with the discussion in §5.4, that hints that the energy $E(C)$ associated to the H^0_ϕ metric may be lower semi continuous when $\phi(c) \geq \text{len}(c)$. The above conformal metric does not entail an unique Riemannian Metric on the space of curves: indeed the choice of λ depends on the homotopy itself.

In the numerical experiments shown in this paper, we chose λ to satisfy (5.15) at time $t = 0$ and found that this was enough to stabilize the flow up to convergence. However, we have no mathematical proof of this phenomenon.

5.3 Numerical results

We note that the minimizing flow (5.12) consists of two stable diffusion terms and a transport term. As such, we have the option to utilize level set methods in the implementation of (5.12).

We represent the evolving homotopy $C(u, v, t)$ as an evolving surface $S(u, v, t)$

$$S(u, v, t) = (C(u, v), v, t)$$

We then perform a *Level Set Embedding* of this surface into a 4D scalar function ψ such that

$$\psi\big(C(u, v, t), v, t\big) = 0.$$

The goal is now to determine an evolution for ψ which yields the evoluton (5.12) for the level sets of each of its 2D cross-sections. Differentiating

$$\frac{d}{dt}\Big(\psi\big(x(u, v, t), y(u, v, t), v, t\big) = 0\Big) \quad \longrightarrow \quad \psi_t + \nabla\psi \cdot C_t = 0$$

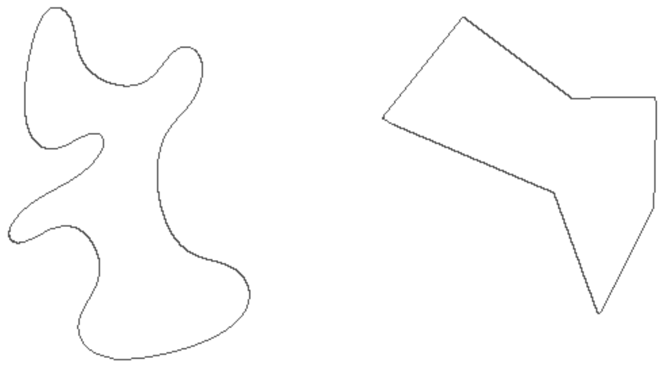
where $\nabla\psi = (\psi_x, \psi_y)$ denotes the 2D spatial gradient of each 2D cross-section of ψ, and substituting (5.12), noting that $N = \nabla\psi / \|\nabla\psi\|$, yields the corresponding

Level Set Evolution.

$$\psi_t = \psi_{vv} - \frac{2\psi_v}{\|\nabla\psi\|^2}(\nabla\psi_v \cdot \nabla\psi) + \frac{\psi_v^2}{\|\nabla\psi\|^4}\left(\nabla^2\psi\nabla\psi\right)\cdot\nabla\psi$$
$$-\frac{1}{2}\left(\frac{\psi_v^2}{\|\nabla\psi\|^2} - \lambda\int_0^L \frac{\psi_v^2}{\|\nabla\psi\|^2}\,ds\right)\nabla\cdot\left(\frac{\nabla\psi}{\|\nabla\psi\|}\right)\|\nabla\psi\| + \lambda L_v\psi_v$$

Note that for simplicity we have dropped the factor $e^{\lambda L}$ from (5.12) since we are guaranteed that this factor is always positive. As a result, we do not change the steady-state of the flow by omitting this factor.

If we numerically compute the geodesic of two curves c_0, c_1 in figure 6, we obtain the geodesic, which is represented, by slicing it in figure 7 on the next page and as a surface in figure 8 on the following page.



Curve c_0 Curve c_1

Figure 6: Curves c_0 and c_1

5.4 Example

This example shows why we think that the conformal energy $E(C)$ may be lower semicontinuous (on planar curves) in the case when $\phi(c) \geq \text{len}(c)$.

Fix $0 < \varepsilon < 1/2$ and $\lambda \geq 0$.

Suppose

$$c(u) = \begin{cases} (u, u\lambda) & \text{if } u \in [0, \varepsilon] \\ (u, (2\varepsilon - u)\lambda) & \text{if } u \in [\varepsilon, 2\varepsilon] \\ (u, 0) & \text{if } u \in [2\varepsilon, 1] \end{cases} \tag{5.18}$$

is the curve in figure 9 on page 45.

We define the homotopy $C : [0,1] \times [0,1] \to \mathbb{R}^2$ by

$$\tilde{C}(u, v) = c(u) + (0, v)$$

Let with $C(u, v) = (u, v)$ be the identity.

We may tesselate, as explained in point 3 in 4.3.1, to build a sequence of homotopies C_h such that $C_h \to_h C$ in L^∞ and

$$\partial_{u,v}C_h \rightharpoonup_h^* \partial_{u,v}C \text{ weakly* in } L^\infty,$$

Figure 7: Slices of homotopy

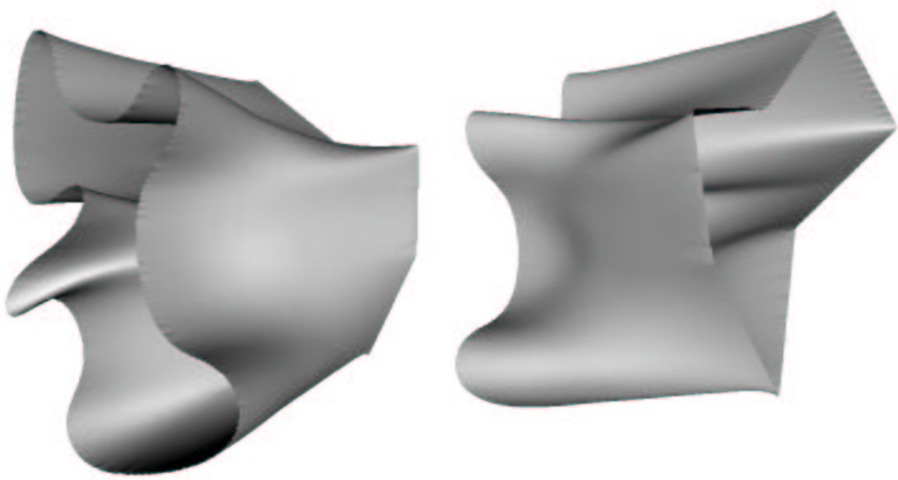


Figure 8: Surface of homotopy

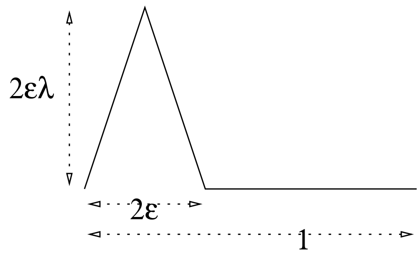


Figure 9: The curve from eq. (5.18)

and $E(C_h) = E(C)$.

Now we compute

$$\operatorname{len}(c) = 1 + 2\varepsilon(\sqrt{1+\lambda^2} - 1) = 1 + 2\varepsilon\alpha$$

where we define $\alpha = \sqrt{1+\lambda^2} - 1$ for convenience. Note that $\alpha \geq 0$.

We compute the energy (using the identity (4.11))

$$E(C_h) = E(\tilde{C}) = \int_0^1 \int_0^1 \frac{\phi(C)}{|\partial_u C|} du dv =$$
$$= (1-2\varepsilon)\phi(C) + 2\varepsilon \frac{\phi(C)}{\sqrt{1+\lambda^2}} = \phi(C)\left(1 - 2\varepsilon + 2\varepsilon\frac{1}{\alpha+1}\right) =$$
$$= \phi(C)\left(1 - 2\varepsilon\frac{\alpha}{\alpha+1}\right) \geq$$
$$\geq (1+2\varepsilon\alpha)\left(1 - 2\varepsilon\frac{\alpha}{\alpha+1}\right) = 1 + 2\varepsilon\left(\alpha - \frac{\alpha}{\alpha+1} - \frac{2\varepsilon\alpha^2}{\alpha+1}\right) =$$
$$= 1 + 2\varepsilon\frac{\alpha^2 - 2\varepsilon\alpha^2}{\alpha+1} = 1 + 2\varepsilon\alpha^2\frac{1-2\varepsilon}{\alpha+1} \geq 1 = E^N(C)$$

A More on Finsler metrics

We now provide two more results on Finsler metrics 1.2, for convenience of the reader. The first result explains the relationship between the length functional $\operatorname{len}(\xi)$ and the energy functional $E(\xi)$

Proposition A.1 *Fix x, y in the following, and let $\mathcal{A}$ be the class of all locally Lipschitz paths $\gamma : [0,1] \to M$ connecting x to y.*

These are known properties of the the length and the energy.

- *If ξ, γ are locally Lipschitz and ϕ is a monotone continuous function such that $\xi = \gamma \circ \phi$ and $\phi(0) = 0, \phi(1) = 1$ then $\operatorname{Len}\gamma = \operatorname{Len}\xi$.*
- *In general, by Hölder inequality, $E(\gamma) \geq \operatorname{Len}(\gamma)^2$.*
- *If γ provides a minimum of $\min_{\mathcal{A}} E(\gamma)$, then it is also a minimum of $\min_{\mathcal{A}} \operatorname{Len}(\gamma)$ in the same class, $E(\gamma) = \operatorname{Len}(\gamma)^2$, and moreover $|\dot\gamma(t)|_{\gamma(t)}$ is constant in t, that is, γ has constant velocity.*

- *If ξ provides a minimum of* $\min_{\mathcal{A}} \mathrm{Len}(\gamma)$*, then there exists a monotone continuous function ϕ and a path γ such that $\xi = \gamma \circ \phi$, and γ is a minimum of* $\min_{\mathcal{A}} E(\gamma)$*.* (44)

Proof. By 2.27, 2.42, 3.8, 3.9 in [Men] and 4.2.1 in [AT00]. □

This second result is the version of the Hopf–Rinow theorem 1.6 to the case of generic metric spaces.

Theorem A.2 (Hopf-Rinow) *Suppose that the metric space (M, d) is locally compact and path-metric, then these are equivalent:*

- *the metric space (M, d) is complete,*
- *closed bounded sets are compact*

and both imply that any two points can be connected by a minimal length geodesic.

A proof is in §1.11 and §1.12 in Gromov's [Gro99], or in Theorem 1.2 in [Men] (which holds also in the asymmetric case).

B Proofs of §2.1.4

Let Z be the set of all $\theta \in L^2([0, 2\pi])$ such that $\theta(s) = a + k(s)\pi$ where $k(s) \in \mathbb{Z}$ is measurable, ($a = 2\pi - \int k$), and

$$|\{k(s) = 0 \bmod 2\}| = |\{k(s) = 1 \bmod 2\}| = \pi$$

Z is closed (by thm. 4.9 in [Bre86]). We see that Z contains the (representations θ of) flat curves ξ, that is, curves ξ whose image is contained in a line; one such curve is

$$\xi_1(s) = \xi_2(s) = \begin{cases} s/\sqrt{2} & s \in [0, \pi] \\ (2\pi - s)/\sqrt{2} & s \in (\pi, 2\pi] \end{cases}, \qquad \theta = \begin{cases} \pi/2 & s \in [0, \pi] \\ 3\pi/2 & s \in (\pi, 2\pi] \end{cases}$$

We provide here the proof of 2.11: $M \setminus Z$ is a manifold.

Proof. Indeed, suppose by contradiction that $\nabla\phi_1, \nabla\phi_2, \nabla\phi_3$ are linear dependant at $\theta \in M$, that is, there exists $a \in \mathbb{R}^3, a \neq 0$ s.t.

$$a_1 \cos(\theta(s)) + a_2 \sin(\theta(s)) + a_3 = 0$$

for almost all s; then, by integrating, $a_3 = 0$, therefore $a_1 \cos(\theta(s)) + a_2 \sin(\theta(s)) = 0$ that means that $\theta \in Z$. See also §3.1 in [KSMJ03]. □

This is the proof of 2.14:

(44) in writing $\xi = \gamma \circ \phi$, we, in a sense, define γ by a *pullback* of ξ: see 2.29 in [Men]. Note that we could not write, in general, $\gamma = \xi \circ \phi^{-1}$: indeed, it is possible that a minimum of $\min_{\mathcal{A}} \mathrm{Len}(\gamma)$ may stay still for an interval of time; that is, we must allow for the case when ϕ is not invertible.

Proof. Fix $\theta_0 \in M \setminus Z$. Let $T = T_{\theta_0} M$ be the tangent at θ_0. T is the vector space orthogonal to $\nabla\phi_i(\theta_0)$ for $i = 1, 2, 3$. Let $e_i = e_i(s) \in L^2 \cap C_c^\infty$ be near $\phi_i(\theta_0)$ in L^2, so that the map $(x, y) : T \times \mathbb{R}^3 \to L^2$

$$(x, y) \mapsto \theta = \theta_0 + x + \sum_{i=1}^{3} e_i y_i \tag{B.1}$$

is an isomorphism. Let M' be M in these coordinates; by the Implicit Function Theorem (5.9 in [Lan99]), there exists an open set $U' \subset T$, $0 \in U'$, an open $V' \subset \mathbb{R}^3$, $0 \in V'$, and a smooth function $f : U \to \mathbb{R}^3$ such that the local part $M' \cap (U' \times V')$ of the manifold M' is the graph of $y = f(x)$.

We immediatly define a smooth projection $\pi : U' \times V' \to M'$ by setting $\pi'(x, y) = (x, f(x))$; this may be expressed in L^2; let $(x(\theta), y(\theta))$ be the inverse of (B.1) and $U = x^{-1}(U')$; we define the projection $\pi : U \to M$ by setting

$$\pi(\theta) = \theta_0 + x + \sum_{i=1}^{3} e_i f_i(x(\theta))$$

Then

$$\pi(\theta)(s) - \theta(s) = \sum_{i=1}^{3} e_i(s) a_i \quad , \quad a_i \doteq (f_i(x(\theta)) - y_i) \in \mathbb{R} \tag{B.2}$$

so if $\theta(s)$ is smooth, then $\pi(\theta)(s)$ is smooth.

Let θ_n be smooth functions such that $\theta_n \to \theta$ in L^2, then $\pi(\theta_n) \to \theta_0$; if we choose them to satisfy $\theta_n(2\pi) - \theta_n(0) = 2\pi h$, then, by the formula (B.2), $\pi(\theta)(2\pi) - \pi(\theta)(0) = 2\pi h$ so that $\pi(\theta_n) \in M$ and it represents a smooth curve with the assigned rotation index h. □

C E^N is ill-posed

The result C.1 following below was inspired from a description of a similar phenomenon, found on page 16 of the slides [Mum] of D. Mumford: it is possible to connect the two segments $c_0(u) = (u, 0)$ and $c_1(u) = (u, 1)$ with a family of homotopies $C_k : [0, 1] \times [0, 1] \to \mathbb{R}^2$ such that $E^N(C_k) \to_k 0$. We represent the idea in figure 10 on the following page.

We use the above idea to show that the distance induced by E^N is zero. (45)

Proposition C.1 (E^N is ill-posed) *Fix c_0 and c_1 to be two regular curves. We want to show that, $\forall \epsilon > 0$, there is a homotopy C connecting c_0 to c_1 such that $E^N(C) < \epsilon$.*

1. *To start, suppose that c_1 is contained in the surface of a sphere, that is, $|c_1| = 1$ is constant. Suppose also that $|\dot{c}_1| = 1$. Consider the linear interpolant, from the origin to c_1:*

$$C(\theta, v) = v c_1(\theta)$$

The image of this homotopy is a cone.

(45) We have recently discovered an identical proposition in [MM]: we anyway propose this proof, since it is more detailed.

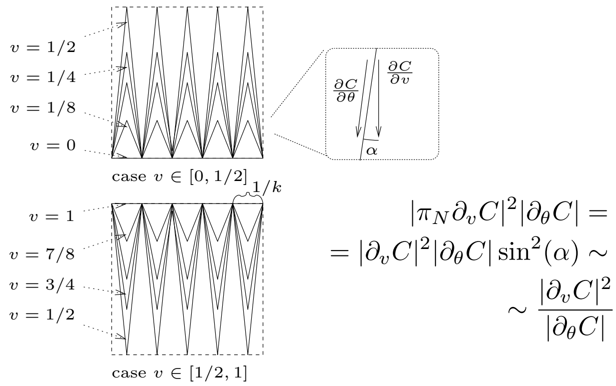


Figure 10: Artistic rendition of the homotopy C_k, from [Mum]

We want to play a bad trick to the linear interpolant: we define a homotopy whose image is the cone, but that moves points with different speeds and times. Let $\epsilon = \pi/k$ in the following; we define the sawtooth $Z : S^1 \to [0, \epsilon]$

$$Z(\theta) = \begin{cases} \theta & \textit{if } \theta \in [0, \epsilon] \\ (2\epsilon - \theta) & \textit{if } \theta \in [\epsilon, 2\epsilon] \\ (\theta - 2\epsilon) & \textit{if } \theta \in [2\epsilon, 3\epsilon] \\ (4\epsilon - \theta) & \textit{if } \theta \in [3\epsilon, 4\epsilon] \\ \dots \end{cases}$$

(note that $Z(\theta) + Z(\theta + \epsilon) = \epsilon$, $Z(\theta) = Z(-\theta)$)

Let

$$C_k(\theta, v) = c_1(\theta) \frac{2v}{\epsilon} Z(\theta)$$

for $v \in [0, 1/2]$, and

$$C_k(\theta, v) = c_1(\theta) \left(1 - 2 \frac{1 - v}{\epsilon} Z(\theta + \epsilon) \right)$$

for $v \in [1/2, 1]$.

The energy $E(C_k)$ is splitted in two parts for v and in $2k$ equal parts in θ, so we compute the energy only for two regions, and then multiply by $2k$.

- *In region $\theta \in [0, \epsilon]$ $v \in [0, 1/2]$, we have*

$$C_k(\theta, v) = c_1(\theta) 2 \frac{1}{\epsilon} v \theta$$

$$\partial_v C_k = c_1(\theta) 2 \frac{1}{\epsilon} \theta$$

$$\partial_\theta C_k = \dot{c}_1(\theta) 2 \frac{v}{\epsilon} \theta + c_1(\theta) 2 \frac{v}{\epsilon}$$

and

$$|\partial_\theta C_k|^2 = 4 v^2 \frac{1}{\epsilon^2} (\theta^2 + 1)$$

Since

$$|\pi_N v|^2 = |v - \langle v,T\rangle T|^2 = |v|^2 - (\langle v,T\rangle)^2$$

then

$$|\pi_N \partial_v C_k|^2 = \left|\frac{2}{\epsilon}\theta \pi_N c_1(\theta)\right|^2 = \frac{4}{\epsilon^2}\theta^2\Big(|c_1(\theta)|^2 - (\langle T, c_1(\theta)\rangle)^2\Big) =$$

$$= \frac{4}{\epsilon^2}\theta^2\left(1 - \langle \partial_\theta C, c_1(\theta)\rangle^2 \frac{1}{|\partial_\theta C|^2}\right) =$$

$$= \frac{4}{\epsilon^2}\theta^2\left(1 - \left\langle c_1(\theta)2\frac{v}{\epsilon}, c_1(\theta)\right\rangle^2 \frac{1}{|\partial_\theta C|^2}\right) = \frac{4}{\epsilon^2}\theta^2\left(1 - 4\frac{1}{\epsilon^2}v^2\frac{1}{|\partial_\theta C|^2}\right) =$$

$$= \frac{4}{\epsilon^2}\theta^2\left(1 - \frac{1}{1+\theta^2}\right) = \frac{4}{\epsilon^2}\theta^4\frac{1}{1+\theta^2}$$

so that the energy for the part $v \in [0, 1/2]$ *of the homotopy is*

$$E^N(C_k) = 4k\int_0^{1/2}\int_0^{\epsilon} |\pi_N\partial_v C_k|^2 \,|\partial_\theta C_k| \;d\theta\; dv =$$

$$= 4k\int_0^{1/2}\int_0^{\epsilon} \frac{4}{\epsilon^2}\theta^4\frac{1}{1+\theta^2}\,|\partial_\theta C_k| \;d\theta\; dv =$$

$$= 16k\frac{1}{\epsilon^2}\int_0^{1/2}\int_0^{\epsilon} \theta^4\frac{1}{1+\theta^2}\sqrt{4v^2\frac{1}{\epsilon^2}(\theta^2+1)} \;d\theta\; dv =$$

$$= 32k\frac{1}{\epsilon^3}\int_0^{1/2}\int_0^{\epsilon} \theta^4\frac{1}{1+\theta^2}v\sqrt{(\theta^2+1)} \;d\theta\; dv =$$

$$= 32k\frac{1}{\epsilon^3}\frac{1}{8}\int_0^{\epsilon}\theta^4\frac{1}{\sqrt{(\theta^2+1)}} \;d\theta \le$$

$$\le 4k\frac{1}{\epsilon^3}\frac{\epsilon^5}{5} = \frac{4}{5}\pi^2\frac{1}{k}$$

- *similarly in region* $\theta \in [0,\epsilon]$ $v \in [1/2, 1]$ *we have*

$$C_k(\theta, v) = c_1(\theta)\left(1 - 2\frac{1-v}{\epsilon}(\epsilon - \theta)\right)$$

but we implicitely change variable $\theta \mapsto \theta - \epsilon$, $v \mapsto v - 1$ *to write*

$$C_k(\theta, v) = c_1(\theta)\left(1 - 2\frac{v}{\epsilon}\theta\right)$$

(this means that we will integrate on $\theta \in [-\epsilon, 0]$ $v \in [-1/2, 0]$*). Then*

$$\partial_v C_k = -c_1(\theta)2\theta\frac{1}{\epsilon}$$

and

$$\partial_\theta C_k = \dot c_1(\theta)\left(1 - 2\frac{v}{\epsilon}\theta\right) - c_1(\theta)2\frac{v}{\epsilon}$$

and

$$|\partial_\theta C_k|^2 = \left(1 - 2\frac{v}{\epsilon}\theta\right)^2 + 4\frac{v^2}{\epsilon^2} = \frac{1}{\epsilon^2}\Big((\epsilon - 2v\theta)^2 + 4v^2\Big)$$

$$|\pi_N \partial_v C_k|^2 = \left|\frac{2}{\epsilon}\theta \pi_N c_1(\theta)\right|^2 = \frac{4}{\epsilon^2}\theta^2\Big(|c_1(\theta)|^2 - \langle T, c_1(\theta)\rangle^2\Big) =$$
$$= \frac{4}{\epsilon^2}\theta^2\left(1 - \left\langle c_1(\theta)2\frac{v}{\epsilon}, c_1(\theta)\right\rangle^2 \frac{1}{|\partial_\theta C_k|^2}\right) =$$
$$= \frac{4}{\epsilon^2}\theta^2\left(1 - 4\frac{v^2}{\epsilon^2}\frac{1}{|\partial_\theta C_k|^2}\right) = \frac{4}{\epsilon^2}\theta^2\left(\frac{(\epsilon - 2v\theta)^2}{(\epsilon - 2v\theta)^2 + 4v^2}\right)$$

Note that

$$(\epsilon - 2v\theta)^2 + 4v^2 \geq \epsilon^2(1 + 2v)^2 + 4v^2 \geq \frac{2\epsilon^4}{(1+\epsilon^2)^2} \qquad (C.1.\star)$$

(the positive minimum is reached at $v = -\epsilon^2/(2 + 2\epsilon^2), \theta = -\epsilon$*)*
Since

$$|\pi_N \partial_v C_k|^2 |\partial_\theta C_k| = \frac{4}{\epsilon^2}\theta^2\left(\frac{(\epsilon - 2v\theta)^2}{(\epsilon - 2v\theta)^2 + 4v^2}\right)\sqrt{\frac{1}{\epsilon^2}\Big((\epsilon - 2v\theta)^2 + 4v^2\Big)} =$$
$$= \frac{4}{\epsilon^3}\theta^2\left(\frac{(\epsilon - 2v\theta)^2}{\sqrt{(\epsilon - 2v\theta)^2 + 4v^2}}\right) = \frac{4}{\epsilon}\tau^2\frac{\epsilon^2(1 - 2v\tau)^2}{\sqrt{\epsilon^2(1 - 2v\tau)^2 + 4v^2}}$$

where $\tau = \theta/\epsilon$ *so the energy for the part* $v \in [1/2, 1]$ *of the homotopy becomes*

$$E^N(C_k) = 2k\int_{-1/2}^0 \int_{-\epsilon}^0 |\pi_N \partial_v C_k|^2 |\partial_\theta C_k|\ d\theta\ dv =$$
$$= 2k\int_{-1/2}^0 \int_{-1}^0 4\tau^2 \frac{\epsilon^2(1 - 2v\tau)^2}{\sqrt{\epsilon^2(1 - 2v\tau)^2 + 4v^2}}\ d\tau\ dv =$$
$$= 8\pi\int_{-1/2}^0 \int_{-1}^0 \tau^2 \frac{(1 - 2v\tau)^2}{\sqrt{(1 - 2v\tau)^2 + 4v^2/\epsilon^2}}\ d\tau\ dv$$

since by (C.1.$\star$*) the integrand is continuous and positive, and it decreases pointwise when* $\epsilon \to 0$*, then so* $E^N(C_k) \to 0$ *as* $\epsilon \to 0$ *(by Beppo-Levi lemma, or Lebesgue theorem).*

2. *as a second step, consider a generic smooth curve* c_1*; we could approximate it with a piecewise smooth curve* c_1' *, where each piece in* C' *is either contained in a sphere, or in a radious exiting from* 0*: by using the above homotopy on each spherical piece, and translating and scaling each radial piece to* 0*, we can have a homotopy*

3. *then, given two generic smooth curves c_0, c_1, we can approximate them as above to obtain c_0', c_1', and build a homotopy*

$$c_0 \to c_0' \to 0 \to c_1' \to c_1$$

this final homotopy can be built with small energy

Remark C.2 *More in general, if $\alpha > \beta > 0$, then the energy*

$$\int_I |\pi_N \partial_v C|^\alpha |\dot{C}|^\beta$$

is ill-defined, as is shown in 4.3.1.

Proposition C.3 (E is ill-posed) *Consider the energy $E(C)$ associated to the metric (2.6). Fix c_0 and c_1 to be two regular curves. $\forall \epsilon > 0$, there is a homotopy $C \in \mathbb{C}$ connecting c_0 to c_1 such that $E(C) < \epsilon$.*

Proof. Consider a homotopy defined as in the previous proposition; if we mollify it, we can obtain a regular homotopy C' such that $\|C' - C\|_{W^{1,3}}$ and $\|C' - C\|_\infty$ are arbitrarily small; then we can reconnect C' to c_0 to c_1 with a small cost, to create C'': by some direct computation, $E(C'') < 2\epsilon$.

By using prop. 3.8 on C'' we obtain a $\widetilde{C}$ such that $E^N(\widetilde{C}) = E^N(C'')$, and since $\pi_{\widetilde{N}} \widetilde{C} = 0$,

$$E(\widetilde{C}) = E^N(\widetilde{C}) = E^N(C'') \leq 2\epsilon$$

□

D Derivation of Flows

In this section we show the details of the calculations of the minimizing flows for both the H^0 and conformal energies.

D.1 Some preliminary calculus

First we develop in the following subsections some of the calculus that we will need to work with the geometric parameters s and v_* introduced in section 5.

D.1.1 Commutation of derivatives

Note that the parameters s and v_* do not form true coordinates and therefore have a non-trivial commutator. The third parameter t will come into play later when we consider a time varying family of homotopies $C(u, v, t)$ and take the resulting time derivative of either the H^0 or the conformal energy along this family.

$$
\begin{aligned}
\frac{\partial}{\partial t}\frac{\partial}{\partial v_*} &= \frac{\partial}{\partial t}\left(\frac{\partial}{\partial v} - \frac{C_u \cdot C_v}{C_u \cdot C_u}\frac{\partial}{\partial u}\right) \\
&= \frac{\partial}{\partial t}\frac{\partial}{\partial v} - \frac{C_u \cdot C_v}{C_u \cdot C_u}\frac{\partial}{\partial t}\frac{\partial}{\partial u} - \left(\frac{C_{ut} \cdot C_v + C_u \cdot C_{vt}}{C_u \cdot C_u} - 2\frac{(C_u \cdot C_v)(C_{ut} \cdot C_u)}{(C_u \cdot C_u)^2}\right)\frac{\partial}{\partial u} \\
&= \left(\frac{\partial}{\partial v} - \frac{C_u \cdot C_v}{C_u \cdot C_u}\frac{\partial}{\partial u}\right)\frac{\partial}{\partial t} - \frac{C_u \cdot \left(C_{tv} - \frac{C_u \cdot C_v}{C_u \cdot C_u}C_{tu}\right) + C_{tu} \cdot \left(C_v - \frac{C_u \cdot C_v}{C_u \cdot C_u}C_u\right)}{C_u \cdot C_u}\frac{\partial}{\partial u} \\
&= \boxed{\frac{\partial}{\partial v_*}\frac{\partial}{\partial t} - \left(C_s \cdot C_{tv_*} + C_{ts} \cdot C_{v_*}\right)\frac{\partial}{\partial s}} \qquad \text{(D.1)}
\end{aligned}
$$

$$
\frac{\partial}{\partial t}\frac{\partial}{\partial s} = \frac{\partial}{\partial t}\left(\frac{1}{\|C_u\|}\frac{\partial}{\partial u}\right) = \frac{1}{\|C_u\|}\frac{\partial}{\partial t}\frac{\partial}{\partial u} - \frac{C_{ut} \cdot C_u}{\|C_u\|^3}\frac{\partial}{\partial u} = \boxed{\frac{\partial}{\partial s}\frac{\partial}{\partial t} - C_{ts} \cdot C_s\frac{\partial}{\partial s}} \qquad \text{(D.2)}
$$

$$
\begin{aligned}
\frac{\partial}{\partial v_*}\frac{\partial}{\partial s} &= \frac{\partial}{\partial v}\left(\frac{1}{\|C_u\|}\frac{\partial}{\partial u}\right) - C_v \cdot C_s\frac{\partial}{\partial s}\left(\frac{1}{\|C_u\|}\frac{\partial}{\partial u}\right) \\
&= \frac{1}{\|C_u\|}\frac{\partial}{\partial v}\frac{\partial}{\partial u} - \frac{C_{uv} \cdot C_u}{\|C_u\|^3}\frac{\partial}{\partial u} - C_v \cdot C_s\frac{\partial}{\partial s}\frac{\partial}{\partial s} \\
&= \frac{\partial}{\partial s}\frac{\partial}{\partial v} - C_{vs} \cdot C_s\frac{\partial}{\partial s} - C_v \cdot C_s\frac{\partial}{\partial s}\frac{\partial}{\partial s} \\
&= \frac{\partial}{\partial s}\left(\frac{\partial}{\partial v} - C_v \cdot C_s\frac{\partial}{\partial s}\right) + C_v \cdot C_{ss}\frac{\partial}{\partial s} = \boxed{\frac{\partial}{\partial s}\frac{\partial}{\partial v_*} + C_{v_*} \cdot C_{ss}\frac{\partial}{\partial s}} \qquad \text{(D.3)}
\end{aligned}
$$

D.1.2 Some identities

Here we write down some useful identities regarding various derivatives of the homotopy with respect to the geometric parameters s and v_*.

1. $C_s \cdot C_s = 1$
2. $C_s \cdot C_{v_*} = 0$
3. $C_{v_* s} \cdot C_{v_*} = C_{s v_*} \cdot C_{v_*} = -C_{v_* v_*} \cdot C_s$
4. $C_{v_* s} \cdot C_s = -C_{ss} \cdot C_{v_*}$
5. $C_{s v_*} \cdot C_s = 0$
6. $C_{ss} \cdot C_s = 0$

D.1.3 Commutation of derivatives with integrals

Finally, we write down how to commute derivatives and integrals when differentiating with respect to t or v_*.

$$\begin{aligned}
\frac{\partial}{\partial t}\int_0^L f\,ds &= \frac{\partial}{\partial t}\int_0^1 f\|C_u\|\,du = \int_0^1 f_t\|C_u\| + f(C_{ut}\cdot C_s)\,du = \int_0^L f_t + f(C_{ts}\cdot C_s)\,ds \\
&= \boxed{\int_0^L f_t - f_s(C_t\cdot C_s) - f(C_t\cdot C_{ss})\,ds} \qquad \text{(D.4)}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial v_*}\int_0^L f\,ds &= \frac{\partial}{\partial v}\int_0^1 f\|C_u\|\,du = \int_0^1 f_v\|C_u\| + f(C_{uv}\cdot C_s)\,du = \int_0^L f_v + f(C_{vs}\cdot C_s)\,ds \\
&= \int_0^L f_v - f_s(C_v\cdot C_s) - f(C_v\cdot C_{ss})\,ds = \boxed{\int_0^L f_{v_*} - f(C_{v_*}\cdot C_{ss})\,ds} \qquad \text{(D.5)}
\end{aligned}$$

D.1.4 Intermediate Expressions

The last step, before begining the flow calculation will be to introduce a few "intermediate" expressions that will help keep the the expressions in the upcoming derivations from becoming too lengthy.

$$\begin{aligned}
m &= C_{v_*}\cdot C_{v_*} && \text{(D.6)}\\
m_t &= 2\,C_{v_*t}\cdot C_{v_*} = 2\,C_{tv_*}\cdot C_{v_*} && \text{(D.7)}\\
m_s &= 2\,C_{v_*s}\cdot C_{v_*} = 2\,C_{sv_*}\cdot C_{v_*} = -2\,C_{v_*v_*}\cdot C_s && \text{(D.8)}\\
m_{v_*} &= 2\,C_{v_*v_*}\cdot C_{v_*} && \text{(D.9)}\\
m_{v_*v_*} &= 2\,C_{v_*v_*v_*}\cdot C_{v_*} + 2\,C_{v_*v_*}\cdot C_{v_*v_*} && \text{(D.10)}
\end{aligned}$$

$$\begin{aligned}
M &= \int_0^L C_{v_*}\cdot C_{v_*} ds && \text{(D.11)}\\
M_t &= \int_0^L 2\,C_{tv_*}\cdot C_{v_*} + 2\,(C_{v_*v_*}\cdot C_s)(C_t\cdot C_s) - m\,(C_t\cdot C_{ss})\,ds && \text{(D.12)}\\
M_{v_*} &= \int_0^L 2\,C_{v_*v_*}\cdot C_{v_*} - m\,(C_{v_*}\cdot C_{ss})\,ds && \text{(D.13)}\\
M_{v_*v_*} &= \int_0^L 2\,C_{v_*v_*v_*}\cdot C_{v_*} + 2\,C_{v_*v_*}\cdot C_{v_*v_*} - m\,(C_{v_*}\cdot C_{ssv_*}) && \text{(D.14)}\\
&\quad -m\,(C_{v_*v_*}\cdot C_{ss}) - 4\,(C_{v_*v_*}\cdot C_{v_*})(C_{v_*}\cdot C_{ss}) + m\,(C_{v_*}\cdot C_{ss})^2\,ds \\
&= \int_0^L 2\,C_{v_*v_*v_*}\cdot C_{v_*} + 2\,C_{v_*v_*}\cdot C_{v_*v_*} + 2\,(C_{sv_*}\cdot C_{v_*})^2 + m\,(C_{sv_*}\cdot C_{sv_*}) \\
&\quad -m\,(C_{v_*v_*}\cdot C_{ss}) - 4\,(C_{v_*v_*}\cdot C_{v_*})(C_{v_*}\cdot C_{ss})\,ds
\end{aligned}$$

$$\begin{aligned}
L &= \int_0^L ds && \text{(D.15)}\\
L_t &= \int_0^L -C_t \cdot C_{ss}\, ds && \text{(D.16)}\\
L_{v_*} &= \int_0^L -C_{v_*} \cdot C_{ss}\, ds && \text{(D.17)}\\
L_{v_*v_*} &= \int_0^L -C_{v_*v_*} \cdot C_{ss} - C_{v_*} \cdot C_{ssv_*} + (C_{v_*} \cdot C_{ss})^2\, ds && \text{(D.18)}\\
&= \int_0^L C_{sv_*} \cdot C_{sv_*}\, ds - C_{v_*v_*} \cdot C_{ss}
\end{aligned}$$

D.2 H^0 flow calculation

We are now ready to begin the flow calculation. We'll start with the case of the H^0 energy in this subsection and then proceed to the conformal case in the followin subsection.

We begin by considering a time-varying family of homotopies $C(u, v, t)$: $[0,1] \times [0,1] \times (0,\infty) \to \mathbb{R}^n$ and write the H^0 energy as

$$E(t) = \int_0^1 \int_0^L \left\|C_{v_*}\right\|^2 ds\, dv = \int_0^1 M dv \qquad (D.19)$$

Then the variation of E is

$$\begin{aligned}
E'(t) &= \int_0^1 M_t\,dv = \int_0^1\int_0^L 2\,C_{tv_*}\cdot C_{v_*} + 2\,(C_{v_*v_*}\cdot C_s)(C_t\cdot C_s) - m\,(C_t\cdot C_{ss})\,ds\,dv \\
&= \int_0^1\int_0^L 2\left(C_{tv} - (C_v\cdot C_s)C_{ts}\right)\cdot C_{v_*}\,ds\,dv + \int_0^1\int_0^L C_t\cdot\left(2\,(C_{v_*v_*}\cdot C_s)C_s - m\,C_{ss}\right)ds\,dv \\
&= 2\int_0^1\int_0^1 C_{tv}\cdot C_{v_*}\|C_u\|\,du\,dv \\
&\quad +2\int_0^1\int_0^L (C_v\cdot C_s)(C_t\cdot C_{v_*s}) + (C_{vs}\cdot C_s + C_v\cdot C_{ss})(C_t\cdot C_{v_*})\,ds\,dv \\
&\quad +\int_0^1\int_0^L C_t\cdot\left(2\,(C_{v_*v_*}\cdot C_s)C_s - m\,C_{ss}\right)ds\,dv \\
&= 2\int_0^1\int_0^1 -(C_t\cdot C_{v_*})(C_{uv}\cdot C_s) - C_t\cdot C_{v_*v}\|C_u\|\,du\,dv \\
&\quad +2\int_0^1\int_0^L (C_v\cdot C_s)(C_t\cdot C_{v_*s}) + (C_{vs}\cdot C_s + C_v\cdot C_{ss})(C_t\cdot C_{v_*})\,ds\,dv \\
&\quad +\int_0^1\int_0^L C_t\cdot\left(2\,(C_{v_*v_*}\cdot C_s)C_s - m\,C_{ss}\right)ds\,dv \\
&= 2\int_0^1\int_0^L C_t\cdot\left(-C_{v_*v} + (C_v\cdot C_s)C_{v_*s} + (C_v\cdot C_{ss})C_{v_*}\right)ds\,dv \\
&\quad +\int_0^1\int_0^L C_t\cdot\left(2\,(C_{v_*v_*}\cdot C_s)C_s - m\,C_{ss}\right)ds\,dv \\
&= -\int_0^1\int_0^L C_t\cdot\left(2C_{v_*v_*} - 2(C_{v_*v_*}\cdot C_s)C_s - 2(C_{v_*}\cdot C_{ss})C_{v_*} + m\,C_{ss}\right)ds\,dv
\end{aligned}$$

In the planar case, C_{v_*} and C_{ss} are linearly dependent (as both are orthogonal to C_s) which means that

$$(C_{v_*}\cdot C_{ss})C_{v_*} = (C_{v_*}\cdot C_{v_*})C_{ss} = mC_{ss} \tag{D.20}$$

and therefore

$$E'(t) = -\int_0^1\int_0^L C_t\cdot\left(2\big(C_{v_*v_*} - (C_{v_*v_*}\cdot C_s)C_s\big) - m\,C_{ss}\right)ds\,dv \tag{D.21}$$

by which we derive the minimization flow

$$C_t = 2\big(C_{v_*v_*} - (C_{v_*v_*}\cdot C_s)C_s\big) - mC_{ss}$$

D.3 Conformal flow calculation

We now define the H^0_ϕ energy as

$$E_\phi(t) = \int_0^1 \phi(L)\int_0^L \left\|C_{v_*}\right\|^2 ds\,dv = \int_0^1 \phi M\,dv \tag{D.22}$$

and compute its derivative as

$$
\begin{aligned}
E'(t) &= \int_0^1 \phi' L_t M + \phi M_t \, dv \\
&= \int_0^1 \left(-\phi' M \int_0^L C_t \cdot C_{ss} \, ds + \phi \int_0^L 2\, C_{tv_*} \cdot C_{v_*} + 2\,(C_{v_*v_*} \cdot C_s)(C_t \cdot C_s) - m\,(C_t \cdot C_{ss}) \, ds \right) d \\
&= \int_0^1 \int_0^L 2\phi \Big(C_{tv} - (C_v \cdot C_s) C_{ts} \Big) \cdot C_{v_*} \, ds \, dv \\
&\quad + \int_0^1 \int_0^L C_t \cdot \Big(2\phi (C_{v_*v_*} \cdot C_s) C_s - (\phi m + \phi' M) C_{ss} \Big) \, ds \, dv \\
&= 2 \int_0^1 \int_0^1 \phi \, C_{tv} \cdot C_{v_*} \|C_u\| \, du \, dv \\
&\quad +2\phi \int_0^1 \int_0^L (C_v \cdot C_s)(C_t \cdot C_{v_*s}) + (C_{vs} \cdot C_s + C_v \cdot C_{ss})(C_t \cdot C_{v_*}) \, ds \, dv \\
&\quad + \int_0^1 \int_0^L C_t \cdot \Big(2\phi (C_{v_*v_*} \cdot C_s) C_s - (\phi m + \phi' M) C_{ss} \Big) \, ds \, dv \\
&= 2 \int_0^1 \int_0^1 -(\phi \, C_t \cdot C_{v_*})(C_{uv} \cdot C_s) - C_t \cdot (\phi' L_v C_{v_*} + \phi \, C_{v_*v}) \|C_u\| \, du \, dv \\
&\quad +2\phi \int_0^1 \int_0^L (C_v \cdot C_s)(C_t \cdot C_{v_*s}) + (C_{vs} \cdot C_s + C_v \cdot C_{ss})(C_t \cdot C_{v_*}) \, ds \, dv \\
&\quad + \int_0^1 \int_0^L C_t \cdot \Big(2\phi (C_{v_*v_*} \cdot C_s) C_s - (\phi m + \phi' M) C_{ss} \Big) \, ds \, dv \\
&= 2 \int_0^1 \int_0^L C_t \cdot \Big(-\phi' L_v C_{v_*} - \phi \, C_{v_*v} + \phi (C_v \cdot C_s) C_{v_*s} + \phi (C_v \cdot C_{ss}) C_{v_*} \Big) \, ds \, dv \\
&\quad + \int_0^1 \int_0^L C_t \cdot \Big(2\phi (C_{v_*v_*} \cdot C_s) C_s - (\phi m + \phi' M) C_{ss} \Big) \, ds \, dv \\
&= -\int_0^1 \int_0^L C_t \cdot \Big(2\phi' L_{v_*} C_{v_*} + 2\phi C_{v_*v_*} - 2\phi (C_{v_*v_*} \cdot C_s) C_s - 2\phi (C_{v_*} \cdot C_{ss}) C_{v_*} + (\phi m + \phi' M)
\end{aligned}
$$

In the planar case, C_{v_*} and C_{ss} are linearly dependent (as both are orthogonal to C_s) which means that

$$
(C_{v_*} \cdot C_{ss}) C_{v_*} = (C_{v_*} \cdot C_{v_*}) C_{ss} = m C_{ss} \tag{D.23}
$$

and therefore

$$
E'(t) = -\int_0^1 \int_0^L C_t \cdot \Big(2\phi \big(C_{v_*v_*} - (C_{v_*v_*} \cdot C_s) C_s \big) + 2\phi' L_{v_*} C_{v_*} + (\phi' M - \phi m) C_{ss} \Big) \, ds \, dv \tag{D.24}
$$

which entails the flow

$$
C_t = 2\phi \big(C_{v_*v_*} - (C_{v_*v_*} \cdot C_s) C_s \big) + 2\phi' L_{v_*} C_{v_*} + (\phi' M - \phi m) C_{ss}
$$

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References

[AFP00] Luigi Ambrosio, Nicola Fusco, and Diego Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 2000. MR MR1857292 (2003a:49002)

[AT00] L. Ambrosio and P. Tilli, *Selected topics in "analysis in metric spaces"*, appunti, edizioni Scuola Normale Superiore, Pisa, 2000.

[Atk75] C. J. Atkin, *The Hopf-Rinow theorem is false in infinite dimensions*, Bull. London Math. Soc. **7** (1975), no. 3, 261–266. MR MR0400283 (53 #4118)

[BCS] D. Bao, S. S. Chern, and Z. Shen, *An introduction to Riemann-Finsler geometry*, (October 1, 1999 version).

[Bre86] H. Brezis, *Analisi funzionale*, Liguori Editore, Napoli, 1986, (italian translation of *Analyse fonctionalle*, Masson, 1983, Paris).

[But89] G. Buttazzo, *Semicontinuity, relaxation and integral representation in the calculus of variation*, scientific & technical, no. 207, Longman, 1989.

[CFK03] G. Charpiat, O. Faugeras, and R. Keriven, *Approximations of shape metrics and application to shape warping and empirical shape statistics*, INRIA report 4820, 2003.

[Dac82] Bernard Dacorogna, *Weak continuity and weak lower semicontinuity of non-linear functionals*, Lecture Notes in Mathematics, vol. 922, Springer-Verlag, 1982.

[EE70] J. Eells and K. D. Elworthy, *Open embeddings of certain Banach manifolds*, Ann. of Math. (2) **91** (1970), 465–485. MR MR0263120 (41 #7725)

[Eke78] Ivar Ekeland, *The Hopf-Rinow theorem in infinite dimension*, J. Differential Geom **13** (1978), no. 2, 287–301.

[Fom90] A.T: Fomenko, *The Plateau problem*, Studies in the development of modern mathematics, Gordon and Breach, 1990.

[Gro99] M. Gromov, *Metric structures for Riemannian and non-Riemannian spaces*, Birkhäuser, 1999.

[Kli82] Wilhelm Klingenberg, *Riemannian Geometry*, W. de Gruyter, Berlin, 1982.

[KSMJ03] Eric Klassen, Anuj Srivastava, Washington Mio, and Shantanu Joshi, *Analysis of planar shapes using geodesic paths on shape spaces*, 2003.

[Lan99] Serge Lang, *Fundamentals of differential geometry*, Springer, 1999.

[Men] A. C. G. Mennucci, *On asymmetric distances*, preprint, `http://cvgmt.sns.it/papers/and04/`.

[MM] Peter W. Michor and David Mumford, *Riemannian geometris of space of plane curves*, http://front.math.ucdavis.edu/math.DG/0312384.

[Mum] D. Mumford, *Slides of the gibbs lectures*, http://www.dam.brown.edu/people/mumford/Papers/Gibbs.pdf.

[Sha49] C. E. Shannon, *A mathematical theory of communication*, The Mathematical Theory of Communication, University of Illinois Press, 1949.

[Sim83] L. Simon, *Lectures on Geometric Measure Theory*, Proc. Center for Mathematical Analysis, vol. 3, Australian National University, Canberra, 1983.

[YM04] A. Yezzi and A. Mennucci, *Conformal riemannian metrics in space of curves*, 2004, MIA,EUSIPCO04, http://www...

[You98] Laurent Younes, *Computable elastic distances between shapes*, SIAM Journal of Applied Mathematics **58** (1998), 565–586.