On Lower Semicontinuity of Integral Functionals in $LD(\Omega)$

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Abstract

In this paper we study a weak convexity property arising in connection with the lower semicontinuity of some integral functionals defined in the space $LD(\Omega)$ of functions $u \in L^1(\Omega, \mathbf{R}^n)$ such that the symmetric distributional derivative Eu is a Radon measure absolutely continuous with respect to the Lebesgue measure.

Key Words. Integral Functional, Lower semicontinuity, symmetric quasi-convexity, function with bounded deformation...
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1 Introduction

We study in this paper the lower semicontinuity of functionals of the type:

$$F(u) = \int_\Omega f\left(\mathcal{E}u(x)\right) dx, \tag{1.1}$$

where $\mathcal{E}u = (\nabla u + \nabla u^T)/2$ and f is a Borel function defined in the space $M_s^{n\times n}$ of symmetric $n \times n$ matrices.
This kind of functionals arises very often in the framework of the mathematical theory of elasto-plasticity. Precisely, they can model the deformation energy of some elasto-plastic body whose reference configuration is Ω.

When we look for the equilibrium configuration of the body Ω in the absence of loads, we need to minimize the functional (1.1) above in some suitable functions space. Very often this is done following the so-called direct method of the calculus of variations, i.e. looking for a topology in which the functional (1.1) is lower semicontinuous and coercive.

Various lower semicontinuity results for functionals (1.1) under the convexity of f have been proved for example in the framework of mathematical formulation of problems arising in perfect plasticity. (See Anzellotti-Giaquinta [4], Suquet [18], [19], Temam [21] etc...). We will introduce and study in the first part of this paper a weak convexity property of the integrand f related to the lower semicontinuity of the functional (1.1) in weak topologies. Precisely, we will say that a function $f : M_s^{n\times n} \to \mathbf{R}_+$ is *symmetric quasi-convex* if

$$\fint_D f\,(A + \mathcal{E}\varphi(x))dx \geq f(A) \tag{1.2}$$

for any bounded open subset D of Ω, $A \in M_s^{n\times n}$ and $\varphi \in W_0^{1,\infty}(D, \mathbf{R}^n)$.

This property is similar and also related to the classical *quasi-convexity* in the sense of Morrey [16]. A variety of results about the classical *quasi-convexity* and its connection to the lower semicontinuity of integral functionals in Sobolev spaces and also in the space BV of functions with *bounded variation* can be found in Acerbi-Fusco [1], Ambrosio-Dal Maso [3], Ball [5], Dacorogna [7], Fonseca-Müller [11], Morrey [16] and the references therein. If the classical *quasi-convexity* is more related to functions spaces whose topologies involve gradients (like Sobolev and BV spaces), our property is related to functions spaces with topologies involving directly the deformation tensor Eu like the so-called space BD of functions with bounded deformation.

We will prove in Theorem 3.1 and Remark 3.2(ii) that the *symmetric quasi-convexity* is a necessary and sufficient condition to the lower semicontinuity of the functional (1.1) in the space $LD(\Omega)$ with respect to the $L^1(\Omega, \mathbf{R}^n)$-norm topology.

Following the result of Šverák [20], we will give in the second part of the paper an example of a non convex function which is *symmetric quasi-convex.*

2 Preliminaries

Let $n \geq 1$ be an integer, Ω be an open subset of $\mathbf{R}^n$. We denote by $M^{n\times n}$ the space of $n\times n$ matrices with euclidean norm $\|\cdot\|$ and $M_s^{n\times n}$ is the subspace of $M^{n\times n}$

made of $n \times n$ symmetric matrices. For any $A \in M^{n\times n}$, A^T is the transposition of A. By $\mathcal{L}^n$ we denote the Lebesgue measure in $\mathbf{R}^n$. For any borel subset E of Ω, $|E|$ indicates the measure of E. For any $x \in \Omega$, $\rho > 0$, we denote by $B(x,\rho)$ the open ball of $\mathbf{R}^n$ centered at x with radius ρ.

Definition 2.1 *We say that a function $u : \Omega \to \mathbf{R}^n$ is with bounded deformation in Ω, and we write $u \in BD(\Omega)$, if $u \in L^1(\Omega, \mathbf{R}^n)$ and*

$$Eu := (Du + Du^T)/2 \in M_b(\Omega, M_s^{n\times n}),$$

where Du is the distributional gradient of u and $M_b(\Omega, M_s^{n\times n})$ is the space of $M_s^{n\times n}$-valued Radon measures with finite total variation in Ω.

This space has been introduced in Matthies-Strang-Christiansen [15] and widely studied by Anzellotti-Giaquinta [4], Kohn [14], Suquet [18], [19], Temam [21], Temam-Strang [22] etc... in connection to problems arising in the theory of plastic bodies.
For any $u \in BD(\Omega)$, the Radon-Nikodym decomposition of the measure Eu gives

$$Eu = E^a u + E^s u,$$

where $E^a u = \mathcal{E}u\mathcal{L}^n\lfloor\Omega$ and $E^s u$ are respectively the absolute continuous part with density $\mathcal{E}u$ and the singular part of Eu with respect to the Lebesgue measure.
The following subspace

$$LD(\Omega) = \{u \in BD(\Omega)\colon\ E^s u = 0\} \tag{2.1}$$

of $BD(\Omega)$ studied in Temam [21] plays the same role in $BD(\Omega)$ as $W^{1,1}(\Omega, \mathbf{R}^n)$ does in the space $BV(\Omega, \mathbf{R}^n)$ of *functions with bounded variation.*
We have $W^{1,1}(\Omega, \mathbf{R}^n) \subset LD(\Omega)$ with strict inclusion, as shown in Temam [21].
The following *approximate differentiability* theorem for BD-functions has been proved in Ambrosio-Coscia-Dal Maso [2] and Hajłasz [13]. (See also Ebobisse [8]).

Theorem 2.2 *Let $u \in BD(\Omega)$. Then for $\mathcal{L}^n$-a.e $x \in \Omega$, there exists $\nabla u(x) \in M^{n\times n}$ such that:*

$$\lim_{\rho\to 0} \frac{1}{\rho^n} \int_{B(x,\rho)} \frac{|u(y) - u(x) - \nabla u(x)(y-x)|}{\rho} dx = 0. \tag{2.2}$$

Moreover, $\mathcal{E}u = (\nabla u + \nabla u^T)/2 \quad \mathcal{L}^n$-a.e. in Ω.

Let us make some observations about the property of convexity we want to study.

Remark 2.3 (i) One can easily see that a function f is *symmetric quasi-convex* if and only if $f \circ \pi$ is *quasi-convex* in the sense of Morrey [16], where π denotes the projection of $M^{n \times n}$ on $M_s^{n \times n}$.

(ii) The *symmetric quasi-convexity* is weaker than convexity, i.e. any convex function is *symmetric quasi-convex.*

(iii) By a density argument one can easily verify that the test functions in (1.2) could also be taken in the space $LD_0(D)$, closure of $C_c^1(D, \mathbf{R}^n)$ in the $BD(D)$-norm topology.

3 The Lower Semicontinuity Theorem

For all what follows C, C_1, C_2, C_3 will denote positive constants.

Theorem 3.1 *Let $f : M_s^{n \times n} \to [0, +\infty[$ be a function which verifies the following conditions:*

(H1) $C_1 \|A\| - C_2 \le f(A) \le C_3(1 + \|A\|)$ *for any $A \in M_s^{n \times n}$;*

(H2) *f is symmetric quasi-convex.*

Then the functional (1.1) is lower semicontinuous in the space $LD(\Omega)$ with respect to the $L^1(\Omega, \mathbf{R}^n)$-norm topology.

Remark 3.2 (i) From Theorem 3.1 we have the lower semicontinuity of (1.1) with respect to the following weak convergences in $LD(\Omega)$:

$u_h \rightharpoonup u$ in $LD(\Omega)$ iff $u_h \rightharpoonup u$ in $L^1(\Omega, \mathbf{R}^n)$ and $Eu_h \rightharpoonup Eu$ in $L^1(\Omega, M_{sym}^{n \times n})$.

$u_h \overset{\star}{\rightharpoonup} u$ in $LD(\Omega)$ iff $u_h \to u$ in $L^1(\Omega, \mathbf{R}^n)$ and $Eu_h \overset{\star}{\rightharpoonup} Eu$ in $M_b(\Omega, M_{sym}^{n \times n})$.

(ii) The *symmetric quasi-convexity* of f is a necessary condition for the lower semicontinuity of the functional (1.1) in $W^{1,\infty}(\Omega, \mathbf{R}^n)$ with respect to the weak $\star$ convergence and also in $W^{1,p}(\Omega, \mathbf{R}^n)$ with respect to the weak convergence, under some additional p-growth assumptions on f.

(iii) In many applications, existence theorems for integral functionals with linear growth and with integrands depending only on the tensor Eu involve the space $BD(\Omega)$. So it would be interesting to study, under the *symmetric* *quasi-convexity* assumption on f, the integral representation of the relaxed functional of (1.1) in the space $BD(\Omega)$ with respect to the $L^1(\Omega, \mathbf{R}^n)$-norm topology. That is, for any $u \in BD(\Omega)$,

$$\bar{F}(u) = \inf \left\{ \liminf_{h \to +\infty} F(u_h) \colon \begin{array}{l} u_h \to u \text{ in } L^1_{loc}(\Omega, \mathbf{R}^n), \\ \{u_h\} \subset C^1(\Omega, \mathbf{R}^n) \end{array} \right\}. \tag{3.1}$$

In other words one would like to obtain, under the *symmetric quasi-convexity* assumption on the integrand f, a BD-version of the result by Ambrosio-Dal Maso [3], Fonseca-Müller [12]. We will study this problem in a forthcoming paper. However, from Theorem 3.1 and by regularization we have

$$\bar{F}(u) = \int_\Omega f(\mathcal{E}u(x))dx \quad \text{for any } u \in LD(\Omega).$$

Proof of Theorem 3.1 : We will divide the proof in three main steps.
Step 1: Under the assumptions of the theorem we show that

$$|\Omega| f(A) \leq \liminf_{h \to +\infty} \int_\Omega f\left(A + \mathcal{E}u_h(x)\right) dx \tag{3.2}$$

whenever $A \in M_s^{n \times n}$, Ω is any bounded open subset of $\mathbf{R}^n$ and $\{u_h\}_h \subset LD(\Omega)$ converges to 0 strongly in $L^1(\Omega, \mathbf{R}^n)$. We can assume without loss of generality that:

$$\liminf_{h \to +\infty} \int_\Omega f\left(A + \mathcal{E}u_h(x)\right) dx = \lim_{h \to +\infty} \int_\Omega f\left(A + \mathcal{E}u_h(x)\right) dx < +\infty \tag{3.3}$$

Then, by the growth condition (H1), the sequence $\{\|\mathcal{E}u_h\|_{L^1}\}$ is bounded. So there exists a positive Radon measure $\mu \in M_b(\Omega)$ such that up to a subsequence,

$$\lim_{h \to +\infty} \int_\Omega \varphi(x) \left\|\mathcal{E}u_h(x)\right\| dx = \int_\Omega \varphi(x) d\mu(x), \tag{3.4}$$

for any $\varphi \in C_0(\Omega)$.
Let $\{\Omega_k\}_k$ be an increasing sequence of subdomains of Ω and $\{\varphi_k\}_k \subset C^1_0(\Omega)$, $0 \leq \varphi_k \leq 1$ such that:

$$\Omega_k \subset\subset \Omega \quad \text{and} \quad \Omega = \bigcup_{k=1}^\infty \Omega_k,$$

$$\varphi_k = 1 \text{ in } \Omega_k \quad \text{and} \quad \varphi_k = 0 \text{ in } \Omega \setminus \bar{\Omega}_{k+1}.$$

Since f is symmetric quasi-convex, we have by Remark 2.3 (iii):

$$\begin{aligned}|\Omega|f(A) \leq \int_\Omega f(A+\mathcal{E}(\varphi_k u_h)(x))dx &= \int_{\Omega\setminus\bar\Omega_{k+1}} f(A)dx + \int_{\Omega_{k+1}\setminus\Omega_k} f(A+\mathcal{E}(\varphi_k u_h)(x))dx \\ &+ \int_{\Omega_k} f(A+\mathcal{E}u_h(x))dx.\end{aligned}$$

Then

$$f(A)|\Omega_{k+1}| \leq \int_{\Omega_{k+1}\setminus\Omega_k} f(A+\mathcal{E}(\varphi_k u_h)(x))dx + \int_{\Omega_k} f(A+\mathcal{E}u_h(x))dx.$$

As f is non negative we have

$$\int_\Omega f(A+\mathcal{E}u_h(x))dx - f(A)|\Omega_{k+1}| \geq -\int_{\Omega_{k+1}\setminus\Omega_k} f(A+\mathcal{E}(\varphi_k u_h)(x))dx. \tag{3.5}$$

On the other hand

$$\begin{aligned}\int_{\Omega_{k+1}\setminus\Omega_k} f(A+\mathcal{E}(\varphi_k u_h)(x))dx &\leq C_3\int_{\Omega_{k+1}\setminus\Omega_k} (1+\|A+\mathcal{E}(\varphi_k u_h)(x)\|)\,dx \\ &\leq C|\Omega_{k+1}\setminus\Omega_k| + C_3\int_{\Omega_{k+1}\setminus\Omega_k}\varphi_k\,\|\mathcal{E}u_h\|\,dx \\ &+ C_3\int_{\Omega_{k+1}\setminus\Omega_k}|u_h|\,\|\nabla\varphi_k\|\,dx.\end{aligned}$$

Following (3.4) and (3.5),

$$\lim_{h\to+\infty}\int_\Omega f(A+\mathcal{E}u_h(x))dx - f(A)|\Omega_{k+1}| \geq -C|\Omega_{k+1}\setminus\Omega_k| - C_3\int_\Omega(\varphi_k-\varphi_{k-1})d\mu(x).$$

Now summing for $k=2,\cdots,i$ and dividing by $i-1$ we obtain

$$\begin{aligned}&\lim_{h\to+\infty}\int_\Omega f(A+\mathcal{E}u_h(x))dx - f(A)\frac{1}{i-1}\sum_{k=2}^{i}|\Omega_{k+1}| \\ &\geq -\frac{C}{i-1}\{|\Omega_{i+1}\setminus\Omega_2| + \int_\Omega(\varphi_i-\varphi_1)d\mu(x)\} \\ &\geq -\frac{C}{i-1}(|\Omega|+\mu(\Omega)).\end{aligned}$$

Then passing to the limit as $i\to+\infty$ we obtain the result of step 1.
Still assuming (3.3) with $A=0$ and $u_h\to u$ in $L^1(\Omega,\mathbf{R}^n)$, then there is a positive Radon measure $\mu\in M_b(\Omega)$ such that:

$$f(\mathcal{E}u_h)\rightharpoonup\mu \quad \text{in} \quad M_b(\Omega) \text{ weakly } * \quad \text{when} \quad h\to+\infty.$$

By the Lebesgue-Nikodym decomposition we have: $\mu = \mu_a \mathcal{L}^n \lfloor \Omega + \mu_s$, where $\mu_s \perp \mathcal{L}^n$ and μ_a is such that:

$$\mu_a(x_0) = \lim_{\rho \to 0^+} \frac{\mu(B(x_0,\rho))}{\mathcal{L}^n(B(x_0,\rho))} < +\infty \tag{3.6}$$

for $\mathcal{L}^n$-almost every $x_0 \in \Omega$.
Step 2: If we suppose that

$$\mu_a(x_0) \geq f(\mathcal{E}u(x_0)) \quad \text{for} \quad \mathcal{L}^n - \text{a.e. } x_0 \in \Omega, \tag{3.7}$$

then the conclusion of the theorem follows. In fact, let $\{\varphi_k\}_k$ be an increasing sequence of smooth cut-off functions with $0 \leq \varphi_k \leq 1$ and $\sup\limits_k \varphi_k = 1$ in Ω. Then

$$\begin{aligned}
\lim_{h \to +\infty} \int_\Omega f(\mathcal{E}u_h(x))dx &\geq \lim_{h \to +\infty} \int_\Omega \varphi_k(x) f(\mathcal{E}u_h(x))dx \\
&= \int_\Omega \varphi_k(x) d\mu(x) \\
&\geq \int_\Omega \varphi_k(x) d\mu_a(x) \\
&\geq \int_\Omega \varphi_k(x) f(\mathcal{E}u(x))dx.
\end{aligned}$$

Letting $k \to +\infty$, the result follows by the monotone convergence theorem.
Step 3: In this step we prove (3.7) using the "blow-up" technique by Fonseca-Müller [11].
Let $B = B(0,1)$ be the open unit ball centered at the origin 0, $B' \subset\subset B$ be a subdomain. Let x_0 be an approximate differentiability point of u, $\mathcal{E}u(x_0) = (\nabla u(x_0) + \nabla u(x_0)^T)/2$ and also such that (3.6) holds true.
Let us now consider the functions u_0, $w_0 : B \to \mathbf{R}^n$ defined by

$$u_0(x) := u(x_0) + \nabla u(x_0)x; \quad w_0(x) := \nabla u(x_0)x$$

and

$$\varphi \in C_0(B) \ , \ 0 \leq \varphi \leq 1 \text{ with } \varphi = 1 \text{ in } B'.$$

Then we have

$$\begin{aligned}
\mu_a(x_0) &= \lim_{\rho \to 0^+} \frac{\mu(B(x_0,\rho))}{|B(x_0,\rho)|} \\
&\geq \limsup_{\rho \to 0^+} \frac{1}{|B(x_0,\rho)|} \int_{B(x_0,\rho)} \varphi(\frac{x-x_0}{\rho}) d\mu(x)
\end{aligned}$$

$$
\begin{aligned}
&= \limsup_{\rho\to 0^+} \lim_{h\to+\infty} \frac{1}{|B(x_0,\rho)|}\int_{B(x_0,\rho)} \varphi\left(\frac{x-x_0}{\rho}\right) f(\mathcal{E}u_h(x))dx \\
&= \limsup_{\rho\to 0^+} \lim_{h\to+\infty} \frac{1}{|B|}\int_{B} \varphi(x) f(\mathcal{E}u_h(x_0+\rho x))dx \\
&\geq \limsup_{\rho\to 0^+} \limsup_{h\to+\infty} \frac{1}{|B|}\int_{B'} f(\mathcal{E}w_{h,\rho}(x))dx,
\end{aligned}
$$

where $w_{h,\rho} : B \to \mathbf{R}^n$ is defined by

$$
w_{h,\rho}(x) := \frac{1}{\rho}(u_h(x_0+\rho x) - u(x_0)) = \frac{1}{\rho}(u_h(x_0+\rho x) - u_0(\rho x)) + w_0(x).
$$

Following (2.2) in Theorem 2.2 we have

$$
\lim_{\rho\to 0^+} \lim_{h\to+\infty} \|w_{h,\rho} - w_0\|_{L^1(B,\mathbf{R}^n)} = 0.
$$

Then by a diagonalization argument we obtain a sequence

$$
w_k = w_{h_k,\rho_k} \quad \text{with} \quad \rho_k \to 0^+ \quad \text{and} \quad h_k \to +\infty \quad \text{as} \quad k\to+\infty
$$

such that $w_k \to w_0$ in $L^1(B, \mathbf{R}^n)$ and

$$
\begin{aligned}
\mu_a(x_0) &\geq \limsup_{k\to+\infty} \frac{1}{|B|}\int_{B'} f(\mathcal{E}w_k(x))dx \\
&\geq \frac{1}{|B|}\liminf_{k\to+\infty} \int_{B'} f(\mathcal{E}w_k(x))dx \\
&= \frac{1}{|B|}\liminf_{k\to+\infty} \int_{B'} f(\mathcal{E}u(x_0) + \mathcal{E}(w_k - w_0))dx \\
&\geq \frac{|B'|}{|B|} f(\mathcal{E}u(x_0))
\end{aligned}
$$

where the last inequality follows from the first step.
Then

$$
\mu_a(x_0) \geq \frac{|B'|}{|B|} f(\mathcal{E}u(x_0)) \qquad \mathcal{L}^n - \text{a.e. } x_0 \in \Omega.
$$

By the arbitrariness of B' we obtain (3.7) and this achieves the proof of Theorem 3.1. $\square$

4 An Example of symmetric quasi-convex function

Let us introduce for a given function $f : M_s^{n\times n} \to [0, +\infty[$, the notion of *symmetric quasi-convex envelope* defined as:

$$SQf(A) = \sup\{g \le f;\ g \text{ is symmetric quasi-convex}\}. \tag{4.1}$$

Following the explicit characterization of the *quasi-convex envelope* by Dacorogna [7], one can show that:

$$SQf(A) = \inf\left\{ \fint_D f(A + \mathcal{E}\varphi(x))dx;\ \varphi \in W_0^{1,\infty}(D, \mathbf{R}^n) \right\}, \tag{4.2}$$

for any $A \in M_s^{n\times n}$ and for any bounded open subset D of $\mathbf{R}^n$. One can also verify that the function defined in (4.2) is independent of the open subset D.
As we have said in the introduction, we are looking for a non convex function which is *symmetric quasi-convex*.

Theorem 4.1 *Let $p \ge 1$, $K = \{I, -I\}$ and $f : M_s^{2\times 2} \to [0, +\infty[$ defined as:*

$$f(A) = d(A, K)^p = \min(\|A - I\|^p, \|A + I\|^p);$$

where I is the identity matrix in $M^{2\times 2}$. Then, $SQf(0) > 0$. Moreover, SQf is not a convex function.

Proof:
Suppose that $SQf(0) = 0$, then by (4.2 there exists for some bounded open subset Ω of $\mathbf{R}^2$ with regular boundary a sequence $\{\varphi_j\} \subset \mathcal{C}_0^\infty(\Omega, \mathbf{R}^2)$ such that

$$\lim_{j\to\infty} \int_\Omega f(\mathcal{E}\varphi_j(x))dx = 0. \tag{4.3}$$

Since

$$d(A, K)^p \ge C_p(|A_{11} - A_{22}|^p + |A_{12}|^p),$$

with $C_p = 2^{-\frac{p+2}{2}}$.By (4.3) we get

$$\lim_{j\to\infty} \int_\Omega (|\partial_1\varphi_{j1} - \partial_2\varphi_{j2}|^p + |\partial_1\varphi_{j2} + \partial_2\varphi_{j1}|^p)dx = 0. \tag{4.4}$$

We also have

$$\begin{cases} \Delta\varphi_{j1} = \partial_1(\partial_1\varphi_{j1} - \partial_2\varphi_{j2}) + \partial_2(\partial_2\varphi_{j1} + \partial_1\varphi_{j2}); \\ \Delta\varphi_{j2} = \partial_1(\partial_2\varphi_{j1} + \partial_1\varphi_{j2}) - \partial_2(\partial_1\varphi_{j1} - \partial_2\varphi_{j2}). \end{cases} \tag{4.5}$$

Since the second members of (4.5) go to zero in $W^{-1,p}(\Omega, \mathbf{R}^2)$ strongly, using the L^p-strong estimates of the Laplace operator for $p > 1$ (see e.g., Morrey [16]), we deduce that $\{\|D\varphi_j\|\}$ converges strongly to zero in $L^p(\Omega)$ for $p > 1$. Then also $\{\|\mathcal{E}\varphi_j\|\}$ strongly converges to zero in $L^p(\Omega)$ and this contradicts (4.3).
In the case $p = 1$ we have from the L^1-weak estimate of the laplacian (see e.g., Stein [17]) that

$$\lim_{j\to\infty} |\{x \in \Omega,\ \|D\varphi_j(x)\| \geq \alpha\}| = 0, \tag{4.6}$$

for any $\alpha > 0$. Then

$$\lim_{j\to\infty} |\{x \in \Omega,\ \|\mathcal{E}\varphi_j(x)\| \geq \alpha\}| = 0. \tag{4.7}$$

On the other hand (4.3) implies

$$\lim_{j\to\infty} \int_\Omega \|\mathcal{E}\varphi_j(x) - A_j(x)\|\, dx = 0 \quad \text{for some } A_j : \Omega \to K.$$

Since

$$\int_{\{\|\mathcal{E}\varphi_j(x)\|\geq\alpha\}} \|\mathcal{E}\varphi_j(x)\|\, dx \leq \int_\Omega \|\mathcal{E}\varphi_j(x) - A_j\|\, dx + \|I\|\, |\{\|\mathcal{E}\varphi_j(x)\| \geq \alpha\}|,$$

we have

$$\lim_{j\to\infty} \int_{\{\|\mathcal{E}\varphi_j(x)\|\geq\alpha\}} \|\mathcal{E}\varphi_j(x)\|\, dx = 0. \tag{4.8}$$

But from (4.7) we have up to a subsequence $\|\mathcal{E}\varphi_j(x)\| \to 0$ for $\mathcal{L}^n$ – a.e. $x \in \Omega$. Then

$$\lim_{j\to\infty} \int_{\{\|\mathcal{E}\varphi_j(x)\|\leq\alpha\}} \|\mathcal{E}\varphi_j(x)\|\, dx = 0. \tag{4.9}$$

From (4.8) and (4.9) we deduce that

$$\lim_{j\to\infty} \int_\Omega \|\mathcal{E}\varphi_j(x)\|\, dx = 0,$$

and this contradicts (4.3). So we conclude that $SQf(0) > 0$.
If SQf were convex, then

$$\begin{aligned} SQf(0) &\leq \frac{1}{2}(SQf(I) + SQf(-I)) \\ &\leq \frac{1}{2}(f(I) + f(-I)) = 0. \end{aligned}$$

So the function SQf is an example of a non convex function which is symmetric quasi-convex. □

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