The homological singularities of maps in trace spaces between manifolds

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Abstract. *We deal with mappings defined between Riemannian manifolds that belong to a trace space of Sobolev functions. The homological singularities of any such map are represented by a current defined in terms of the boundary of its graph. Under suitable topological assumptions on the domain and target manifolds, we show that the non triviality of the singular current is the only obstruction to the strong density of smooth maps. Moreover, we obtain an upper bound for the minimal integral connection of the singular current that depends on the fractional norm of the mapping.*

1 Introduction

In the last years there has been a growing interest in studying the fractional Sobolev classes of mappings defined between manifolds. In this framework, [6, 9, 10, 11, 12, 13, 35] deal with properties of general fractional spaces. As to trace spaces, we address to [5, 27, 32] for the so called *extension problem*, to [8, 25] for the analysis of the *minimal connections* of the *singularities*, and to [7, 8] for the lifting problem. Moreover, we refer to [4, 8, 15, 21, 19, 22, 23, 34, 36, 38, 41] for questions about *density of smooth maps*, relaxed energies and related variational problems. Finally, topological compactness theorems are presented in [33, 39].

In this paper, we let $\mathcal{X}$ and $\mathcal{Y}$ be two smooth, connected, compact, oriented Riemannian manifolds that are isometrically embedded into $\mathbb{R}^l$ and $\mathbb{R}^N$, respectively. We shall equip $\mathcal{X}$ and $\mathcal{Y}$ with the metric induced by the Euclidean norms on the ambient spaces, and we let $n := \dim \mathcal{X}$. We shall also assume that the target manifold $\mathcal{Y}$ is without boundary, the model case being $\mathcal{Y} = \mathbb{S}^{\mathfrak{p}-1}$, the unit $(\mathfrak{p}-1)$-sphere in $\mathbb{R}^{\mathfrak{p}}$. The domain manifold $\mathcal{X}$ may have a (possibly empty) smooth boundary $\partial\mathcal{X}$, a manifold of dimension $n-1$, the model cases being $\mathcal{X} = B^n$, the unit n-ball, or $\mathcal{X} = \mathbb{S}^n$, the unit n-sphere.

For the sake of simplicity, in the sequel we shall always denote

$$W^{1/p} := W^{1-1/p,p}, \qquad p > 1\,,$$

and we recall, see e.g. [1], that for any real exponent $p > 1$ the fractional Sobolev space $W^{1/p}(\mathcal{X})$ is the Banach space of real valued functions u in $L^p(\mathcal{X})$ which have finite $W^{1/p}$-seminorm

$$|u|^p_{1/p,\mathcal{X}} := \int_{\mathcal{X}}\int_{\mathcal{X}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+p-1}}\, d\mathcal{H}^n(x)\, d\mathcal{H}^n(y) < \infty\,,$$

where $\mathcal{H}^k$ is the k-dimensional *Hausdorff measure*, endowed with the norm

$$\|u\|^p_{1/p,\mathcal{X}} := \|u\|^p_{L^p(\mathcal{X})} + |u|^p_{1/p,\mathcal{X}}\,. \tag{1.1}$$

$W^{1/p}(\mathcal{X},\mathbb{R}^N)$ is the space of vector valued maps $u = (u^1,\dots,u^N)$ such that $u^j \in W^{1/p}(\mathcal{X})$ for every $j = 1,\dots,N$. If $\mathcal{X} = \partial\mathcal{M}$ for some smooth manifold $\mathcal{M}$, e.g., $\mathcal{X} = \mathbb{S}^n$, then $W^{1/p}(\partial\mathcal{M},\mathbb{R}^N)$ can be characterized as the space of functions u that are *traces* on $\partial\mathcal{M}$ of functions U in the Sobolev space $W^{1,p}(\mathcal{M},\mathbb{R}^N)$. More generally, since $\mathcal{X} \subset \mathbb{R}^l$, denoting by $\mathcal{C}^{n+1}$ the $(n+1)$-dimensional "cylinder"

$$\mathcal{C}^{n+1} := \mathcal{X} \times I \subset \mathbb{R}^l \times \mathbb{R}\,, \qquad I := [0,1]\,,$$

$W^{1/p}(\mathcal{X},\mathbb{R}^N)$ can be seen as the space of functions u that are traces on $\mathcal{X}\times\{0\}$ of functions U in the Sobolev space $W^{1,p}(\mathcal{C}^{n+1},\mathbb{R}^N)$. Since $\mathcal{Y} \subset \mathbb{R}^N$, we also let

$$W^{1/p}(\mathcal{X},\mathcal{Y}) := \{u \in W^{1/p}(\mathcal{X},\mathbb{R}^N) \mid u(x) \in \mathcal{Y} \quad \text{for } \mathcal{H}^n\text{-a.e. } x \in \mathcal{X}\}\,.$$

THE EXTENSION PROBLEM. The non-trivial topology of the domain and target manifolds plays a role is the so called *extension problem.* It is well-known that the class of functions $u : \mathcal{X} \to \mathcal{Y}$ that are traces on $\mathcal{X} \times \{0\}$ of Sobolev maps in $W^{1,1}(\mathcal{C}^{n+1}, \mathbb{R}^N)$ agrees with the class $L^1(\mathcal{X}, \mathcal{Y})$. Moreover, by Gagliardo's theorem [18] it turns out that each map $u \in L^1(\mathcal{X}, \mathcal{Y})$ is the trace on $\mathcal{X} \times \{0\}$ of a Sobolev map $U \in W^{1,1}(\mathcal{C}^{n+1}, \mathcal{Y})$. For this reason we shall restrict to the case of exponents $p > 1$. However, for e.g. $\mathcal{X} = \partial\mathcal{M}$, setting

$$T^{1/p}(\partial\mathcal{M}, \mathcal{Y}) := \{u \in W^{1/p}(\partial\mathcal{M}, \mathcal{Y}) \mid u = U_{|\partial\mathcal{M}} \quad \text{for some} \quad U \in W^{1,p}(\mathcal{M}, \mathcal{Y})\},$$

in general the strict inclusion $T^{1/p}(\partial\mathcal{M}, \mathcal{Y}) \subsetneq W^{1/p}(\partial\mathcal{M}, \mathcal{Y})$ holds, for $p > 1$.

This is related to the so called *extension property*: we say that property $\mathcal{P}(\mathcal{M}, \mathcal{Y})$ holds if *every continuous map* $u : \partial\mathcal{M} \to \mathcal{Y}$ *admits a continuous extension* $U : \mathcal{M} \to \mathcal{Y}$. In fact, extending results by Hardt-Lin [27], Bethuel-Demengel [5] showed that for $p \geq n + 1 = \dim(\mathcal{M})$

$$T^{1/p}(\partial\mathcal{M}, \mathcal{Y}) = W^{1/p}(\partial\mathcal{M}, \mathcal{Y}) \quad \iff \quad \mathcal{P}(\mathcal{M}, \mathcal{Y}) \quad \text{holds}.$$

Moreover, in the case $1 < p < n + 1$, the above equality holds provided that $\mathcal{Y}$ is $(\mathfrak{p} - 1)$-connected, where

$$\mathfrak{p} := [p] \qquad \text{the integer part of } p.$$

More precisely, denoting by $\pi_k(\mathcal{Y})$ the k-dimensional *free homotopy group* of $\mathcal{Y}$, they showed that if $\pi_k(\mathcal{Y}) = 0$ for every $k = 0, \ldots, \mathfrak{p} - 1$, then $T^{1/p}(\partial\mathcal{M}, \mathcal{Y}) = W^{1/p}(\partial\mathcal{M}, \mathcal{Y})$. Moreover, if $\pi_k(\mathcal{Y}) \neq 0$ for some $k = 0, \ldots, \mathfrak{p} - 1$, and $1 < p < n + 1$, they also showed the existence of a manifold $\mathcal{M}$ of dimension $n + 1$ for which the strict inclusion $T^{1/p}(\partial\mathcal{M}, \mathcal{Y}) \subsetneq W^{1/p}(\partial\mathcal{M}, \mathcal{Y})$ holds.

We remark that it is a difficult task to solve the extension problem $\mathcal{P}(\mathcal{M}, \mathcal{Y})$, see e.g. [31]. Of course, for $\mathcal{M} = B^{n+1}$ we have that $\mathcal{P}(B^{n+1}, \mathcal{Y})$ holds if and only if $\pi_n(\mathcal{Y})$ is trivial. We also address to [32] for the analysis of the topological obstructions to the above mentioned extension problem.

THE $\mathcal{E}_{1/p}$-ENERGY. In the sequel, instead of working with the $W^{1/p}$-norm (1.1), we shall work with the equivalent energy $\mathcal{E}_{1/p}(u)$ defined as follows. We define

$$\text{Ext}(u) \in W^{1,p}(\mathcal{C}^{n+1}, \mathbb{R}^N),$$

the *extension* of a map u in $W^{1/p}(\mathcal{X}, \mathcal{Y})$, as the Hölder continuous function which minimizes the *p-energy integral*

$$\mathbf{D}_p(U) := \frac{1}{p^{p/2}} \int_{\mathcal{C}^{n+1}} |DU(x,t)|^p \, d\mathcal{H}^{n+1}(x,t)$$

among all functions $U \in W^{1,p}(\mathcal{C}^{n+1}, \mathbb{R}^N)$ that agree with u on $\mathcal{X} \times \{0\}$.

We also set

$$\mathcal{E}_{1/p}(u) := \mathbf{D}_p(\text{Ext}(u)),$$

so that clearly $\mathcal{E}_{1/p}(u) \simeq ||u||_{1/p,\mathcal{X}}$. More precisely, by uniform convexity, and since $\mathcal{Y}$ is compact, it is readily checked that for maps in $W^{1/p}(\mathcal{X}, \mathcal{Y})$ the strong convergence $u_k \to u$ in $W^{1/p}$ is equivalent to the a.e. convergence plus the convergence of the energies $\mathcal{E}_{1/p}(u_k) \to \mathcal{E}_{1/p}(u)$.

MINIMAL CONNECTIONS. Let Ω be an open subset of $\mathbb{R}^n$ and $\mathbb{S}^{\mathfrak{p}-1}$ be the unit $(\mathfrak{p}-1)$-sphere in $\mathbb{R}^{\mathfrak{p}}$, so that $W^{1/\mathfrak{p}}(\Omega, \mathbb{S}^{\mathfrak{p}-1})$ agrees with the subclass of vector valued maps $u \in L^{\mathfrak{p}}(\Omega, \mathbb{R}^{\mathfrak{p}})$ such that $|u(x)| = 1$ for $\mathcal{L}^n$-a.e. $x \in \Omega$ and each component u^j is the trace on $\Omega \times \{0\}$ of some Sobolev map $U^j \in W^{1,\mathfrak{p}}(\Omega \times [0,1])$.

For $n \geq \mathfrak{p} \geq 2$ integers, Hang-Lin [25] defined the *singularities* of a map $u \in W^{1/\mathfrak{p}}(\Omega, \mathbb{S}^{\mathfrak{p}-1})$ by the $(n - \mathfrak{p})$-dimensional *current* J_u in $\mathcal{D}_{n-\mathfrak{p}}(\Omega)$ acting on compactly supported smooth $(n - \mathfrak{p})$-forms in Ω as

$$J_u(\phi) := \frac{1}{|B^{\mathfrak{p}}|} \int_{\Omega \times [0,1]} d\widetilde{\phi} \wedge U^\#(dy^1 \wedge \cdots \wedge dy^{\mathfrak{p}}), \quad \phi \in \mathcal{D}^{n-\mathfrak{p}}(\Omega). \tag{1.2}$$

Here $|B^{\mathfrak{p}}|$ is the measure of the unit $\mathfrak{p}$-ball, $U := \text{Ext}(u) \in W^{1,\mathfrak{p}}(\Omega \times [0,1], \mathbb{R}^{\mathfrak{p}})$, and $d\widetilde{\phi}$ denotes the *differential* of any smooth $(n - \mathfrak{p})$-form $\widetilde{\phi}$ in $\Omega \times [0,1[$ that extends ϕ, i.e., such that $\widetilde{\phi}_{|\Omega} = \phi$.

They showed that the *minimal integral connection* of the singularities of u is bounded in terms of the $W^{1/\mathfrak{p}}$-seminorm of u. More precisely, they proved that for every $u \in W^{1/\mathfrak{p}}(\Omega, \mathbb{S}^{\mathfrak{p}-1})$ there exists an *integer multiplicity* (say i.m.) *rectifiable current* $L \in \mathcal{R}_{n-\mathfrak{p}+1}(\Omega \times [0,1])$ such that

$$(\partial L) \llcorner (\Omega \times [0,1[) = J_u \quad \text{and} \quad \mathbf{M}(L) \le c\, |u|_{1/\mathfrak{p},\Omega}\,, \tag{1.3}$$

where $c = c(n, \mathfrak{p}) > 0$ is an absolute constant.

As to the minimal integral connections of the singularities of $W^{1/2}$-maps with values into the unit circle $\mathbb{S}^1$, we also refer to [8].

Density properties. Another relevant question recently studied has been to determine whether smooth maps from $\mathcal{X}$ to $\mathcal{Y}$ are sequentially dense in $W^{1/p}(\mathcal{X}, \mathcal{Y})$ with respect to the $W^{1/p}$-norm. Denoting

$$H_S^{1/p}(\mathcal{X}, \mathcal{Y}) := \{u \in W^{1/p}(\mathcal{X}, \mathcal{Y}) \quad | \quad \begin{array}{l}\text{there exists } \{u_k\} \subset C^\infty(\mathcal{X}, \mathcal{Y}) \\ \text{such that } u_k \to u \text{ strongly in } W^{1/p}\}\,,\end{array} \tag{1.4}$$

it is well-known, see [4, 8], that

$$H_S^{1/p}(\mathcal{X}, \mathcal{Y}) = W^{1/p}(\mathcal{X}, \mathcal{Y}) \qquad \text{if} \quad p \ge n+1\,. \tag{1.5}$$

On the other hand, in case of higher dimension $n > p-1$, in general the strict inclusion

$$H_S^{1/p}(\mathcal{X}, \mathcal{Y}) \subsetneq W^{1/p}(\mathcal{X}, \mathcal{Y})$$

holds. More precisely, Bethuel [4] noticed that if $\pi_{\mathfrak{p}-1}(\mathcal{Y}) \neq 0$, and $n+1 > p > 1$, even for $\mathcal{X} = B^n$ or $\mathcal{X} = \mathbb{S}^n$ there exist functions $u \in W^{1/p}(\mathcal{X}, \mathcal{Y})$ which cannot be approximated in $W^{1/p}$ by sequences of smooth maps in $W^{1/p}(\mathcal{X}, \mathcal{Y})$.

In order to obtain a suitable dense class of "smooth" maps, in the case $n+1 > p > 1$, i.e., $n \ge \mathfrak{p}$, Bethuel introduced in [4] the classes $\mathcal{R}^\infty_{1/p}(\mathcal{X}, \mathcal{Y})$ and $\mathcal{R}^0_{1/p}(\mathcal{X}, \mathcal{Y})$. They are given by all the maps $u \in W^{1/p}(\mathcal{X}, \mathcal{Y})$ which are smooth, respectively continuous, except on a singular set $\Sigma(u)$ of the type

$$\Sigma(u) = \bigcup_{i=1}^r \Sigma_i\,, \qquad r \in \mathbb{N}\,, \tag{1.6}$$

where Σ_i is a smooth $(n-\mathfrak{p})$-dimensional subset of $\mathcal{X}$ with smooth boundary, if $n \ge \mathfrak{p}+1$, and Σ_i is a point if $n = \mathfrak{p}$. The following density property holds true:

Theorem 1.1 *For every* $1 < p < n+1$*, where* $n = \dim(\mathcal{X})$*, the class* $\mathcal{R}^\infty_{1/p}(\mathcal{X}, \mathcal{Y})$ *is sequentially dense in* $W^{1/p}(\mathcal{X}, \mathcal{Y})$*.*

In the case $n = p = 2$, Theorem 1.1 was proved in [41], compare also [8], for $\mathcal{X} = \mathbb{S}^2$ and with $\mathcal{Y} = \mathbb{S}^1$, the standard unit circle. For $p = 2$, it was extended in [22] to the case $\mathcal{X} = B^n$ or $\mathbb{S}^n$, in higher dimension $n \ge 2$ and for general target manifolds $\mathcal{Y}$, see also [24]. A complete proof in the general case is given in [38]. Moreover, in [38] we also proved:

Proposition 1.2 *If* $n \le p < n+1$*, and* $p > 1$*, then* $H_S^{1/p}(\mathcal{X}, \mathcal{Y}) = W^{1/p}(\mathcal{X}, \mathcal{Y})$ *if and only if* $\pi_{n-1}(\mathcal{Y}) = 0$*.*

In case of higher dimension $n > p$, i.e., $n \ge \mathfrak{p}+1$, following observations by Hang-Lin [26], we showed in [38] that the possibly non-trivial topology of the domain manifold $\mathcal{X}$ plays a role. To this purpose, we recall that $\mathcal{X}$ is said to satisfy the *k-extension property with respect to* $\mathcal{Y}$, where $k \in \mathbb{N}$, if for any given CW-complex X on $\mathcal{X}$, denoting by X^k its k-dimensional skeleton, any continuous map $f : X^{k+1} \to \mathcal{Y}$ is such that its restriction to X^k can be extended to a continuous map from $\mathcal{X}$ into $\mathcal{Y}$. In [38] we obtained the following characterization, that we state here for the case $\partial\mathcal{X} = \emptyset$.

Theorem 1.3 *If* $n > p > 1$*, smooth maps in* $C^\infty(\mathcal{X}, \mathcal{Y})$ *are sequentially dense in* $W^{1/p}(\mathcal{X}, \mathcal{Y})$*, i.e.,* $H_S^{1/p}(\mathcal{X}, \mathcal{Y}) = W^{1/p}(\mathcal{X}, \mathcal{Y})$*, if and only if* $\pi_{\mathfrak{p}-1}(\mathcal{Y}) = 0$ *and* $\mathcal{X}$ *satisfies the* $(\mathfrak{p}-1)$*-extension property with respect to* $\mathcal{Y}$*.*

As a consequence of Theorem 1.3 we also have:

Corollary 1.4 *If* $n > p > 1$ *and* $\pi_k(\mathcal{Y}) = 0$ *for every integer* $k = \mathfrak{p}-1, \dots, n-1$*, then* $H^{1/p}_S(\mathcal{X},\mathcal{Y}) = W^{1/p}(\mathcal{X},\mathcal{Y})$.

Corollary 1.5 *Let* $n > p \geq 2$ *and* $k = 1, \dots, \mathfrak{p}-1$ *integer. If* $\pi_i(\mathcal{X}) = 0$ *for every* $i = 0, \dots, k-1$ *and* $\pi_j(\mathcal{Y}) = 0$ *for every* $j = k, \dots, \mathfrak{p}-1$*, then* $H^{1/p}_S(\mathcal{X},\mathcal{Y}) = W^{1/p}(\mathcal{X},\mathcal{Y})$.

In particular, in the model case $\mathcal{X} = \mathbb{S}^n$ we have:

Corollary 1.6 *If* $n+1 > p > 1$*, smooth maps in* $C^\infty(\mathbb{S}^n,\mathcal{Y})$ *are sequentially dense in* $W^{1/p}$*, i.e.,* $H^{1/p}_S(\mathbb{S}^n,\mathcal{Y}) = W^{1/p}(\mathbb{S}^n,\mathcal{Y})$*, if and only if* $\pi_{\mathfrak{p}-1}(\mathcal{Y}) = 0$.

Assume now that $1 < p < 2$. According to Proposition 1.2 and Theorem 1.3, since $\mathcal{Y}$ is connected, we have that $\pi_0(\mathcal{Y}) = 0$ and that $\mathcal{X}$ trivially satisfies the 0-extension property with respect to $\mathcal{Y}$. Therefore, we immediately obtain:

Corollary 1.7 *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be two smooth, connected, compact, oriented Riemannian manifolds, with* $n := \dim(\mathcal{X}) \geq 1$ *and* $\mathcal{Y}$ *without boundary. Then for every* $1 < p < 2$ *we have*

$$H^{1/p}_S(\mathcal{X},\mathcal{Y}) = W^{1/p}(\mathcal{X},\mathcal{Y}) .$$

Finally, if the manifold $\mathcal{X}$ has a non-zero boundary, analogous density results can be obtained for maps in $W^{1/p}(\mathcal{X},\mathcal{Y})$ with prescribed boundary data, see Remark 6.6 below.

Plan of the paper. On account of Corollary 1.7, in this paper we shall assume that $p \geq 2$.

In Sec. 2, using some background from Geometric Measure Theory [16, 42], and from the theory of Cartesian currents by Giaquinta-Modica-Souček [20, 21], we shall introduce the class of *n-currents* G_u *in* $\mathcal{X}\times\mathcal{Y}$ *carried by the graph* of a function $u \in W^{1/p}(\mathcal{X},\mathcal{Y})$, Definition 2.2. They are actually "semi-currents", i.e., linear functionals acting on compactly supported smooth n-forms $\omega = \omega(x,y)$ in $\mathcal{X}\times\mathcal{Y}$ that contain *at most* $\mathfrak{p}-1$ differentials in the "vertical" y-directions.

In Sec. 3, we shall then introduce the current $\mathbb{P}(u)$ that describes the homological singularities of u; it will be defined in terms of the *boundary* ∂G_u of the current G_u. Of course, due to the density property (1.5), we shall restrict our analysis to the higher dimension $n > p-1$, i.e., $n \geq \mathfrak{p} := [p] \geq 2$.

Denoting by $\mathcal{H}^{sph}_{\mathfrak{p}-1}(\mathcal{Y})$ the *spherical subgroup* of the *singular homology group* $\mathcal{H}_{\mathfrak{p}-1}(\mathcal{Y})$, see (3.10), we shall always *assume that both* $\mathcal{H}_{\mathfrak{p}-1}(\mathcal{Y})$ *and the quotient space* $\mathcal{H}_{\mathfrak{p}-1}(\mathcal{Y})/\mathcal{H}^{sph}_{\mathfrak{p}-1}(\mathcal{Y})$ *are torsion-free*, compare [20, Vol. II, Sec. 5.4.2]. We shall define the homological singularities $\mathbb{P}(u)$ of a map $u \in W^{1/p}(\mathcal{X},\mathcal{Y})$ as a homology map in $\mathcal{D}_{n-\mathfrak{p}}(\mathcal{X};\mathcal{H}_{\mathfrak{p}-1}(\mathcal{Y};\mathbb{R}))$, i.e., an $(n-\mathfrak{p})$-current on $\mathcal{X}$ with values in the *real homology group* $\mathcal{H}_{\mathfrak{p}-1}(\mathcal{Y};\mathbb{R})$, see (3.1) and (3.2). In the model case $\mathcal{X} = \Omega \subset \mathbb{R}^n$ open, and $\mathcal{Y} = \mathbb{S}^{\mathfrak{p}-1}$, our definition of homological singularities agrees with the one given by Hang-Lin [25], see (1.2).

Sec. 4 is then dedicated to the subclass of maps $u \in W^{1/p}(\mathcal{X},\mathcal{Y})$ satisfying the condition $\mathbb{P}(u) = 0$. This condition is equivalent to requiring that the graph of u has no "holes", i.e., that the boundary current ∂G_u is zero when tested on a suitable subclass of compactly supported $(n-1)$-forms in $\mathcal{X}\times\mathcal{Y}$. We are therefore led introduce the class of *Cartesian maps* in $W^{1/p}(\mathcal{X},\mathcal{Y})$

$$\mathrm{cart}^{1/p}(\mathcal{X},\mathcal{Y}) := \{u \in W^{1/p}(\mathcal{X},\mathcal{Y}) \mid \mathbb{P}(u) = 0\} .$$

We shall first show, Theorem 4.2, that any map $u \in W^{1/p}(\mathcal{X},\mathcal{Y})$ with no homological singularities, i.e., such that $\mathbb{P}(u) = 0$, can be strongly approximated by maps $u_k \in R^0_{1/p}(\mathcal{X},\mathcal{Y})$ satisfying the same condition $\mathbb{P}(u_k) = 0$, compare Theorem 1.1.

Trivially $\mathbb{P}(u) = 0$ if u is smooth. Moreover, the strong convergence $u_k \to u$ in $W^{1/p}$ implies the weak convergence $G_{u_k} \rightharpoonup G_u$ as currents, which preserves the condition $\mathbb{P}(u_k) = 0$, see Remark 2.4 below. On account of (1.4), this clearly yields

$$H^{1/p}_S(\mathcal{X},\mathcal{Y}) \subset \mathrm{cart}^{1/p}(\mathcal{X},\mathcal{Y}) . \tag{1.7}$$

Of course, the possible occurrence of the equality in (1.7) is due to the fact that the current $\mathbb{P}(u)$, i.e., *the homological singularities describe all the obstructions to the density of smooth maps.* We recall that the first result in this direction was obtained by Bethuel [3] for the class of Sobolev maps $W^{1,2}(B^3,\mathbb{S}^2)$.

In fact, Theorem 4.3, under suitable topological hypotheses on $\mathcal{X}$ and $\mathcal{Y}$ we shall obtain in any dimension $n\geq\mathfrak{p}:=[p]\geq 2$ that

$$H_S^{1/p}(\mathcal{X},\mathcal{Y})=\mathrm{cart}^{1/p}(\mathcal{X},\mathcal{Y})\,.$$

More precisely, since we use arguments from Theorem 1.3, in Theorem 4.3 we shall assume (in the case of dimension $n\geq\mathfrak{p}+1$) that $\mathcal{X}$ satisfies the $(\mathfrak{p}-1)$-extension property with respect to $\mathcal{Y}$. Moreover, in the case $\mathfrak{p}\geq 3$, we shall assume that for any base point $y_0\in\mathcal{Y}$ the *Hurewicz homomorphism* from the $(\mathfrak{p}-1)^{th}$ homotopy group $\pi_{\mathfrak{p}-1}(\mathcal{Y};y_0)$ onto the $(\mathfrak{p}-1)^{th}$ real homology group $\mathcal{H}_{\mathfrak{p}-1}(\mathcal{Y};\mathbb{R})$ is injective. Alternatively, in the case $\mathfrak{p}=2$ we shall also assume that the first homotopy group $\pi_1(\mathcal{Y})$ is commutative.

Notice that if the injectivity hypothesis on the Hurewicz maps fails to hold (or if $\pi_1(\mathcal{Y})$ is not commutative, for $\mathfrak{p}=2$), even in the case $\mathcal{X}=B^{\mathfrak{p}}$, there exist functions u in $\mathrm{cart}^{1/p}(B^{\mathfrak{p}},\mathcal{Y})$, where $\mathfrak{p}:=[p]\geq 2$, smooth outside the origin, which cannot be approximated strongly in $W^{1/p}$ by smooth maps $u_k:B^{\mathfrak{p}}\to\mathcal{Y}$, i.e., such that $u\notin H_S^{1/p}(B^{\mathfrak{p}},\mathcal{Y})$, whence the strict inclusion holds in (1.7), see Example 4.4. Such maps have a *topological singularity* at the origin that cannot be seen by the homology, and similar examples with topological singularities of codimension $\mathfrak{p}$ can be obtained for any $n\geq\mathfrak{p}+1$. We address to [29, 30] for recent results in the analysis of the topological singular set of Sobolev maps.

In Sec. 5, we shall first recall the notion of real and integral mass, collecting some general facts about the connections of the homological singularities of maps in $W^{1/p}(\mathcal{X},\mathcal{Y})$. We then extend above mentioned result by Hang-Lin [25] about the minimal connection of the singularities J_u of maps u in $W^{1/\mathfrak{p}}(\Omega,\mathbb{S}^{\mathfrak{p}-1})$. More precisely, we will show, Proposition 5.5, that property (1.3) holds true for maps $u\in W^{1/p}(\mathcal{X},\mathbb{S}^{\mathfrak{p}-1})$, where $\mathcal{X}$ is a more general domain manifold of dimension $n\geq\mathfrak{p}$. Moreover, the integral connection L in (1.3) may be chosen with support in $\mathcal{X}$, provided that the upper bound of its mass contains an extra term, namely

$$\mathbf{M}(L)\leq c\left(\mathcal{E}_{1/p}(u)+\|\,\mathrm{Ext}(u)\|^p_{L^p(\mathcal{C}^{n+1})}\right), \tag{1.8}$$

where $\mathrm{Ext}(u)\in W^{1,p}(\mathcal{C}^{n+1},\mathbb{R}^{\mathfrak{p}})$ is the extension of u. Notice that

$$\mathcal{E}_{1/\mathfrak{p}}(u_k)+\|\,\mathrm{Ext}(u_k)\|^p_{L^p(\mathcal{C}^{n+1})}\to\mathcal{E}_{1/\mathfrak{p}}(u)+\|\,\mathrm{Ext}(u)\|^p_{L^p(\mathcal{C}^{n+1})}$$

provided that $u_k\to u$ strongly in $W^{1/p}$. In Example 5.9 below, we shall describe the geometric construction of the minimal connection. Moreover, we shall also consider the case of maps with prescribed boundary data, Proposition 5.7.

In Sec. 6, we shall then solve the same problem for more general target manifolds $\mathcal{Y}$ (the nontrivial case being the one of dimension $\dim(\mathcal{Y})\geq\mathfrak{p}-1$). Making use of some techniques from Pakzad-Rivière [40], we shall see, Theorem 6.1, that the homological singularities of every map u in $\mathcal{R}^\infty_{1/p}(\mathcal{X},\mathcal{Y})$ can be closed by an i.m. rectifiable $(n-\mathfrak{p}+1)$-current L in $\mathcal{X}$ with mass satisfying a bound as in (1.8).

Finally, in dimension $n=\mathfrak{p}$, or $n\geq\mathfrak{p}=2$, we are able to extend Theorem 6.1 to the whole class of functions $u\in W^{1/p}(\mathcal{X},\mathcal{Y})$, see Proposition 6.4. As we shall explain in Remark 5.4 below, it is a nontrivial matter to extend Theorem 6.1 to functions $u\in W^{1/p}(\mathcal{X},\mathcal{Y})$, for general target manifolds $\mathcal{Y}$, in the case $n\geq\mathfrak{p}+1$, when $\mathfrak{p}\geq 3$. However, in a forthcoming paper we will show that Theorem 6.1, in conjunction with a strong density result, yields the boundedness of the relaxed energy of maps in $W^{1/\mathfrak{p}}(\mathcal{X},\mathcal{Y})$.

2 Graphs of maps with finite $\mathbf{W^{1/p}}$-energy

In this section we define the current G_u carried by the graph of a function $u\in W^{1/p}(\mathcal{X},\mathcal{Y})$. We let $p\geq 2$ and denote $\mathfrak{p}:=[p]$. Moreover, we shall assume $n:=\dim(\mathcal{X})\geq\mathfrak{p}-1$.

If $U\in W^{1,p}(\mathcal{C}^{n+1},\mathbb{R}^N)$, and $\mathcal{H}^k$ is the k-dimensional *Hausdorff measure* in $\mathcal{C}^{n+1}$, we denote by

$$\mathbf{D}_p(U):=\frac{1}{p^{p/2}}\int_{\mathcal{C}^{n+1}}|DU(z)|^p\,d\mathcal{H}^{n+1}(z)\,,\qquad z=(x,t)$$

the p-*energy* of u. For maps $u \in W^{1/p}(\mathcal{X}, \mathbb{R}^N)$ and $U \in W^{1,p}(\mathcal{C}^{n+1}, \mathbb{R}^N)$, we write $\mathbf{T}(U) = u$ if $U = u$ on $\mathcal{X} \times \{0\}$. Also, for $u \in W^{1/p}(\mathcal{X}, \mathbb{R}^N) \cap L^\infty$, we shall denote by $\mathrm{Ext}(u)$ a function in $W^{1,p}(\mathcal{C}^{n+1}, \mathbb{R}^N) \cap L^\infty$ that minimizes the p-energy $\mathbf{D}_p(U)$ among all Sobolev maps $U \in W^{1,p}(\mathcal{C}^{n+1}, \mathbb{R}^N) \cap L^\infty$ such that $\mathbf{T}(U) = u$. Notice that $W^{1/p}(\mathcal{X}, \mathcal{Y}) \subset W^{1/p}(\mathcal{X}, \mathbb{R}^N) \cap L^\infty$, as $\mathcal{Y} \subset \mathbb{R}^N$ is compact.

$\mathcal{D}_{k,r}$-CURRENTS. Recall that $n = \dim \mathcal{X}$ and set $M := \dim(\mathcal{Y})$. Every compactly supported smooth differential k-form $\omega \in \mathcal{D}^k(\mathcal{X} \times \mathcal{Y})$, where $k \leq n$, splits as a sum $\omega = \sum_{j=0}^{\underline{k}} \omega^{(j)}$, $\underline{k} := \min(k, M)$, where the $\omega^{(j)}$'s are the k-forms that contain exactly j differentials in the vertical $\mathcal{Y}$ variables. For fixed $r = 1, \dots, \underline{k}$ we denote by $\mathcal{D}^{k,r}(\mathcal{X} \times \mathcal{Y})$ the subspace of $\mathcal{D}^k(\mathcal{X} \times \mathcal{Y})$ of k-forms of the type $\omega = \sum_{j=0}^r \omega^{(j)}$, and by $\mathcal{D}_{k,r}(\mathcal{X} \times \mathcal{Y})$ the dual space of $\mathcal{D}^{k,r}(\mathcal{X} \times \mathcal{Y})$. Of course we have $\mathcal{D}_{k,k} = \mathcal{D}_k$, the space of all k-currents. Moreover, a sequence $\{T_k\} \subset \mathcal{D}_{k,r}(\mathcal{X} \times \mathcal{Y})$ is said to converges *weakly in* $\mathcal{D}_{k,r}$, say $T_k \rightharpoonup T$, if $T_k(\omega) \to T(\omega)$ for every $\omega \in \mathcal{D}^{k,r}(\mathcal{X} \times \mathcal{Y})$. The class $\mathcal{D}_{k,r}(\mathcal{X} \times \mathcal{Y})$ is closed under the weak convergence in $\mathcal{D}_{k,r}$. A similar notation holds by replacing $\mathcal{X}$ and $\mathcal{Y}$ with $\mathcal{C}^{n+1}$ and $\mathbb{R}^N$, respectively.

Example 2.1 If $U \in W^{1,p}(\mathcal{C}^{n+1}, \mathbb{R}^N)$, then G_U is a well-defined $(n+1, \mathfrak{p})$-current in $\mathcal{D}_{n+1,\mathfrak{p}}(\mathcal{C}^{n+1} \times \mathbb{R}^N)$ and, in an approximate sense, $G_U := (Id \bowtie U)_\# [\![\mathcal{C}^{n+1}]\!]$, where $(Id \bowtie U)(z) := (z, U(z))$, compare [20]. If e.g. $\omega = \gamma \wedge \eta \in \mathcal{D}^{n+1}(\mathcal{C}^{n+1} \times \mathbb{R}^N)$, where $\gamma \in \mathcal{D}^{n+1-h}(\mathcal{C}^{n+1})$, $\eta \in \mathcal{D}^h(\mathbb{R}^N)$, and $0 \leq h \leq \min\{n+1, M, \mathfrak{p}\}$, we have

$$G_U(\gamma \wedge \eta) = [\![\mathcal{C}^{n+1}]\!]((Id \bowtie U)^\#(\gamma \wedge \eta)) = [\![\mathcal{C}^{n+1}]\!](\gamma \wedge U^\# \eta) = \int_{\mathcal{C}^{n+1}} \gamma \wedge U^\# \eta \,.$$

Setting moreover

$$\|G_U\| := \sup\{G_U(\omega) \mid \omega \in \mathcal{D}^{n+1,\mathfrak{p}}(\mathcal{C}^{n+1} \times \mathbb{R}^N),\ \|\omega\| \leq 1\}\,,$$

where $\|\omega\|$ is the *comass* norm of ω, by the parallelogram inequality we infer that

$$\|G_U\| \leq C\,(\mathcal{H}^n(\mathcal{X}) + \mathbf{D}_p(U))$$

for some absolute constant $C = C(n, p, \mathcal{X}) > 0$, not depending on U.

BOUNDARIES. The exterior differential of forms in $\mathcal{X} \times \mathcal{Y}$ splits into a horizontal and a vertical differential, $d = d_x + d_y$. Of course $\partial_x T(\omega) := T(d_x \omega)$ defines a horizontal boundary operator $\partial_x : \mathcal{D}_{k,r}(\mathcal{X} \times \mathcal{Y}) \to \mathcal{D}_{k-1,r}(\mathcal{X} \times \mathcal{Y})$. However, the vertical differential $d_y \omega$ of any form $\omega \in \mathcal{D}^{k-1,r}(\mathcal{X} \times \mathcal{Y})$ belongs to $\mathcal{D}^{k,r}(\mathcal{X} \times \mathcal{Y})$ if and only if $d_y \omega^{(r)} = 0$. Therefore, for every $T \in \mathcal{D}_{k,r}(\mathcal{X} \times \mathcal{Y})$ the vertical boundary operator $\partial_y T$ makes sense only as an element of the dual space of $\mathcal{Z}^{k-1,r}(\mathcal{X} \times \mathcal{Y})$, where

$$\mathcal{Z}^{k,r}(\mathcal{X} \times \mathcal{Y}) := \{\omega \in \mathcal{D}^{k,r}(\mathcal{X} \times \mathcal{Y}) \mid d_y \omega^{(r)} = 0\}\,.$$

For future use, we also set

$$\mathcal{B}^{k,r}(\mathcal{X} \times \mathcal{Y}) := \{\omega \in \mathcal{D}^{k,r}(\mathcal{X} \times \mathcal{Y}) \mid \exists\, \eta \in \mathcal{D}^{k-1,r-1}(\mathcal{X} \times \mathcal{Y}) \ : \ \omega^{(r)} = d_y \eta\}$$

and

$$\mathcal{H}^{k,r}(\mathcal{X} \times \mathcal{Y}) := \frac{\mathcal{Z}^{k,r}(\mathcal{X} \times \mathcal{Y})}{\mathcal{B}^{k,r}(\mathcal{X} \times \mathcal{Y})}\,,$$

and recall, see e.g. [24, Prop. 4.23], that

$$\mathcal{H}^{k,r}(\mathcal{X} \times \mathcal{Y}) \simeq \mathcal{D}^{k-r}(\mathcal{X}) \otimes \mathcal{H}^r_{dR}(\mathcal{Y})\,, \tag{2.1}$$

where $\mathcal{H}^r_{dR}(\mathcal{Y})$ is the r-th *de Rham cohomology group*.

GRAPHS OF $W^{1/p}$-MAPS. Recall that $\mathfrak{p} := [p] \geq 2$.

Definition 2.2 *To any map* $u \in W^{1/p}(\mathcal{X}, \mathbb{R}^N) \cap L^\infty$ *we associate an* $(n, \mathfrak{p}-1)$*-current* G_u *in* $\mathcal{D}_{n,\mathfrak{p}-1}(\mathcal{X} \times \mathbb{R}^N)$ *by setting*

$$G_u := (-1)^{n-1} \partial G_U \llcorner ((\mathcal{X} \times \{0\}) \times \mathbb{R}^N) \qquad \text{on} \quad \mathcal{D}^{n,\mathfrak{p}-1}(\mathcal{X} \times \mathbb{R}^N)\,, \tag{2.2}$$

where $U := \mathrm{Ext}(u) \in W^{1,p}(\mathcal{C}^{n+1}, \mathbb{R}^N)$*, see Example* 2.1.

In particular, if $u \in W^{1/p}(\mathcal{X},\mathcal{Y})$, by Federer's support theorem [16], we infer that *the current* G_u *in Definition* 2.2 *belongs to* $\mathcal{D}_{n,\mathfrak{p}-1}(\mathcal{X}\times\mathcal{Y})$.

In order to write more explicitly the formula (2.2), we first observe that by Stokes theorem, since $U = \mathrm{Ext}(u)$ is "smooth" in the interior of $\mathcal{C}^{n+1}$, we have

$$\partial G_U \llcorner \mathrm{int}(\mathcal{C}^{n+1})\times\mathbb{R}^N = 0\,. \tag{2.3}$$

Remark 2.3 Definition 2.2 does not depend on the choice of the Sobolev function $U \in W^{1,p}(\mathcal{C}^{n+1},\mathbb{R}^N)$ such that $\mathbf{T}(U)=u$, provided that (2.3) holds true.

Let $\eta:[0,1]\to[0,1]$ be a given smooth decreasing map such that $\eta(t)=1$ for $t\in[0,1/4]$ and $\eta(t)=0$ for $t\in[3/4,1]$. To every k-form $\varphi\in\mathcal{D}^k(\mathcal{X})$ we will associate the smooth k-form $\widetilde{\varphi}$ in $\mathcal{C}^{n+1}$ defined by

$$\widetilde{\varphi} := \varphi\wedge\eta\,.$$

Therefore, if $k+h=n$ and $h\le\mathfrak{p}-1$, by (2.3) and Example 2.1 we infer that for every $\widehat{\omega}\in\mathcal{D}^h(\mathbb{R}^N)$

$$(-1)^{n-1}G_u(\varphi\wedge\widehat{\omega}) = \partial G_U(\widetilde{\varphi}\wedge\widehat{\omega}) \tag{2.4}$$

whereas (2.4) *does not depend* on the choice of the cut-off function η.

In particular, if $u\in W^{1/p}(\mathcal{X},\mathcal{Y})$, since G_u belongs to $\mathcal{D}_{n,\mathfrak{p}-1}(\mathcal{X}\times\mathcal{Y})$, denoting by $i:\mathcal{Y}\hookrightarrow\mathbb{R}^N$ the injection map, for every $\varphi\in\mathcal{D}^k(\mathcal{X})$ and $\omega\in\mathcal{D}^h(\mathcal{Y})$, with k and h as above, we have that

$$G_u(\varphi\wedge\omega) = (-1)^{n-1}\partial G_U(\widetilde{\varphi}\wedge\widehat{\omega})\,, \tag{2.5}$$

where $\widehat{\omega}$ is *any* h-form in $\mathcal{D}^h(\mathbb{R}^N)$ such that $i^{\#}\widehat{\omega}=\omega$.

Remark 2.4 In the case $\mathfrak{p}=2$, Definition 2.2 is equivalent to the one from [21], that makes use of the theory of distributions, see also [23]. If $\mathfrak{p}\ge 3$, it is not clear if it may be given a definition of G_u in terms of distributions, i.e., that does not depend on the use of the extension map $\mathrm{Ext}(u)$. However, if $u\in W^{1/p}(\mathcal{X},\mathcal{Y})$ is smooth, then G_u agrees with the usual definition of current carried by the graph of u, i.e.,

$$G_u(\omega) = \int_{\mathcal{X}}(\mathrm{Id}\bowtie u)^{\#}\omega \qquad \forall\,\omega\in\mathcal{D}^{n,\mathfrak{p}-1}(\mathcal{X}\times\mathcal{Y})\,,$$

where $(Id\bowtie u)(x) := (x,u(x))$. We also remark that in general G_u is not an i.m. rectifiable current in $\mathcal{X}\times\mathcal{Y}$, even for $p=2$ and $n=1$.

Finally, we observe that if $\{u_k\}\subset W^{1/p}(\mathcal{X},\mathcal{Y})$ is a sequence that converges to u strongly in $W^{1/p}$, then $U_k := \mathrm{Ext}(u_k)$ converges to $U := \mathrm{Ext}(u)$ strongly in $W^{1,p}(\mathcal{C}^{n+1},\mathbb{R}^N)$. This yields that G_{U_k} converges to G_U weakly in $\mathcal{D}_{n+1,\mathfrak{p}}(\mathcal{C}^{n+1}\times\mathbb{R}^N)$ and G_{u_k} converges to G_u weakly in $\mathcal{D}_{n,\mathfrak{p}-1}(\mathcal{X}\times\mathcal{Y})$, by (2.2).

3 The homological singularities of $\mathbf{W}^{1/\mathfrak{p}}$-maps

In this section we define the current $\mathbb{P}(u)$ that describes the homological singularities of a map u in $W^{1/p}(\mathcal{X},\mathcal{Y})$. We shall assume that $n\ge\mathfrak{p}:=[p]\ge 2$.

THE BOUNDARY OF THE GRAPH OF $W^{1/p}$-MAPS. Since the $(\mathfrak{p}-1)^{th}$ homology group $\mathcal{H}_{\mathfrak{p}-1}(\mathcal{Y})$ is torsion-free, we may and will denote by $[\gamma_1],\dots,[\gamma_{\overline{s}}]$ a family of generators of $\mathcal{H}_{\mathfrak{p}-1}(\mathcal{Y})$. More precisely, the γ_s's are integral cycles (with finite mass) in $\mathcal{Z}_{\mathfrak{p}-1}(\mathcal{Y})$, such that

$$\mathcal{H}_{\mathfrak{p}-1}(\mathcal{Y}) = \left\{\sum_{s=1}^{\overline{s}} n_s\,[\gamma_s] \mid n_s\in\mathbb{Z}\right\}.$$

By using the de Rham duality between the $(\mathfrak{p}-1)^{th}$ real homology group and the $(\mathfrak{p}-1)^{th}$ cohomology group $\mathcal{H}^{\mathfrak{p}-1}_{dR}(\mathcal{Y})$, we denote by $[\sigma^1],\dots,[\sigma^{\overline{s}}]$ a dual basis in $\mathcal{H}^{\mathfrak{p}-1}_{dR}(\mathcal{Y})$, so that $\gamma_s(\sigma^r)=\delta_{sr}$.

The following proposition collects the main properties of the current ∂G_u.

Proposition 3.1 *Let* $u \in W^{1/p}(\mathcal{X}, \mathcal{Y})$. *Then for every form* $\alpha \in \mathcal{D}^{n-1,\mathfrak{p}-2}(\mathcal{X} \times \mathcal{Y})$ *we have* $\partial G_u(\alpha) = 0$, *and* $\partial_y \partial G_u(\widetilde{\alpha}) = 0$ *for every form* $\widetilde{\alpha} \in \mathcal{D}^{n-2,\mathfrak{p}-2}(\mathcal{X} \times \mathcal{Y})$.

PROOF: If $U := \mathrm{Ext}(u)$, by a standard density argument we infer that $\partial G_U = 0$ on forms in $\mathcal{D}^{n,\mathfrak{p}-1}(\mathcal{C}^{n+1} \times \mathbb{R}^N)$, i.e., with at most $\mathfrak{p}-1$ vertical differentials. Therefore, if $\alpha = \varphi \wedge \omega$, with $\varphi \in \mathcal{D}^k(\mathcal{X})$, $\omega \in \mathcal{D}^h(\mathcal{Y})$, $k + h = n - 1$, and $h \leq \mathfrak{p} - 2$, by (2.5) we have

$$\begin{aligned} \partial G_u(\alpha) &= G_u(d\varphi \wedge \omega) + (-1)^k G_u(\varphi \wedge d\omega) \\ &= (-1)^{n-1} \partial G_U(d\varphi \wedge \eta \wedge \widehat{\omega}) + (-1)^{n-1+k} \partial G_U(\varphi \wedge \eta \wedge d\widehat{\omega}) = 0 \,. \end{aligned}$$

Moreover, if $\widetilde{\alpha} = \varphi \wedge \omega$ as above, where this time $k + h = n - 2$ and $h \leq \mathfrak{p} - 2$,

$$\begin{aligned} \partial_y \partial G_u(\widetilde{\alpha}) &= (-1)^k \partial G_u(\varphi \wedge d\omega) = (-1)^k G_u(d\varphi \wedge d\omega) \\ &= (-1)^{n-1+k} \partial G_U(d\varphi \wedge \eta \wedge d\widehat{\omega}) = 0 \,. \end{aligned}$$

The assertion follows by linearity and density of forms $\varphi \wedge \omega$. □

Similarly to [20, Vol. II, Sec. 5.4.2], see also [24, Sec. 4.2], by Proposition 3.1 we infer that $\partial G_u = 0$ on $\mathcal{B}^{n-1,\mathfrak{p}-1}(\mathcal{X} \times \mathcal{Y})$, whence $\partial G_u(\omega)$ depends only on the cohomology class of $\omega \in \mathcal{Z}^{n-1,\mathfrak{p}-1}(\mathcal{X} \times \mathcal{Y})$. As a consequence ∂G_u induces a functional $(\partial G_u)_*$ on $\mathcal{H}^{n-1,\mathfrak{p}-1}(\mathcal{X} \times \mathcal{Y})$ given by

$$(\partial G_u)_*(\omega + \mathcal{B}^{n-1,\mathfrak{p}-1}) := \partial G_u(\omega + \mathcal{B}^{n-1,\mathfrak{p}-1}) = \partial G_u(\omega) \,, \quad \omega \in \mathcal{Z}^{n-1,\mathfrak{p}-1} \,,$$

and by (2.1) the *homology map* $(\partial G_u)_*$ is uniquely represented as an element of $\mathcal{D}_{n-\mathfrak{p}}(\mathcal{X}; \mathcal{H}_{\mathfrak{p}-1}(\mathcal{Y}; \mathbb{R}))$. More explicitly, let $\pi : \mathcal{X} \times \mathbb{R}^N \to \mathcal{X}$ and $\widehat{\pi} : \mathcal{X} \times \mathbb{R}^N \to \mathbb{R}^N$ denote the orthogonal projections onto the first and second factor, respectively. If $\phi \in \mathcal{D}^{n-\mathfrak{p}}(\mathcal{X})$, we have $[(\partial G_u)_*(\phi)] \in \mathcal{H}_{\mathfrak{p}-1}(\mathcal{Y}; \mathbb{R})$ and for $s = 1, \ldots, \overline{s}$

$$\langle (\partial G_u)_*(\phi), [\sigma^s] \rangle = \partial G_u(\pi^\# \phi \wedge \widehat{\pi}^\# \sigma^s) \,,$$

$\langle , \rangle$ denoting the de Rham duality between $\mathcal{H}_{\mathfrak{p}-1}(\mathcal{Y}; \mathbb{R})$ and $\mathcal{H}^{\mathfrak{p}-1}_{dR}(\mathcal{Y})$. Notice that in general $(\partial G_u)_*$ is non-trivial.

THE CURRENT $\mathbb{P}(u)$ OF THE HOMOLOGICAL SINGULARITIES. Similarly to [23], we set

$$\mathbb{P}(u) := (\partial G_u)_* \in \mathcal{D}_{n-\mathfrak{p}}(\mathcal{X}; \mathcal{H}_{\mathfrak{p}-1}(\mathcal{Y}; \mathbb{R})) \,. \tag{3.1}$$

For each $\sigma \in [\sigma] \in \mathcal{H}^{\mathfrak{p}-1}_{dR}(\mathcal{Y})$ we also define the current $\mathbb{P}(u; \sigma) \in \mathcal{D}_{n-\mathfrak{p}}(\mathcal{X})$ by

$$\mathbb{P}(u; \sigma) := (-1)^{\mathfrak{p}} \, (-1)^{(n-\mathfrak{p})(\mathfrak{p}-1)} \pi_\#((\partial G_u) \llcorner \widehat{\pi}^\# \sigma) \,,$$

so that for any $\phi \in \mathcal{D}^{n-\mathfrak{p}}(\mathcal{X})$

$$\begin{aligned} \mathbb{P}(u; \sigma)(\phi) &= (-1)^{\mathfrak{p}} \, (-1)^{(n-\mathfrak{p})(\mathfrak{p}-1)} \partial G_u(\widehat{\pi}^\# \sigma \wedge \pi^\# \phi) \\ &= (-1)^{\mathfrak{p}} \partial G_u(\pi^\# \phi \wedge \widehat{\pi}^\# \sigma) \,. \end{aligned} \tag{3.2}$$

Also, for every closed $(\mathfrak{p}-1)$-form $\sigma \in \mathcal{Z}^{\mathfrak{p}-1}(\mathcal{Y})$ we define the current $\mathbb{D}(u; \sigma) \in \mathcal{D}_{n-\mathfrak{p}+1}(\mathcal{X})$ by

$$\mathbb{D}(u; \sigma) := (-1)^{\mathfrak{p}(n-\mathfrak{p}+1)} \, \pi_\#(G_U \llcorner \widehat{\pi}^\# d\widehat{\sigma}) \,, \qquad U := \mathrm{Ext}(u) \,,$$

where $\widehat{\sigma} \in \mathcal{D}^{\mathfrak{p}-1}(\mathbb{R}^N)$ satisfies $i^\# \widehat{\sigma} = \sigma$, so that for any $\gamma \in \mathcal{D}^{n-\mathfrak{p}+1}(\mathcal{X})$

$$\mathbb{D}(u; \sigma)(\gamma) = (-1)^{\mathfrak{p}(n-\mathfrak{p}+1)} \, G_U(\widehat{\pi}^\# d\widehat{\sigma} \wedge \pi^\# \widetilde{\gamma}) = G_U(\pi^\# \widetilde{\gamma} \wedge \widehat{\pi}^\# d\widehat{\sigma}) \,. \tag{3.3}$$

Proposition 3.2 *For every* $u \in W^{1/p}(\mathcal{X}, \mathcal{Y})$ *the following properties hold:*

(i) *for* $s = 1, \ldots, \overline{s}$

$$\mathbb{P}(u; \sigma^s)(\phi) = (-1)^{\mathfrak{p}} \langle \mathbb{P}(u)(\phi), [\sigma^s] \rangle \,,$$

i.e., $\mathbb{P}(u; \sigma^s)$ *does not depend on the representative in the cohomology class* $[\sigma^s]$;

(ii) $\partial\,\mathbb{P}(u) \llcorner \operatorname{int}(\mathcal{X}) = 0$ *and* $\mathbb{P}(u) = (-1)^{\mathfrak{p}} \sum_{s=1}^{\overline{s}} \mathbb{P}(u;\sigma^s) \otimes [\gamma_s]$, *hence it does not depend on the choice of* γ_s *in* $[\gamma_s]$*;*

(iii) *for each representative* $\sigma \in \mathcal{Z}^{\mathfrak{p}-1}(\mathcal{Y})$ *in* $[\sigma]$ *we have*

$$\partial\,\mathbb{D}(u;\sigma) = \mathbb{P}(u;\sigma) \qquad on \quad \mathcal{D}^{n-\mathfrak{p}}(\mathcal{X})\,. \tag{3.4}$$

PROOF: Properties (i) and (ii) are easily checked, compare e.g. [23]. Moreover, we observe that for every $\sigma \in \mathcal{Z}^{\mathfrak{p}-1}(\mathcal{Y})$ and $\phi \in \mathcal{D}^{n-\mathfrak{p}}(\mathcal{X})$, since $d\widetilde{\phi}$ is compactly supported in $\mathcal{X} \times [0,3/4]$, by (2.2), (2.3), and (2.5) we have

$$\begin{aligned} \partial G_U(\pi^\# d\widetilde{\phi} \wedge \widehat{\pi}^\# \widehat{\sigma}) &= (-1)^{n-1} G_u(\pi^\#(d_x\widetilde{\phi} + d_t\widetilde{\phi})_{|t=0} \wedge \widehat{\pi}^\# \sigma) \\ &= (-1)^{n-1} G_u(\pi^\# d\phi \wedge \widehat{\pi}^\# \sigma)\,. \end{aligned}$$

On account of (3.2) and (3.3), and since $d\widehat{\pi}^\#\sigma = \widehat{\pi}^\# d\sigma = 0$, we then compute

$$\begin{aligned} \mathbb{P}(u;\sigma)(\phi) &= (-1)^{\mathfrak{p}} G_u(d\pi^\#\phi \wedge \widehat{\pi}^\#\sigma) = (-1)^{\mathfrak{p}} G_u(\pi^\# d\phi \wedge \widehat{\pi}^\# \sigma) \\ &= (-1)^{n-\mathfrak{p}+1} \partial G_U(\pi^\# d\widetilde{\phi} \wedge \widehat{\pi}^\# \widehat{\sigma}) = G_U(\pi^\# d\widetilde{\phi} \wedge d\widehat{\pi}^\# \widehat{\sigma}) \\ &= G_U(\pi^\# d\widetilde{\phi} \wedge \widehat{\pi}^\# d\widehat{\sigma}) = \mathbb{D}(u;\sigma)(d\phi) = \partial\,\mathbb{D}(u;\sigma)(\phi)\,, \end{aligned}$$

that yields (3.4). □

As a consequence of Proposition 3.2, we set

$$\mathbb{D}_s(u) := \mathbb{D}(u;\sigma^s)\,, \qquad \mathbb{P}_s(u) := \mathbb{P}(u;\sigma^s) \tag{3.5}$$

for every $s = 1, \ldots, \overline{s}$, so that by (3.4) we have

$$\mathbb{P}_s(u) = \partial\,\mathbb{D}_s(u) \quad \text{on} \quad \mathcal{D}^{n-\mathfrak{p}}(\mathcal{X}) \qquad \forall\, s\,. \tag{3.6}$$

We finally notice that $\mathbb{D}_s(u)$ is a current of *finite mass* in $\mathcal{D}_{n-\mathfrak{p}+1}(\mathcal{X})$, as $U = \operatorname{Ext}(u)$ is a $W^{1,p}$-function and $d\widehat{\sigma}^s \in \mathcal{D}^{\mathfrak{p}}(\mathbb{R}^N)$, with $\mathfrak{p} = [p]$, see Example 2.1.

THE MODEL CASE. Assume that $\mathcal{Y} = \mathbb{S}^{\mathfrak{p}-1}$, and let $\omega_{\mathbb{S}^{\mathfrak{p}-1}}$ denote the *normalized volume* $(\mathfrak{p}-1)$*-form*

$$\omega_{\mathbb{S}^{\mathfrak{p}-1}} := \frac{1}{\alpha_{\mathfrak{p}}} \sum_{j=1}^{\mathfrak{p}} (-1)^{j-1} y^j\, dy^1 \wedge \cdots \wedge dy^{j-1} \wedge dy^{j+1} \wedge \cdots \wedge dy^{\mathfrak{p}}\,, \tag{3.7}$$

where $\alpha_{\mathfrak{p}} := \mathcal{H}^{\mathfrak{p}-1}(\mathbb{S}^{\mathfrak{p}-1})$, so that $[\![\, \mathbb{S}^{\mathfrak{p}-1}]\!](\omega_{\mathbb{S}^{\mathfrak{p}-1}}) = \int_{\mathbb{S}^{\mathfrak{p}-1}} \omega_{\mathbb{S}^{\mathfrak{p}-1}} = 1$.

Therefore, by (3.2), for every $u \in W^{1/p}(\mathcal{X}, \mathbb{S}^{\mathfrak{p}-1})$ the currents $\mathbb{P}_s(u)$ simply reduce to the current $\mathbf{P}(u) \in \mathcal{D}_{n-\mathfrak{p}}(\mathcal{X})$ given by

$$\mathbf{P}(u)(\phi) := (-1)^{\mathfrak{p}}\, \partial G_u(\pi^\#\phi \wedge \widehat{\pi}^\# \omega_{\mathbb{S}^{\mathfrak{p}-1}})\,, \qquad \phi \in \mathcal{D}^{n-\mathfrak{p}}(\mathcal{X})\,,$$

and $\mathbb{D}_s(u)$, by (3.3), to the current $\mathbb{D}(u) \in \mathcal{D}_{n-\mathfrak{p}+1}(\mathcal{X})$ given by

$$\mathbb{D}(u)(\gamma) := G_U(\pi^\#\widetilde{\gamma} \wedge \widehat{\pi}^\# d\widehat{\omega}_{\mathbb{S}^{\mathfrak{p}-1}})\,, \qquad \gamma \in \mathcal{D}^{n-\mathfrak{p}+1}(\mathcal{X})\,, \tag{3.8}$$

where $U := \operatorname{Ext}(u) \in W^{1,p}(\mathcal{C}^{n+1}, \mathbb{R}^{\mathfrak{p}})$ takes values into $\overline{B}^{\mathfrak{p}}$.

We may and do choose $\widehat{\omega}_{\mathbb{S}^{\mathfrak{p}-1}} \in \mathcal{D}^{\mathfrak{p}-1}(\mathbb{R}^{\mathfrak{p}})$ in such a way that it agrees with the right-hand side of (3.7) on the closure $\overline{B}^{\mathfrak{p}}$ of the unit ball. Since $\alpha_{\mathfrak{p}} = \mathfrak{p}\,|B^{\mathfrak{p}}|$, this yields that

$$U^\#\, d\widehat{\omega}_{\mathbb{S}^{\mathfrak{p}-1}} = U^\# \omega_{B^{\mathfrak{p}}}\,, \qquad \omega_{B^{\mathfrak{p}}} := \frac{1}{|B^{\mathfrak{p}}|}\, dy^1 \wedge \cdots \wedge dy^{\mathfrak{p}}\,. \tag{3.9}$$

As a consequence, since by (3.4) we have $\mathbf{P}(u)(\phi) = \mathbb{D}(u)(d\phi)$, on account of (3.8) and of Example 2.1 we obtain that

$$\mathbf{P}(u)(\phi) = \frac{1}{|B^{\mathfrak{p}}|} \int_{\mathcal{C}^{n+1}} d\widetilde{\phi} \wedge U^\#(dy^1 \wedge \cdots \wedge dy^{\mathfrak{p}}) \qquad \forall\, \phi \in \mathcal{D}^{n-\mathfrak{p}}(\mathcal{X})\,.$$

Therefore, if $\mathcal{X} = \Omega$ for some open set $\Omega \subset \mathbb{R}^n$ we conclude that for every $u \in W^{1/\mathfrak{p}}(\mathcal{X}, \mathbb{S}^{\mathfrak{p}-1})$ the current $\mathbf{P}(u)$ agrees with the current J_u in (1.2) introduced by Hang-Lin [25].

Remark 3.3 For general target manifolds $\mathcal{Y}$, we similarly obtain that for every $u \in W^{1/p}(\mathcal{X}, \mathcal{Y})$

$$\mathbb{P}_s(u)(\phi) = \mathbb{D}_s(u)(d\phi) = G_U(\pi^\# d\widetilde{\phi} \wedge \widehat{\pi}^\# d\widehat{\sigma}^s) = \int_{\mathcal{C}^{n+1}} d\widetilde{\phi} \wedge U^\#(d\widehat{\sigma}^s)$$

for every $\phi \in \mathcal{D}^{n-\mathfrak{p}}(\mathcal{X})$ and $s = 1, \ldots, \overline{s}$.

Spherical cycles. We finally observe, Proposition 3.6, that the current $\mathbb{P}(u)$ carrying the singularities of maps $u \in \mathcal{R}^\infty_{1/p}(\mathcal{X}, \mathcal{Y})$ is an *integral flat chain*, and that it actually only depends on the *spherical subgroup* $\mathcal{H}^{sph}_{\mathfrak{p}-1}(\mathcal{Y})$ of $\mathcal{H}_{\mathfrak{p}-1}(\mathcal{Y})$.

Definition 3.4 *We say that an integral* $(\mathfrak{p}-1)$*-cycle* $C \in \mathcal{Z}_{\mathfrak{p}-1}(\mathcal{Y})$ *is* of spherical type *if its homology class contains a Lipschitz image of the* $(\mathfrak{p}-1)$*-sphere* $\mathbb{S}^{\mathfrak{p}-1}$*, i.e., if there exist a* $(\mathfrak{p}-1)$*-cycle* $Z \in \mathcal{Z}_{\mathfrak{p}-1}(\mathcal{Y})$*, an i.m. rectifiable* $\mathfrak{p}$*-current* $R \in \mathcal{R}_\mathfrak{p}(\mathcal{Y})$*, and a Lipschitz function* $\phi : \mathbb{S}^{\mathfrak{p}-1} \to \mathcal{Y}$*, such that*

$$C - Z = \partial R \qquad \text{and} \qquad Z = \phi_\# [\![\mathbb{S}^{\mathfrak{p}-1}]\!] .$$

Denoting then

$$\mathcal{H}^{sph}_{\mathfrak{p}-1}(\mathcal{Y}) := \{[\gamma] \in \mathcal{H}_{\mathfrak{p}-1}(\mathcal{Y}) \mid \exists\, \phi \in \mathrm{Lip}(\mathbb{S}^{\mathfrak{p}-1}, \mathcal{Y}) \mid \phi_\# [\![\mathbb{S}^{\mathfrak{p}-1}]\!] \in [\gamma]\} , \tag{3.10}$$

since the quotient $\mathcal{H}_{\mathfrak{p}-1}(\mathcal{Y})/\mathcal{H}^{sph}_{\mathfrak{p}-1}(\mathcal{Y})$ is assumed to be torsion-free, we may and do choose the γ_s's in such a way that $[\gamma_1], \ldots, [\gamma_{\widetilde{s}}]$ generate the spherical homology classes in $\mathcal{H}^{sph}_{\mathfrak{p}-1}(\mathcal{Y})$ for some $\widetilde{s} \leq \overline{s}$. Therefore, the dual basis of *spherical* $(\mathfrak{p}-1)$*-forms* in $\mathcal{H}^{\mathfrak{p}-1}_{sph}(\mathcal{Y})$ is given by $[\sigma^1], \ldots, [\sigma^{\widetilde{s}}]$.

Remark 3.5 In the case $\mathfrak{p} = 2$, clearly every integral 1-cycle in $\mathcal{Z}_1(\mathcal{Y})$ is of $\mathbb{S}^1$-type.

Proposition 3.6 *If* $n \geq \mathfrak{p} := [p]$ *and* $u \in \mathcal{R}^\infty_{1/p}(\mathcal{X}, \mathcal{Y})$ *we have:*

i) $\mathbb{P}(u;\sigma)$ *is an* $(n-\mathfrak{p})$*-dimensional i.m. rectifiable current for each* $\sigma \in [\sigma] \in H^{\mathfrak{p}-1}_{sph}(\mathcal{Y})$*; moreover,* $\mathbb{P}(u;\sigma) = 0$ *if* $[\sigma]$ *does not belong to* $H^{\mathfrak{p}-1}_{sph}(\mathcal{Y})$*.*

ii) $\mathbb{P}(u)$ *is an i.m. rectifiable* $(n-\mathfrak{p})$*-current in* $\mathcal{X}$ *with values in the subgroup* $\mathcal{H}^{sph}_{\mathfrak{p}-1}(\mathcal{Y})$*, i.e.,* $\mathbb{P}(u)$ *belongs to* $\mathcal{R}_{n-\mathfrak{p}}(\mathcal{X}; \mathcal{H}^{sph}_{\mathfrak{p}-1}(\mathcal{Y}))$*, and*

$$\mathbb{P}(u) = (-1)^\mathfrak{p} \sum_{s=1}^{\widetilde{s}} \mathbb{P}_s(u) \otimes [\gamma_s] , \qquad \mathbb{P}_s(u) := \mathbb{P}(u;\sigma^s) \in \mathcal{R}_{n-\mathfrak{p}}(\mathcal{X}) .$$

iii) *if* $n = \mathfrak{p}$*, then* $\mathbb{P}(u)$ *is a finite combination, with integer coefficients* $d_{i,s} \in \mathbb{Z}$*, of Dirac measures at points* $a_i \in \mathcal{X}$*,*

$$\mathbb{P}(u) = \sum_{s=1}^{\widetilde{s}} \sum_{i=1}^{I_s} d_{i,s} \delta_{a_i} \otimes [\gamma_s] .$$

Proof: On account of (3.2) and (3.4), the proof is obtained by an adaptation, with minor modifications, of the one of [24, Thm. 4.32], see also [20, Vol. II, Sec. 5.4.2]. For this reason, we omit any further detail. □

4 Removing homologically trivial singularities

In this section we analyze the subclass of maps $u \in W^{1/p}(\mathcal{X}, \mathcal{Y})$, for $p \geq 2$, which have no homological singularities, i.e., such that $\mathbb{P}(u) = 0$. We first show, Theorem 4.2, that any $W^{1/p}$-map satisfying $\mathbb{P}(u) = 0$ can be strongly approximated by maps $u_k \in R^0_{1/p}(\mathcal{X}, \mathcal{Y})$ satisfying the same condition $\mathbb{P}(u_k) = 0$.

Under suitable hypotheses on the topology of $\mathcal{X}$ and $\mathcal{Y}$, we then show, Theorem 4.3, that any $W^{1/p}$-map satisfying $\mathbb{P}(u) = 0$ can be strongly approximated by smooth maps in $W^{1/p}(\mathcal{X}, \mathcal{Y})$. Moreover, we shall see that the additional topological assumption turns out to be optimal, see Example 4.4.

Cartesian maps. On account of the definitions from the previous sections, see also Proposition 3.1, it is readily checked that a map $u \in W^{1/p}(\mathcal{X},\mathcal{Y})$ has zero homological singularities, i.e., satisfies $\mathbb{P}(u)=0$, if and only if the current G_u associated to its graph has no inner boundary, i.e.,

$$\partial G_u = 0 \quad \text{on} \quad \mathcal{Z}^{n-1,\mathfrak{p}-1}(\mathcal{X}\times\mathcal{Y}), \qquad \mathfrak{p}:=[p]\geq 2. \tag{4.1}$$

For this reason, we give the following

Definition 4.1 *Let $p\geq 2$. A map $u:\mathcal{X}\to\mathcal{Y}$ is said to be a* Cartesian map *in the class* $\mathrm{cart}^{1/p}(\mathcal{X},\mathcal{Y})$ *if u belongs to $W^{1/p}(\mathcal{X},\mathcal{Y})$ and satisfies the null-boundary condition* (4.1).

Trivially, condition $\mathbb{P}(u)=0$ holds true if u is smooth, say Lipschitz. Moreover, if $\mathcal{Y}$ has dimension lower than $\mathfrak{p}-1$, we have $\mathcal{H}_{\mathfrak{p}-1}(\mathcal{Y})=0$ and hence trivially $\mathbb{P}(u)=0$, whence $W^{1/p}(\mathcal{X},\mathcal{Y})=\mathrm{cart}^{1/p}(\mathcal{X},\mathcal{Y})$. Therefore, in the sequel we shall tacitly assume that $\dim(\mathcal{Y})\geq\mathfrak{p}-1$ and that the homology group $\mathcal{H}_{\mathfrak{p}-1}(\mathcal{Y})$ is non-trivial, so that in general $\mathbb{P}(u)\neq 0$, i.e., the strict inclusion $\mathrm{cart}^{1/p}(\mathcal{X},\mathcal{Y})\subsetneq W^{1/p}(\mathcal{X},\mathcal{Y})$ holds.

Moreover, since the null-boundary condition (4.1) is preserved by the weak convergence in $\mathcal{D}_{n,\mathfrak{p}-1}$, and the strong convergence $u_k\to u$ in $W^{1/p}(\mathcal{X},\mathcal{Y})$ yields the weak convergence $G_{u_k}\rightharpoonup G_u$ in $\mathcal{D}_{n,\mathfrak{p}-1}$, see Remark 2.4, according to (1.4) we immediately obtain that

$$H^{1/p}_S(\mathcal{X},\mathcal{Y})\subset\mathrm{cart}^{1/p}(\mathcal{X},\mathcal{Y}). \tag{4.2}$$

The case of exponents $p<2$. If $1<p<2$, of course the definitions from Secs. 2 and 3 continue to hold. However, for $\mathfrak{p}:=[p]=1$, the manifold $\mathcal{Y}$ being connected (the model case $\mathcal{Y}=\mathbb{S}^{\mathfrak{p}-1}$ cannot be considered), we infer that $\mathcal{H}_0(\mathcal{Y})\simeq\mathbb{Z}$ and $\mathcal{H}^0_{dR}(\mathcal{Y})\simeq\mathbb{Z}$. As a consequence, compare [24, Prop. 4.23], for $k=0,\dots,n$ we have

$$\mathcal{Z}^{k,0}(\mathcal{X},\mathcal{Y})=\mathcal{D}^k(\mathcal{X}), \quad \mathcal{B}^{k,0}(\mathcal{X},\mathcal{Y})=\{0\}$$

and hence, by (2.1),

$$\mathcal{H}^{k,0}(\mathcal{X}\times\mathcal{Y})\simeq\mathcal{D}^k(\mathcal{X})\otimes\mathbb{Z}.$$

In particular, according to (3.1) and (3.2), the homological singularities of a map $u\in W^{1/p}(\mathcal{X},\mathcal{Y})$ are described by the current $\mathbb{P}(u)\in\mathcal{D}_{n-1}(\mathcal{X})$ given by

$$\mathbb{P}(u)(\varphi):=-\partial G_u(\pi^{\#}\varphi), \qquad \varphi\in\mathcal{D}^{n-1}(\mathcal{X}).$$

Therefore, according to Definition 4.1, a map $u\in W^{1/p}(\mathcal{X},\mathcal{Y})$ belongs to the class $\mathrm{cart}^{1/p}(\mathcal{X},\mathcal{Y})$ if the current $G_u\in\mathcal{D}_{n,0}(\mathcal{X}\times\mathcal{Y})$ satisfies

$$\partial G_u(\pi^{\#}\varphi)=0 \qquad \forall\,\varphi\in\mathcal{D}^{n-1}(\mathcal{X}).$$

However, by (2.3) and (2.5) we infer that

$$\partial G_u(\varphi)=G_u(d\varphi)=(-1)^{n-1}\partial G_U(d\varphi\wedge\eta)=0.$$

This yields that for every $1<p<2$ the class $W^{1/p}(\mathcal{X},\mathcal{Y})$ agrees with the class of Cartesian maps $\mathrm{cart}^{1/p}(\mathcal{X},\mathcal{Y})$, that is, *every map $u\in W^{1/p}(\mathcal{X},\mathcal{Y})$ has no homological singularities*:

$$\forall\,u\in W^{1/p}(\mathcal{X},\mathcal{Y}), \quad 1<p<2, \quad \text{we have} \quad \mathbb{P}(u)=0.$$

Of course, this last property can be seen as a consequence of Corollary 1.7, on account of (4.2).

Density results in $\mathrm{cart}^{1/p}$. Assume now that $p\geq 2$. Using arguments taken from the proof of Theorem 1.1 from [38], we shall first prove the following

Theorem 4.2 *For every Cartesian map $u\in\mathrm{cart}^{1/p}(\mathcal{X},\mathcal{Y})$ there exists a sequence of maps $\{u_k\}\subset R^0_{1/p}(\mathcal{X},\mathcal{Y})\cap\mathrm{cart}^{1/p}(\mathcal{X},\mathcal{Y})$ such that $u_k\to u$ strongly in $W^{1/p}$.*

We shall then prove that under suitable hypotheses on $\mathcal{X}$ and $\mathcal{Y}$ *every map in* $\mathrm{cart}^{1/p}(\mathcal{X},\mathcal{Y})$ *can be approximated by sequences of smooth maps in* $W^{1/p}$. On account of (1.5), we shall assume $n\geq\mathfrak{p}\geq 2$.

Theorem 4.3 *Let $p \geq 2$ and $n \geq \mathfrak{p} := [p]$. In the case $\mathfrak{p} \geq 3$, assume that for any base point $y_0 \in \mathcal{Y}$ the* Hurewicz homomorphism *from the $(\mathfrak{p}-1)^{th}$ homotopy group $\pi_{\mathfrak{p}-1}(\mathcal{Y}; y_0)$ onto the $(\mathfrak{p}-1)^{th}$ real homology group $\mathcal{H}_{\mathfrak{p}-1}(\mathcal{Y}; \mathbb{R})$ is injective. Alternatively, in the case $\mathfrak{p} = 2$, assume that the* first homotopy group $\pi_1(\mathcal{Y})$ *is commutative. Moreover, if $n \geq \mathfrak{p}+1$, assume that $\mathcal{X}$ satisfies the $(\mathfrak{p}-1)$-extension property with respect to $\mathcal{Y}$. Then*

$$\text{cart}^{1/p}(\mathcal{X}, \mathcal{Y}) = H_S^{1/p}(\mathcal{X}, \mathcal{Y}).$$

Similarly to [23], we now see that even in the case $\mathcal{X} = B^n$, and $n = \mathfrak{p}$, the injectivity hypothesis of the Hurewicz maps, in the case $\mathfrak{p} \geq 3$, or the commutativity hypothesis of the first homotopy group, in the case $\mathfrak{p} = 2$, cannot be dropped from the statement of Theorem 4.3.

Example 4.4 Assume that the target manifold $\mathcal{Y}$ does not satisfy the injectivity hypothesis on the Hurewicz maps (or that $\pi_1(\mathcal{Y})$ is not commutative, for $\mathfrak{p} = 2$). We claim that there exist functions u in $\text{cart}^{1/p}(B^{\mathfrak{p}}, \mathcal{Y})$, where $\mathfrak{p} := [p] \geq 2$, which cannot be approximated strongly in $W^{1/p}$ by smooth maps $u_k : B^{\mathfrak{p}} \to \mathcal{Y}$, i.e., such that $u \notin H_S^{1/p}(B^{\mathfrak{p}}, \mathcal{Y})$, whence the strict inclusion holds in (1.7).

In fact, for any such target manifold $\mathcal{Y}$ there exists a Lipschitz function $\widetilde{\varphi} : \mathbb{S}^{\mathfrak{p}-1} \to \mathcal{Y}$ such that $\widetilde{\varphi}$ is not homotopic to a constant map in $\mathcal{Y}$, but such that $\widetilde{\varphi}$ is homologically trivial. Arguing as e.g. in the proof of [24, Thm. 5.3.6], we then find a Lipschitz function $\varphi : \mathbb{S}^{\mathfrak{p}-1} \to \mathcal{Y}$ that is homotopic to $\widetilde{\varphi}$ in $\mathcal{Y}$, but such that the image current $\varphi_{\#}[\![\mathbb{S}^{\mathfrak{p}-1}]\!] = 0$.

Consider the map $u := \varphi(x/|x|)$. Clearly u belongs to $W^{1/p}(B^{\mathfrak{p}}, \mathcal{Y})$, as $p < \mathfrak{p}+1$. Since moreover, compare [20, Vol. I, Sec. 3.2.2],

$$(\partial G_u) \llcorner B^{\mathfrak{p}} \times \mathcal{Y} = -\delta_0 \times \varphi_{\#}[\![\mathbb{S}^{\mathfrak{p}-1}]\!],$$

where δ_0 is the unit Dirac mass at the origin, condition $\varphi_{\#}[\![\mathbb{S}^{\mathfrak{p}-1}]\!] = 0$ yields that $\partial G_u = 0$ in $B^{\mathfrak{p}} \times \mathcal{Y}$, i.e., $\mathbb{P}(u) = 0$, whence $u \in \text{cart}^{1/p}(B^{\mathfrak{p}}, \mathcal{Y})$. Now, if u were approximable by smooth maps from $B^{\mathfrak{p}}$ into $\mathcal{Y}$ strongly in $W^{1/p}$, whence strongly in $W^{1/\mathfrak{p}}$, since the strong $W^{1/\mathfrak{p}}$-convergence preserves the $(\mathfrak{p}-1)$-*homotopy type*, see [4, Lemma 1], we would obtain that φ is homotopically trivial, a contradiction. □

PROOF OF THEOREM 4.2: We follow the lines of the proof taken from [38, Sec. 2] of Theorem 1.1 above, where we denoted $d := [p]$, to which we refer for the notation and for further details.

For this reason, we denote $\mathcal{Q}^n :=]0,1[^n$ and let u be a map in $\text{cart}^{1/p}(\mathcal{Q}^n, \mathcal{Y})$. We can improve the slicing argument at the beginning of the proof of [38, Thm. 1], choosing for every $m \in \mathbb{N}^+$ the grid of size $1/m$ in such a way that the following properties are satisfied:

i) the restriction u_F to each k-face F of the k-skeleton $C_m^{(k)}$ of the grid belongs to $\text{cart}^{1/p}$, i.e.,

$$\partial G_{u_F} = 0 \quad \text{on} \quad \mathcal{Z}^{k,\mathfrak{p}-1}(F \times \mathcal{Y}),$$

for $k = \mathfrak{p}-1, \ldots, n$;

ii) if F_1, F_2 are $(\mathfrak{p}-1)$-faces of $C_m^{(\mathfrak{p}-1)}$ that intersect in a $(\mathfrak{p}-2)$-face I, then

$$\partial G_{u_{F_1}} \llcorner I \times \mathcal{Y} = -\partial G_{u_{F_2}} \llcorner I \times \mathcal{Y} \quad \text{on} \quad \mathcal{D}^{\mathfrak{p}-2}(\mathcal{X} \times \mathcal{Y}).$$

We first consider the case $n = \mathfrak{p}$.

THE CASE $n = \mathfrak{p}$. We recall that $\mathcal{Y} \subset \mathbb{R}^N$, and set

$$\mathcal{Y}_\varepsilon := \overline{U_\varepsilon(\mathcal{Y})}, \tag{4.3}$$

where $U_\varepsilon(A) := \{y \in \mathbb{R}^N \mid \text{dist}(y, A) < \varepsilon\}$ is the ε-neighborhood of $A \subset \mathbb{R}^N$. Since $\mathcal{Y}$ is smooth and compact, there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leq \varepsilon_0$ the nearest point projection Π_ε of $\mathcal{Y}_\varepsilon$ onto $\mathcal{Y}$ is a well defined Lipschitz map with Lipschitz constant $\text{Lip}(\Pi_\varepsilon) \leq (1 + c\,\varepsilon) \to 1^+$ as $\varepsilon \to 0^+$. In particular, for $0 < \varepsilon \leq \varepsilon_0$, the ε-neighborhood $\mathcal{Y}_\varepsilon$ is equivalent to $\mathcal{Y}$ in the sense of the algebraic topology.

It is readily checked that the assertion follows if we show that we can find a sequence $\{h_j\} \searrow 0$ such that the traces $\mathbf{T}(W_{h_j}^{(m)})$ of the approximating maps $W_{h_j}^{(m)}$ from [38, Thm. 1] are functions in $\text{cart}^{1/p}(\mathcal{Q}_m^n, \mathcal{Y}_{\varepsilon_0})$.

We recall that $\{C_l\}_{l=1}^{(m-1)^n}$ is a list of the $(n+1)$-cubes in $\mathcal{F}_m$, and we denote by F_l the n-cube given by the intersection of C_l with $\mathcal{Q}^n \times \{0\}$. The approximating map $W_h^{(m)}$ has been defined on C_l by

$$W_h^{(m)}(z) := V_h^{(m)}\Big[f_l^{-1}\Big(\frac{f_l(z)}{2m\|f_l(z)\|_{n+1}}\Big)\Big]\,,$$

where f_l is a suitable bilipschitz homeomorphism between C_l and the $(n+1)$-cube $[-1/(2m), 1/(2m)]^{n+1}$, and $V_h^{(m)}$ is given by [38, Prop. 3] in correspondence of a grid satisfying i) and ii). As a consequence, setting $v_h^{(m)} := \mathbf{T}(V_h^{(m)})$, in order to show that the trace $\mathbf{T}(W_h^{(m)})$ belongs to $\mathrm{cart}^{1/p}(\mathcal{Q}^n_m, \mathcal{Y}_{\varepsilon_0})$, and conclude the proof of Theorem 4.2 in the case $n = \mathfrak{p}$, it suffices to prove Proposition 4.5. We are not able to find of a more direct argument, see Remark 4.7 below.

Proposition 4.5 *There exists a sequence* $\{h_j\} \searrow 0$ *such that every* l *and* j *the* $(n-1)$*-cycle* $v^{(m)}_{h_j\,\#}[\![\partial F_l]\!]$ *is homologically trivial in* $\mathcal{Y}_{\varepsilon_0}$.

PROOF: Since $\mathcal{Y}_{\varepsilon_0}$ is equivalent to $\mathcal{Y}$ in the sense of the algebraic topology, for every s there exists a *closed* $(\mathfrak{p}-1)$-form $\widetilde{\sigma}^s$ in $\mathcal{Y}_{\varepsilon_0}$ that agrees with σ^s on $\mathcal{Y}$ and such that $\{\widetilde{\sigma}^s\}_{s=1}^{\widetilde{s}}$ is a basis of the subgroup $\mathcal{H}^{\mathfrak{p}-1}_{sph}(\mathcal{Y}_{\varepsilon_0})$ of spherical $(\mathfrak{p}-1)$-forms in $\mathcal{H}^{\mathfrak{p}-1}_{dR}(\mathcal{Y}_{\varepsilon_0})$, see Definition 3.4. Therefore, since $n = \mathfrak{p}$, the assertion follows if we show that

$$v^{(m)}_{h_j\,\#}[\![\partial F_l]\!](\widetilde{\sigma}^s) = 0 \qquad \forall\, s = 1, \ldots, \widetilde{s}\,.$$

Looking at the proof of [38, Prop. 2], we observe that $\Sigma(P_0, h)$ intersects the $(n-1)$-skeleton $\Sigma_m^{(n-1)} \times \{0\}$ if $P_0 \in \Sigma_m^{(n-1)} \times]-h/2, h/2[$. Therefore, the same argument used in the above mentioned proof yields that the approximating sequence $\{U_h^{(m)}\}_h$ actually satisfies $U_h^{(m)}(x,t) \in \mathcal{Y}_{\varepsilon_0}$ for every $(x,t) \in \partial F_l \times]-h/2, h/2[$ and for every l, provided that $h < h_\varepsilon$.

As a consequence, we readily infer that the approximating sequence $\{V_h^{(m)}\}_h$ given by [38, Prop. 3] satisfies that same condition, i.e., $V_h^{(m)}(\partial F_l \times]-h/2, h/2[) \subset \mathcal{Y}_{\varepsilon_0}$ for every l, if $h < h_\varepsilon$.

Setting now $V_h^l := V^{(m)}_{h\,|\partial F_l \times]0,h/2[}$, since the differential $d\widetilde{\sigma}^s = 0$ and the map V_h^l is smooth, we have

$$\partial V^l_{h\,\#}([\![\partial F_l]\!] \times [\![(0, h/2)]\!])(\widetilde{\sigma}^s) = V^l_{h\#}([\![\partial F_l]\!] \times [\![(0, h/2)]\!])(d\widetilde{\sigma}^s) = 0\,.$$

Therefore, if $\phi_h^{(m)}(x) := V_h^{(m)}(x, \delta)$ for some suitable $0 < \delta < h/2$ to be chosen, by a standard homotopy argument we infer that

$$v^{(m)}_{h\,\#}[\![\partial F_l]\!](\widetilde{\sigma}^s) = \phi^{(m)}_{h\,\#}[\![\partial F_l]\!](\widetilde{\sigma}^s)\,. \tag{4.4}$$

Now, let $\widehat{\sigma}^s$ be an $(n-1)$-form in $\mathcal{D}^{n-1}(\mathbb{R}^N)$ such that $i^\#\widehat{\sigma}^s = \sigma^s$ and $j^\#\widehat{\sigma}^s = \widetilde{\sigma}^s$, where $j : \mathcal{Y}_{\varepsilon_0} \to \mathbb{R}^N$ is the injection map. Since $u \in \mathrm{cart}^{1/p}(\mathcal{Q}^n, \mathcal{Y})$, by (4.1) for every test function $\varphi \in C_c^\infty(\mathcal{Q}^n)$ we have

$$G_u(d\varphi \wedge \sigma^s) = \partial G_u(\varphi \wedge \sigma^s) = 0\,.$$

Taking the cut-off function $\eta : [0,1] \to [0,1]$ in (2.5) in such a way that $\eta'(t) < 0$ for $t \in]0, h_\varepsilon[$, this yields that

$$G_U(d\varphi \wedge d(\eta \wedge \widehat{\sigma}^s)) = -\partial G_U(d\varphi \wedge \eta \wedge \widehat{\sigma}^s) = 0\,, \qquad U := \mathrm{Ext}(u)\,.$$

Therefore, choosing a suitable sequence of test functions $\{\varphi_k\} \subset C_c^\infty(\mathcal{Q}^n)$ that strongly converges in L^1 to the characteristic function of F_l, since by Example 2.1

$$G_U(d\varphi_k \wedge d(\eta \wedge \widehat{\sigma}^s)) = \int_{\mathcal{Q}^n \times [0,1]} d\varphi_k \wedge \big(d\eta \wedge U^\#\widehat{\sigma}^s + \eta \wedge U^\# d\widehat{\sigma}^s\big)\,,$$

by a standard argument we obtain that

$$G_{U_l}(d(\eta \wedge \widehat{\sigma}^s)) = 0\,, \qquad \text{where} \quad U_l := U_{|\partial F_l \times [0,1]}\,.$$

By the strong convergence of $V^{(m)}_{h\,|\partial F_l \times [0,1]}$ to U_l as $h \to 0$, on account of properties i) and ii) above, this gives that for every sequence $\{\widetilde{h}_j\} \searrow 0$ and for every l we can find a subsequence $\{h_j^{(l)}\} \searrow 0$ such that

$$G_{V_h^l}(d(\eta \wedge \widehat{\sigma}^s)) = 0 \qquad \text{for} \quad h = h_j^{(l)}\,, \quad \forall\, j\,, \quad \forall\, s\,.$$

Moreover, since $V_h^l(\partial F_l \times]0,h/2[) \subset \mathcal{Y}_{\varepsilon_0}$ and $j^\# \widehat{\sigma}^s = \widetilde{\sigma}^s$, with $d\widetilde{\sigma}^s = 0$, this gives

$$G_{V_h^l}(d\eta \wedge \widetilde{\sigma}^s) = G_{V_h^l}(d(\eta \wedge \widetilde{\sigma}^s)) = G_{V_h^l}(d(\eta \wedge \widehat{\sigma}^s)) = 0\,.$$

Therefore, setting $\phi_j^l(x) := V^{(m)}_{h_j^{(l)}}(x, \delta_j^{(l)})$ for a suitable $0 < \delta_j^{(l)} < h_j^{(l)}/2$, by a slicing argument we find that

$$\phi^l_{j\,\#}[\![\, \partial F_l \,]\!](\widetilde{\sigma}^s) = 0\,.$$

Finally, a diagonal argument on $l = 1, \ldots, (m-1)^n$ yields the assertion, by (4.4). $\square$

THE CASE $n \geq \mathfrak{p}+1$. The proof is an adaptation of the one of [38, Thm. 1], using the same argument as above. In fact, when extending $W_h^{(m)}$ to the $(\mathfrak{p}+1)$-cubes of the grid, we argue as in the case $n = \mathfrak{p}$. Moreover, when extending $W_h^{(m)}$ to the $(k+1)$-cubes of the grid, for $k = \mathfrak{p}+1, \ldots, n$, we see that actually *no boundary is "produced"*. This is essentially due to the following lemma, that concludes the proof.

Lemma 4.6 *Let $k = \mathfrak{p}+1, \ldots, n$ integer and $u : B^k \to \mathcal{Y}$ be given by $u(x) := v(x/|x|)$ for some $v \in W^{1/p}(\partial B^k, \mathcal{Y})$. Then $u \in \mathrm{cart}^{1/p}(B^k, \mathcal{Y})$.*

PROOF: Since $k > p$, trivially $u \in W^{1/p}(B^k, \mathcal{Y})$. Moreover, if $w : B^k \to \mathbb{R}^N$ is a smooth $W^{1,p}$-map such that $w_{|\partial B^k} = u$, and $R \in \mathcal{R}_k(\mathbb{R}^N)$ is the i.m. rectifiable current $R := w_\# [\![\, B^k \,]\!]$, we have

$$\partial G_u \llcorner B^k \times \mathcal{Y} = -\delta_0 \times \partial R\,. \tag{4.5}$$

Since the *integral flat cycle* ∂R has dimension $k-1 \geq \mathfrak{p}$, the property (4.5) gives automatically that $\partial G_u(\omega) = 0$ for every $\omega \in \mathcal{Z}^{k-1,\mathfrak{p}-1}(B^k \times \mathcal{Y})$, i.e., the null-boundary condition (4.1), whence Lemma 4.6 is proved, as required. $\square$

Remark 4.7 Lemma 4.6 is false in dimension $k = \mathfrak{p}$, even if $u(x) = v(x/|x|)$ for some map $v : \Omega \to \mathcal{Y}$ in $\mathrm{cart}^{1/p}(\Omega, \mathcal{Y})$, where $B^\mathfrak{p} \subset\subset \Omega \subset \mathbb{R}^p$, see Definition 4.1. Actually, if v is smooth, then (4.5) holds with $R := v_\# [\![\, B^\mathfrak{p} \,]\!]$, an i.m. rectifiable current in $\mathcal{R}_\mathfrak{p}(\mathcal{Y})$. Therefore, the integral flat $(\mathfrak{p}-1)$-cycle ∂R is *homologically trivial*, i.e., $\partial R(\sigma) = R(d\sigma) = 0$ for every closed form $\sigma \in \mathcal{Z}^{\mathfrak{p}-1}(\mathcal{Y})$, whence by (4.5)

$$\partial G_u = 0 \qquad \text{on} \quad \mathcal{Z}^{\mathfrak{p}-1,\mathfrak{p}-1}(B^\mathfrak{p} \times \mathcal{Y})\,, \tag{4.6}$$

i.e., $u \in \mathrm{cart}^{1/p}(B^\mathfrak{p}, \mathcal{Y})$. However, this argument fails to hold if v is a generic map in $\mathrm{cart}^{1/p}(\Omega, \mathcal{Y})$, since in general we cannot conclude that $v_\# [\![\, B^\mathfrak{p} \,]\!]$ is an i.m. rectifiable current in $\mathcal{R}_\mathfrak{p}(\mathcal{Y})$, even in the case $\mathfrak{p} = p = 2$.

PROOF OF THEOREM 4.3: By Theorem 4.2, it suffices to show that every map $u \in \mathcal{R}^0_{1/p}(\mathcal{X}, \mathcal{Y}) \cap \mathrm{cart}^{1/p}$ is the strong $W^{1/p}$-limit of a sequence of smooth functions in $W^{1/p}(\mathcal{X}, \mathcal{Y}) \cap C^\infty$. We distinguish two cases.

THE CASE $n = \mathfrak{p}$. Every map $u \in \mathcal{R}^0_{1/p}(\mathcal{X}, \mathcal{Y}) \cap \mathrm{cart}^{1/p}$ is continuous outside a discrete set, see (1.6). Since we use a local argument, we may assume that $u \in \mathrm{cart}^{1/p}(B^\mathfrak{p}, \mathcal{Y})$ and u is continuous outside the origin. In order to remove the singularity of u, using the same argument given for the case $\mathfrak{p} = p = 2$ in [23, Prop. 5.1], it suffices to show that for $r > 0$ small the set

$$\{w \in W^{1/p}(B_r^\mathfrak{p}, \mathcal{Y}) \cap C^0(\overline{B}_r^\mathfrak{p}, \mathcal{Y}) \mid w_{|\partial B_r^\mathfrak{p}} = u_{|\partial B_r^\mathfrak{p}}\}$$

is non-empty. By the assumption on the Hurewicz maps, this holds true if we have

$$du_{|\partial B_r^\mathfrak{p}}{}^\# \sigma^s = 0 \qquad \forall\, s = 1, \ldots, \overline{s}\,.$$

As in [23, Prop. 5.1], this follows from the null-boundary condition (4.1), i.e., from (4.6), that is equivalent to the property $\mathbb{P}_s(u) = 0$ for every s, see (3.5). A standard convolution and projection argument as e.g. in [4] yields the assertion.

THE CASE $n \geq \mathfrak{p}+1$. Let $u \in \mathcal{R}^0_{1/p}(\mathcal{X}, \mathcal{Y}) \cap \mathrm{cart}^{1/p}$ and let X be a cubeulation of $\mathcal{X}$. Without loss of generality, by using a slicing argument, we assume that X is in *dual position* with respect to u, compare [38]. More precisely, we may and do assume that the $(\mathfrak{p}-1)$-skeleton $X^{\mathfrak{p}-1}$ is disjoint from the $(n-\mathfrak{p})$-dimensional singular set $\Sigma(u)$ of u, see (1.6), and that the properties i) and ii) at the beginning of the proof of Theorem 4.2 are satisfied. Arguing as in the case $n=\mathfrak{p}$, the above properties, in conjunction with the injectivity on the Hurewicz maps $\pi_{\mathfrak{p}-1}(\mathcal{Y};y_0)\to\mathcal{H}_{\mathfrak{p}-1}(\mathcal{Y};\mathbb{R})$, for $\mathfrak{p}\geq 3$, or with the commutativity of $\pi_1(\mathcal{Y})$, for $\mathfrak{p}=2$, yields that the restriction $u_{|X^{\mathfrak{p}-1}}$ has a continuous extension $g:X^{\mathfrak{p}}\to\mathcal{Y}$.

Therefore, by the $(\mathfrak{p}-1)$-extension property, the restriction $u_{|X^{\mathfrak{p}-1}}$ can be extended to a continuous map from $\mathcal{X}$ into $\mathcal{Y}$. By applying [38, Thm. 3] we then obtain that u is the strong $W^{1/p}$-limit of a smooth sequence in $W^{1/p}(\mathcal{X},\mathcal{Y})\cap C^\infty$, as required. □

5 Minimal connections of maps in $\mathbf{W^{1/p}}$

In this section we discuss the minimal integral connection of the homological singularities $\mathbb{P}(u)$ of a $W^{1/p}$-map u from an n-dimensional manifold $\mathcal{X}$ into the sphere $\mathbb{S}^{\mathfrak{p}-1}$, giving also an explicit example. The next section will be dedicated to the case of more general target manifolds. First, we collect the notion of real and integral mass, and prove some general properties of the current $\mathbb{P}(u)$.

Real and integral mass. We let $n\geq\mathfrak{p}:=[p]\geq 2$ and $\Omega\subset\mathcal{X}$ be an open set. Recall:

Definition 5.1 *For every $\Gamma\in\mathcal{D}_{n-\mathfrak{p}}(\Omega)$ we denote by*

$$\begin{aligned} m_{r,\Omega}(\Gamma)&:=\inf\{\mathbf{M}(D)\mid D\in\mathcal{D}_{n-\mathfrak{p}+1}(\Omega)\,,\quad(\partial D)\llcorner\Omega=\Gamma\}\\ m_{i,\Omega}(\Gamma)&:=\inf\{\mathbf{M}(L)\mid L\in\mathcal{R}_{n-\mathfrak{p}+1}(\Omega)\,,\quad(\partial L)\llcorner\Omega=\Gamma\}\end{aligned}$$

the real mass *and* integral mass *of* Γ relative to Ω, *respectively. In case $m_{i,\Omega}(\Gamma)<\infty$, an i.m. rectifiable current $L\in\mathcal{R}_{n-\mathfrak{p}+1}(\Omega)$ is an* integral minimal connection for the mass *of* Γ allowing connections to the boundary *of* Ω *if* $(\partial L)\llcorner\Omega=\Gamma$ *and* $\mathbf{M}(L)=m_{i,\Omega}(\Gamma)$.

We first show that $\mathbb{P}(u)$ is an $(n-\mathfrak{p})$-dimensional *real flat chain.*

Proposition 5.2 *For every $u\in W^{1/p}(\mathcal{X},\mathcal{Y})$ the current $\mathbb{P}(u)$ is the real flat limit of the currents $\mathbb{P}(u_k)$ in $\mathcal{R}_{n-\mathfrak{p}}(\mathcal{X};\mathcal{H}^{sph}_{\mathfrak{p}-1}(\mathcal{Y}))$, where $\{u_k\}\subset\mathcal{R}^\infty_{1/p}(\mathcal{X},\mathcal{Y})$ is a sequence that strongly converges in $W^{1/p}$ to u, and*

$$\mathbb{P}(u)(\phi)=(-1)^{\mathfrak{p}}\sum_{s=1}^{\widetilde{s}}\mathbb{P}_s(u)(\phi)\,[\gamma^s]\in\mathcal{H}^{sph}_{\mathfrak{p}-1}(\mathcal{Y};\mathbb{R})\qquad\forall\,\phi\in\mathcal{D}^{n-\mathfrak{p}}(\mathcal{X})\,.$$

In particular, $\mathbb{P}_s(u)=0$ for $s=\widetilde{s}+1,\dots,\overline{s}$.

Proof: Using Theorem 1.1 and Proposition 3.6, the proof is obtained as e.g. in [20, Vol. II, Sec. 4.5.2]. □

In dimension $n=\mathfrak{p}$, moreover, we obtain that $\mathbb{P}(u)$ is an *integral flat chain.*

Proposition 5.3 *Let $n=\mathfrak{p}$. Let $u\in W^{1/p}(\mathcal{X},\mathcal{Y})$ and $\{u_k\}$ be a sequence of maps in $\mathcal{R}^\infty_{1/p}(\mathcal{X},\mathcal{Y})$ that strongly converges in $W^{1/p}$ to u. Then we have:*

(i) $\mathbf{M}(\mathbb{D}_s(u_k)-\mathbb{D}_s(u))\to 0$ *as* $k\to\infty$ *for each* $s=1,\dots,\widetilde{s}$;

(ii) *there exists a current $L\in\mathcal{R}_1(\mathcal{X};\mathcal{H}^{sph}_{\mathfrak{p}-1}(\mathcal{Y}))$, with $\mathbf{M}(L)<\infty$, such that $\mathbb{P}(u)=(\partial L)\llcorner\operatorname{int}(\mathcal{X})$; in particular, $\mathbb{P}(u)$ is an integral flat chain;*

(iii) *if $L^s_{u_k,u}$ denotes an i.m. rectifiable current in $\mathcal{R}_1(\mathcal{X})$ of least mass such that*

$$(\partial L^s_{u_k,u})\llcorner\operatorname{int}(\mathcal{X})=\mathbb{P}_s(u)-\mathbb{P}_s(u_k)\,,\qquad s=1,\dots,\widetilde{s}\,,\tag{5.1}$$

then $\mathbf{M}(L^s_{u_k,u})\to 0$ as $k\to\infty$;

(iv) *if* $\partial\mathcal{X} = \emptyset$*, or* $u = \varphi$ *on* $\partial\mathcal{X}$ *for some smooth* $W^{1/p}$*-map* $\varphi : \mathcal{X} \to \mathcal{Y}$*, then for each* $s = 1, \dots, \widetilde{s}$ *there exist points* $a_i, b_i \in \mathcal{X}$ *such that*

$$\mathbb{P}_s(u) = \sum_{i=1}^{\infty} (\delta_{a_i} - \delta_{b_i}) \,, \qquad \sum_{i=1}^{\infty} \mathrm{dist}_{\mathcal{X}}(a_i, b_i) < \infty \,,$$

where $\mathrm{dist}_{\mathcal{X}}$ *is the geodesic distance in* $\mathcal{X}$*.*

PROOF: Using (3.3), the proof of property (i) is similar to the one in [23, Prop. 1.4], and holds true even in higher dimension $n \geq \mathfrak{p}+1$. As to the rest of the theorem, we observe that $\mathbb{P}_s(u_k)$ is a $(n-\mathfrak{p})$-dimensional i.m. rectifiable current in $\mathcal{R}_{n-\mathfrak{p}}(\mathcal{X})$. By Federer's theorem [17], for $n = \mathfrak{p}$ we then have that

$$m_{i,\mathrm{int}(\mathcal{X})}(\mathbb{P}_s(u_k)) = m_{r,\mathrm{int}(\mathcal{X})}(\mathbb{P}_s(u_k)) \qquad \forall\, s = 1, \dots, \widetilde{s} \,, \tag{5.2}$$

see Definition 5.1. Therefore, (i) and (3.6) give $m_{i,\mathrm{int}(\mathcal{X})}(\mathbb{P}_s(u_k) - \mathbb{P}_s(u)) \to 0$, and the claims follow. □

Remark 5.4 The above argument fails to hold in higher dimension $n \geq \mathfrak{p}+1$, for any integer $\mathfrak{p} \geq 3$. In this case, in fact, we do not know whether (5.2) holds true, compare [37, 43], or even if

$$m_{i,\mathrm{int}(\mathcal{X})}(\mathbb{P}_s(u_k)) \leq c \cdot m_{r,\mathrm{int}(\mathcal{X})}(\mathbb{P}_s(u_k))$$

for some absolute constant $c > 0$, not depending on u_k, a weaker condition that would give the assertion of Proposition 5.3. We recall that in the case $\mathfrak{p} = 2$, Hardt-Pitts' theorem [28] yields (5.2) and hence Proposition 5.3, for any $n \geq 2$, see [23].

INTEGRAL CONNECTIONS. Assume now that $\mathcal{Y} = \mathbb{S}^{\mathfrak{p}-1}$ and that $\mathcal{X}$ *has no boundary.* We show that the current $\mathbb{P}(u)$ carrying the singularities of any map $u \in W^{1/p}(\mathcal{X}, \mathbb{S}^{\mathfrak{p}-1})$ is an integral flat chain.

Proposition 5.5 *Let* $n \geq \mathfrak{p} \geq 2$*. For every* $u \in W^{1/p}(\mathcal{X}, \mathbb{S}^{\mathfrak{p}-1})$ *there exists an i.m. rectifiable current* $L \in \mathcal{R}_{n-\mathfrak{p}+1}(\mathcal{X})$ *such that*

$$\partial L = \mathbf{P}(u) \quad \textit{and} \quad \mathbf{M}(L) \leq C \left(\mathcal{E}_{1/p}(u) + \| \mathrm{Ext}(u) \|^p_{L^p(\mathcal{C}^{n+1})} \right),$$

where $\mathrm{Ext}(u) \in W^{1,p}(\mathcal{C}^{n+1}, \mathbb{R}^N)$ *is the extension of* u *and* $C > 0$ *is an absolute constant, not depending on* u*.*

Remark 5.6 This property (and its local version) was proved by Hang-Lin [25] for $p \geq 2$ integer and for $\mathcal{X} = \mathbb{R}^n$, using the coarea formula and the degree theory developed by Brezis-Nirenberg [14]. In the sequel we shall give a similar proof based on arguments from Sec. 2. It turns out that the extra term $\| \mathrm{Ext}(u) \|^p_{L^p(\mathcal{C}^{n+1})}$ in the above formula can be removed if we require that the integral connection L belongs to $\mathcal{R}_{n-\mathfrak{p}+1}(\mathcal{C}^{n+1})$, as in [25].

If the boundary $\partial\mathcal{X}$ is nonempty, for every smooth function $\varphi : \overline{\mathcal{X}} \to \mathbb{S}^{\mathfrak{p}-1}$ we denote

$$W^{1/p}_{\varphi}(\mathcal{X}, \mathbb{S}^{\mathfrak{p}-1}) := \{ u \in W^{1/p}(\mathcal{X}, \mathbb{S}^{\mathfrak{p}-1}) \mid u = \varphi \text{ on } \partial\mathcal{X} \} \,.$$

Similarly to Proposition 5.5, we also obtain:

Proposition 5.7 *Let* $n \geq \mathfrak{p} \geq 2$*. For every* $u \in W^{1/p}_{\varphi}(\mathcal{X}, \mathbb{S}^{\mathfrak{p}-1})$ *there exists an i.m. rectifiable current* $L \in \mathcal{R}_{n-\mathfrak{p}+1}(\mathcal{X})$ *such that* $\partial L = \mathbf{P}(u)$ *and*

$$\mathbf{M}(L) \leq C \left(\mathcal{E}_{1/p}(u) + \| \mathrm{Ext}(u) \|^p_{L^p(\mathcal{C}^{n+1})} + \mathcal{E}_{1/p}(\varphi) + \| \mathrm{Ext}(\varphi) \|^p_{L^p(\mathcal{C}^{n+1})} \right),$$

where $C > 0$ *is an absolute constant, not depending on* u *and* φ*.*

We may similarly prove a local version of Proposition 5.5, concerning the integral mass of the singularity $\mathbf{P}(u)$ relative to any open set $\Omega \subset \mathcal{X}$, see Definition 5.1. This means that we look for an integral minimal connection for the mass of the restriction $\mathbf{P}(u) \llcorner \Omega$, allowing connections to the boundary of Ω. We readily obtain:

Corollary 5.8 *For every* $u \in W^{1/p}(\mathcal{X}, \mathbb{S}^{\mathfrak{p}-1})$ *and every open set* $\Omega \subset \mathcal{X}$ *we have*

$$m_{i,\Omega}(\mathbf{P}(u)) \leq C \, \| \operatorname{Ext}(u) \|^p_{W^{1,p}(\Omega \times [0,1])} \,.$$

An example. We now give an explicit example that may clarify the statement of Proposition 5.5. We follow an idea that goes back to Bethuel [4, 3.1].

Example 5.9 Consider the $(n+1)$-dimensional open cylinder $\widetilde{\mathcal{C}}^{n+1} := B^{\mathfrak{p}} \times B^{n-\mathfrak{p}+1}$, where $n \geq \mathfrak{p} \geq 2$, and let $\mathcal{X}$ be the boundary n-manifold $\mathcal{X} := \partial \widetilde{\mathcal{C}}^{n+1}$. We shall denote by 0_d and $| \cdot |_d$ the origin and the Euclidean norm on $\mathbb{R}^d$, respectively.

Let $u : \mathcal{X} \to \mathbb{S}^{\mathfrak{p}-1}$ be given by

$$u(z) := \frac{x}{|x|_{\mathfrak{p}}} \,, \qquad z = (x, \widetilde{x}) \in \mathbb{R}^{\mathfrak{p}} \times \mathbb{R}^{n-p+1} \simeq \mathbb{R}^{n+1} \,.$$

It turns out that $u \in R^{\infty}_{1/p}(\mathcal{X}, \mathbb{S}^{\mathfrak{p}-1})$, with singular set $\Sigma(u) = \{0_{\mathfrak{p}}\} \times \mathbb{S}^{n-\mathfrak{p}}$, see (1.6), and homological singularities given by the i.m. rectifiable current

$$\mathbf{P}(u) = [\![\, \{0_{\mathfrak{p}}\} \times \mathbb{S}^{n-\mathfrak{p}} \,]\!] \in \mathcal{R}_{n-\mathfrak{p}}(\mathcal{X}) \,.$$

We have to show that u is the trace on $\partial \widetilde{\mathcal{C}}^{n+1}$ of a smooth function $U : \widetilde{\mathcal{C}}^{n+1} \to B^{\mathfrak{p}}$ that belongs to the Sobolev space $W^{1,p}(\widetilde{\mathcal{C}}^{n+1}, \mathbb{R}^{\mathfrak{p}})$, the other properties being readily checked.

To this purpose, consider the map $V : B^{\mathfrak{p}} \times [0,1] \to B^{\mathfrak{p}}$ given by

$$V(x,\rho) := \begin{cases} \widetilde{\Pi}\left(\dfrac{(x, \rho - 1)}{|(x, \rho-1)|_{\mathfrak{p}+1}} \right) & \text{if} \quad |(x,\rho-1)|_{\mathfrak{p}+1} < 1 \\ x & \text{otherwise} \,, \end{cases}$$

where $\rho \in [0,1]$ and $\widetilde{\Pi} : \mathbb{R}^{\mathfrak{p}} \times \mathbb{R} \to \mathbb{R}^{\mathfrak{p}}$ is the orthogonal projection onto the first $\mathfrak{p}$ coordinates. Clearly V is smooth outside the point $(0_{\mathfrak{p}}, 1)$, and V belongs to $W^{1,q}(B^{\mathfrak{p}} \times [0,1[, \mathbb{R}^{\mathfrak{p}})$ for every $1 \leq q < \mathfrak{p}+1$, in particular for $q = p$, as $\mathfrak{p} = [p]$. Moreover, the trace of V satisfies

$$V(x,\rho) = \frac{x}{|x|_{\mathfrak{p}}} \qquad \text{on} \quad (\partial B^{\mathfrak{p}} \times [0,1[) \cup (B^{\mathfrak{p}} \times \{1\}) \,.$$

It then clearly suffices to define U by means of a rotation on the x-variables, i.e.,

$$U(x, \widetilde{x}) := V(x, |\widetilde{x}|_{n-\mathfrak{p}+1}) \,.$$

We now observe that

$$U^{-1}(0_{\mathbb{R}^{\mathfrak{p}}}) = \{0_{\mathbb{R}^{\mathfrak{p}}}\} \times B^{n-\mathfrak{p}+1} \,.$$

In general, for every $y \in B^{\mathfrak{p}}$ the pull-back $U^{-1}(y)$ is an $(n-\mathfrak{p}+1)$-surface given by the rotation on the x-variables (reflection, for $n = \mathfrak{p}$) of the 1-dimensional subset $V^{-1}(y)$ of $B^{\mathfrak{p}} \times [0,1[$, and we have

$$V^{-1}(y) = I^y_1 \cup I^y_2 \,,$$

where I^y_i is the line segment connecting the points P^y_0 with P^y_i, for $i = 1, 2$, and

$$P^y_0 := \big(y, 1 - \sqrt{1 - |y|^2_{\mathfrak{p}}} \, \big) \,, \qquad P^y_1 := (0_{\mathfrak{p}}, 1) \,, \qquad P^y_2 := (y, 0) \,.$$

In particular, for every y the boundary of $U^{-1}(y)$ agrees with the $(n-\mathfrak{p})$-sphere $\{0_{\mathbb{R}^{\mathfrak{p}}}\} \times \mathbb{S}^{n-\mathfrak{p}}$, i.e., with the singular set $\Sigma(u)$ of u.

Let L^U_y denote the i.m. rectifiable current in $\mathcal{R}_{n-\mathfrak{p}+1}(\widetilde{\mathcal{C}}^{n+1})$ given by the integration of forms in $\mathcal{D}^{n-\mathfrak{p}+1}(\widetilde{\mathcal{C}}^{n+1})$ over the naturally oriented $(n-\mathfrak{p}+1)$-surface $U^{-1}(y)$, see (5.5) below. It turns out that

$$\partial L^U_y = \mathbf{P}(u) \,.$$

Finally, choosing y so that $|y|_{\mathfrak{p}} > 1/2$, and denoting by $\widehat{\Pi} : \widetilde{\mathcal{C}}^{n+1} \setminus \{0_{n+1}\} \to \partial\widetilde{\mathcal{C}}^{n+1} = \mathcal{X}$ the projection

$$\widehat{\Pi}(z) := \frac{z}{\max\{|x|_{\mathfrak{p}},\, |\widetilde{x}|_{n-\mathfrak{p}+1}\}}\,, \qquad z = (x, \widetilde{x})\,,$$

since $\widehat{\Pi}_{\#}\mathbf{P}(u) = \mathbf{P}(u)$ we conclude that the current $\widehat{\Pi}_{\#} L_y^U \in \mathcal{R}_{n-\mathfrak{p}+1}(\mathcal{X})$ satisfies all the requirements of Proposition 5.5.

PROOFS. We now give the proof of Propositions 5.5 and 5.7.

PROOF OF PROPOSITION 5.5: Let $U(x,t) := \mathrm{Ext}(u)(x,t)\cdot\eta(t) \in W^{1,p}(\mathcal{C}^{n+1}, \mathbb{R}^{\mathfrak{p}})$, where $\eta : [0,1] \to [0,1]$ is a smooth decreasing function such that $\eta(t) = 1$ for $t \in [0, 1/4]$, $\eta(t) = 0$ for $t \in [3/4, 1]$ and $\|\eta'\|_\infty \le 4$. Notice that we have

$$\mathbf{D}_{\mathfrak{p}}(U) \le c_1(p,\mathcal{X})\, \mathbf{D}_p(U) \le c_2(p,\mathcal{X})\, \big(\mathbf{D}_p(\mathrm{Ext}(u)) + \|\,\mathrm{Ext}(u)\|^p_{L^p(\mathcal{C}^{n+1})}\big)$$

for some absolute constants $c_i(p,\mathcal{X}) > 0$, not depending on $\mathrm{Ext}(u)$, and recall that $\mathcal{E}_{1/p}(u) := \mathbf{D}_p(\mathrm{Ext}(u))$. By a projection argument we may assume that the image of U is contained in the closure $\overline{B}^{\mathfrak{p}}$ of the unit $\mathfrak{p}$-ball. Moreover, by definition U is smooth on $\mathcal{X}\times]0,1]$, and $U(x,0) = u(x)$, $U(x,1) \equiv 0_{\mathbb{R}^{\mathfrak{p}}}$.

Denote by $J_{\mathfrak{p}}(U)$ the $\mathfrak{p}$-dimensional Jacobian of U, so that

$$\frac{1}{\mathfrak{p}^{\mathfrak{p}/2}} |DU(z)|^{\mathfrak{p}} \ge J_{\mathfrak{p}}(U)(z) \qquad \forall\, z \in \mathcal{X}\times]0,1]\,.$$

By the coarea formula, as in [2] we have

$$\mathbf{D}_{\mathfrak{p}}(U) \ge \int_{\mathcal{C}^{n+1}} J_{\mathfrak{p}}(U)\, d\mathcal{H}^{n+1} = \int_{B^{\mathfrak{p}}} \mathcal{H}^{n-\mathfrak{p}+1}(U^{-1}(y))\, d\mathcal{L}^{\mathfrak{p}}(y)\,.$$

Therefore, we find a regular value $y \in B^{\mathfrak{p}} \setminus \{0_{\mathbb{R}^{\mathfrak{p}}}\}$ of U such that

$$\mathcal{H}^{n-\mathfrak{p}+1}(U^{-1}(y)) \le \frac{2}{|B^{\mathfrak{p}}|}\, \mathbf{D}_{\mathfrak{p}}(U)\,.$$

Define the current $\mathbb{D}(U) \in \mathcal{D}_{n-\mathfrak{p}+1}(\mathcal{X})$ by

$$\mathbb{D}(U)(\gamma) = \int_{\mathcal{C}^{n+1}} \widetilde{\gamma} \wedge U^{\#}\omega_{B^{\mathfrak{p}}}\,, \qquad \gamma \in \mathcal{D}^{n-\mathfrak{p}+1}(\mathcal{X})\,,$$

where $\widetilde{\gamma} := \gamma \wedge \eta \in \mathcal{D}^{n-\mathfrak{p}+1}(\mathcal{C}^{n+1})$ and $\omega_{B^{\mathfrak{p}}}$ is given by (3.9). Arguing as in (3.6) for (3.8), we have

$$\mathbf{P}(u) = \partial\, \mathbb{D}(U) \qquad \text{on} \quad \mathcal{D}^{n-\mathfrak{p}}(\mathcal{X})\,. \tag{5.3}$$

Similarly to [20, Vol. II, Sec. 5.2.1], we now define the smooth $(n-\mathfrak{p}+1)$-vector field $D(U)$ as the dual to $U^{\#}\omega_{B^{\mathfrak{p}}}$, i.e., in local coordinates,

$$\langle \eta, D(U)(z)\rangle\, dz := \eta \wedge U^{\#}\omega_{B^{\mathfrak{p}}}(z) \qquad \forall\, \eta \in \Lambda^{n-\mathfrak{p}+1}(\mathbb{R}^{n+1})\,.$$

More precisely, $D(U)$ may be identified with $\star U^{\#}\omega_{B^{\mathfrak{p}}}$, where $\star$ is the *Hodge operator*. We thus have

$$\mathbb{D}(U)(\gamma) = \int_{\mathcal{C}^{n+1}} \langle \widetilde{\gamma}, D(U)\rangle\, d\mathcal{H}^{n+1}(z) \qquad \forall\, \gamma \in \mathcal{D}^{n-\mathfrak{p}+1}(\mathcal{X})\,. \tag{5.4}$$

Also, if $U(z) = y$ the $(n-\mathfrak{p}+1)$-vector $D(U)(z)$ is tangent to the naturally oriented level $(n-\mathfrak{p}+1)$-surface

$$U^{-1}(y) := \{z \in \mathcal{C}^{n+1} \mid U(z) = y\}\,.$$

As a consequence, the $(n-\mathfrak{p}+1)$-current

$$L_y^U := \tau\Big(U^{-1}(y), 1, \frac{D(U)}{|D(U)|}\Big) \tag{5.5}$$

turns out to be an i.m. rectifiable current $L_y^U \in \mathcal{R}_{n-\mathfrak{p}+1}(\mathcal{C}^{n+1})$ with mass

$$\mathbf{M}(L_y^U) = \mathcal{H}^{n-\mathfrak{p}+1}(U^{-1}(y))\,.$$

Moreover, since $U(x,1) \equiv 0_{\mathbb{R}^\mathfrak{p}}$, and $y \neq 0_{\mathbb{R}^\mathfrak{p}}$, by (5.3) and (5.4) we infer that

$$(\partial L_y^U) \llcorner \mathcal{X} \times [0,1] = \mathbf{P}(u)\,.$$

Setting $L := \Pi_\# L_y^U$, where $\Pi : \mathcal{X} \times [0,1] \to \mathcal{X}$ is the projection map $\Pi(x,t) := x$, the assertion readily follows. □

Proof of Proposition 5.7: Let $\Phi(x,t) := \mathrm{Ext}(\varphi)(x,t)\cdot\eta(t) \in W^{1,p}(\mathcal{C}^{n+1},\mathbb{R}^N)$ and $L_y^\Phi \in \mathcal{R}_{n-\mathfrak{p}+1}(\mathcal{C}^{n+1})$ be given by

$$L_y^\Phi := \tau\Big(\Phi^{-1}(y), 1, \frac{D(\Phi)}{|D(\Phi)|}\Big)\,,$$

so that $\mathbf{M}(L_y^\Phi) = \mathcal{H}^{n-\mathfrak{p}+1}(\Phi^{-1}(y))$. Since $u = \varphi$ on $\partial\mathcal{X}$ and $\mathbf{P}(\varphi) = 0$, this time we have

$$\partial(L_y^U - L_y^\Phi) = \mathbf{P}(u)\,.$$

Similarly to Proposition 5.5, we readily prove the assertion. □

6 The case of general target manifolds

In this section we extend Propositions 5.5 and 5.7 to more general target manifolds $\mathcal{Y}$ as in Sec. 4. More precisely, in the case $\mathfrak{p} = 2$ we shall *assume that* $\pi_1(\mathcal{Y})$ *is commutative*, whereas in the case $\mathfrak{p} \geq 3$, we shall *assume that* $\pi_1(\mathcal{Y}) = 0$ *and that the Hurewicz homomorphism from the* $(\mathfrak{p}-1)^{th}$ *free homotopy group* $\pi_{\mathfrak{p}-1}(\mathcal{Y})$ *onto the* $(\mathfrak{p}-1)^{th}$ *real homology group* $\mathcal{H}_{\mathfrak{p}-1}(\mathcal{Y};\mathbb{R})$ *is injective.*

If the boundary $\partial\mathcal{X}$ is nonempty, for every smooth function $\varphi : \overline{\mathcal{X}} \to \mathcal{Y}$ we shall denote as above

$$\mathcal{R}^\infty_{1/p,\varphi}(\mathcal{X},\mathcal{Y}) := \{u \in \mathcal{R}^\infty_{1/p}(\mathcal{X},\mathcal{Y}) \mid u = \varphi \text{ on } \partial\mathcal{X}\}\,.$$

Theorem 6.1 *Let $n \geq \mathfrak{p} := [p] \geq 2$. Let $\varphi : \overline{\mathcal{X}} \to \mathcal{Y}$ be a smooth function and let $u \in \mathcal{R}^\infty_{1/p,\varphi}(\mathcal{X},\mathcal{Y})$. Then for every $s = 1,\dots,\widetilde{s}$ there exists an i.m. rectifiable current $L_s \in \mathcal{R}_{n-\mathfrak{p}+1}(\mathcal{X})$ such that $\partial L_s = \mathbb{P}_s(u)$ and the mass*

$$\mathbf{M}(L_s) \leq C\left(\mathcal{E}_{1/p}(u) + \|\mathrm{Ext}(u)\|^p_{L^p(\mathcal{C}^{n+1})} + \mathcal{E}_{1/p}(\varphi) + \|\mathrm{Ext}(\varphi)\|^p_{L^p(\mathcal{C}^{n+1})}\right),$$

where $C > 0$ is an absolute constant, not depending on u and φ. Moreover, if $\partial\mathcal{X} = \emptyset$ and $u \in \mathcal{R}^\infty_{1/p}(\mathcal{X},\mathcal{Y})$, we have

$$\mathbf{M}(L_s) \leq C\left(\mathcal{E}_{1/p}(u) + \|\mathrm{Ext}(u)\|^p_{L^p(\mathcal{C}^{n+1})}\right).$$

Remark 6.2 As in Remark 5.6, from the proof of Theorem 6.1 we infer that in the above estimates for the mass of L_s we can remove the extra terms $C\,(\|\mathrm{Ext}(u)\|^p_{L^p(\mathcal{C}^{n+1})} + \|\mathrm{Ext}(\varphi)\|^p_{L^p(\mathcal{C}^{n+1})})$ provided that we require that the integral connections L_s belong to $\mathcal{R}_{n-\mathfrak{p}+1}(\mathcal{C}^{n+1})$. Moreover, since $\mathcal{Y}$ is compact, if the boundary datum φ is constant we have $\mathcal{E}_{1/p}(\varphi) = 0$ and $\|\mathrm{Ext}(\varphi)\|^p_{L^p(\mathcal{C}^{n+1},\mathbb{R}^N)} \leq C$.

Let $\Omega \subset \mathcal{X}$ be an open set. According to Definition 5.1, we may look for an integral minimal connection for the mass of $\mathbb{P}_s(u) \llcorner \Omega$ allowing connections to the boundary of Ω. We thus readily extend Corollary 5.8 as follows:

Corollary 6.3 *For every $u \in \mathcal{R}^\infty_{1/p}(\mathcal{X},\mathcal{Y})$ and every open set $\Omega \subset \mathcal{X}$ we have*

$$m_{i,\Omega}(\mathbb{P}_s(u)) \leq C\,\|\mathrm{Ext}(u)\|^p_{W^{1,p}(\Omega\times[0,1])} \qquad \forall\, s = 1,\dots\widetilde{s}\,.$$

In the case $n = \mathfrak{p}$, or $n \geq \mathfrak{p} = 2$, we finally obtain:

Proposition 6.4 *If $n=\mathfrak{p}$ or $\mathfrak{p}=2$, Theorem* 6.1 *and Corollary* 6.3 *hold true for the whole classes of maps in $W^{1/p}_\varphi(\mathcal{X},\mathcal{Y})$ or in $W^{1/p}(\mathcal{X},\mathcal{Y})$.*

PROOF: Assume $u\in W^{1/p}_\varphi(\mathcal{X},\mathcal{Y})$, and let $\{u_k\}\subset\mathcal{R}^\infty_{1/p,\varphi}(\mathcal{X},\mathcal{Y})$ converge strongly in $W^{1/p}$ to u, see Remark 6.6. For $s=1,\ldots,\widetilde{s}$, as in Proposition 5.3, we have that $\mathbf{M}(\mathbb{D}_s(u_k)-\mathbb{D}_s(u))\to 0$ as $k\to\infty$; whence, if $n=\mathfrak{p}$, there exists $L^s_{u_k,u}\in\mathcal{R}_1(\mathcal{X})$ such that (5.1) holds and $\mathbf{M}(L^s_{u_k,u})\to 0$ as $k\to\infty$. By applying Theorem 6.1 to each u_k we find $L^k_s\in\mathcal{R}_{n-\mathfrak{p}+1}(\mathcal{X})$ such that $\partial L^k_s=\mathbb{P}_s(u_k)$ and

$$\mathbf{M}(L^k_s)\le C\left(\mathcal{E}_{1/p}(u_k)+\|\operatorname{Ext}(u_k)\|^p_{L^p(\mathcal{C}^{n+1})}+\mathcal{E}_{1/p}(\varphi)+\|\operatorname{Ext}(\varphi)\|^p_{L^p(\mathcal{C}^{n+1})}\right).$$

Since $\mathcal{E}_{1/p}(u_k)\to\mathcal{E}_{1/p}(u)$ and $\|\operatorname{Ext}(u_k)\|^p_{L^p(\mathcal{C}^{n+1})}\to\|\operatorname{Ext}(u)\|^p_{L^p(\mathcal{C}^{n+1})}$ as $k\to\infty$, the assertion follows by taking $L_s:=L^s_{u_k,u}+L^k_s$ for k large. Moreover, if $\mathfrak{p}=2$, Hardt-Pitts' theorem [28] yields (5.2), whence Proposition 5.3 holds in any dimension $n\ge 2$, and we proceed as above. Finally, the extension of Corollary 6.3 is proved in a similar way. □

PROOF OF THEOREM 6.1. The rest of this section is dedicated to the proof of Theorem 6.1. We shall make use of arguments by Pakzad-Rivière [40], to which we refer for further details.

We first observe that by the hypotheses, all the homotopy groups $\pi_{\mathfrak{p}-1}(\mathcal{Y};y_0)$ are canonically isomorphic, and that there exists an isomorphism $\rho_\mathfrak{p}$ between $\mathcal{H}_{\mathfrak{p}-1}(\mathcal{Y})$ and $\pi_{\mathfrak{p}-1}(\mathcal{Y})$. Since $[\gamma_1],\ldots,[\gamma_{\widetilde{s}}]$ generate the spherical subgroup $\mathcal{H}^{sph}_{\mathfrak{p}-1}(\mathcal{Y})$, we infer that the equivalence classes $\Gamma_s:=\rho_\mathfrak{p}[\gamma_s]\in\pi_{\mathfrak{p}-1}(\mathcal{Y})$, for $s=1,\ldots,\widetilde{s}$, generate the subgroup $\rho_\mathfrak{p}(\mathcal{H}^{sph}_{\mathfrak{p}-1}(\mathcal{Y}))$ of $\pi_{\mathfrak{p}-1}(\mathcal{Y})$.

According to Proposition 3.6, and following the notation from [40, Def. 2.7], for any given map $u\in\mathcal{R}^\infty_{1/p}(\mathcal{X},\mathcal{Y})$, if $\Sigma(u)\subset B=\cup_{i=1}^\mu\sigma_i$, where this time the σ_i's are $(n-\mathfrak{p})$-dimensional (and curvilinear) non-overlapping polyhedra, we have

$$(-1)^\mathfrak{p}\,\mathbb{P}(u)=\sum_{s=1}^{\widetilde{s}}\mathbb{P}_s(u)\otimes[\gamma_s]\,,\qquad \mathbb{P}_s(u)=\sum_{i=1}^\mu m_{i,s}\,[\![\,\sigma_i\,]\!]\in\mathcal{R}_{n-\mathfrak{p}}(\mathcal{X})\,,$$

for some integers $m_{i,s}\in\mathbb{Z}$. Moreover, we have

$$c(n,\mathfrak{p})\cdot\rho_\mathfrak{p}\Big(\sum_{s=1}^{\widetilde{s}}m_{i,s}\,[\gamma_s]\Big)=[u,\sigma_i]:=[u_{|\Sigma_{a,\delta}}]_{\pi_{\mathfrak{p}-1}(\mathcal{Y})}\,,$$

for some given constant sign $c(n,\mathfrak{p})=\pm 1$, only depending on n and $\mathfrak{p}$.

We also recall from [40, Def. 2.8] that the current $\mathbf{S}_u\in\mathcal{R}_{n-\mathfrak{p}}(\mathcal{X};\pi_{\mathfrak{p}-1}(\mathcal{Y}))$ given by

$$\mathbf{S}_u:=\sum_{i=1}^\mu[\![\,\sigma_i\,]\!]\otimes[u,\sigma_i] \tag{6.1}$$

describes the *topological singularity* of u. Notice that the induced homomorphism $\rho_{\mathfrak{p}*}:\mathcal{R}(\mathcal{X};\mathcal{H}_{\mathfrak{p}-1}(\mathcal{Y}))\to\mathcal{R}(\mathcal{X};\pi_{\mathfrak{p}-1}(\mathcal{Y}))$ satisfies

$$\rho_{\mathfrak{p}*}((-1)^\mathfrak{p}\,\mathbb{P}(u)):=\sum_{s=1}^{\widetilde{s}}\mathbb{P}_s(u)\otimes\rho_\mathfrak{p}[\gamma_s]$$

and we thus have

$$\mathbf{S}_u=\sum_{s=1}^{\widetilde{s}}\mathbf{T}_s(u)\otimes\Gamma_s\,,\qquad \mathbf{T}_s(u):=c(n,\mathfrak{p})\;\mathbb{P}_s(u)\,. \tag{6.2}$$

Remark 6.5 In the model case $\mathcal{Y}=\mathbb{S}^{p-1}$, for $u\in\mathcal{R}^\infty_{1/p}(\mathcal{X},\mathbb{S}^{p-1})$, we have

$$\mathbf{P}(u)=\sum_{i=1}^\mu m_i\,[\![\,\sigma_i\,]\!]\in\mathcal{R}_{n-\mathfrak{p}}(\mathcal{X})\,,$$

with $m_i \in \mathbb{Z}$, whereas the topological singularity is simply defined by

$$\mathbf{S}_u := \sum_{i=1}^{\mu} c(n,\mathfrak{p}) \cdot m_i \, [\![\, \sigma_i \,]\!] \,, \qquad c(n,\mathfrak{p}) \cdot m_i = [u_{|\Sigma_{a,\delta}}]_{\pi_{\mathfrak{p}-1}(\mathbb{S}^{p-1})} \in \mathbb{Z} \,. \tag{6.3}$$

We divide the rest of the proof in eight steps.

STEP 1: For $\mathfrak{p} \le l \le M+1$, where $M := \dim(\mathcal{Y})$, let $\mathcal{Y}^{l-1}$ denote the $(l-1)$-skeleton of some finite (curvilinear) triangulation of $\mathcal{Y}$, so that $\mathcal{Y}^M = \mathcal{Y}$. For $X = C^\infty$, $W^{1/p}$, $\mathcal{R}^\infty_{1/p}$, or $\mathcal{R}^\infty_{1/p,\varphi}$, where $\varphi : \mathcal{X} \to \mathcal{Y}^{l-1}$ is a smooth $W^{1/p}$-function, we shall denote

$$X(\mathcal{X}, \mathcal{Y}^{l-1}) := \{ u \in X(\mathcal{X}, \mathcal{Y}) \mid u(x) \in \mathcal{Y}^{l-1} \text{ for } \mathcal{H}^n\text{-a.e. } x \in \mathcal{X} \} \,.$$

Remark 6.6 If $\partial\mathcal{X} = \emptyset$, in this proof we shall identify $\mathcal{R}^\infty_{1/p} = \mathcal{R}^\infty_{1/p,\varphi}$ for some *constant* map $\varphi : \mathcal{X} \to \mathcal{Y}$. Similarly to Theorem 1.1, it is not difficult to show that for every $\mathfrak{p} \le l \le M+1$ the class $\mathcal{R}^\infty_{1/p,\varphi}(\mathcal{X}, \mathcal{Y}^{l-1})$ is dense in $W^{1/p}_\varphi(\mathcal{X}, \mathcal{Y}^{l-1})$ with respect to the strong $W^{1/p}$-topology.

Let $i^l : \mathcal{Y}^{l-1} \hookrightarrow \mathcal{Y}^l$ denote the injection map from $\mathcal{Y}^{l-1}$ into $\mathcal{Y}^l$. Since the homomorphism $i^l_* : \pi_{\mathfrak{p}-1}(\mathcal{Y}^{l-1}) \to \pi_{\mathfrak{p}-1}(\mathcal{Y}^l)$ induced by i^l is onto, we infer that $\pi_{\mathfrak{p}-1}(\mathcal{Y}^{l-1})$ is finitely generated.

As a consequence, we may and do define the topological singularity of a map $v = u^{l-1} \in \mathcal{R}^\infty_{1/p}(\mathcal{X}, \mathcal{Y}^{l-1})$ as the current $\mathbf{S}^{l-1}_v \in \mathcal{R}_{n-\mathfrak{p}}(\mathcal{X}; \pi_{\mathfrak{p}-1}(\mathcal{Y}^{l-1}))$

$$\mathbf{S}^{l-1}_v := \sum_{i=1}^{\mu} [\![\, \sigma_i \,]\!] \otimes [v, \sigma_i] \,, \qquad [v, \sigma_i] := [v_{|\Sigma_{a,\delta}}]_{\pi_{\mathfrak{p}-1}(\mathcal{Y}^{l-1})} \,.$$

Of course, $\mathbf{S}^{l-1}_u$ agrees with $\mathbf{S}_u$ from (6.1) in the case $l = M+1$, i.e., for maps u in $\mathcal{R}^\infty_{1/\mathfrak{p}}(\mathcal{X}, \mathcal{Y})$.

Finally, we shall denote by $\widetilde{\chi}^l_* : \mathcal{R}_{n-\mathfrak{p}}(\mathcal{X}; \pi_{\mathfrak{p}-1}(\mathcal{Y}^{l-1})) \to \mathcal{R}_{n-\mathfrak{p}}(\mathcal{X}; \pi_{\mathfrak{p}-1}(\mathcal{Y}^l))$ the corresponding homomorphism induced by i^l, so that

$$\widetilde{\chi}^l_*(\mathbf{S}^{l-1}_v) = \sum_{i=1}^{\mu} [\![\, \sigma_i \,]\!] \otimes i^l_*[v, \sigma_i] \,, \qquad i^l_*[v, \sigma_i] \in \pi_{\mathfrak{p}-1}(\mathcal{Y}^l) \,.$$

STEP 2: As in (4.3), for $\mathfrak{p} \le l \le M$ we denote by

$$\mathcal{Y}^l_\varepsilon := \overline{U_\varepsilon(\mathcal{Y}^l)}$$

the ε-neighborhood of $\mathcal{Y}^l$ in $\mathbb{R}^N$. Since the triangulation is finite, we can find $\varepsilon_l > 0$ and a Lipschitz projection Π_l of $\mathcal{Y}^l_{\varepsilon_l}$ onto $\mathcal{Y}^l$ satisfying the following properties:

i) $\mathcal{H}^N(\Pi_l^{-1}(w)) = 0$ for every $w \in \mathcal{Y}^l \setminus \mathcal{Y}^{l-1}$;

ii) if $y, y_0 \in \mathcal{Y}^l_{\varepsilon_l} \setminus \mathcal{Y}^l$, with $y \neq y_0$, satisfy

$$\frac{y - y_0}{|y - y_0|} = \frac{y - \Pi_l(y)}{d_l(y)} \,, \qquad d_l(y) := |\Pi_l(y) - y| \,, \tag{6.4}$$

then $\Pi_l(y) = \Pi_l(y_0) \in \mathcal{Y}^l$;

iii) setting for every $y_0 \in \mathcal{Y}^l_{\varepsilon_l} \setminus \mathcal{Y}^l$

$$\Delta_l(y_0) := \max\{ d_l(y) \mid y \in \mathcal{Y}^l_{\varepsilon_l} \setminus \mathcal{Y}^l \text{ satisfies (6.4)} \} \,,$$

the function $\Delta_l : \mathcal{Y}^l_{\varepsilon_l} \setminus \mathcal{Y}^l \to \mathbb{R}^+$ is Lipschitz continuous, with Lipschitz constant $\mathrm{Lip}(\Delta_l) \le c \; \mathrm{Lip}(\Pi_l)$, and $\Delta_l(y_0) = d_l(y_0)$ if $y_0 \in \partial\mathcal{Y}^l_{\varepsilon_l}$.

STEP 3: For $\mathfrak{p} \le l \le M$, let

$$U^l := \{(x,y) \in B^l \times B^l \mid x \ne y\}$$

and $p_l : U^l \times U^l \to \partial B^l$ be such that $p_l(x,y)$ is the unique point on the boundary ∂B^l which is on the ray from x to y, see [40, Def. 2.9]. We recall that for every $0 < \delta < 1$ we have

$$\int_{B^l(0,1-\delta)} |D_y p_l(x,y_0)|^p \, dx \le C(l,p,\delta) < \infty \qquad \forall\, y_0 \in B^l \,, \tag{6.5}$$

where the constant $C(l,p,\delta)$ does not depend on y_0.

As in [40, Sec. 4], write $\mathcal{Y}^l = \bigcup_{i=1}^{s_l} N_i^l$, where $\xi_i^l : B^l \to N_i^l := \xi_i^l(B^l)$ are diffeomorphisms and each two different N_i^l's either are pairwise disjoint or intersect on a lower dimensional face in $\mathcal{Y}^{l-1}$. For $w := (w_1, \dots, w_{s_l})$, where $w_i \in N_i^l \setminus \mathcal{Y}^{l-1}$ for every $i = 1, \dots, s_l$, let $p_w^l : \mathcal{Y}^l \setminus \{w_1, \dots, w_{s_l}\} \to \mathcal{Y}^{l-1}$ be the map

$$p_w^l(y) := \begin{cases} \xi_i^l\big(p_l({\xi_i^l}^{-1}(w_i), {\xi_i^l}^{-1}(y))\big) & \text{if } \; y \in N_i^l \setminus \mathcal{Y}^{l-1} \,, \; i = 1, \dots, s_l \,, \\ y & \text{otherwise}\,. \end{cases}$$

Similarly to [40, Lemma 4.1], we obtain:

i) p_w^l is well defined and locally Lipschitz;

ii) for every $(\mathfrak{p}-1)$-cycle C in $\mathcal{Y}^l$, with support $\operatorname{spt} C \subset \mathcal{Y}^l \setminus \{w_1, \dots, w_{s_l}\}$, we have

$$i_*^l([p_w^l(C)]_{\pi_{\mathfrak{p}-1}(\mathcal{Y}^{l-1})}) = [C]_{\pi_{\mathfrak{p}-1}(\mathcal{Y}^l)} \,; \tag{6.6}$$

iii) setting $N_{i,\varepsilon}^l := \xi_i^l(B^l(0,1-\varepsilon))$ and $N_\varepsilon^l := N_{1,\varepsilon}^l \times \cdots \times N_{s_l,\varepsilon}^l$, for every $0 < \varepsilon < 1$ and $y \in \mathcal{Y}^l$ we have

$$\int_{N_\varepsilon^l} |Dp_w^l(y)|^p \, d\mathcal{H}^{ls_l}(w) \le C(p,l,\varepsilon) < \infty \,. \tag{6.7}$$

STEP 4: For $\mathfrak{p} \le l \le M$, using the projection map Π_l from Step 2, we extend p_w^l to the map

$$P_w^l : \mathcal{Y}_{\varepsilon_l}^l \setminus \bigcup_{i=1}^{s_l} \Pi_l^{-1}(w_i) \to \mathbb{R}^N$$

defined for every $y \in \operatorname{dom}(P_w^l) \setminus \mathcal{Y}^l$ by

$$P_w^l(y) := \frac{d_l(y)}{\Delta_l(y)}\, y + \left(1 - \frac{d_l(y)}{\Delta_l(y)}\right) p_w^l(\Pi_l(y)) \,.$$

Since $|D[p_w^l(\Pi_l(y)]| \le C\,(\operatorname{Lip}\Pi_l)\,|Dp_w^l(\Pi_l(y))|$ and $|Dd_l(y)| \le C\,(\operatorname{Lip}\Pi_l)$, whereas Δ_l is Lipschitz continuous, it turns out that P_w^l is locally Lipschitz, too. Moreover, since $P_w^l(y) = y$ for $y \in \partial\mathcal{Y}_{\varepsilon_l}^l$, we may and do extend P_w^l to a locally Lipschitz map equal to the identity on $\mathbb{R}^N \setminus \operatorname{int}(\mathcal{Y}_{\varepsilon_l}^l)$.

By using (6.7), we similarly obtain that for every $y \in \mathbb{R}^N$ and for $\varepsilon > 0$ small

$$\int_{N_\varepsilon^l} |DP_w^l(y)|^p \, d\mathcal{H}^{ls_l}(w) \le \frac{1}{3}\, C(p,\varepsilon,\varepsilon_l,\mathcal{Y}) < \infty \,, \tag{6.8}$$

where the constant $C(p,\varepsilon,\varepsilon_l,\mathcal{Y}) > 0$ does not depend on $y \in \mathbb{R}^N$.

In fact, as in [40, Lemma 3.1], if $y \in \Pi_l^{-1}(N_{\varepsilon/2}^l)$ we infer that (6.8) follows from the smoothness of P_w^l, the definition of p_w^l, and (6.5). If $y \in \mathbb{R}^N \setminus \Pi_l^{-1}(N_{\varepsilon/2}^l)$, then (6.8) follows from the fact that $|DP_w^l(y)| \le K$ for every $w \in N_\varepsilon^l$, where $K > 0$ is an absolute constant.

Now, for every $v \in W^{1/p}(\mathcal{X}, \mathcal{Y}^l)$, where $l \ge \mathfrak{p}$, we denote by V the extension $V := \operatorname{Ext}(v) \in W^{1,p}(\mathcal{C}^{n+1}, \mathbb{R}^N)$, where $\mathcal{C}^{n+1} := \mathcal{X} \times [0,1]$. By (6.8) and Fubini's theorem we have

$$\int_{N_\varepsilon^l} \int_{\mathcal{C}^{n+1}} |D(P_w^l(V(z)))|^p \, d\mathcal{H}^{n+1}(z)\, d\mathcal{H}^{ls_l}(w) \le \frac{1}{3}\, C(p,\varepsilon,\varepsilon_l,\mathcal{Y}) \int_{\mathcal{C}^{n+1}} |DV|^p \, d\mathcal{H}^{n+1}.$$

Since by the definition $\mathbf{T}(P^l_w \circ V) = p^l_w \circ v$, this yields that $P^l_w \circ V$ belongs to $W^{1,p}(\mathcal{C}^{n+1}, \mathbb{R}^N)$, whence $p^l_w \circ v$ belongs to $W^{1/p}(\mathcal{X}, \mathcal{Y}^{l-1})$ for $\mathcal{H}^{ls_l}$-a.e. $w \in N^l_\varepsilon$. Moreover, we find a positive $\mathcal{H}^{ls_l}$-measurable set $W \subset N^l_\varepsilon$, with positive measure

$$\mathcal{H}^{ls_l}(W) \geq \frac{2}{3}\,\mathcal{H}^{ls_l}(N^l_\varepsilon)\,, \tag{6.9}$$

such that for every $w \in W$

$$\mathcal{E}_{1/p}(p^l_w \circ v) \leq \mathbf{D}_p(P^l_w \circ V) \leq \frac{C(p,\varepsilon,\varepsilon_l,\mathcal{Y})}{\mathcal{H}^{ls_l}(N^l_\varepsilon)}\,\mathbf{D}_p(V)\,, \tag{6.10}$$

where $\mathcal{E}_{1/p}(v) := \mathbf{D}_p(V)$.

Finally, by the definition of P^l_w, and by the compactness of $\mathcal{Y}$, for any such w we also have

$$|P^l_w \circ V(z)| \leq C \cdot |V(z)| \qquad \text{for } \mathcal{H}^{n+1}\text{-a.e. } z \in \mathcal{C}^{n+1}\,, \tag{6.11}$$

where $C = C(l,\mathcal{Y}) > 0$ is an absolute constant.

STEP 5: Let $\mathfrak{p} \leq l \leq M+1$, and recall, Theorem 1.1 and Remark 6.6, that the class $\mathcal{R}^\infty_{1/p,\varphi}(\mathcal{X},\mathcal{Y}^{l-1})$ is dense in $W^{1/p}_\varphi(\mathcal{X},\mathcal{Y}^{l-1})$, for every smooth $W^{1/p}$-function $\varphi : \mathcal{X} \to \mathcal{Y}^{l-1}$. Similarly to [40, Sec. 2.2], we now show that a suitable subclass of *radial* maps in $\mathcal{R}^\infty_{1/p,\varphi}$ is dense in $W^{1/p}_\varphi$.

To this purpose, since we use a local argument, and $\mathcal{X}$ is compact, taking a local coordinate chart we may and do assume that $\mathcal{X} = \mathcal{Q}^n := [0,1]^n$, and $v \in \mathcal{R}^\infty_{1/p,\varphi}(\mathcal{Q}^n,\mathcal{Y}^{l-1})$. We then find a compact set $B \subset \text{int}(\mathcal{Q}^n)$ of the type $B = \cup_{i=1}^{\mu}\sigma_i$, where the σ_i's are non-overlapping $(n-\mathfrak{p})$-dimensional polyhedra, such that the singular set $\Sigma(v) \subset B$, see (1.6), and any two different faces of B intersect only on their boundaries. We set V^δ, σ^δ_i, $\pi_i : \sigma^\delta_i \to \sigma_i$, and $\pi : V^{\delta_0} \to B$ as in [40, Sec. 2.2], and we define $v_\delta : \mathcal{Q}^n \to \mathcal{Y}^{l-1}$ by

$$v_\delta(x) := \begin{cases} v(h_\delta(x)) & \text{if } \; x \in V^\delta \\ v(x) & \text{otherwise}\,, \end{cases} \tag{6.12}$$

where $h_\delta(x) \in \partial V^\delta$ is the unique point on the ray from $\pi(x)$ to x. Notice that the δ-neighborhood V^δ is contained in $\text{int}(\mathcal{Q}^n)$, provided that $\delta > 0$ is sufficiently small. Also, for $\mathcal{X} = \mathcal{Q}^n$, we let

$$R^\infty_{1/p,\varphi}(\mathcal{X},\mathcal{Y}^{l-1}) := \{v_\delta \mid v \in \mathcal{R}^\infty_{1/p,\varphi}(\mathcal{X},\mathcal{Y}^{l-1})\}$$

denote the subclass of *radial* maps in $\mathcal{R}^\infty_{1/p,\varphi}$. Similarly as for [40, eq. (2.3)], we observe that for $\delta_1 > 0$ sufficiently small, there is some constant K, depending only on B, for which

$$\begin{aligned} \int_{\partial V^\delta \times I} |D(\text{Ext}\, v)|^p \, d\mathcal{H}^n &\leq \frac{K}{\delta_1} \int_{V^{\delta_1} \times I} |D(\text{Ext}\, v)|^p \, d\mathcal{H}^{n+1}, \\ \int_{V^\delta \times I} |D(\text{Ext}\, v_\delta)|^p \, d\mathcal{H}^{n+1} &\leq \delta\, K \int_{\partial V^\delta \times I} |D(\text{Ext}\, v)|^p \, d\mathcal{H}^n \end{aligned} \tag{6.13}$$

for $\delta \in I_0$, a positive measure subset of $[0,\delta_1]$.

Since $\mathcal{X}$ is compact, repeating the argument for a finite cover of local charts of $\mathcal{X}$, we obtain that (6.13) holds true for every $v \in \mathcal{R}^\infty_{1/p,\varphi}(\mathcal{X},\mathcal{Y}^{l-1})$, where this time V^δ is a suitable "δ-neighborhood" of a compact set $B \subset \text{int}(\mathcal{X})$ that contains the singular set $\Sigma(v)$ of v and is given by a finite union of non-overlapping $(n-\mathfrak{p})$-dimensional curvilinear polyhedra. By using Theorem 1.1 and Remark 6.6, property (6.13) yields that also $R^\infty_{1/p,\varphi}(\mathcal{X},\mathcal{Y}^{l-1})$ is dense in $W^{1/p}_\varphi(\mathcal{X},\mathcal{Y}^{l-1})$.

STEP 6: Similarly to [40, Lemma 4.2], to which we refer for further details, we now prove:

Lemma 6.7 *Let $\mathfrak{p} \leq l \leq M$ and $u^l \in \mathcal{R}^\infty_{1/p,\varphi^l}(\mathcal{X},\mathcal{Y}^l)$ for some smooth $W^{1/p}$-map $\varphi^l : \mathcal{X} \to \mathcal{Y}^l$. Then there exists a map $u^{l-1} : \mathcal{X} \to \mathcal{Y}^{l-1}$, a smooth $W^{1/p}$-map $\varphi^{l-1} : \mathcal{X} \to \mathcal{Y}^{l-1}$, and a constant $C > 0$, independent of u^l and φ^l, such that:*

(a) $u^{l-1} \in \mathcal{R}^\infty_{1/p,\varphi^{l-1}}(\mathcal{X},\mathcal{Y}^{l-1})$;

(b) $\mathcal{E}_{1/p}(v^{l-1}) \leq C \cdot \mathcal{E}_{1/p}(v^l)$ *for both* $v = u$ *and* $v = \varphi$;

(c) $\| \operatorname{Ext}(v^{l-1})\|_{L^p(\mathcal{C}^{n+1},\mathbb{R}^N)} \le C\cdot \| \operatorname{Ext}(v^l)\|_{L^p(\mathcal{C}^{n+1},\mathbb{R}^N)}$ *for both* $v=u$ *and* $v=\varphi$*;*

(d) *according to Step 1, we have* $\widetilde{\chi}^l_*(\mathbf{S}^{l-1}_{u^{l-1}}) = \mathbf{S}^l_{u^l}$*.*

Proof: Let $U^l := \operatorname{Ext}(u^l) \in W^{1,p}(\mathcal{C}^{n+1},\mathbb{R}^N)$. Using (6.10) and (6.13), with $v := u^l$, as in [40, eq. (4.7)] we fix $\varepsilon_1,\varepsilon_2,\varepsilon_3>0$ and $0<\delta<\delta_1$ such that

$$\frac{C(p,\varepsilon,\varepsilon_l,\mathcal{Y})}{\mathcal{H}^{ls_l}(N^l_\varepsilon)}\Big(K^2\int_{V^{\delta_1}\times I}|DU^l|^p\,d\mathcal{H}^{n+1}+\delta K\varepsilon_2+\varepsilon_1\Big)+\varepsilon_3 \le \int_{\mathcal{C}^{n+1}}|DU^l|^p\,d\mathcal{H}^{n+1}. \tag{6.14}$$

Moreover, since $u^l\in\mathcal{R}^\infty_{1/p}(\mathcal{X},\mathcal{Y}^l)$, for $\mathcal{H}^{ls_l}$-a.e. $w=(w_1,\dots,w_{s_l})\in W$, where $W=W(u)\subset N^l_\varepsilon$ is the positive measure subset constructed in Step 4, see (6.9), we obtain that ${u^l}^{-1}(w_i)\cap(\mathcal{X}\setminus V^\delta)$ is a finite mass smooth submanifold of $\mathcal{X}\setminus V^\delta$ of dimension $n-l$, with smooth boundary contained in ∂V^δ, for every $i=1,\dots,s_l$. For any such w, and for $\varepsilon'>0$, we then find a Lipschitz diffeomorphism $f_{\varepsilon'}$ of $\mathcal{X}$ such that $f_{\varepsilon'}$ is the identity outside a small neighborhood of $\bigcup_{i=1}^{s_l}{u^l}^{-1}(w_i)$, and we have:

i) $f_{\varepsilon'}(V^\delta)=V^\delta$, $f_{\varepsilon'}(\partial V^\delta)=\partial V^\delta$;

ii) $(u^l\circ f_{\varepsilon'})^{-1}(w_i)\cap(\mathcal{X}\setminus V^\delta)$ is a polyhedral $(n-l)$-chain of $\mathcal{X}\setminus V^\delta$;

iii) $(u^l\circ f_{\varepsilon'})^{-1}(w_i)\cap(\partial V^\delta)$ is a polyhedral $(n-l-1)$-chain of ∂V^δ;

iv) $\displaystyle\int_{\mathcal{C}^{n+1}}|D\big(U^l\circ(f_{\varepsilon'}\bowtie Id_I)\big)-DU^l|^p\,d\mathcal{H}^{n+1}<\varepsilon'$;

v) $\displaystyle\int_{\partial V^\delta\times I}|D\big(U^l\circ(f_{\varepsilon'}\bowtie Id_I)\big)-DU^l|^p\,d\mathcal{H}^{n}<\varepsilon'$.

Setting $\varepsilon':=\min\{\varepsilon_1,\varepsilon_2\}$ and $v^l:=(u^l\circ f_{\varepsilon'})_\delta$, see (6.12), we infer that v^l has the same topological singularity as u^l on components of B. Moreover, setting $V^l:=\operatorname{Ext}(v^l)$, by iv) and v) above, and by (6.13), as in [40, eq. (4.9)] we obtain

$$\int_{\mathcal{C}^{n+1}}|DV^l|^p\,d\mathcal{H}^{n+1}\le\int_{\mathcal{C}^{n+1}}|DU^l|^p\,d\mathcal{H}^{n+1}+\Big(K^2\int_{V^{\delta_1}\times I}|DU^l|^p\,d\mathcal{H}^{n+1}+\delta K\varepsilon_2+\varepsilon_1\Big) \tag{6.15}$$

whereas by (6.10) we have

$$\int_{\mathcal{C}^{n+1}}|D(P^l_w\circ V^l)|^p\,d\mathcal{H}^{n+1}\le\frac{C(p,\varepsilon,\varepsilon_l,\mathcal{Y})}{\mathcal{H}^{ls_l}(N^l_\varepsilon)}\int_{\mathcal{C}^{n+1}}|DV^l|^p\,d\mathcal{H}^{n+1}. \tag{6.16}$$

Since $\mathbf{T}(P^l_w\circ V^l)=p^l_w\circ v^l$, from the above we infer that $v^{l-1}:=p^l_w\circ v^l$ belongs to $W^{1/p}(\mathcal{X},\mathcal{Y}^{l-1})$, and is locally Lipschitz away from

$$\Sigma(v^{l-1}):=\bigcup_{i=1}^{s_l}(u^l\circ f_{\varepsilon'})_\delta^{-1}(w_i)\cup B\,.$$

We now essentially repeat the previous construction with φ^l instead of u^l. By using (6.9), with $W=W(v)\subset N^l_\varepsilon$ corresponding to both $v=u^l$ and $v=\varphi^l$, we may and do choose

$$w\in W(u^l)\cap W(\varphi^l)\,,$$

so that actually v^{l-1} belongs to $W^{1/p}_{\varphi^{l-1}}(\mathcal{X},\mathcal{Y}^{l-1})$, where $\varphi^{l-1}:\mathcal{X}\to\mathcal{Y}^{l-1}$ is smooth.

Also, by the construction we may and do find a map $u^{l-1}\in\mathcal{R}^\infty_{1/p,\varphi^{l-1}}(\mathcal{X},\mathcal{Y}^{l-1})$ that has the same topological singularity as v^{l-1}, i.e., $\mathbf{S}^{l-1}_{u^{l-1}}=\mathbf{S}^{l-1}_{v^{l-1}}$, and a Sobolev function $U^{l-1}\in W^{1,p}(\mathcal{C}^{n+1},\mathbb{R}^N)$ such that $\mathbf{T}(U^{l-1})=u^{l-1}$ and

$$\int_{\mathcal{C}^{n+1}}|DU^{l-1}-D(P^l_w\circ V^l)|^p\,d\mathcal{H}^{n+1}\le\varepsilon_3\,.$$

Using (6.14), (6.15), and (6.16), we finally get:

$$\mathcal{E}_{1/p}(u^{l-1}) \le \mathbf{D}_p(U^{l-1}) \le \left(\frac{C(p,\varepsilon,\varepsilon_l,\mathcal{Y})}{\mathcal{H}^{ls_l}(N^l_\varepsilon)} + 1 \right) \mathbf{D}_p(U^l)\,,$$

where $\mathbf{D}_p(U^l) =: \mathcal{E}_{1/p}(u^l)$, and we similarly obtain that

$$\mathcal{E}_{1/p}(\varphi^{l-1}) \le C\,\mathcal{E}_{1/p}(\varphi^l)\,.$$

The above yields the proof of (a) and (b), whereas property (c) follows from (6.11) and from the compactness of $\mathcal{X}$. Finally, property (d) is a direct consequence of (6.6) and of the construction of u^{l-1}, compare Steps 1 and 2. □

Step 7: Since $\pi_{\mathfrak{p}-1}(\mathcal{Y}^{\mathfrak{p}-1})$ is finitely generated, we let $\{g_s\}_{s=1}^{\beta}$ be a set of its generators. As in [40, Lemma 4.3], but this time using Propositions 5.5 and 5.7, we now prove:

Lemma 6.8 *Let $\psi : \mathcal{X} \to \mathcal{Y}^{\mathfrak{p}-1}$ be a smooth $W^{1/p}$-map and $v \in \mathcal{R}^\infty_{1/p,\psi}(\mathcal{X},\mathcal{Y}^{\mathfrak{p}-1})$. Then there exists a current $\widetilde{L} \in \mathcal{R}_{n-\mathfrak{p}+1}(\mathcal{X};\pi_{\mathfrak{p}-1}(\mathcal{Y}^{\mathfrak{p}-1}))$, say $L = \sum_{s=1}^{\beta} \widetilde{L}_s \otimes g_s$, where $\widetilde{L}_s \in \mathcal{R}_{n-\mathfrak{p}-1}(\mathcal{X})$, such that $\partial\widetilde{L} = \mathbf{S}^{\mathfrak{p}-1}_v$ and for every s*

$$\mathbf{M}(\widetilde{L}_s) \le C\,(\mathcal{E}_{1/p}(v) + \|\operatorname{Ext}(v)\|^p_{L^p(\mathcal{C}^{n+1},\mathbb{R}^N)} + \mathcal{E}_{1/p}(\psi) + \|\operatorname{Ext}(\psi)\|^p_{L^p(\mathcal{C}^{n+1},\mathbb{R}^N)})\,,$$

where the absolute constant $C>0$ does not depend on v and ψ. Moreover, if $\partial\mathcal{X} = \emptyset$ and $v \in \mathcal{R}^\infty_{1/p}(\mathcal{X},\mathcal{Y}^{\mathfrak{p}-1})$ we have

$$\mathbf{M}(\widetilde{L}_s) \le C\,(\mathcal{E}_{1/p}(v) + \|\operatorname{Ext}(v)\|^p_{L^p(\mathcal{C}^{n+1},\mathbb{R}^N)})\,.$$

Proof: For $s = 1,\dots,\beta$, let $\alpha_s : \pi_{\mathfrak{p}-1}(\mathcal{Y}^{\mathfrak{p}-1}) \to \mathbb{Z}$ be such that for every homotopy class $a \in \pi_{\mathfrak{p}-1}(\mathcal{Y}^{\mathfrak{p}-1})$ we have $a = \sum_{s=1}^{\beta} \alpha_s(a)\,g_s$. Moreover, for every s we can find a smooth map $p_s : \mathcal{Y}^{\mathfrak{p}-1} \to \mathbb{S}^{\mathfrak{p}-1}$ such that for any $(\mathfrak{p}-1)$-cycle C in $\mathcal{Y}^{\mathfrak{p}-1}$

$$[p_s(C)]_{\pi_{\mathfrak{p}-1}(\mathbb{S}^{\mathfrak{p}-1})} = \alpha_s([C]_{\pi_{\mathfrak{p}-1}(\mathcal{Y}^{\mathfrak{p}-1})})\,. \tag{6.17}$$

Now, $p_s \circ v$ belongs to $\mathcal{R}^\infty_{1/p,\psi_s}(\mathcal{X},\mathbb{S}^{\mathfrak{p}-1})$ for every $v \in \mathcal{R}^\infty_{1/p,\psi}(\mathcal{X},\mathcal{Y}^{\mathfrak{p}-1})$, where $\psi_s := p_s \circ \psi : \mathcal{X} \to \mathbb{S}^{\mathfrak{p}-1}$ is a smooth $W^{1/p}$-map. Moreover, the one-to-one group homomorphisms $k^s : \mathbb{Z} \to \pi_{\mathfrak{p}-1}(\mathcal{Y}^{\mathfrak{p}-1})$ defined by $k^s(n) := n\,g_s$ satisfy

$$\sum_{s=1}^{\beta} k^s(\alpha_s(a)) = a \qquad \forall\, a \in \pi_{\mathfrak{p}-1}(\mathcal{Y}^{\mathfrak{p}-1})\,.$$

By (6.17), this gives that

$$\sum_{s=1}^{\beta} k^s_*(\mathbf{S}_{p_s\circ v}) = \mathbf{S}^{\mathfrak{p}-1}_v\,,$$

where for any map $u \in \mathcal{R}^\infty_{1/p}(\mathcal{X},\mathbb{S}^{p-1})$ satisfying (6.3) we have set

$$k^s_*(\mathbf{S}_u) := \sum_{i=1}^{\mu} [\![\,\sigma_i\,]\!] \otimes k^s(c(n,\mathfrak{p})\cdot m_i) = c(n,\mathfrak{p}) \sum_{i=1}^{\mu} m_i [\![\,\sigma_i\,]\!] \otimes g_s\,.$$

By applying Proposition 5.7 to $p_s \circ v$, for every s we find $L_s \in \mathcal{R}_{n-\mathfrak{p}+1}(\mathcal{X})$ such that $\partial L_s = \mathbf{P}(p_s\circ v)$ and

$$\mathbf{M}(L_s) \le C\left(\mathcal{E}_{1/p}(p_s\circ v) + \|\operatorname{Ext}(p_s\circ v)\|^p_{L^p(\mathcal{C}^{n+1})} + \mathcal{E}_{1/p}(\psi_s) + \|\operatorname{Ext}(\psi_s)\|^p_{L^p(\mathcal{C}^{n+1})}\right).$$

Since $\mathbf{S}_{p_s\circ v} = c(n,\mathfrak{p})\,\mathbf{P}(p_s\circ v)$, see Remark 6.5, setting $\widetilde{L}_s := c(n,\mathfrak{p})\,L_s$, the current

$$\widetilde{L} := \sum_{s=1}^{\widetilde{s}} k^s_*(\widetilde{L}_s) = \sum_{s=1}^{\widetilde{s}} \widetilde{L}_s \otimes g_s \in \mathcal{R}_{n-\mathfrak{p}+1}(\mathcal{X};\pi_{\mathfrak{p}-1}(\mathcal{Y}^{\mathfrak{p}-1}))$$

satisfies

$$\partial\widetilde{L} = \sum_{s=1}^{\widetilde{s}} k_*^s(\partial\widetilde{L}_s) = \sum_{s=1}^{\widetilde{s}} k_*^s(\mathbf{S}_{p_s\circ v}) = \mathbf{S}_v^{\mathfrak{p}-1}\,.$$

The last assertion is similarly obtained by using Proposition 5.5. □

STEP 8: We finally prove the assertion.

As in [40, Prop. 4.1], using Lemma 6.7 iteratively, with $l=\mathfrak{p},\dots,M:=\dim(\mathcal{Y})$, we find a map $u^{\mathfrak{p}-1}\in R^\infty_{1/p,\varphi^{\mathfrak{p}-1}}(\mathcal{X},\mathcal{Y}^{\mathfrak{p}-1})$ and a smooth $W^{1/p}$-function $\varphi^{\mathfrak{p}-1}:\mathcal{X}\to\mathcal{Y}^{\mathfrak{p}-1}$ such that for both $v=u$ and $v=\varphi$ we have

$$\mathcal{E}_{1/p}(v^{\mathfrak{p}-1})\le C_1\,\mathcal{E}_{1/p}(v) \quad\text{and}\quad \|\operatorname{Ext}(v^{\mathfrak{p}-1})\|_{L^p(\mathcal{C}^{n+1},\mathbb{R}^N)}\le C_1\,\|\operatorname{Ext}(v)\|_{L^p(\mathcal{C}^{n+1},\mathbb{R}^N)}\,, \tag{6.18}$$

where $C_1>0$ is an absolute constant. Also, property (d) in Lemma 6.7 yields

$$\widetilde{\chi}_*(\mathbf{S}^{\mathfrak{p}-1}_{u^{\mathfrak{p}-1}})=\mathbf{S}_u\,,$$

where $\widetilde{\chi}_*:\mathcal{R}_{n-\mathfrak{p}}(\mathcal{X};\pi_{\mathfrak{p}-1}(\mathcal{Y}^{\mathfrak{p}-1}))\to\mathcal{R}_{n-\mathfrak{p}}(\mathcal{X};\pi_{\mathfrak{p}-1}(\mathcal{Y}))$ denotes the homomorphism induced by the injection map $\widetilde{i}:\mathcal{Y}^{\mathfrak{p}-1}\hookrightarrow\mathcal{Y}$.

Applying Lemma 6.8 to $v=u^{\mathfrak{p}-1}$, with $\psi=\varphi^{\mathfrak{p}-1}$, we find a current $\widetilde{L}=\sum_{r=1}^{\beta}\widetilde{L}_r\otimes g_r$, where $\widetilde{L}_r\in\mathcal{R}_{n-\mathfrak{p}-1}(\mathcal{X})$, i.e., $\widetilde{L}\in\mathcal{R}_{n-\mathfrak{p}+1}(\mathcal{X};\pi_{\mathfrak{p}-1}(\mathcal{Y}^{\mathfrak{p}-1}))$, such that $\partial\widetilde{L}=\mathbf{S}^{\mathfrak{p}-1}_{u^{\mathfrak{p}-1}}$. Setting $L:=\widetilde{\chi}_*(\widetilde{L})\in\mathcal{R}_{n-\mathfrak{p}+1}(\mathcal{X};\pi_{\mathfrak{p}-1}(\mathcal{Y}))$, we have

$$L=\sum_{r=1}^{\beta}\widetilde{L}_r\otimes\widetilde{i}_*g_r\,,\qquad \partial L=\widetilde{\chi}_*(\partial\widetilde{L})=\widetilde{\chi}_*(\mathbf{S}^{\mathfrak{p}-1}_{u^{\mathfrak{p}-1}})=\mathbf{S}_u\,.$$

Moreover, it turns out that each $\widetilde{i}_*g_r$ belongs to $\rho_{\mathfrak{p}}(\mathcal{H}^{sph}_{\mathfrak{p}-1}(\mathcal{Y}))$. We thus can find some integers $\{\lambda^r_s\}_{s=1}^{\widetilde{s}}\subset\mathbb{Z}$ such that

$$\widetilde{i}_*g_r=\sum_{s=1}^{\widetilde{s}}\lambda^r_s\,\Gamma_s\qquad\forall\,r=1,\dots,\beta\,,$$

whence

$$L=\sum_{s=1}^{\widetilde{s}}\widehat{L}_s\otimes\Gamma_s\,,\qquad \widehat{L}_s:=\sum_{r=1}^{\beta}\lambda^r_s\,\widetilde{L}_r\,.$$

On account of the notation from (6.2), property $\partial L=\mathbf{S}_u$ means that

$$\partial\widehat{L}_s=\mathbf{T}_s(u)\qquad\forall\,s=1,\dots,\widetilde{s}\,.$$

Therefore, the currents $L_s:=c(n,\mathfrak{p})\,\widehat{L}_s$ satisfy the assertion, as the mass estimates follow from Lemma 6.8 and from (6.18). □

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