A Viscosity Property of Minimizing Micromagnetic Configurations

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I Introduction

The object of study of the present paper is the space of divergence free unit vector fields u of a bounded domain Ω of $\mathbb{R}^2$ admitting a lifting ϕ in $L^\infty(\Omega;\mathbb{R})$ ($u = e^{i\phi}$) such that the distribution U in $\mathcal{D}'(\Omega\times\mathbb{R})$ defined by

$$< U_\phi; \psi(x,a) >:= -\int_{a\in\mathbb{R}}\int_\Omega e^{i\phi\wedge a}\cdot\nabla_x\psi \tag{I.1}$$

is a bounded Radon Measure in $\Omega\times\mathbb{R}$.
Here we use the notation $\phi(x)\wedge a := \inf\{\phi(x); a\}$ and integrals with respect to the parameter a will always be understood with respect to the Lebesgue measure on $\mathbb{R}$.

We call $\mathcal{M}_{div}(\Omega)$ the space of divergence free unit vector fields in Ω satisfying (I.1) for a suitable lifting ϕ.

Observe that if u admits a sufficiently smooth lifting ϕ (say ϕ is Lipschitz) then $\forall a\in\mathbb{R}$ $div(e^{i\phi\wedge a}) = 0$ in $\mathcal{D}'(\Omega)$ for all $a\in\mathbb{R}$ and therefore $U_\phi = 0$ (this condition is in fact necessary and sufficient for the existence of a Lipschitz lifting, see [LR]). Therefore the measure U_ϕ is related to the "jump set" of u (or, more precisely, the "jump set" of ϕ). This expectation is confirmed by the explicit expression of U_ϕ given in [RS1] under the *a priori* assumption that ϕ is a BV function.

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The Space $\mathcal{M}_{div}(\Omega)$ was introduced in [RS1] and [RS2] by the third author and S.Serfaty in connection with the a variational problem related to micromagnetism that we briefly describe. For a given simply connected bounded domain Ω of $\mathbb{R}^2$ and a positive parameter $\varepsilon > 0$, we consider the following energy, for unit vector-field u in $W^{1,2}(\Omega; S^1)$,

$$E_\varepsilon(u) := \varepsilon \int_\Omega |\nabla u|^2 + \frac{1}{\varepsilon}\int_{\mathbb{R}^2} |H|^2 \quad , \tag{I.2}$$

where $H = -\nabla(G * \hat{u})$, $\hat{u} = u$ on Ω, $\hat{u} = 0$ in $\mathbb{R}^2 \setminus \Omega$ and G is the Kernel of the Laplacian on $\mathbb{R}^2$ so that, in particular, we have $div(\hat{u} + H) = 0$ in $\mathcal{D}'(\mathbb{R}^2)$. It was proved in [RS1], [RS2] that, from any sequence $u_{\varepsilon_n} \in W^{1,2}(\Omega, S^1)$ such that $\varepsilon_n \to 0$ and $E_{\varepsilon_n}(u_{\varepsilon_n}) \le C$ independently of n, one can extract a subsequence $u_{\varepsilon_{n'}}$ converging strongly in $L^p(\Omega)$ for any $p < +\infty$ to a limit u in $\mathcal{M}_{div}(\Omega)$. Moreover

$$E_0(\phi) := 2\int_{a\in\mathbb{R}}\int_\Omega |div(e^{i\phi\wedge a})|(\Omega) \le \liminf E_{\varepsilon_{n'}}(u_{\epsilon_{n'}}) \tag{I.3}$$

is the expected Gamma-Limit of E_ε as $\varepsilon \to 0$ (notice also that it is not hard to show that $E_0(\phi)$ equals $|U_\phi|(\Omega)$, but this is not strictly relevant in our discussion). One can for instance prove that, see [RS1], [RS2],

$$\lim_{\varepsilon\to 0} inf_{u\in W^{1,2}} E_\varepsilon(u) = 2 \inf_{u\in\mathcal{M}_{div(\Omega)}} \int_{a\in\mathbb{R}}\int_\Omega |div(e^{i\phi\wedge a})| = 2|\partial\Omega| \tag{I.4}$$

where $|\partial\Omega|$ is the perimeter of Ω, if Ω has a Lipschitz boundary. One of the questions left open in [RS1], [RS2] was to describe the limit of minimizing configurations of E_ε. It was conjectured in these works that, representing u as $-\nabla^\perp g$ for a suitable 1-Lipschitz function g, the function g is a viscosity solution of $|\nabla g|^2 - 1 = 0$ in Ω. Since g is equal to 0 on $\partial\Omega$, this fact would imply that: $g(x) = dist(x; \partial\Omega)$, the unique viscosity solution of the boundary value problem. It was proved in [RS2] that any limit u of minimizing sequences of E_ε has to satisfy

$$div(e^{i\phi\wedge a}) \ge 0 \qquad \forall a \in \mathbb{R}. \tag{I.5}$$

The following theorem, which is the main result of the present paper, answers positively to the above problem left open in [RS2].

Theorem I.1 *Let $u = -\nabla^{\perp} g$ be a divergence free unit vector-field in the space $\mathcal{M}_{div}(\Omega)$. If (I.5) holds then g is a viscosity solution of $|\nabla g|^2 - 1 = 0$ in Ω. In particular we have*

$$g(x) = \inf_{y \in \partial\Omega} \{g(y) + |x - y|\} \tag{I.6}$$

and therefore g is locally semiconcave in Ω and $u \in BV_{loc}(\Omega; S^1)$.

It was proved in [RS2] that the maps in $\mathcal{M}_{div}(\Omega)$ satisfy the kinetic equation

$$ie^{ia} \cdot \nabla_x \chi_a = -\partial_a(div(e^{i\phi \wedge a})) \qquad \mathcal{D}'(\Omega \times \mathbb{R}) \tag{I.7}$$

where $\chi_a(x) = \chi(\phi(x) - a)$ and χ is the characteristic function of $\mathbb{R}_-$. It seems that the standard theory of kinetic equations does not directly apply here to deduce from the sign condition of the jumps (I.5) the uniqueness result (I.6), whereas the viscosity methods are very efficient. This being said, due to the natural kinetic meaning of (I.5), it would be nice to have a direct proof of theorem I.1 via the kinetic equation approach.

As we mentioned above a consequence of theorem I.2 is the following corollary.

Corollary I.1 *Let u_ε be a family of unit vector-fields over a bounded smooth simply connected domain Ω of $\mathbb{R}^2$ such that*

$$\lim_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon) = \lim_{\varepsilon \to 0} \inf \left\{ E_\varepsilon(u) \ ; \ u \in W^{1,2}(\Omega, S^1) \right\} = 2|\partial\Omega|$$

and $\int_{\partial\Omega} u_\varepsilon \cdot \tau \le 0$, where τ is the unit tangent to $\partial\Omega$. Then

$$u_\varepsilon \longrightarrow u_\star := \nabla^{\perp} dist(x, \partial\Omega) \qquad in\ L^p(\Omega)$$

for any $p < +\infty$.

A similar problem to (I.2) is the so called Aviles-Giga problem where, instead of constraining u to be unit, one requires iu to be a gradient, say $u = -\nabla^{\perp} g \in W^{1,2}(\Omega; \mathbb{R}^2)$, and one introduces the Ginzburg-Landau Functional

$$GL(\nabla^{\perp} g) := \varepsilon \int_\Omega |\nabla^2 g| + \frac{1}{2\varepsilon} \int_\Omega (|\nabla g|^2 - 1)^2 \tag{I.8}$$

The asymptotic analysis of this problem, as ε tends to 0, has been extensively developped in [ADM], [AG1], [AG2], [DKMO1], [DKMO2], [JP], although a complete Γ-convergence result is still not available. One of the relevant facts showing up when one tries to compare these two, apparently very similar, relaxations ((I.2) and (I.8)) of the eikonal equation $|\nabla g|^2 - 1 = 0$ is that non viscosity solution can sometimes realise a fundamental state of the limiting Aviles-Giga Functional. This result, shown by W.Jin and R.Kohn in [JK], is in striking contrast with Corollary I.1.

The apparently similar problems (I.2) and (I.8) carry a major difference: in the Aviles-Giga problem there is no obstruction for ∇g to have vortices, this is truly a *vectorial* problem, whereas in (I.2) the energy $|U_\phi|$ is not 0 near to vortices, since a jump in the lifting is forced. In this sense the latter problem is a *scalar* one. In [ARS] the formulation (I.2) is slightly "relaxed" in order to release the topological constraint not to have vortices. It has to be noticed, see the results in [ARS], that in this more general framework the viscosity solution is not minimizing anymore in the limit $\varepsilon \to 0$.

Our theorem I.1 has been obtained by us within a more general research project aimed to show the countable rectifiability of the "jump set" of vector fields $u \in \mathcal{M}_{div}(\Omega)$ and an explicit representation, in terms of the jumps of ϕ, of U_ϕ. Indeed, it can be proven by the construction in [ADM] that the vector fields $u \in \mathcal{M}_{div}(\Omega)$ need not be in $BV(\Omega)$, not even locally, and therefore this regularity result for the jump set has to be established independently from the BV theory.

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II Proof of theorem I.1.

Take $u = e^{i\phi} = -\nabla^\perp g$ satisfying the assumptions of theorem I.1 : $u \in \mathcal{M}_{div}(\Omega)$ and u satisfies the sign condition (I.5). We claim that g is a viscosity solution of

$$|\nabla g|^2 - 1 = 0 \tag{II.9}$$

We recall the definition for g to be a viscosity solution of (II.9) (see for instance [CEL], or [AD] definition 6.1).

(a) g is a viscosity subsolution of (II.9) in Ω if for any $x_0 \in \Omega$ and any $\psi \in C_c^\infty(\Omega)$ such that $g - \psi$ has a local maximum at x_0, ψ satisfies

$$|\nabla\psi|(x_0) \leq 1. \tag{II.10}$$

(b) g is a viscosity supersolution of (II.9) in Ω if for any $x_0 \in \Omega$, and any $\psi \in C_c^\infty(\Omega)$ such that $g - \psi$ has a local minimum at x_0, ψ satisfies

$$|\nabla\psi|(x_0) \geq 1. \tag{II.11}$$

Finally, g is a viscosity solution of (II.9) if g is both a viscosity subsolution and a viscosity supersolution of (II.9).

Step 1 : It is standard to prove that g is a viscosity subsolution of (II.9). Assuming $g - \psi$ admits a local maximum at $x_0 \in \Omega$, we have for any $t \geq 0$ small enough and $\omega \in S^1$

$$0 \leq [g(x_0) - \psi(x_0)] - [g(x_0 + t\omega) - \psi(x_0 + t\omega)]$$

using the fact that g is lipschitz, with a lipschitz constant equal to 1, we then have

$$0 \leq t + \psi(x_0 + t\omega) - \psi(x_0) \quad .$$

Therefore

$$-t \leq t\nabla\psi(x_0) \cdot \omega + o(t).$$

Choosing $\omega = -\nabla\psi(x_0)/|\nabla\psi(x_0)|$ (we assume $|\nabla\psi|(x_0) \neq 0$, otherwise (II.10) is already true) we get (II.10).

Step 2 : We shall prove now that g is a viscosity supersolution of (II.9). This requires some more work and the crucial use of the sign condition (I.5).

Let $\psi \in C_c^\infty(\Omega)$ and assume that $g - \psi$ admits a local minimum at $x_0 \in \Omega$. Without loss of generality we assume that $g(x_0) = \psi(x_0)$. We shall prove that $|\nabla\psi|(x_0) \geq 1$. We first proceed to a dilation process which is not absolutely unavoidable in the proof below, but makes the presentation cleaner.

Let $\chi \in C_c^\infty(B_2)$ with $\chi \equiv 1$ in B_1. Because of the sign condition (I.5) we have

$$\begin{aligned}
&\limsup_{r\to 0} \frac{1}{r} \int_{a\in\mathbb{R}} \int_{B_r(x_0)} |div(e^{i\phi\wedge a})| \leq \limsup_{r\to 0} \frac{1}{r} \int_{a\in\mathbb{R}} \int_{B_{2r}(x_0)} div(e^{i\phi\wedge a}) \chi_{x_0,r} \\
&= \limsup_{r\to 0} -\frac{1}{r^2} \int_{a=-\|\phi\|_\infty}^{\|\phi\|_\infty} \int_{B_{2r}(x_0)} e^{i\phi\wedge a} \cdot \nabla\chi_{x_0,r} \leq C\ \|\phi\|_\infty < +\infty
\end{aligned} \tag{II.12}$$

where $\chi_{x_0,r} := \chi((\cdot - x_0)/r)$. This being granted, we define for any $r > 0$,

$$\begin{array}{rcl} \phi_r(x) & = & \phi(x_0 + rx) \\ u_r(x) & = & u(x_0 + rx) \\ g_r(x) & = & \frac{1}{r}(g(x_0 + rx) - g(x_0)) \\ \psi_r(x) & = & \frac{1}{r}(\psi(x_0 + rx) - \psi(x_0)) \quad . \end{array}$$

Using (II.12) we have

$$\limsup_{r\to 0} \int_{a\in\mathbb{R}} \int_{B_1} |div(e^{i\phi_r \wedge a})| < +\infty. \tag{II.13}$$

Therefore we may use the compactness result of [RS2] (in fact some modified version of it, see [Le], saying that from any sequence $u_j = e^{i\phi_j}$ in $\mathcal{M}_{div}(B_1)$ such that $|U_{\phi_j}|$ and $\|\phi_j\|_\infty$ are uniformly bounded, one can extract a strongly converging sequence $\phi_{j'}$ in $L^p(B_1)$, $\forall p < +\infty$) to deduce that there exists a sequence r_i tending to zero such that

$$\phi_{r_i} \to \phi_\infty \qquad \text{in } L^p(B_1) \quad \forall p < +\infty \quad . \tag{II.14}$$

As a consequence u_{r_i} converge to $u_\infty := e^{i\phi_\infty}$ in $L^p(B_1)$ for any $p < \infty$ and the Sobloev embedding theorem implies that g_{r_i} uniformly converge in B_1 to a 1-Lipschitz function g_∞. We immediately obtain that $\nabla^\perp g_\infty = -e^{i\phi_\infty}$ and

$$\begin{array}{ll} div\, e^{i\phi_\infty \wedge a} \geq 0 & \forall a \in \mathbb{R} \\ & \\ g_\infty - \psi_\infty & \text{admits a global minimum on } B_1 \text{ at } 0 \end{array} \tag{II.15}$$

where $\psi_\infty := \lambda\omega \cdot x$, $\lambda = |\nabla\psi|(x_0)$ and $\omega := \nabla\psi(x_0)/|\nabla\psi(x_0)|$ whenever $\lambda \neq 0$ (if $\lambda = 0$ we choose any $\omega \in S^1$).

Observe that after this linearization our goal is to prove now that $\lambda \geq 1$. We argue by contradiction and assume that $\lambda < 1$.

In order to ensure that 0 is a strict minimum of $g_\infty - \psi_\infty$ we add to this function a small quadratic quantity $\delta\|x\|^2$, where $\delta > 0$ will be chosen so that $\tilde{\lambda} := \lambda + 2\delta < 1$. We then denote for any $\delta > 0$

$$f_\delta(x) := g_\infty(x) - \psi_\infty(x) + \delta\|x\|^2 \quad .$$

Observe that $f_\delta \geq 0$ on B_1, that $f_\delta(0) = 0$ and that this minimum is achieved only at 0. Moreover, our choice of δ ensures that $|\nabla f_\delta| > 0$ in B_1.

Let $\tau > 0$ be the minimum of f_δ on ∂B_1 and let a^* be the essential supremum of ϕ_∞ on $\{f_\delta < \tau\}$. We choose $a < a^*$ sufficiently close to a^*, so that $a^* - a < \pi$ and

$$1 + \sin\left(\frac{\pi}{2} + a^* - a\right) > 2\tilde{\lambda}^2 \quad .$$

Notice that all level sets $E_s := \{f_\delta = s\}$ are, for $0 \le s < \tau$, compact subsets of B_1. Moreover, as $\{\phi_\infty > a\}$ has strictly positive Lebesgue measure, the co-area formula for Lipschitz functions and the non-degeneracy of ∇f_δ imply that the set

$$S := \big\{s \in (0,\tau) : \ \mathcal{H}^1(E_s \cap \{\phi_\infty > a\}) > 0, \ \mathcal{H}^1(E_s \cap \{\phi_\infty > a^*\}) = 0\big\}$$

has strictly positive Lebesgue measure. Setting $\omega^\perp := (-\omega_2, \omega_1)$, $x^\perp := (-x_2, x_1)$ and $\nu = \nabla f_\delta / |\nabla f_\delta|$, we choose $s \in S$ where the function

$$s \mapsto l(s) := \int_{E_s} (e^{i\phi_\infty \wedge a} + \lambda\omega^\perp - 2\delta x^\perp) \cdot \nu$$

is left approximately continuous, i.e.

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{s-\varepsilon}^{s} l(s')\, ds' = l(s) \quad .$$

By applying the coarea formula, we have in particular

$$\begin{aligned} &\lim_{\varepsilon \to 0} \int_{B_1} (e^{i\phi_\infty \wedge a} + \lambda\omega^\perp - 2\delta x^\perp) \cdot \nabla\zeta_\varepsilon \\ &= -\int_{E_s} (e^{i\phi_\infty \wedge a} + \lambda\omega^\perp - 2\delta x^\perp) \cdot \nu \quad , \end{aligned}$$

where $\zeta_\varepsilon(x) := 1 \wedge (s - f_\delta)^+/\varepsilon$.

Applying the sign assumption, we get

$$0 \le -\int_{B_1} \nabla\zeta_\varepsilon \cdot (e^{i\phi_\infty \wedge a} + \lambda\omega^\perp - 2\delta x^\perp) \to \int_{E_s} (e^{i\phi_\infty \wedge a} + \lambda\omega^\perp - 2\delta x^\perp) \cdot \nu \quad . \tag{II.16}$$

We decompose E_s into $\Gamma_s^+ := E_s \cap \{\phi_\infty > a\}$ and $\Gamma_s^- := E_s \cap \{\phi_\infty \le a\}$. Since $e^{i\phi_\infty} + \lambda\omega^\perp - 2\delta x^\perp = -\nabla^\perp f_\delta$, combining this fact with (II.16) we finally obtain that

$$0 \le \int_{\Gamma_s^+} (e^{ia} + \lambda\omega^\perp - 2\delta x^\perp) \cdot (\nabla g_\infty - \lambda\omega + 2\delta x). \tag{II.17}$$

On the other hand, using the identity $a \cdot b^\perp = -a^\perp \cdot b$, on Γ_s^+ we get

$$\begin{aligned}&(e^{ia} + \lambda\omega^\perp - 2\delta x^\perp) \cdot (\nabla g_\infty - \lambda\omega + 2\delta x)\\&= e^{ia} \cdot \nabla g_\infty - (e^{ia} + \nabla^\perp g_\infty) \cdot (\lambda\omega - 2\delta x)\\&\le e^{ia} \cdot \nabla g_\infty + \tilde{\lambda}\|\nabla^\perp g_\infty + e^{ia}\| \quad .\end{aligned}$$

Therefore

$$0 \le \int_{\Gamma_s^+} e^{ia} \cdot \nabla g_\infty + \tilde{\lambda}\|\nabla^\perp g_\infty + e^{ia}\| \quad . \tag{II.18}$$

For $a \le \phi_\infty \le a^*$ we have $\pi/2 \le \phi_\infty + \pi/2 - a \le \pi/2 + a^* - a \le \frac{3\pi}{2}$ and therefore

$$e^{ia} \cdot \nabla g_\infty = \cos\left(\phi_\infty + \frac{\pi}{2} - a\right) \le 0. \tag{II.19}$$

Observe now that $\|\nabla^\perp g_\infty + e^{ia}\|^2 = 2 + 2\nabla^\perp g_\infty \cdot e^{ia} = 2(1 - \sin(\phi_\infty + \frac{\pi}{2} - a))$ and, by our choice of a, we have

$$\begin{aligned}&(e^{ia} \cdot \nabla g_\infty)^2 = \cos^2\left(\phi_\infty + \frac{\pi}{2} - a\right)\\&= 1 - \sin^2\left(\phi_\infty + \frac{\pi}{2} - a\right) > 2\tilde{\lambda}^2\left(1 - \sin\left(\phi_\infty + \frac{\pi}{2} - a\right)\right)\\&= \tilde{\lambda}^2\|\nabla^\perp g_\infty + e^{ia}\|^2 \quad .\end{aligned} \tag{II.20}$$

Combining (II.18), (II.19) and (II.20) we get a contradiction, since Γ_s^+ by our choice of s has strictly positive $\mathcal{H}^1$-measure. Thus g is a viscosity solution of $|\nabla g|^2 - 1 = 0$ in Ω.

Finally, since the right hand side in (I.6) is a viscosity solution in Ω with the same boundary values, the uniqueness theorem for first order viscosity solutions with convex Hamiltonians leads to (I.6). Since

$$D^2|\cdot - y| + g(y) \le C(B)I \quad \text{on } B \text{ for any } y \in \partial\Omega$$

on any ball $B \subset\subset \Omega$, it turns out that $D^2 g \le C(B)I$ in B as well. Therefore g is locally semi-concave in Ω. It is well known that gradients of concave or semi-concave functions are locally BV.

Remark II.1 *Observe that the above presented proof gives a new approach for the first part of the proof of the main result in [LR] saying that any $u = e^{i\phi}$ in $\mathcal{M}_{div}$ satisfying $U_\phi = 0$ is Lipshitz.*

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