Monotonicity of transport plans

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Abstract

We study monotonicity properties for minimizers of transport problems. In the one-dimensional case, we present an algorithm to construct minimizing monotone transport plans by "monotonizing" a given minimizing transport plan. This method applies in particular to the case of the L^1-Wasserstein metric where we prove the existence of monotone minimizers for arbitrary marginals. We find that monotone transport plans are in a certain sense close to monotone transport maps.

Keywords: Transport problems, covariance, Wasserstein distance, cyclical monotonicity.

1 Introduction

The classical transport problem, introduced by Monge, is to find a map $\psi \colon \mathbb{R}^n \to \mathbb{R}^n$ ($n \in \mathbb{N}$ given) minimizing the functional

$$C(\psi) := \int_{\mathbb{R}^n} |x - \psi(x)| g_0(x)\, dx, \tag{1}$$

such that

$$\int_{\psi^{-1}(B)} g_0(x)\, dx = \int_B g_1(y)\, dy$$

for all Borel sets $B \subset \mathbb{R}^n$, where g_0, g_1 are integrable functions on $\mathbb{R}^n$. This corresponds to an optimal transport of a pile of matter described by the function g_0 into a hole described by g_1. This problem has been subsequently relaxed and generalized to the form we are studying in this article:

Definition 1.1 (Transport problem)**.** *Let* $\mu, \nu \in \mathcal{P}(\mathbb{R}^n)$ *and let* $c\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *be a lower semicontinuous function (the* cost function*). The* transport problem *consists then of finding a probability measure* $T \in \mathcal{P}(\mathbb{R}^n \times \mathbb{R}^n)$ *which minimizes*

$$C(T) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} c(x,y) dT(x,y), \tag{2}$$

such that the marginals of T *are given by* μ *and* ν*, i.e.,*

$$\pi_1 T = \mu, \quad \pi_2 T = \nu,$$

where π_1 *is the projection on the first n coordinates and* π_2 *the projection on the second n coordinates, i.e.*

$$\pi_1 T := \int_{\mathbb{R}^n} dT(\cdot, y), \quad \pi_2 T := \int_{\mathbb{R}^n} dT(x, \cdot).$$

(We remark that this definition and our subsequent results can be naturally extended to arbitrary positive Radon measure of fixed total measure.)
It is well known that the above transport problem admits a solution, see, e.g., [1]. In particular the case where $c(x,y) := |x-y|^p$ with $p \geq 1$ has been studied extensively. It leads to the definition of the *Wasserstein distance*

$$W_p(\mu,\nu) := \left(\inf\left\{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |x-y|^p dT(x,y);\ T \in \mathcal{P}(\mathbb{R}^n, \mathbb{R}^n),\ \pi_1 T = \mu,\ \pi_2 T = \nu\right\}\right)^{1/p}$$

which has important applications in various fields of mathematics, compare [1].
In this article we want to take a closer look on certain properties of the optimal transport plan T of a transport problem given by Definition 1.1. This is motivated by two applications. One of them stems from asset pricing, compare [4, 3] where, e.g., the covariance of two probability measures has to be optimized. It follows from the results in this article that for arbitrary given probability measures (also if they are not absolutely continuous) the conjoint probability (which can be considered as a transport plan) is monotone in the sense of our Definition 2.1, below. This turns out to be useful for the explicit computation of optimal assets. The second motivation stems from a model for damage in solid bodies which had been introduced in [6]. The mathematical analysis of this model requires a good understanding of gradient flows induced by the L^1-Wasserstein metric. In [7], we consider a time discretized problem that naturally leads to the minimization of the L^1-Wasserstein metric. At this point it is necessary to know that monotone transport plans exist. We refer the reader to [7] for details on this application.

This article generalizes some of the classical results by Gangbo and McCann [2] on transport plans.
In the following section we will first give a precise definition of what we call a monotone transport plan and will demonstrate that this notion naturally extends the notion of monotonicity for functions in $\mathbb{R}$. We will then prove the existence of such monotone transport plans in a constructive way that can be used for direct numerical computations.
In Section 3 we give general results on the monotonicity of transport plans which solve certain transport problems and apply them to the applications we have sketched above.
In the final Section 4 we generalize our results to higher dimensions and connect them to the theory of cyclically monotone functions.

2 Monotone transport plans – definition and existence

In this section we define what we understand by the "monotonicity" of a transport plan and compare it with the notion of "cyclical montonicity".

Definition 2.1 (Monotonicity of transport plans)**.** *Let* $T \in \mathcal{P}(\mathbb{R}, \mathbb{R})$ *be a transport plan with marginals* $\mu, \nu \in \mathcal{P}(\mathbb{R})$*. Then* T *is called* monotone increasing *if the following condition holds:*
For all Borel sets $A, B \subset \mathbb{R}$ *with* $\mu(A) > 0, \mu(B) > 0$ *and* $\inf\{x \in B\} > \sup\{x \in A\}$ *we define*

$$A' := \operatorname{supp} \pi_2(T|_{A\times\mathbb{R}}), \quad B' := \operatorname{supp} \pi_2(T|_{B\times\mathbb{R}}).$$

Then $\inf\{x \in B'\} \geq \sup\{x \in A'\}$.
T *is called* monotone decreasing *if we have instead* $\inf\{x \in A'\} \geq \sup\{x \in B'\}$.

The definition of monotonicity is a natural extension of the usual notion in the following sense:

Proposition 2.2. *If* T *can be expressed as a transport map rather than a transport plan, then its monotonicity corresponds to the monotonicity of the transport map in the usual sense of a function in* $\mathbb{R}$*. More precisely, if there exists a Borel map* $\tilde{T} : \operatorname{supp} \mu \to \mathbb{R}$ *such that* $T = (Id \times \tilde{T})_{\#}\mu$*, then* T *is monotone iff* $\tilde{T}$ *is* μ*-a.e. monotone.*

We remark that it is necessary to allow $\tilde{T}$ to be non-monotone on a set N with $\mu(N) = 0$, since $\tilde{T}$ can be defined arbitrarily on such sets.

Proof. Let $\tilde{T}$ be monotone increasing. Choose subsets $A, B \subset \operatorname{supp}\mu \subset \mathbb{R}$ with $\mu(A) > 0, \mu(B) > 0$ and $\inf\{x \in B\} < \sup\{x \in A\}$ and define $T := (Id \times \tilde{T})_{\#}\mu$. Then $\pi_2(T|_{A\times\mathbb{R}}) = (\tilde{T}_{\#}\mu)(A)$. Therefore $\operatorname{supp}\pi_2(T|_{A\times\mathbb{R}}) \subset \tilde{T}(A)$ (and analogously for B). Since $\tilde{T}$ is monotone increasing, we have $\inf\{x \in \tilde{T}(B)\} \leq \sup\{x \in \tilde{T}(A)\}$ and the monotonicity of T follows immediately.

On the other hand let T be monotone increasing and of the form $T = (Id \times \tilde{T})_{\#}\mu$. Suppose $\tilde{T}$ is not monotone increasing on a set of positive μ-measure, then there are sets $A, B \in \mathbb{R}$ with $\mu(A), \mu(B) > 0$, $\sup\{x \in A\} < \sup\{x \in B\}$ and $\sup\{y \in \tilde{T}(A)\} > \inf\{y \in \tilde{T}(B)\}$. Hence $A' := \operatorname{supp}\pi_2(T|_{A\times\mathbb{R}}) = \tilde{T}(A)$ and $B' := \operatorname{supp}\pi_2(T|_{B\times\mathbb{R}}) = \tilde{T}(B)$ do not satisfy the monotonicity condition and therefore T fails to be monotone. □

In the following we need to compare monotonicity with the classical notion of cyclical monotonicity. We denote by $\langle\cdot,\cdot\rangle$ the scalar product in $\mathbb{R}^n$.

Definition 2.3. *Let $T \in \mathcal{P}(\mathbb{R}^n \times \mathbb{R}^n)$ be a transport plan with support S. The set S is called* cyclically monotone *if for any finite set of points $(x_i, y_i)_{i=1,\dots,k} \subset S$ we have*

$$\langle y_1, x_2 - x_1\rangle + \langle y_2, x_3 - x_2\rangle + \cdots + \langle y_k, x_1 - x_k\rangle \leq 0.$$

In this case we call T cyclically monotone.

Every cyclical monotone transport plan is also monotone. In Section 4 we will see that this extends naturally to higher dimensions.

It is at first glance not obvious that for any marginals a monotone transport plan exists, but its existence has been proved in [5, Theorem 6] for the case of cyclical monotonicity in arbitrary dimensions.

In the one-dimensional case, one can also construct a monotone transport plan explicitly. The following result gives an approximation for the one-dimensional case that can be used for numerical computations of monotone transport plans.

Theorem 2.4. *Let $\mu, \nu \in \mathcal{P}(\mathbb{R})$ then there exists an explicit approximation for a monotone increasing (or decreasing) transport plan with marginals μ and ν.*

Proof. Let $\varepsilon > 0$. We first choose bounded intervals $D = (d_1, d_2)$, $E = (e_1, e_2) \subset \mathbb{R}$ such that $\mu(D) = \nu(E) \geq 1-\varepsilon$ and $\mu((-\infty, d_1)) = \nu((-\infty, e_1))$. Then we decompose $D\times E \subset \mathbb{R}\times\mathbb{R}$ into squares. For simplicity we assume that $d_1 = e_1 = 0$. We define $I_i^k := [(i-1)2^{-k}, i2^{-k})$ for $1 \leq i \leq N := [1 + \max(d_2, e_2) \cdot 2^k]$ and

$$m_i := 2^k \int_{I_i^k} \mu|_D, \quad n_i := 2^k \int_{I_i^k} \nu|_E.$$

Denote the midpoints of the intervals I_i^k by z_i^k. We define sequences of measures μ_k and ν_k by

$$\mu_k := \left(\sum_{i=1}^{N} m_i \delta_{z_i^k}\right), \quad \nu_k := \left(\sum_{i=1}^{N} n_i \delta_{z_i^k}\right).$$

We choose $\varepsilon = 1/k$ and construct a monotone transport plan T_k for μ_k and ν_k: Define $T_k := \sum_{i,j=1}^{N} a_{i,j}^k \delta_{(z_i^k, z_j^k)}$ where the constants $a_{i,j}^k$ are determined by the following algorithm:

Set $i = j = 1$, $L = m_1$.
While $i \leq N$ or $j \leq N$:
{
If $L > n_j$ then $L = L - n_j$, $a_{i,j}^k = n_j$.
If $L \leq n_j$ then $L = 0$, $a_{i,j}^k = L$.
If $L = 0$ then $i = i + 1$, $L = m_i$, otherwise $j = j + 1$.
}

The algorithm terminates since $\sum_{i=1}^{N} m_i = \mu(D) = \nu(E) = \sum_{j=1}^{N} n_j$. The resulting transport plan $T_k = \sum_{i,j=1}^{N} a_{i,j}^k \delta_{z_i^k}$ is monotone increasing by construction.
Finally we let $k \to \infty$. The resulting monotone increasing transport plans T_k converge weakly-$\star$ to a limit $T \in \mathcal{P}(\mathbb{R} \times \mathbb{R})$, which is still monotone increasing.
The same argument can be used to construct a monotone decreasing transport plan. □

We conclude this section with a look at the uniqueness of monotone transport plans. A uniqueness result (for cyclical monotonicity) has already been found in [5, Corollary 14] (under the additional condition that one of the marginal measures has no concentrations).

Proposition 2.5. *Let $T_1, T_2 \in \mathcal{P}(\mathbb{R} \times \mathbb{R})$ be monotone increasing transport plans with marginals μ and ν. Then $T_1 = T_2$.*

Proof. We discretize T_1 and T_2 similarly as before: Define $a_{ij}^k := \int_{I_i^k \times I_j^k} dT_1(x, y)$, $b_{ij}^k := \int_{I_i^k \times I_j^k} dT_2(x, y)$ and $T_1^k := \sum_{i,j=1}^{k} a_{ij}^k \delta_{(z_i^k, z_j^k)}$ and $T_2^k := \sum_{i,j=1}^{k} b_{ij}^k \delta_{(z_i^k, z_j^k)}$ analogously, where I_i^k and z_i^k are defined as in the proof of Theorem 2.4.
Then for $k \to \infty$ we have again $T_1^k \overset{\star}{\rightharpoonup} T_1$ and $T_2^k \overset{\star}{\rightharpoonup} T_2$. Suppose that $T_1 \neq T_2$. Then for k sufficiently large, there are i_0, j_0 such that $a_{i_0 j_0} < b_{i_0 j_0}$. Then since $\sum_i a_{i j_0} = \sum_i b_{i j_0}$ (T_1 and T_2 have the same marginals), there exists a $i_1 \neq i_0$ such that $a_{i_1 j_0} < b_{i_1 j_0}$. Let us assume without loss of generality that $i_1 < i_0$. Since $\sum_j a_{i_1 j} = \sum_j b_{i_1 j}$, there exists a $j_1 \neq j_0$ such that $a_{i_1 j_1} > b_{i_1 j_1}$. By monotonicity we must hence have $j_1 < j_0$. Since $\sum_i a_{i j_1} = \sum_i b_{i j_1}$, there exists a $i_2 \neq i_1$ such that $a_{i_2 j_1} < b_{i_2 j_1}$, and by monotonicity we must have $i_2 < i_1$. Iterating this argument, we get an infinite sequence of i_k with $i_{k+1} < i_k$. Since i_k are indices from a finite index set, this is a contradiction. □

3 Optimality of monotone transport plans

3.1 General results

The aim of this section is to find conditions under which a *transport problem* has a monotone transport plan as minimizer. We generalize the classical results on cyclical monotone transport plans by Gangbo and McCann [2, 5] in two ways: First, we prove that minimizers are monotone even if we allow the marginals to have concentrations. Second, we prove that the existence of monotone minimizers can be generalized to a larger class of admissible functionals that encompasses, in particular, the L^1-Wasserstein metric. For the latter result we need to introduce some novel "monotonizing" method that transforms an arbitrary minimizing transport plan into a monotone transport plan of the same energy.

Theorem 3.1. *Let c be a continuous function satisfying for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$ with $x_1 < x_2$ and $y_1 < y_2$*

$$c(x_1, y_1) + c(x_2, y_2) < c(x_1, y_2) + c(x_2, y_1), \tag{3}$$

then the transport problem of Definition 1.1 with cost function c admits a unique minimizer, and this minimizer is monotone increasing.

Proof. The existence of a minimizer is, as has already been pointed out, well established. Suppose a minimizer T is not monotone increasing. Then there are sets $A, B \subset \mathbb{R}$ with $\mu(A), \mu(B) > 0$ and $\inf\{x \in B\} > \sup\{x \in A\}$ such that for

$$A' := \operatorname{supp} \pi_2(T|_{A\times\mathbb{R}}), \quad B' := \operatorname{supp} \pi_2(T|_{B\times\mathbb{R}})$$

we have $\inf\{x \in B'\} < \sup\{x \in A'\}$. Therefore there are sets $D, E \in \mathbb{R} \times \mathbb{R}$ such that $T(D), T(E) > 0$ and such that for all $(x_1, y_1) \in D$ and $(x_2, y_2) \in E$ we have $x_1 \leq x_2$ and $y_1 > y_2$.

We choose compact subsets $R \subset D$ and $S \subset E$ such that $\min\{T(R), T(S)\} =: K > 0$. Then we can choose a nonnegative Radon measure $\tau \leq T$ with $\operatorname{supp} \tau = R \cup S$ and $\tau(R) = \tau(S) = K$. We will demonstrate (by use of an approximation argument) that there exists a measure $\tau' \neq \tau$ such that $T' := T - \tau + \tau'$ is a transport plan with marginals μ and ν satisfying $C(T') < C(T)$.

First, we notice that since R and S are compact, the continuous function

$$e(x_1, y_1, x_2, y_2) := -c(x_1, y_1) - c(x_2, y_2) + c(x_1, y_2) + c(x_2, y_1)$$

admits a minimizer e_0 on $(x_1, y_1) \in R$, $(x_2, y_2) \in S$. Since by assumption e is positive on this set, this minimizer must be positive as well, and we have $e_0 > 0$.

Now we approximate $\tau|_R$ and $\tau|_S$ by sums of Dirac measures, such that, for $(x_1^{iN}, y_1^{iN}) \in R$ and $(x_2^{iN}, y_2^{iN}) \in S$, we have

$$\begin{aligned}\tau_N^R &:= \sum_{i=1}^N \frac{K}{N}\delta_{(x_1^{iN}, y_1^{iN})} \overset{\star}{\rightharpoonup} \tau|_R,\\ \tau_N^S &:= \sum_{i=1}^N \frac{K}{N}\delta_{(x_2^{iN}, y_2^{iN})} \overset{\star}{\rightharpoonup} \tau|_S,\end{aligned}$$

as $N \to \infty$. (Such an approximation is possible, compare [5, Lemma 7].)
We define $\tau_N := \tau_N^R + \tau_N^S$. For a fixed N, we construct τ'_N as follows:

$$\tau'_N := \sum_{i=1}^N \frac{K}{N}\left(\delta_{(x_1^{iN}, y_2^{iN})} + \delta_{(x_2^{iN}, y_1^{iN})}\right).$$

It is easy to see that τ'_N has the same marginals as τ_N. Define $T_N := T - \tau + \tau_N$ and $T'_N := T - \tau + \tau'_N$. By construction, $T_N \overset{\star}{\rightharpoonup} T$. The sequence (T'_N) is obviously tight, hence it converges (up to a subsequence) to a limit $T' \in \mathcal{P}(\mathbb{R} \times \mathbb{R})$. Using the weak-$\star$ convergence and the continuity of c we have $C(T_N) \to C(T)$ and $C(T'_N) \to C(T')$ for $N \to \infty$. By (3), we have

$$\begin{aligned}C(T'_N) - C(T_N) &= C(\tau'_N) - C(\tau_N)\\ &= -\sum_{i=1}^N \frac{K}{N} e(x_1^{iN}, y_1^{iN}, x_2^{iN}, y_2^{iN})\\ &\leq -Ke_0.\end{aligned}$$

This estimate is uniform in N. Hence we can take the limit $N \to \infty$ to obtain $C(T') < C(T)$. Since the marginals also converge, we have found a transport plan T' with marginals μ and ν but lower cost. Hence T cannot be a minimizer of the transport problem. Uniqueness follows then from Proposition 2.5. □
If we study the important class of cost functions of the form $c(x, y) = k(|x - y|)$ where k is a convex function we are lead to the following theorem:

Theorem 3.2. *Let c be a cost function of the form $c(x, y) = k(|x - y|)$ with $k\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ strictly convex and monotone increasing, then the transport problem of Definition 1.1 admits a minimizer which is unique and monotone increasing.*

Proof. Let $x_1, x_2, y_1, y_2 \in \mathbb{R}$ with $x_1 < x_2$ and $y_1 < y_2$. We want to show that

$$c(x_1, y_1) + c(x_2, y_2) < c(x_1, y_2) + c(x_2, y_1), \tag{4}$$

since we can then simply apply Theorem 3.1.

We need to distinguish three cases. (Other situations follow by the symmetry of the problem.)
CASE 1: $x_1 < x_2 \leq y_1 < y_2$
We use the following auxiliary statement: If $x < y$ and $0 < \varepsilon \leq (x+y)/2$ we have $k(x) + k(y) > k(x+\varepsilon) + k(y-\varepsilon)$. To prove this statement we estimate (using the strict convexity of k)

$$\begin{aligned} k(x+\varepsilon) + k(y-\varepsilon) &= k(x) + \int_x^{x+\varepsilon} k'(\xi) d\xi + k(y) - \int_{y-\varepsilon}^{y} k'(\xi) d\xi \\ &< k(x) + k'(x+\varepsilon) + k(y) - k'(y-\varepsilon) \\ &\leq k(x) + k(y). \end{aligned}$$

Now we apply this statement with $x := y_1 - x_2$, $y := y_2 - x_1$ and $\varepsilon := y_2 - y_1$ to derive

$$k(|x_1 - y_1|) + k(|x_2 - y_2|) < k(|x_1 - y_2|) + k(|x_2 - y_1|).$$

CASE 2: $x_1 < y_1 \leq x_2 < y_2$
We have $k(|y_2 - x_1|) \geq k(|y_1 - x_1|) + \int_{y_1}^{y_1+(y_2-x_2)} k'(\xi)\, d\xi$. Since k is strictly convex, we can estimate

$$\int_{y_1}^{y_1+(y_2-x_2)} k'(\xi)\, d\xi > \int_0^{y_2-x_2} k'(\xi)\, d\xi = k(|y_2 - x_2|).$$

Taking both together we get the desired estimate.
CASE 3: $x_1 \leq y_1 < y_2 \leq x_2$ Here the inequality follows from the fact that k is monotone and hence $k(|y_2 - x_1|) > k(|y_1 - x_1|)$ and $k(|x_2 - y_1|) > k(|x_2 - y_2|)$. □

Remark 3.3. *This proves in particular the well-known result that transport* maps *(as far as they exist) of L^p-Waserstein distances with $p > 1$ are (μ-a.e.) monotone.*

3.2 "Monotonizing" non-monotone transport plans

In the following we want to generalize our existence result. However, the classical "book shifting example" shows that we cannot expect all minimizers of transport problems to be monotone:
Let $c(x,y) := |x-y|$ and $\mu := \chi_{[0,1]}dx$, $\nu = \chi_{[0.5,1.5]}dx$. Then the transport map

$$\psi := \begin{cases} x+1, & \text{if } x \leq 0.5, \\ x, & \text{if } x > 0.5 \end{cases}$$

can be easily shown to be a minimizer of the associated transport problem. However, ψ is not monotone.
Nevertheless, we will prove that in this (and various other situations) at least an *alternative* minimizer exists which is monotone:

Theorem 3.4. *Let c be a continuous cost function satisfying for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$ with $x_1 < x_2$ and $y_1 < y_2$*

$$c(x_1, y_1) + c(x_2, y_2) \leq c(x_1, y_2) + c(x_2, y_1), \tag{5}$$

then the transport problem of Definition 1.1 admits a monotone increasing minimizer (and possibly other minimizers which do not need to be monotone).

Proof. Let T be some minimizer of the transport problem. We approximate T by a sequence of measures which are finite sums of Dirac masses on a grid. To be more precise we define for $i, j = -4^k, \ldots, 4^k$ the squares

$$Q^k_{ij} := [i2^{-k}, (i+1)2^{-k}) \times [j2^{-k}, (j+1)2^{-k}).$$

We denote the midpoint of the square Q^k_{ij} by M^k_{ij}. Then we can define T_k by

$$T_k := \sum_{i,j=-4^k}^{4^k} 2^k T(Q^k_{ij}) \delta_{M^k_{ij}}. \tag{6}$$

As above, $T_k \stackrel{\star}{\rightharpoonup} T$ for $k \to \infty$. Now we can “monotonize” T_k, i.e. we can perform a finite number of manipulations on T_k which lead to a modified transport plan T'_k which is monotone increasing and satisfies $C(T'_k) \leq C(T_k)$. This is equivalent to convert the matrix a given by

$$a_{i,j} := T(Q^k_{ij})$$

(where we omit for simplicity the index k) iteratively into a matrix $a'_{i,j}$ where for all $i_1 < i_2$ and $j_1 > j_2$ either $a'_{i_1,j_1} = 0$ or $a'_{i_2,j_1} = 0$ whereby not changing the sums over rows or columns, only allowing for nonnegative entries and not increasing the corresponding cost $C(T'_k)$ of the transport plan T'_k defined by

$$T'_k := \sum_{i,j=-4^k}^{4^k} 2^k a'_{i,j} \delta_{M^k_{ij}}.$$

This can be achieved by the following iteration, illustrated in Example 3.6: In each step we transform a given matrix $a^m = (a^{(m)}_{i,j}) \in \mathbb{R}^{p \times q}_{\geq 0}$ (with associated transport plan $T^{(m)}_k$) with

$$a^{(m)}_{i,j} = \begin{pmatrix} a^{(m)}_{1,1} & a^{(m)}_{1,2} & \ldots & a^{(m)}_{1,q} \\ \vdots & & & \vdots \\ a^{(m)}_{p,1} & a^{(m)}_{p,2} & \ldots & a^{(m)}_{p,q} \end{pmatrix}$$

into a new matrix $a^{(m+1)}$ being of "row" or "column form", i.e. either

$$a_{i,j}^{(m+1)} = \begin{pmatrix} a_{1,1}^{(m+1)} & 0 & \dots & 0 \\ a_{2,1}^{(m+1)} & a_{2,2}^{(m+1)} & \dots & a_{2,q}^{(m+1)} \\ \vdots & \vdots & & \vdots \\ a_{p,1}^{(m+1)} & a_{p,2}^{(m+1)} & \dots & a_{p,q}^{(m+1)} \end{pmatrix} \quad \text{or} \quad a_{i,j}^{(m+1)} = \begin{pmatrix} a_{1,1}^{(m+1)} & a_{1,2}^{(m+1)} & \dots & a_{1,q}^{(m+1)} \\ 0 & a_{2,2}^{(m+1)} & \dots & a_{2,q}^{(m+1)} \\ \vdots & \vdots & & \vdots \\ 0 & a_{p,2}^{(m+1)} & \dots & a_{p,q}^{(m+1)} \end{pmatrix},$$

such that the sums over rows and columns are preserved and the associated transport plan $T_k^{(m+1)}$ has no higher cost than $T_k^{(m)}$, i.e., $C(T_k^{(m+1)}) \leq C(T_k^{(m)})$. In this way, we have reduced the problem to a pure algebraic statement for matrices, its proof is given below, see Lemma 3.5.

We can now apply the same lemma in the next iteration step to proceed from $(m+1)$ to $(m+2)$ where we apply it only for all but the first row of $a^{(m+1)}$ (if $a^{(m+1)}$ is of row form) or all but the first column (if $a^{(m+1)}$ is of column form).

Starting with $a^{(0)} := a_{i,j}$, the iteration stops after finitely many steps when the remaining matrix has been reduced to a vector (since in every step the matrix gets reduced by either a row or a column). The result a' of this iteration is of the desired form: It satisfies by construction the condition that for all $i_1 < i_2$ and $j_1 > j_2$ either $a'_{i_1,j_1} = 0$ or $a'_{i_2,j_1} = 0$, and hence its associated transport plan T'_k is monotone increasing. Moreover, since in every iteration step the sums over rows and columns of $a^{(m)}$ are preserved, the transport plan T'_k has the same marginals as T_k, and finally $C(T'_k) \leq C(T_k)$.

We now take the limit $k \to \infty$. There exists a T' such that (at least for a subsequence) $T'_k \overset{\star}{\rightharpoonup} T'$. Since (T'_k) is tight and $\|T'_k\| \to 1$, we obtain $T' \in \mathcal{P}(\mathbb{R} \times \mathbb{R})$. Since $\pi_1 T - \pi_1 T'_k = \pi_1 T - \pi_1 T_k \to 0$ when $k \to \infty$, and the same holds for π_2, we have constructed a transport plan T' with marginals μ and ν. Due to the weak-$\star$ convergence we also have $C(T) - C(T') = \lim_{k\to\infty} C(T_k) - C(T'_k) \geq 0$. Therefore T' is a minimizing transport plan. It remains to show that T' is monotone increasing. Suppose that it is not, then there must be sets $D, E \subset \mathbb{R} \times \mathbb{R}$ with $T'(D), T'(E) > 0$ and such that for all $(x_1, y_1) \in D$ and $(x_2, y_2) \in E$ we have $x_1 < x_2$ and $y_1 > y_2$. We can assume that D and E are such that we can choose squares S_D, S_E in $\{Q_{ij}^k\}$ with $S_D \subset D$ and $S_E \subset E$ and $T'(S_D), T'(S_E) > 0$, but this leads to a contradiction: By the weak-$\star$ convergence we would have that also $T'_k(S_D), T'_k(S_E) > 0$ for k large enough. This would be a contradiction to the monotonicity of T'_k. Hence T' is a monotone increasing minimizer. □

We conclude with the algebraic lemma used in the above proof:

Lemma 3.5. *Let $A = (a_{i,j}) \in \mathbb{R}^{n\times m}$ be a matrix with nonnegative entries. Let $c\colon \{1,\dots,n\} \times \{1,\dots m\} \to \mathbb{R}$ be a function satisfying the inequality (5). Define*

$$C(A) := \sum_{i=1}^{n} \sum_{j=1}^{m} c(i,j) a_{i,j}.$$

Then there exists a matrix $B = (b_{i,j}) \in \mathbb{R}^{n\times m}$ *with the following properties:*

(i) $b_{i,j} \geq 0$ *for all* i, j,

(ii) $\sum_{i=1}^{n} b_{i,j} = \sum_{i=1}^{n} a_{i,j}$ *for all* j *and* $\sum_{j=1}^{m} b_{i,j} = \sum_{j=1}^{m} a_{i,j}$ *for all* i,

(iii) $C(B) \leq C(A)$,

(iv) either $b_{i,1} = 0$ *for all* $i = 2, \ldots, n$
or $b_{1,j} = 0$ *for all* $j = 2, \ldots, m$.

Proof. The proof is constructive, in fact we give a simple algorithm that computes B for a given matrix A. Since property (iv) is (as we have seen in the proof of Theorem 3.4) directly connected to the monotonicity of a corresponding transport plan, we say that this algorithm “monotonizes” a given matrix A.
A key feature in the monotonization will be the following construction (which has essentially been already applied in the proof of Theorem 3.1) which we call a *switch* of (i_1, j_1) and (i_2, j_2):
Take $i_1, i_2 \in \{1, \ldots, n\}$ and $j_1, j_2 \in \{1, \ldots, m\}$ with $i_1 < i_2$, $j_2 < j_1$. Define $\beta := \min\{a_{i_1,j_1}, a_{i_2,j_2}\}$ and

$$\begin{aligned} b_{i_1,j_1} &:= a_{i_1,j_1} - \beta, \\ b_{i_2,j_2} &:= a_{i_2,j_2} - \beta, \\ b_{i_1,j_2} &:= a_{i_1,j_2} + \beta, \\ b_{i_2,j_1} &:= a_{i_2,j_1} + \beta, \\ b_{i,j} &:= a_{i,j} \quad \text{for all other pairs } (i, j). \end{aligned}$$

A small calculation shows that the matrix $B := (b_{i,j})$ satisfies the properties (i)-(iii) in the statement of this lemma and that moreover either $b_{i_1,j_1} = 0$ or $b_{i_2,j_2} = 0$ (or both).
Now we just need to find a sequence of switches that transforms A into a matrix B satisfying property (iv) and we have proved the lemma. This can be achieved with the help of the following algorithm:

```
Set i = n and j = m.
While i > 1 and j > 1:
{
    Switch (i, 1) and (1, j). (The result will again be called A.)
    If a_{1,j} = 0 then j = j − 1.
    If a_{i,1} = 0 then i = i − 1.
}
Set B = (a_{i,j}).
```

The properties which the switch is satisfying ensure that the algorithm terminates and that its result B satisfies the properties (i)-(iii). A closer look at the algorithm reveals furthermore that in each processing of the while loop either $a_{i,1}$ or $a_{1,j}$ is set to zero. From this it follows in particular that B also satisfies (iv). This proves the lemma. □

Example 3.6. *To demonstrate the above algorithm let us consider the matrix*

$$A := \begin{pmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 4 & 5 & 6 \end{pmatrix}.$$

We will show that A can be monotonized (in the sense above) to

$$A_{mon} := \begin{pmatrix} 6 & 0 & 0 \\ 6 & 15 & 3 \\ 0 & 0 & 15 \end{pmatrix}.$$

Proof. We apply the algorithm of Lemma 3.5. Set $i = 3$ and $j = 3$ and switch $(3,1)$ and $(1,3)$ to get

$$A' = \begin{pmatrix} 4 & 2 & \underline{0} \\ 7 & 8 & 9 \\ \underline{1} & 5 & 9 \end{pmatrix},$$

where we underlined the "switched" entries. Since $a_{3,1} = 0$, we reduce i by one. In the next step we accordingly switch $(2,1)$ and $(1,3)$ leading to

$$A'' = \begin{pmatrix} 4 & \underline{1} & 0 \\ 7 & 8 & 9 \\ \underline{0} & 6 & 9 \end{pmatrix}.$$

This time $a_{1,3} = 0$, hence j gets reduced to $j = 2$. We then switch $(2,1)$ and $(1,2)$ and arrive at

$$B = \begin{pmatrix} 6 & \underline{0} & 0 \\ \underline{6} & 9 & 9 \\ 0 & 6 & 9 \end{pmatrix}.$$

This matrix satisfies the condition (iv). (It is of "row form".) We can check the sums over the columns and rows and we see that they are still unchanged.
In the next step in the monotonization process, we look at the remaining matrix

$$\begin{pmatrix} 6 & 9 & 9 \\ 0 & 6 & 9 \end{pmatrix}.$$

Incidentally, this matrix is already of column form and we can go on by applying Lemma 3.5 to the matrix

$$\begin{pmatrix} 9 & 9 \\ 6 & 9 \end{pmatrix}.$$

The same procedure as above yields

$$\begin{pmatrix} 15 & 3 \\ 0 & 15 \end{pmatrix}.$$

Taking everything together, we have found the monotonized matrix A_{mon}. □

3.3 Useful properties and applications

To conclude this section, we collect some useful properties that can be derived quite easily for monotone transport plans. First, we show that every monotone transport plan can be (mostly) described by a set-valued map and that this map in a certain sense corresponds to a transport map:

Proposition 3.7. *Let T be a monotone increasing transport plan. Then the set-valued function $\psi(x) := (\operatorname{supp} T) \cap (\{x\} \times \mathbb{R})$ is itself monotone increasing when we define $\psi(x_1) \geq \psi(x_2)$ iff for all $a \in \psi(x_1), b \in \psi(x_2)$ we have $a \geq b$.*
Moreover there is a sequence of monotone increasing transport plans T_ε induced by transport maps ψ_ε with $T_\varepsilon \overset{\star}{\rightharpoonup} T$. If $\operatorname{supp} T$ is compact, the transport maps ψ_ε converge to ψ.

Remark 3.8. *The fact that we can approximate a transport plan by transport maps is actually not a property only of* monotone *transport plans. Indeed, one can approximate* any *transport plan by transport maps using a similar construction as in the proof of Theorem 3.4.*

Proof of Proposition 3.7:
The first statement follows directly from the definition. The approximating transport plans T_ε can be chosen as

$$T_\varepsilon(x, y) := T(x + \varepsilon y, y).$$

Supposing that T_ε cannot be represented by a transport map would imply that the map $\psi_\varepsilon(x) := (\operatorname{supp} T_\varepsilon) \cap (\{x\} \times \mathbb{R})$ were indeed set-valued, i.e. that there exists some $x \in \mathbb{R}$ with $\operatorname{card}(\psi_\varepsilon(x)) > 1$. Take $y_1, y_2 \in \psi_\varepsilon(x)$ with $y_1 > y_2$ then $x_1 := x - \varepsilon y_1 > x_2 := x - \varepsilon y_2$ and hence the points (x_1, y_1) and (x_2, y_2) violate the monotonicity condition. □
Finally we prove a very natural property about the behavior of maps induced by transport plans:

Proposition 3.9. *Let T be a transport plan with marginals μ and ν. Let $T_\# : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ be a map with*

$$T_\#(\tau) := \pi_2(T|_A),$$

where T is a transport plan and $A \subset \operatorname{supp} T$ with $\pi_1(T|_A) \geq \tau$ (i.e. for every measurable set $X \subset \mathbb{R}$ we have $\pi_1(T|_A)(X) \geq \tau(X)$).
Take some $x \in \mathbb{R}$. Let $\mu \in \mathcal{P}(\mathbb{R})$ and $\tau := \mu|_{[x,\infty)}$. If T is a monotone increasing transport plan with $\pi_1(T) = \mu$ then

$$\inf \operatorname{supp} T_\#(\tau) \geq \sup \operatorname{supp} T_\#(\mu - \tau).$$

Proof. Suppose the opposite. Then there are $a \in \operatorname{supp} T_\#(\tau)$ and $b \in \operatorname{supp} T_\#(\mu-\tau)$ such that $a < b$. Therefore $\operatorname{supp} T \cap ([x,\infty) \times \{a\}) \neq \emptyset$ and also $\operatorname{supp} T \cap ((-\infty, x)) \times \{b\}) \neq \emptyset$. Hence we can choose $y_1 := a$ and $y_2 := b$ and have found points (x_1, y_1) and (x_2, y_2) in $\operatorname{supp} T$ violating the monotonicity condition, since $y_1 < y_2$ and $x_1 \geq x > x_2$. □

Remark 3.10. *While $T_\#$ is in general not uniquely determined, in the specific case of τ chosen as above, it is unique.*

All results of this and the last section carry over directly to the L^1-Wasserstein distance. We have seen in the "book shifting example" that in this case, i.e., if $c(x,y) := |x - y|$, we cannot expect a minimizer of the transport problem to be monotone. Theorem 3.4, however, shows that there exists at least one monotone increasing minimizer:

Corollary 3.11. *The transport problem associated to the L^1-Wasserstein distance in $\mathbb{R}^1$ admits a monotone increasing minimizer.*

This, together with Proposition 3.9 can be directly applied to the above mentioned model for damage, compare [7].

4 Higher dimensional problems

In this section we generalize the definition of monotonicity and some of our results from the previous sections to higher dimensional problems.

Definition 4.1. *Let $T \in \mathcal{P}(\mathbb{R}^n \times \mathbb{R}^n)$ be a transport plan. Then we call T* monotone increasing *if the set $S := \operatorname{supp} T$ satisfies the following condition:*
For all $x_1, x_2 \in S$, we have $\langle x_2 - x_1, y_2 - y_1 \rangle \geq 0$.

It can be easily shown that in the case $n = 1$, this coincides with Definition 2.1.
The next (simple) lemma draws a connection from this definition to the classical notion of *cyclical monotonicity*: If a transport plan has a cyclically monotone support, then it is montone increasing.

Lemma 4.2. *Let $T \in \mathcal{P}(\mathbb{R}^n \times \mathbb{R}^n)$ be a transport plan with cyclically monotone support S. Then T is monotone increasing.*

We can now generalize Theorem 3.1 to the higher dimensional case.

Theorem 4.3. *Let c be a Borel function satisfying for all $x_1, x_2, y_1, y_2 \in \mathbb{R}^n$ with $\langle x_2 - x_1, y_2 - y_1 \rangle > 0$*

$$c(x_1, y_1) + c(x_2, y_2) < c(x_1, y_2) + c(x_2, y_1), \tag{7}$$

then the transport problem of Definition 1.1 with cost function c admits a minimizer which is monotone increasing.

Proof. The proof follows the same outline as the proof of Theorem 3.1. □

Remark 4.4. *Uniqueness for the case that $\mu(E) = 0$ whenever $\mathcal{H}^{n-1}(E) = 0$ and that $\operatorname{supp} T$ is cyclically monotone has been proved in [5, Corollary 14].*

The additional condition on μ which is needed to prove uniqueness in Theorem 4.3 cannot be removed as the following classical example demonstrates:

Example 4.5. *Let $n = 2$. Define $\mu := \frac{1}{2}\delta_{(-1,0)} + \frac{1}{2}\delta_{(1,0)}$ and $\nu := \frac{1}{2}\delta_{(0,-1)} + \frac{1}{2}\delta_{(0,1)}$. Then $T_1(x, y) = \frac{1}{2}\delta_{(-1,0),(0,-1)} + \frac{1}{2}\delta_{(1,0),(0,1)}$ and $T_2(x, y) = \frac{1}{2}\delta_{(-1,0),(0,1)} + \frac{1}{2}\delta_{(1,0),(0,-1)}$ are both monotone increasing transport plans. Moreover, for any cost function of the form $c(x, y) = k(|x - y|)$, they are both minimizers of the transport problem of Definition 1.1.*

It is easy to generalize Theorem 3.4 to higher dimensions in the sense that a transport plan exists which is monotone increasing along a given line. However, it seems to be difficult to prove existence of a transport plan which is monotone increasing in the sense of Definition 4.1.

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References

[1] Luigi Ambrosio. Lecture notes on optimal transport problems. In P. Colli and C.F. Rodrigues, editors, *CIME summer school in Madeira*, *Lecture Notes in Mathematics*, volume 1813, pages 1–52. Springer Verlag, 2003.

[2] Wilfrid Gangbo and Robert J. McCann. The geometry of optimal transportation. *Acta Math.*, 177(2):113–161, 1996.

[3] Thorsten Hens and Marc Oliver Rieger. Following the market. in preparation.

[4] Thorsten Hens, Marc Oliver Rieger, and Mei Wang. Optimal product design for Prospect Theory investors: A CAPM approach. NCCR – Financial Valuation and Risk Management, 2006.

[5] Robert J. McCann. Existence and uniqueness of monotone measure-preserving maps. *Duke Math. J.*, 80(2):309–323, 1995.

[6] Marc Oliver Rieger. A model for hysteresis in mechanics using local minimizers of Young measures. In *Progress in Nonlinear Differential Equations and Their Applications*, volume 63, pages 363–374. Birkhäuser Verlag Basel, 2005.

[7] Marc Oliver Rieger and Johannes Zimmer. Young measure flow as a model for damage. Preprint 11, Bath Institute for Complex Systems, Bath, UK, 2005.