Convexity and Semiconvexity along Vector Fields

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Abstract

Given a family of vector fields we introduce a notion of convexity and of semiconvexity of a function along the trajectories of the fields and give infinitesimal characterizations in terms of inequalities in viscosity sense for the matrix of second derivatives with respect to the fields. We also prove that such functions are Lipschitz continuous with respect to the Carnot-Carathéodory distance associated to the family of fields and have a bounded gradient in the directions of the fields. This extends to Carnot-Carathéodory metric spaces several results for the Heisenberg group and Carnot groups obtained by a number of authors.

1 Introduction.

Consider a smooth vector field X in $\mathbb{R}^n$ and its trajectories, i.e., the solutions of $\dot{x}(t) = X(x(t))$. It is natural to say that a function $u : \Omega \to \mathbb{R}$, $\Omega \subseteq \mathbb{R}^n$ open, is convex along the vector field X if its restriction to each trajectory of X is convex, that is, $t \mapsto u(x(t))$ is convex for all $x(\cdot)$. When u is smooth, one observes that $\frac{d^2}{dt^2} u(x(t)) = X^2 u(x(t))$ and then such a function is convex along X if and only if $X^2 u \geq 0$. If u is not smooth a natural question is whether its convexity can be characterized by some weak version of the inequality $X^2 u \geq 0$. One of the results of this paper is that

$$u \in USC(\overline{\Omega}) \text{ is convex along } X \iff -X^2 u \leq 0 \text{ in viscosity sense.}$$

Moreover, we prove that this property implies a local gradient estimate in the direction of X

$$|Xu| \leq C(K)\|u\|_\infty \quad \forall\, K \subset\subset \Omega.$$

More generally, we consider a family of m vector fields $\mathcal{X} = \{X_1, \ldots, X_m\}$ of class C^2 and say that u is convex along them if $t \mapsto u(x(t))$ is convex on all trajectories of $\mathrm{Span}(\mathcal{X})$, i.e., for all $x(\cdot)$ solving

$$\dot{x}(t) = \sum_{i=1}^{m} \alpha_i X_i(x(t))$$

for some $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$. For u smooth a computation gives

$$\frac{d^2}{dt^2} u(x(t)) = \alpha^T D^2_{\mathcal{X}} u(x(t))\alpha,$$

where

$$D^2_{\mathcal{X}} u(x) := \big(X_i(X_j u(x))\big)_{i,j}$$

is the $m \times m$ Hessian matrix of u with respect to the vector fields, so the positive semi-definiteness of this matrix is equivalent to the convexity of u along $\mathcal{X}$. Our main result is that

$$u \in USC(\overline{\Omega}) \text{ is convex along } X_1, \ldots, X_m \iff -D^2_{\mathcal{X}} u \leq 0 \text{ in viscosity sense,}$$

which means that

$$D^2_{\mathcal{X}} \phi(x) \geq 0 \text{ for all smooth } \phi \text{ and } x \in \arg\max(u - \phi). \tag{1}$$

We also define the weaker notion of *semi-convexity* with respect to the vector fields, in the trajectory and in the viscosity sense, prove their equivalence, and show that this property implies the local boundedness of the gradient of u with respect to the vector fields $D_{\mathcal{X}} u := (X_1 u, \ldots, X_m u)$.

The problem we study is related to the search of a notion of convexity in Carnot groups with useful properties. This issue received considerable attention recently in the sub-elliptic and sub-Riemannian communities. Monti and Rickly [27] proved that all geodetically convex functions in the Heisenberg group are constants, so this notions appears useless outside Riemannian geometry. A notion of *horizontal convexity* in the Heisenberg group, that seems to have been first conceived by Caffarelli, was introduced and studied independently by Lu, Manfredi, and Stroffolini [24] and by Danielli, Garofalo, and Nhieu [16] (in more general Carnot groups and with the name of weak H-convexity). It uses convex combinations built by the group operation and dilations, and does not need any a priori regularity or boundedness of the function. Lu, Manfredi, and Stroffolini introduced also the notion of convexity in viscosity sense. It requires a stratification of the Lie algebra associated to the Carnot group and the choice of a basis of the first layer, that is the horizontal subspace, formed by left-invariant vector fields $\mathcal{X} = \{X_1, \ldots, X_m\}$. Then $u \in USC(\overline{\Omega})$ is called *v-convex* if the horizontal Hessian $D^2_{\mathcal{X}} u$ is positive semi-definite in the viscosity sense, i.e., (1) holds [24, 23]. They proved in the Heisenberg group the Lipschitz continuity of a v-convex function with respect to the Carnot-Carathéodory distance, a bound on the horizontal gradient $D_{\mathcal{X}} u$, and the v-convexity of any horizontally convex function. The full equivalence of the two notions turned out to be harder and was settled by Wang [33], Magnani [25], and Juutinen, Lu, Manfredi, and Stroffolini [23] with different proofs. The Lipschitz continuity was also studied under different assumptions in [7], [16], [28], [25], [31]. A nice survey of these results is in the book [11]. Further properties of convex functions in Carnot groups, including the existence of second derivatives a.e. and the horizontal Monge-Ampère operator, were studied by [21], [22], [16], [17], [20], [25], [13]. See also [32] for a study of Hessian measures associated to Hörmander vector fields with step 2. The present paper seems to be the first to introduce the notion of semiconvexity in this setting.

We believe our results could be the first step of a theory of convex functions and sets in general metric spaces of Carnot-Carathéodory type, without the algebraic structure of Carnot groups. We recall that a Carnot-Carathéodory (briefly C-C) space is a manifold M endowed with a distribution $\mathcal{X}$, and the C-C distance $d(x, y)$ between two point $x, y \in M$ is the minimum time taken by a trajectory of the control system associated to $X_1, \dots, X_m$ to join x and y (see Definition (20)). If $d(x, y) < +\infty$ everywhere, then (M, d) is a metric space. We refer to [9] and [26] for a general presentation of the subject. If $M = \mathbb{R}^n$ it looks natural to say that a function $u : M \to \mathbb{R}$ is convex if it is convex along the vector fields $\mathcal{X}$. Then our results give an infinitesimal characterization of this property in terms of the second order sub-jet [14] and a bound on the $\mathcal{X}$-components of the (viscosity) sub-differential [3]. We also prove that $\mathcal{X}$-semiconvex functions are Lipschitz continuous with respect to the C-C distance d. We refer to [12] for semi-convexity in the Euclidean setting and its many applications. We believe that this notion can play a similar important role for functions in metric spaces.

We remark that our proofs of the main results are completely different from those in Carnot groups of the cited papers, that exploit the deep algebraic and geometric properties of that setting. We use instead methods of the theory of viscosity solutions, in particular an idea of straightening the trajectories and reducing inequalities in $\mathbb{R}^n$ to inequalities along curves that is inspired by the proof in [15, 3] that u is nondecreasing along the trajectories of X if and only if $Xu \geq 0$ in viscosity sense.

A second motivation of the present paper comes from subelliptic partial differential equations of Monge-Ampère type, i.e., of the form

$$-\det\left(D^2_{\mathcal{X}}u\right) + H(x, u, D_{\mathcal{X}}u) = 0, \quad \text{in } \Omega.$$

In fact, these equations are degenerate elliptic exactly on v-convex functions, and the well-posedness of their Dirchlet problem was studied among such functions in Carnot groups by the first-named author and Mannucci [4, 5, 6]. The gradient bound for v-convex functions is particularly useful in that context. We give an example of Comparison Principle for Monge-Ampère type equations that extends a result in [5] from Carnot groups to general vector fields.

The paper is organized as follows. In Section 2 we introduce the definitions of $\mathcal{X}$-convexity and v-convexity with some comments and simple examples. Section 3 contains the proof of the main result about the equivalence of the two notions. Section 4 recalls some basic facts about Carnot-Carathéodory metric spaces, Carnot groups, and the Heisenberg group, including the notion of horizontal convexity, and gives several examples of $\mathcal{X}$-convex functions. Section 5 introduces $\mathcal{X}$-semiconvexity along with some simple properties and examples. Section 6 is about the gradient bounds and d-Lipschitz continuity of $\mathcal{X}$-semiconvex functions. Finally, Section 7 gives an application to PDEs of Monge-Ampère type.

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2 Notions of convexity.

Throughout the paper $\mathcal{X} = \{X_1(x), \dots, X_m(x)\}$ is a family of C^2-vector fields in $\mathbb{R}^n$ and $\Omega \subset \mathbb{R}^n$ is open and connected.

Definition 2.1. *We call* $\mathcal{X}$-line *any absolutely continuous curve, satisfying*

$$\dot{x}(t) = \sum_{i=1}^{m} \alpha_i X_i(x(t)) = \sigma(x(t))\alpha, \quad t \in [T_1, T_2], \tag{2}$$

for some $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m$*, where* $\sigma(x)$ *is the* $n \times m$*-matrix having the vectors* $X_1(x), \dots, X_m(x)$ *as columns.*

Remark 2.1. Note that, up to a simple reparametrization, we can always assume $|\alpha|^2 = \sum_{i=1}^m \alpha_i^2 = 1$.

Using the $\mathcal{X}$-lines, we can introduce the following definition.

Definition 2.2. *We say that* $u : \Omega \to \mathbb{R}$ *is* convex along the vector fields $X_1(x), \dots, X_m(x)$*, briefly* $\mathcal{X}$-convex*, if for every curve* $x : [T_1, T_2] \to \Omega$ *satisfying* (2)*, the composition* $t \mapsto u(x(t))$ *is a real-valued convex function on* $[T_1, T_2]$.

We say that u *is* $\mathcal{X}$-concave *if* $-u$ *is* $\mathcal{X}$*-convex.*

Remark 2.2. If $\mathcal{Y} = \{Y_1(x), \dots, Y_m(x)\}$ is another family of C^2-vector fields such that any $\mathcal{X}$-line is also a $\mathcal{Y}$-line and viceversa, then u is $\mathcal{X}$-convex if and only if it is $\mathcal{Y}$-convex. This is the case, for instance, if we make a change of basis of *span*$\mathcal{X}$ independent of x, i.e., $Y_j(x)$ is the $j-$th column of $\sigma(x)A$ for some $m \times m$ matrix A with $\det A \neq 0$.

Remark 2.3. The preceding remark suggests that we can also think of $\mathcal{X}$-convexity as the usual convexity on a family of curves with suitable properties. The Euclidean convexity in $\mathbb{R}^n$ is recovered by taking for $\mathcal{X}$ the canonical basis, so the $\mathcal{X}-$lines are the usual straight lines. Incidentally, in $(\mathbb{R}^n, |\cdot|)$ the straight lines are the geodesics, so $\mathcal{X}-$convexity coincides with geodetic convexity. But this is not the case in general, for instance in the Heisenberg group, by a result of Monti and Rickly [27].

For the second definition we introduce the matrix of the second-order derivatives w.r.t. the family of vector fields $\mathcal{X}$, that we call *intrinsic Hessian*, or *$\mathcal{X}$-Hessian*.

We always indicate by $A = (a_{i\,j})_{i\,j}$ the matrix having $a_{i\,j}$ as element of position (i, j) and as v^k the kth-component of a vector v, but we denote the points in $\mathbb{R}^n$ by the usual notation $x = (x_1, \dots, x_n)$. The intrinsic Hessian (or $\mathcal{X}$-Hessian) of a C^2 function u is the $m \times m$-matrix.

$$\begin{aligned} D^2_{\mathcal{X}} u(x) &:= \big(X_i(X_j u(x))\big)_{i\,j} = \sigma^T(x) D^2 u(x) \sigma(x) + \left(\nabla_{X_i} X_j(x) \cdot Du(x)\right)_{i\,j} \\ &= \sigma^T(x) D^2 u(x) \sigma(x) + \left(\sum_{k=1}^{n} \nabla_{X_i} X_j^k(x)\, u_{x_k}(x) \right)_{i\,j}, \end{aligned} \tag{3}$$

for $i,j = 1,\dots,m$, where $\cdot$ indicates the standard inner product in $\mathbb{R}^n$ and $\nabla_{X_i}X_j$ the derivative of the vector field X_j along the vector field X_i. More precisely, $\nabla_{X_i}X_j$ is the vector in $\mathbb{R}^n$ whose k-component is $\nabla_{X_i}X_j^k(x) = X_i(X_j^k(x))$, where $X_j^k(x)$ is the k-component of the vector field $X_j(x)$.

In other words, $\nabla_{X_i}X_j = DX_jX_i$ where DX_j is the Jacobian-matrix of the map $x \mapsto X_j(x)$.

Note that, if $n = m$ and $\mathcal{X}$ is the canonical basis in $\mathbb{R}^n$, then (3) is the usual Hessian, while for $X_1(x),\dots,X_m(x)$ satisfying the Hörmander condition (3) is the so-called horizontal Hessian, and its trace is the horizontal Laplacian(called also sub-Laplacian).

Proposition 2.1. *If $u \in C^2(\overline{\Omega})$ and $x(\cdot)$ is the $\mathcal{X}$-line corresponding to $\alpha \in \mathbb{R}^m$ (i.e., it satisfies (2)) then*

$$\frac{d^2}{dt^2}u(x(t)) = \alpha^T D^2_{\mathcal{X}}u(x(t))\,\alpha\,. \tag{4}$$

Therefore u is $\mathcal{X}$-convex if and only if $D^2_{\mathcal{X}}u \geq 0$ in Ω.

Remark 2.4. By $D^2_{\mathcal{X}}u \geq 0$ we mean $(D^2_{\mathcal{X}}u)^* \geq 0$, where

$$(D^2_{\mathcal{X}}u)^*(x) := \sigma^T(x)D^2u(x)\sigma(x) + \left(\frac{\nabla_{X_i}X_j(x) + \nabla_{X_j}X_i(x)}{2}\cdot Du(x)\right)_{i\,j}$$

is the symmetrized matrix of the intrinsic Hessian. In fact, $a^TD^2_{\mathcal{X}}u(x)a = a^T(D^2_{\mathcal{X}}u)^*(x)a$. In this paper we often prefer to use the non-symmetrized intrinsic Hessian to simplify the calculations.

Proof. We compute

$$\begin{aligned}\frac{d^2}{dt^2}u(x(t)) = \frac{d}{dt}\big[Du\ \cdot\ \dot{x}(t)\big] &= \frac{d}{dt}\big[Du\ \cdot\ \sigma(x(t))\alpha\big]\\ &= D^2u\ \dot{x}(t)\ \cdot\ \sigma(x(t))\,\alpha + \alpha^T\left(\frac{d}{dt}\sigma^T(x(t))\right)Du.\end{aligned}$$

The first term on the right-hand side can be written as $\alpha^T\sigma^T(x(t))D^2u\ \sigma(x(t))\alpha$. Next we calculate the second term:

$$\begin{aligned}\sum_i \alpha_i\frac{d}{dt}\sigma_{i\,j}(x(t))u_{x_j} &= \sum_{i,k}\alpha_iu_{x_j}\sigma_{i\,j\,x_k}\dot{x}_k(t) = \sum_{i,h,l}\alpha_iu_{x_j}\big(DX_i\big)_{j\,k}\alpha_l\sigma_{k\,l}\\ &= \sum_{i,l}\alpha_i\ \alpha_lu_{x_j}\big(DX_iX_l\big)_j = \sum_{i,l}\alpha_iDX_iX_l\cdot Du\ \alpha_l = \alpha^T\big(DX_iX_j\cdot Du\big)_{i\,j}\alpha\end{aligned}$$

Therefore

$$\frac{d^2}{dt^2}u(x(t)) = \alpha^T\left(\sigma^T(x(t))D^2u\ \sigma(x(t)) + \big(DX_iX_j\cdot Du\big)_{i\,j}\right)\alpha = \alpha^TD^2_{\mathcal{X}}u\ \alpha\,.$$

$\square$

This result motivates the following definition for non-smooth functions.

Definition 2.3. *We say that* $u \in USC(\overline{\Omega})$ *is* convex in the viscosity sense *with respect to the fields* $\mathcal{X}$*, briefly* v-convex*, if*

$$-D^2_{\mathcal{X}} u \leq 0 \quad \text{in } \Omega \tag{5}$$

in the viscosity sense, i.e., for any $\varphi \in C^2$ *and* $x \in \Omega$ *such that* $u - \varphi$ *has a local maximum at* x*, we have that* $a^T D^2_{\mathcal{X}} \varphi(x)\, a \geq 0$*, for all* $a \in \mathbb{R}^m$*.*

Remark 2.5. It is easy to see, as in [24], that the definition of v-convexity is equivalent to each of the following statements:
i) u is a viscosity subsolution of the linear PDE

$$-\text{trace}\,\left(A(D^2_{\mathcal{X}} u)^*\right) = 0, \quad \text{in } \Omega,$$

for every (constant) $m \times m$ symmetric and positive definite matrix A;
ii) u is a viscosity subsolution of the fully nonlinear PDE

$$F\left(x, u, Du, (D^2_{\mathcal{X}} u)^*\right) = 0, \quad \text{in } \Omega,$$

for every continuous F with $F(x, z, p, 0) = 0$, non-decreasing in the second entry, and degenerate elliptic, i.e., non-increasing in the last entry with respect to the usual partial order of symmetric matrices.

Remark 2.6. The v-convexity of u can also be characterized by a single scalar inequality, that is,

$$-\lambda_{\min}\left((D^2_{\mathcal{X}} u)^*\right) \leq 0$$

in viscosity sense, where $\lambda_{\min}(M)$ denotes the minimal eigenvalue of the symmetric matrix M. Note that this can also be written as the Hamilton-Jacobi-Bellman inequality

$$\max\left\{-\text{trace}\,\left(A(D^2_{\mathcal{X}} u)^*\right) \mid A_{ij} = a_i a_j,\ |a| = 1\right\} \leq 0.$$

Example 2.1 (*Canonical vector fields*)**.** Let $1 \leq m \leq n$ and $\mathcal{X} = \{\mathrm{e}_1, ..., \mathrm{e}_m\}$, where e_i is ith-unit-vector of the canonical Euclidean basis. In this particular case, the equivalence between Definition 2.2 and Definition 2.3 tells that u is convex in the first m components if and only if the Hessian matrix of u with respect to the first m variables $x_1, \dots, x_m$ is positive semidefinite in the viscosity sense. This result is known, at least for $m = n$ [1].

Example 2.2 ($n = 1$)**.** Consider in dimension $n = 1$ the vector field $X(x) = \sigma(x) = bx$, for some constant b. Then a C^2 function u is X-convex if and only if $x^2 u'' + x u' \geq 0$. Then

1. $u(x) = \mathrm{e}^{\alpha x}$ with $\alpha > 0$ is X-convex only in the half-lines $[0, +\infty)$ and $(-\infty, -1/\alpha]$, so Euclidean convexity does not imply X-convexity;

2. $u'' \geq 0$, $u' \geq 0$ for $x > 0$, $u' \leq 0$ for $x < 0$ imply X-convexity;

3. $u(x) = |x|^\alpha / \alpha$ is X-convex for all $\alpha > 0$ although it is not Euclidean convex for $\alpha < 1$.

Example 2.3 (*Convexity on circles*)**.** In $\mathbb{R}^2$ consider the vector field $X = y\,\partial_x - x\,\partial_y$. Then

1. any (non-constant) linear function is X-convex on a half-plane and X-concave on the complement;
2. a quadratic form is X-convex in $\mathbb{R}^2$ if and on only if it is of the form $u(x,y) = a(x^2 + y^2)$ for some $a \in \mathbb{R}$,

which is consistent with the fact that the X-lines run infinitely many times on circles centered at the origin. In Remark 6.3 we show that in fact any X-convex function is constant on every such circle.

Example 2.4 (*Convexity on hyperbola)***.** In $\mathbb{R}^2$ take the vector field $X = y\,\partial_x + x\,\partial_y$, whose trajectories are hyperbola. Then

1. any (non-constant) linear function is X-convex on a half-plane and X-concave on the complement;
2. a quadratic form $u(x,y) = ax^2/2 + by^2/2 + kxy$ is X-convex in $\mathbb{R}^2$ if and on only if $a + b \geq 2|k|$.

Note that any (Euclidean) convex quadratic form is X-convex because $ab \geq k^2$ implies the previous inequality. On the other hand, an X-convex quadratic form might be strictly concave in either x or y.

In Section 4, we present our main motivating examples, i.e. Carnot-Carathéodory metric spaces and Carnot groups.

3 Viscosity characterization of convexity along vector fields

In this section we will show the equivalence between Definition 2.2 and Definition 2.3, in the general case of upper semicontinuous functions. As in Carnot groups, the necessity of v-convexity is much easier to prove.

Proposition 3.1. *Convexity along vector fields implies v-convexity.*

Proof. We assume that, for any curve $x(t)$ as in Definition 2.2, $u(x(t))$ is convex in $[T_1, T_2]$. Without loss of generality, we can assume that $0 \in (T_1, T_2)$. To prove that Definition 2.3 holds, we have to show that $D^2_{\mathcal{X}}\varphi(x^0) \geq 0$ for any smooth φ such that $u - \varphi$ has a local maximum at $x^0 = x(0) \in \Omega$. We can assume that such a maximum is equal to 0. Then we look at the second-order Taylor expansion of the real function $\varphi \circ x$ centered at 0 and evaluated, respectively, at t and $-t$ (for sufficient small t), which are

$$\varphi(x(\pm t)) = \varphi(x^0) \pm \frac{d}{dt}\varphi(x(t))\big|_{t=0} t + \frac{1}{2}\frac{d^2}{dt^2}\varphi(x(t))\big|_{t=0} t^2 + o(t^2).$$

We recall that $u(x^0) = \varphi(x^0)$ and $u(x) \leq \varphi(x)$ for x near x^0.

By Proposition 2.1

$$\frac{u(x(t)) + u(x(-t))}{2} \leq \frac{\varphi(x(t)) + \varphi(x(-t))}{2} = u(x^0) + \frac{1}{2}\frac{d^2}{dt^2}\varphi(x(t))\big|_{t=0} t^2 + o(t^2)$$
$$= u(x^0) + \frac{1}{2}\alpha^T D^2_{\mathcal{X}}\varphi(x^0)\alpha t^2 + o(t^2). \quad (6)$$

By Definition 2.2 $u \circ x$ is convex in $[T_1, T_2]$, so that

$$\frac{u(x(t)) + u(x(-t))}{2} \geq u(x(0)) = u(x^0).$$

Then (6) gives

$$u(x^0) \leq u(x^0) + \frac{1}{2}\alpha^T D^2_{\mathcal{X}}\varphi(x^0)\alpha t^2 + o(t^2).$$

Dividing by t^2 and letting $t \to 0$ we conclude

$$\alpha^T D^2_{\mathcal{X}}\varphi(x^0)\alpha \geq 0 \quad \forall\, \alpha \in \mathbb{R}^m.$$

□

The other implication is more difficult. We use the idea of straightening the $\mathcal{X}$-lines, then reducing inequalities on $\mathbb{R}^n$ to inequalities on Euclidean lines. This is inspired by the proof in [15, 3] that u is nondecreasing along the trajectories of X if and only if $X \cdot Du \geq 0$ in viscosity sense.

We start with the (Euclidean) case $n = 1$.

Proposition 3.2. *Let $u \in USC\big([T_1, T_2]\big)$. If $-u'' \leq 0$ in (T_1, T_2), in the viscosity sense, then u is convex in $[T_1, T_2]$.*

Proof. We prove the result by contradiction. Let us suppose that there exist $T_1 \leq t_1 < t_3 \leq T_2$ and $\lambda \in (0, 1)$ such that

$$u(t_2) > \lambda u(t_1) + (1 - \lambda)u(t_3),$$

with $t_2 = \lambda t_1 + (1 - \lambda)t_3$. Note that it is possible to find $\varphi \in C^2$ with $\varphi'' < 0$ in (t_1, t_3) such that

$$\begin{aligned} \varphi(t_1) &= u(t_1), \\ \varphi(t_3) &= u(t_3), \\ \varphi(t_2) &< u(t_2). \end{aligned}$$

Therefore there exists a $t \in (t_1, t_3)$ which is (local) maximum point for $u - \varphi$. Since by hypotesis $-u'' \leq 0$ in the viscosity sense, then $-\varphi''(t) \leq 0$ but this contradicts the fact that φ is a strictly concave function in (t_1, t_3). □

The following proposition makes clear the one dimensional reduction.

Proposition 3.3. *Let $\Omega \subset \mathbb{R}^n$ open and connected and $u \in USC(\overline{\Omega})$. Let $x = (x_1, z)$ with $z \in \mathbb{R}^{n-1}$ and u^z the real function defined on $\Omega_z = \{x_1 \in \mathbb{R} \,|\, (x_1, z) = \Omega\}$ as $u^z(x_1) := u(x_1, z)$. Then the following properties are equivalent:*

1. *$-(u^z)''(x_1) \leq 0$ in Ω_z, in the viscosity sense, for any $z \in \mathbb{R}^{n-1}$,*
2. *$-u_{x_1 x_1}(x) \leq 0$ in Ω, in the viscosity sense.*

Proof. The proof uses the same argument as Lemma II.5.17 of [3]. The fact that the second statement follows from the first one is trivial: we have just to remark that, given a test function φ such that $u-\varphi$ has a local maximum at $x^0=(x_1^0,z^0)\in\Omega$, then $\eta(x_1):=\varphi(x_1,z^0)$ is such that $u^{z^0}(x_1)-\eta(x_1)$ has a local maximum at $x_1^0\in\Omega_{z^0}$ and moreover $\eta''(x_1^0)=\varphi_{x_1\,x_1}(x^0)$.

To show the reverse implication, we take $\eta\in C^2(\Omega_{z^0})$ such that $u^{z^0}(x_1)-\eta(x_1)$ has a local maximum at x_1^0, where $z^0\in\mathbb{R}^{n-1}$ is fixed, and we can assume the maximum is strict in some interval $I=[x_1^0-R,x_1^0+R]$. Moreover, by adding a constant, we may also assume that $\eta\geq 1$ on I. Then, we set $\mathbb{R}^n\ni x=(x_1,\dots,x_n)=(x_1,z)\in\mathbb{R}\times\mathbb{R}^{n-1}$ and for any $\varepsilon>0$ we define

$$\varphi^\varepsilon(x):=\eta(x_1)\left(1+\frac{|z-z^0|^2}{\varepsilon}\right).$$

Let $x^\varepsilon=(x_1^\varepsilon,z^\varepsilon)$ be a maximum point for $u-\varphi^\varepsilon$ in $\overline{B}:=\overline{B_R}(x^0)$. Then

$$u(x^\varepsilon)-\varphi^\varepsilon(x^\varepsilon)\geq u(x^0)-\varphi^\varepsilon(x^0)=u^{z^0}(x_1^0)-\eta(x_1^0) \tag{7}$$

From $\eta\geq 1$ and $x^\varepsilon\in\overline{B}$ one can deduce that there exists a constant $C>0$ independent of $\varepsilon>0$ such that

$$\frac{|z^\varepsilon-z^0|^2}{\varepsilon}\leq u(x^\varepsilon)-\eta(x_1^\varepsilon)-u^{z^0}(x_1^0)+\eta(x_1^0)\leq C \tag{8}$$

fo all $\varepsilon>0$. Hence $z^\varepsilon\to z^0$.

Next we take a subsequence such that $x_1^\varepsilon\to\bar{x}_1$. By (7)

$$u(\bar{x}_1,z^0)-\eta(\bar{x}_1)\geq u^{z^0}(x_1^0)-\eta(x_1^0),$$

so $\bar{x}_1=x_1^0$ because x_1^0 is a point of strict maximum for $u^{z^0}-\eta$ in I. Now letting $\varepsilon\to 0$ in (8) we get $|z^\varepsilon-z^0|^2/\varepsilon\to 0$.

To conclude, it is sufficient to remark that

$$\frac{\partial^2\varphi^\varepsilon}{\partial x_1^2}(x^\varepsilon)=\eta''(x_1^\varepsilon)\left(1+\frac{|z^\varepsilon-z^0|^2}{\varepsilon}\right)\to\eta''(x_1^0),\quad \text{as } \varepsilon\to 0^+,$$

so $-\varphi^\varepsilon_{x_1x_1}(x^\varepsilon)\leq 0$ implies $-\eta''(x_1^0)\leq 0$, as desired. □

Remark 3.1. Note that if Ω is convex then Ω_z is a real interval. For general connected domains in $\mathbb{R}^n$, the real set Ω_z can be written as disjoint union of intervals. We say that a real function is convex in a disjoint union of intervals if it is convex in each interval.

Now we tackle the general n-dimensional case. We fix $\alpha\in\mathbb{R}^m$ and consider the ODE for the corresponding $\mathcal{X}$-line, starting at $t=0$ from $x^0\in\Omega$, i.e.,

$$\begin{cases}\dot{x}(t)=\sigma(x(t))\alpha, & t\in\mathbb{R},\\ x(0)=x^0.\end{cases} \tag{9}$$

We set $f(x):=\sigma(x)\alpha$ and assume first that $f(x^0)\neq 0$ (as we will see later, the other case is trivial). Then it is known that there exists a C^2-diffeomorphism $\Phi:U\to V\subset\mathbb{R}^n$ defined on a neighbourhood U of x^0 such that, in the corresponding new coordinates, the vector field $f(x)$ can be locally rewritten as e_1, the first vector of the canonical basis of $\mathbb{R}^n$ (see, e.g., [2]). This means that, in the new coordinates $\xi := \Phi(x)$, the ODE (9) can be rewritten (for small t) in the canonical form:

$$\begin{cases} \dot{\xi}(t) = \mathrm{e}_1, & t \text{ near } 0, \\ \xi(0) = \xi^0 := \Phi(x^0). \end{cases} \tag{10}$$

Of course, the local diffeomorphism Φ depends on the parameters α and x^0 but we will not write explicitly this dependence.

Proposition 3.4. *Let $\Omega \subset \mathbb{R}^n$ open, $u \in USC(\overline{\Omega})$ and $\alpha \in \mathbb{R}^m$ and let Φ be the local diffeomorphism introduced above in a neighborhood U of x^0. If we define $v(\xi) := u \circ \Phi^{-1}(\xi)$, then*

$$-\alpha^T D^2_{\mathcal{X}} u(x)\alpha \leq 0 \ \textit{ in } U, \quad \textit{in the viscosity sense} \tag{11}$$

if and only if

$$-v_{\xi_1\xi_1}(\xi) \leq 0 \ \textit{ in } \Phi(U), \quad \textit{in the viscosity sense.} \tag{12}$$

Proof. Let us first recall that the first derivatives of a diffeomorphism $\Phi = (\Phi^1, \dots, \Phi^n)$ can be expressed by the $n \times n$-Jacobian-matrix, which is

$$D\Phi = D_x\Phi = \begin{pmatrix} \Phi^1_{x_1} & \dots & \Phi^1_{x_n} \\ \dots & \dots & \dots \\ \Phi^n_{x_1} & \dots & \Phi^n_{x_n} \end{pmatrix}.$$

Let $x(t)$ be a curve satisfying (9) for our choice of $\alpha \in \mathbb{R}^m$ and with initial datum $x(0) = x$, then for x near x^0, we consider the local deffeomorphism Φ, and we know that $\xi(t) := \Phi(x(t))$ satisfies (10) with initial datum $\xi(0) = \xi = \Phi(x)$ for any t close enough to 0 (and with ξ near to $\xi^0 := \Phi(x^0)$). In particular the ODE holds for $t = 0$, which implies

$$\mathrm{e}_1 = \dot{\xi}(0) = \frac{d}{dt}\left[\Phi(x(t))\right]\Big|_{t=0} = \big[D\Phi(x(t))\dot{x}(t)\big]\big|_{t=0} = D\Phi(x)\,\sigma(x)\alpha.$$

Since α is fixed, for sake of semplicity, we may set $Y(x) := \sigma(x)\,\alpha$; therefore the previous identity can be written as $\mathrm{e}_1 = D\Phi(x)Y(x)$ or, equivalently,

$$Y(x) = \big(D\Phi(x)\big)^{-1}\mathrm{e}_1. \tag{13}$$

By definition of $Y(x)$, we have that $Yu = \sum_{i=1}^m \alpha_i X_i u$, therefore

$$Y^2u = Y\big(Yu\big) = Y\left(\sum_{i=1}^m \alpha_i X_i u\right) = \sum_{i,j=1}^m \alpha_i\alpha_j X_i(X_j u) = \alpha^T D^2_{\mathcal{X}} u\,\alpha. \tag{14}$$

We first show the result in the regular case, i.e. for $u \in C^2(\overline{\Omega})$.
Let us consider $v(\xi) = u \circ \Phi^{-1}(\xi)$, then

$$Dv(\xi) = (D\Phi^{-1}(\xi))^T Du(\Phi^{-1}(\xi)) = \Big[\big((D\Phi)^{-1}\big)^T Du\Big] \circ \Phi^{-1}(\xi). \tag{15}$$

Moreover

$$v_{\xi_1}(\xi) = \frac{\partial}{\partial_{\xi_1}} v(\xi) = \big\langle Dv, e_1 \big\rangle(\xi) = \big\langle \big((D\Phi)^{-1}\big)^T Du, e_1 \big\rangle(\Phi^{-1}(\xi)) =$$
$$= \big\langle Du, (D\Phi)^{-1} e_1 \big\rangle \circ \Phi^{-1}(\xi) = \big\langle Du, Y \big\rangle \circ \Phi^{-1}(\xi) = (Yu) \circ \Phi^{-1}(\xi). \quad (16)$$

The same formula holds for any $w \in C^1$ and $\zeta = w \circ \Phi^{-1}$, i.e.

$$\frac{\partial}{\partial_{\xi_1}} \zeta(\xi) = (Yw) \circ \Phi^{-1}(\xi).$$

Let us choose $w = Yu$ and $\zeta = v_{\xi_1}$, then (16) gives

$$v_{\xi_1 \xi_1}(\xi) = \zeta_{\xi_1}(\xi) = (Yw) \circ \Phi^{-1}(\xi) = \big(Y(Yu)\big) \circ \Phi^{-1}(\xi) = \alpha^T D^2_{\mathcal{X}} u \ (\Phi^{-1}(\xi))\ \alpha. \quad (17)$$

where we have used (14).
The identity (17) concludes the proof in the smooth case (i.e. under the assumptions $u \in C^2(\Omega)$).

The general case of $u \in USC(\overline{\Omega})$ is treated by taking a smooth test function ψ such that $v - \psi$ has a local maximum at ξ^0 and defining $\varphi(x) = \psi(\Phi(x))$. Then $u - \varphi$ has a local maximum at $x^0 = \Phi^{-1}(\xi^0)$ and the preceding calculation with u and v replaced by φ and ϕ, respectively, gives the conclusion. □

Remark 3.2. An alternative way to prove the previous result, in particular the identity

$$v_{\xi_1 \xi_1}(\xi) = \alpha^T D^2_{\mathcal{X}} u(\Phi^{-1}(\xi))\ \alpha. \quad (18)$$

is by direct computations. The outline is the following. First one shows that

$$D^2 v(\xi) = (D\Phi^{-1}(\xi))^T D^2 u(\Phi^{-1}(\xi)) D\Phi^{-1}(\xi) + \Big(\Phi^{-1}_{\xi_l\, \xi_k}(\xi) \cdot Du(\Phi^{-1}(\xi))\Big)_{lk},$$

which implies

$$D^2 u = D\Phi^T D^2 v\, D\Phi + D\Phi^T \Big[\big((D\Phi)^{-1}\big)^T \big(\big[(D\Phi)^{-1}\Phi_{x_l\, x_k}\big] \cdot Du\big)_{lk} (D\Phi)^{-1}\Big]\, D\Phi$$
$$= D\Phi^T D^2 v\, D\Phi + \big(\big[(D\Phi)^{-1}\Phi_{x_l\, x_k}\big] \cdot \big[D\Phi^T Dv\big]\big)_{lk},$$

where both the sides are calculated at the point $x = \Phi^{-1}(\xi)$. An explicit but nontrivial computation shows that:

$$\begin{aligned} &\alpha^T\, D^2_{\mathcal{X}} u\ \alpha = \\ &= \mathrm{e}_1^T D^2 v\ \mathrm{e}_1 + \alpha^T \sigma^T \left(\Phi_{x_l x_k} \cdot Dv\right)_{lk} \sigma\alpha + \alpha^T \left(\left[D\Phi\ \nabla_{X_l} X_k\right] \cdot Dv\right)_{lk} \alpha \\ &= \mathrm{e}_1^T D^2 v \mathrm{e}_1 + \alpha^T \left(X_l \cdot \left[\left(\Phi^k_{x_i\, x_j}\right)_{ij} \sigma\alpha\right] + \sum_{j=1}^m \alpha_j \left(D\Phi \nabla_{X_l} X_j\right)^k \right)_{lk} Dv. \end{aligned} \quad (19)$$

By differentiating the identity (13) along the vector fields $X_1, \dots, X_m$, it is possible to show that, for any $l = 1, \dots, m$,

$$0 = \nabla_{X_l} \mathrm{e}_1 = X_l \cdot \left[\Big(\Phi^k_{x_i x_j}(x)\Big)_{ij} \sigma(x)\, \alpha\right] + \sum_{j=1}^m \alpha_j \big(D\Phi(x)\ \nabla_{X_l} X_j(x)\big)^k.$$

Plugging this identity in (19), we find exactly (18).

Using Proposition 3.4, we can give the main result of the paper.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$ be open and connected and $\mathcal{X} = \{X_1, \ldots, X_m\}$ be a family of C^2 vector fields on $\mathbb{R}^n$. Then a function $u \in USC(\overline{\Omega})$ is convex along the vector fields (Definition 2.2) if and only if u is v-convex (Definition 2.3).*

Proof. We have already shown that Definition 2.2 implies Definition 2.3 (see Proposition 3.1). We need just to prove the opposite implication. So we assume that

$$-D^2_{\mathcal{X}} u(x) \leq 0 \quad \text{in } \Omega, \quad \text{in the viscosity sense,}$$

and let $x(t)$ be a curve satisfying the ODE (9). First we consider the case $f(x^0) = 0$. Since x^0 is an equilibrium point for (9), the trajectory-solution is constant (i.e. $x(t) = x^0$, for any $t \in \mathbb{R}$). So $u(x(t)) = u(x^0)$ is constant and therefore trivially convex.

On the other hand, when $f(x^0) \neq 0$ we can build a diffeomorphism Φ such that the ODE (9) can be rewritten as the ODE (10). For sake of simplicity we assume that Φ is globally defined on Ω (if it is not we have just to write the following proof in a neighborhood U of x^0). Proposition 3.4 tells that $v(\xi) := u(\Phi^{-1}(\xi))$ satisfies

$$-v_{\xi_1\xi_1} \leq 0, \quad \text{in } \Omega^\Phi =: \Omega', \quad \text{in the viscosity sense.}$$

By Proposition 3.3, for any $z \in \mathbb{R}^{n-1}$, $v^z(s) := v(s,z)$ satisfies $-(v^z)''(s) \leq 0$ in $\Omega'_z = \{s \in \mathbb{R} \,|\, (s,z) \in \Omega\}$, in the viscosity sense. Applying the 1-dimensional result (Proposition 3.2) to the real function v^z, we get that it is convex in Ω'_z (which means convex in any interval contained in Ω'_z).

Let us now consider a curve $x(t)$ and the corresponding interval $[T_1, T_2]$ as in Definition 2.2. Without loss of generality, we can assume that $0 \in (T_1, T_2)$ so that $x(0) = x^0 \in \Omega$. Let $z = (\xi_2^0, \ldots, \xi_n^0) = \big(\Phi(x^0)_2, \ldots, \Phi(x^0)_n\big)$, so that

$$v^z(s) = v(s, \xi_2^0, \ldots, \xi_n^0) = v(\xi(t)) = u(x(t)),$$

with $t = s - \xi_1^0$. Then $u \circ x$ is convex in $\Omega'_z + \xi_1^0 := \{t \in \mathbb{R} \,|\, t - \xi_1^0 \in \Omega'_z\}$.

To conclude the proof, we need just to check that the interval $(T_1, T_2) \subset \Omega'_z + \xi^0$. In fact, we know that, for any $t \in (T_1, T_2)$, $x(t) \in \Omega$ that means $\xi(t) \in \Omega'$. Since $\xi(t) = (t + \xi_1^0, \xi_2^0, \ldots, \xi_n^0)$, we get $s = t + \xi_1^0 \in \Omega'_x$, and so $t \in \Omega'_z + \xi^0$. □

Remark 3.3. Proposition 3.4 can be stated in a more general form because the proof does not depend on the fact that we choose the particular local diffeomorphism Φ given by the Rectification Theorem. Suppose we have two families of vector fields $\mathcal{X}^1, \mathcal{X}^2$, with corresponding matrices σ_1, σ_2, and for a fixed $\alpha \in \mathbb{R}^m$ consider $Y_1(x) = \sigma_1(x)\,\alpha$ and $Y_2(\xi) = \sigma_2(\xi)\,\alpha$. For a given point $x_0 \in \mathbb{R}^n$ consider a local C^2-diffeomorphism $\xi = F(x)$ on a neighborhood of x_0, which brings the solutions of the ODE

$$\dot{x}(t) = Y_1(x(t)), \qquad x(0) = x_0,$$

into the solutions of the ODE

$$\dot{\xi}(t) = Y_2(\xi(t)), \qquad \xi(0) = \xi_0.$$

Then identity (14) still holds, i.e., $v := u \circ F^{-1}$ satisfies

$$\alpha^T D^2_{\mathcal{X}^1} u\big(F^{-1}(\xi)\big)\, \alpha = \alpha^T D^2_{\mathcal{X}^2} v(\xi)\, \alpha.$$

in the viscosity sense.

This allows to give a definition of convexity of a function u from a n-dimensional manifold M with respect to a family of vector fields $\mathcal{X}$ defined on M independent of the choice of the charts. In fact, by means of local charts, we can always associate to the vector fields $X_1, \ldots, X_m$ a family of vectors fields on $\mathbb{R}^n$, $\widetilde{X}_1, \ldots, \widetilde{X}_m$, and then apply Definition 2.2 to such a family. Then the definition of convexity along $\mathcal{X}$-lines is charts-invariant because a (smooth) change of charts leads to a local diffeomorphism F with the properties above.

4 Carnot-Carathéodory metric spaces

We mentioned in the Introduction that our main motivation comes from the theory of Carnot-Carathéodory metric spaces. Let us recall that the Carnot-Carathéodory (briefly, C-C), or sub-Riemannian distance d on $\mathbb{R}^n$ associated to a familiy of vector fields $\mathcal{X}$ is

$$d(x,y) := \inf \{T \geq 0 \,|\, \exists\, \gamma \text{ admissible in } [0,T] \text{ with } \gamma(0) = x,\ \gamma(T) = y\}, \quad (20)$$

with the convention $\inf \emptyset = +\infty$, where a curve γ is admissible if it is absolutely continuous in $[0,T]$ and there exist measurable functions $\alpha_i(t)$, $i = 1, \ldots, m$, such that $|\alpha(t)|^2 = \sum_{i=1}^m \alpha_i^2(t) = 1$ and

$$\dot\gamma(t) = \sum_{i=1}^m \alpha_i(t) X_i(\gamma(t)), \quad \text{a.e. } t \in [0,T]. \quad (21)$$

The pair $(\mathbb{R}^n, d)$ is a Carnot-Carathéodory metric spaces, or sub-Riemannian geometry, if $d(x,y) < +\infty$ for all $x, y \in \mathbb{R}^n$, and the vector fields $\mathcal{X}$ are its generators.

In Section 6 we will use the assumption

$$\text{the identity map } Id : (\mathbb{R}^n, d) \to (\mathbb{R}^n, |\cdot|) \text{ is a homeomorphism.} \quad (22)$$

A classical sufficient condition for it to hold is that the vector fields $\mathcal{X} = \{X_1(x), \ldots, X_m(x)\}$ are smooth and satisfy the Hörmander condition, i.e, the Lie algebra they generate has full rank at any point.

In a C-C space a function u is called d-Lipschitz continuous in Ω if there is $L \geq 0$ such that

$$|u(x) - u(y)| \leq L d(x,y), \quad \forall\, x, y \in \Omega. \quad (23)$$

See, e.g., [26] and [9] for a general presentation of C-C spaces and [18, 19] for the properties of d-Lipschitz functions.

Important examples of such spaces are the Carnot groups, where v-convexity was first introduced [24, 23].

Example 4.1 (*Carnot groups)***.** A Carnot group $(\mathbb{G}, *)$ is a Lie group, nilpotent and simply connected, whose Lie algebra admits a stratification $g = \oplus_{i=1}^k V_i$. Any such group is isomorphic to a homogeneous Carnot group on $\mathbb{R}^n$, that is, a triple $(\mathbb{R}^n, *, \delta_\lambda)$ where $\mathbb{R}^n = \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k}$, $*$ is a group operation whose identity is 0 and such that $(x, y) \mapsto y^{-1} * x$ is smooth, the dilation $\delta_\lambda : \mathbb{R}^n \to \mathbb{R}^n$

$$\delta_\lambda(x) = \delta_\lambda(x^{(1)}, \dots, x^{(r)}) := (\lambda x^{(1)}, \lambda^2 x^{(2)}, \dots, \lambda^r x^{(r)}), \quad x^{(i)} \in \mathbb{R}^{n_i}$$

is an automorphism of the group $(\mathbb{R}^n, *)$ for all $\lambda > 0$, and there are $m = n_1$ smooth vector fields $X_1, \dots, X_m$ on $\mathbb{R}^n$ invariant with respect to the left translations $\tau_\beta(x) := \beta * x$ for all $\beta \in \mathbb{R}^n$ and such that the Lie algebra they generate has rank n at every point $x \in \mathbb{R}^n$. The fields $\mathcal{X} = \{X_1, \dots, X_m\}$ are called the generators of the Carnot group and can be written in the form

$$X_j = \frac{\partial}{\partial x_j} + \sum_{i=m+1}^{n} \sigma_{ij}(x) \frac{\partial}{\partial x_i}, \quad j = 1, \dots, m, \tag{24}$$

with $\sigma_{ij}(x) = \sigma_{ij}(x_1, \dots, x_{i-1})$ homogeneous polynomials of a degree $\leq n - m$. For the proofs and more information on these geometries we refer to [9, 26, 11].

It is easy to check that any function depending only on $x_1, \dots, x_m$ and convex in $\mathbb{R}^m$ is $\mathcal{X}$-convex (see also Example 4.4).

Example 4.2 (*Heisenberg group*)**.** The most known Carnot group is the Heisenberg group. The (n-dimensional) Heisenberg group $\mathbb{H}^n$ is a Carnot group of step 2 (i.e., $k = 2$ in the stratification), defined on $\mathbb{R}^{2n+1}$. If $n = 1$, the stratification is $V_1 \oplus V_2$ where $V_1 \equiv \mathbb{R}^2$ and $V_2 \equiv \mathbb{R}$. The group operation is

$$x * y := \left(x_1 + y_1, x_2 + y_2, x_3 + y_3 + \frac{x_1 y_2 - x_2 y_1}{2} \right)$$

and the generators are the two vector fields:

$$X_1(x) = \begin{pmatrix} 1 \\ 0 \\ \frac{x_2}{2} \end{pmatrix}, \quad X_2(x) = \begin{pmatrix} 0 \\ 1 \\ -\frac{x_1}{2} \end{pmatrix}. \tag{25}$$

It was proved in [16] that the homogeneus norm

$$\|x\|_0 := \left((x_1^2 + x_2^2)^2 + 16 x_3^2 \right)^{\frac{1}{4}}$$

is horizontally convex (see the definition below), so it is v-convex by a result of [24] or by Proposition 3.1 and Lemma 4.1 below. The homogenous norm gives a distance equivalent to the Carnot-Carathéodory one, but their convexity properties are very different, as we show next.

The Carnot-Carathéodory distance form the origin $u(x) = d(x, 0)$ is not v-convex near the center of the Heisenberg group, that is, the x_3-axis. In fact, it was recently proved in [10] that u is not a viscosity subsolution for the horizontal infinite Laplace equation $-\Delta_{\mathcal{X},\infty} u = 0$ at any point $(0, 0, z) \in \mathbb{R}^3$ with $z \neq 0$. Since this PDE is degenerate elliptic and homogeneous in the Hessian matrix entry, a result of [24] says that all v-convex functions must be subsolutions, see Remark 2.5 above. Therefore u is not v-convex in any set containing points of the center of the group, and neither $\mathcal{X}$-convex nor horizontally convex (by Lemma 4.1 below). It could be interesting to understand if some power of u is horizontally convex in the whole space.

Our final remark is that in the Heisenberg group Euclidean convexity implies convexity along the vector fields (but of course the converse does not hold, as the homogeneous norm shows). In fact $\nabla_{X_i} X_j + \nabla_{X_j} X_i = 0$, for $i, j = 1, 2$. Therefore

$$D^2_{\mathcal{X}} u(x) = \sigma^T(x) D^2 u(x) \sigma(x),$$

which is positive semi-definite whenever the Eulidean Hessian $D^2 u(x)$ is so. The same property can be proved in any Carnot group with step 2 (once they are written in their canonical form, via exponential coordinates).

In Carnot groups the following notion of convexity was introduced in [24], [23] and [16], see also [11].

Definition 4.1. *Let $(\mathbb{G}, *)$ be a Carnot group and $\Omega \subset \mathbb{G}$ open and connected. We indicate as $\mathbb{V}$ the horizontal space defined by V_1 at the origin. A function $u : \Omega \to \mathbb{R}$ is called* horizontally convex *(briefly,* h-convex*) if, for any $p \in \Omega$ and $h \in \mathbb{V}$ such that $p * \delta_t(h) \in \Omega$ for all $t \in (-1, 1)$, the function*

$$t \mapsto u\big(p * \delta_t(h)\big)$$

is convex for $-1 < t < 1$.

In the homogeneous Carnot group on $\mathbb{R}^n$ isomorphic to $\mathbb{G}$ we can write

$$\mathbb{V} = span\,\{X_1(0), \dots, X_m(0)\},$$

where $X_1, \dots, X_m$ are the generators of the group. In the Heisenberg group

$$\mathbb{V} = \mathcal{H}_0 := \{\, h = (h_1, h_2, 0) \in \mathbb{R}^3 \,\} \quad \text{and} \quad h^{-1} = (-h_1, -h_2, -h_3).$$

The next Lemma states the equivalence of $\mathcal{X}$-convexity and h-convexity. It is a mere restatement of Proposition 8.3.17 in [11], because the notion (2) introduced there coincides with Definition 2.2 above, up to a reparametrization of the $\mathcal{X}$-lines.

Lemma 4.1. *Let $(\mathbb{G}, *)$ be a Carnot group on $\mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ open and connected, and $u : \Omega \to \mathbb{R}$ upper semicontinuous. Then u is convex along the generators $\mathcal{X}$ of $\mathbb{G}$ (Definition 2.2) if and only if u is horizontally convex (Definition 4.1).*

By putting together this Lemma and Theorem 3.1 we get the following result related to the work of several authors [24, 7, 33, 25, 23].

Corollary 4.1. *In Carnot groups v-convexity, horizontal convexity, and convexity along the generators $\mathcal{X}$ of the group are equivalent.*

Next we consider sub-Riemannian geometries which are not Carnot groups.

Example 4.3 (*Grušin plane*)**.** Consider the sub-Riemannian structure induced on $\mathbb{R}^2$ by the vector fields

$$X_1(x, y) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad X_2(x, y) = \begin{pmatrix} 0 \\ x \end{pmatrix}.$$

The symmetrized intrinsic Hessian is

$$\big(D^2_{\mathcal{X}} u\big)^* = \begin{pmatrix} u_{xx} & x u_{xy} + \frac{u_y}{2} \\ x u_{xy} + \frac{u_y}{2} & x^2 u_{yy} \end{pmatrix}.$$

The contribution of the first-oder part is in general strong enough to make non-convex along the vector fields many functions which are convex in the Euclidean sense.

Consider first $u(x,y) = f(x) + g(y)$. Then $(D^2_{\mathcal{X}}u)^* \geq 0$ if and only if $f'' \geq 0$ and

$$\det\left(D^2_{\mathcal{X}}u\right)^* = x^2 f'' g'' - \frac{(g')^2}{4} \geq 0,$$

so that $f'', g'' > 0$ is not a sufficient condition. For instance, $u(x,y) = x^2 + y^2$, is $\mathcal{X}$-convex only in the set $\{(x,y)\,|\,2x - y > 0, 2x + y > 0\} \cup \{(x,y)\,|\,2x - y < 0, 2x + y < 0\}$ while it is (strictly) $\mathcal{X}$-concave in the complement.

By analogy with the Heisenberg case consider next the homogenous norm

$$u(x,y) = |(x,y)|_0 = \left(x^4 + cy^2\right)^{\frac{1}{4}},$$

for $c > 0$. Then $u_{xx} = 3cx^2y^2(x^4 + cy^2)^{-\frac{7}{4}} > 0$ (for $xy \neq 0$), while

$$\det\left(D^2_{\mathcal{X}}u\right)^* = -\frac{c^2}{16}y^2\left(x^4 + cy^2\right)^{-\frac{3}{2}} < 0,$$

for any $c > 0$ and $y \neq 0$. This means that for any choice of $c > 0$ the homogeneous norm is nowhere $\mathcal{X}$-convex, different from the Heisenberg case (where the homogenous norm is $\mathcal{X}$-convex for $c = 16$, as we recalled before).

Example 4.4 (*Vector fields of Carnot type)***.** Consider vector fields of the form

$$X_j = \frac{\partial}{\partial x_j} + \sum_{i=m+1}^{n} a_{ij}(x)\frac{\partial}{\partial x_i}, \quad j = 1, \ldots, m. \tag{26}$$

They have in common with the generators of Carnot groups the special structure of the matrix σ, whose first m lines are the identity matrix. Take a function $u(x) = v(x_1, \ldots, x_m)$ depending only on the first m variables. An easy calculation shows that

$$D^2_{\mathcal{X}}u = D^2 v.$$

Therefore u is $\mathcal{X}$-convex if and only if v is (Euclidean) convex in $\mathbb{R}^m$.

5 Semiconvexity along vector fields

In this section we extend to the geometry associated with a family of vector fields the notions of semiconvexity and semiconcavity. They are classical in the Euclidean setting and very useful for PDE as well as optimal control problems, see the survey in [12] and the references therein.

Definition 5.1. *A function* $u : \Omega \to \mathbb{R}$ *is* $\mathcal{X}$-semiconvex *if there exists* $C \geq 0$ *such that, for every curve* $x(t) \in \Omega$ *satisfying equation* (2)*,* $u \circ x$ *is a real-valued semiconvex function with constant* C *in* $[T_1, T_2]$*, i.e., for any* $t, t+s, t-s \in [T_1, T_2]$*,*

$$2u(x(t)) - u(x(t+s)) - u(x(t-s)) \leq Cs^2. \tag{27}$$

A function is $\mathcal{X}$*-semiconcave if* $-u$ *is* $\mathcal{X}$*-semiconvex.*

Example 5.1. If $X_i(x) = e_i$, $i = 1, \dots, m$, then a function is $\mathcal{X}$-semiconvex if it is semiconvex w.r.t. the first m variables $x_1, \dots, x_m$.

Example 5.2. If $u(x) = v(x) + w(x)$ with v $\mathcal{X}$-convex in Ω and $w \in C^2(\Omega)$ with $\|D^2 u\|_\infty \leq C$, then u is $\mathcal{X}$-semiconvex with constant C. This sufficient condition is also necessary for the class of vector fields of Carnot type of Example 4.4, see Proposition 5.2.

Example 5.3 (*Marginal functions*)**.** It is easy to see from the Definition 5.1 that the supremum of $\mathcal{X}$-semiconvex functions with the same constant C is $\mathcal{X}$-semiconvex with constant C. In particular, the marginal function

$$u(x) = \sup_\beta v_\beta(x),$$

with v_β twice differentiable with respect to the fields and $X_i X_j v_\beta$ uniformly bounded in Ω, is $\mathcal{X}$-semiconvex in Ω.

As in the case of convexity (i.e., $C = 0$) we can give an infinitesimal version of the semiconvexity property.

Definition 5.2. *A function $u \in USC(\overline{\Omega})$ is* v-semiconvex *if there exists a constant $C \geq 0$ such that*

$$-D^2_{\mathcal{X}} u \leq CI, \quad \textit{in the viscosity sense in } \Omega, \tag{28}$$

where I denotes the identity $m \times m$ matrix.

Remark 5.1. This matrix condition can also be expressed by the scalar partial differential inequality

$$-\lambda_{\min}((D^2_{\mathcal{X}} u)^*) \leq C$$

in the usual viscosity sense.

Proposition 5.1. *A function $u \in USC(\overline{\Omega})$ is $\mathcal{X}$-semiconvex if and only if it is v-semiconvex.*

Proof. The strategy is the same used in the convex case (i.e., for $C = 0$). We need to show only the analogue of Proposition 3.2, that is, the Euclidean 1-dimensional case, because all the other steps hold without modifications.

We assume $-u'' \leq C$ in the viscosity sense and rescale the time so that $t = 0$ in (27). Then $\widetilde{u}(s) := u(s) + \frac{C}{2} s^2$ satisfies $-\widetilde{u}'' \leq 0$ in the viscosity sense, so it is convex in $[T_1, T_2]$ by Proposition 3.2. This gives

$$\widetilde{u}(0) \leq \frac{\widetilde{u}(s) + \widetilde{u}(-s)}{2}.$$

By the definition of $\widetilde{u}(s)$ we get

$$u(s) + \frac{C}{2} s^2 + u(-s) + \frac{C}{2} s^2 - 2u(0) \leq 0,$$

that proves the Euclidean 1-dimensional case. □

Proposition 5.2. *Assume the vector fields are of Carnot type, i.e., they have the form*

$$X_j = \frac{\partial}{\partial x_j} + \sum_{i=m+1}^{n} a_{ij}(x) \frac{\partial}{\partial x_i}, \quad j = 1, \dots, m.$$

Then $u \in USC(\overline{\Omega})$ is $\mathcal{X}$-semiconvex with constant C if and only if

$$u(x) + \frac{C}{2} \sum_{i=1}^{m} x_i^2 \quad \text{is } \mathcal{X}\text{-convex.}$$

Proof. An easy calculation as in Example 4.4 gives

$$D^2_{\mathcal{X}} \left(u(x) + \frac{C}{2} \sum_{i=1}^{m} x_i^2 \right) = D^2_{\mathcal{X}} u(x) + CI,$$

where I is the $m \times m$ identity matrix, and the conclusion follows from the definitions. ☐

In the particular case of a Carnot group a notion of semiconvexity can be also introduced as follows.

Definition 5.3. *Let $(\mathbb{G}, *)$ be a Carnot group and $\mathbb{V}$ be the horizontal space as in Definition 4.1. A function $u : \mathbb{G} \to \mathbb{R}$ is called* h-semiconvex *if there exists $C \geq 0$ such that, for any $p \in \mathbb{G}$ and $h \in \mathbb{V}$,*

$$2u(p) - u(p * h) - u(p * h^{-1}) \leq C|h|^2,$$

where h^{-1} is the inverse element of h w.r.t. to the operation $*$ and $|\cdot|$ is the Euclidean norm in $\mathbb{R}^m$.

By the same argument of Lemma 4.1 we get the following.

Lemma 5.1. *If $X_1, \dots, X_m$ are the generators of a Carnot group on $\mathbb{R}^n$, then $u \in USC(\Omega)$ is $\mathcal{X}$-semiconvex if and only if it is h-semiconvex.*

6 Bounds on the gradient and d-Lipschitz continuity

In the Euclidean setting it is known that if a function is semiconvex or semiconcave then it is locally Lipschitz continuous and there is a bound on its (generalized) gradient, see, e.g., [3, 12]. This property has many applications. In this section we first show the corresponding gradient estimate for $\mathcal{X}$-semiconvex functions, and then deduce from it the Lipschitz continuity with respect to the C-C distance d. We recall that $\sigma(x)$ is the matrix whose columns are the coefficients of the vector fields, see Definition 2.1.

Proposition 6.1. *Let $u \in USC(\overline{\Omega})$ be $\mathcal{X}$-semiconvex (or $\mathcal{X}$-semiconcave) and bounded, with $\Omega \subset \mathbb{R}^n$ open and bounded. Then, for any open $\Omega_1 \subset\subset \Omega$,*

$$|D_{\mathcal{X}} u(x)| := |\sigma^T(x) Du(x)| \leq L \quad \text{in the viscosity sense in } \Omega_1, \tag{29}$$

for some $L = L(\|u\|_\infty, \delta, C) < +\infty$, where C is the constant of $\mathcal{X}$-semiconvexity (or $\mathcal{X}$-semiconcavity) of u and

$$\delta = d(\Omega_1, \partial\Omega) := \inf\{d(x, y) \mid x \in \Omega_1, y \in \partial\Omega\}.$$

Remark 6.1. If Ω is unbounded, we need first to fix $\Omega' \subset \Omega$ bounded and then $\Omega_1 \subset\subset \Omega'$. In such a case $L = L(\sup_{\Omega'} |u|, \delta', C)$, where $\delta' = d(\Omega_1, \partial\Omega')$.

Proof. Let us fix $x^0 \in \Omega_1$, $\alpha \in \mathbb{R}^m$ with $|\alpha| = 1$, and a $\mathcal{X}$-line $x_\alpha(t)$ as in (2) such that $x_\alpha(0) = x^0 \in \Omega_1$. Now fix $R > 0$ such that $x_\alpha(t) \in \Omega$ for all $t \in [-R, R]$. Since $u \circ x_\alpha$ is semiconvex in this interval, the standard Lipschitz estimate for Euclidean semiconvex functions [12] gives

$$\sup_{t,s\in[-R,R]} \frac{|u(x_\alpha(t)) - u(x_\alpha(s))|}{|t-s|} \leq \frac{2}{R} \sup_{[-R,R]} |u \circ x_\alpha| + 2RC.$$

Let us note that we can assume $R \geq \delta$ for all $x^0 \in \Omega_1$ and α with $|\alpha| = 1$, because x_α is also a trajectory of (21) and therefore it cannot reach $\partial\Omega$ in a time less than δ. Moreover, since the vector fields are locally bounded, $d(x, y) \geq C(K)|x - y|$ in any compact set $K \subset \mathbb{R}^n$. That implies $\delta > 0$ because $\Omega_1 \subset\subset \Omega$.

So if $\delta < +\infty$ we take $R = \delta$, while if $\delta = +\infty$ we can choose any $R \in \mathbb{R}$, e.g., $R = 1$. In both cases we find $L < +\infty$ and $\overline{R} > 0$, depending just on $\|u\|_\infty$, C and δ, such that

$$u(x^0) - u(x(t)) \leq Lt, \quad \text{for all } 0 \leq t \leq \overline{R}.$$

Now fix a test function ϕ such that $u - \phi$ has a max at x^0. Then the last inequality holds with u replaced by ϕ, and we can divide by t and let $t \to 0$ to get

$$-D\phi(x^0) \cdot (\sigma(x^0)\alpha) \leq L.$$

Next we take the maximum over α, $|\alpha| = 1$, and find

$$|\sigma^T(x^0)D\phi(x^0)| \leq L.$$

Since $x^0 \in \Omega_1$ is arbitrary, we get the conclusion. □

Remark 6.2. If $\delta < +\infty$, then

$$L = \frac{2\,\|u\|_\infty}{\delta} + 2\delta C,$$

which is the same as the known Lipschitz estimate in the Euclidean case, with the usual distance replaced by d in the definition of δ. If, instead, $\delta = +\infty$ and u is $\mathcal{X}$-convex ($C = 0$) then we can take $L = 0$.

Remark 6.3. If u is $\mathcal{X}$-convex (or $\mathcal{X}$-concave) in Ω and $\Omega_1 \subset\subset \Omega$ is invariant for the $\mathcal{X}$-lines (i.e., they never leave Ω_1 if they start in Ω_1), then u is constant on each $\mathcal{X}$-line in Ω_1. In fact, we can let $R \to +\infty$ in the previous proof and get $L = 0$, so that u has null directional derivative along all vector fields X_i. In particular, this shows that in the Example 2.3 any X-convex function in a disc of radius r centered at the origin is constant on all circles of radius less than r centered at the origin.

A similar argument can be used, for u $\mathcal{X}$-convex, if $\overline{x}$ is a stationary point of the ODE for some α (i.e. $\sigma(\overline{x})\overline{\alpha} = 0$, with $|\overline{\alpha}| = 1$). If u is continuous at $\overline{x}$ then u is constant in the whole domain of attraction of $\overline{x}$, i.e., on all points x^0 such that, for some α, the $\mathcal{X}$-line $x_\alpha(t)$ with $x_\alpha(0) = x^0$ tends to $\overline{x}$ at t tends either to $+\infty$ or to $-\infty$.

Next we prove that the horizontal gradient estimate of the last Proposition implies the d-Lipschitz continuity of a $\mathcal{X}$-semiconvex function u. The d-Lipschitz continuity of convex functions in Heisenberg and Carnot groups was studied in various ways by several authors [24, 7, 16, 28, 25, 31]. Our proof for general C-C metric spaces is completely different.

Theorem 6.1. *Let d be the C-C distance defined by* (20) *and $u \in USC(\overline{\Omega})$ be $\mathcal{X}$-semiconvex and locally bounded. Then, for any open $\Omega_1 \subset\subset \Omega$,*

(i) u is d-Lipschitz continuous in Ω_1;

(ii) if, moreover, $X_1(x), \dots, X_m(x)$ are the generators of a C-C metric space with the property (22)*, then the distributional derivatives $X_j u$ exist a.e. and $X_j u \in L^\infty(\Omega_1)$.*

Remark 6.4. The statement *(ii)* can also be written as

$$|D_{\mathcal{X}} u(x)| \leq L, \quad \text{a.e. in } \Omega_1,$$

where $L = L(\sup_{\Omega'} |u|, \Omega', \Omega_1, C)$ (C is the $\mathcal{X}$-semiconvexity constant), for some bounded $\Omega' \subset \Omega$ such that $\Omega_1 \subset\subset \Omega'$, and the horizontal gradient $D_{\mathcal{X}} u := (X_1 u, \dots, X_m u)$ is meant in the sense of distributions.

Remark 6.5. If the property (22) holds d is continuous, so the statement *(i)* implies that a $\mathcal{X}$-semiconvex function is continuous. In particular, if the vector fields $\mathcal{X}$ satisfy the Hörmander condition, then there are $\alpha \in]0, 1]$ and $C' \geq 0$ such that $d(x, y) \leq C'|x - y|^\alpha$ for all $x, y \in \Omega_1$. In this case any $\mathcal{X}$-semiconvex $u \in USC(\overline{\Omega})$ is Hölder continuous in each $\Omega_1 \subset\subset \Omega$.

The proof of Theorem 6.1 borrows from a recent paper by Soravia [30] the use of a suboptimality principle for a Hamilton-Jacobi-Bellman inequality, that we state next.

Lemma 6.1. *If $\mathcal{O} \subseteq \mathbb{R}^n$ is open and bounded, $u \in USC(\overline{\mathcal{O}})$ is a viscosity subsolution of*

$$|\sigma^T(x) Du(x)| \leq l(x) \qquad \text{in } \mathcal{O}, \tag{30}$$

$l \in C(\bar{\mathcal{O}})$, then

$$u(x) \leq \int_0^t l(\gamma(s))\, ds + u(\gamma(t)) \tag{31}$$

for any $x \in \mathcal{O}$, $t > 0$, and $\gamma(\cdot)$ trajectory of the control system

$$\dot{\gamma}(t) = \sigma(\gamma(t))\alpha(t), \quad \alpha(\cdot) \text{ measurable }, |\alpha(t)| = 1, \tag{32}$$

such that $\gamma(0) = x$ and $\gamma(s) \in \mathcal{O}$ for all $s < t$.

Proof. The proof is based on rewriting (30) as the H-J-B inequality

$$\max_{\alpha \in \mathbb{R}^m, |\alpha| = 1} \{-(\sigma(x)\alpha) \cdot Du - l(x)\} \leq 0, \quad \text{in } \mathcal{O}$$

associated with the control system (32) and the integral cost functional on trajectories whose running cost is l. With this formulation the statement is contained in Theorem II.5.21 of [3] if $u \in C(\mathcal{O})$. For u merely u.s.c. it is a special case of Theorem 3.2 of Soravia [29], that has a different and less direct proof. For the reader's convenience we outline the proof of [3] and show it extends easily to $u \in USC(\overline{\mathcal{O}})$. In fact, it uses the same tools we employed in Section 3.

Step 1. For an open interval $I \subseteq \mathbb{R}$ and $w \in USC(I)$, v is nondecreasing if and only if $-w' \leq 0$ in I, in viscosity sense. The proof is Exercise V.1.9 of [3].

Step 2. With the notations of Proposition 3.3

$$-(u^z)'(x_1) \leq l(x_1, z) \qquad \text{in } \mathcal{O}_z, \text{ in viscosity sense, } \forall\, z,$$

is equivalent to $-u_{x_1}(x) \leq l(x)$ in $\mathcal{O}$ in viscosity sense. The proof is the same of Proposition 3.3.

Step 3. For a fixed $\alpha \in \mathbb{R}^m$, $|\alpha| = 1$, the partial differential inequality

$$-(\sigma(x)\alpha) \cdot Du(x) \leq l(x) \qquad \text{in } \mathcal{O}, \text{ in viscosity sense,} \tag{33}$$

implies the inequality (31) for the trajectory of (32) corresponding to the constant control $\alpha(t) \equiv \alpha$. This is proved, as in Proposition II.5.18 of [3] and similar to Proposition 3.4 and Theorem 3.1, by fixing x^0 and taking a diffeomorphism $\xi = \Phi(x)$ that trasform the ODE into the canonical form $\dot{\xi}(t) = \mathrm{e}_1$. In the new coordinates (33) becomes

$$-v_{\xi_1}(\xi) \leq l(\Phi^{-1}(\xi)), \qquad v(\xi) := u(\Phi^{-1}(\xi)).$$

Then Step 2 and Step 1 applied to $w(t) := u(\Phi^{-1}(\xi(t))) + \int_0^t l(\Phi^{-1}(\xi(s)))\, ds$, with $\xi(s) = \Phi(x^0) + (s, 0, \ldots, 0)$, gives $w(0) \leq w(t)$ and therefore (31) holds for this trajectory.

Step 4. A repeated application of Step 3 gives the inequality (31) for all trajectories corresponding to piecewise constant controls $\alpha(\cdot)$. A measurable control $\alpha(\cdot)$ can be approximated by piecewise constant controls α_n such that the corresponding trajectories γ_n converge uniformly to γ on bounded intervals, see, e.g., Lemma II.5.20 of [3]. Then

$$u(\gamma(t)) \geq \limsup_n u(\gamma_n(t)) \geq u(x) + \int_0^t l(\gamma(s))\, ds,$$

which completes the proof. □

Proof of Theorem 6.1. To prove *(i)* we choose $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$ and apply Lemma 6.1 in Ω_2 to get the inequality

$$u(x) \leq Lt + u(\gamma(t)) \tag{34}$$

for any $x \in \Omega_2$, $t > 0$, and $\gamma(\cdot)$ trajectory of (21) such that $\gamma(0) = x$ and $\gamma(s) \in \Omega_2$ for all $s < t$. Next we fix $x, y \in \Omega_1$. If $d(x, y) < +\infty$, for each $\varepsilon > 0$ there is a trajectory γ of (21) such that $\gamma(0) = x$, $\gamma(t) = y$ and $t = d(x, y) + \varepsilon$. We claim that $\gamma(s) \in \Omega_2$ for all $0 < s < t$ if $d(x, y)$ is smaller than a constant $\overline{t}$ independent of the choice of $x, y \in \Omega_1$. Then (34) gives

$$u(x) \leq L(d(x, y) + \varepsilon) + u(y),$$

and by reversing the roles of x and y and letting $\varepsilon \to 0$ we get

$$|u(x) - u(y)| \leq Ld(x, y) \tag{35}$$

for all $x, y \in \Omega_1$ with $d(x,y) \le \bar{t}$. On the other hand, if $d(x,y) \ge \bar{t}$ the inequality (35) holds with L replaced by $\sup_{\Omega_1} |u|/\bar{t}$, so we reach the desired conclusion.

To prove the claim we set

$$\delta := \operatorname{dist}(\overline{\Omega}_1, \partial\Omega_2), \quad S := \sup_{\Omega_2} |\sigma|,$$

where dist denotes the Euclidean distance. Since γ starts and ends in Ω_1, if it ever reaches $\partial\Omega_2$ then

$$\operatorname{length}(\gamma \cap \Omega_2) > 2\delta.$$

On the other hand, by integrating in time the equation for γ (21), we get

$$\operatorname{length}(\gamma \cap \Omega_2) \le tS \sup_{[0,t]} |\alpha| \le (d(x,y) + \varepsilon)S.$$

This gives a contradiction if $d(x,y) + \varepsilon \le 2\delta/S$, so the claim is proved by taking any $\bar{t}$ such that

$$\bar{t} < \frac{2\delta}{S}.$$

This completes the proof of *(i)*.

The statement *(ii)* follows from *(i)* by a result of [18] and [19]. $\square$

We end this section with a Liouville-type property of $\mathcal{X}$-convex functions.

Corollary 6.1. *Assume $d(x,y) < +\infty$ for all $x, y \in \mathbb{R}^n$. Then any $u \in USC(\mathbb{R}^n)$ bounded and $\mathcal{X}$-convex must be constant.*

Proof. By Remark 6.2 we can take $L = 0$ in the preceding proof and get $u(x) \le u(\gamma(t))$ for all $x \in \mathbb{R}^n$, $t > 0$, and $\gamma(\cdot)$ trajectory of (21) such that $\gamma(0) = x$. Given $y \in \mathbb{R}^n$, the assumption on d ensures the existence of such a trajectory that reaches y at some time t. Then $u(x) \le u(y)$ and we reach the conclusion by the arbitrariness of x and y,

$\square$

7 Application to PDEs of Monge-Ampère type

This section gives an application of Proposition 6.1 to fully nonlinear partial differential equations of Monge-Ampère type of the form

$$-\det\left(D^2_{\mathcal{X}} u\right) + H(x, u, D_{\mathcal{X}} u) = 0, \quad \text{in } \Omega, \tag{36}$$

with $\Omega \subset \mathbb{R}^n$ open and bounded. These are classical PDEs in the Euclidean setting ($\mathcal{X}$ = the canonical basis of $\mathbb{R}^n$), they were studied in Carnot groups by [16] and [21] and for more general vector fields by [4, 5, 6]. They are elliptic if one restricts to convex functions, in the Euclidean case, and to v-convex functions in the current generality. That is one of the main reasons of our interest in the equivalence between v-convexity and $\mathcal{X}$-convexity.

The following result is a comparison principle for this equation under mild assumptions on the Hamiltonian H. It implies that the Dirichlet problem for (36) has at most one viscosity solution. In the case of vector fields $\mathcal{X}$ generators of a Carnot group the result was proved in [5]. Proposition 6.1 allows to extend it to general vector fields. The recent paper [6] exploits the same proposition to prove other comparison principles for equations of the form (36), in particular for problems lacking the strict monotonicity assumption (37) below, such as the equation of prescribed horizontal Gauss curvature.

We recall from [4, 5] that a function u is called *uniformly* v-convex or $\mathcal{X}$-convex if for some $\gamma > 0$ it satisfies the inequality

$$-D^2_{\mathcal{X}} u + \gamma\, I \leq 0$$

in the viscosity sense, and that a viscosity supersolution of (36) is required to satisfy the usual inequality only for C^2 test functions ϕ with $D^2_{\mathcal{X}}\phi > 0$.

Corollary 7.1. *Suppose the Hamiltonian $H : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^m \to (0, +\infty)$ is continuous, $u \in USC(\overline{\Omega})$ is a bounded, uniformly $\mathcal{X}$-convex viscosity subsolution of* (36)*, and $v \in LSC(\overline{\Omega})$ is a bounded viscosity supersolution of* (36)*. Assume also that H satisfies for some $\lambda > 0$*

$$H(x, r, q) - H(x, s, q) \geq \lambda(r - s), \quad \forall\, x \in \overline{\Omega},\ q \in \mathbb{R}^m,\ r, s \in [-M, M], \tag{37}$$

where $M = \max\{\|u\|_\infty, \|v\|_\infty\}$. Then

$$\sup_{\Omega}(u - v) \leq \max_{\partial\Omega}(u - v)^+.$$

Proof. The proof is exactly the same as that of Theorem 3.1 in [5], but for one step that we now explain. Since the subsolution u is assumed $\mathcal{X}$-convex, it satisfies the local bound (29) on the $\mathcal{X}$-gradient by Proposition 6.1. This fact was known so far only for the horizontal gradient in Carnot groups from the results of [25], [28], and [23]. For this reason Theorem 3.1 in [5] assumed that the vector fields $\mathcal{X}$ were generators of a Carnot group. Now this assumption can be removed and the comparison principle holds for any family $\mathcal{X}$ of C^2 vector fields. □

References

[1] Alvarez, O.; Lasry, J.-M.; Lions, P.-L.; *Convex viscosity solutions and state constraints.* J. Math. Pures Appl. (9) 76 (1997), no. 3, 265–288.

[2] Arnold, V.I.; *Ordinary differential equations.* Springer-Verlag, Berlin, 2006

[3] Bardi, M.; Capuzzo Dolcetta, I.; *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations.* Birkhäuser, Boston, 1997.

[4] Bardi, M.; Mannucci, P.; *Comparison principles for subelliptic equations of Monge-Ampère type.* Boll. Unione Mat. Ital. 9, no.1 (2008), 2, 489–495.

[5] Bardi, M.; Mannucci, P.; *Comparison Principles for Monge-Ampère type equation in Carnot groups: a direct proof.* Lecture Notes of Seminario Interdisciplinare di Matematica Vol. 7 (2008), 41–51.

[6] Bardi, M.; Mannucci, P.; *Comparison principles and Dirichlet problem for equations of Monge-Ampère type associated to vector fields.* Submitted, available at http://cvgmt.sns.it/people/bardi/

[7] Balogh, Z.; Rickly, M.; *Regularity of convex functions on Heisenberg groups.* Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 2 (2003), no. 4, 847–868.

[8] Beckenbach, E.F.; *Convex functions.* Bull. Amer. Math. Soc. 54 (1948), 439–460.

[9] Bellaïche, A.; Risler, J.J. (Editors); *Sub-Riemannian geometry.* Progress in Mathematics 144, Birkhuser Verlag, Basel, 1996.

[10] Bieske, T.; Dragoni, F.; Manfredi, J.J.; *The Carnot-Carathéodory distance and the infinite Laplacian.* J. Geom. Anal. 19 (2009), 737–754.

[11] Bonfiglioli, A.; Lanconelli, E.; Uguzzoni, F.; *Stratified Lie Groups and Potential Theory for Their Sub-Laplacians.* Springer, Berlin 2007.

[12] Cannarsa, P.; Sinestrari, C.; *Semiconcave Functions, Hamilton-Jacobi Equations, and Optimal Control.* Progress in Nonlinear Differential Equations and their Applications, 58. Birkhuser Boston, 2004.

[13] Capogna, L.; Pauls, S.D.; Tyson, J.T.; *Convexity and a horizontal second fundamental form for hypersurfaces in Carnot groups.* to appear in Trans. Amer. Math. Soc. (2008)

[14] Crandall, M.G.; Ishii, H.; Lions, P.-L.; *Users guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. (N.S.) 27 (1992), 1–67.

[15] Crandall, M.G.; Lions, P.-L.; *Viscosity solutions of Hamiltonian-Jacobi equations.* Trans. Amer. Math. Soc. 277 (1983), 1–42.

[16] Danielli, D.; Garofalo, N,; Nhieu, D.M.; *Notions of convexity in Carnot groups.* Comm. Anal. Geom. 11 (2003), no. 2, 263–341.

[17] Danielli, D.; Garofalo, N,; Nhieu, D.M.; Tournier, F.; *The theorem of Busemann-Feller-Alexandrov in Carnot groups.* Comm. Anal. Geom. 12 (2004), no. 4, 853–886.

[18] Franchi, B.; Serapioni, R.; Serra Cassano, F.; *Approximation and imbedding theorems for weighted Sobolev spaces associated with Lipschitz continuous vector fields.* Boll. Un. Mat. Ital. B (7) 11 (1997), no. 1, 83–117.

[19] Garofalo, N.; Nhieu, D.-M.; *Lipschitz continuity, global smooth approximations and extension theorems for Sobolev functions in Carnot-Carathodory spaces.* J. Anal. Math. 74 (1998), 67–97.

[20] Garofalo, N.; Tournier, F.; *New properties of convex functions in the Heisenberg group.* Trans. Amer. Math. Soc. 358 , no. 5, (2006), 2011–2055 (electronic).

[21] Gutiérrez, C.E.; Montanari, A.; *Maximum and comparison principles for convex functions on the Heisenberg group.* Comm. Partial Differential Equations 29, no. 9-10, (2004),1305–1334.

[22] Gutiérrez, C.E.; Montanari, A.; *On the second order derivatives of convex functions on the Heisenberg group.* Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 3 (2004), no. 2, 349–366.

[23] Juutinen, P.; Lu, G,; Manfredi, J.J.; Stroffolini, B.; *Convex functions on Carnot groups.* Rev. Mat. Iberoam. 23 (2007), no. 1, 191–200.

[24] Lu, G,; Manfredi, J.J.; Stroffolini, B.; *Convex functions in the Heisenberg group.* Calc. Var. Partial Differential Equations 19 (2004), no. 1, 1–22.

[25] Magnani, V.; *Lipschitz continuity, Aleksandrov theorem and characterizations for H-convex functions.* Math. Ann. 334 (2006), no. 1, 199–233.

[26] Montgomery, R.; *A tour of subriemannian geometries, their geodesics and applications.* Mathematical Surveys and Monographs 91, American Mathematical Society, Providence, RI, 2002.

[27] Monti, R.; Rickly, M.; *Geodetically convex sets in the Heisenberg group.* J. Convex Anal. 12 (2005), no. 1, 187–196.

[28] Rickly, M.; *First order regularity of convex functions on Carnot groups.* J. Geom. Anal. 16 (2006), 679–702.

[29] Soravia, P.; *Optimality principles and representation formulas for viscosity solutions of Hamilton-Jacobi equations. II. Equations of control problems with state constraints.* Differential Integral Equations 12 (1999), no. 2, 275–293.

[30] Soravia, P.; *Viscosity and almost everywhere solutions of first-order subelliptic Hamilton-Jacobi equations.* Boll. Unione Mat. Ital. to appear.

[31] Sun, M.; Yang, X.; *Lipschitz continuity for H-convex functions in Carnot groups.* Commun. Contemp. Math. 8 (2006), no. 1, 1–8.

[32] Trudinger, N.S.; *On Hessian Measure for non-commuting vectro fields.* Pure and Applied Math. Quarterly, 2 (2006), no.1, 147–161.

[33] Wang, C.; *Viscosity convex functions on Carnot groups.* Proc. Amer. Math. Soc. 133 (2005), no. 4, 1247–1253.