Homogenization of periodic multi-dimensional structures

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Riassunto. Si studia il comportamento asintotico di una classe di funzionali integrali che possono dipendere da misure concentrate su strutture periodiche multidimensionali, quando tale periodo tende a 0. Il problema viene ambientato in spazi di Sobolev rispetto a misure periodiche. Si dimostra, sotto ipotesi generali, che un appropriato limite può venire definito su uno spazio di Sobolev usuale usando tecniche di Γ-convergenza. Il limite viene espresso come un funzionale integrale il cui integrando è caratterizzato da opportune formule.

1 Introduction

In this paper we deal with the asymptotic behaviour of integral functionals which may model energies concentrated on multidimensional structures. The model example we have in mind is that of composite elastic bodies composed of n-dimensional elastic grains interacting through contact forces depending on the relative displacements of their common boundaries (see Example 3.1). In a general setting, following the approach of Ambrosio, Buttazzo and Fonseca [2], we consider integrals of the form

$$F_\varepsilon(u) = \int_\Omega f(\frac{x}{\varepsilon}, \frac{dDu}{d\mu_\varepsilon}) d\mu_\varepsilon,$$

defined on the space $\mathrm{W}^{1,p}_{\mu_\varepsilon}(\Omega; \mathbf{R}^m)$ of Sobolev functions with respect to the measure μ_ε, which is the set of L^p-functions of Ω whose distributional derivative is a measure absolutely continuous with respect to μ_ε with p-summable densities. We study the limit as $\varepsilon \to 0$ of such functionals under the hypotheses that f is a Borel function 1-periodic in the first variable satisfying a standard growth condition of order p, and

$$\mu_\varepsilon(B) = \varepsilon^n \mu(\frac{1}{\varepsilon} B)$$

where μ is a fixed 1-periodic Radon measure. We show (Theorem 3.6) that under suitable requirements on the measure μ, the family (F_ε) Γ-converges as $\varepsilon \to 0$ to a functional of the form

$$F_{\rm hom}(u) = \int_\Omega f_{\rm hom}(Du)\, dx$$

on $W^{1,p}(\Omega;\mathbf{R}^m)$, where the function $f_{\rm hom}$ is described by an asymptotic formula that generalizes the usual one, corresponding to the case when μ is the Lebesgue measure (see Braides [4] and Müller [15]). This problem had been studied in the case when μ is the restriction of the Lebesgue measure to a periodic set whose complement is composed by well separated bounded sets by Braides and Garroni [6] (media with stiff inclusions). Another meaningful case is when μ is the $(n-1)$-dimensional Hausdorff measure restricted to the union of the boundaries of a periodic partition of $\mathbf{R}^n$. In this case the functions in $W^{1,p}_\mu(\Omega;\mathbf{R}^m)$ are piecewise constant and the functionals F_ε can be interpreted as a finite-difference approximation of the homogenized functional (Section 5, see also Kozlov [13], Pankov [16] and Davini [8]).

The approach described above is somehow complementary to the "smooth approach" where the functionals F_ε are defined as

$$F_\varepsilon(u) = \int_\Omega f(\frac{x}{\varepsilon}, \nabla u) d\mu_\varepsilon$$

on $C^\infty(\Omega;\mathbf{R}^m)$, whose homogenization is studied by Zhikov [18] (see also Braides and Chiadò Piat [5] for the case $\mu = \chi_E$ with E periodic, and Bouchitté, Buttazzo and Seppecher [3] for relaxation results in the case of general μ).

2 Notation and preliminaries

Let Ω be a bounded open subset of $\mathbf{R}^n$; we will use standard notation for the Sobolev and Lebesgue spaces $W^{1,p}(\Omega;\mathbf{R}^m)$ and $L^p(\Omega;\mathbf{R}^m)$; p' and p^* denoting the conjugate and Sobolev exponent of $p \geq 1$, respectively. The L^∞-norm of a function u is denoted simply by $\|u\|_\infty$. We denote by $\mathcal{A}(\Omega)$ the family of all open subsets of Ω; $\mathbb{M}^{m\times n}$ stands for the space of $m \times n$ matrices. The letter c will denote a strictly positive constant independent of the parameters under consideration, whose value may vary from line to line. The Hausdorff k-dimensional measure in $\mathbf{R}^n$ is denoted by $\mathcal{H}^k$. We write $|E|$ for the Lebesgue measure of E. If E is a subset of $\mathbf{R}^n$ then χ_E is its *characteristic function*.

Given a vector-valued measure μ on Ω, we adopt the notation $|\mu|$ for its total variation (see Federer [12]). We say that $u \in L^1(\Omega;\mathbf{R}^m)$ is a *function of bounded variation*, and we write $u \in BV(\Omega;\mathbf{R}^m)$, if all its distributional first derivatives $D_i u_j$ are signed measure on Ω. We denote by Du the $\mathbb{M}^{m\times n}$-valued measure whose entries are $D_i u_j$. For the general exposition of the theory of functions of bounded variation we refer to Federer [12], Evans and Gariepy [11], and Ziemer [17].

If $u \in L^1(\Omega;\mathbf{R}^m)$, we denote by $\tilde{u}$ the *precise representative* of u, whose components are defined by

$$\tilde{u}_i(x) = \limsup_{\rho\to 0^+} ⨍_{B(x,\rho)} u_i(y)\, dy\,, \tag{1}$$

where $B(x,\rho)$ denotes the open ball of centre x and radius ρ.

2.1 Γ-convergence

We recall the definition of De Giorgi's Γ-*convergence* in L^p spaces. If for all $j \in \mathbf{N}$ $F_j : \mathrm{L}^p(\Omega; \mathbf{R}^m) \to [0, +\infty]$ is a functional, then, for $u \in \mathrm{L}^p(\Omega; \mathbf{R}^m)$, we define

$$\Gamma(\mathrm{L}^p)\text{-}\liminf_{j\to+\infty} F_j(u) = \inf\{\liminf_{j\to+\infty} F_j(u_j) \;:\; u_j \stackrel{\mathrm{L}^p}{\to} u\},$$

and

$$\Gamma(\mathrm{L}^p)\text{-}\limsup_{j\to+\infty} F_j(u) = \inf\{\limsup_{j\to+\infty} F_j(u_j) \;:\; u_j \stackrel{\mathrm{L}^p}{\to} u\};$$

if these two quantities coincide their common value will be called the Γ-*limit* of the sequence (F_j) in u, and will be denoted by $\Gamma(\mathrm{L}^p)\text{-}\lim_{j\to+\infty} F_j(u)$.

It is easy to check that $l = \Gamma(\mathrm{L}^p)\text{-}\lim_{j\to+\infty} F_j(u)$ if and only if

(a) for every sequence (u_j) converging to u we have

$$l \leq \liminf_{j\to+\infty} F_j(u_j);$$

(b) there exists a sequence (u_j) converging to u such that

$$l \geq \limsup_{j\to+\infty} F_j(u_j).$$

We say that (F_ε) Γ-converges to l at u as $\varepsilon \to 0$ if for every sequence of positive numbers (ε_j) converging to 0 there exists a subsequence (ε_{j_k}) for which we have

$$l = \Gamma(\mathrm{L}^p)\text{-}\lim_{k\to+\infty} F_{\varepsilon_{j_k}}(u).$$

We recall that the Γ-upper and lower limits defined above are L^p-lower semicontinuous functions. For all properties of Γ-convergence and its importance in the theory of homogenization we refer to the book of Dal Maso [9].

2.2 Sobolev spaces with respect to a measure

The following notion of Sobolev space with respect to a measure has been introduced by Ambrosio, Buttazzo and Fonseca [2].

Definition 2.1 *Let λ be a finite Borel positive measure on the open set $\Omega \subset \mathbf{R}^n$, and let $1 \leq p \leq +\infty$. The* Sobolev space with respect to λ, $\mathrm{W}^{1,p}_\lambda(\Omega; \mathbf{R}^m)$, *is defined as*

$$\mathrm{W}^{1,p}_\lambda(\Omega; \mathbf{R}^m) = \left\{ u \in \mathrm{L}^p(\Omega; \mathbf{R}^m) : \; Du << \lambda, \; \frac{dDu}{d\lambda} \in \mathrm{L}^p_\lambda(\Omega; \mathbb{M}^{m\times n}) \right\}, \qquad (2)$$

where $\mathrm{L}^p_\lambda(\Omega; \mathbf{R}^N)$ stands for the usual Lebesgue space of p-summable $\mathbf{R}^N$-valued functions with respect to λ.

Remark 2.2 By definition, functions in $\mathrm{W}^{1,p}_\lambda(\Omega;\mathbf{R}^m)$ are functions of bounded variation. From the properties of the space $BV(\Omega;\mathbf{R}^m)$ the following two facts can be easily deduced, that are used in the sequel.

(a) $\mathrm{W}^{1,p}_\lambda(\Omega;\mathbf{R}^m)$ is embedded in $\mathrm{L}^{n/(n-1)}(\Omega;\mathbf{R}^m)$.

(b) If $u\in \mathrm{W}^{1,p}_\lambda(\Omega;\mathbf{R}^m)$ and $v\in \mathrm{W}^{1,\infty}_\lambda(\Omega)$ then $uv\in \mathrm{W}^{1,p}_\lambda(\Omega;\mathbf{R}^m)$, and

$$\frac{dD(uv)}{d\lambda}=\tilde v\frac{dDu}{d\lambda}+\tilde u\otimes\frac{dDv}{d\lambda}. \tag{3}$$

Note that in (3) it is necessary to consider the precise representatives, since the measure λ may take into account also sets of zero Lebesgue measure.

If $u\in \mathrm{W}^{1,p}_\lambda(\Omega;\mathbf{R}^m)$ then $Du(B)=0$ if B is a set of zero $(n-1)$-Hausdorff measure. Hence, $\mathrm{W}^{1,p}_\lambda(\Omega;\mathbf{R}^m)=\mathrm{W}^{1,p}_{\lambda'}(\Omega;\mathbf{R}^m)$ if $\lambda-\lambda'$ is concentrated on a set of Hausdorff dimension lower than $n-1$; e.g., points in $\mathbf{R}^3$.

Properties of lower semicontinuity and relaxation for functionals defined on Sobolev spaces with respect to a measure have been studied in [2].

3 Statement of the main result

Let μ be a non-zero positive Radon measure on $\mathbf{R}^n$ which is 1-periodic; i. e.,

$$\mu(B+e_i)=\mu(B)$$

for all Borel subsets B of $\mathbf{R}^n$ and for all $i=1,\dots,n$. The measure μ will be fixed throughout the paper. We will assume the normalization

$$\mu([0,1)^n)=1\,. \tag{4}$$

For all $\varepsilon>0$ we define the ε-periodic positive Radon measure μ_ε by

$$\mu_\varepsilon(B)=\varepsilon^n\,\mu(\frac{1}{\varepsilon}B) \tag{5}$$

for all Borel sets B. Note that by (4) the family (μ_ε) converges locally weakly* in the sense of measures to the Lebesgue measure as $\varepsilon\to 0$.

In the sequel $f:\mathbf{R}^n\times\mathbb{M}^{m\times n}\to[0,+\infty)$ will be a fixed Borel function 1-periodic in the first variable and satisfying the growth condition of order $p\geq 1$: there exist $0<\alpha\leq\beta$ such that

$$\alpha|A|^p\leq f(x,A)\leq\beta(1+|A|^p) \tag{6}$$

for all $x\in\mathbf{R}^n$ and $A\in\mathbb{M}^{m\times n}$.

For every bounded open set Ω, we define the functionals at scale $\varepsilon>0$ as

$$F_\varepsilon(u,\Omega)=\begin{cases}\displaystyle\int_\Omega f(\frac{x}{\varepsilon},\frac{dDu}{d\mu_\varepsilon})d\mu_\varepsilon & \text{if } u\in \mathrm{W}^{1,p}_{\mu_\varepsilon}(\Omega;\mathbf{R}^m)\\ +\infty & \text{otherwise.}\end{cases} \tag{7}$$

Example 3.1 (a) (Perfectly-rigid bodies connected with springs) We take

$$E = \{y \in \mathbf{R}^n : \ \exists i \in \{1, \dots, n\} \text{ such that } \ y_i \in \mathbf{Z}\},$$

that is, the union of all the boundaries of cubes $Q_i = i + (0,1)^n$ with $i \in \mathbf{Z}^n$. E is an $(n-1)$-dimensional set in $\mathbf{R}^n$. We take

$$\mu(B) = \frac{1}{n}\mathcal{H}^{n-1}(B \cap E)$$

for all Borel sets B, where $\mathcal{H}^{n-1}$ stands for the $(n-1)$-dimensional surface measure. For every $\varepsilon > 0$ we have

$$\mu_\varepsilon(B) = \frac{1}{n}\varepsilon\mathcal{H}^{n-1}(B \cap \varepsilon E)\,.$$

In this case $\mathrm{W}^{1,p}_{\mu_\varepsilon}$ consists of functions which are constant on every connected component of each $\varepsilon Q_i \cap \Omega$, since we must have $Du = 0$ on these sets. In the case that u is constant on each $\varepsilon Q_i \cap \Omega$, e.g. if Ω is convex, we have

$$\frac{dDu}{d\mu_\varepsilon} = \frac{n}{\varepsilon}\frac{dDu}{d\mathcal{H}^{n-1}} = \frac{n}{\varepsilon}(u_i - u_j) \otimes (i - j) \text{ on } \partial(\varepsilon Q_i) \cap \partial(\varepsilon Q_j) \cap \Omega\,,$$

where u_i is the value of u on εQ_i. In this case the functionals F_ε take the form

$$\varepsilon \int_{\Omega \cap \varepsilon E} g(\frac{x}{\varepsilon}, \frac{1}{\varepsilon}\frac{dDu}{d\mathcal{H}^{n-1}}) d\mathcal{H}^{n-1}.$$

Note that if Ω is bounded then $\mathrm{W}^{1,p}_{\mu_\varepsilon}(\Omega; \mathbf{R}^m) = \mathrm{W}^{1,\infty}_{\mu_\varepsilon}(\Omega; \mathbf{R}^m)$ for all p if the number of connected components of each $\Omega \cap \varepsilon Q_i$ is finite.

(b) (Elastic media connected with springs) Let E be as above and let

$$\begin{aligned} \mu(B) &= \frac{1}{n+1}\Big(|B| + \mathcal{H}^{n-1}(E \cap B)\Big) \\ \mu_\varepsilon(B) &= \frac{1}{n+1}\Big(|B| + \varepsilon\mathcal{H}^{n-1}((\varepsilon E) \cap B)\Big)\,. \end{aligned}$$

In this case the functions in $\mathrm{W}^{1,p}_{\mu_\varepsilon}(\Omega; \mathbf{R}^m)$ are functions whose restriction to each $\varepsilon Q_i \cap \Omega$ belongs to $\mathrm{W}^{1,p}(\varepsilon Q_i \cap \Omega; \mathbf{R}^m)$, and such that the difference of the traces on both sides of $\partial(\varepsilon Q_i) \cap \partial(\varepsilon Q_j) \cap \Omega$ is p-summable for every $i, j \in \mathbf{Z}^n$. The functionals F_ε take the form

$$\frac{1}{n+1}\int_\Omega f(\frac{x}{\varepsilon}, \frac{dDu}{dx})\, dx + \varepsilon \int_{\Omega \cap \varepsilon E} g(\frac{x}{\varepsilon}, \frac{1}{\varepsilon}\frac{dDu}{d\mathcal{H}^{n-1}}) d\mathcal{H}^{n-1}.$$

In order to obtain a meaningful limit of the functionals F_ε as $\varepsilon \to 0$, some requirements have to be made so that the limit functionals admit an integral representation on $\mathrm{W}^{1,p}(\Omega; \mathbf{R}^m)$.

Definition 3.2 *A 1-periodic positive Radon measure μ on $\mathbf{R}^n$ will be called p-*homogenizable *if the following properties hold:*

(i) (Poincaré inequality) *there exist a constant c such that for all $k \in \mathbf{N}$*

$$\int_{(0,k)^n} |u|^p dx \leq ck^p \int_{(0,k)^n} \Big|\frac{dDu}{d\mu}\Big|^p d\mu \tag{8}$$

for all $u \in \mathrm{W}^{1,p}_\mu((0,k)^n)$ with $\int_{(0,k)^n} u dx = 0$;

(ii) (existence of cut-off functions) *there exist $K > 0$ and $\delta > 0$ such that for all $\varepsilon > 0$, for all pairs U, V of open subsets of $\mathbf{R}^n$ with $U \subset\subset V$, and* $\mathrm{dist}\,(U, \partial V) \geq \delta\varepsilon$, *and for all $u \in \mathrm{W}^{1,p}_{\mu_\varepsilon}(V)$ there exists $\phi \in \mathrm{W}^{1,\infty}_{\mu_\varepsilon}(V)$ with $0 \leq \phi \leq 1$, $\phi = 1$ on U, $\phi = 0$ in a neighbourhood of ∂V, such that*

$$\int_V \Big|\frac{dD\phi}{d\mu_\varepsilon}\tilde{u}\Big|^p d\mu_\varepsilon \leq \frac{K}{\big(\mathrm{dist}\,(U, \partial V)\big)^p} \int_{V\setminus U} |u|^p\, dx\,. \tag{9}$$

Such a ϕ will be called a cut-off function between U and V*;*

(iii) (existence of periodic test-functions) *for all $i = 1, \ldots, n$, there exists $z_i \in \mathrm{W}^{1,p}_{\mu,\mathrm{loc}}(\mathbf{R}^n)$ such that $x \mapsto z_i(x) - x_i$ is 1-periodic.*

Remark 3.3 Note that the Lebesgue measure satisfies trivially all the properties of Definition 3.2. Property (ii) depends on μ and p.

Example 3.4 (a) The measure μ in Example 3.1(a) is p-homogenizable for all $p \geq 1$. In fact, (i) follows from the Appendix. To prove (ii) let $\delta = 5\sqrt{n}$. Fixed $\varepsilon > 0$, set $U_\varepsilon = \bigcup\{\varepsilon Q_i :\ \varepsilon Q_i \cap U \neq \emptyset\}$. Note that $U_\varepsilon \subset\subset V$. Choose (we use the notation $[t]$ for the integer part of t)

$$\phi(x) = 1 - \Big(\frac{1}{C}\Big[\frac{1}{\varepsilon}\inf\{|x-y|_\infty : y \in U_\varepsilon\}\Big] \wedge 1\Big),$$

where $|x - y|_\infty = \max_{1\leq i\leq n} |x_i - y_i|$, and

$$C = \Big[\frac{1}{\varepsilon}\inf\Big\{|x-y|_\infty : x \in U_\varepsilon,\ y \in \partial V\Big\}\Big] - 2\,.$$

Note that $|dD\phi/d\mu_\varepsilon| \leq n/(C\varepsilon) \leq c/\mathrm{dist}\,(U, \partial V)$ for some constant c independent of U and V. Moreover, if $u \in \mathrm{W}^{1,p}_{\mu_\varepsilon}(V)$ then u is equal to a constant u_i on each cube εQ_i such that $D\phi \neq 0$ on $\partial(\varepsilon Q_i)$. Hence, for two such cubes

$$\varepsilon \int_{\partial\varepsilon Q_i \cap \partial\varepsilon Q_j} |\widetilde{u}|^p\, d\mathcal{H}^{n-1} \leq \varepsilon \int_{\partial\varepsilon Q_i \cap \partial\varepsilon Q_j} (|u_i|^p + |u_j|^p)\, d\mathcal{H}^{n-1} = \int_{\varepsilon Q_i \cup \varepsilon Q_j} |u|^p\, dx$$

so that

$$\begin{aligned}\int_V \Big|\frac{dD\phi}{d\mu_\varepsilon}\widetilde{u}\Big|^p d\mu_\varepsilon &\leq \frac{c^p\varepsilon}{\mathrm{dist}\,(U,\partial V)^p}\int_{(V\setminus U)\cap\varepsilon E\cap \mathrm{spt} D\phi} |\widetilde{u}|^p\, d\mathcal{H}^{n-1}\\ &\leq 2n\frac{c^p}{\mathrm{dist}\,(U,\partial V)^p}\int_{V\setminus U} |u|^p\, dx\,.\end{aligned}$$

The proof of (ii) is then complete. To verify (iii) take simply $z_i(x) = [x_i]$.

(b) The measure μ in Example 3.1(b) is p-homogenizable for all $p \geq 1$. In fact, (i) follows from the Appendix . The proof of (ii) and (iii) is trivial since the Lebesgue measure is absolutely continuous with respect to μ.

The homogenization theorem for functionals in (7) takes the following form.

Theorem 3.5 *Let μ be a p-homogenizable measure, and for every bounded open subset Ω of $\mathbf{R}^n$ let $F_\varepsilon(\cdot, \Omega)$ be defined on $\mathrm{L}^p(\Omega; \mathbf{R}^m)$ by (7). Then the Γ-limit*

$$F_{\mathrm{hom}}(u, \Omega) = \Gamma(\mathrm{L}^p)\text{-}\lim_{\varepsilon \to 0} F_\varepsilon(u, \Omega) \tag{10}$$

exists for all bounded open subsets Ω with Lipschitz boundary and for all $u \in \mathrm{W}^{1,p}(\Omega; \mathbf{R}^m)$, and it can be represented as

$$F_{\mathrm{hom}}(u, \Omega) = \int_\Omega f_{\mathrm{hom}}(Du)\, dx\,, \tag{11}$$

where the homogenized integrand satisfies the asymptotic formula

$$\begin{aligned} f_{\mathrm{hom}}(A) \quad = \quad & \lim_{k \to +\infty} \inf\Big\{ \frac{1}{k^n} \int_{[0,k)^n} f(x, \frac{dDu}{d\mu}) d\mu : \\ & \qquad u \in \mathrm{W}^{1,p}_{\mu,\mathrm{loc}}(\mathbf{R}^n; \mathbf{R}^m),\ u - Ax \ \ k\text{-periodic}\Big\}\,. \end{aligned} \tag{12}$$

If $p > 1$ then $F_{\mathrm{hom}}(u, \Omega) = +\infty$ if $u \in \mathrm{L}^p(\Omega; \mathbf{R}^m) \setminus \mathrm{W}^{1,p}(\Omega; \mathbf{R}^m)$. Furthermore, if f is convex then the cell-problem formula holds

$$\begin{aligned} f_{\mathrm{hom}}(A) \quad = \quad & \inf\Big\{ \int_{[0,1)^n} f(x, \frac{dDu}{d\mu})\, d\mu : \\ & \qquad u \in \mathrm{W}^{1,p}_{\mu,\mathrm{loc}}(\mathbf{R}^n; \mathbf{R}^m),\ u - Ax\ 1\text{-periodic}\Big\} \end{aligned} \tag{13}$$

for all $A \in \mathbb{M}^{m \times n}$.

Remark 3.6 In formulas (12) and (13) we cannot replace the sets $[0, k)^n$ and $[0, 1)^n$ by the sets $(0, k)^n$ and $(0, 1)^n$, respectively, if μ charges $[0, 1)^n \setminus (0, 1)^n$.

Remark 3.7 If μ is not a p-homogenizable measure then f_{hom} may be equal to $+\infty$ for all non-zero matrices A. As an example, take

$$\mu(B) = \sum_{i \in \mathbf{Z}^n} \lambda(i + B)\,, \tag{14}$$

where λ is any probability measure with $\operatorname{spt} \lambda$ contained in $(0, 1)^n$. Then test-functions u in (12) must be constant on a periodic connected component of $\mathbf{R}^n$, and hence we get that $f_{\mathrm{hom}}(A) = +\infty$ if $A \neq 0$.

Remark 3.8 Contrary to the usual homogenization results in the framework of ordinary Sobolev spaces, the hypothesis that Ω has a Lipschitz boundary (which will be used in an essential way in Step 3 of Proposition 4.3) cannot be removed from Theorem 3.5. To check this, take simply $n = 2$ and

$$\Omega = \Big(\bigcup_{i=1}^{\infty}(q_i - 2^{-i-3}, q_i + 2^{-i-3}) \times (0,1)\Big) \cup \Big(\bigcup_{i=1}^{\infty}(0,1) \times (q_i - 2^{-i-3}, q_i + 2^{-i-3})\Big),$$

where (q_i) is a numbering of $\mathbf{Q} \cap (0,1)$. Take as μ the measure of Example 3.1(a) and any f in Theorem 3.5. Note that, as $\Omega \cap \frac{1}{k}Q_i$ is connected for all sub-cubes $\frac{1}{k}Q_i$ of $(0,1)^2$, each function $u \in \mathrm{W}^{1,p}_{\mu_{1/k}}(\Omega \cap (0,1)^2; \mathbf{R}^m)$ is constant on each such $\Omega \cap \frac{1}{k}Q_i$. Hence, the two spaces $\mathrm{W}^{1,p}_{\mu_{1/k}}(\Omega \cap (0,1)^2; \mathbf{R}^m)$ and $\mathrm{W}^{1,p}_{\mu_{1/k}}((0,1)^2; \mathbf{R}^m)$ are equivalent, and, as $\frac{1}{k}E \cap (0,1)^2 \subset \Omega \cap (0,1)^2$,

$$F_{1/k}(u, \Omega \cap (0,1)^2) = F_{1/k}(u, (0,1)^2).$$

If the thesis of Theorem 3.5 were true, then we would easily conclude that for all $v \in \mathrm{W}^{1,p}(\Omega \cap (0,1)^2; \mathbf{R}^m)$ with $F_{\mathrm{hom}}(u, \Omega \cap (0,1)^2) < +\infty$ there exists $u \in \mathrm{W}^{1,p}((0,1)^2; \mathbf{R}^m)$ with $u = v$ on $\Omega \cap (0,1)^2$ and

$$F_{\mathrm{hom}}(v, \Omega \cap (0,1)^2) = F_{\mathrm{hom}}(u, (0,1)^2),$$

which is not possible for example if $f \geq 1$ since $|\Omega \cap (0,1)^2| \neq |(0,1)^2|$.

4 Proof of the homogenization theorem

The proof of Theorem 3.5 will be obtained at the end of the section, as a consequence of the following propositions, which adapt to this case the usual methods for the homogenization by Γ-convergence. While the usual compactness and integral representation results in Dal Maso [9] hold with minor modification also in this case, a more complex proof for the so-called fundamental estimate, for the growth condition from above and for the homogenization formula is necessary.

¿From now on, Ω will be a fixed bounded open subset of $\mathbf{R}^n$ with Lipschitz boundary.

Proposition 4.1 (Fundamental Estimate) *For every $\sigma > 0$ there exists ε_σ and $M > 0$ such that for all U, U', V open subsets of Ω with $U' \subset U$ and $\operatorname{dist}(U', V \setminus U) > 0$, for all $\varepsilon < \varepsilon_\sigma \operatorname{dist}(U', V \setminus U)$ and for all $u \in \mathrm{W}^{1,p}_{\mu_\varepsilon}(\Omega; \mathbf{R}^m)$, $v \in \mathrm{W}^{1,p}_{\mu_\varepsilon}(\Omega; \mathbf{R}^m)$ there exists a cut-off function between U' and U, $\phi \in \mathrm{W}^{1,\infty}_{\mu_\varepsilon}(U \cup V)$, such that*

$$\begin{aligned} F_\varepsilon(\phi u + (1-\phi)v, U' \cup V) \quad &\leq \quad (1+\sigma)(F_\varepsilon(u, U) + F_\varepsilon(v, V)) \\ &+ \frac{M}{\big(\operatorname{dist}(U', V \setminus U)\big)^p} \int_{(U \cap V) \setminus U'} |u - v|^p dx + \sigma \mu_\varepsilon((U \cap V) \setminus U'). \end{aligned} \tag{15}$$

PROOF. Let $K>0$ and $\delta>0$ be the constants given by Definition 3.2(ii), let $N\in\mathbf{N}$ be such that $N\delta\varepsilon\le \mathrm{dist}\,(U',V\setminus U)$, and let $U_k=\{x\in U: N\mathrm{dist}\,(x,U')< k\,\mathrm{dist}\,(U',V\setminus U)\}$, $U_0=U'$. For each $k=1,\dots,N$ let ϕ_k be a cut-off function between U_{k-1} and U_k, satisfying (9), which exists since $\mathrm{dist}\,(U_{k-1},\partial U_k)\ge\delta\varepsilon$. We have, using Remark 2.2(b), (6) and (9)

$$\begin{aligned}
&F_\varepsilon(\phi_k u+(1-\phi_k)v,U'\cup V)\\
=\;&\int_{U'\cup V} f(\frac{x}{\varepsilon},\tilde\phi_k\frac{dDu}{d\mu_\varepsilon}+(1-\tilde\phi_k)\frac{dDv}{d\mu_\varepsilon}+(\tilde u-\tilde v)\otimes\frac{dD\phi_k}{d\mu_\varepsilon})d\mu_\varepsilon\\
\le\;&\int_U f(\frac{x}{\varepsilon},\frac{dDu}{d\mu_\varepsilon})d\mu_\varepsilon+\int_V f(\frac{x}{\varepsilon},\frac{dDv}{d\mu_\varepsilon})d\mu_\varepsilon\\
&\quad+4^p\beta\int_{(U_k\setminus U_{k-1})\cap V}\Big(1+\Big|\frac{dDu}{d\mu_\varepsilon}\Big|^p+\Big|\frac{dDv}{d\mu_\varepsilon}\Big|^p\Big)d\mu_\varepsilon\\
&\quad+4^p\beta\int_{(U_k\setminus U_{k-1})\cap V}\Big|(\tilde u-\tilde v)\otimes\frac{dD\phi_k}{d\mu_\varepsilon}\Big|^p d\mu_\varepsilon\\
\le\;&F_\varepsilon(u,U)+F_\varepsilon(v,V)\\
&\quad+4^p\beta\int_{(U_k\setminus U_{k-1})\cap V}\Big(1+\Big|\frac{dDu}{d\mu_\varepsilon}\Big|^p+\Big|\frac{dDv}{d\mu_\varepsilon}\Big|^p\Big)d\mu_\varepsilon\\
&\quad+4^p\beta\frac{KN^p}{\big(\mathrm{dist}\,(U',V\setminus U)\big)^p}\int_{(U_k\setminus U_{k-1})\cap V}|u-v|^p\,dx
\end{aligned}$$

where K is the constant appearing in (9).

Choose k such that

$$\begin{aligned}
&\int_{(U_k\setminus U_{k-1})\cap V}(1+\Big|\frac{dDu}{d\mu_\varepsilon}\Big|^p+\Big|\frac{dDv}{d\mu_\varepsilon}\Big|^p)d\mu_\varepsilon\\
&\qquad+\frac{KN^p}{\big(\mathrm{dist}\,(U',V\setminus U)\big)^p}\int_{(U_k\setminus U_{k-1})\cap V}|u-v|^p\,dx\\
&\le\frac{1}{N}\Big(\int_{(U\cap V)\setminus U'}(1+\Big|\frac{dDu}{d\mu_\varepsilon}\Big|^p+\Big|\frac{dDv}{d\mu_\varepsilon}\Big|^p)d\mu_\varepsilon\\
&\qquad+\frac{KN^p}{\big(\mathrm{dist}\,(U',V\setminus U)\big)^p}\int_{(U\cap V)\setminus U'}|u-v|^p\,dx\Big).
\end{aligned}$$

Then, taking into account also (6),

$$\begin{aligned}
&F_\varepsilon(\phi_k u+(1-\phi_k)v,U'\cup V)\\
\le\;&F_\varepsilon(u,U)+F_\varepsilon(v,V)\\
&\quad+\frac{4^p\beta}{N\alpha}\Big(\int_{(U\cap V)\setminus U'}f(\frac{x}{\varepsilon},\frac{dDu}{d\mu_\varepsilon})d\mu_\varepsilon+\int_{(U\cap V)\setminus U'}f(\frac{x}{\varepsilon},\frac{dDv}{d\mu_\varepsilon})d\mu_\varepsilon\Big)\\
&\quad+4^p\beta\frac{KN^{p-1}}{\big(\mathrm{dist}\,(U',V\setminus U)\big)^p}\int_{(U\cap V)\setminus U'}|u-v|^p dx+\frac{4^p\beta}{N}\mu_\varepsilon((U\cap V)\setminus U')
\end{aligned}$$

$$
\begin{aligned}
\leq \quad & \Big(1+\frac{4^p\beta}{N\alpha}\Big)\Big(F_\varepsilon(u,U)+F_\varepsilon(v,V)\Big) \\
& \qquad +4^p\beta\frac{KN^{p-1}}{\big(\mathrm{dist}\,(U',V\setminus U)\big)^p}\int_{(U\cap V)\setminus U'}|u-v|^p dx+\frac{4^p\beta}{N}\mu_\varepsilon((U\cap V)\setminus U')\,.
\end{aligned}
$$

We can choose ε_σ satisfying

$$
\frac{4^p\beta}{\sigma\min\{1,\alpha\}}+1=\frac{1}{\delta\varepsilon_\sigma},
$$

so that we can find N, depending only on σ and on the constants of the problem, in such a way that (15) holds, with $M=4^pK\beta N^{p-1}$. $\square$

Proposition 4.2 *For every $A\in\mathbb{M}^{m\times n}$ there exists $z_A\in \mathrm{W}^{1,p}_{\mu,\mathrm{loc}}(\mathbf{R}^n;\mathbf{R}^m)$ such that z_A-Ax is 1-periodic and satisfies*

$$
\int_{[0,1)^n}\Big|\frac{dDz_A}{d\mu}\Big|^p d\mu\leq c|A|^p. \tag{16}
$$

PROOF. Define $z_A=\sum_{i=1}^m\sum_{j=1}^n A_{ij}z_je_i$, where z_i are as in Definition 3.2(iii). Inequality (16) is trivial. $\square$

We fix an infinitesimal sequence (ε_j). We define

$$
F'(u,U)=\Gamma(\mathrm{L}^p)\text{-}\liminf_{j\to+\infty}F_{\varepsilon_j}(u,U)
$$

$$
F''(u,U)=\Gamma(\mathrm{L}^p)\text{-}\limsup_{j\to+\infty}F_{\varepsilon_j}(u,U)
$$

for all $u\in L^p(\Omega;\mathbf{R}^m)$ and for all open subsets U of Ω.

Proposition 4.3 (Growth Condition) *We have*

$$
F''(u,U)\leq c\int_U(1+|Du|^p)dx
$$

for all $u\in \mathrm{W}^{1,p}(\Omega;\mathbf{R}^m)$ and for all open subsets U of Ω with $|\partial U|=0$.

PROOF. *Step 1: we have $F''(Ax,U)\leq c|\overline{U}|(1+|A|^p)$ for all $A\in\mathbb{M}^{m\times n}$ and for all $U\in\mathcal{A}(\Omega)$.*

Let z_A be given by Proposition 4.2. We may assume that z_j-x_j has mean value 0 in the periodicity cell, so that the functions $z^\varepsilon_A(x)=\varepsilon z_A(x/\varepsilon)$ converge in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^n;\mathbf{R}^m)$ to Ax, and

$$
\begin{aligned}
F''(Ax,U) \quad &\leq \quad \limsup_{\varepsilon\to0+}\int_U f\big(\frac{x}{\varepsilon},\frac{dDz^\varepsilon_A}{d\mu_\varepsilon}\big)d\mu_\varepsilon \\
&\leq \quad \beta\limsup_{\varepsilon\to0+}\int_U\big(1+\Big|\frac{dDz^\varepsilon_A}{d\mu_\varepsilon}\Big|^p\big)d\mu_\varepsilon\leq c|\overline{U}|\,(1+|A|^p).
\end{aligned}
$$

Step 2: we have $F''(u,U) \le c\int_U (1+|Du|^p)dx$ for all piecewise affine function $u \in \mathrm{W}^{1,p}(\Omega;\mathbf{R}^m)$ and for all open subsets $U \subseteq \Omega$ with $|\partial U| = 0$.

We write $u = \sum_{i=1}^N \chi_{U_i} u_i$, where $U_1, \dots, U_N$ are disjoint open subsets of U such that $|U \setminus \bigcup_i U_i| = 0$ and $|\overline{U}_i| = |U_i|$, and $u_i(x) = A_i x + c_i$ for some $A_i \in \mathbb{M}^{m\times n}$ and $c_i \in \mathbf{R}^m$. For each i we set $u_i^\varepsilon(x) = z_{A_i}^\varepsilon(x) + c_i$, as from Step 1.

We will prove Step 2 by finite induction. First, we give an estimate on $U_1 \cup U_2$. For all ε sufficiently small, we can apply Proposition 4.1 choosing the sets

$$U_2^\eta = \{x \in U : \ \mathrm{dist}\,(x, U_2) < \eta\},$$

U_2 and U_1 as the sets U, U' and V in its statement, respectively, where $\eta = \eta_\varepsilon > 0$ will be determined later, and taking $\sigma = 1$, $u = u_2^\varepsilon$ and $v = u_1^\varepsilon$. We obtain then a cut-off function $\phi = \phi_\varepsilon$ between U_2 and U_2^η such that

$$\begin{aligned} F_\varepsilon(\phi_\varepsilon u_2^\varepsilon + (1-\phi_\varepsilon)u_1^\varepsilon, U_1 \cup U_2) \quad &\le \quad 2(F_\varepsilon(u_1^\varepsilon, U_1) + F_\varepsilon(u_2^\varepsilon, U_2^\eta)) \\ &\quad + \frac{M}{\eta^p}\int_{U_1\cap U_2^\eta} |u_2^\varepsilon - u_1^\varepsilon|^p\,dx + \mu_\varepsilon(U_1 \cap U_2^\eta)\,. \end{aligned}$$

The constant M is the one given by Proposition 4.1 with $\sigma = 1$. We can choose now $\eta = \eta_\varepsilon$, tending to 0 as $\varepsilon \to 0$, in such a way that

$$\lim_{\varepsilon\to 0} \frac{1}{\eta_\varepsilon^p} \int_{U_1 \cap U_2^{\eta_\varepsilon}} |u_2^\varepsilon - u_1^\varepsilon|^p\,dx = 0\,,$$

taking into account that

$$\lim_{\varepsilon\to 0} \int_{U_1\cap U_2^\eta} |u_2^\varepsilon - u_1^\varepsilon|^p\,dx = \int_{U_1\cap U_2^\eta} |u_2 - u_1|^p\,dx \le c\|Du\|_\infty^p \eta^{p+1}$$

since u_i are affine and $u_2 = u_1$ on $\partial U_1 \cap \partial U_2$. If we define $w_1^\varepsilon = \phi_\varepsilon u_2^\varepsilon + (1-\phi_\varepsilon)u_1^\varepsilon$, we have $w_1^\varepsilon \to u$ in $\mathrm{L}^p(U_1 \cup U_2; \mathbf{R}^m)$ and

$$\limsup_{\varepsilon\to 0} F_\varepsilon(w_1^\varepsilon, U_1\cup U_2) \le c\int_{U_1\cup U_2} (1 + |Du|^p)\,dx$$

as in the proof of Step 1.

We can proceed now by induction, repeating at each step the previous argument replacing U_1 by $U_1 \cup \ldots \cup U_j$, U_2 by U_{j+1}, u_1^ε by the w_j^ε constructed in the preceding step, and u_2^ε by u_{j+1}^ε.

Step 3: conclusion.

To conclude the proof it suffices to recall that $F''(\cdot, U)$ is weakly lower semicontinuous and piecewise affine functions are dense in $\mathrm{W}^{1,p}(\Omega;\mathbf{R}^m)$. □

Proposition 4.4 *There exists a subsequence of (ε_j) (not relabeled) such that for all open subsets U of Ω there exists the Γ-limit*

$$\Gamma\text{-}\lim_{j\to+\infty} F_{\varepsilon_j}(u,U) = F(u,U)\,,$$

and there exists a function $\varphi : \mathbb{M}^{m\times n} \to \mathbf{R}$ such that

$$F(u,U) = \int_U \varphi(Du)dx$$

for all $u \in \mathrm{W}^{1,p}(\Omega;\mathbf{R}^m)$ and $U \subset \Omega$ with $|\partial U| = 0$.

Proof. The proof of this proposition can be obtained using the methods of Γ-convergence, for which we refer to the book by Dal Maso [9], outlining the necessary modifications.

Using the compactness of Γ-convergence (see Theorem 8.5 in [9]) and a diagonal procedure, we extract a subsequence (not relabeled) such that the Γ-limit

$$\Gamma(\mathrm{L}^p)\text{-}\lim_{j\to+\infty} F_{\varepsilon_j}(u,U) = F(u,U)$$

exists for all $u \in L^p(\Omega;\mathbf{R}^m)$ and for all sets U in the countable family $\mathcal{R}$ of all finite unions of open rectangles of Ω with rational vertices.

Now, observe that for all open subsets $U \subseteq \Omega$ with $|\partial U| = 0$ we have

$$F''(u,U) = \sup\{F''(u,V):\ V \subset\subset U,\ V \text{open}\},$$

$$F'(u,U) = \sup\{F'(u,V):\ V \subset\subset U,\ V \text{open}\}.$$

This can be shown modifying the proof of [9] Proposition 18.6 for functionals that satisfy the conclusions of Proposition 4.1 and Proposition 4.3.

Next, we note that the Γ-limit $F(u,U) = \Gamma\text{-}\lim_{j\to+\infty} F_{\varepsilon_j}(u,U)$ exists for all $U \in \mathcal{A}(\Omega)$ with $|\partial U| = 0$, and for all $u \in \mathrm{W}^{1,p}(\Omega;\mathbf{R}^m)$ the function $F(u,\cdot)$ is the restriction to the family these open sets of a Borel measure on Ω. This result can be obtained by [9] Proposition 16.4 and by the De Giorgi-Letta measure criterion ([9] Theorem 14.23), noting that the proof of [9] Proposition 18.3 can be repeated using Proposition 4.1.

Eventually, the existence of $\varphi : \mathbb{M}^{m\times n} \to \mathbf{R}$ such that

$$F(u,U) = \int_U \varphi(Du)dx$$

for all $u \in \mathrm{W}^{1,p}(\Omega;\mathbf{R}^m)$ and for all $U \in \mathcal{A}(\Omega)$ with $|\partial U| = 0$ follows from the integral representation Theorem 4.3.2 in [7], observing that translation invariance in x can be obtained, e.g., as in [9] Theorem 24.1 (see also [14] Lemma 4.2). $\square$

Proposition 4.5 (Homogenization Formula) *For all $A \in \mathbb{M}^{m\times n}$ there exists the limit in* (12) *and we have $\varphi(A) = f_{\mathrm{hom}}(A)$.*

Proof. In order to simplify the proof of formula (12), we can suppose that $\mu([0,1)^n \setminus (0,1)^n) = 0$, which holds up to a translation. For all $A \in \mathbb{M}^{m\times n}$ and $k \in \mathbf{N}$ we define

$$g_k(A) = \inf\Big\{\frac{1}{k^n}\int_{(0,k)^n} f(x, \frac{dDu}{d\mu})d\mu : u \in \mathrm{W}^{1,p}_{\mu,\mathrm{loc}}(\mathbf{R}^n;\mathbf{R}^m),\ u - Ax \ \ k\text{-periodic}\Big\}.$$

Fixed $A \in \mathbb{M}^{m\times n}$ let $u \in \mathrm{W}^{1,p}_{\mu,\mathrm{loc}}(\mathbf{R}^n;\mathbf{R}^m)$ with $u - Ax$ k-periodic and with mean value 0 on $(0,k)^n$. Define the sequence $u_j(x) = \varepsilon_j\, u(x/\varepsilon_j)$, and note that $u_j \to Ax$ in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^n;\mathbf{R}^m)$. We have then

$$\varphi(A) = F(Ax,(0,1)^n) \le \liminf_{j\to+\infty} F_{\varepsilon_j}(u_j,(0,1)^n) = \frac{1}{k^n}\int_{(0,k)^n} f(x,\frac{dDu}{d\mu})d\mu.$$

Hence, $\varphi(A) \le g_k(A)$, so that

$$\varphi(A) \le \liminf_{k\to+\infty} g_k(A). \tag{17}$$

Conversely, let $w_j \to Ax$ be such that

$$\varphi(A) = F(Ax,(0,1)^n) = \lim_{j\to+\infty} F_{\varepsilon_j}(w_j,(0,1)^n).$$

Let $\sigma > 0$. Let $T_j = 1/\varepsilon_j$ and let $u_j(x) = T_j w_j(x/T_j)$. We use the notation $K_j = [T_j] + 1$.

If j is large enough and $N > 4$, we can use Proposition 4.1 with $\varepsilon = 1$, $U = (0,T_j)^n$, $V = (0,K_j)^n \setminus (2T_j/N, T_j - 2(T_j/N))^n$, $U' = (T_j/N, T_j - (T_j/N))^n$, $u = u_j$, and $v = z_A$. We get then

$$\begin{aligned}
& F_1(\phi u + (1-\phi)v,(0,K_j)^n) \\
= \;& F_1(\phi u + (1-\phi)v, U'\cup V) \\
\le \;& (1+\sigma)(F_1(u,U) + F_1(v,V)) \\
& +MN^pT_j^{-p}\int_{(U\cap V)\setminus U'} |u-v|^p dx + \sigma\mu((U\cap V)\setminus U').
\end{aligned} \tag{18}$$

Since $\phi u + (1-\phi)v - Ax$ is K_j-periodic, we obtain

$$\begin{aligned}
& K_j^n\, g_{K_j}(A) \\
\le \;& (1+\sigma)(F_1(u_j,(0,T_j)^n) + F_1(z_A,V)) \\
& +MN^pT_j^{-p}\int_{(0,T_j)^n\setminus(T_j/N,T_j-(T_j/N))^n} |u_j - z_A|^p dx + \sigma\mu((U\cap V)\setminus U') \\
\le \;& (1+\sigma)(T_j^n F_{\varepsilon_j}(w_j,(0,1)^n) + c\frac{K_j^n}{N}(1+|A|^p) \\
& +MN^pT_j^n\int_{(0,1)^n} |w_j - z_j|^p dx + \sigma c K_j^n,
\end{aligned}$$

where $z_j(x) = T_j^{-1} z_A(T_j x)$. Note that $z_j \to Ax$ in $\mathrm{L}^p((0,1)^n; \mathbf{R}^m)$; hence

$$\lim_{j\to+\infty} \int_{(0,1)^n} |w_j - z_j|^p dx = 0\,.$$

Dividing the estimate above by K_j^n, and letting first $j \to +\infty$ and then $\sigma \to 0$ and $N \to +\infty$, we get

$$\limsup_{j\to+\infty} g_{K_j}(A) \leq \varphi(A). \tag{19}$$

By (17) and (19) we obtain then

$$\varphi(A) = \liminf_{k\to+\infty} g_k(A) = \lim_{j\to+\infty} g_{K_j}(A).$$

The first equality shows that φ is independent of the sequence (ε_j). Repeating the reasoning then with a sequence (ε_j) such that

$$\lim_{j\to+\infty} g_{K_j}(A) = \limsup_{k\to+\infty} g_k(A)$$

the proof is complete. □

Proof of Theorem 3.5. The previous propositions show that the limit in (10) exists and (11) holds with f_{hom} given by (12). Formula (13) in the convex case follows as in [15].

It remains to check that $F_{\mathrm{hom}}(u, \Omega) = +\infty$ if $u \in \mathrm{L}^p(\Omega; \mathbf{R}^m) \setminus \mathrm{W}^{1,p}(\Omega; \mathbf{R}^m)$ when $p > 1$. Clearly, it suffices to prove this for $f(A) = |A|^p$. In this case, F_{hom} is convex, hence it is determined by its behaviour on $\mathrm{W}^{1,p}(\Omega; \mathbf{R}^m)$ (see [9] Chapter 23). It will be enough then to prove that $f_{\mathrm{hom}}(A) \geq c|A|^p$. Since f_{hom} is positively homogeneous of degree p, it is sufficient to check that $f_{\mathrm{hom}}(A) \neq 0$ if $A \neq 0$. To this aim, let $u_\varepsilon \to Ax$ be such that $F_\varepsilon(u_\varepsilon, (0,1)^n) \to f_{\mathrm{hom}}(A)$. If $f_{\mathrm{hom}}(A) = 0$ then by Definition 3.2(i) and a scaling argument we obtain that $u_{\frac{1}{k}}$ tends to a constant, and a contradiction. □

5 Limits of a class of difference schemes

In this section we show how some energies depending on finite differences can be seen as a particular case of functionals defined on Sobolev spaces with respect to the measures introduced in Example 3.1(a). For the sake of illustration we deal only with the case of integrands independent of x. We remark that in the case of quadratic functionals (i.e., $\psi_k(\xi) = c_k \xi^2$ below), our result can be framed in the theory of difference operators elaborated by Kozlov [13], where a compactness and representation theorem is given for a general class of operators.

Let $\Omega \subseteq R^n$ be an open set with Lipschitz boundary, and let

$$I_\varepsilon = \{i \in \mathbf{Z}^n : \varepsilon i + [0, \varepsilon]^n \subseteq \Omega\}\,.$$

Let $\psi_1....\psi_n$ be convex functions such that

$$|\xi|^p \leq \psi_k(\xi) \leq c(1+|\xi|^p)$$

for all $\xi \in M^{m\times n}$ and $k = 1,\ldots,n$. We define A_ε the set of functions

$$u : (\mathbf{Z}^n \cap \frac{1}{\varepsilon}\Omega) \to \mathbf{R}^m$$

and for all $u \in A_\varepsilon$

$$\Psi_\varepsilon(u) = \sum_{k=1}^{n}\sum_{i\in I_\varepsilon} \varepsilon^n \psi_k\left(\frac{u(i+e_k)-u(i)}{\varepsilon}\right).$$

If $u \in A_\varepsilon$ then we can associate to u the piecewise constant function $v_u : \Omega \longrightarrow \mathbf{R}^m$ defined by

$$v_u(x) = \begin{cases} u(i) & x \in \varepsilon i + [0,\varepsilon)^n \quad \varepsilon i \in \Omega \cap \varepsilon\mathbf{Z}^n \\ \\ 0 & \text{otherwise} \end{cases}.$$

Definition 5.1 *Let $u_j \in A_{\varepsilon_j}$. We say that u_j converges to $u \in \mathrm{L}^p(\Omega)$ if and only if v_{u_j} converges to u in $\mathrm{L}^p(\Omega)$.*

Theorem 5.2 *The functionals Ψ_ε Γ-converge as $\varepsilon \to 0$ to*

$$\Psi(u) = \begin{cases} \displaystyle\sum_{k=1}^{n}\int_\Omega \psi_k(\frac{\partial u}{\partial x_k})dx & u \in \mathrm{W}^{1,p}(\Omega;\mathbf{R}^m) \\ \\ +\infty & u \in \mathrm{L}^p(\Omega;\mathbf{R}^m)\setminus \mathrm{W}^{1,p}(\Omega;\mathbf{R}^m) \end{cases}$$

with respect to the convergence in $\mathrm{L}^p(\Omega)$ as in Definition 5.1.

Proof. Let $f : M^{m\times n} \longrightarrow [0,+\infty)$ be defined by

$$f(\xi) = n\sum_{k=1}^{n}\psi_k(\frac{\xi_k}{n})$$

where $\xi_k = \xi e_k$. If we consider μ as in Example 3.1(a), since f is convex, by formula (13) it follows that

$$f_{\mathrm{hom}}(\xi) = \frac{1}{n}f(n\xi) = \sum_{k=1}^{n}\psi_k(\xi_k).$$

In fact,the computation of (13) is trivial, since $u(x) = \sum_{k=1}^{n}\xi_k[x_k]$ is the unique function $u \in \mathrm{W}^{1,p}_{\mu,\mathrm{loc}}(\mathbf{R}^n;\mathbf{R}^m)$, up to translations, such that $u - \xi x$ is 1-periodic. By formula (11)

$$F_{\mathrm{hom}}(u,\Omega) = \begin{cases} \displaystyle\int_\Omega \sum_{k=1}^{n}\psi_k(\frac{\partial u}{\partial x_k})dx & u \in \mathrm{W}^{1,p}(\Omega;\mathbf{R}^m) \\ \\ +\infty & u \in \mathrm{L}^p(\Omega;\mathbf{R}^m)\setminus \mathrm{W}^{1,p}(\Omega;\mathbf{R}^m) \end{cases}$$

and $F_{\mathrm{hom}}(u,\Omega)=\Psi(u)$.

For all $U\subset\subset\Omega$ open set with $\partial U|=0$ and $\varepsilon>0$, let

$$F_\varepsilon(u,U)=\int_U f(\frac{dDu}{d\mu_\varepsilon})d\mu_\varepsilon\,,$$

and let $u_j\in A_{\varepsilon_j}$ converge to $u\in \mathrm{L}^p(\Omega)$. Then

$$\begin{aligned}
\liminf_{j\to+\infty}\Psi_{\varepsilon_j}(u_j) &= \liminf_{j\to+\infty}\varepsilon_j^n\sum_{k=1}^n\sum_{i\in I_{\varepsilon_j}}\psi_k\left(\frac{u_j(i+e_k)-u_j(i)}{\varepsilon_j}\right)\\
&\geq \liminf_{j\to+\infty}\sum_{k=1}^n\varepsilon_j\int_U\psi_k(\frac{1}{n}\frac{dDv_{u_j}}{d\mu_{\varepsilon_j}})d\mathcal{H}^{n-1}\\
&= \liminf_{j\to+\infty}\int_U f(\frac{dDv_{u_j}}{d\mu_{\varepsilon_j}})d\mu_{\varepsilon_j}\\
&= \liminf_{j\to+\infty}F_{\varepsilon_j}(v_{u_j},U)\\
&\geq F_{\mathrm{hom}}(u,U)
\end{aligned}$$

by formula (10) and the definition of Γ-convergence, so that

$$\liminf_{j\to+\infty}\Psi_{\varepsilon_j}(u_j)\geq\sup_{U\subset\subset\Omega}F_{\mathrm{hom}}(u,U)=\Psi(u)\,.$$

By the arbitrariness of u_j

$$\Gamma(\mathrm{L}^p)\text{-}\liminf_{\varepsilon\to0}\Psi_\varepsilon(u)\geq\Psi(u).$$

Conversely, suppose that $v_j\in \mathrm{W}^{1,p}_{\mu_{\varepsilon_j}}(\Omega;\mathbf{R}^m)$ converges to u in $\mathrm{L}^p(\Omega)$ and define

$$u_j(i)=\limsup_{\rho\to0^+}⨍_{B(0,\rho)\cap[0,\varepsilon_j)^n}v_j(x-\varepsilon_j i)\,dx \tag{20}$$

for all $i\in\mathbf{Z}^n\cap\frac{1}{\varepsilon}\Omega$. Note that if $i\in I_\varepsilon$ or $i-e_k\in I_\varepsilon$ for some k then the average in (20) is constant for ρ small enough.

By definition, u_j converges to $u\in\mathrm{L}^p(\Omega)$ and

$$\limsup_{j\to+\infty}\Psi_{\varepsilon_j}(u_j)\leq\limsup_{j\to+\infty}F_{\varepsilon_j}(v_j,\Omega);$$

there follows that

$$\Gamma(\mathrm{L}^p)\text{-}\limsup_{\varepsilon\to0}\Psi_\varepsilon(u)\leq\Gamma(\mathrm{L}^p)\text{-}\limsup_{\varepsilon\to0}F_\varepsilon(u,\Omega)=\Psi(u)\,,$$

so that

$$\Gamma(\mathrm{L}^p)\text{-}\lim_{\varepsilon\to0}\Psi_\varepsilon(u)=\Psi(u)\,,$$

and the proof is concluded. $\square$

6 Appendix: Sobolev inequalities in $\mathrm{W}^{1,p}_\mu$

In this appendix we include some results about Sobolev inequalities in the spaces $\mathrm{W}^{1,p}_\mu$. In particular, we will prove that the measures in Example 3.1 satisfy the Poincaré inequality in Definition 3.2(i).

Proposition 6.1 *Let μ be the measure in Example 3.1(b). Then for all $1 \le q \le n(np-2p+1)/(n-p)(n-1)$ (for any $q \ge 1$ if $p \ge n$) and for all $k \in \mathbf{N}$ there exists a constant $C(k)$ such that for all $u \in \mathrm{W}^{1,p}_\mu((0,k)^n)$ with $\int_{(0,k)^n} u\,dx = 0$ we have*

$$\Big(\int_{(0,k)^n} |u|^q dx\Big)^{1/q} \le C(k)\Big(\int_{(0,k)^n}\Big|\frac{dDu}{d\mu}\Big|^p d\mu\Big)^{1/p}. \tag{21}$$

Moreover, if $q = p$ then we can take $C(k) = c\,k$ with c a fixed constant.

Proof. If $n = 1$ then (21) follows from the Sobolev inequality for BV functions (see Remark 6.4). We will deal only with the case $p < n$ and $q > p$, which again is not a restriction. The other cases can be derived from this by applying Hölder's inequality.

We set $U = (0,k)^n$. We start by considering an inequality involving the median of a function rather than the mean. We recall that the set of the *medians of* u (in U), $\mathrm{med}(u)$, is the set of real numbers t such that

$$|U \cap \{u > t\}| \le \frac{1}{2}|U| \qquad \text{and} \qquad |U \cap \{u < t\}| \le \frac{1}{2}|U|.$$

Let $u \in \mathrm{W}^{1,p}_\mu(U)$. By the Poincaré inequality for BV functions, there exists a constant $c = c(U)$ such that for $u \in BV(U)$ and $t \in \mathrm{med}(u)$

$$\|u - t\|_{L^{n/(n-1)}(U)} \le c|Du|(U) \tag{22}$$

(see [17] Theorem 5.12.10). By a scaling argument it can be easily checked that c may be chosen independent of k. From now on, we denote c any constant which satisfies this property.

Let first $q \ge np/(n-1)$, and set $v = u|u|^{r-1}$ with $r > 1$. If $0 \in \mathrm{med}(u)$ then $0 \in \mathrm{med}(v)$; hence, by (22),

$$\|v\|_{L^{n/(n-1)}(U)} \le c|Dv|(U).$$

We then get, by Hölder's and Minkowski's inequalities,

$$\begin{aligned}\Big(\int_U |u|^{rn/(n-1)}dx\Big)^{(n-1)/n} \quad &\le \quad c\int_U |u|^{r-1}|\nabla u|dx \\ &\quad +c\int_{U\cap E} |u^+ - u^-|(|u^+|^{r-1} + |u^-|^{r-1})d\mathcal{H}^{n-1}\end{aligned}$$

$$\begin{aligned}
&\leq c\|\nabla u\|_p \left(\int_U |u|^{p'(r-1)} dx\right)^{1/p'} \\
&\quad + c\left(\int_{U\cap E} |u^+ - u^-|^p d\mathcal{H}^{n-1}\right)^{1/p} \\
&\times\left(\left(\int_{U\cap E} |u^+|^{p'(r-1)} d\mathcal{H}^{n-1}\right)^{1/p'} + \left(\int_{U\cap E} |u^-|^{p'(r-1)} d\mathcal{H}^{n-1}\right)^{1/p'}\right).
\end{aligned}$$

Let $q = rn/(n-1)$ and $\alpha = p'(r-1)$; then we can rewrite the estimate above as

$$\begin{aligned}
\left(\int_U |u|^q dx\right)^{r/q} &\leq c\|\nabla u\|_p \left(\int_U |u|^\alpha dx\right)^{(r-1)/\alpha} \\
&\quad + c\left(\int_{U\cap E} |u^+ - u^-|^p d\mathcal{H}^{n-1}\right)^{1/p} \\
&\times\left(\left(\int_{U\cap E} |u^+|^\alpha d\mathcal{H}^{n-1}\right)^{(r-1)/\alpha} + \left(\int_{U\cap E} |u^-|^\alpha d\mathcal{H}^{n-1}\right)^{(r-1)/\alpha}\right).
\end{aligned}$$

Interpreting $u^\pm$ as traces of Sobolev functions defined on each cube of $U \setminus E$, we have

$$\left(\int_{U\cap E} |u^\pm|^\alpha d\mathcal{H}^{n-1}\right)^{1/\alpha} \leq c\|u\|_{W^{1,p}(U\setminus E)} \tag{23}$$

for $p \leq \alpha \leq p(n-1)/(n-p)$ (see [1] Theorem 7.58). Hence,

$$\begin{aligned}
\|u\|_q^r &\leq c\|\nabla u\|_p \|u\|_\alpha^{r-1} \\
&\quad + c\left(\int_{U\cap E} |u^+ - u^-|^p d\mathcal{H}^{n-1}\right)^{1/p} \left(\|u\|_p^{r-1} + \|\nabla u\|_p^{r-1}\right).
\end{aligned}$$

Note that $\alpha < q \leq n(np-2p+1)/(n-1)(n-p)$. By Hölder's inequality

$$\|u\|_\alpha^{r-1} \leq \|u\|_q^{r-1}|U|^{(r-1)(\frac{1}{\alpha}-\frac{1}{q})} \qquad \text{and} \qquad \|u\|_p^{r-1} \leq \|u\|_q^{r-1}|U|^{(r-1)(\frac{1}{p}-\frac{1}{q})}.$$

If we denote $c_1 = |U|^{(r-1)(\frac{1}{\alpha}-\frac{1}{q})}$ and $c_2 = |U|^{(r-1)(\frac{1}{p}-\frac{1}{q})}$, we get

$$\begin{aligned}
\|u\|_q^r &\leq c_1 c\Big(\int_U \Big|\frac{dDu}{d\mu}\Big|^p d\mu\Big)^{1/p} \|u\|_q^{r-1} + c\Big(\int_U \Big|\frac{dDu}{d\mu}\Big|^p d\mu\Big)^{1/p} \\
&\quad \times\left(c_2\|u\|_q^{r-1} + \Big(\int_U \Big|\frac{dDu}{d\mu}\Big|^p d\mu\Big)^{(r-1)/p}\right) \qquad (24) \\
&\leq (c_1+c_2)c\Big(\int_U \Big|\frac{dDu}{d\mu}\Big|^p d\mu\Big)^{1/p} \|u\|_q^{r-1} + c\Big(\int_U \Big|\frac{dDu}{d\mu}\Big|^p d\mu\Big)^{r/p}.
\end{aligned}$$

By Young's inequality

$$\begin{aligned}
& (c_1+c_2)c\Big(\int_U\Big|\frac{dDu}{d\mu}\Big|^p d\mu\Big)^{1/p}\|u\|_q^{r-1} \\
\leq\ & \frac{1}{r}\left(\left(\frac{2(r-1)}{r}\right)^{(r-1)/r}(c_1+c_2)c\Big(\int_U\Big|\frac{dDu}{d\mu}\Big|^p d\mu\Big)^{1/p}\right)^r \\
& \quad +\frac{r-1}{r}\left(\|u\|_q^{r-1}\left(\frac{r}{2(r-1)}\right)^{(r-1)/r}\right)^{r/(r-1)} \\
=\ & \left(\frac{2(r-1)}{r}\right)^{r-1}\frac{((c_1+c_2)c)^r}{r}\Big(\int_U\Big|\frac{dDu}{d\mu}\Big|^p d\mu\Big)^{r/p}+\frac{1}{2}\|u\|_q^r\,,
\end{aligned}$$

so that, by (24),

$$\|u\|_q\leq c_4c\left(\int_U\Big|\frac{dDu}{d\mu}\Big|^p d\mu\right)^{1/p},$$

where $c_4=1+c_1+c_2$. In particular, we have that, for a general u and $t\in\mathrm{med}(u)$,

$$\|u-t\|_q\leq c_4c\Big(\int_U\Big|\frac{dDu}{d\mu}\Big|^p d\mu\Big)^{1/p}. \tag{25}$$

By Minkowski's inequality and (25)

$$\begin{aligned}
\|u\|_q &\leq \|u-t\|_q+|t||U|^{1/q} \\
&\leq c_4c\Big(\int_U\Big|\frac{dDu}{d\mu}\Big|^p d\mu\Big)^{1/p}+|t||U|^{1/q}\,.
\end{aligned} \tag{26}$$

Suppose in addition that $\int_U u dx=0$. We then can estimate

$$\begin{aligned}
|t| &= |\fint_U u dx-t|\leq\fint_U|u-t|dx\leq\left(\fint_U|u-t|^{n/(n-1)}dx\right)^{(n-1)/n} \\
&\leq \frac{c}{|U|^{(n-1)/n}}\int_U\Big|\frac{dDu}{d\mu}\Big|d\mu\leq c\frac{|U|^{1/p'}}{|U|^{(n-1)/n}}\Big(\int_U\Big|\frac{dDu}{d\mu}\Big|^p d\mu\Big)^{1/p} \\
&= c|U|^{(p-n)/np}\Big(\int_U\Big|\frac{dDu}{d\mu}\Big|^p d\mu\Big)^{1/p},
\end{aligned}$$

by (22) and Jensen's and Hölder's inequalities. Finally, by (26),

$$\Big(\int_U|u|^q dx\Big)^{1/q}\leq c\Big(c_4+|U|^{1/q+(p-n)/np}\Big)\Big(\int_U\Big|\frac{dDu}{d\mu}\Big|^p d\mu\Big)^{1/p}.$$

To conclude the proof set

$$\begin{aligned}
C(k) &= c\Big(c_4+|U|^{1/q+(p-n)/np}\Big) \\
&= c\Big(1+k^{n(r-1)(\frac{1}{\alpha}-\frac{1}{q})}+k^{n(r-1)(\frac{1}{p}-\frac{1}{q})}+k^{n/q+(p-n)/p}\Big).
\end{aligned} \tag{27}$$

In particular if $q = np/(n-1)$ we have $\alpha = r = p$ and $C(k) = c(1 + 3k^{(p-1)/p})$. If $q < np/(n-1)$ an application of Hölder's inequality yields that we can take $C(k) = ck^{((p-n)/p+(n/q))}$. We obtain the last statement of the proposition when $p = q$. $\square$

Remark 6.2 The last statement of the previous proposition proves the Poincaré inequality in Definition 3.2(i) for the measures μ in Example 3.1. In fact, the Poincaré inequality for the measures in Example 3.1(a) is a particular case of that for the measures in Example 3.1(b).

Remark 6.3 Proposition 6.1, and hence also p-homogenizability, can be proved for measures of the more general form

$$\mu(B) = \frac{1}{1 + \mathcal{H}^{n-1}(E \cap [0,1)^n)} (|B| + \mathcal{H}^{n-1}(B \cap E)),$$

provided that E is a 1-periodic closed set of σ-finite $n-1$-dimensional Hausdorff measure and that $[0,1]^n \setminus E$ has a finite number of connected component, each one with a Lipschitz boundary. The proof follows the same line, remarking that the particular form of E was used only in (23).

Remark 6.4 The validity of a Sobolev inequality for a general μ depends on the measure μ itself and p. In particular it always holds if $n = 1$ for all p and q, or if $p < n/(n-1)$ with $q = n/(n-1)$. In fact, in this case, by the Sobolev inequality for BV-functions and Hölder's inequality

$$\begin{aligned} \Big(\int_U |u|^{n/(n-1)} dx\Big)^{(n-1)/n} &\leq c|Du|(U) = c\int_U \Big|\frac{dDu}{d\mu}\Big| d\mu \\ &\leq c\Big(\int_U \Big|\frac{dDu}{d\mu}\Big|^p d\mu\Big)^{1/p} \mu(U)^{(p-1)/p}. \end{aligned}$$

Conversely, if $q > p \geq n/(n-1)$, take a 1-periodic function $u \in (BV_{\rm loc}(\mathbf{R}^n) \cap \mathrm{L}^p((0,1)^n) \setminus \mathrm{L}^q((0,1)^n))$, and set $\mu = |Du|$. Clearly $|dDu/d\mu| = 1$, so that $u \in \mathrm{W}^{1,p}_\mu(U)$ for all subsets U of $\mathbf{R}^n$, but we have $\int_U |u|^q dx = +\infty$ for each U sufficiently large.

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