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# A MODEL FOR THE QUASI-STATIC GROWTH OF BRITTLE FRACTURES BASED ON LOCAL MINIMIZATION

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ABSTRACT. We study a variant of the variational model for the quasi-static growth of brittle fractures proposed by Francfort and Marigo in [9]. The main feature of our model is that, in the discrete-time formulation, in each step we do not consider absolute minimizers of the energy, but, in a sense, we look for local minimizers which are sufficiently close to the approximate solution obtained in the previous step. This is done by introducing in the variational problem an additional term which penalizes the $L^2$-distance between the approximate solutions at two consecutive times. We study the continuous-time version of this model, obtained by passing to the limit as the time step tends to zero, and show that it satisfies (for almost every time) some minimality conditions which are slightly different from those considered in [9] and [8], but are still enough to prove (under suitable regularity assumptions on the crack path) that the classical Griffith's criterion holds at the crack tips. We prove also that, if no initial crack is present and if the data of the problem are sufficiently smooth, no crack will develop in this model, provided the penalization term is large enough.

**Keywords:** variational models, energy minimization, free-discontinuity problems, crack propagation, quasi-static evolution, brittle fractures, Griffith's criterion, stress intensity factor.

**2000 Mathematics Subject Classification:** 35R35, 74R10, 49Q10, 35A35, 35B30, 35J25.

## 1. INTRODUCTION

In this paper we present a variational model for the irreversible quasi-static growth of brittle fractures in the two-dimensional antiplane case, subject to a time dependent boundary displacement. The reference configuration is a bounded Lipschitz domain $\Omega$ of the plane, whose boundary $\partial\Omega$ is divided into two disjoint locally connected subsets $\partial_D\Omega$ and $\partial_N\Omega$, where we prescribe a nonhomogeneous Dirichlet condition and a homogeneous Neumann condition, respectively. According to Griffith's theory, the energy considered in the model is given by

$$E(u,K) := \int_{\Omega\setminus K} |\nabla u|^2\,dx + \mathcal{H}^1(K)\,, \tag{1.1}$$

where the compact set $K \subset \overline{\Omega}$ represents the crack in the reference configuration, the scalar function $u$ represents the displacement orthogonal to the plane of $\Omega$, and $\mathcal{H}^1$ is the one-dimensional Hausdorff measure.

For technical reasons, due to the behaviour of the solutions of Neumann problems in domains with cracks (see [3], [8]), we impose an a priori bound on the number of connected components of $K$.

Given a time dependent energy functional $\mathcal{F}(z,t)$, defined for $z$ in a Banach space $Z$ and for $t \in [0,T]$, a quasi-static evolution corresponding to $\mathcal{F}$ is a function $t \mapsto z(t)$ which satisfies the equality $\nabla_z\mathcal{F}(z(t),t) = 0$ for every $t \in [0,T]$. A standard way to obtain this function is by singular perturbation. We consider the $\varepsilon$-gradient flow

$$\varepsilon\dot{z}^\varepsilon(t) + \nabla_z\mathcal{F}(z^\varepsilon(t),t) = 0\,, \tag{1.2}$$

starting from a local minimizer $z_0$ of $\mathcal{F}(\cdot,0)$. Under suitable assumptions on $\mathcal{F}$ the solutions $z^\varepsilon(t)$ converge, as $\varepsilon\to 0$, to a function $z(t)$ which satisfies the equation

$$\nabla_z \mathcal{F}(z(t),t)=0\,; \tag{1.3}$$

moreover, due to the choice of the sign in (1.2), it turns out that $z(t)$ is a local minimizer of $\mathcal{F}(\cdot,t)$ for a generic $t\in[0,T]$.

Heuristically, the potential well of $\mathcal{F}(\cdot,0)$ corresponding to $z_0$ will be slightly deformed for $t$ small, and the solution $z(t)$ of (1.3) obtained by this approximation method follows the local minimizer of the deformed potential well. It may happen that for some critical value $t_0$ this potential well disappears, and for this special time $z(t_0)$ will be only a critical point of $\mathcal{F}(\cdot,t_0)$; in general, in this case $z(t)$ is discontinuous at $t_0$ and jumps to another point $z(t_0+)$, which is a local minimizer of $\mathcal{F}(\cdot,t_0)$; the evolution continues then in this new potential well. By a simple rescaling argument we see that $z(t_0+)$ can also be obtained from $z(t_0)$ by solving the gradient flow (1.2) with $\varepsilon=1$ and with initial conditions close to $z(t_0)$, and taking then the limit as $t\to+\infty$.

We want to adapt these ideas to the case of the energy (1.1) with the time dependent Dirichlet boundary condition $u(t)=g(t)$ on $\partial_D\Omega\backslash K(t)$ and with initial condition $(u_0,K_0)$. We assume that $g(t)$ is the trace on $\partial_D\Omega$ of a function of $H^1(\Omega)$, still denoted by $g(t)$, and that the map $t\mapsto g(t)$ belongs to $AC([0,T];H^1(\Omega))\cap L^\infty([0,T];L^\infty(\Omega))$.

Since we are looking for equilibria, it is natural to assume that $u_0$ minimizes

$$\int_{\Omega\backslash K_0}|\nabla u|^2\,dx$$

among all functions $u\in H^1(\Omega\backslash K_0)$ with $u=g(0)$ on $\partial_D\Omega\backslash K_0$.

The main difficulty in the definition of the $\varepsilon$-gradient flow for (1.1) is that this energy it is neither differentiable nor convex, due to the presence of the term $\mathcal{H}^1(K)$, and therefore we can not rely on a notion of (sub)differential. Following [2] and [6], we define the $\varepsilon$-gradient flow of the energy (1.1) using an approximation by a discrete-time process based on an implicit scheme.

Let us fix an integer $m\geq 1$ and let $\mathcal{K}_m(\overline{\Omega})$ be the set of all compact subsets $K$ of $\overline{\Omega}$ with at most $m$ connected components and with $\mathcal{H}^1(K)<+\infty$. We consider also the set $\mathcal{K}'_m(\overline{\Omega})$ of all $K\in\mathcal{K}_m(\overline{\Omega})$ without isolated points, and we assume that the initial crack $K_0$ belongs to $\mathcal{K}'_m(\overline{\Omega})$.

Given the time step $\delta>0$, for any integer $i\geq 0$ let $t_i^\delta:=i\delta$, and, for $t_i^\delta\leq T$, let $g_i^\delta:=g(t_i^\delta)$. We define $(u_i^{\varepsilon,\delta},K_i^{\varepsilon,\delta})$ inductively as follows: $(u_0^{\varepsilon,\delta},K_0^{\varepsilon,\delta}):=(u_0,K_0)$; for $i\geq 1$ we define $(u_i^{\varepsilon,\delta},K_i^{\varepsilon,\delta})$ as a solution of the minimum problem

$$\min_{(u,K)}\big\{E(u,K)+\frac{\varepsilon}{\delta}\,\|u-u_{i-1}^{\varepsilon,\delta}\|^2\big\}\,, \tag{1.4}$$

where $\|\cdot\|$ denotes the $L^2$-norm in $\Omega$, and the minimum is taken over all pairs $(u,K)$ such that $K\in\mathcal{K}'_m(\overline{\Omega})$, $K\supset K_{i-1}^{\varepsilon,\delta}$, $u\in H^1(\Omega\backslash K)$, $u=g_i^\delta$ on $\partial_D\Omega\backslash K$. The constraint $K\supset K_{i-1}^{\varepsilon,\delta}$ reflects the irreversibility of the fracture process.

We define now the step functions $u^{\varepsilon,\delta}(t)$ and $K^{\varepsilon,\delta}(t)$ on $[0,T]$ by setting $u^{\varepsilon,\delta}(t):=u_i^{\varepsilon,\delta}$ and $K^{\varepsilon,\delta}(t):=K_i^{\varepsilon,\delta}$, for $t_i^\delta\leq t<t_{i+1}^\delta$.

The limit $(u^\varepsilon(t),K^\varepsilon(t))$ of $(u^{\varepsilon,\delta}(t),K^{\varepsilon,\delta}(t))$ along a suitable sequence $\delta_k\to 0$ is by definition the $\varepsilon$-gradient flow for the energy (1.1). In order to obtain the quasi-static evolution for this energy by the singular perturbation approach, we should consider now the limit of $(u^\varepsilon(t),K^\varepsilon(t))$ as $\varepsilon\to 0$ along a suitable sequence. This can be done, but we are not able to prove satisfactory properties of the limit evolution process $(u(t),K(t))$.

Therefore we prefer to consider a variant of the singular perturbation method. We study the limit of $(u^{\varepsilon,\delta}(t),K^{\varepsilon,\delta}(t))$ as $\varepsilon$ and $\delta$ tend to zero simultaneously, with $\varepsilon$ proportional to $\delta$. In particular, given $\lambda>0$, we assume that the coefficient $\varepsilon/\delta$ which appears in (1.4) is equal to $\lambda$. As $\varepsilon = \lambda\delta$, we can use the simplified notation $u_i^\delta := u_i^{\varepsilon,\delta}$ and $K_i^\delta := K_i^{\varepsilon,\delta}$. Note that $(u_0^\delta, K_0^\delta) := (u_0, K_0)$ and for every $i \geq 1$ $(u_i^\delta, K_i^\delta)$ is a solution of the minimum problem

$$\min_{(u,K)} \left\{ E(u,K) + \lambda \|u - u_{i-1}^\delta\|^2 \right\}, \tag{1.5}$$

where the minimum is taken over all pairs $(u,K)$ such that $K \in \mathcal{K}'_m(\overline{\Omega})$, $K \supset K_{i-1}^\delta$, $u \in H^1(\Omega \backslash K)$, $u = g_i^\delta$ on $\partial_D \Omega \backslash K$.

The term containing $\lambda$ is the main difference with respect to the discrete-time formulation of the model proposed by Francfort and Marigo in [9], which corresponds to the case $\lambda = 0$. The fact that $\lambda$ is greater than 0 penalizes the $L^2$-distance between $u_i^\delta$ and $u_{i-1}^\delta$ and avoids some unnatural jumps which may occur in the continuous-time formulation for $\lambda = 0$. In a sense, when $\lambda$ is large, local minimizers (close to $u_{i-1}^\delta$) are preferred to global minimizers.

We introduce as before the piecewise constant interpolation $(u^\delta(t), K^\delta(t))$ defined by $u^\delta(t) := u_i^\delta$ and $K^\delta(t) := K_i^\delta$, for $t_i^\delta \leq t < t_{i+1}^\delta$.

We prove (Lemma 4.5) that there exists a left-continuous increasing function $K : [0,T] \to \mathcal{K}_m(\overline{\Omega})$ such that for every $t \in [0,T]$, with

$$K(t) = K(t+) := \textstyle\bigcap_{s>t} K(s),$$

$K^\delta(t)$ converges to $K(t)$ as $\delta \to 0$ along a suitable sequence independent of $t$. In the rest of this discussion we always refer to this sequence when we write $\delta \to 0$.

For every $t \in [0,T]$ let $u(t)$ be a minimizer of

$$\int_{\Omega\backslash K(t)} |\nabla u|^2 \, dx \tag{1.6}$$

among all functions $u \in H^1(\Omega \backslash K(t))$ with $u = g(t)$ on $\partial_D \Omega \backslash K(t)$. We prove (Lemma 4.8) that

$$E(u(t), K(t)) \leq E(u,K) + \lambda \|u - u(t)\|^2 \tag{1.7}$$

for every $0 < t \leq T$, for every $K \in \mathcal{K}_m(\overline{\Omega})$ with $K \supset K(t)$, and for every $u \in H^1(\Omega \backslash K)$ with $u = g(t)$ on $\partial_D \Omega \backslash K$. Moreover we prove (Lemma 4.9) that

$$E(u(t), K(t)) - E(u(s), K(s)) \leq 2 \int_s^t \int_{\Omega\backslash K(\tau)} \nabla u(\tau) \nabla \dot{g}(\tau) \, dx \, d\tau, \tag{1.8}$$

where $\dot{g}(t)$ is the time derivative of the function $g(t)$.

This inequality shows that $t \mapsto E(u(t), K(t))$ is a function with bounded variation and that

$$\frac{d}{dt} E(u(t), K(t)) \leq 2 \int_{\Omega\backslash K(t)} \nabla u(t) \nabla \dot{g}(t) \, dx$$

for a.e. $t \in [0,T]$ (Remark 3.5), and this leads to the existence of a function $\omega(s,t)$, defined for $0 \leq s < t \leq T$, with

$$\lim_{s \to t-} \frac{\omega(s,t)}{t-s} = 0 \qquad \text{for every } t \in (0,T),$$

such that for a.e. $t \in [0,T]$ and every $s < t$ we have

$$E(u(t), K(t)) \leq E(u, K(s)) + \omega(s,t) \tag{1.9}$$

for every $u \in H^1(\Omega \backslash K(s))$ with $u = g(t)$ on $\partial_D \Omega \backslash K(s)$ (Proposition 3.6 and Remark 3.7).

The minimality properties (1.7) and (1.9) are used in Section 6 to prove that the classical Griffith's criterion holds at the crack tips for a.e. $t \in [0,T]$, provided $K(t)$ satisfies suitable regularity conditions.

The fact that $\lambda > 0$ in (1.5) leads to an additional condition on the possible discontinuites of $(u(t), K(t))$. For every $t \in [0,T)$, for every $K \in \mathcal{K}_m(\overline{\Omega})$, and for every $u \in H^1(\Omega \backslash K)$ with $u = g(t)$ on $\partial_D \Omega \backslash K$, we determine a set $\mathcal{R}^t(u,K)$ (Definitions 3.1

and 3.2), depending on the boundary condition $g(\cdot)$, such that for every $t \in [0,T)$ we have $(u(t+), K(t+)) \in \mathcal{R}^t(u(t), K(t))$, where $u(t+)$ is a minimizer of (1.6) with $K(t)$ replaced by $K(t+)$ (Lemmas 4.10 and 4.11). In Section 5 we show that, if $\Omega$ and $g(t)$ are sufficiently regular and no initial crack is present, i.e., $K_0 = \emptyset$, then, for $\lambda$ large enough, no crack will appear, i.e., $K(t) = \emptyset$ for every $t \in [0,T]$, and $\mathcal{R}^t(u(t), \emptyset) = \{(u(t), \emptyset)\}$. Note that the model by Francfort and Marigo [9], based on global minimization, gives, in general, a different result.

## 2. NOTATION AND PRELIMINARIES

Throughout the paper $\Omega$ is a fixed bounded connected open subset of $\mathbb{R}^2$ with Lipschitz boundary. Let $\mathcal{K}(\overline{\Omega})$ be the set of all compact subsets of $\overline{\Omega}$. Given an integer $m \geq 1$, let $\mathcal{K}_m(\overline{\Omega})$ be the set of all compact subsets $K$ of $\overline{\Omega}$ with at most $m$ connected components and such that $\mathcal{H}^1(K) < +\infty$. We shall consider also the set $\mathcal{K}'_m(\overline{\Omega})$ of all $K \in \mathcal{K}_m(\overline{\Omega})$ without isolated points.

We recall that the *Hausdorff distance* between $K_1$, $K_2 \in \mathcal{K}(\overline{\Omega})$ is defined by

$$d_H(K_1, K_2) := \max\Big\{ \sup_{x \in K_1} \operatorname{dist}(x, K_2), \sup_{y \in K_2} \operatorname{dist}(y, K_1) \Big\},$$

with the conventions $\operatorname{dist}(x, \emptyset) = \operatorname{diam}(\Omega)$ and $\sup \emptyset = 0$, so that $d_H(\emptyset, K) = 0$ if $K = \emptyset$, and $d_H(\emptyset, K) = \operatorname{diam}(\Omega)$ if $K \neq \emptyset$. We say that $K_n \to K$ in the Hausdorff metric if $d_H(K_n, K) \to 0$. The following compactness theorem is well-known (see, e.g., [13, Blaschke's Selection Theorem]).

**Theorem 2.1.** *The metric space $(\mathcal{K}(\overline{\Omega}), d_H)$ is compact.*

It is well-known that, in general, the Hausdorff measure is not lower semicontinuous on $\mathcal{K}(\overline{\Omega})$ with respect to the convergence in the Hausdorff metric. The following result, which is a consequence of the Gołąb theorem, shows that on the class $\mathcal{K}_m(\overline{\Omega})$ the Hausdorff measure is lower semicontinuous. We refer to [8, Corollary 3.3] for a proof.

**Theorem 2.2.** *Let $K_n$ be compact sets in $\overline{\Omega}$ with a uniformly bounded number of connected components. If $K_n$ converge to $K$ in the Hausdorff metric, then*

$$\mathcal{H}^1(K \cap U) \leq \liminf_{n \to \infty} \mathcal{H}^1(K_n \cap U)$$

*for every open set $U \subset \mathbb{R}^2$.*

In the rest of the paper $\partial\Omega$ is the union of two (possibly empty) disjoint sets $\partial_D\Omega$ and $\partial_N\Omega$, with a finite number of connected components, on (part of) which we impose a nonhomogeneous Dirichlet boundary condition and a homogeneous Neumann boundary condition, respectively.

Given a function $u \in H^1(\Omega \backslash K)$ for some $K \in \mathcal{K}(\overline{\Omega})$, we always extend $u$ and $\nabla u$ to $\Omega$ by setting $u = 0$ and $\nabla u = 0$ a.e. on $K$. Note that, however, $\nabla u$ is the distributional gradient of $u$ only in $\Omega \backslash K$, and, in general, $\nabla u$ does not coincide in $\Omega$ with the gradient of an extension of $u$.

For every $K \in \mathcal{K}(\overline{\Omega})$ we consider the space

$$H^1_0(\Omega \backslash K, \partial_D\Omega \backslash K) := \{u \in H^1(\Omega \backslash K) : u = 0 \text{ on } \partial_D\Omega \backslash K\},$$

where the equality on the boundary is intended in the sense of traces. The following definition reformulates in our particular case a general notion of convergence studied by Mosco in [12].

**Definition 2.3.** *Let $K_n, K \in \mathcal{K}(\overline{\Omega})$. We say that the spaces $H^1_0(\Omega \backslash K_n, \partial_D\Omega \backslash K_n)$ converge to the space $H^1_0(\Omega \backslash K, \partial_D\Omega \backslash K)$ in the sense of Mosco if the following properties hold:*

(M$_1$) *for every $u \in H^1_0(\Omega \backslash K, \partial_D\Omega \backslash K)$ there exists $u_n \in H^1_0(\Omega \backslash K_n, \partial_D\Omega \backslash K_n)$ such that $u_n \to u$ strongly in $L^2(\Omega)$ and $\nabla u_n \to \nabla u$ strongly in $L^2(\Omega; \mathbb{R}^2)$;*

($\mathrm{M}_2$) *if* $u_n \in H^1_0(\Omega \backslash K_n, \partial_D \Omega \backslash K_n)$ *for every* $n$ *and* $\liminf_n \|u_n\|_{H^1(\Omega \backslash K_n)} < +\infty$, *then there exist a subsequence* $u_{n_k}$ *and a function* $u \in H^1_0(\Omega \backslash K, \partial_D \Omega \backslash K)$ *such that* $u_{n_k} \rightharpoonup u$ *weakly in* $L^2(\Omega)$ *and* $\nabla u_{n_k} \rightharpoonup \nabla u$ *weakly in* $L^2(\Omega; \mathbb{R}^2)$.

The following theorem shows the connection between Mosco convergence of the spaces $H^1_0(\Omega \backslash K_n, \partial_D \Omega \backslash K_n)$ and convergence in the Hausdorff metric of the sets $K_n$.

**Theorem 2.4.** *Let* $K_n, K$ *be compact sets in* $\overline{\Omega}$ *with a uniformly bounded number of connected components, such that* $K_n \to K$ *in the Hausdorff metric and* $\mathrm{meas}(K_n) \to \mathrm{meas}(K)$. *Then* $H^1_0(\Omega \backslash K_n, \partial_D \Omega \backslash K_n)$ *converges to* $H^1_0(\Omega \backslash K, \partial_D \Omega \backslash K)$ *in the sense of Mosco.*

*Proof.* Under these hypotheses Bucur and Varchon proved in [4] the Mosco convergence of $H^1(\Omega \backslash K_n)$ to $H^1(\Omega \backslash K)$. The extension to the case when boundary conditions are imposed is due to Chambolle [5] (see also [7, Theorem 6.3]). □

Throughout the paper $\lambda$ is a fixed constant, with $\lambda > 0$. We use the notation $(\cdot|\cdot)$ and $\|\cdot\|$ for the scalar product and the norm in $L^2(\Omega)$ or in $L^2(\Omega; \mathbb{R}^2)$, according to the context. We have often to minimize energies of the form (1.1) among pairs $(u, K)$, where $K \in \mathcal{K}_m(\overline{\Omega})$ and $u \in H^1(\Omega \backslash K)$, with a prescribed boundary condition $u = \phi$ on $\partial_D \Omega \backslash K$. We prefer to minimize first with respect to $u$ and then with respect to $K$. This leads to the following definitions.

**Definition 2.5.** *If* $K \in \mathcal{K}_m(\overline{\Omega})$ *for some* $m \geq 1$, $\phi \in H^1(\Omega \backslash K) \cap L^\infty(\Omega)$, *and* $w \in L^2(\Omega)$, *we define*

$$\mathcal{E}(\phi, K) := \min_{v \in \mathcal{V}(\phi, K)} \{\|\nabla v\|^2 + \mathcal{H}^1(K)\}, \tag{2.1}$$

$$\mathcal{E}_\lambda(\phi, K, w) := \min_{v \in \mathcal{V}(\phi, K)} \{\|\nabla v\|^2 + \mathcal{H}^1(K) + \lambda \|v - w\|^2\}, \tag{2.2}$$

*where*

$$\mathcal{V}(\phi, K) := \{v \in H^1(\Omega \backslash K) : v = \phi \quad \text{on } \partial_D \Omega \backslash K\}. \tag{2.3}$$

**Remark 2.6.** By minimality, a solution $u$ of (2.1) satisfies the inequality $\|\nabla u\|^2 \leq \|\nabla \phi\|^2$. A truncation argument shows that there exists a minimizing sequence $u_n$ of (2.1) such that $\|u_n\|_\infty \leq \|\phi\|_\infty$, where $\|\cdot\|_\infty$ denotes the norm in $L^\infty(\Omega)$. By the direct method of the calculus of variations we can then prove that there exists a solution $u$ of (2.1) with $\|u\|_\infty \leq \|\phi\|_\infty$. It is easy to see that the solution is unique on the connected components of $\Omega \backslash K$ whose boundaries meet $\partial_D \Omega \backslash K$, while on the other connected components it is given by an arbitrary constant. This shows that two solutions have the same gradient. If $u$ is a solution of the minimum problem (2.1), then $\mathcal{E}_\lambda(\phi, K, u) = \mathcal{E}(\phi, K)$.

**Remark 2.7.** By minimality, the solution $u$ of (2.2) satisfies $\|\nabla u\|^2 + \lambda \|u - w\|^2 \leq \|\nabla \phi\|^2 + \lambda \|\phi - w\|^2$. If $w$ belongs to $L^\infty(\Omega)$, then an easy truncation argument shows that $u$ belongs to $L^\infty(\Omega)$ and $\|u\|_\infty \leq \max\{\|\phi\|_\infty, \|w\|_\infty\}$.

**Remark 2.8.** If $w$ is constant on a connected component $U$ of $\Omega \backslash K$ whose boundary does not meet $\partial_D \Omega \backslash K$, then the minimizer $u$ of (2.2) coincides with $w$ on $U$. Therefore the value $\mathcal{E}_\lambda(\phi, K, w)$ does not depend on the constant value of $w$ on $U$.

**Remark 2.9.** A function $u$ is a minimizer of (2.1) if and only if

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \backslash K, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial_N \Omega \cup K, \\ u = \phi & \text{on } \partial_D \Omega \backslash K, \end{cases} \tag{2.4}$$

i.e., $u$ satisfies the following conditions

$$\begin{cases} u - \phi \in H^1_0(\Omega \backslash K, \partial_D \Omega \backslash K), \\ (\nabla u | \nabla v) = 0 \qquad \forall\, v \in H^1_0(\Omega \backslash K, \partial_D \Omega \backslash K). \end{cases} \tag{2.5}$$

Similarly, $u$ is the minimizer of (2.2) if and only if $u$ is the solution of the problem

$$\begin{cases} \Delta u = \lambda(u-w) & \text{in } \Omega\backslash K\,, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial_N\Omega \cup K\,, \\ u = \phi & \text{on } \partial_D\Omega\backslash K\,, \end{cases} \tag{2.6}$$

i.e., $u$ satisfies the following conditions

$$\begin{cases} u - \phi \in H^1_0(\Omega\backslash K, \partial_D\Omega\backslash K)\,, \\ (\nabla u|\nabla v) + \lambda(u-w|v) = 0 \qquad \forall\, v \in H^1_0(\Omega\backslash K, \partial_D\Omega\backslash K)\,. \end{cases} \tag{2.7}$$

This implies that, if the minimizer $u$ of (2.2) is equal to $w$, then $u$ is also a minimizer of (2.1).

We consider now the stability of the solutions to problems (2.1) and (2.2) when $\phi$, $K$, and $w$ vary.

**Theorem 2.10.** *Let $m \geq 1$, and let $\phi_n, \phi \in H^1(\Omega) \cap L^\infty(\Omega)$, $K_n, K \in \mathcal{K}_m(\overline{\Omega})$, and $w_n, w \in L^2(\Omega)$. Let $u_n$ and $u$ be the solutions of the minimum problems (2.2) which define $\mathcal{E}_\lambda(\phi_n, K_n, w_n)$ and $\mathcal{E}_\lambda(\phi, K, w)$, respectively. Assume that $\phi_n \rightharpoonup \phi$ weakly in $H^1(\Omega)$, $K_n \to K$ in the Hausdorff metric, and $w_n \rightharpoonup w$ weakly in $L^2(\Omega)$. Then $u_n \rightharpoonup u$ weakly in $L^2(\Omega)$, $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^2(\Omega;\mathbb{R}^2)$, and*

$$\mathcal{E}_\lambda(\phi, K, w) \leq \liminf_{h\to\infty} \mathcal{E}_\lambda(\phi_n, K_n, w_n)\,. \tag{2.8}$$

*If $\phi_n$ and $w_n$ are uniformly bounded in $L^\infty(\Omega)$ and $\phi_n \to \phi$ strongly in $H^1(\Omega)$, then $u_n \to u$ strongly in $L^2(\Omega)$ and $\nabla u_n \to \nabla u$ strongly in $L^2(\Omega;\mathbb{R}^2)$.*

*Proof.* By Remark 2.7 the norms $\|u_n\|_{H^1(\Omega\backslash K_n)}$ are uniformly bounded. By Theorem 2.4 there exists $u^* \in H^1(\Omega\backslash K)$, with $u^* = \phi$ on $\partial_D\Omega\backslash K$, such that, up to a subsequence, $u_n \rightharpoonup u^*$ weakly in $L^2(\Omega)$ and $\nabla u_n \rightharpoonup \nabla u^*$ weakly in $L^2(\Omega;\mathbb{R}^2)$. By (2.7) we have

$$(\nabla u_n|\nabla v_n) + \lambda(u_n - w_n|v_n) = 0 \tag{2.9}$$

for every $v_n \in H^1_0(\Omega\backslash K_n, \partial_D\Omega\backslash K_n)$. If $v \in H^1_0(\Omega\backslash K, \partial_D\Omega\backslash K)$, by Theorem 2.4 there exist $v_n \in H^1_0(\Omega\backslash K_n, \partial_D\Omega\backslash K_n)$ such that $v_n \to v$ strongly in $L^2(\Omega)$ and $\nabla v_n \to \nabla v$ strongly in $L^2(\Omega;\mathbb{R}^2)$, and passing to the limit in (2.9) we obtain that $u^*$ is a solution of (2.7). By uniqueness, $u^* = u$, and the convergence holds for the whole sequence. Inequality (2.8) follows now by lower semicontinuity (Theorem 2.2).

Assume that $\phi_n$ and $w_n$ are uniformly bounded in $L^\infty(\Omega)$. Then the same is true for the solutions $u_n$ (Remark 2.7). To prove the strong convergence in $L^2(\Omega)$ of $u_n$, let $U \subset\subset \Omega\backslash K$ be an open set with boundary of class $C^1$. As $K_n \to K$ in the Hausdorff metric, we have $U \subset\subset \Omega\backslash K_n$ for $n$ large enough. Since $u_n$ is bounded in $H^1(U)$ uniformly with respect to $n$ (Remark 2.7), by the Rellich theorem $u_n \to u$ strongly in $L^2(U)$. As the functions $u_n$ are uniformly bounded in $L^\infty(\Omega)$, the norms $\|u_n\|_{L^2(\Omega\backslash U)}$ can be made arbitrarily small by taking $U$ arbitrarily close to $\Omega\backslash K$. Therefore $u_n \to u$ strongly in $L^2(\Omega)$.

If, in addition, $\phi_n \to \phi$ strongly in $H^1(\Omega)$, taking $v_n := u_n - \phi_n$ as test function in (2.9) we can easily prove that $\|\nabla u_n\| \to \|\nabla u\|$, which implies the strong convergence of the gradients. ☐

The following corollary will be used in Section 4.

**Corollary 2.11.** *Let $m \geq 1$, and let $\phi_n, \phi \in H^1(\Omega) \cap L^\infty(\Omega)$, $K_n, K \in \mathcal{K}_m(\overline{\Omega})$, and $w_n \in L^\infty(\Omega)$. Let $u_n$ be the solutions of the minimum problems (2.2) which define $\mathcal{E}_\lambda(\phi_n, K_n, w_n)$, and let $u$ be a solution of the minimum problem (2.1) which defines $\mathcal{E}(\phi, K)$. Assume that $\phi_n$ and $w_n$ are uniformly bounded in $L^\infty(\Omega)$, and that $\phi_n \to \phi$ strongly in $H^1(\Omega)$, $K_n \to K$ in the Hausdorff metric, and $u_n - w_n \to 0$ strongly in $L^2(\Omega)$. Then $\nabla u_n \to \nabla u$ strongly* *in* $L^2(\Omega;\mathbb{R}^2)$. *Moreover there exist a subsequence* $u_{n_k}$ *of* $u_n$ *and a solution* $u^*$ *of the minimum problem (2.1) which defines* $\mathcal{E}(\phi,K)$ *(possibly different from* $u$*) such that* $u_{n_k}\to u^*$ *strongly in* $L^2(\Omega)$.

*Proof.* As $w_n$ is bounded in $L^\infty(\Omega)$, there exists a subsequence $w_{n_k}$ which converges weakly in $L^2(\Omega)$ to a function $w$. Let $u^*$ be the solution of the minimum problem (2.2) which defines $\mathcal{E}_\lambda(\phi,K,w)$. By Theorem 2.10 we have $u_{n_k}\to u^*$ strongly in $L^2(\Omega)$ and $\nabla u_{n_k}\to\nabla u^*$ strongly in $L^2(\Omega;\mathbb{R}^2)$. As $u_n-w_n\to 0$ strongly in $L^2(\Omega)$, the functions $u^*$ and $w$ are equal. By Remark 2.9 this implies that $u^*$ is a solution of the minimum problem (2.1) which defines $\mathcal{E}(\phi,K)$, and by the uniqueness of the gradients we have $\nabla u=\nabla u^*$ a.e. in $\Omega$. Since we can repeat this argument for an arbitrary subsequence, we conclude that the whole sequence $\nabla u_n$ converges to $\nabla u$ strongly in $L^2(\Omega;\mathbb{R}^2)$. ☐

## 3. IRREVERSIBLE QUASI-STATIC EVOLUTION

In this section we define a continuous-time evolution of a cracked body by investigating the properties of the limits of the discrete-time evolution described in the introduction.

Let us fix the boundary displacement $g\in AC([0,T];H^1(\Omega))\cap L^\infty([0,T];L^\infty(\Omega))$ and an integer $m\geq 1$. Given an initial crack $K_0\in\mathcal{K}'_m(\overline{\Omega})$, we shall construct an increasing function $K\colon[0,T]\to\mathcal{K}_m(\overline{\Omega})$ satisfying suitable minimality conditions. We define

$$K(t-):=\mathrm{cl}\big(\textstyle\bigcup_{s<t}K(s)\big)\qquad\text{for } 0<t\leq T\,,\tag{3.1}$$

$$K(t+):=\textstyle\bigcap_{s>t}K(s)\qquad\text{for } 0\leq t<T\,,\tag{3.2}$$

where cl denotes the closure. We say that $t\mapsto K(t)$ is left-continuous if $K(t-)=K(t)$ for every $t\in(0,T]$. It is easy to see that

$$K(t-)\subset K(t)\subset K(t+)\qquad\text{for } 0<t<T\,,\tag{3.3}$$

$$K(t-)=\mathrm{cl}\big(\textstyle\bigcup_{s<t}K(s-)\big)\qquad\text{for } 0<t\leq T\,,\tag{3.4}$$

$$K(t+)=\textstyle\bigcap_{s>t}K(s+)\qquad\text{for } 0\leq t<T\,.\tag{3.5}$$

Let $\Theta$ be the set of points $t\in(0,T)$ such that $K(t-)=K(t+)$. By [8, Theorem 6.1] the set $[0,T]\backslash\Theta$ is at most countable.

For every $t\in[0,T]$ let $u(t)$ (resp. $u(t-)$, $u(t+)$) be a solution of the minimum problem (2.1) corresponding to $\phi=g(t)$ and $K=K(t)$ (resp. $K=K(t-)$, $K=K(t+)$). By Remark 2.6 $\|\nabla u(t)\|$ is bounded uniformly with respect to $t$. By [8, Theorem 5.1] and by (3.1) and (3.2),

$$\text{for } 0<t\leq T\qquad \nabla u(s)\to\nabla u(t-)\qquad\text{strongly in } L^2(\Omega;\mathbb{R}^2)\text{ as } s\to t-\,,\tag{3.6}$$

$$\text{for } 0\leq t<T\qquad \nabla u(s)\to\nabla u(t+)\qquad\text{strongly in } L^2(\Omega;\mathbb{R}^2)\text{ as } s\to t+\,.\tag{3.7}$$

This implies in particular that $t\mapsto\nabla u(t)$ is continuous from $[0,T]$ into $L^2(\Omega;\mathbb{R}^2)$ at every point $t\in\Theta$. Therefore the first estimate of Remark 2.6 implies that

$$t\mapsto\nabla u(t)\quad\text{belongs to}\quad L^\infty([0,T];L^2(\Omega;\mathbb{R}^2))\,.\tag{3.8}$$

Although the boundary displacement $g(t)$ is continuous with respect to $t$, the continuous-time evolution that we shall obtain as limit of the discrete-time evolutions may exhibit some jump discontinuities of the pair $(u(t),K(t))$. Given a time step $\delta>0$, the approximation procedure considered in the introduction uses sequences $(v_i^\delta,H_i^\delta)$ with the property that, for every $i\geq 1$, $H_i^\delta$ is a solution of the minimum problem

$$\min_K\big\{\mathcal{E}_\lambda(g_i^\delta,K,v_{i-1}^\delta):K\in\mathcal{K}'_m(\overline{\Omega}),\ K\supset H_{i-1}^\delta\big\}\,,\tag{3.9}$$

and $v_i^\delta$ is the solution of the minimum problem (2.2) defining $\mathcal{E}_\lambda(g_i^\delta,H_i^\delta,v_{i-1}^\delta)$. We recall that $g_i^\delta:=g(t_i^\delta)$, with $t_i^\delta:=i\delta$. The existence of a solution to (3.9) is proved in Lemma 4.1.

Let us consider first the discontinuities that may occur at the initial time $t=0$.

**Definition 3.1.** *Given a pair* $(u,K)$ *with* $K\in\mathcal{K}_m'(\overline{\Omega})$, $u\in H^1(\Omega\backslash K)$, *and* $u=g(0)$ *on* $\partial_D\Omega\backslash K$, *we define* $\mathcal{R}^0(u,K)$ *as the set of all pairs* $(v,H)$ *such that*

(a) $H\in\mathcal{K}_m(\overline{\Omega})$, $v\in H^1(\Omega\backslash H)$, $v=g(0)$ *on* $\partial_D\Omega\backslash H$,

(b) *there exist a sequence* $\delta_n\to 0+$, *a sequence of integers* $l_n\to\infty$, *with* $l_n\delta_n\to 0$, *and a sequence* $(v_i^\delta,H_i^\delta)$ *satisfying (3.9), such that*

($\mathrm{b}_1$) $v_0^{\delta_n}=u$ *and* $H_0^{\delta_n}=K$ *for every* $n$,

($\mathrm{b}_2$) $H_{l_n}^{\delta_n}\to H$ *in the Hausdorff metric and* $\nabla v_{l_n}^{\delta_n}\to\nabla v$ *strongly in* $L^2(\Omega;\mathbb{R}^2)$.

We will prove that the continuous-time evolution satisfies

$$(u(0+),K(0+))\in\mathcal{R}^0(u(0),K(0))\,.$$

This shows in particular that $(u(0+),K(0+))=(u(0),K(0))$ when $\mathcal{R}^0(u(0),K(0))$ contains only $(u(0),K(0))$ (see Section 5).

The definition of $\mathcal{R}^t(u,K)$ at time $t>0$ is more complex, since the approximation procedure described in the introduction forces us to replace ($\mathrm{b}_1$) by a more technical condition.

**Definition 3.2.** *Given* $t\in(0,T)$ *and a pair* $(u,K)$ *with* $K\in\mathcal{K}_m'(\overline{\Omega})$, $u\in H^1(\Omega\backslash K)$, *and* $u=g(t)$ *on* $\partial_D\Omega\backslash K$, *we define* $\mathcal{R}^t(u,K)$ *as the set of all pairs* $(v,H)$ *such that*

(a) $H\in\mathcal{K}_m(\overline{\Omega})$, $v\in H^1(\Omega\backslash H)$, $v=g(t)$ *on* $\partial_D\Omega\backslash H$,

(b) *there exist a sequence* $\delta_n\to 0+$, *three sequences of integers* $h_n$, $k_n$, $l_n$ *converging to* $\infty$, *with* $h_n\delta_n\to t-$, $k_n\delta_n\to t-$, $l_n\delta_n\to t+$, $k_n-h_n\to\infty$, $l_n-k_n\to\infty$, *and a sequence* $(v_i^\delta,H_i^\delta)$ *satisfying (3.9), such that*

($\mathrm{b}_1$) *for every sequence* $\sigma_n$ *with* $h_n\le\sigma_n\le k_n$ *we have* $H_{\sigma_n}^{\delta_n}\to K$ *in the Hausdorff metric and* $\nabla v_{\sigma_n}^{\delta_n}\to\nabla u$ *strongly in* $L^2(\Omega;\mathbb{R}^2)$,

($\mathrm{b}_2$) $H_{l_n}^{\delta_n}\to H$ *in the Hausdorff metric and* $\nabla v_{l_n}^{\delta_n}\to\nabla v$ *strongly in* $L^2(\Omega;\mathbb{R}^2)$.

We will prove that the continuous-time evolution satisfies

$$(u(t+),K(t+))\in\mathcal{R}^t(u(t-),K(t-))\,.$$

This gives a restriction on the possible jumps and shows in particular that $(u(t),K(t))$ is continuous at time $t$ whenever $\mathcal{R}^t(u(t-),K(t-))$ contains only $(u(t-),K(t-))$ (see Section 5).

We are now in a position to state the main result of the paper, which provides a continuous-time variational model for the quasi-static growth of brittle fractures.

**Theorem 3.3.** *Let* $T>0$, $\lambda>0$, $m\ge 1$, *let* $g\in AC([0,T];H^1(\Omega))\cap L^\infty([0,T];L^\infty(\Omega))$, *let* $\dot g\in L^1([0,T];H^1(\Omega))$ *be its time derivative, and let* $K_0\in\mathcal{K}_m'(\overline{\Omega})$. *Then there exists a function* $K\colon[0,T]\to\mathcal{K}_m(\overline{\Omega})$ *such that, if* $u(t)$ *is a solution of the minimum problem (2.1) corresponding to* $\phi=g(t)$ *and* $K=K(t)$, *the following conditions are satisfied:*

(a) $$K(0)=K_0\quad\textit{and}\quad K(s)\subset K(t)\quad\textit{for } 0\le s\le t\le T\,,$$

(b) $$\textit{for } 0<t\le T\quad \mathcal{E}_\lambda(g(t),K(t),u(t))\le\mathcal{E}_\lambda(g(t),K,u(t))\quad\forall\,K\in\mathcal{K}_m(\overline{\Omega}),\ \ K\supset K(t)\,,$$

(c) $$\textit{for } 0\le s\le t\le T\quad \mathcal{E}(g(t),K(t))-\mathcal{E}(g(s),K(s))\le 2\int_s^t(\nabla u(\tau)|\nabla\dot g(\tau))\,d\tau\,,$$

(d) $$(u(0+),K(0+))\in\mathcal{R}^0(u(0),K(0))\,,$$

(e) $$\textit{for } 0<t<T\quad (u(t+),K(t+))\in\mathcal{R}^t(u(t-),K(t-))\,,$$

*where* $K(t-)$ *and* $K(t+)$ *are defined by (3.1) and (3.2), while* $u(t-)$ *and* $u(t+)$ *are solutions of the minimum problems (2.1), with* $\phi=g(t)$, *corresponding to* $K=K(t-)$ *and* $K=K(t+)$.

**Remark 3.4.** Since $t\mapsto\nabla u(t)$ belongs to $L^\infty([0,T];L^2(\Omega;\mathbb{R}^2))$ (see (3.8)) and $t\mapsto\nabla\dot g(t)$ belongs to $L^1([0,T];L^2(\Omega;\mathbb{R}^2))$, the function $t\mapsto(\nabla u(t)|\nabla\dot g(t))$ is integrable on $[0,T]$.

**Remark 3.5.** By Remark 3.4, condition (c) of Theorem 3.3 implies that

(f) the function $t \mapsto \mathcal{E}(g(t), K(t))$ has bounded variation on $[0,T]$, and its positive variation is absolutely continuous on $[0,T]$;

$$\text{(g)} \qquad \frac{d}{dt}\mathcal{E}(g(t), K(t)) \leq 2(\nabla u(t)|\nabla \dot{g}(t)) \quad \text{for a.e. } t \in [0,T].$$

Conversely, (c) follows from (f) and (g).

**Proposition 3.6.** *Under the assumptions of Theorem 3.3, if* $K : [0,T] \to \mathcal{K}_m(\overline{\Omega})$ *satisfies (a) and (c), then*

$$\text{(h)} \qquad \lim_{s \to t} \frac{\mathcal{E}(g(t), K(s)) - \mathcal{E}(g(t), K(t))}{s - t} \leq 0 \quad \textit{for a.e. } t \in [0,T].$$

*Conversely, if* $t \mapsto K(t)$ *satisfies (a) of Theorem 3.3, (f) of Remark 3.5, and*

$$\text{(h}'\text{)} \qquad \limsup_{s \to t-} \frac{\mathcal{E}(g(t), K(s)) - \mathcal{E}(g(t), K(t))}{s - t} \leq 0 \quad \textit{for a.e. } t \in [0,T].$$

*then* $t \mapsto K(t)$ *satisfies also (g) of Remark 3.5; therefore it satisfies condition (c) of Theorem 3.3.*

*Proof.* We notice that $\mathcal{E}(g(t), K(t)) - \mathcal{E}(g(s), K(s))$ can be written as

$$[\mathcal{E}(g(t), K(t)) - \mathcal{E}(g(t), K(s))] + [\mathcal{E}(g(t), K(s)) - \mathcal{E}(g(s), K(s))]. \tag{3.10}$$

Let $u(t,s)$ be a solution of the minimum problem (2.1) corresponding to $\phi = g(t)$ and $K = K(s)$. Then taking $u(t,s) - u(s) - g(t) + g(s)$ as test function in the equations satisfied by $u(t,s)$ and $u(s)$ we obtain that

$$\mathcal{E}(g(t), K(s)) - \mathcal{E}(g(s), K(s)) = (\nabla u(t,s) + \nabla u(s)|\nabla g(t) - \nabla g(s)). \tag{3.11}$$

Let $\Theta$ be the set of points $t \in (0,T)$ such that $K(t+) = K(t-)$. By [8, Proposition 6.1] we have that $[0,T]\backslash\Theta$ is at most countable. Assume that $t \in \Theta$. As $K(s) \to K(t)$ in the Hausdorff metric for $s \to t$, by [8, Theorem 5.1] both $\nabla u(t,s)$ and $\nabla u(s)$ converge to $\nabla u(t)$ strongly in $L^2(\Omega;\mathbb{R}^2)$ as $s \to t$. We now divide (3.10) and (3.11) by $t - s$ and pass to the limit as $s \to t-$. If (c) is satisfied, from condition (g) of Remark 3.5 we get (h) for all $t \in \Theta$ such that $\frac{d}{dt}\mathcal{E}(g(t), K(t))$ and $\nabla\dot{g}(t)$ exist. Conversely, if (f) and (h$'$) are satisfied, then (g) holds for all $t \in \Theta$ such that $\frac{d}{dt}\mathcal{E}(g(t), K(t))$ and $\nabla\dot{g}(t)$ exist. ☐

**Remark 3.7.** Condition (h$'$) of Proposition 3.6 is equivalent to the existence of a function $\omega(s,t)$, defined for $0 \leq s < t \leq T$, with

$$\lim_{s \to t-} \frac{\omega(s,t)}{t - s} = 0 \qquad \text{for every } t \in (0,T),$$

such that for a.e. $t \in [0,T]$ and every $s < t$ the energy $E(u,K)$ defined in (1.1) satisfies

$$E(u(t), K(t)) \leq E(u, K(s)) + \omega(s,t)$$

for every $u \in H^1(\Omega\backslash K(s))$ with $u = g(t)$ on $\partial_D\Omega\backslash K(s)$.

## 4. PROOF OF THE MAIN RESULT

In this section we prove Theorem 3.3 by a time discretization process. Let us fix a solution $u_0 \in H^1(\Omega\backslash K_0) \cap L^\infty(\Omega)$ of the minimum problem (2.1) corresponding to $\phi = g(0)$ and to $K = K_0$. By Remark 2.6 we may assume that $\|u_0\|_\infty \leq \|g(0)\|_\infty$. Given $\delta > 0$, we define $(u_i^\delta, K_i^\delta)$ inductively as follows: $u_0^\delta := u_0$ and $K_0^\delta := K_0$; for $i \geq 1$ we define $K_i^\delta$ as a solution of the minimum problem

$$\min_K \{\mathcal{E}_\lambda(g_i^\delta, K, u_{i-1}^\delta) : K \in \mathcal{K}'_m(\overline{\Omega}),\ K \supset K_{i-1}^\delta\}, \tag{4.1}$$

and $u_i^\delta$ as the solution of the minimum problem (2.2) defining $\mathcal{E}_\lambda(g_i^\delta, K_i^\delta, u_{i-1}^\delta)$.

**Lemma 4.1.** *There exists a solution $K_i^\delta$ of the minimum problem (4.1). Moreover*

$$\mathcal{E}_\lambda(g_i^\delta, K_i^\delta, u_{i-1}^\delta) \le \mathcal{E}_\lambda(g_i^\delta, K, u_{i-1}^\delta) \tag{4.2}$$

*for every $K \in \mathcal{K}_m(\overline{\Omega})$ with $K \supset K_{i-1}^\delta$.*

*Proof.* By hypothesis $K_0^\delta := K_0 \in \mathcal{K}'_m(\overline{\Omega})$. Assume by induction that $K_{i-1}^\delta \in \mathcal{K}'_m(\overline{\Omega})$ and $u_{i-1}^\delta \in H^1(\Omega \backslash K_{i-1}^\delta) \cap L^\infty(\Omega)$. Consider a minimizing sequence $K_n$ of problem (4.1). We may assume that $\mathcal{H}^1(K_n)$ is uniformly bounded. By compactness (Theorem 2.1), passing to a subsequence we may assume that $K_n$ converges in the Hausdorff metric to some compact set $K^*$ containing $K_{i-1}^\delta$. By Theorem 2.2 we have $K^* \in \mathcal{K}_m(\overline{\Omega})$. Let $K_i^\delta$ be the set of nonisolated points of $K^*$. Then $K_i^\delta \in \mathcal{K}'_m(\overline{\Omega})$ and $K^* \backslash K_i^\delta$ has a finite number of points. Since $K^* \supset K_{i-1}^\delta$ and $K_{i-1}^\delta$ has no isolated points, we have $K_i^\delta \supset K_{i-1}^\delta$. By Theorem 2.10 we conclude that $\mathcal{E}_\lambda(g_i^\delta, K_i^\delta, u_{i-1}^\delta) = \mathcal{E}_\lambda(g_i^\delta, K^*, u_{i-1}^\delta) \le \liminf_n \mathcal{E}_\lambda(g_i^\delta, K_n, u_{i-1}^\delta)$. Since $K_n$ is a minimizing sequence, this proves that $K_i^\delta$ is a solution of the minimum problem (4.1).

To prove (4.2) it is enough to observe that if $K \in \mathcal{K}_m(\overline{\Omega})$ and $K \supset K_{i-1}^\delta$, the set $K'$ of nonisolated points of $K$ belongs to $\mathcal{K}'_m(\overline{\Omega})$ and contains $K_{i-1}^\delta$ (since this set has no isolated points). As $K \backslash K'$ has a finite number of points, from (4.1) we obtain $\mathcal{E}_\lambda(g_i^\delta, K_i^\delta, u_{i-1}^\delta) \le \mathcal{E}_\lambda(g_i^\delta, K', u_{i-1}^\delta) = \mathcal{E}_\lambda(g_i^\delta, K, u_{i-1}^\delta)$. ☐

**Remark 4.2.** If $M$ is a constant such that $\|g(t)\|_\infty \le M$ for every $t \in [0,T]$, then $\|u_0\|_\infty \le M$ and Remark 2.7, applied inductively, gives $\|u_i^\delta\|_\infty \le M$ for every $\delta > 0$ and every $i \ge 0$ with $t_i^\delta \le T$. By the minimality of $u_i^\delta$ we have

$$\|\nabla u_i^\delta\|^2 \le \|\nabla g_i^\delta\|^2 + \lambda \|g_i^\delta - u_{i-1}^\delta\|^2 ,$$

which shows that $\nabla u_i^\delta$ is bounded in $L^2(\Omega, \mathbb{R}^2)$ uniformly with respect to $\delta$ and $i$.

We define now the step functions $g^\delta(t)$, $K^\delta(t)$, and $u^\delta(t)$ on $[0,T]$ by setting $g^\delta(t) := g_i^\delta$, $K^\delta(t) := K_i^\delta$, and $u^\delta(t) := u_i^\delta$ for $t_i^\delta \le t < t_{i+1}^\delta$.

**Lemma 4.3.** *There exists a positive function $\rho(\delta)$, converging to zero as $\delta \to 0$, such that*

$$\begin{aligned} &\|\nabla u_j^\delta\|^2 + \mathcal{H}^1(K_j^\delta) + \lambda \sum_{h=i+1}^{j} \|u_h^\delta - u_{h-1}^\delta\|^2 \le \\ &\le \|\nabla u_i^\delta\|^2 + \mathcal{H}^1(K_i^\delta) + 2\int_{t_i^\delta}^{t_j^\delta} (\nabla u^\delta(\tau) | \nabla \dot{g}(\tau))\, d\tau + \rho(\delta) \end{aligned} \tag{4.3}$$

*for $0 \le i < j$ with $t_j^\delta \le T$.*

*Proof.* Let us fix an integer $r$ with $i \le r < j$. From the absolute continuity of $t \mapsto g(t)$ we have

$$g_{r+1}^\delta - g_r^\delta = \int_{t_r^\delta}^{t_{r+1}^\delta} \dot{g}(\tau)\, d\tau , \tag{4.4}$$

where the integral is a Bochner integral for functions with values in $H^1(\Omega)$. This implies that

$$\nabla g_{r+1}^\delta - \nabla g_r^\delta = \int_{t_r^\delta}^{t_{r+1}^\delta} \nabla \dot{g}(\tau)\, d\tau , \tag{4.5}$$

where the integral is now a Bochner integral for functions with values in $L^2(\Omega; \mathbb{R}^2)$.

As $u_r^\delta + g_{r+1}^\delta - g_r^\delta \in H^1(\Omega \backslash K_r^\delta)$ and $u_r^\delta + g_{r+1}^\delta - g_r^\delta = g_{r+1}^\delta$ on $\partial_D \Omega \backslash K_r^\delta$, we have

$$\mathcal{E}_\lambda(g_{r+1}^\delta, K_r^\delta, u_r^\delta) \le \|\nabla u_r^\delta + \nabla g_{r+1}^\delta - \nabla g_r^\delta\|^2 + \mathcal{H}^1(K_r^\delta) + \lambda \|g_{r+1}^\delta - g_r^\delta\|^2 . \tag{4.6}$$

By the minimality of $u^\delta_{r+1}$ and of $K^\delta_{r+1}$ (see (4.1)) we have

$$\|\nabla u^\delta_{r+1}\|^2 + \mathcal{H}^1(K^\delta_{r+1}) + \lambda\,\|u^\delta_{r+1} - u^\delta_r\|^2 = \mathcal{E}_\lambda(g^\delta_{r+1}, K^\delta_{r+1}, u^\delta_r) \le \mathcal{E}_\lambda(g^\delta_{r+1}, K^\delta_r, u^\delta_r)\,. \tag{4.7}$$

From (4.4), (4.5), (4.6), and (4.7) we obtain

$$\begin{aligned}
&\|\nabla u^\delta_{r+1}\|^2 + \mathcal{H}^1(K^\delta_{r+1}) + \lambda\,\|u^\delta_{r+1} - u^\delta_r\|^2 \le \\
&\le \|\nabla u^\delta_r + \nabla g^\delta_{r+1} - \nabla g^\delta_r\|^2 + \mathcal{H}^1(K^\delta_r) + \lambda\,\|g^\delta_{r+1} - g^\delta_r\|^2 \le \\
&\le \|\nabla u^\delta_r\|^2 + \mathcal{H}^1(K^\delta_r) + 2\int_{t^\delta_r}^{t^\delta_{r+1}} (\nabla u^\delta_r | \nabla \dot g(\tau))\, d\tau + \\
&+ \Big(\int_{t^\delta_r}^{t^\delta_{r+1}} \|\nabla \dot g(\tau)\|\, d\tau\Big)^2 + \lambda\Big(\int_{t^\delta_r}^{t^\delta_{r+1}} \|\dot g(\tau)\|\, d\tau\Big)^2 \le \\
&\le \|\nabla u^\delta_r\|^2 + \mathcal{H}^1(K^\delta_r) + 2\int_{t^\delta_r}^{t^\delta_{r+1}} (\nabla u^\delta(\tau) | \nabla \dot g(\tau))\, d\tau + \\
&+ \sigma(\delta)\Big(\int_{t^\delta_r}^{t^\delta_{r+1}} \|\nabla \dot g(\tau)\|\, d\tau + \lambda\int_{t^\delta_r}^{t^\delta_{r+1}} \|\dot g(\tau)\|\, d\tau\Big)\,,
\end{aligned}$$

where

$$\sigma(\delta) := \max_r \int_{t^\delta_r}^{t^\delta_{r+1}} (\|\nabla \dot g(\tau)\| + \|\dot g(\tau)\|)\, d\tau \;\longrightarrow\; 0$$

by the absolute continuity of the integral. Iterating now this inequality for $i \le r < j$ we get (4.3) with $\rho(\delta) := \sigma(\delta)\int_0^T (\|\nabla \dot g(\tau)\| + \lambda\,\|\dot g(\tau)\|)\, d\tau$. □

**Lemma 4.4.** *There exists a constant $M$, depending only on $g$, $K_0$, $u_0$, and $\lambda$, such that*

$$\|\nabla u^\delta_i\|^2 \le M\,, \qquad \sum_{0<t^\delta_h\le T} \|u^\delta_h - u^\delta_{h-1}\|^2 \le M\,, \quad \text{and} \quad \mathcal{H}^1(K^\delta_i) \le M \tag{4.8}$$

*for every $\delta > 0$ and for every $i \ge 0$ with $t^\delta_i \le T$.*

*Proof.* From the previous lemma we get

$$\begin{aligned}
&\|\nabla u^\delta_i\|^2 + \mathcal{H}^1(K^\delta_i) + \lambda\sum_{h=1}^{i} \|u^\delta_h - u^\delta_{h-1}\|^2 \le \\
&\le \|\nabla u^\delta_0\|^2 + \mathcal{H}^1(K^\delta_0) + 2\int_{t^\delta_0}^{t^\delta_i} (\nabla u^\delta(\tau) | \nabla \dot g(\tau))\, d\tau + \rho(\delta) = \qquad (4.9)\\
&= \|\nabla u_0\|^2 + \mathcal{H}^1(K_0) + 2\int_0^{t^\delta_i} (\nabla u^\delta(\tau) | \nabla \dot g(\tau))\, d\tau + \rho(\delta)\,.
\end{aligned}$$

The first inequality in (4.8) is proved in Remark 4.2. The other inequalities follow now from (4.9). □

**Lemma 4.5.** *There exists an increasing left-continuous function $K\colon [0,T] \to \mathcal{K}_m(\overline{\Omega})$ such that, for every $t \in (0,T)$ with $K(t) = K(t+)$, $K^\delta(t)$ converges to $K(t)$ in the Hausdorff metric as $\delta \to 0$ along a suitable sequence independent of $t$.*

*Proof.* By [8, Theorem 6.3] there exists an increasing function $\hat K\colon [0,T] \to \mathcal{K}(\overline{\Omega})$ such that, for every $t \in [0,T]$, $K^\delta(t)$ converges to $\hat K(t)$ in the Hausdorff metric as $\delta \to 0$ along a suitable sequence independent of $t$. By Lemma 4.4 we have $\mathcal{H}^1(K^\delta(t)) \le M$ for every $t \in [0,T]$, $\delta > 0$, and by Theorem 2.2 this implies $\hat K(t) \in \mathcal{K}_m(\overline{\Omega})$ for every $t \in [0,T]$. Let

$K\colon [0,T] \to \mathcal{K}_m(\overline{\Omega})$ be the left-continuous regularization of $\hat{K}(t)$, defined by $K(0) = \hat{K}(0)$ and $K(t) := \hat{K}(t-)$ for every $t \in (0,T]$. Then $K(t)$ is left-continuous by (3.4), and by (3.3) $K^\delta(t)$ converges to $K(t)$ for every $t \in (0,T)$ with $K(t) = K(t+)$. □

In the rest of this section, when we write $\delta \to 0$ we always refer to the sequence given by Lemma 4.5. Let $\Theta$ be the set of points $t \in (0,T)$ such that $K(t) = K(t+)$. Then $[0,T]\backslash\Theta$ is at most countable (see [8, Proposition 6.1]), and $K(t_n) \to K(t)$ in the Hausdorff metric for every $t \in \Theta$ and every sequence $t_n$ in $[0,T]$ converging to $t$.

**Lemma 4.6.** *For every $t \in (0,T]$ there exist two sequences of integers $h_\delta$ and $k_\delta$ such that $k_\delta - h_\delta \to \infty$, $h_\delta\delta \to t-$, $k_\delta\delta \to t-$, $K^\delta(h_\delta\delta)$ and $K^\delta(k_\delta\delta)$ converge to $K(t)$ in the Hausdorff metric, and*

$$\sum_{h=h_\delta}^{k_\delta} \|u_h^\delta - u_{h-1}^\delta\|^2 \to 0 \tag{4.10}$$

*as $\delta \to 0$. In particular, setting $t_\delta = h_\delta\delta$, we have that $u^\delta(t_\delta) - u^\delta(t_\delta - \delta) \to 0$ strongly in $L^2(\Omega)$.*

*Proof.* Let $\tau_k \to t-$ be such that $\tau_k \in \Theta$. Then $K^\delta(\tau_k) \to K(\tau_k)$ in the Hausdorff metric as $\delta \to 0$ by Lemma 4.5. We choose a strictly decreasing sequence $\delta_k \searrow 0$ such that for every $\delta \le \delta_k$

$$d_H(K^\delta(\tau_k), K(\tau_k)) < \frac{1}{k}\,.$$

For $\delta_{k+1} \le \delta < \delta_k$ let $s_\delta = \tau_k$. Then $s_\delta \to t-$ and

$$d_H(K^\delta(s_\delta), K(t)) \le \frac{1}{k} + d_H(K(s_\delta), K(t)) \qquad \text{for} \quad \delta_{k+1} \le \delta < \delta_k\,.$$

Let $a_\delta$ and $b_\delta$ be integers such that $a_\delta\delta \le s_\delta - \delta^{\frac{1}{3}} < (a_\delta + 1)\delta$ and $b_\delta = a_\delta + [\delta^{-\frac{1}{6}}][\delta^{-\frac{1}{2}}]$, where $[\cdot]$ denotes the integer part. By construction we have that $a_\delta\delta \to t-$ and $b_\delta\delta \to t-$. From the estimate in Lemma 4.4 between $b_\delta$ and $a_\delta$ we obtain

$$\sum_{h=a_\delta+1}^{b_\delta} \|u_h^\delta - u_{h-1}^\delta\|^2 \le M\,.$$

Then we divide the above sum into $[\delta^{-\frac{1}{6}}]$ groups of $[\delta^{-\frac{1}{2}}]$ consecutive terms, and we find that the sum of one of these groups must be less than or equal to $M/[\delta^{-\frac{1}{6}}]$. Therefore there exist two integers $h_\delta$ and $k_\delta$ such that $a_\delta < h_\delta < k_\delta \le b_\delta$, $k_\delta - h_\delta = [\delta^{-\frac{1}{2}}]$, and

$$\sum_{h=h_\delta}^{k_\delta} \|u_h^\delta - u_{h-1}^\delta\|^2 \le \frac{M}{[\delta^{-\frac{1}{6}}]}\,.$$

It is then obvious that $k_\delta - h_\delta \to \infty$, $h_\delta\delta$ and $k_\delta\delta$ converge to $t-$, and (4.10) is satisfied.

Let us fix $s \in \Theta$ with $s < t$. Then

$$K^\delta(s) \subset K^\delta(h_\delta\delta) \subset K^\delta(k_\delta\delta) \subset K^\delta(s_\delta)\,,$$

being $s < h_\delta\delta < k_\delta\delta \le s_\delta$ for $\delta$ small enough. By compactness (Theorem 2.1) we may assume that $K^\delta(h_\delta\delta) \to K'$ and $K^\delta(k_\delta\delta) \to K''$ in the Hausdorff metric. Since $K^\delta(s) \to K(s)$ and $K^\delta(s_\delta) \to K(t)$ in the Hausdorff metric, we have

$$K(s) \subset K' \subset K'' \subset K(t)\,.$$

Passing to the limit as $s \to t-$ we get

$$K(t) = K(t-) \subset K' \subset K'' \subset K(t)\,,$$

which implies that $K^\delta(h_\delta\delta)$ and $K^\delta(k_\delta\delta)$ converge to $K(t)$ in the Hausdorff metric. □

**Lemma 4.7.** *For every $t \in \Theta$ we have $u^\delta(t) - u^\delta(t-\delta) \to 0$ strongly in $L^2(\Omega)$ and $\nabla u^\delta(t) \to \nabla u(t)$ strongly in $L^2(\Omega;\mathbb{R}^2)$. Moreover there exists a solution $u^*(t)$ of problem (2.1) corresponding to $\phi = g(t)$ and $K = K(t)$ (possibly different from $u(t)$) such that a subsequence of $u^\delta(t)$ (possibly depending on $t$) converges to $u^*(t)$ strongly in $L^2(\Omega)$.*

*Proof.* Let $t \in \Theta$. By Lemma 4.6 there exists a sequence $t_\delta \to t-$ such that $K^\delta(t_\delta) \to K(t)$ in the Hausdorff metric and $u^\delta(t_\delta) - u^\delta(t_\delta - \delta) \to 0$ strongly in $L^2(\Omega)$. By the same argument, we can also construct $t'_\delta = l_\delta \delta$, with $l_\delta$ integer, such that $t'_\delta \to t+$, $K^\delta(t'_\delta)$ converge to $K(t+) = K(t)$ in the Hausdorff metric, and $u^\delta(t'_\delta) - u^\delta(t'_\delta - \delta) \to 0$ strongly in $L^2(\Omega)$.

By Remark 4.2 the sequence $u^\delta(t_\delta - \delta)$ is bounded in $L^\infty(\Omega)$, and by construction $u^\delta(t_\delta)$ is the solution of the minimum problem (2.2) which defines $\mathcal{E}_\lambda(g(t_\delta), K^\delta(t_\delta), u^\delta(t_\delta - \delta))$. Therefore Corollary 2.11 implies that $\nabla u^\delta(t_\delta) \to \nabla u(t)$ strongly in $L^2(\Omega;\mathbb{R}^2)$. The same argument shows that $\nabla u^\delta(t'_\delta) \to \nabla u(t)$ strongly in $L^2(\Omega;\mathbb{R}^2)$.

Then the estimate in Lemma 4.3 between $h_\delta$ and $l_\delta$ gives

$$\|\nabla u^\delta(t'_\delta)\|^2 + \mathcal{H}^1(K^\delta(t'_\delta)) + \lambda \sum_{h=h_\delta+1}^{l_\delta} \|u_h^\delta - u_{h-1}^\delta\|^2 \le$$

$$\le \|\nabla u^\delta(t_\delta)\|^2 + \mathcal{H}^1(K^\delta(t_\delta)) + 2\int_{t_\delta}^{t'_\delta} (\nabla u^\delta(\tau)|\nabla \dot g(\tau))d\tau + \rho(\delta)\,.$$

Passing now to the limit as $\delta \to 0$ we get

$$\sum_{h=h_\delta+1}^{l_\delta} \|u_h^\delta - u_{h-1}^\delta\|^2 \to 0\,. \tag{4.11}$$

Let $i_\delta$ be the integer such that $i_\delta \delta \le t < (i_\delta + 1)\delta$. As $h_\delta < i_\delta \le l_\delta$, by (4.11) we obtain that $u^\delta(t) - u^\delta(t-\delta) = u_{i_\delta}^\delta - u_{i_\delta - 1}^\delta \to 0$ strongly in $L^2(\Omega)$.

Since $K^\delta(t) \to K(t)$ in the Hausforff metric and $u^\delta(t)$ is the solution of the minimum problem (2.2) which defines $\mathcal{E}_\lambda(g^\delta(t), K^\delta(t), u^\delta(t-\delta))$, from Corollary 2.11 we obtain that $\nabla u^\delta(t) \to \nabla u(t)$ strongly in $L^2(\Omega;\mathbb{R}^2)$ and that there exists a solution $u^*(t)$ of problem (2.1) corresponding to $\phi = g(t)$ and $K = K(t)$ (possibly different from $u(t)$) such that a subsequence of $u^\delta(t)$ (possibly depending on $t$) converges to $u^*(t)$ strongly in $L^2(\Omega)$. ☐

We show now that the increasing left-continuous function $K\colon [0,T] \to \mathcal{K}(\overline{\Omega})$ satisfies all conditions of Theorem 3.3. The following lemma proves condition (b).

**Lemma 4.8.** *For every $t \in (0,T]$ we have*

$$\mathcal{E}_\lambda(g(t), K(t), u(t)) \le \mathcal{E}_\lambda(g(t), K, u(t)) \quad \forall\, K \in \mathcal{K}_m(\overline{\Omega})\,,\ K \supset K(t)\,. \tag{4.12}$$

*Proof.* Let us consider first the case $t \in \Theta$. By Lemma 4.7, $\nabla u^\delta(t) \to \nabla u(t)$ strongly in $L^2(\Omega;\mathbb{R}^2)$ and, passing to a subsequence (which may depend on $t$), we may assume that $u^\delta(t) \to u^*(t)$ strongly in $L^2(\Omega)$, for some solution $u^*(t)$ of the minimum problem (2.1) corresponding to $\phi = g(t)$ and $K = K(t)$. Then $\nabla u^*(t) = \nabla u(t)$ a.e. on $\Omega$ and $u^*(t) = u(t)$ a.e. on the connected components of $\Omega \backslash K(t)$ whose boundaries meet $\partial_D \Omega \backslash K(t)$, while on the other connected components $u^*(t)$ and $u(t)$ are constant (Remark 2.6). Moreover, $\mathcal{E}_\lambda(g(t), K(t), u^*(t)) = \mathcal{E}_\lambda(g(t), K(t), u(t)) = \mathcal{E}(g(t), K(t))$. By Lemma 4.7 we have that $u^\delta(t-\delta) \to u^*(t)$ strongly in $L^2(\Omega)$, and by Remark 4.2 $u^\delta(t-\delta)$ is bounded in $L^\infty(\Omega)$.

Let $K \in \mathcal{K}_m(\overline{\Omega})$ with $K \supset K(t)$. Since $K^\delta(t)$ converges to $K(t)$ in the Hausdorff metric as $\delta \to 0$, by [8, Lemma 3.5] there exists a sequence $K^\delta$ in $\mathcal{K}_m(\overline{\Omega})$, converging to $K$ in the Hausdorff metric, such that $K^\delta \supset K^\delta(t)$ and $\mathcal{H}^1(K^\delta \backslash K^\delta(t)) \to \mathcal{H}^1(K \backslash K(t))$ as $\delta \to 0$. By Lemma 4.4 this implies that $\mathcal{H}^1(K^\delta)$ is bounded as $\delta \to 0$. Let $u^\delta$ and $u$ be the solutions of the minimum problems (2.2) which define $\mathcal{E}_\lambda(g^\delta(t), K^\delta, u^\delta(t-\delta))$ and $\mathcal{E}_\lambda(g(t), K, u^*(t))$, respectively. By Theorem 2.10 $\nabla u^\delta \to \nabla u$ strongly in $L^2(\Omega;\mathbb{R}^2)$.

The minimality of $K^\delta(t)$ expressed by (4.2) in Lemma 4.1 gives

$$\mathcal{E}_\lambda(g^\delta(t), K^\delta(t), u^\delta(t-\delta)) \le \mathcal{E}_\lambda(g^\delta(t), K^\delta, u^\delta(t-\delta)),$$

which implies

$$\|\nabla u^\delta(t)\|^2 + \lambda \|u^\delta(t) - u^\delta(t-\delta)\|^2 \le \|\nabla u^\delta\|^2 + \mathcal{H}^1(K^\delta \setminus K^\delta(t)) + \lambda \|u^\delta - u^\delta(t-\delta)\|^2 .$$

Passing now to the limit as $\delta \to 0$ and using Lemma 4.7 we get $\|\nabla u(t)\|^2 \le \|\nabla u\|^2 + \mathcal{H}^1(K \setminus K(t)) + \lambda \|u - u^*(t)\|^2$. Adding $\mathcal{H}^1(K(t))$ to both sides we obtain

$$\mathcal{E}_\lambda(g(t), K(t), u(t)) \le \mathcal{E}_\lambda(g(t), K, u^*(t)).$$

As each connected component of $\Omega \setminus K$ is contained in a connected component of $\Omega \setminus K(t)$, by Remark 2.8 we have $\mathcal{E}_\lambda(g(t), K, u^*(t)) = \mathcal{E}_\lambda(g(t), K, u(t))$. Therefore the previous inequality gives (4.12) for $t \in \Theta$.

Let us consider now the general case $t \in (0, T]$, which is obtained by approximation. We fix $t \in (0, T]$ and a compact set $K \in \mathcal{K}_m(\overline{\Omega})$ with $K \supset K(t)$. Let $t_k \to t-$, with $t_k \in \Theta$. Then $\nabla u(t_k) \to \nabla u(t)$ strongly in $L^2(\Omega; \mathbb{R}^2)$ by (3.6). Arguing as in the proof of Theorem 2.10, we may assume, passing to a subsequence, that $u(t_k) \to u^*$ strongly in $L^2(\Omega)$, for some solution $u^*$ of the minimum problem (2.1) corresponding to $\phi = g(t)$ and $K = K(t)$. Then $\nabla u^* = \nabla u(t)$ a.e. on $\Omega$ and $u^* = u(t)$ a.e. on the connected components of $\Omega \setminus K(t)$ whose boundaries meet $\partial_D \Omega \setminus K(t)$, while $u^*$ and $u(t)$ are constant on the other connected components. Moreover we have that $\mathcal{E}_\lambda(g(t), K(t), u^*) = \mathcal{E}_\lambda(g(t), K(t), u(t)) = \mathcal{E}(g(t), K(t))$. By the first part of the proof $\mathcal{E}_\lambda(g(t_k), K(t_k), u(t_k)) \le \mathcal{E}_\lambda(g(t_k), K, u(t_k))$. Passing now to the limit as $k \to \infty$ thanks to Theorem 2.10 we get

$$\begin{aligned} \mathcal{E}_\lambda(g(t), K(t), u(t)) = \mathcal{E}_\lambda(g(t), K(t), u^*) &\le \liminf_{k\to\infty} \mathcal{E}_\lambda(g(t_k), K(t_k), u(t_k)) \le \\ \le \lim_{k\to\infty} \mathcal{E}_\lambda(g(t_k), K, u(t_k)) &= \mathcal{E}_\lambda(g(t), K, u^*) = \mathcal{E}_\lambda(g(t), K, u(t)), \end{aligned}$$

where the last equality follows from Remark 2.8. ☐

The following lemma proves condition (c) of Theorem 3.3.

**Lemma 4.9.** *For every $s$, $t$ with $0 \le s < t \le T$*

$$\|\nabla u(t)\|^2 + \mathcal{H}^1(K(t)) \le \|\nabla u(s)\|^2 + \mathcal{H}^1(K(s)) + 2\int_s^t (\nabla u(\tau) | \nabla \dot g(\tau)) d\tau . \tag{4.13}$$

*Proof.* Let us fix $s, t \in \Theta$ with $0 \le s < t \le T$. Given $\delta > 0$, let $i$ and $j$ be the integers such that $t_i^\delta \le s < t_{i+1}^\delta$ and $t_j^\delta \le t < t_{j+1}^\delta$. Let us define $s_\delta := t_i^\delta$ and $t_\delta := t_j^\delta$. Applying Lemma 4.3 we obtain

$$\|\nabla u^\delta(t)\|^2 + \mathcal{H}^1(K^\delta(t) \setminus K^\delta(s)) \le \|\nabla u^\delta(s)\|^2 + 2\int_{s_\delta}^{t_\delta} (\nabla u^\delta(\tau) | \nabla \dot g(\tau))\, d\tau + \rho(\delta), \tag{4.14}$$

with $\rho(\delta)$ converging to zero as $\delta \to 0$. By Lemma 4.7 for every $\tau \in \Theta$ we have $\nabla u^\delta(\tau) \to \nabla u(\tau)$ strongly in $L^2(\Omega, \mathbb{R}^2)$ as $\delta \to 0$, and by Lemma 4.4 we have $\|\nabla u^\delta(\tau)\| \le M$ for every $\tau \in [0, T]$. By [8, Corollary 3.4] we get

$$\mathcal{H}^1(K(t) \setminus K(s)) \le \liminf_{\delta \to 0} \mathcal{H}^1(K^\delta(t) \setminus K^\delta(s)).$$

Passing now to the limit in (4.14) as $\delta \to 0$ we obtain (4.13) for every $s, t \in \Theta$ with $0 \le s < t \le T$.

In the general case we consider two sequences $s_k \to s-$ and $t_k \to t-$ with $s_k, t_k \in \Theta$. Then $K(s_k) \to K(s)$ and $K(t_k) \to K(t)$ in the Hausdorff metric, while $\nabla u(s_k) \to \nabla u(s)$ and $\nabla u(t_k) \to \nabla u(t)$ strongly in $L^2(\Omega; \mathbb{R}^2)$ by (3.6). By the first part of the proof we have that

$$\|\nabla u(t_k)\|^2 + \mathcal{H}^1(K(t_k) \setminus K(s_k)) \le \|\nabla u(s_k)\|^2 + 2\int_{s_k}^{t_k} (\nabla u(\tau) | \nabla \dot g(\tau))\, d\tau . \tag{4.15}$$

Passing now to the limit in (4.15) as $k \to \infty$ and using again [8, Corollary 3.4] we obtain (4.13). $\square$

The following lemma proves condition (d) of Theorem 3.3.

**Lemma 4.10.** *We have* $(u(0+), K(0+)) \in \mathcal{R}^0(u_0, K_0)$.

*Proof.* By the definition of $u(0+)$ it follows that condition (a) in Definition 3.1 is satisfied. We now take $(v_i^\delta, H_i^\delta) := (u_i^\delta, K_i^\delta)$. By the argument used in the proof of Lemma 4.7, we can construct a sequence of integers $l_\delta \to \infty$ such that $l_\delta \delta \to 0+$, $K^\delta(l_\delta \delta)$ converges to $K(0+)$ in the Hausdorff metric, and $\nabla u^\delta(l_\delta\delta) \to \nabla u(0+)$ strongly in $L^2(\Omega; \mathbb{R}^2)$ as $\delta \to 0$. This proves that $(u(0+), K(0+)$ satisfies condition (b) in Definition 3.1. $\square$

The following lemma proves condition (e) of Theorem 3.3.

**Lemma 4.11.** *For* $0 < t < T$ *we have* $(u(t+), K(t+)) \in \mathcal{R}^t(u(t), K(t))$.

*Proof.* Fix $0 < t < T$. By the definition of $u(t+)$ it follows that condition (a) in Definition 3.2 is satisfied. We now take $(v_i^\delta, H_i^\delta) := (u_i^\delta, K_i^\delta)$. Let $h_\delta$ and $k_\delta$ be the sequences of integers given by Lemma 4.6. Since $\sum_{h=h_\delta}^{k_\delta} \|u_h^\delta - u_{h-1}^\delta\|^2 \to 0$ as $\delta \to 0$, we have $u_{\sigma_\delta}^\delta - u_{\sigma_\delta - 1}^\delta \to 0$ strongly in $L^2(\Omega)$ for every sequence $\sigma_\delta$ of integers between $h_\delta$ and $k_\delta$. Since both $K^\delta(h_\delta\delta)$ and $K^\delta(k_\delta\delta)$ converge to $K(t)$ in the Hausdorff metric, $K^\delta(\sigma_\delta\delta)$ converges to $K(t)$ in the Hausdorff metric. Therefore $\nabla u^\delta(\sigma_\delta\delta) \to \nabla u(t)$ strongly in $L^2(\Omega;\mathbb{R}^2)$ by Corollary 2.11. This shows that condition $(b_1)$ in Definition 3.2 is satisfied.

By the same argument as in the proof of Lemma 4.7, we can construct a sequence of integers $l_\delta \to \infty$ such that $l_\delta\delta \to t+$, $l_\delta - k_\delta \to \infty$, $K^\delta(l_\delta\delta)$ converges to $K(t+)$ in the Hausdorff metric, and $\nabla u^\delta(l_\delta\delta) \to \nabla u(t+)$ strongly in $L^2(\Omega;\mathbb{R}^2)$. This shows that condition $(b_2)$ in Definition 3.2 is satisfied. $\square$

## 5. Example

In this section we consider in detail the particular case when no initial crack is present, i.e., $K_0 = \emptyset$. We prove that, if $\Omega$ and $g(t)$ are sufficiently regular, no crack will appear in our model, provided $\lambda$ is large enough. More precisely, under these conditions we prove that $K(t) = \emptyset$ is the unique function which satisfies conditions (a)–(e) of Theorem 3.3.

**Theorem 5.1.** *Assume that* $\partial\Omega$ *is of class* $C^2$, $\partial_D\Omega = \partial\Omega$, *and* $g \in AC([0,T]; H^1(\Omega)) \cap L^\infty([0,T]; C^{1,\alpha}(\overline{\Omega})) \cap C^0([0,T]; C^0(\overline{\Omega}))$ *for some* $0 < \alpha < 1$. *If* $K_0 = \emptyset$ *and* $\lambda$ *is larger than the constant* $\lambda_0$ *given by (5.8), then* $K(t) = \emptyset$ *is the unique function* $K\colon [0,T] \to \mathcal{K}_m(\overline{\Omega})$ *which satisfies conditions (a)–(e) of Theorem 3.3. Moreover,*

$$\mathcal{R}^t(u(t), \emptyset) = \{(u(t), \emptyset)\} \tag{5.1}$$

*for every* $t \in [0,T)$.

To prove Theorem 5.1 we need some estimates on the solutions of the Dirichlet problems

$$\begin{cases} \Delta u = f & \text{in } \Omega\,, \\ u = \phi & \text{on } \partial\Omega\,, \end{cases} \tag{5.2}$$

and

$$\begin{cases} \Delta v = \lambda(v - w) & \text{in } \Omega\,, \\ v = \psi & \text{on } \partial\Omega\,. \end{cases} \tag{5.3}$$

If $\partial\Omega \in C^{1,\alpha}$ and $\phi \in C^{1,\alpha}(\overline{\Omega})$, for $0 < \alpha < 1$, and $f \in L^\infty(\Omega)$, then the solution $u$ of (5.2) belongs to $C^{1,\alpha}(\overline{\Omega})$ (see, e.g., [10, Corollary 8.35]) and there exists a constant $C$, independent of $f$ and $\phi$, such that

$$\|\nabla u\|_\infty \le C\,(\|f\|_\infty + \|\nabla\phi\|_{0,\alpha})\,, \tag{5.4}$$

where $\|\cdot\|_{0,\alpha}$ denotes the norm in $C^{0,\alpha}(\overline{\Omega};\mathbb{R}^2)$ and $\|\cdot\|_\infty$ denotes the norm in $L^\infty(\Omega)$ or in $L^\infty(\Omega;\mathbb{R}^2)$, according to the context.

If $w\in L^2(\Omega)$ and $\psi\in H^1(\Omega)\cap L^\infty(\Omega)$, then the solution $v$ of (5.3) belongs to $H^1(\Omega)\cap L^\infty(\Omega)$ and

$$\|v\|_\infty\le\|\psi\|_\infty+C_\lambda\,\|w\|\,, \tag{5.5}$$

where the constant $C_\lambda$ depends on $\lambda$, but not on $w$ and $\psi$ (see, e.g., [10, Theorem 8.16]).

*Proof of Theorem 5.1.* We begin by proving that $K(t)=\emptyset$ satisfies condition (b). Since every $K\in\mathcal{K}_m(\overline{\Omega})$ can be approximated in the Hausdorff metric by a sequence of compact sets contained in $\Omega$, with convergence of the lenghts, taking Theorem 2.10 into account it is enough to prove that for every $0<t\le T$ we have

$$\mathcal{E}_\lambda(g(t),\emptyset,u(t))\le\mathcal{E}_\lambda(g(t),K,u(t)) \tag{5.6}$$

for every compact set $K\subset\Omega$.

To this end we use the calibration constructed in [1, Section 5.3]. In that section the Neumann condition on $\partial\Omega$ is used only to obtain that $\phi^x(x,t)\,\nu=0$ for $x\in\partial\Omega$, which is not needed in our case, where we prescribe a Dirichlet boundary condition on $\partial\Omega$ (see [1, Theorem 3.3]). This calibration can be constructed provided we are able to prove inequality (5.12) of [1], which in our case reduces to

$$2^7\,\|\nabla u(t)\|_\infty^4<\lambda\,. \tag{5.7}$$

By (5.4) there exists a constant $C$ such that

$$\|\nabla u(t)\|_\infty\le C\,G_\alpha\,,$$

where

$$G_\alpha:=\sup_{t\in[0,T]}\|\nabla g(t)\|_{0,\alpha}\,.$$

Therefore (5.7) is satisfied if

$$\lambda>\lambda_0:=2^7\,C^4\,G_\alpha^4\,, \tag{5.8}$$

and in this case the calibration constructed in [1, Section 5.3] proves (5.6) for every compact set $K\subset\Omega$, which implies condition (b) of Theorem 3.3.

Let us prove now that $\mathcal{R}^0(u(0),\emptyset)=\{(u(0),\emptyset)\}$. As $\mathcal{R}^0(u(0),\emptyset)\neq\emptyset$ (see, e.g., Lemma 4.10), it is enough to show that $\mathcal{R}^0(u(0),\emptyset)\subset\{(u(0),\emptyset)\}$.

Let $(v,H)\in\mathcal{R}^0(u(0),\emptyset)$, let $\delta_n$, $l_n$, $v_i^{\delta_n}$, and $H_i^{\delta_n}$ be the sequences, the functions, and the sets which appear in condition (b) of Definition 3.1, and let

$$\varepsilon_n:=\sup_{t\in[\delta_n,T]}\|g(t)-g(t-\delta_n)\|_\infty\,. \tag{5.9}$$

As $g\in C^0([0,T],C^0(\overline{\Omega}))$, the sequence $\varepsilon_n$ tends to $0$. Starting from $w_0^{\delta_n}:=u(0)$, we consider also the sequence $w_i^{\delta_n}$, $0<i\le l_n$, of the solutions of the Dirichlet problems

$$\begin{cases}\Delta w_i^{\delta_n}=\lambda(w_i^{\delta_n}-w_{i-1}^{\delta_n}) & \text{in }\Omega\,,\\ w_i^{\delta_n}=g_i^{\delta_n} & \text{on }\partial\Omega\,.\end{cases} \tag{5.10}$$

By using the calibration constructed in [1, Section 5.3], we will prove by induction on $i$ that $H_i^{\delta_n}=\emptyset$ and $v_i^{\delta_n}=w_i^{\delta_n}$ for $\lambda>\lambda_0$ and for $n$ large enough. This calibration can be constructed provided we are able to prove inequality (5.12) of [1], which in this case reads

$$\|\nabla w_i^{\delta_n}\|_\infty(\sqrt{\lambda}\,\|w_i^{\delta_n}-w_{i-1}^{\delta_n}\|_\infty+\sqrt{2}\,\|\nabla w_i^{\delta_n}\|_\infty)<\frac{1}{8}\sqrt{\lambda}\,. \tag{5.11}$$

To obtain (5.11) we prove by induction on $i$ that

$$\|\Delta w_i^{\delta_n}\|_\infty\le\lambda\,\varepsilon_n\,. \tag{5.12}$$

This inequality is true for $i=0$ since $w_0^{\delta_n}=u(0)$, which is harmonic. Assume that (5.12) holds for $i-1$. Then $w:=w_{i-1}^{\delta_n}+\varepsilon_n$ is a super-solution of problem (5.10), in the sense that

$$\begin{cases} \Delta w \le \lambda(w-w_{i-1}^{\delta_n}) & \text{in } \Omega\,, \\ w \ge g_i^{\delta_n} & \text{on } \partial\Omega\,. \end{cases}$$

Indeed, $\Delta w = \Delta w_{i-1}^{\delta_n} \le \lambda\,\varepsilon_n = \lambda(w-w_{i-1}^{\delta_n})$ in $\Omega$ by the inductive hypothesis, while $w=g_{i-1}^{\delta_n}+\varepsilon_n \ge g_i^{\delta_n}$ on $\partial\Omega$ by (5.9). Therefore $w_i^{\delta_n}\le w_{i-1}^{\delta_n}+\varepsilon_n$ in $\Omega$. Similarly, $w_{i-1}^{\delta_n}-\varepsilon_n$ is a sub-solution of (5.10); this implies that $w_i^{\delta_n}\ge w_{i-1}^{\delta_n}-\varepsilon_n$, which, together with the previous inequality gives

$$\|w_i^{\delta_n}-w_{i-1}^{\delta_n}\|_\infty \le \varepsilon_n\,. \tag{5.13}$$

By (5.10) we have $\|\Delta w_i^{\delta_n}\|_\infty \le \lambda\,\varepsilon_n$, concluding the proof of (5.12).

From (5.4) and (5.12) we obtain

$$\|\nabla w_i^{\delta_n}\|_\infty \le C\,(\lambda\,\varepsilon_n + G_\alpha)\,. \tag{5.14}$$

By (5.13) and (5.14) for $n$ large enough we have

$$\begin{aligned} &\|\nabla w_i^{\delta_n}\|_\infty(\sqrt{\lambda}\,\|w_i^{\delta_n}-w_{i-1}^{\delta_n}\|_\infty+\sqrt{2}\,\|\nabla w_i^{\delta_n}\|_\infty)\le \\ &\le C\,(\lambda\,\varepsilon_n+G_\alpha)\,(\sqrt{\lambda}\varepsilon_n+\sqrt{2}\,C\,(\lambda\,\varepsilon_n+G_\alpha)) < \frac{1}{8}\sqrt{\lambda}\,, \end{aligned}$$

where the last inequality follows from (5.8) and from the fact that $\varepsilon_n\to 0$. This proves (5.11) for $n$ large enough.

Therefore, using the calibration constructed in [1, Section 5.3], we can prove that $(w_i^{\delta_n},\emptyset)$ is the unique minimizer of the functional

$$F_{i-1}^{\delta_n}(u,K):=\|\nabla u\|^2+\mathcal{H}^1(K)+\lambda\,\|u-w_{i-1}^{\delta_n}\|^2$$

among all pairs $(u,K)$ with $K\in\mathcal{K}'_m(\overline{\Omega})$, $u\in H^1(\Omega\backslash K)$, $u=g_i^{\delta_n}$ on $\partial\Omega\backslash K$.

As $v_0^{\delta_n}=u(0)=w_0^{\delta_n}$ and $H_0^{\delta_n}=K(0)=\emptyset$, both $(v_1^{\delta_n},H_1^{\delta_n})$ and $(w_1^{\delta_n},\emptyset)$ minimize $F_0^{\delta_n}$ with the same Dirichlet condition $g_1^{\delta_n}$, hence the uniqueness result gives $v_1^{\delta_n}=w_1^{\delta_n}$ and $H_1^{\delta_n}=\emptyset$. In the same way, by induction we prove that $v_i^{\delta_n}=w_i^{\delta_n}$ and $H_i^{\delta_n}=\emptyset$ for every $i$.

By condition ($\mathrm{b}_2$) of Definition 3.1 we have $H=\emptyset$ and $\nabla w_{l_n}^{\delta_n}\to\nabla v$ strongly in $L^2(\Omega;\mathbb{R}^2)$. As $w_{l_n}^{\delta_n}-w_{l_n-1}^{\delta_n}\to 0$ strongly in $L^2(\Omega)$ by (5.13) and $g_{l_n}^{\delta_n}=g(l_n\delta_n)\to g(0)$ strongly in $H^1(\Omega)$, the continuous dependence of the solutions of (5.10) on the data implies that $w_{l_n}^{\delta_n}$ converges to $u(0)$ strongly in $H^1(\Omega)$. This shows that $v=u(0)$ and concludes the proof of the inclusion $\mathcal{R}^0(u(0),\emptyset)\subset\{(u(0),\emptyset)\}$.

Let us prove now that $\mathcal{R}^t(u(t),\emptyset)=\{(u(t),\emptyset)\}$ for every $0<t<T$. As $\mathcal{R}^t(u(t),\emptyset)\ne\emptyset$ (see, e.g., Lemma 4.11), it is enough to show that $\mathcal{R}^t(u(t),\emptyset)\subset\{(u(t),\emptyset)\}$. Let $(v,H)\in\mathcal{R}^t(u(t),\emptyset)$ and let $\delta_n$, $h_n$, $k_n$, $l_n$, $v_i^{\delta_n}$, and $H_i^{\delta_n}$ be the sequences, the functions and the sets which appear in condition (b) of Definition 3.2. As $\emptyset$ is isolated in the Hausdorff metric, by ($\mathrm{b}_1$) we may assume that $H_{k_n}^{\delta_n}=\emptyset$, and by the monotonicity of $H_i^{\delta_n}$ we deduce that $H_i^{\delta_n}=\emptyset$ for $0\le i\le k_n$. It follows that $v_i^{\delta_n}$ belongs to $H^1(\Omega)$ and solves the Dirichlet problem

$$\begin{cases} \Delta v_i^{\delta_n}=\lambda(v_i^{\delta_n}-v_{i-1}^{\delta_n}) & \text{in } \Omega\,, \\ v_i^{\delta_n}=g_i^{\delta_n} & \text{on } \partial\Omega\,, \end{cases} \tag{5.15}$$

for $1\le i\le k_n$ .

In order to prove that $H_i^{\delta_n}=\emptyset$ for $k_n<i\le l_n$ we apply the calibration method as in the case $t=0$. We define $w_{k_n}^{\delta_n}:=v_{k_n}^{\delta_n}$ and we consider the sequence $w_i^{\delta_n}$, $k_n<i\le l_n$, defined inductively by the solutions of (5.10). We can construct a calibration for $w_i^{\delta_n}$ provided (5.11)

holds. As before, it is enough to show that $\Delta w_i^{\delta_n} \to 0$ in $L^\infty(\Omega)$ as $n \to \infty$, uniformly for $k_n \le i \le l_n$. The inductive argument used to prove (5.12) shows that

$$\|\Delta w_i^{\delta_n}\|_\infty \le \max\{\lambda\,\varepsilon_n, \|\Delta w_{k_n}^{\delta_n}\|_\infty\} = \max\{\lambda\,\varepsilon_n, \|\Delta v_{k_n}^{\delta_n}\|_\infty\}$$

for $k_n \le i \le l_n$. Therefore, in order to obtain (5.11), it is enough to show that $\Delta v_{k_n}^{\delta_n} \to 0$ in $L^\infty(\Omega)$.

By (5.15) we have $\Delta v_{k_n}^{\delta_n} = \lambda(v_{k_n}^{\delta_n} - v_{k_n-1}^{\delta_n})$. To estimate $\|v_{k_n}^{\delta_n} - v_{k_n-1}^{\delta_n}\|_\infty$ we note that the difference satisfies

$$\begin{cases} \Delta(v_{k_n}^{\delta_n} - v_{k_n-1}^{\delta_n}) = \lambda((v_{k_n}^{\delta_n} - v_{k_n-1}^{\delta_n}) - (v_{k_n-1}^{\delta_n} - v_{k_n-2}^{\delta_n})) & \text{in } \Omega\,, \\ v_{k_n}^{\delta_n} - v_{k_n-1}^{\delta_n} = g_{k_n}^{\delta_n} - g_{k_n-1}^{\delta_n} & \text{on } \partial\Omega\,. \end{cases} \tag{5.16}$$

As $g_{k_n-1}^{\delta_n} - g_{k_n-2}^{\delta_n} \to 0$ strongly in $H^1(\Omega)$, and $\nabla v_{k_n-1}^{\delta_n} - \nabla v_{k_n-2}^{\delta_n} \to 0$ strongly in $L^2(\Omega;\mathbb{R}^2)$ (by condition ($\mathrm{b}_1$) of Definition 3.2), using the Poincaré inequality we conclude that $v_{k_n-1}^{\delta_n} - v_{k_n-2}^{\delta_n} \to 0$ strongly in $L^2(\Omega)$.

Since $g_{k_n}^{\delta_n} - g_{k_n-1}^{\delta_n} \to 0$ in $L^\infty(\Omega)$, estimate (5.5) for (5.16) implies that $v_{k_n}^{\delta_n} - v_{k_n-1}^{\delta_n} \to 0$ in $L^\infty(\Omega)$. By (5.15) this implies that $\Delta v_{k_n}^{\delta_n} \to 0$ in $L^\infty(\Omega)$.

Therefore, arguing as in the case $t = 0$, we can construct now a calibration for $w_i^{\delta_n}$, which shows that $H_i^{\delta_n} = \emptyset$ and $v_i^{\delta_n} = w_i^{\delta_n}$ for $k_n \le i \le l_n$, and leads to the conclusion of the proof of (5.1).

So far we have proved that $K(t) = \emptyset$ satisfies conditions (a), (b), (d), and (e) of Theorem 3.3. As $\mathcal{E}(g(t), K(s)) = \mathcal{E}(g(t), \emptyset)$, condition (h′) of Proposition 3.6 is trivial. Condition (f) of Remark 3.5 follows from the smooth dependence of the energy on the boundary data. By Proposition 3.6 conditions (f) and (h′) imply condition (c) of Theorem 3.3.

Let us prove now the uniqueness. Let $\tilde{K}\colon [0,T] \to \mathcal{K}_m(\overline{\Omega})$ be another function which satisfies conditions (a)–(e) of Theorem 3.3, and let $\tilde{u}(t)$ be a solution of the minimum problem (2.1) corresponding to $\phi = g(t)$ and $K = \tilde{K}(t)$. Assume by contradiction that there exists an instant $t \in [0,T]$ such that $\tilde{K}(t) \ne \emptyset$ and let $t_0$ be the infimum of such instants. By the finite intersection property we have $\tilde{K}(t_0+) \ne \emptyset$. We will show that properties (a), (d), and (e), together with (5.1), imply that $\tilde{K}(t_0+) = \emptyset$. This contradiction proves that $\tilde{K}(t) = \emptyset$ for every $t \in [0,T]$.

If $t_0 = 0$, by properties (a) and (d), and by (5.1) we have

$$(\tilde{u}(0+), \tilde{K}(0+)) \in \mathcal{R}^0(\tilde{u}(0), \tilde{K}(0)) = \mathcal{R}^0(u(0), \emptyset) = \{(u(0), \emptyset)\}\,,$$

hence $\tilde{K}(0+) = \emptyset$.

If $t_0 > 0$, we have $\tilde{K}(t) = \emptyset$ and $\tilde{u}(t) = u(t)$ for $0 \le t < t_0$. Hence $\tilde{K}(t_0-) = \emptyset$ and $\tilde{u}(t_0-) = u(t_0-)$. By property (e) and by (5.1) we have

$$(\tilde{u}(t_0+), \tilde{K}(t_0+)) \in \mathcal{R}^{t_0}(\tilde{u}(t_0-), \tilde{K}(t_0-)) = \mathcal{R}^{t_0}(u(t_0-), \emptyset) = \{(u(t_0), \emptyset)\}\,,$$

hence $\tilde{K}(t_0+) = \emptyset$. This concludes the proof of the uniqueness. □

## 6. Behaviour Near the Tips

In this section, given $g \in AC([0,T]; H^1(\Omega)) \cap L^\infty([0,T]; L^\infty(\Omega))$, we consider a function $K\colon [0,T] \to \mathcal{K}_m(\overline{\Omega})$ which satisfies conditions (a)–(e) of Theorem 3.3, and study the behaviour of the solutions $u(t)$ near the "tips" of the sets $K(t)$. Under some natural assumptions on the geometry of the sets $K(t)$, we shall see that $K(t)$ satisfies Griffith's criterion for crack growth.

More precisely, let $0 \le t_0 < t_1 \le T$. Suppose that the following structure condition is satisfied: there exists a finite family of simple arcs $\Gamma_i$, $i = 1, \ldots, p$, contained in $\Omega$ and parametrized by arc length by $C^2$ paths $\gamma_i\colon[\sigma_i^0,\sigma_i^1]\to\Omega$, such that, for $t_0<t<t_1$,

$$K(t)=K(t_0)\cup\bigcup_{i=1}^{p}\Gamma_i(\sigma_i(t))\,, \tag{6.1}$$

where $\Gamma_i(\sigma):=\{\gamma_i(\tau):\sigma_i^0\le\tau\le\sigma\}$ and $\sigma_i\colon[t_0,t_1]\to[\sigma_i^0,\sigma_i^1]$ are nondecreasing functions with $\sigma_i(t_0)=\sigma_i^0$ and $\sigma_i^0<\sigma_i(t)<\sigma_i^1$ for $t_0<t<t_1$. Assume also that the arcs $\Gamma_i$ are pairwise disjoint, and that $\Gamma_i\cap K(t_0)=\{\gamma_i(\sigma_i^0)\}$. For $i=1,\dots,p$ and $\sigma_i^0<\sigma<\sigma_i^1$ let $\kappa_i(u,\sigma)$ be the stress intensity factor defined by (8.2) in [8] with $\gamma=\gamma_i$ and $B$ equal to a sufficiently small ball centred at $\gamma_i(\sigma)$.

We are now in a position to state the main result of this section, which expresses Griffith's criterion in our model.

**Theorem 6.1.** *Let $T>0$, $\lambda>0$, $m\ge1$, and $g\in AC([0,T];H^1(\Omega))\cap L^\infty([0,T];L^\infty(\Omega))$. Let $K\colon[0,T]\to\mathcal{K}_m(\overline{\Omega})$ be a function which satisfies conditions (a)–(e) of Theorem 3.3, and let $u(t)$ be a solution of the minimum problem (2.1) defining $\mathcal{E}(g(t),K(t))$. Given $0\le t_0<t_1\le T$, assume that (6.1) is satisfied for $t_0<t<t_1$, and that the arcs $\Gamma_i$ and the functions $\sigma_i$ satisfy all properties considered above. Then*

$$\dot\sigma_i(t)\ge0\qquad\text{for a.e. } t\in(t_0,t_1)\,, \tag{6.2}$$

$$1-\kappa_i(u(t),\sigma_i(t))^2\ge0\qquad\text{for every } t\in(t_0,t_1)\,, \tag{6.3}$$

$$\big\{1-\kappa_i(u(t),\sigma_i(t))^2\big\}\,\dot\sigma_i(t)=0\qquad\text{for a.e. } t\in(t_0,t_1)\,, \tag{6.4}$$

*for $i=1,\dots,p$.*

The proof of Theorem 6.1 is obtained by adapting the proof of Theorem 8.4 of [8]. We indicate here only the changes to be done.

First of all, we need a localized version of the energies $\mathcal{E}$ and $\mathcal{E}_\lambda$. If $A$ is a bounded open set in $\mathbb{R}^2$ with Lipschitz boundary, $K$ is a compact set in $\mathbb{R}^2$, $\phi\colon\partial A\backslash K\to\mathbb{R}$ is a bounded function, and $w\in L^2(A)$, we define

$$\mathcal{E}(\phi,K,A):=\min_{v\in\mathcal{V}(\phi,K,A)}\Big\{\int_{A\backslash K}|\nabla v|^2\,dx+\mathcal{H}^1(K\cap\overline{A})\Big\}\,, \tag{6.5}$$

$$\mathcal{E}_\lambda(\phi,K,A,w):=\min_{v\in\mathcal{V}(\phi,K,A)}\Big\{\int_{A\backslash K}|\nabla v|^2\,dx+\mathcal{H}^1(K\cap\overline{A})+\lambda\int_{A\backslash K}|v-w|^2\,dx\Big\}\,, \tag{6.6}$$

where

$$\mathcal{V}(\phi,K,A):=\{v\in H^1(A\backslash K):v=\phi\quad\text{on }\partial A\backslash K\}\,.$$

Then we can prove the following result for $\mathcal{E}_\lambda$, arguing as in [8, Lemma 8.5].

**Lemma 6.2.** *Let $m\ge1$, $\lambda>0$, let $H\in\mathcal{K}_m(\overline{\Omega})$ with $h$ connected components, let $\phi\in H^1(\Omega)$, $w\in L^2(\Omega)$, and let $u$ be the solution of the minimum problem (2.2) which defines $\mathcal{E}_\lambda(\phi,H,w)$. Given an open subset $A$ of $\Omega$, with Lipschitz boundary, such that $H\cap\overline{A}\ne\emptyset$, let $q$ be the number of connected components of $H$ which meet $\overline{A}$. Assume that*

$$\mathcal{E}_\lambda(\phi,H,w)\le\mathcal{E}_\lambda(\phi,K,w)\qquad\forall\,K\in\mathcal{K}_m(\overline{\Omega}),\ \ K\supset H\,. \tag{6.7}$$

*Then*

$$\mathcal{E}_\lambda(u,H,A,w)\le\mathcal{E}_\lambda(u,K,A,w)\qquad\forall\,K\in\mathcal{K}_{q+m-h}(\overline{A}),\ \ K\supset H\cap\overline{A}\,. \tag{6.8}$$

*Proof of Theorem 6.1.* We now consider in detail the changes needed in the proof of Theorem 8.4 of [8]. Inequality (8.12) must be replaced by

$$\frac{d}{d\sigma}\mathcal{E}_\lambda(u(t),\Gamma_i(\sigma),B_i,u(t))\Big|_{\sigma=\sigma_i(t)}\ge0\,, \tag{6.9}$$

which can be derived, arguing as in [8], from Lemma 6.2 and from the minimality property (b) of Theorem 3.3.

On the other hand, we can show that

$$\frac{d}{d\sigma}\mathcal{E}_\lambda(u(t),\Gamma_i(\sigma),B_i,u(t))\Big|_{\sigma=\sigma_i(t)} = 1-\kappa_i(u(t),\sigma_i(t))^2$$

by adapting the proof of [11, Theorem 6.4.1]. This equality, together with (6.9), proves (6.3).

To obtain (6.4) we continue the proof of Theorem 8.4 of [8], noting that the inequality in condition (h′) of Proposition 3.6 is enough to conclude the proof. □

**Acknowledgements.** The work of Gianni Dal Maso is part of the Project "Calculus of Variations", supported by SISSA and by the Italian Ministry of Education, University, and Research. The work of Rodica Toader is part of the Project "Methods and Problems in Real Analysis", supported by the University of Trieste and by the Italian Ministry of Education, University, and Research.

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