A SINGULAR PERTURBATION APPROACH TO A TWO PHASE PARABOLIC FREE BOUNDARY PROBLEM ARISING IN COMBUSTION THEORY

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Abstract. We study the uniform properties of solutions to a singular perturbation problem associated to a general second order parabolic operator. In particular, our main results show that, under suitable assumptions, the limit function is a pointwise solution to a free boundary problem that naturally arises in combustion theory.

1. Introduction

In recent times there has been a resurgence of interest in the regularity of two phase free boundary problems, especially in the difficult parabolic case. These problems are often approximated by regularizing ones. To obtain information about the solution to the original problem one tries to establish results for the approximating ones which carry over in the limit. In this work we are concerned with a free boundary problem for a large class of parabolic partial differential equations. It consists of the determination of a function $u(x,t)$, defined in a space-time domain $\mathcal{D} \subset \mathbb{R}^{m+1}$, which represents a temperature and is a weak solution to

$$Lu = div A(x,t)\nabla u - \partial_t u + \mathbf{b}(x,t) \cdot \nabla u + c(x,t)u = 0 \quad \text{in } \mathcal{D} \setminus \partial\{u > 0\}. \tag{FBP1}$$

We assume that $A(x,t) = (a_{ij}(x,t))_{i,j} \in C^1(\mathbb{R}^{m+1})$ is an $m \times m$ real, symmetric, uniformly elliptic matrix, with bounded L^∞-norm, and $\mathbf{b}, c \in L^\infty(\mathbb{R}^{m+1})$. The notion of weak solution is described in [LSU]. Two conditions are given on the a priori unknown moving interface $\mathcal{D} \cap \partial\{u > 0\}$, also called the *free boundary*:

$$u = 0, \tag{FBP2}$$

$$\langle A\nabla u^+, \eta\rangle^2 - \langle A\nabla u^-, \eta\rangle^2 = 2M, \tag{FBP3}$$

where M is a positive constant, η denotes the inward spacial normal to $\mathcal{D} \cap \partial\{u > 0\}$, $u^+ = max(u,0)$, and $u^- = max(-u,0)$. This problem arises in a natural way in combustion theory, to describe the propagation in non-homogeneous media of deflagration flames in the limit of high activation energy. It is obtained via an asymptotic method which simplifies the complicated system of nonlinear equations (conservation laws) describing the process of combustion on the basis of physically sound approximations. The main assumption is that of taking to the limit the activation energy of the chemical reaction. For further details we refer the reader to [BL], [BE], and [V]. The very way the problem is derived, as a simplified asymptotic model, suggests viewing it as the limit of approximating singular perturbation problems consisting of the semilinear equations

$$Lu^\varepsilon = \beta_\varepsilon(u^\varepsilon) \tag{SPP}$$

where $\varepsilon \to 0$ (which corresponds to letting the activation energy go to infinity). In order to approximate problem (FBP1-3) as $\varepsilon \to 0$, the term $\beta_\varepsilon(u^\varepsilon)$, which represents a reaction term for the temperature, has to satisfy certain conditions. Specifically, we will assume

$$\beta_\varepsilon(s) = \frac{1}{\varepsilon}\beta\left(\frac{s}{\varepsilon}\right),$$

with β a nonnegative, Lipschitz continuous function, supported in $[0,1]$, such that

$$\|\beta\|_{L^\infty(\mathbb{R})} \le M_0 < \infty \quad \text{and} \quad \int_{\mathbb{R}} \beta\, ds = M.$$

The elliptic version of this problem in the one phase case (i.e. when $u^\varepsilon \ge 0$) and, in particular, the issue of convergence for traveling waves, have been studied in the pioneering work [BCN]. The first results in the parabolic context, again in the one phase setting, were obtained in the important paper [CV], where the authors dealt with the initial value problem associated to (SPP). When L is the heat operator $H = \Delta - \partial_t$, they proved that the functions u^ε converge to a function u which is a weak solution of the free boundary problem

$$\begin{cases} Hu = 0 & \text{in } \{u > 0\}, \\ u = 0,\ u_\eta = \sqrt{2M} & \text{on } \partial\{u > 0\}, \end{cases} \tag{1.1}$$

under suitable assumptions on the initial data u_0^ε. In (1.1), u_η denotes the derivative of u with respect to the inward spacial normal η to the free boundary. More recently, Caffarelli, Lederman and Wolanski ([CLW1], [CLW2]) have continued with the local study of the equation $Hu^\varepsilon = \beta_\varepsilon(u^\varepsilon)$, in a domain $\mathcal{D} \subset \mathbb{R}^{m+1}$, in the more general two phase setting (that is, the solutions are allowed to change sign). They have shown that the limit function u satisfies

$$\begin{cases} Hu = 0 & \text{in } \mathcal{D} \setminus \partial\{u > 0\}, \\ u = 0, \quad (u_\eta^+)^2 - (u_\eta^-)^2 = 2M & \text{on } \mathcal{D} \cap \partial\{u > 0\} \end{cases} \tag{1.2}$$

in a pointwise sense at "regular" free boundary points and in a viscosity sense when $\{u \equiv 0\}^\circ = \emptyset$.

The aim of the present work is to extend these results to the general operator L which appears in (FBP1). We consider a family $\{u^\varepsilon\}$ of solutions to (SPP) in a domain $\mathcal{D} \subset \mathbb{R}^{m+1}$. It has been shown in [CK] that if $\{u^\varepsilon\}$ is uniformly bounded, then it is locally uniformly Lipschitz continuous in space. Section 2 is devoted to the construction of a family of uniformly bounded solutions, obtained by solving a Neumann-type boundary value problem with uniformly bounded initial data. In Section 3 we prove that uniformly bounded solutions are also locally uniformly Hölder continuous in time, with exponent 1/2, and that they converge uniformly on compact subsets of $\mathcal{D}$ to a function u which is a solution to

$$Lu = \mu,$$

where μ is a nonnegative measure supported on $\mathcal{D} \cap \partial\{u > 0\}$. In particular, u satisfies (FBP1) and (FBP2). Next, we address the following central question: *Is the free boundary condition* (FBP3) *satisfied?* The strategy to provide a positive answer is to investigate the local behavior of the limit function u around a free boundary point $(x_0, t_0) \in \mathcal{D} \cap \partial\{u > 0\}$. Our main results, Theorems 1.1 and 1.2, show that, under suitable assumptions, u has an asymptotic development around (x_0, t_0) which implies that both $\langle A(x,t)\nabla u^+, \nu\rangle$ and $\langle A(x,t)\nabla u^-, \nu\rangle$ exist, and that condition (FBP3) holds in a pointwise sense. We introduce first the relevant definitions.

Definition 1.1. *A unit vector $\eta \in \mathbb{R}^m$ is said to be the* inward unit spacial normal in the parabolic measure theoretic sense *to the free boundary $\partial\{u > 0\}$ at a point $(x_0, t_0) \in \partial\{u > 0\}$ if*

$$\lim_{r\to 0} \frac{1}{r^{m+2}} \iint_{Q_r(x_0,t_0)} |\chi_{\{u>0\}} - \chi_{\{(x,t)|\langle x-x_0,\eta\rangle>0\}}|\, dx\, dt = 0. \tag{1.3}$$

Definition 1.2. *Let u be a continuous function in a domain $\mathcal{D} \subset \mathbb{R}^{m+1}$. A point $(x_0, t_0) \in \mathcal{D} \cap \partial\{u > 0\}$ is said to be* regular from the positive side *if there is a cylinder $Q_\rho(y, s) \subset \{u > 0\}$ such that $(x_0, t_0) \in \partial Q_\rho(y, s)$.*

Definition 1.3. *Let $E, \Gamma \subset \mathbb{R}^{m+1}$. We say that E has* uniform positive density *on Γ if there exists $c, r_0 > 0$ such that*

$$\frac{|E \cap Q_r(x,t)|}{|Q_r(x,t)|} \geq c \qquad \text{for } 0 < r < r_0,\ (x,t) \in \Gamma.$$

Definition 1.4. *Let v be a continuous function in a domain $\mathcal{D} \subset \mathbb{R}^{m+1}$. We say that v is* nondegenerate *at a point $(x_0, t_0) \in \mathcal{D} \cap \{v = 0\}$ if there exist c, $r_0 > 0$ such that*

$$\frac{1}{r^{m+2}} \iint_{Q_r^-(x_0,t_0)} v\, dx\, dt \geq c\, r \qquad \text{for any } r \in (0, r_0).$$

Our first hypothesis concerns the regularity of the free boundary at (x_0, t_0):

(H1) *$\partial\{u > 0\}$ has at (x_0, t_0) an inward unit spacial normal η in the parabolic measure theoretic sense.*

Moreover, without loss of generality, we may assume:

(H2) $A(x_0, t_0) = I$.

The following theorems precisely describe the above mentioned asymptotic expansion. Due to their intrinsically different natures, it is necessary to treat the one- and two-phase cases separately.

Theorem 1.1. (One-phase case) *Let u^{ε_j} be solutions to (SPP) in a domain $\mathcal{D} \subset \mathbb{R}^{m+1}$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of $\mathcal{D}$ and $\varepsilon_j \to 0$. Assume that $(x_0, t_0) \in \mathcal{D} \cap \partial\{u > 0\}$ satisfies (H1) and (H2), and that $u \geq 0$ in $\mathcal{D}$. Let $\Gamma \subset \mathcal{D} \cap \partial\{u > 0\}$ denote the set of free boundary points that are regular from the positive side and suppose that:*
(H3) *There exists $\delta > 0$ such that the set $\{u = 0\}$ has uniform positive density on $\Gamma \cap Q_\delta(x_0, t_0)$;*
(H4) *u is nondegenerate at (x_0, t_0).*
Under these assumptions, we have

$$u(x,t) = \sqrt{2M}\langle x - x_0, \eta\rangle^+ + o(|x - x_0| + |t - t_0|^{1/2}). \tag{1.4}$$

We would like to emphasize that Theorem 1.1 is new even in the special case $L = H$, and complements the results already available in the literature. In fact, in [CLW2] the authors prove in particular that nonnegative limit solutions to $Hu^\varepsilon = \beta_\varepsilon(u^\varepsilon)$ satisfy (1.4) under the assumption that the set $\{u = 0\}$ have vanishing density at (x_0, t_0), i.e. $\lim_{r\to 0} \frac{|\{u=0\}\cap Q_r(x_0,t_0)|}{|Q_r(x_0,t_0)|} = 0$. The corresponding result in the two phase setting reads as follows.

Theorem 1.2. (Two-phase case) *Let u^{ε_j} be solutions to (SPP) in a domain $\mathcal{D} \subset \mathbb{R}^{m+1}$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of $\mathcal{D}$, and $\varepsilon_j \to 0$. Let $(x_0, t_0) \in \mathcal{D} \cap \partial\{u > 0\}$ satisfy (H1) and (H2). Assume moreover:*
(H4') *u^- is nondegenerate at (x_0, t_0).*
Then there exist $\alpha, \gamma > 0$ such that

$$u(x,t) = \alpha\langle x - x_0, \eta\rangle^+ - \gamma\langle x - x_0, \eta\rangle^- + o(|x - x_0| + |t - t_0|^{1/2}), \tag{1.5}$$

with

$$\alpha^2 - \gamma^2 = 2M.$$

We explicitely observe that Theorems 1.1 and 1.2 are of a local nature. In fact, on the one hand, the u^ε are not forced to be globally defined, nor to take on prescribed initial or boundary values. On the other hand, all of the hypotheses are made only at the point (x_0, t_0). The one exception is given by (H3), which can be interpreted as a nondegeneracy condition on the vanishing part of u in a (small) neighborhood of (x_0, t_0). Conditions of the type (H3), (H4), and (H4') first appeared in the study of free boundary problems in [AC] and [ACF], and they seem to be natural assumptions in order to prove the regularity of the interface, see the discussion below.

The key ingredient in proving Theorems 1.1 and 1.2 is a blow-up argument. Precisely, one performs a parabolic scaling of u around (x_0, t_0) by letting $u_\lambda(x,t) = \frac{1}{\lambda}u(x_0 + \lambda x, t_0 + \lambda^2 t)$, and studies the limit U obtained when $\lambda \to 0$. The core of the proof is then to show that U is piecewise linear, with the "right" gradient jump. From this fact, the desired conclusion readily

follows. In order to carry out this plan, it is necessary to understand a very special case, namely when the limit of solutions to a rescaled problem is given by the difference of two hyperplanes, see Propositions 3.5 and 3.6. Section 4 is devoted to the proof of Theorem 1.1. One of the main ingredients in the proof consists in the study of the precise behavior of ∇u near free boundary points, see Theorem 4.1. The proof of Theorem 1.2 is contained in Section 5. We mention here that the nondegeneracy assumption on u^- allows to apply a beautiful two-phase monotonicity formula, due to Caffarelli and Kenig [CK], combined with an important convexity property of eigenvalues proved by Beckner, Kenig and Pipher [BKP]. Finally, we show in Theorem 5.1 that if the free boundary is given by a differentiable hypersurface in a neighborhood of (x_0, t_0), then the conclusion of Theorem 1.2 continues to hold with $\gamma \geq 0$, and assumptions (H3), (H4), and (H4') replaced by a single weaker nondegeneracy condition:
(H5) If $\liminf_{r\to 0} \frac{|\{u<0\}\cap Q_r(x_0,t_0)|}{|Q_r(x_0,t_0)|} = 0$, then u^+ is nondegenerate at (x_0, t_0).

The results presented in this paper constitute the first crucial step in investigating the regularity properties of the free boundary $\partial\{u > 0\}$ for the problem (FBP1-3). Let us remark that there exist limit functions which do not satisfy the free boundary condition (FBP3) in the classical sense on any portion of the interface. Hence, extra hypotheses need to be made in order to obtain regularity results. In the elliptic setting, this problem has been treated in [LW] for the Laplace equation, using the fundamental regularity theory, developed by Caffarelli ([C1], [C2], [C3]) for viscosity solutions of a class of elliptic free boundary problems which includes the one under consideration. The main results in [LW] show that, if u^+ is locally uniformly nondegenerate and the set $\{u \leq 0\}$ has locally uniform positive density on $\partial\{u > 0\}$, then there is a subset of the interface, whose complement has vanishing $(m-1)$-dimensional Hausdorff measure, which is locally a $C^{1,\alpha}$ surface. In addition, if u^- is locally uniformly nondegenerate on $\partial\{u > 0\}$, then the free boundary is locally a $C^{1,\alpha}$ surface, and therefore there are no singularities. These results, in particular, show that assumptions of the type (H3)-(H5) above are natural for this type of free boundary problems. Moreover, an asymptotic expansion analogous to (1.5) plays a crucial role in their analysis. The study of the regularity properties of the interface for solutions to (FBP1-3) is the object of forthcoming work.

Notations. The following notations will be used. We let $\nu, \Lambda > 0$ be such that $\nu|\xi|^2 \leq \langle A(x,t)\xi, \xi\rangle \leq \nu^{-1}|\xi|^2$ for every $(x,t) \in \mathbb{R}^{m+1}$ and $\xi \in \mathbb{R}^m$, and $\|\mathbf{b}\|_\infty + \|c\|_\infty \leq \Lambda$. Given $K \subset\subset \mathbb{R}^{m+1}$, $\omega = \omega(K)$ denotes a positive constant such that $\max_{i,j=1,\dots,m} \|\nabla_{x,t} a_{ij}\|_{L^\infty(K)} \leq \omega$. For any $r > 0$, $(x_0, t_0) \in \mathbb{R}^{m+1}$, and $K \subset \mathbb{R}^{m+1}$ we set: $B_r(x_0) = \{x \in \mathbb{R}^m \mid |x - x_0| < r\}$, $Q_r(x_0, t_0) = B_r(x_0) \times (t_0 - r^2, t_0 + r^2)$, $Q_r^-(x_0, t_0) = Q_r(x_0, t_0) \cap \{t \leq t_0\}$, $Q_r = Q_r(0, 0)$, $Q_r^- = Q_r^-(0, 0)$, $N_r(K) = \{(x, t) \mid (x, t) \in Q_r(x_0, t_0)$ for some $(x_0, t_0) \in K\}$, $N_r^-(K) = \{(x, t) \mid (x, t) \in Q_r^-(x_0, t_0)$ for some $(x_0, t_0) \in K\}$. We also let $d_p((x, t), (y, s)) = \max\left(|x - y|, |t - s|^{1/2}\right)$, and for a set $E \subset \mathbb{R}^{m+1}$, $d_p((x,t),E) = \inf_{(y,s)\in E} d_p((x,t),(y,s))$. The symbol ∂_p will denote the parabolic boundary. We define

$$B_\varepsilon(s) = \int_0^s \beta_\varepsilon(\tau)\, d\tau.$$

A function v is in the class $Lip_{loc}(1,1/2)$ in a domain $\mathcal{D} \subset \mathbb{R}^{m+1}$ if for any $K \subset\subset \mathcal{D}$ there exists a constant $L = L(K)$ such that $|v(x,t) - v(y,s)| \le L\left(|x-y| + |t-s|^{1/2}\right)$ for any $(x,t),(y,s) \in K$. Finally, C will denote an all purpose constant.

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2. Global uniform estimates

In this section we consider an initial-boundary value problem for the equation (SPP) and we prove that if the initial data u_0^ε are uniformly bounded, so are the solutions u^ε to the problem. In the sequel Ω will denote a bounded domain in $\mathbb{R}^m$ satisfying the interior sphere condition at every point of the boundary. We let $\Omega_T = \Omega \times (0,T)$ for some $T > 0$, and consider the problem

$$\begin{cases} Lu^\varepsilon = \beta_\varepsilon(u^\varepsilon) & \text{in } \Omega_T, \\ A\nabla u^\varepsilon \cdot \eta = 0 & \text{on } \partial\Omega \times (0,T), \\ u^\varepsilon(x,0) = u_0^\varepsilon(x) & x \in \Omega, \end{cases} \tag{2.1}$$

where η denotes the outward unit normal to Ω.

We begin our study with a suitable version of the maximum principle.

Theorem 2.1. *Assume v is a weak L-subsolution in Ω_T, i.e.*

$$Lv \ge 0 \qquad \text{in } \Omega_T, \tag{2.2}$$

with boundary conditions

$$\begin{cases} A\nabla v \cdot \eta = 0 & \text{on } \partial\Omega \times (0,T), \\ v(x,0) = v_0(x) & x \in \Omega. \end{cases} \tag{2.3}$$

Suppose there exists a positive constant A_0 such that $v_0 \le A_0$ in Ω. Then

$$v(x,t) \le A_0 \qquad \text{for a.e. } (x,t) \in \Omega_T.$$

The proof of Theorem 2.1 is inspired to that in [AS, Theorem 1] for the Dirichlet problem. In the sequel, we will need the following modification of [AS, Lemma 6].

Lemma 2.1. *Suppose that w is weakly differentiable with respect to t in Ω_T and vanishes in a neighborhood of the boundary set $\{t = 0\}$. Then*

$$\int_0^\tau \int_\Omega w_t \, dx \, dt = \int_\Omega w \mid_{t=\tau} \, dx$$

for a.e. $\tau \in (0, T)$.

Proof. Let $\psi = \psi(t) \in C_0^\infty(0, T)$, $\phi = \phi(x) \in C_0^\infty(\Omega)$. Then

$$\int_0^T \psi_t \left(\int_\Omega w\phi \, dx \right) dt = \iint_{\Omega_T} \psi_t \phi \, w \, dx \, dt = -\iint_{\Omega_T} \psi\phi \, w_t \, dx \, dt = -\int_0^T \psi \left(\int_\Omega w_t \phi \, dx \right) dt,$$

and thus, since ψ is arbitrary,

$$\left(\int_\Omega w\phi \, dx \right)_t = \int_\Omega w_t \phi \, dx. \tag{2.4}$$

On the other hand,

$$\int_0^\tau \left(\int_\Omega w\phi \, dx \right)_t dt = \left(\int_\Omega w\phi \, dx \right) \Big|_{t=\tau} = \int_\Omega w \mid_{t=\tau} \phi \, dx \tag{2.5}$$

for a.e. $\tau \in (0, T)$. Combining (2.4) and (2.5) we obtain

$$\int_\Omega \left(\int_0^\tau w_t \, dt \right) \phi \, dx = \int_0^\tau \int_\Omega w_t \phi \, dx \, dt = \int_\Omega w \mid_{t=\tau} \phi \, dx,$$

for a.e. $\tau \in (0, T)$. Since ϕ is also arbitrary, we conclude

$$\int_0^\tau w_t \, dt = w \mid_{t=\tau} \qquad \text{a.e. in } \Omega, \text{ for a.e. } \tau \in (0, T),$$

and the lemma is proved. □

Proof of Theorem 2.1. We begin making the temporary assumption that v has a weak derivative $v_t \in L^{2,2}_{loc}(\Omega_T)$.

Suppose first $A_0 < 0$. Set $\overline{v} = max\{v, 0\}$ and for $0 < \tau < T$ define $\varphi(x, t) = \overline{v}(x, t)\chi(t, \tau)$, where $\chi(t, \tau)$ denotes the characteristic function of the interval $(0, \tau)$. Under the current assumptions φ is an admissible test function. We thus have

$$\iint_{\Omega_T} A\nabla v \cdot \nabla\varphi \, dx \, dt - \iint_{\Omega_T} v\varphi_t \, dx \, dt - \iint_{\Omega_T} \mathbf{b} \cdot \nabla v \, \varphi \, dx \, dt - \iint_{\Omega_T} c \, v \, \varphi \, dx \, dt$$
$$\leq \int_\Omega v_0(x)\varphi(x, 0) \, dx = 0.$$

Using the chain rule for weakly differentiable functions (see, e.g., [AS, Lemma 5]) and Lemma 2.1, this inequality can be rewritten as

$$\iint_{\Omega_T} A\nabla v \cdot \nabla\varphi \, dx \, dt + \frac{1}{2} \int_\Omega \overline{v}^2 \mid_{t=\tau} \, dx - \iint_{\Omega_T} \mathbf{b} \cdot \nabla v \, \varphi \, dx \, dt - \iint_{\Omega_T} c \, v \, \varphi \, dx \, dt \leq 0 \tag{2.6}$$

for a.e. $\tau \in (0,T)$. Next, note that on the set where $\varphi > 0$ one has $\nabla v = \nabla \overline{v}$ and $|v| = v$. On this set,

$$A\nabla v \cdot \nabla \varphi - \mathbf{b} \cdot \nabla v\, \varphi - c\, v\, \varphi = A\nabla \overline{v} \nabla \overline{v} - \mathbf{b} \cdot \nabla \overline{v}\, \overline{v} - c\overline{v}^2 \geq \nu |\nabla \overline{v}|^2 - \Lambda |\nabla \overline{v}| \overline{v} - \Lambda \overline{v}^2. \tag{2.7}$$

Observing that

$$\Lambda |\nabla \overline{v}| \overline{v} \leq \frac{\nu}{2} |\nabla \overline{v}|^2 + \frac{\Lambda^2}{2\nu} \overline{v}^2$$

we infer from (2.7)

$$A\nabla v \cdot \nabla \varphi - \mathbf{b} \cdot \nabla v\, \varphi - c\, v\, \varphi \geq \frac{\nu}{2} |\nabla \overline{v}|^2 - \frac{\tilde{\Lambda}}{2} \overline{v}^2, \tag{2.8}$$

where $\tilde{\Lambda} = 2\Lambda \left(\frac{\Lambda}{2\nu} + 1\right)$. The latter inequality holds also on the set where $\varphi = 0$. We infer from (2.6) and (2.8) that

$$\nu \int_0^\tau \int_\Omega |\nabla \overline{v}|^2 \, dx\, dt + \int_\Omega \overline{v}^2 \,|_{t=\tau}\, dx \leq \tilde{\Lambda} \int_0^\tau \int_\Omega \overline{v}^2 \, dx\, dt.$$

If $\chi(t, \tau_1, \tau_2)$ denotes the characteristic function of the interval $0 < \tau_1 < \tau_2$, $0 < \tau_1 < \tau_2 < T$, a slight modification of the above arguments allows to show the fundamental inequality

$$\nu \int_{\tau_1}^{\tau_2} \int_\Omega |\nabla \overline{v}|^2 \, dx\, dt + \int_\Omega \overline{v}^2 \Big|_{t=\tau_1}^{t=\tau_2} dx \leq \tilde{\Lambda} \int_{\tau_1}^{\tau_2} \int_\Omega \overline{v}^2 \, dx\, dt \tag{2.9}$$

for a.e. $\tau_1, \tau_2 \in (0,T)$. Using Cauchy–Schwarz's inequality, the integral appearing in the right hand side of (2.9) can be estimated as follows:

$$\begin{aligned} \int_{\tau_1}^{\tau_2} \int_\Omega \overline{v}^2 \, dx\, dt &\leq \left(\int_{\tau_1}^{\tau_2} \left(\int_\Omega \overline{v}^2 \, dx \right)^2 dt \right)^{1/2} (\tau_2 - \tau_1)^{1/2} \\ &\leq (\tau_2 - \tau_1)^{1/2} \left(\sup_{t \in (\tau_1, \tau_2)} \|\overline{v}(\cdot, t)\|^2_{L^2_x(\Omega)} \right)^{1/2} \left(\int_{\tau_1}^{\tau_2} \int_\Omega \overline{v}^2 \, dx\, dt \right)^{1/2}. \end{aligned} \tag{2.10}$$

If $\|\overline{v}\|_{L^2_{x,t}(\Omega \times (\tau_1, \tau_2))} = 0$, clearly $\overline{v} = 0$ a.e. in $\Omega \times (\tau_1, \tau_2)$ and so $v \leq 0$ a.e. in this cylinder. Assume instead $\|\overline{v}\|_{L^2_{x,t}(\Omega \times (\tau_1, \tau_2))} > 0$. Then it follows from (2.9) and (2.10) that for a.e. $\tau_1, \tau_2 \in (0,T)$

$$\int_\Omega \overline{v}^2 \Big|_{t=\tau_1}^{t=\tau_2} dx \leq \tilde{\Lambda}(\tau_2 - \tau_1) \sup_{t \in (\tau_1, \tau_2)} \|\overline{v}(\cdot, t)\|^2_{L^2_x(\Omega)}. \tag{2.11}$$

Now let s be a time variable over the interval $I = (\tau_1, \tau_1 + \frac{1}{2\tilde{\Lambda}})$, and set $\mathcal{X}(s) = \int_\Omega \overline{v}^2(x, s)\, dx$. Replacing τ_2 with s, we then deduce from (2.11) that for a.e. $s \in I$ it holds

$$\mathcal{X}(s) \leq \tilde{\Lambda}(s - \tau_1) \sup_{t \in (\tau_1, s)} \|\overline{v}(\cdot, t)\|^2_{L^2_x(\Omega)} + \mathcal{X}(\tau_1),$$

which in turn yields

$$\operatorname*{ess\,sup}_{s \in I} \mathcal{X}(s) \leq 2\mathcal{X}(\tau_1).$$

Iteration of this inequality gives

$$\mathcal{X}(s) \leq 2^{1+2\tau_1/\mu}\mathcal{X}(0) = 0$$

for a.e. $s \in I$. But this implies $\overline{v} = 0$ in $\Omega \times I$, which is absurd. In conclusion, we have proved that if $A_0 < 0$, then $v \leq 0$ a.e. in $\Omega \times (\tau_1, \tau_2)$. Since τ_1, τ_2 were arbitrarily chosen in $(0,T)$, we conclude that $v \leq 0$ a.e. in Ω_T. At this point, we can dispose of the assumption $A_0 < 0$. Let $\delta > 0$ and set $V = v - A_0 - \delta$. Then V is a solution to the problem (2.2)–(2.3), with $V < 0$ on $\{t = 0\}$. By virtue of our previous conclusion, $V \leq 0$ a.e. in Ω_T, and thus $v \leq A_0 + \delta$ a.e. in Ω_T. Letting $\delta \to 0$ we obtain $v \leq A_0$ a.e. in Ω_T.

Finally, we need only to remove the assumption that v has a weak derivative $v_t \in L^{2,2}_{loc}(\Omega_T)$. To this end we let $\tau \in (0,T)$, $h \in (0, T-\tau)$, and $w_h(x,t) = 1/h \int_t^{t+h} w(x,s)\,ds$ be the Steklov average of w. Observe that $\varphi_h(x,t) = \overline{v_h}\chi(t,\tau)$ is an admissible test function in the weak formulation of (2.2) and that if $v < 0$ near $\{t = 0\}$, then φ_h there vanishes provided h is small enough. Under this assumption, a simple calculation shows

$$\iint_{\Omega_T} (A\nabla v)_h \cdot \nabla\varphi_h \,dx\,dt + \iint_{\Omega_T} \partial_t v_h\, \varphi_h \,dx\,dt - \iint_{\Omega_T} (\mathbf{b}\cdot\nabla v)_h\, \varphi_h \,dx\,dt - \iint_{\Omega_T} (c\,v)_h \varphi_h \,dx\,dt$$
$$\leq \int_\Omega v_0(x)\varphi_h(x,0)\,dx = 0.$$

We can now integrate with respect to t and infer that

$$\iint_{\Omega_T} (A\nabla v)_h \cdot \nabla\varphi_h \,dx\,dt + \frac{1}{2}\int_\Omega \overline{v_h}^2 \mid_{t=\tau} \,dx - \iint_{\Omega_T} (\mathbf{b}\cdot\nabla v)_h\, \varphi_h \,dx\,dt - \iint_{\Omega_T} (c\,v)_h \varphi_h \,dx\,dt \leq 0.$$

Sending $h \to 0$ we find that (2.6) holds, and therefore we can repeat the above arguments. The proof is concluded. □

We also need to recall the following result from [CK].

Theorem 2.2. *Let $\theta \in C_0^\infty(\mathbb{R}^m)$, $0 \leq \theta \leq 1$, $\theta \equiv 1$ in $B_{1/4}$ and $supp\,\theta \subset B_{1/2}$. For $\rho > 0$ define $\psi_\rho(x,t) = \theta(x/\rho)$. There exist $\rho_0 = \rho_0(m,\nu,\Lambda,\omega)$ and $C = C(m,\nu,\Lambda,\omega)$ such that if $\rho \leq \rho_0$, $Lv = 0$ in $R_\rho = B_\rho \times [0,\rho^2]$, and $v\mid_{\partial_p R_\rho} = \psi_\rho \mid_{\partial_p R_\rho}$, then*

$$\frac{\partial v}{\partial \eta} \geq C > 0 \qquad \text{on } \partial_p R_\rho \cap \{t \geq \rho^2/2\}.$$

Here $\frac{\partial}{\partial\eta}$ denotes differentiation in the direction of the unit normal.

At this point we are ready to prove that if the initial data u_0^ε in (2.1) are uniformly bounded, so are the solutions u^ε.

Proposition 2.1. *Let $u^\varepsilon \in C(\overline{\Omega_T}) \cap C^{1,1/2}(\Omega_T)$ be a family of solutions to (2.1), with $c \leq 0$. If $\|u_0^\varepsilon\|_{L^\infty(\Omega)} \leq A_0$ for some $A_0 \geq 0$, then $\|u^\varepsilon\|_{L^\infty(\Omega_T)} \leq A_0$.*

Proof. Since $\beta_\varepsilon \geq 0$, the assertion $u^\varepsilon \leq A_0$ in Ω_T clearly follows from Theorem 2.1. We now want to prove that $u^\varepsilon \geq -A_0$ in Ω_T. By contradiction, assume that u^ε attains an interior minimum $-A_1 < -A_0$ at $(x_0, t_0) \in \Omega_T$. If we let $\|u^\varepsilon\|_{C^{1,1/2}(\Omega_T)} = L$ and $r_0 = \min\left\{\frac{A_1}{2L}, \frac{1}{\sqrt{2}} d_p\left((x_0,t_0), \partial_p\Omega_T\right)\right\}$, then $Q_0 = Q^-_{r_0}(x_0,t_0) \subset\subset \Omega_T$, and $u^\varepsilon \leq 0$ in Q_0. It follows that

$$L(u^\varepsilon + A_1) = L(u^\varepsilon) + c\, A_1 \leq 0 \qquad \text{in } Q_0,$$

since c is nonpositive and $\beta_\varepsilon(s) = 0$ for $s < 0$. By the strong minimum principle, $u^\varepsilon = -A_1$ in Q_0. It is now possible to repeat this argument with (x_0, t_0) replaced by $(x_0, t_0 - r_0^2)$. Iterating this procedure, one finds sequences $t_k = t_{k-1} - r_{k-1}^2$ and $r_k = \min\left\{\frac{A_1}{2L}, \frac{1}{\sqrt{2}} d_p\left((x_0,t_k), \partial_p\Omega_T\right)\right\}$ such that $u^\varepsilon = -A_1$ in $Q^-_{r_k}(x_0,t_k)$. In particular, $u^\varepsilon(x_0,t_k) = -A_1$ for all $k = 0, 1, \ldots$. We explicitely observe that there exists $\overline{k} \geq 0$ such that $r_k = \sqrt{\frac{t_k}{2}}$ for all $k \geq \overline{k}$, and thus $t_k \to 0$ as $k \to \infty$. We may conclude, letting $k \to \infty$ and recalling that $u^\varepsilon \in C(\overline{\Omega_T})$, that $u^\varepsilon(x_0, 0) = -A_1$. But this contradicts the hypothesis $u_0^\varepsilon \geq -A_0 > -A_1$ in Ω, and therefore u^ε has no interior minimum.

Next, we assume that u^ε attains its minimum $-A_1 < -A_0$ on the lateral boundary $\partial\Omega \times (0,T)$. Let $\tau = \inf\{t \in (0,T) \mid \text{There exists } x \in \partial\Omega \text{ such that } u^\varepsilon(x,t) = -A_1\}$. Since $u_0^\varepsilon \geq -A_0$ in Ω, we have $\tau > 0$. Let $P = (\xi, \tau)$, with $\xi \in \partial\Omega$, be such that $u^\varepsilon(\xi,\tau) = -A_1$. Without loss of generality we may assume $A(\xi,\tau) = I$. We note that $d_p(P, \{u > 0\}) > 0$, and the fact that Ω satisfies an interior sphere condition at every boundary point, imply the existence of $\xi_0 \in \Omega$ and $\rho > 0$ such that $|\xi - \xi_0| = \rho$, $\widetilde{Q} = Q^-_\rho(\xi_0, \tau) \subset \{u \leq 0\} \cap \Omega_T$. In $\widetilde{Q}$ the function $u^\varepsilon + A_1$ satisfies the equation $L(u^\varepsilon + A_1) \leq 0$. Moreover, since we have already ruled out the possibility that u^ε attains its minimum value $-A_1$ in the interior of Ω_T, and $u^\varepsilon > -A_1$ on $\partial\Omega \times (0,\tau)$, we have that

$$\alpha = \min_{\overline{B_\rho(\xi_0) \times \{t = \tau - \rho^2\}}} (u^\varepsilon + A_1) > 0.$$

Let v be a solution to $Lv = 0$ in $\widetilde{Q}$ such that $v \mid_{\partial_p \widetilde{Q}} = \psi_\rho \mid_{\partial_p \widetilde{Q}}$, where ψ_ρ is as in Theorem 2.2. By Theorem 2.2, there exists $C = C(m, \nu, \Lambda, \omega)$ such that

$$\frac{\partial v}{\partial \eta} \geq C > 0 \qquad \text{on } \partial_p \widetilde{Q} \cap \{t \geq \tau - \rho^2/2\}. \tag{2.12}$$

By the maximum principle, $u^\varepsilon + A_1 - \alpha v \geq 0$ in $\widetilde{Q}$ and therefore, by (2.12),

$$\frac{\partial u^\varepsilon}{\partial \eta}(P) = \frac{\partial (u^\varepsilon + A_1)}{\partial \eta} \geq \alpha \frac{\partial v}{\partial \eta}(P) > 0.$$

But u^ε is a solution to the problem (2.1) and so, in particular,

$$\frac{\partial u^\varepsilon}{\partial \eta}(P) = 0.$$

This contradiction shows that the minimum of u^ε cannot be attained on the lateral boundary of Ω_T, and thus

$$u^\varepsilon \geq -A_0 \qquad \text{in } \Omega_T.$$

This completes the proof of the proposition. $\square$

3. Local uniform estimates and passage to the limit as $\varepsilon \to 0$

In this section we prove uniform estimates for an uniformly bounded family $\{u_\varepsilon\}$ of solutions to (SPP) in a domain $\mathcal{D} \subset \mathbb{R}^{m+1}$, and then we establish some convergence results when $\varepsilon \to 0$. These results, in particular, apply to the family of solutions constructed in Section 2. As a consequence, we show that the limit function u is a solution of the free boundary problem (FBP1-2) in a weak sense. Our first goal is to show that if the family $\{u^\varepsilon\}$ of solutions to (SPP) is uniformly bounded in the L^∞-norm in $\mathcal{D}$, it is also locally uniformly bounded in the $Lip(1, \frac{1}{2})$-seminorm in $\mathcal{D}$.

Theorem 3.1. *Let $\{u^\varepsilon\}$ be a family of solutions to (SPP) in a domain $\mathcal{D} \subset \mathbb{R}^{m+1}$ such that $\|u^\varepsilon\|_{L^\infty(\mathcal{D})} \leq A_0$, for some $A_0 > 0$. Let $K \subset \mathcal{D}$ be a compact set and let $\tau > 0$ be such that $N_{2\tau}(K) \subset \mathcal{D}$. There exists a positive constant $L = L(m, \nu, \Lambda, \omega, M_0, A_0, \tau)$ such that*

$$|u^\varepsilon(x,t) - u^\varepsilon(y,s)| \leq L\left(|x-y| + |t-s|^{1/2}\right) \qquad \text{for } (x,t), (y,s) \in K.$$

We begin by recalling the local bound on the gradient established in [CK]:

Theorem 3.2. *Suppose that (SPP) holds in Q_4. If $\|u\|_{L^\infty(Q_4)} \leq A_0$, then there exists $C = C(m, \nu, \Lambda, \omega, M_0, A_0) > 0$ such that*

$$\|\nabla u\|_{L^\infty(Q_{1/2})} \leq C.$$

As an immediate corollary, we have the following result.

Proposition 3.1. *Let $\{u^\varepsilon\}$ be a family of solutions to (SPP) in a domain $\mathcal{D} \subset \mathbb{R}^{m+1}$. Assume there exists a constant $A_0 > 0$ independent of ε such that $\|u_\varepsilon\|_{L^\infty(\mathcal{D})} \leq A_0$. Let $K \subset \mathcal{D}$ be a compact set such that $N^-_\tau(K) \subset \mathcal{D}$ for some $\tau > 0$. There exists a positive constant $A_1 = A_1(m, \nu, \Lambda, \omega, M_0, A_0, \tau)$ such that*

$$|\nabla u^\varepsilon(x,t)| \leq A_1 \qquad \text{for any } (x,t) \in K.$$

Proof. Apply Theorem 3.2 to $v^\varepsilon_\tau(x,t) = \frac{1}{\tau} u^\varepsilon(x_0 + \tau x, t_0 + \tau^2 t)$, with $(x_0, t_0) \in K$. $\square$

In order to prove Theorem 3.1, we will also need the following proposition.

Proposition 3.2. *Let u be a solution to $Lu = f$ in $Q_1^-(0,0)$, with $\|f\|_\infty < M_0$ in $\{u<0\}\cup\{u>1\}$. If $\|u\|_\infty \le A_0$ and $\|\nabla u\|_\infty \le A_1$, then there exist two positive constant $C_1 = C_1(m)$ and $C_2 = C_2(m,\nu,\Lambda,M_0)$ such that*

$$|u(0,-t) - u(0,0)| \le C_1\Lambda A_0 + C_2 A_1 \qquad \textit{for all } t \in (0,1).$$

Proof. We begin by showing that if the cylinder $B_1 \times (s,t)$, with $-1 < s < t \le 0$, is contained in $\{u<0\}\cup\{u>1\}$, then

$$|u(0,s) - u(0,t)| \le C_1\Lambda A_0 + C_2 A_1.$$

Without loss of generality we may assume $A_1 > 1$. Taking $\varphi \in C_0^\infty(B_1)$, with $\int \varphi\, dx = 1$, as test function in the weak formulation of $Lu = f$, we obtain

$$\int_{B_1} u(x,t)\varphi\, dx - \int_{B_1} u(x,s)\varphi\, dx = -\iint_{B_1\times(s,t)} A\nabla u\cdot\nabla\varphi\, dx\, d\tau$$
$$+ \iint_{B_1\times(s,t)} \mathbf{b}\cdot\nabla u\,\varphi\, dx\, d\tau + \iint_{B_1\times(s,t)} c\, u\,\varphi\, dx\, d\tau - \iint_{B_1\times(s,t)} f\varphi\, dx\, d\tau.$$

We thus obtain

$$\left|\int_{B_1} u(x,t)\varphi\, dx - \int_{B_1} u(x,s)\varphi\, dx\right| \le C_m\left(\nu^{-1}A_1 + \Lambda(A_1+A_0) + M_0\right) \le C_1\Lambda A_0 + C_2 A_1.$$

By Taylor's formula, $u(x,s) = u(0,s) + O(|x|)$ and $u(x,t) = u(0,t) + O(|x|)$. Hence,

$$|u(0,s) - u(0,t)| \le C_1\Lambda A_0 + C_2 A_1. \qquad \text{for } -1 < s < t \le 0. \tag{3.1}$$

Consider now the cylinder $B_1(0)\times(-t,0)$, for $0 < t < 1$. If it is contained in $\{u<0\}\cup\{u>1\}$ we simply apply (3.1) to obtain the desired conclusion. If not, let $t_1 = \inf\{s \in (-t,0) \mid \text{There exists } x_1 \in B_1 \text{ such that } 0 \le u(x_1,s) \le 1\}$ and let $t_2 = \sup\{s \in (-t,0) \mid \text{There exists } x_2 \in B_1 \text{ such that } 0 \le u(x_2,s) \le 1\}$. Assume first $t_1 > -t$, $t_2 = 0$. We observe that, under the current assumption, $B_1(0)\times(-t,t_1)$ is contained in $\{u<0\}\cup\{u>1\}$.

1. $u(0,0) \in [0,1]$.

   We have

$$\begin{aligned}|u(0,-t) - u(0,0)| &\le |u(0,-t) - u(0,t_1)| + |u(0,t_1) - u(x_1,t_1)| + |u(x_1,t_1)| + u(0,0)\\ \text{(by (3.1))}\quad &\le C_1\Lambda A_0 + C_2A_1 + A_1|x_1| + 2 \le C_1\Lambda A_0 + C_2' A_1.\end{aligned}$$

2. $u(0,0) \notin [0,1]$.

   In this case,

$$\begin{aligned}|u(0,-t) - u(0,0)| &\le |u(0,-t) - u(0,t_1)| + |u(0,t_1) - u(x_1,t_1)| + |u(x_1,t_1)|\\ &\quad + |u(0,0) - u(x_2,0)| + |u(x_2,0)|\\ \text{(by (3.1) again)}\quad &\le C_1\Lambda A_0 + C_2A_1 + A_1(|x_1|+|x_2|) + 2 \le C_1\Lambda A_0 + C_2' A_1.\end{aligned}$$

The cases i) $t_1 = -t, t_2 = 0$, ii) $t_1 = -t$, $t_2 < 0$, and iii) $t_1 > -t$, $t_2 < 0$ can be dealt with analogously. The proof is thus complete. □

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. Let $(x_0, t_0) \in K$ and define, for $(x,t) \in Q_1^-$, $w_\lambda^\varepsilon(x,t) = \frac{1}{\lambda} u^\varepsilon(x_0 + \lambda x, t_0 + \lambda^2 t)$. If $0 < \lambda < \tau$, w_λ^ε satisfies in Q_1^- the equation

$$L_\lambda w_\lambda^\varepsilon = div\, A^\lambda(x,t)\nabla w_\lambda^\varepsilon + \mathbf{b}^\lambda(x,t)\cdot\nabla w_\lambda^\varepsilon + c^\lambda(x,t) w_\lambda^\varepsilon - \partial_t w_\lambda^\varepsilon = \beta_{\varepsilon/\lambda}(w_\lambda^\varepsilon),$$

with $A^\lambda(x,t) = A(x_0+\lambda x, t_0+\lambda^2 t)$, $\mathbf{b}^\lambda(x,t) = \lambda \mathbf{b}(x_0+\lambda x, t_0+\lambda^2 t)$, $c^\lambda(x,t) = \lambda^2 c(x_0+\lambda x, t_0+\lambda^2 t)$. We have $\|w_\lambda^\varepsilon\|_{L^\infty(Q_1^-)} \le A_0/\lambda$, $\|\nabla w_\lambda^\varepsilon\|_{L^\infty(Q_1^-)} \le \|\nabla u^\varepsilon\|_{L^\infty(K)} \le A_1$ by Proposition 3.1. We now want to show that $\beta_{\varepsilon/\lambda}(w_\lambda^\varepsilon)$ is bounded in $\{w_\lambda^\varepsilon < 0\} \cup \{w_\lambda^\varepsilon > 1\}$. Clearly, if $w_\lambda^\varepsilon < 0$, or $w_\lambda^\varepsilon > 1$ with $\varepsilon/\lambda \le 1$, then $\beta_{\varepsilon/\lambda}(w_\lambda^\varepsilon) = 0$ because $supp\, \beta_{\varepsilon/\lambda} \subset [0, \varepsilon/\lambda]$. If instead $w_\lambda^\varepsilon > 1$ and $\varepsilon/\lambda > 1$ then $\beta_{\varepsilon/\lambda}(w_\lambda^\varepsilon) \le M_0$. We may thus apply Proposition 3.2 to obtain

$$|w_\lambda^\varepsilon(0,t) - w_\lambda^\varepsilon(0,0)| \le C \qquad \text{for any } t \in (-1,0),$$

or

$$|u^\varepsilon(x_0, t_0 + \lambda^2 t) - u^\varepsilon(x_0,t_0)| \le C\lambda \qquad \text{for any } t \in (-1,0). \tag{3.2}$$

The desired conclusion follows from Proposition 3.1 and (3.2). □

Next, we want to show that an uniformly bounded family of solutions to (SPP) converges to a limit function u which is a solution to (FBP1-2) in a weak sense. In order to do so, first we need to establish the existence of the limit, along with some convergence properties of its derivatives.

Lemma 3.1. *Let u^ε be a family of solutions to (SPP) in a domain $\mathcal{D} \subset \mathbb{R}^{m+1}$. Assume $\|u^\varepsilon\|_{L^\infty(\mathcal{D})} \le A_0$ for some $A_0 > 0$. For every sequence $\varepsilon_j \to 0$ there exist a subsequence $\varepsilon_{j'} \to 0$ and $u \in Lip(1,1/2)$ in $\mathcal{D}$ such that:*
i) $u^{\varepsilon_{j'}} \to u$ uniformly on compact subsets of $\mathcal{D}$;
ii) $\nabla u^{\varepsilon_{j'}} \to \nabla u$ in $L^2_{loc}(\mathcal{D})$;
iii) $\partial_t u^{\varepsilon_{j'}} \rightharpoonup \partial_t u$ weakly in $L^2_{loc}(\mathcal{D})$, and for all compact sets $K \subset \mathcal{D}$ there exists a positive constant C_K such that $\|\partial_t u^{\varepsilon_{j'}}\|_{L^2(K)} \le C_K$;
iv) $Lu = 0$ in $\mathcal{D} \setminus \partial\{u > 0\}$.

Proof. Let $K \subset \mathcal{D}$ be compact and fix $\tau > 0$ such that $N_{2\tau}(K) \subset \mathcal{D}$. By Theorem 3.1, there exists $L = L(m, \nu, \Lambda, \omega, M_0, A_0, \tau) > 0$ such that

$$|u^\varepsilon(x,t) - u^\varepsilon(y,s)| \le L(|x-y| + |t-s|^{1/2}) \qquad \text{for all } (x,t),(y,s) \in N_\tau(K).$$

By Ascoli-Arzelà's theorem, for any given sequence $\varepsilon_j \to 0$ there exists a subsequence $\varepsilon_{j'} \to 0$ and a function $u \in Lip(1,1/2)$ in $N_\tau(K)$ such that $u^{\varepsilon_{j'}} \to u$ uniformly in $N_\tau(K)$. This proves i). In order to prove iii), fix $(x_0,t_0)\in K$ and let $\psi\geq 0$, $\psi\in C_0^\infty(B_\tau(x_0))$, $\psi\equiv 1$ in $B_{\tau/2}(x_0)$. If we choose $\partial_t u^\varepsilon \psi^2$ as test function in the weak formulation of (SPP) in $Q_\tau(x_0,t_0)$ and integrate by parts, we obtain

$$\frac{1}{2}\int_{B_\tau(x_0)} A\nabla u^\varepsilon\cdot\nabla u^\varepsilon\,|_{t=t_0+\tau^2}\;\psi^2\,dx-\frac{1}{2}\int_{B_\tau(x_0)} A\nabla u^\varepsilon\cdot\nabla u^\varepsilon\,|_{t=t_0-\tau^2}\;\psi^2\,dx$$
$$-\frac{1}{2}\iint \partial_t A\,\nabla u^\varepsilon\cdot\nabla u^\varepsilon\psi^2+2\iint A\nabla u^\varepsilon\cdot\nabla\psi\,\partial_t u^\varepsilon\psi$$
$$+\iint(\partial_t u^\varepsilon)^2\psi^2-\iint \mathbf{b}\cdot\nabla u^\varepsilon\,\partial_t u^\varepsilon\psi^2-\iint c\,u^\varepsilon\partial_t u^\varepsilon\psi^2=$$
$$-\iint\beta_\varepsilon(u^\varepsilon)\,\partial_t u^\varepsilon\psi^2,$$

where all the "double" integrals are performed over the set $Q_\tau(x_0,t_0)$. Now, let $\omega>0$ be such $\|\partial_t A\|_{L^\infty(Q_\tau(x_0,t_0))}\leq\omega$. Recalling that $\|A\|_\infty\leq\nu^{-1}$, $\|\mathbf{b}\|_\infty+\|c\|_\infty\leq\Lambda$, $\|u^\varepsilon\|_\infty\leq A_0$, and $|\nabla u^\varepsilon|\leq A_1$ in K by Proposition 3.1, and applying Cauchy–Schwarz's inequality we infer

$$\iint(\partial_t u^\varepsilon)^2\psi^2\leq CA_1^2+\frac{1}{\sigma}\iint(A\nabla u^\varepsilon\cdot\nabla\psi)^2+\sigma\iint(\partial_t u^\varepsilon\,\psi)^2+\frac{\Lambda}{2\sigma}\iint|\nabla u^\varepsilon|^2\psi^2$$
$$+\frac{\Lambda\sigma}{2}\iint(\partial_t u^\varepsilon\,\psi)^2+\frac{\Lambda}{2\sigma}\iint|u^\varepsilon|^2\psi^2+\frac{\Lambda\sigma}{2}\iint(\partial_t u^\varepsilon\,\psi)^2$$
$$-\int_{B_\tau(x_0)}B_\varepsilon(u^\varepsilon)(x,t_0+\tau^2)\psi^2\,dx+\int_{B_\tau(x_0)}B_\varepsilon(u^\varepsilon)(x,t_0-\tau^2)\psi^2\,dx,$$

where $C=C(m,\tau,\nu,\omega)$, $\sigma>0$ and $B_\varepsilon(v)=\int_0^v\beta_\varepsilon(s)ds\leq M$. In particular, if we choose $\sigma>0$ such that $\sigma(1+\Lambda)=1/2$, we conclude

$$\iint(\partial_t u^\varepsilon)^2\psi^2\leq C=C(m,\nu,\omega,\Lambda,M_0,A_0,\tau).$$

This yields

$$\int_{t_0-\tau^2}^{t_0+\tau^2}\int_{B_{\tau/2}(x_0)}(\partial_t u^\varepsilon)^2\leq C,$$

and therefore, via a compactness argument,

$$\int_K(\partial_t u^\varepsilon)^2\;\leq C,$$

with C independent of ε. As a consequence, for any given sequence $\varepsilon_j\to 0$ there exists a subsequence $\varepsilon_{j'}\to 0$ such that

$$\partial_t u^{\varepsilon_{j'}}\rightharpoonup\partial_t u\quad\text{weakly in } L^2(K).$$

The proof of iii) is thus completed. We now turn our attention to iv). Since u is continuous, the sets $\{u>0\}$ and $\{u<0\}$ are open. Let $\varphi\in C_0^\infty(\mathbb{R}^{m+1})$ be supported in $\{u>0\}$ and let $(x,t)\in supp\,\varphi$. Fix $\varepsilon_j\to 0$. There exists an open neighborhood $\mathcal{U}_{(x,t)}$ of (x,t) such that

$u^{\varepsilon_j} \geq \frac{u(x,t)}{2} > 0$ in $\mathcal{U}_{(x,t)}$ and thus, if ε_j is small enough, $\beta_{\varepsilon_j}(u^{\varepsilon_j}) = 0$ in $\mathcal{U}_{(x,t)}$. Using a partition of unity argument, we find

$$\iint A\nabla u^{\varepsilon_j} \cdot \nabla\varphi + \iint \partial_t u^{\varepsilon_j} \varphi - \iint \mathbf{b} \cdot \nabla u^{\varepsilon_j} \varphi - \iint c\, u^{\varepsilon_j} \varphi = 0. \tag{3.3}$$

By Proposition 3.1, we know that $\|\nabla u^{\varepsilon}\|_{L^\infty(N_\tau(K))} \leq A_1$, and therefore we may assume $\nabla u^{\varepsilon_{j'}} \rightharpoonup \nabla u$ weakly in $L^2(N_\tau(K))$. Passing to the limit in (3.3) yields

$$\iint A\nabla u \cdot \nabla\varphi + \iint \partial_t u\, \varphi - \iint \mathbf{b} \cdot \nabla u\, \varphi - \iint c\, u\, \varphi = 0.$$

Hence $Lu = 0$ in $\{u > 0\}$ and, analogously, in $\{u < 0\}$. Moreover, u is a supersolution in $\{u \leq 0\}^\circ$. On the other hand, since $Lu^{\varepsilon_j} \geq 0$ in $\mathcal{D}$, we have $Lu \geq 0$ in $\mathcal{D}$, and therefore $Lu = 0$ in $\{u \leq 0\}^\circ \cup \{u > 0\}$. Finally, we need to prove ii). Let $\delta > 0$ and take $(u-\delta)^+\psi$, where ψ is as above, as test function. Since $Lu = 0$ in the positivity set of u, integrating by parts we obtain

$$\begin{aligned}
\iint_{\{u>\delta\}} A\nabla u \cdot \nabla u\, \psi = &- \iint_{\{u>\delta\}} A\nabla u \cdot \nabla\psi\, u + \delta \iint_{\{u>\delta\}} A\nabla u \cdot \nabla\psi \\
&- \frac{1}{2} \int_{\{u>\delta\}} (u-\delta)^2(x, t_0 + \tau^2)\psi\, dx \\
&+ \frac{1}{2} \int_{\{u>\delta\}} (u-\delta)^2(x, t_0 - \tau^2)\psi\, dx \\
&+ \iint_{\{u>\delta\}} \mathbf{b} \cdot \nabla u\, (u-\delta)\psi + \iint_{\{u>\delta\}} c\, u(u-\delta)\psi.
\end{aligned} \tag{3.4}$$

Analogously,

$$\begin{aligned}
\iint_{\{u<-\delta\}} A\nabla u \cdot \nabla u\, \psi = &- \iint_{\{u<-\delta\}} A\nabla u \cdot \nabla\psi\, u - \delta \iint_{\{u<-\delta\}} A\nabla u \cdot \nabla\psi \\
&- \frac{1}{2} \int_{\{u<-\delta\}} (u+\delta)^2(x, t_0 + \tau^2)\psi\, dx \\
&+ \frac{1}{2} \int_{\{u<-\delta\}} (u+\delta)^2(x, t_0 - \tau^2)\psi\, dx \\
&+ \iint_{\{u<-\delta\}} \mathbf{b} \cdot \nabla u\, (u+\delta)\psi + \iint_{\{u<-\delta\}} c\, u(u+\delta)\psi.
\end{aligned} \tag{3.5}$$

Adding equations (3.4) and (3.5) and letting $\delta \to 0$ we find

$$\begin{aligned}
\iint_{\{u\neq 0\}} A\nabla u \cdot \nabla u\, \psi = &- \iint_{\{u\neq 0\}} A\nabla u \cdot \nabla\psi\, u - \frac{1}{2} \int_{\{u\neq 0\}} u^2(x, t_0 + \tau^2)\psi\, dx \\
&+ \frac{1}{2} \int_{\{u\neq 0\}} u^2(x, t_0 - \tau^2)\psi\, dx + \iint_{\{u\neq 0\}} \mathbf{b} \cdot \nabla u\, u\psi \\
&+ \iint_{\{u\neq 0\}} c\, u^2\psi.
\end{aligned} \tag{3.6}$$

On the other hand, the observation $\beta_\varepsilon(u^\varepsilon)u^\varepsilon \geq 0$ yields

$$\iint A\nabla u^\varepsilon \cdot \nabla u^\varepsilon\, \psi \leq -\iint A\nabla u^\varepsilon \cdot \nabla\psi\, u^\varepsilon - \frac{1}{2}\int_{B_\tau(x_0)} (u^\varepsilon)^2(x, t_0+\tau^2)\psi\, dx + \frac{1}{2}\int_{B_\tau(x_0)} (u^\varepsilon)^2(x, t_0-\tau^2)\psi\, dx + \iint \mathbf{b}\cdot\nabla u^\varepsilon\, u^\varepsilon\psi + \iint c(u^\varepsilon)^2\psi, \tag{3.7}$$

where all the "double" integrals are performed over $Q_\tau(x_0, t_0)$. Using the uniform convergence of u^ε to u and the weak convergence of ∇u^ε to ∇u in $Q_\tau(x_0, t_0)$, we infer from (3.6) and (3.7) that

$$\limsup_{j'\to\infty} \iint A\nabla u^{\varepsilon_{j'}} \cdot \nabla u^{\varepsilon_{j'}}\, \psi \leq \iint_{Q_\tau(x_0,t_0)} A\nabla u \cdot \nabla u\, \psi. \tag{3.8}$$

Now, since the matrix A is symmetric and positive definite, there exists a matrix B, with bounded L^∞-norm, such that $B^2 = A$. Recalling that $\|B\nabla u^{\varepsilon_j}\|_{L^2(N_\tau(K))} \leq C$ by Proposition 3.1, we have

$$B\nabla u^{\varepsilon_{j'}} \rightharpoonup B\nabla u \qquad \text{weakly in } L^2(N_\tau(K)), \tag{3.9}$$

and therefore

$$\iint_{Q_\tau(x_0,t_0)} A\nabla u \cdot \nabla u\, \psi \leq \liminf_{j'\to\infty} \iint_{Q_\tau(x_0,t_0)} A\nabla u^{\varepsilon_{j'}} \cdot \nabla u^{\varepsilon_{j'}}\, \psi. \tag{3.10}$$

It follows from (3.8), (3.9), and (3.10) that

$$\iint \psi |B(\nabla u^{\varepsilon_{j'}} - \nabla u)|^2 \to 0 \qquad \text{as } j' \to \infty.$$

Finally, the ellipticity of A and a simple compactness argument yield

$$\nabla u^{\varepsilon_{j'}} \to \nabla u \qquad \text{in } L^2(K).$$

The conclusion of part ii) is proved, and so is the lemma. □

A slight modification of the above arguments yields the following variant of Lemma 3.1.

Lemma 3.2. *Let u^n be a family of solutions to*

$$L_n u^n = div\, A^n \nabla u^n + \mathbf{b^n}\cdot\nabla u^n + c^n u^n - \partial_t u^n = \beta_{\varepsilon_n}(u^n)$$

in a domain $\mathcal{D} \subset \mathbb{R}^{m+1}$, where $A^n(x,t) = A(x_n + \lambda_n x, t_n + \lambda_n^2 t)$, $\mathbf{b^n}(x,t) = \lambda_n \mathbf{b}(x_n + \lambda_n x, t_n + \lambda_n^2 t)$, $c^n(x,t) = \lambda_n^2(x_n + \lambda_n x, t_n + \lambda_n^2 t)$, $(x_n, t_n) \to (x_0, t_0) \in \mathcal{D}$ and $\varepsilon_n, \lambda_n \to 0$ as $n \to \infty$, with $A(x_0, t_0) = I$. Assume $\|u^n\|_{L^\infty(\mathcal{D})} \leq A_0$ for some $A_0 > 0$. There exists a subsequence $\{u^{n'}\}$ of $\{u^n\}$ and $u \in Lip_{loc}(1, 1/2)$ in $\mathcal{D}$ such that:

i) $u^{n'} \to u$ uniformly on compact subsets of $\mathcal{D}$;

ii) $\nabla u^{n'} \to \nabla u$ in $L^2_{loc}(\mathcal{D})$;

iii) $\partial_t u^{n'} \rightharpoonup \partial_t u$ weakly in $L^2_{loc}(\mathcal{D})$. Moreover, for any compact set $K \subset \mathcal{D}$ there exists a constant $C_K > 0$ such that $\|\partial_t u^{n'}\|_{L^2(K)} \leq C_K$;

iv) u is a solution to the heat equation in $\mathcal{D} \setminus \partial\{u > 0\}$.

The following approximation result will play a crucial in role in the implementation of blow-up arguments in the sequel.

Lemma 3.3. *Let $\{u^{\varepsilon_j}\}$ be a family of solutions to (SPP) in a domain $\mathcal{D} \subset \mathbb{R}^{m+1}$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of $\mathcal{D}$ and $\varepsilon_j \to 0$ as $j \to \infty$. Let (x_0, t_0), $(x_n, t_n) \in \mathcal{D} \cap \partial\{u > 0\}$ be such that $(x_n, t_n) \to (x_0, t_0)$ as $n \to \infty$. Assume $A(x_0, t_0) = I$. Let $\lambda_n \to 0$, $u_{\lambda_n}(x, t) = \frac{1}{\lambda_n} u(x_n + \lambda_n x, t_n + \lambda_n^2 t)$, and $(u^{\varepsilon_j})_{\lambda_n} = \frac{1}{\lambda_n} u^{\varepsilon_j}(x_n + \lambda_n x, t_n + \lambda_n^2 t)$. Suppose that $u_{\lambda_n} \to U$ as $n \to \infty$ uniformly on compact sets of $\mathbb{R}^{m+1}$. There exists $j(n) \to \infty$ such that for every $j_n \geq j(n)$ there holds that $\frac{\varepsilon_{j_n}}{\lambda_n} \to 0$ and*
i) $(u^{\varepsilon_{j_n}})_{\lambda_n} \to U$ uniformly on compact sets of $\mathbb{R}^{m+1}$;
ii) $\nabla(u^{\varepsilon_{j_n}})_{\lambda_n} \to \nabla U$ in $L^2_{loc}(\mathbb{R}^{m+1})$;
iii) $\partial_t(u^{\varepsilon_{j_n}})_{\lambda_n} \rightharpoonup \partial_t U$ weakly in $L^2_{loc}(\mathbb{R}^{m+1})$;
iv) $\nabla u_{\lambda_n} \to \nabla U$ in $L^2_{loc}(\mathbb{R}^{m+1})$;
v) $\partial_t u_{\lambda_n} \rightharpoonup \partial_t U$ weakly in $L^2_{loc}(\mathbb{R}^{m+1})$.

Proof. The proof is along the lines of the one of Lemma 3.2 in [CLW1]. We discuss here only the relevant modifications. For simplicity we assume $(x_n, t_n) = (x_0, t_0)$. Proceeding as in the cited reference, one can show that i) holds. The functions $(u^{\varepsilon_{j_n}})_{\lambda_n}$ are solutions to

$$L_n(u^{\varepsilon_{j_n}})_{\lambda_n} = \beta_{\frac{\varepsilon_{j_n}}{\lambda_n}}\left((u^{\varepsilon_{j_n}})_{\lambda_n}\right)$$

in Q_k, where k is a fixed positive number and L_n is as in Lemma 3.2. By Lemma 3.2 there exists a subsequence, still denoted by j_n, such that $\nabla(u^{\varepsilon_{j_n}})_{\lambda_n} \to \nabla U$ in $L^2(Q_k)$, and $\partial_t(u^{\varepsilon_{j_n}})_{\lambda_n} \rightharpoonup \partial_t U$ weakly in $L^2(Q_k)$. Then also ii) and iii) hold. In order to prove iv), let $\delta > 0$ and consider

$$\|\nabla u_{\lambda_n} - \nabla U\| \leq \|\nabla u_{\lambda_n} - \nabla(u^{\varepsilon_j})_{\lambda_n}\| + \|\nabla(u^{\varepsilon_j})_{\lambda_n} - \nabla U\| = I + II,$$

where all the norms are in $L^2(Q_k)$. We already know that $II < \delta$ if $j \geq j_n$ and n is sufficiently large. Moreover, by virtue of Lemma 3.1 it holds

$$\begin{aligned} I^2 &= \iint_{Q_k} |\nabla u - \nabla u^{\varepsilon_j}|^2 (x_0 + \lambda_n x, t_0 + \lambda_n^2 t) \\ &= \frac{1}{\lambda_n^{m+2}} \iint_{Q_{\lambda_n k}(x_0, t_0)} |\nabla u - \nabla u^{\varepsilon_j}|^2 (x, t) \, < \delta^2 \end{aligned}$$

if j and n are sufficiently large. This suffices to prove iv). Finally, in order to prove v) one needs to apply Lemma 3.2 to show that, for j and n large enough, $\|\partial_t(u^{\varepsilon_j})_{\lambda_n}\|_{L^2(Q_k)} \leq C$, and then proceed as in [CLW1, Lemma 3.2]. □

Remark 3.1. *The conclusion of Lemma 3.3 continues to hold when the operator L is replaced by $L_k = div\, A^k \nabla - \partial_t + \mathbf{b^k} \cdot \nabla + c^k$, where $A^k(x, t) = A(x_k + \tau_k x, t_k + \tau_k^2 t)$, $\mathbf{b^k}(x, t) = \tau_k \mathbf{b}(x_k + \tau_k x, t_k + \tau_k^2 t)$, $c^k(x, t) = \tau_k^2 c(x_k + \tau_k x, t_k + \tau_k^2 t)$, $(x_k, t_k) \to (\overline{x}, \overline{t}) \in \mathcal{D}$ and $\tau_k \to 0$ as $k \to \infty$, provided the assumption $A(x_0, t_0) = I$ is replaced by $A(\overline{x}, \overline{t}) = I$.*

We can now prove that limit functions are solutions to (FBP1-2) in a weak sense.

Proposition 3.3. *Let u^{ε_j} be a family of solutions to (SPP) in a domain $\mathcal{D} \subset \mathbb{R}^{m+1}$. If $u^{\varepsilon_j} \to u$ uniformly on compact subsets of $\mathcal{D}$ as $\varepsilon_j \to 0$, then there exists a locally finite measure μ supported on the free boundary $\mathcal{D} \cap \partial\{u > 0\}$ such that $\beta_{\varepsilon_j}(u^{\varepsilon_j}) \rightharpoonup \mu$ weakly in $\mathcal{D}$. In particular, $Lu = \mu$ in $\mathcal{D}$, i.e.*

$$\iint_{\mathcal{D}} A\nabla u \cdot \nabla\phi \, dx\, dt - \iint_{\mathcal{D}} u\, \partial_t \phi \, dx\, dt - \iint_{\mathcal{D}} (\mathbf{b} \cdot \nabla u + c\, u)\phi \, dx\, dt = -\iint_{\mathcal{D}} \phi \, d\mu \tag{3.11}$$

for all $\phi \in C_0^\infty(D)$.

Proof. By definition of weak solution to (SPP), if $\phi \in C_0^\infty(\mathcal{D})$, one has

$$\iint_{\mathcal{D}} A\nabla u^\varepsilon \cdot \nabla\phi - \iint_{\mathcal{D}} u^\varepsilon \partial_t \phi - \iint_{\mathcal{D}} (\mathbf{b} \cdot \nabla u^\varepsilon + c\, u^\varepsilon)\phi = -\iint_{\mathcal{D}} \beta_\varepsilon(u^\varepsilon)\phi. \tag{3.12}$$

Since $u^{\varepsilon_j} \to u$ uniformly on compact subsets of $\mathcal{D}$, by Lemma 3.1 we know $\nabla u^{\varepsilon_j} \to \nabla u$ in $L^2_{loc}(\mathcal{D})$, and so the left-hand side of (3.12) converges to the left-hand side of (3.11). Now let $K \subset \mathcal{D}$ be compact, and pick $\phi \in C_0^\infty(\mathcal{D})$, $\phi \geq 0$, $\phi \equiv 1$ in K. The sequence $\left\{\iint_{\mathcal{D}} \beta_{\varepsilon_j}(u^{\varepsilon_j})\phi \, dx\, dt\right\}_{j\in\mathbb{N}}$ is convergent, and therefore it is bounded. Hence

$$\iint_K \beta_{\varepsilon_j}(u^{\varepsilon_j}) \, dx\, dt \leq \iint_{\mathcal{D}} \beta_{\varepsilon_j}(u^{\varepsilon_j})\phi \, dx\, dt \leq C.$$

This implies that there exists a locally finite measure μ such that, passing to a subsequence (still denoted by ε_j) if necessary, $\beta_{\varepsilon_j}(u^{\varepsilon_j}) \to \mu$ as measures in $\mathcal{D}$. Passing to the limit in (3.12), we get (3.11). Moreover, since $Lu = 0$ in $\mathcal{D} \setminus \partial\{u > 0\}$ by Lemma 3.1, we conclude that μ is supported in $\mathcal{D} \cap \partial\{u > 0\}$. The proof is thus complete. □

In a similar fashion, we can also show that the solutions u^ε to the global problem (2.1) converge to a (weak) solution of the corresponding free boundary problem, if the initial data u_0^ε are uniformly bounded.

Proposition 3.4. *Let $u^\varepsilon \in C(\overline{\Omega_T}) \cap C^{1,1/2}(\overline{\Omega} \times (0,T))$ be a family of solutions to (2.1), with $c \leq 0$ and $\|u_0^\varepsilon\|_{L^\infty(\Omega)} \leq A_0$ for some $A_0 \geq 0$. Let $\varepsilon_j \to 0$, $u_0 \in L^\infty(\Omega)$, and $u \in Lip_{loc}(1, 1/2)$ in Ω_T be such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω_T, and $u_0^{\varepsilon_j} \to u_0$ $*$-weakly in $L^\infty(\Omega)$. Then there exists a locally finite measure μ in $\overline{\Omega_T}$ such that $\beta_{\varepsilon_j} \to \mu$ as measures in $\overline{\Omega_T}$. Moreover, $supp\, \mu \cap \Omega_T \subset \partial\{u > 0\}$ and*

$$\begin{cases} Lu = \mu & \text{in } \Omega_T, \\ A\nabla u \cdot \eta = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & x \in \Omega, \end{cases}$$

in the following weak sense:

$$\int_0^t \int_\Omega (u\phi_s - A\nabla u \cdot \nabla\phi + \mathbf{b}\cdot\nabla u\,\phi + c\,u\,\phi)\; dx\,ds + \int_\Omega u_0(x)\phi(x,0)\,dx = \int_0^t \int_\Omega \phi\,d\mu$$

for every $\phi \in C_0^\infty(\overline{\Omega}\times[0,t))$.

Proof. By definition of weak solution to (2.1), if $\phi \in C_0^\infty(\overline{\Omega}\times[0,t))$ we have

$$\begin{aligned}\int_0^t \int_\Omega (u^\varepsilon\phi_s - A\nabla u^\varepsilon\cdot\nabla\phi + \mathbf{b}\cdot\nabla u^\varepsilon\,\phi + c\,u^\varepsilon\phi)\; dx\,ds + \int_\Omega u_0^\varepsilon(x)\phi(x,0)\,dx \\ = \int_0^t \int_\Omega \beta_\varepsilon(u^\varepsilon)\phi\,dx\,ds.\end{aligned} \tag{3.13}$$

By virtue of Proposition 2.1, we know that $u^{\varepsilon_j} \to u$ $*$-weakly in $L^\infty(\Omega_T)$. Let us see that $\nabla u^{\varepsilon_j} \rightharpoonup \nabla u$ weakly in $L^2_{loc}(\overline{\Omega}\times[0,T))$. Choosing $u^\varepsilon\psi^2$ as test function in the weak formulation of (SPP), with $\psi\in C_0^\infty(\overline{\Omega})$, we have for $0<t_0<t_1<T$

$$\begin{aligned}\int_{t_0}^{t_1}\int_K A\nabla u^\varepsilon\cdot\nabla(u^\varepsilon\psi^2) - \int_{t_0}^{t_1}\int_K (\mathbf{b}\cdot\nabla u^\varepsilon + c\,u^\varepsilon)\,u^\varepsilon\psi^2 + \int_{t_0}^{t_1}\int_K \partial_t u^\varepsilon\, u^\varepsilon\psi^2 \\ = -\int_{t_0}^{t_1}\int_K \beta_\varepsilon(u^\varepsilon)\,u^\varepsilon\psi^2 \le 0\end{aligned}$$

because $\beta_\varepsilon(s)s \ge 0$ for every $s\in\mathbb{R}$. Here $K = supp\,\psi$. We thus obtain, using Proposition 2.1 again and Cauchy–Schwarz's inequality,

$$\begin{aligned}\nu\int_{t_0}^{t_1}\int_K |\nabla u^\varepsilon|^2\psi^2 &\le -2\int_{t_0}^{t_1}\int_K A\nabla u^\varepsilon\cdot\nabla\psi\,u^\varepsilon\psi + \int_{t_0}^{t_1}\int_K (\mathbf{b}\cdot\nabla u^\varepsilon + c\,u^\varepsilon)\,u^\varepsilon\psi^2 \\ &\quad - \frac{1}{2}\int_{t_0}^{t_1}\int_K \partial_t(u^\varepsilon)^2\psi^2 \\ &\le C_1(\psi,A_0,\nu,\Lambda)\left(\int_{t_0}^{t_1}\int_K |\nabla u^\varepsilon|^2\psi^2\right)^{\frac{1}{2}}.\end{aligned}$$

This gives

$$\int_{t_0}^{t_1}\int_K |\nabla u^\varepsilon|^2\psi^2 \le C_2(\psi,A_0,\nu,\Lambda) \tag{3.14}$$

and therefore $\nabla u^{\varepsilon_j}\rightharpoonup\nabla u$ weakly in $L^2_{loc}(\overline{\Omega}\times[0,T))$. Next we want to show that $\beta_{\varepsilon_j}(u^{\varepsilon_j})\to\mu$ as measures. To this aim, we estimate $\beta_\varepsilon(u^\varepsilon)$ in $L^1_{loc}(\overline{\Omega_T})$. Let $\varphi\in C_0^\infty(\overline{\Omega})$, $\varphi\ge 0$. By definition of weak solution, it holds for a.e. $t>0$

$$\begin{aligned}\int_0^t\int_\Omega \beta_\varepsilon(u^\varepsilon)\varphi = -\int_\Omega u^\varepsilon(x,t)\varphi\,dx + \int_\Omega u_0^\varepsilon(x)\varphi(x)\,dx \\ + \int_0^t\int_\Omega (-A\nabla u^\varepsilon\cdot\nabla\varphi + \mathbf{b}\cdot\nabla u^\varepsilon\,\varphi + c\,u^\varepsilon\varphi)\,.\end{aligned}$$

We have

$$\int_0^t\int_\Omega A\nabla u^\varepsilon\cdot\nabla\varphi=\int_0^t\int_{\partial\Omega}A\nabla\varphi\cdot\eta\, u^\varepsilon\, d\sigma\, ds-\int_0^t\int_\Omega \partial_{x_j}a_{ij}u^\varepsilon\partial_{x_i}\varphi$$
$$-\int_0^t\int_\Omega a_{ij}u^\varepsilon\partial^2_{x_ix_j}\varphi\le C(A_0,\varphi,\nu,\omega)$$

and, using (3.14),

$$\int_0^t\int_\Omega \mathbf{b}\cdot\nabla u^\varepsilon\varphi\le\Lambda\left(\int_0^t\int_\Omega|\nabla u^\varepsilon|^2\varphi^2\right)^{1/2}|\Omega_T|^{1/2}\le C(\varphi,A_0,\Lambda,\nu).$$

If $K\subset\overline{\Omega}$ is a compact set and $\varphi\equiv 1$ in K, we finally obtain

$$\int_0^t\int_K\beta_\varepsilon(u^\varepsilon)\le C(\varphi,A_0,\nu,\omega,\Lambda).$$

The passage to the limit as $\varepsilon_j\to 0$ in (3.13) can now be carried out as in the proof of Proposition 3.3. □

We close this section turning our attention to the study of the case when the limit function u is piecewise linear. As we will see in Section 3, this analysis is one of the key ingredients in understanding the local behavior of general limit solutions.

Proposition 3.5. *Let $\mathcal{D}$ be a domain in $\mathbb{R}^{m+1}$, $(x_0,t_0)\in\mathcal{D}$, and $(x_n,t_n)\to(x_0,t_0)$ as $n\to\infty$. Assume $A(x_0,t_0)=I$. Let u^{ε_n} be solutions to*

$$L_nu^{\varepsilon_n}=\beta_{\varepsilon_n}(u^{\varepsilon_n})\tag{3.15}$$

in $\mathcal{D}$, where L_n is as in Lemma 3.2. If u^{ε_n} converge to $\alpha(x-x_0)_1^+-\gamma(x-x_0)_1^-$ uniformly on compact subsets of $\mathcal{D}$, with $\alpha\in\mathbb{R}$, $\gamma>0$, and $\varepsilon_n\to 0$ as $n\to\infty$, then

$$\alpha^2-\gamma^2=2M.$$

Proof. Without loss of generality, we may assume $(x_0,t_0)=(0,0)$. By Lemma 3.2, we know that $\partial_tu^{\varepsilon_n}\rightharpoonup 0$ weakly in $L^2_{loc}(\mathcal{D})$, and $\nabla u^{\varepsilon_n}\to\alpha\chi_{\{x_1>0\}}\mathbf{e_1}+\gamma\chi_{\{x_1<0\}}\mathbf{e_1}$ in $L^2_{loc}(\mathcal{D})$. Moreover, $A^n(x,t)\to I$, $\mathbf{b^n}\to 0$, and $c^n\to 0$ uniformly on compact subsets of $\mathcal{D}$. This suffices to show that u is subcaloric in $\mathcal{D}$. Assume $\alpha\le 0$. Then $u\le 0$ in $\mathcal{D}$, $u<0$ in $\{x_1<0\}$, and $u(0,0)=0$, but this is not possible because u is subcaloric in $\mathcal{D}$. Hence necessarily $\alpha>0$.

Now, let $\psi\in C_0^\infty(\mathcal{D})$. Choosing $\partial_{x_1}u^{\varepsilon_n}\psi$ as test function in the weak formulation of (3.15) and integrating by parts we obtain

$$\begin{aligned}-\frac12\iint_{\mathcal{D}}\partial_{x_1}A^n\nabla u^{\varepsilon_n}\cdot\nabla u^{\varepsilon_n}\,\psi-\frac12\iint_{\mathcal{D}}A^n\nabla u^{\varepsilon_n}\cdot\nabla u^{\varepsilon_n}\,\partial_{x_1}\psi+\iint_{\mathcal{D}}\partial_tu^{\varepsilon_n}\partial_{x_1}u^{\varepsilon_n}\psi\\ +\iint_{\mathcal{D}}A^n\nabla u^{\varepsilon_n}\cdot\nabla\psi\,\partial_{x_1}u^{\varepsilon_n}-\iint_{\mathcal{D}}\mathbf{b^n}\cdot\nabla u^{\varepsilon_n}\,\partial_{x_1}u^{\varepsilon_n}\psi\\ -\iint_{\mathcal{D}}c^nu^{\varepsilon_n}\partial_{x_1}u^{\varepsilon_n}\psi=\iint_{\mathcal{D}}B_{\varepsilon_n}(u^{\varepsilon_n})\partial_{x_1}\psi.\end{aligned}\tag{3.16}$$

Next, we show that

$$B_{\varepsilon_n}(u^{\varepsilon_n}) \to M\chi_{\{x_1>0\}} \qquad \text{in } L^1_{loc}(\mathcal{D}). \tag{3.17}$$

In fact, if $(y,s) \in \mathcal{D} \cap \{x_1 > 0\}$, then $u^{\varepsilon_n} \geq \alpha y_1/2$ in a neighborhood of (y,s) for n sufficiently large. Hence, if $u^{\varepsilon_n}(x,t) \geq \varepsilon_n$ we have

$$B_{\varepsilon_n}(u^{\varepsilon_n})(x,t) = \int_0^{\frac{u^{\varepsilon_n}(x,t)}{\varepsilon_n}} \beta(s)\, ds = M.$$

Analogously, if $(y,s) \in \mathcal{D} \cap \{x_1 < 0\}$, then $B_{\varepsilon_n}(u^{\varepsilon_n}) = 0$ in a neighborhood of (y,s) for n large enough. To prove (3.17), it suffices to observe that $0 \leq B_{\varepsilon_n}(s) \leq M$. Passing to the limit in (3.16) as $n \to \infty$ we thus find

$$-\frac{1}{2}\iint_{\{x_1>0\}} \alpha^2 \partial_{x_1}\psi - \frac{1}{2}\iint_{\{x_1<0\}} \gamma^2 \partial_{x_1}\psi + \iint_{\{x_1>0\}} \alpha^2 \partial_{x_1}\psi + \iint_{\{x_1<0\}} \gamma^2 \partial_{x_1}\psi$$
$$= M \iint_{\{x_1>0\}} \partial_{x_1}\psi.$$

Integration by parts yields

$$M \iint_{\{x_1=0\}} \psi\, dx'\, dt = \frac{\alpha^2}{2} \iint_{\{x_1=0\}} \psi\, dx'\, dt - \frac{\gamma^2}{2} \iint_{\{x_1=0\}} \psi\, dx'\, dt.$$

Since the choice of $\psi \in C_0^\infty(\mathcal{D})$ is arbitrary, we infer $\alpha^2 - \gamma^2 = 2M$. □

Proposition 3.6. *Let $\mathcal{D}$ be a domain in $\mathbb{R}^{m+1}$, $(x_0,t_0) \in \mathcal{D}$, and $(x_n,t_n) \to (x_0,t_0)$ as $n \to \infty$. Assume $A(x_0,t_0) = I$. Let u^{ε_n} be solutions to*

$$L_n u^{\varepsilon_n} = \beta_{\varepsilon_n}(u^{\varepsilon_n})$$

in $\mathcal{D}$, where L_n is as in Lemma 3.2. Suppose u^{ε_n} converge to $\alpha(x-x_0)_1^+$ uniformly on compact subsets of $\mathcal{D}$, with $\alpha \in \mathbb{R}$, and $\varepsilon_n \to 0$ as $n \to \infty$. Then

$$0 \leq \alpha \leq \sqrt{2M}.$$

Proof. Without loss of generality, assume $(x_0,t_0) = (0,0)$. Arguing as in the proof of Proposition 3.5, we see that necessarily $\alpha \geq 0$. Since $0 \leq B_{\varepsilon_n}(s) \leq M$, there exists $M(x,t) \in L^\infty(\mathcal{D})$, $0 \leq M(x,t) \leq M$, such that on a subsequence (still denoted by B_{ε_n}) $B_{\varepsilon_n}(u^{\varepsilon_n}) \to M(x,t)$ $*$-weakly in $L^\infty(\mathcal{D})$. Arguments similar to the ones employed in the proof of Proposition 3.5 show that $M(x,t) \equiv M$ in $\mathcal{D} \cap \{x_1 > 0\}$. Moreover, using Propositions 3.1 and 3.3 it is immediate to recognize that

$$\nabla B_{\varepsilon_n}(u^{\varepsilon_n}) = \beta_{\varepsilon_n}(u^{\varepsilon_n})\nabla u^{\varepsilon_n} \to 0$$

in $L^1_{loc}(\mathcal{D}\cap\{x_1<0\})$. Hence $M(x,t)=M(t)$ in $\mathcal{D}\cap\{x_1<0\}$. Passing to the limit in (3.16) yields

$$M\iint_{\{x_1>0\}}\partial_{x_1}\psi+\iint_{\{x_1<0\}}M(t)\partial_{x_1}\psi=\frac{\alpha^2}{2}\iint_{\{x_1>0\}}\partial_{x_1}\psi,$$

and integrating by parts we find

$$M\iint_{\{x_1=0\}}\psi\,dx'\,dt-\iint_{\{x_1=0\}}M(t)\partial_{x_1}\psi\,dx'\,dt=\frac{\alpha^2}{2}\iint_{\{x_1=0\}}\psi\,dx'\,dt.$$

The arbitrariness of $\psi\in C_0^\infty(\mathcal{D})$ allows to conclude $\frac{\alpha^2}{2}=M-M(t)\le M$, because $M(t)\ge 0$. Hence $\alpha^2\le 2M$, and the proof is complete. □

4. Asymptotic Behavior of Limit Solutions: One Phase Case

In this and the next section we prove that the limit solution u has an asymptotic expansion at any "regular" free boundary point which implies, in particular, that both $\langle A\nabla u^+,\eta\rangle$ and $\langle A\nabla u^-,\eta\rangle$ exist, and that the free boundary condition

$$\langle A\nabla u^+,\eta\rangle^2-\langle A\nabla u^-,\eta\rangle^2=2M$$

is satisfied. We treat the one phase situation first.

The following gradient estimate near the free boundary plays a fundamental role in the proof of Theorem 1.1.

Theorem 4.1. *Let u and Γ be as in Theorem 1.1. If $(x_0,t_0)\in\mathcal{D}\cap\partial\{u>0\}$ is such that (H2) and (H3) hold, then*

$$\limsup_{(x,t)\to(x_0,t_0)}|\nabla u(x,t)|\le\sqrt{2M}.$$

Proof. Let $\alpha=\limsup_{(x,t)\to(x_0,t_0)}|\nabla u(x,t)|$. Since $u\in Lip_{loc}(1,1/2)$ in $\mathcal{D}$, clearly $\alpha<\infty$. If $\alpha=0$ there is nothing to prove, so we may assume $\alpha>0$. There exists a sequence $(x_n,t_n)\to(x_0,t_0)$ such that $|\nabla u(x_n,t_n)|\to\alpha$ and $u(x_n,t_n)>0$. Let $(z_n,s_n)\in\mathcal{D}\cap\partial\{u>0\}$ be such that $d_n=\max\{|z_n-x_n|,|s_n-t_n|^{1/2}\}=d_p((x_n,t_n),\partial\{u>0\})$. Define $u_{d_n}(x,t)=\frac{1}{d_n}u(z_n+d_nx,s_n+{d_n}^2t)$. Since $u\in Lip_{loc}(1,1/2)$ in $\mathcal{D}$ and $u_{d_n}(0,0)=0$ for every n, $\{u_{d_n}\}$ is uniformly bounded on compact subsets of $\mathbb{R}^{m+1}$, and therefore for a subsequence (still denoted by d_n) $u_{d_n}\to u_0$ uniformly on compact subsets of $\mathbb{R}^{m+1}$, where $u_0\in Lip_{loc}(1,1/2)$ in $\mathbb{R}^{m+1}$. Now set

$$\overline{x}_n=\frac{x_n-z_n}{d_n}\quad \overline{t}_n=\frac{t_n-s_n}{{d_n}^2}.$$

Clearly $(\overline{x}_n,\overline{t}_n)\in\partial Q_1$, and thus we may choose the subsequence d_n so that $(\overline{x}_n,\overline{t}_n)\to(\overline{x},\overline{t})\in\partial Q_1$. Without loss of generality, we may assume $\overline{x}=\mathbf{e_1}$, $\overline{t}=0$. Using Lemma 3.1, (iv) and Lemma 3.3, (iv) and (v), it is easy to recognize that u_0 is caloric and nonnegative in $Q_1(\mathbf{e_1},0)$. We now claim that $|\nabla u_0(\mathbf{e_1},0)|=\alpha$. In order to prove the claim, it will suffice to show that $\nabla u_{d_n}\to$

∇u_0 uniformly on compact subsets of $Q_1(\mathbf{e_1}, 0)$. For the sake of brevity, let $v_{n,m} = u_{d_n} - u_{d_m}$. Let K be a compact set in $Q_1(\mathbf{e_1}, 0)$ and let $\tau > 0$ be such that $N_{2\tau}(K) \subset Q_1(\overline{x}_n, \overline{t}_n)$ for n large. Letting $A^l(x,t) = A(z_l + d_l x, s_l + d_l^2 t)$, $B^l(x,t) = I - A^l(x,t)$, $\mathbf{b^l}(x,t) = d_l \mathbf{b}(z_l + d_l x, s_l + d_l^2 t)$, and $c^l(x,t) = d_l^2 c(z_l + d_l x, s_l + d_l^2 t)$, we have

$$Hv_{n,m} = div[B^n \nabla u_{d_n} - B^m \nabla u_{d_m}] + [\mathbf{b^m} \cdot \nabla u_{d_m} - \mathbf{b^n} \cdot \nabla u_{d_n}] + [c^m u_{d_m} - c^n u_{d_n}]$$

in $N_\tau(K)$, for n, m large enough. As observed above, $\{u_{d_n}\}$ is an uniformly bounded family on compact subsets of $\mathbb{R}^{m+1}$ and so $\|c^m u_{d_m} - c^n u_{d_n}\|_{L^\infty(N_\tau(K))} \leq C({d_n}^2 + {d_m}^2)\Lambda$ for some positive constant C. As a consequence of Theorem 3.2, there exists $A_1 > 0$ such that $\|\mathbf{b^n} \cdot \nabla u_{d_n} - \mathbf{b^m} \cdot \nabla u_{d_m}\|_{L^\infty(N_\tau(K))} \leq (d_n + d_m)\Lambda A_1$ and $\|B^n \nabla u_{d_n} - B^m \nabla u_{d_m}\|_{L^\infty(N_\tau(K))} \leq (\|B_n\|_\infty + \|B_m\|_\infty)A_1$. If we define $\mathbf{g}(x,t) = B^n(x,t)\nabla u_{d_n}(x,t) - B^m(x,t)\nabla u_{d_m}(x,t)$, we have

$$|\mathbf{g}(x,t) - \mathbf{g}(y,s)| \leq \Omega_n + \Omega_m,$$

with

$$\begin{aligned} \Omega_l &= |B^l(x,t)\nabla u_{d_l}(x,t) - B^l(y,s)\nabla u_{d_l}(y,s)| \\ &\leq \|B^l(x,t) - B^l(y,s)\||\nabla u_{d_l}(x,t)| + \|B^l(y,s)\||\nabla u_{d_l}(x,t) - \nabla u_{d_l}(y,s)|. \end{aligned}$$

Now, since u_{d_l} is a solution to

$$div\, A^l \nabla u_{d_l} + \mathbf{b^l} \cdot \nabla u_{d_l} + c^l u_{d_l} - \partial_t u_{d_l} = 0$$

in $N_\tau(K)$, with $A \in C^1(\mathbb{R}^{m+1})$ and $\mathbf{b}$, $c \in L^\infty(\mathbb{R}^{m+1})$, $\nabla u_{d_l} \in C^\gamma$ for some $\gamma > 0$. We may conclude

$$\Omega_l \leq C(A_1 d_l \omega + \|B^l\|_\infty)(|x-y| + |t-s|^{1/2})^\gamma,$$

and thus

$$|\mathbf{g}(x,t) - \mathbf{g}(y,s)| \leq C(d_n + d_m + \|B^n\|_\infty + \|B^m\|_\infty)(|x-y| + |t-s|^{1/2})^\gamma.$$

By [CK, Corollary 1.2.22],

$$\begin{aligned} \|\nabla u_{d_n} - \nabla u_{d_m}\|_{L^\infty(K)} &= \|\nabla v_{n,m}\|_{L^\infty(K)} \\ &\leq C\left(\|v_{n,m}\|_{L^2(N_\tau(K))} + d_n + d_m + \|B^n\|_\infty + \|B^m\|_\infty\right). \end{aligned}$$

Since $v_{n,m}$, B^n and B^m are uniformly convergent to 0 in $N_\tau(K)$, we infer that $\nabla u_{d_n} \to \nabla u_0$ uniformly in K. Next, it is easy to show that $|\nabla u_0| \leq \alpha$ in $\mathbb{R}^{m+1}$. Indeed, let $R > 1$, $\delta > 0$. There exists $\tau_0 > 0$ such that $|\nabla u(x,t)| \leq \alpha + \delta$ for any $(x,t) \in Q_{\tau R}(x_0, t_0)$ if $\tau \leq \tau_0$. Set $\tau_n = \max\{|x_n - x_0|, |t_n - t_0|^{1/2}\}$. One has $Q_{d_n R}(z_n, s_n) \subset Q_{3\tau_n R}(x_0, t_0)$, and therefore $|\nabla u_{d_n}(x,t)| \leq \alpha + \delta$ for any $(x,t) \in Q_R$ for n large enough. In particular, $\nabla u_{d_n} \to \nabla u_0$ $*$-weakly in $L^\infty(Q_R)$ and thus $|\nabla u_0| \leq \alpha + \delta$ in Q_R. Since δ and R are arbitrary, we conclude

$$|\nabla u_0| \leq \alpha \qquad \text{in } \mathbb{R}^{m+1}.$$

Let $\upsilon = \frac{\nabla u_0(\mathbf{e_1},0)}{|\nabla u_0(\mathbf{e_1},0)|}$ and set $w = \frac{\partial u_0}{\partial \upsilon}$, which is caloric in $Q_1(\mathbf{e_1}, 0)$ and satisfies $w \le \alpha$ in $Q_1^-(\mathbf{e_1}, 0)$, $w(\mathbf{e_1}, 0) = \alpha$. By the strong maximum principle, $w \equiv \alpha$ in $Q_1^-(\mathbf{e_1}, 0)$ and so $u_0(x,t) = \alpha\langle x, \upsilon\rangle + b(t)$ in $Q_1^-(\mathbf{e_1}, 0)$. But since u_0 is caloric in $Q_1(\mathbf{e_1}, 0)$, and $u_{d_n}(0,0) = 0$ for any $n \in \mathbb{N}$, necessarily $u_0(x,t) = \alpha\langle x, \upsilon\rangle$ in $Q_1^-(\mathbf{e_1}, 0)$. At this point we observe that $|\nabla u_0(\mathbf{e_1}, 0)| = \alpha$ forces $u_0 > 0$ in $B_1(\mathbf{e_1}) \times \{0\}$. From this we infer that $\upsilon = \mathbf{e_1}$ and

$$u_0(x,t) = \alpha x_1 \qquad \text{in } Q_1^-(\mathbf{e_1}, 0).$$

It is not difficult to see that

$$u_0(x,t) = \alpha x_1 \qquad \text{in } \{x_1 \ge 0, t \le 0\}.$$

We now apply Corollary A.1 in [CLW1] to u_0 in $\{x_1 < 0\} \cap \{t \le 0\}$ and obtain

$$u_0(x,t) = \gamma x_1^- + o(|x| + |t|^{1/2}) \qquad \text{in } \{x_1 < 0, t \le 0\}$$

for some $\gamma \ge 0$. Define, for $\lambda > 0$, $(u_0)_\lambda(x,t) = \frac{1}{\lambda} u_0(\lambda x, \lambda^2 t)$. There exist a sequence $\lambda_k \to 0$ and a function $v_0 \in Lip(1, 1/2)$ in $\mathbb{R}^{m+1}$ such that $(u_0)_{\lambda_k} \to v_0$ uniformly on compact sets of $\mathbb{R}^{m+1}$. We have $v_0(x,t) = \alpha x_1^+ + \gamma x_1^-$ in $\mathbb{R}^m \times \{t \le 0\}$. Since the set $\{u \equiv 0\}$ has locally uniform positive density on $\Gamma \cap Q_\delta(x_0, t_0)$, it follows

$$0 < c \le \frac{|\{u \equiv 0\} \cap Q^-_{d_n r}(z_n, s_n)|}{|Q^-_{d_n r}(z_n, s_n)|}$$

for $r > 0$ and n sufficiently large. A change of variable gives

$$c \le \frac{|\{u_{d_n} \equiv 0\} \cap Q_r^-|}{|Q_r^-|}$$

and therefore, passing to the limit as $n \to \infty$, we obtain

$$c \le \frac{|\{u_0 \equiv 0\} \cap Q_r^-|}{|Q_r^-|}.$$

Rescaling and letting $k \to \infty$ we obtain

$$c \le \frac{|\{v_0 \equiv 0\} \cap Q_1^-|}{|Q_1^-|}.$$

This implies $\gamma = 0$, and $v_0(x,t) = \alpha x_1^+$ in $\mathbb{R}^m \times \{t \le 0\}$. By Lemma 3.3, there exists a sequence $d_n \to 0$ such that u^{δ_n} is a solution to

$$L_n u^{\delta_n} = div\, A^n \nabla u^{\delta_n} - \partial_t u^{\delta_n} + \mathbf{b^n} \cdot \nabla u^{\delta_n} + c^n u^{\delta_n} = \beta_{\delta_n}(u^{\delta_n})$$

in Q_1^-, where A^n, $\mathbf{b^n}$, and c^n are as above. Moreover, $u^{\delta_n} \to u_0$ uniformly on compact subsets of Q_1^-, and $(u_0)_{\lambda_n} \to v_0$ uniformly on compact sets of $\mathbb{R}^{m+1}$. Applying Lemma 3.3 again, see also Remark 3.1, we find a sequence $\tilde{\delta_n} \to 0$ and solutions $u^{\tilde{\delta_n}}$ to

$$div\, A^{j_n}(\lambda_n x, {\lambda_n}^2 t) \nabla u^{\tilde{\delta_n}} - \partial_t u^{\tilde{\delta_n}} + \mathbf{b^{j_n}}(\lambda_n x, {\lambda_n}^2 t) \cdot \nabla u^{\tilde{\delta_n}} + c^{j_n}(\lambda_n x, {\lambda_n}^2 t) u^{\tilde{\delta_n}} = \beta_{\tilde{\delta_n}}(u^{\tilde{\delta_n}})$$

in Q_1^- such that $u^{\tilde{\delta_n}} \to v_0 = \alpha x_1^+$ uniformly on compact subsets of Q_1^-. Finally, we may apply Proposition 3.6 to conclude $\alpha \le \sqrt{2M}$. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Without loss of generality we may assume $(x_0, t_0) = (0, 0)$ and $\eta = \mathbf{e_1}$. Define, for $\lambda > 0$, $u_\lambda(x,t) = \frac{1}{\lambda} u(\lambda x, \lambda^2 t)$, and let $\rho > 0$ be such that $Q_\rho \subset\subset \mathcal{D}$. Since $u_\lambda \in Lip(1, 1/2)$ in $Q_{\rho/\lambda}$ uniformly in λ, and $u_\lambda(0,0) = 0$, there exist a subsequence $\lambda_n \to 0$ and a function $U \in Lip(1, 1/2)$ in $\mathbb{R}^{m+1}$ such that $u_{\lambda_n} \to U$ uniformly on compact subsets of $\mathbb{R}^{m+1}$. It is easy to show, using Lemmas 3.1 and 3.3, that u_λ is a solution to

$$L_\lambda u_\lambda = div\, A^\lambda \nabla u_\lambda - \partial_t u_\lambda + \mathbf{b}^\lambda \cdot \nabla u_\lambda + c^\lambda u_\lambda = 0$$

in $\{u_\lambda > 0\}$, and that U is caloric in $\{U > 0\}$. Here $A^\lambda(x,t) = A(\lambda x, \lambda^2 t)$, $\mathbf{b}^\lambda(x,t) = \lambda \mathbf{b}(\lambda x, \lambda^2 t)$, $c^\lambda(x,t) = \lambda^2 c(\lambda x, \lambda^2 t)$. On the other hand, rescaling (1.3) we see that, for every $k > 0$

$$|\{u_\lambda > 0\} \cap \{x_1 < 0\} \cap Q_k| \to 0 \qquad \text{as } \lambda \to 0.$$

We deduce that U is nonnegative in $\{x_1 > 0\}$, caloric in $\{U > 0\}$, and vanishes in $\{x_1 < 0\}$. By Corollary A.1 in [CLW1], for any point $(0, \overline{x}', \overline{t})$, $\overline{x}' \in \mathbb{R}^{m-1}$, $\overline{t} \in \mathbb{R}$ there exists $\alpha \ge 0$ such that

$$U(x,t) = \alpha x_1^+ + o(|(x_1, x') - (0, \overline{x}')| + |t - \overline{t}|^{1/2}) \qquad \text{in } \{x_1 > 0\} \cap \{t \le \overline{t}\}.$$

By virtue of Lemma 3.3, there exists a sequence $j_n \to \infty$ such that $\delta_n = \frac{\varepsilon_{j_n}}{\lambda_n} \to 0$ and $u^{\delta_n} = (u^{\varepsilon_{j_n}})_{\lambda_n} \to U$ uniformly on compact sets of $\mathbb{R}^{m+1}$ as $n \to \infty$. Here $(u^{\varepsilon_{j_n}})_{\lambda_n}(x,t) = \frac{1}{\lambda_n} u^{\varepsilon_{j_n}}(\lambda_n x, {\lambda_n}^2 t)$. It is readily seen that u^{δ_n} is a solution to

$$L_{\lambda_n} u^{\delta_n} = \beta_{\delta_n}(u^{\delta_n}),$$

where L_{λ_n} is as above. Next, define $U_\tau(x,t) = \frac{1}{\tau} U(\tau x_1, \tau x' + \overline{x}', \tau^2 t + \overline{t})$. Then $U_\tau \to \alpha x_1^+$ uniformly on compact subsets of $\{t \le 0\}$ as $\tau \to 0$. We may apply Lemma 3.3 again (see also Remark 3.1), to conclude that there exist a sequence $\sigma_n = \frac{\delta_{j_n}}{\tau_n}$ and solutions u^{σ_n} to

$$L_n u^{\sigma_n} = div\, A^n \nabla u^{\sigma_n} - \partial_t u^{\sigma_n} + \mathbf{b^n} \cdot \nabla u^{\sigma_n} + c^n u^{\sigma_n} = \beta_{\sigma_n}(u^{\sigma_n}) \tag{4.1}$$

such that $u^{\sigma_n} \to \alpha x_1^+$ uniformly on compact subsets of $\{t \le 0\}$, and

$$\begin{aligned} &\nabla u^{\sigma_n} \to \alpha \chi_{\{x_1 > 0\}} \mathbf{e_1} \qquad \text{in } L^2_{loc}(\{t \le 0\}), \\ &\partial_t u^{\sigma_n} \rightharpoonup 0 \qquad \text{weakly in } L^2_{loc}(\{t \le 0\}) \end{aligned} \tag{4.2}$$

by virtue of Lemma 3.2. Here $A^n(x,t) = A^{\lambda_{j_n}}(\tau_n x_1, \tau_n x' + \overline{x}', \tau_n^2 t + \overline{t})$, $\mathbf{b^n}(x,t) = \tau_n \mathbf{b}^{\lambda_{\mathbf{j_n}}}(\tau_n x_1, \tau_n x' + \overline{x}', \tau_n^2 t + \overline{t})$, $c^n(x,t) = \tau_n^2(\tau_n x_1, \tau_n x' + \overline{x}', \tau_n^2 t + \overline{t})$. Let $\psi \in C_0^\infty(\{t \le 0\})$ and choose $\partial_{x_1} u^{\sigma_n} \psi$ as test function in the weak formulation of (4.1). Integrating by parts we obtain (3.16), with ε_n replaced by σ_n. Next, since $0 \le B_{\sigma_n}(s) \le M$, there exists $M(x,t) \in L^\infty(\mathbb{R}^{m+1})$, $0 \le M(x,t) \le M$, such that on a subsequence, still denoted by B_{σ_n}, $B_{\sigma_n}(u^{\sigma_n})(x,t) \to M(x,t)$ $*$-weakly in $L^\infty(\mathbb{R}^{m+1})$. Proceeding as in the proof of Proposition 3.6, we see that $M(x,t) = M\chi_{\{x_1>0\}} + \overline{M}\chi_{\{x_1<0\}}$ and $\alpha^2 = 2(M - \overline{M})$, where $\overline{M}$ is a nonnegative constant. We now claim that either $\overline{M} = 0$ or $\overline{M} = M$. Let $\eta_1, \eta_2 > 0$, and let $Q = Q_r(y,s) \subset\subset \{x_1 < 0\}$. There exists $0 < \mu < 1$ such that

$$
\begin{aligned}
|Q \cap \{\eta_1 < B_{\sigma_n}(u^{\sigma_n}) < M - \eta_2\}| &\leq \left| Q \cap \left\{ \mu < \frac{u^{\sigma_n}}{\sigma_n} < 1 - \mu \right\} \right| \\
&\leq \left| Q \cap \left\{ (\beta_{\sigma_n}(u^{\sigma_n}) \geq \frac{1}{\sigma_n} \inf_{[\mu, 1-\mu]} \beta \right\} \right| \to 0
\end{aligned}
\tag{4.3}
$$

as $n \to \infty$, since $\beta_{\sigma_n}(u^{\sigma_n}) \to 0$ in $L^1(Q)$ by Proposition 3.3. The claim will thus follow from the following lemma, whose proof we postpone for a moment.

Lemma 4.1. *The sequence $\{B_{\sigma_n}(u^{\sigma_n})\}$ is precompact in $L^1(Q)$.*

Let us see that $\alpha > 0$. By virtue of the nondegeneracy assumption on u at $(0,0)$, for every $r > 0$ and n sufficiently large,

$$
\frac{1}{r^{m+2}} \iint_{Q_r^-} u_{\lambda_n} \geq cr,
$$

and passing to the limit as $n \to \infty$,

$$
\frac{1}{r^{m+2}} \iint_{Q_r^-} U \geq cr.
$$

Clearly, this forces $\alpha > 0$, and as a consequence $\overline{M} = 0$ and $\alpha = \sqrt{2M}$. We have thus shown that for every point $(0, \overline{x}', \overline{t})$

$$
U(x,t) = \begin{cases} \sqrt{2M}x_1 + o(|(x_1,x') - (0,\overline{x}')| + |t - \overline{t}|^{1/2}), & t \leq \overline{t},\ x_1 > 0; \\ 0, & x_1 \leq 0. \end{cases}
\tag{4.4}
$$

Next, let us show that $|\nabla U| \leq \sqrt{2M}$ in $\mathbb{R}^{m+1}$. By Theorem 4.1,

$$
\limsup_{(x,t)\to(0,0)} |\nabla u(x,t)| \leq \sqrt{2M}.
$$

Let R, $\sigma > 0$ be fixed. There exists $\lambda_0 > 0$ such that $|\nabla u(x,t)| \leq \sqrt{2M} + \sigma$ for any $(x,t) \in Q_{\lambda R}$ if $\lambda \leq \lambda_0$. Therefore, $|\nabla u_{\lambda_n}(x,t)| \leq \sqrt{2M} + \sigma$ for any $(x,t) \in Q_R$ if n is sufficiently large. Moreover, $\nabla u_{\lambda_n} \to \nabla U$ $*$-weakly in $L^\infty(Q_R)$ and so $|\nabla U(x,t)| \leq \sqrt{2M} + \sigma$ for any $(x,t) \in Q_R$. Since R and σ were arbitrarily chosen, we have that $|\nabla U| \leq \sqrt{2M}$ in $\mathbb{R}^{m+1}$.

At this point it suffices to observe that $U \equiv 0$ on $\{x_1 = 0\}$ to conclude that $U \leq \sqrt{2M}x_1$ in $\{x_1 > 0\}$. Applying Hopf's maximum principle to the function $v(x,t) = U(x,t) - \sqrt{2M}x_1$, we see that necessarily

$$
U(x,t) = \sqrt{2M}x_1 \qquad \text{in } \{x_1 > 0\}.
$$

As an immediate consequence we finally obtain

$$
u(x,t) = \sqrt{2M}x_1^+ + o(|x| + |t|^{1/2}).
$$

To conclude the proof, we need to prove Lemma 4.1.

Proof of Lemma 4.1. The proof of this result is inspired by an idea in [W, Proposition 4.1]. Thanks to Proposition 3.1, it suffices to show the precompactness of $v^{\sigma_n} = \langle A^n \nabla u^{\sigma_n}, \nabla u^{\sigma_n}\rangle + 2B_{\sigma_n}(u^{\sigma_n})$. Let $\{\phi_\delta\}_{\delta\in(0,1)}$ be a family of approximations to the identity in space, i.e. $\phi_\delta \in C_0^\infty(\mathbb{R}^m)$, *supp* $\phi_\delta \subset B_\delta, \phi_\delta \geq 0, \int \phi_\delta = 1$. For $(x,t) \in Q$ and $0 < \delta < 1$ fixed, we compute

$$\begin{aligned}\partial_t(v^{\sigma_n} * \phi_\delta)(x,t) &= \int_{\mathbb{R}^m} \Big(\partial_t \langle A^n \nabla u^{\sigma_n}, \nabla u^{\sigma_n}\rangle + 2\beta_{\sigma_n}(u^{\sigma_n})\partial_t u^{\sigma_n}\Big)(x-y,t)\ \phi_\delta(y)\,dy\\ &= \int_{\mathbb{R}^m} \langle \partial_t A^n \nabla u^{\sigma_n}, \nabla u^{\sigma_n}\rangle(x-y,t)\ \phi_\delta(y)\,dy\\ &+ 2\int_{\mathbb{R}^m} \Big(\langle A^n \partial_t \nabla u^{\sigma_n}, \nabla u^{\sigma_n}\rangle + \beta_{\sigma_n}(u^{\sigma_n})\partial_t u^{\sigma_n}\Big)(x-y,t)\ \phi_\delta(y)\,dy = I_1 + 2I_2.\end{aligned}$$

First we estimate

$$\begin{aligned}I_2 &= \int_{\mathbb{R}^m} (A^n_{ij}\partial_t\partial_j u^{\sigma_n}\partial_i u^{\sigma_n} + \beta_{\sigma_n}(u^{\sigma_n})\partial_t u^{\sigma_n})(x-y,t)\ \phi_\delta(y)\ dy\\ &= \int_{\mathbb{R}^m} \left(-\partial_j(A^n_{ij}\partial_i u^{\sigma_n})\partial_t u^{\sigma_n} + \beta_{\sigma_n}(u^{\sigma_n})\partial_t u^{\sigma_n}\right)(z,t)\ \phi_\delta(x-z)\ dz\\ &+ \int_{\mathbb{R}^m} A^n_{ij}\partial_t u^{\sigma_n}\partial_i u^{\sigma_n}(z,t)\partial_j\phi_\delta(x-z)\ dz\\ &= \int_{\mathbb{R}^m} (\mathbf{b^n}\cdot\nabla u^{\sigma_n} + c^n u^{\sigma_n})\partial_t u^{\sigma_n}(z,t)\phi_\delta(x-z)\ dz - \int_{\mathbb{R}^m} (\partial_t u^{\sigma_n})^2(z,t)\phi_\delta(x-z)\ dz\\ &+ \int_{\mathbb{R}^m} A^n_{ij}\partial_t u^{\sigma_n}\partial_i u^{\sigma_n}(z,t)\partial_j\phi_\delta(x-z)\ dz\\ &\leq \tau_n\left(\int_{\mathbb{R}^m} (\nabla u^{\sigma_n} + u^{\sigma_n})^2(z,t)\phi_\delta(x-z)\ dz\right)^{\frac12}\left(\int_{\mathbb{R}^m} (\partial_t u^{\sigma_n})^2(z,t)\phi_\delta(x-z)\ dz\right)^{\frac12}\\ &+\ \|\nabla\phi_\delta\|_\infty\|A\|_\infty\|\nabla u^{\sigma_n}\|_2\|\partial_t u^{\sigma_n}\|_2,\end{aligned}$$

where all the norms are on $B_\delta(x)$. On the other hand,

$$I_1 \leq \tau_n^2\|\partial_t A\|_\infty\|\nabla u^{\sigma_n}\|_\infty^2,$$

where again the norms are on $B_\delta(x)$. By virtue of Proposition 3.1 and Lemma 3.1, (iii), we conclude that

$$\|\partial_t(v^{\sigma_n} * \phi_\delta)\|_{L^1(Q)} \leq C_1(\delta).$$

Also,

$$\|\nabla(v^{\sigma_n} * \phi_\delta)\|_{L^1(Q)} \leq \|v^{\sigma_n}\|_{L^\infty(Q)}\|\nabla\phi_\delta\|_{L^1(Q)} \leq C_2(\delta),$$

and both constants C_1 and C_2 are independent of σ_n. Thus for each $0 < \delta < 1$ the sequence $\{v^{\sigma_n} * \phi_\delta\}_{n\in\mathbb{N}}$ is precompact in $L^1(Q)$. At this point, we estimate

$$
\begin{aligned}
\|B_{\sigma_n}(u^{\sigma_n}) - B_{\sigma_n}(u^{\sigma_n} * \phi_\delta)\|_{L_1(Q)} &\le \sup_{y\in B_\delta} \int_Q |B_{\sigma_n}(u^{\sigma_n})(x,t) - B_{\sigma_n}(u^{\sigma_n})(x-y,t)|\, dx\, dt \\
&\le \delta \sup_{y\in B_\delta} \sup_{s\in[0,1]} \int_Q |\nabla(B_{\sigma_n}(u^{\sigma_n}))(x-sy,t)|\, dx\, dt \\
&\le C\delta \sup_{y\in B_\delta} \sup_{s\in[0,1]} \int_Q \beta_{\sigma_n}(u^{\sigma_n})(x-sy,t)|\, dx\, dt \\
&\le C\delta.
\end{aligned} \tag{4.5}
$$

Here we have used that $\nabla B_{\sigma_n}(u^{\sigma_n}) = \beta_{\sigma_n}(u^{\sigma_n})\nabla u^{\sigma_n}$ and Proposition 3.1 again. Finally, we observe that the precompactness of $\{|\nabla u^{\sigma_n}|^2\}_{n\in\mathbb{N}}$ in $L^1(Q)$ implies

$$
\|\langle A^n \nabla u^{\sigma_n}, \nabla u^{\sigma_n}\rangle - \langle A^n \nabla u^{\sigma_n}, \nabla u^{\sigma_n}\rangle * \phi_\delta\|_{L^1(Q)} \to 0 \tag{4.6}
$$

as $\delta \to 0$. From the precompactness of $\{v^{\sigma_n} * \phi_\delta\}_{n\in\mathbb{N}}$ and estimates (4.5), (4.6), the desired conclusion follows. The lemma is thus proved. □

□

5. Asymptotic Behavior of Limit Solutions: Two Phase Case

In this section we study the local behavior of limit solutions in the two phase case. In particular, we present here the proof of Theorem 1.2. The following results will play a crucial role in the sequel.

Lemma 5.1. *Let $\{u^{\varepsilon_j}\}$ be a family of solutions to (SPP) in a domain $\mathcal{D} \subset \mathbb{R}^{m+1}$ such that u^{ε_j} converges to u uniformly on compact subsets of $\mathcal{D}$ and $\varepsilon_j \to 0$. Let $(x_0,t_0) \in \mathcal{D} \cap \partial\{u > 0\}$, and assume $A(x_0,t_0) = I$. Define $u_\lambda(x,t) = \frac{1}{\lambda} u(x_0 + \lambda x, t_0 + \lambda^2 t)$. There exists $\delta \ge 0$ such that if for a sequence $\lambda_n \to 0$, $u_{\lambda_n} \to U$ uniformly on compact sets of $\mathbb{R}^{m+1}$, then*

$$
J_U(t) = \frac{1}{t^2}\left(\int_{-t}^0 \int_{\mathbb{R}^m} |\nabla U^+|^2 G(x,-s)\, dx\, ds\right)\left(\int_{-t}^0 \int_{\mathbb{R}^m} |\nabla U^-|^2 G(x,-s)\, dx\, ds\right) = \delta
$$

for every $t > 0$, where $G(x,t) = \frac{1}{(4\pi t)^{m/2}} e^{-\frac{|x|^2}{4t}}$.

Proof. Without loss of generality, we may assume $(x_0,t_0) = (0,0)$, with $A(0,0) = I$, and that $Q_1^- \subset\subset \mathcal{D}$. Let $\varphi \in C_0^\infty(B_1(0))$, $\varphi \equiv 1$ in $B_{1/2}$ and define

$$
J_u(\varphi,t) = \frac{1}{t^2}\left(\int_{-t}^0 \int_{\mathbb{R}^m} |\nabla(u^+\varphi)|^2 G(x,-s)\, dx\, ds\right)\left(\int_{-t}^0 \int_{\mathbb{R}^m} |\nabla(u^-\varphi)|^2 G(x,-s)\, dx\, ds\right)
$$

for $0 < t < 1$. Let $\delta = \lim_{t\to 0^+} J_u(\varphi, t)$. Since u^+ and u^- are L-subsolutions in $\mathcal{D}$, the proof of Theorem 2.3.1 in [CK] shows that $\delta < \infty$. Now, define $\varphi_\lambda(x) = \varphi(\lambda x)$. A simple computation shows that $J_{u_\lambda}(\varphi_\lambda, t) = J_u(\varphi, \lambda^2 t)$, and therefore we have

$$\lim_{\lambda\to 0^+} J_{u_\lambda}(\varphi_\lambda, t) = \delta \tag{5.1}$$

for any $t > 0$. Now, fix $t > 0$ and let $\lambda_n \to 0$ be such that $u_{\lambda_n} \to U$ uniformly on compact subsets of $\mathbb{R}^{m+1}$. By Lemma 3.3 we know that $\nabla u_{\lambda_n} \to \nabla U$ in $L^2_{loc}(\mathbb{R}^{m+1})$, so that for a subsequence (still denoted by λ_n) $\nabla u_{\lambda_n} \to \nabla U$ a.e. in $\mathbb{R}^{m+1}$. Moreover, since $u \in Lip_{loc}(1, 1/2)$ in $\mathcal{D}$, there exists $L > 0$ such that $|\nabla u| \le L$ in Q_1^-. It is then easy to recognize that $\nabla(u_{\lambda_n}\varphi_{\lambda_n}) \to \nabla U$ a.e. and $\|\nabla(u^\pm_{\lambda_n}\varphi_{\lambda_n})\|_{L^\infty(\mathbb{R}^m\times(-t,0])} \le C$ for n sufficiently large. By Lebesgue dominated convergence theorem,

$$\lim_{n\to\infty} J_{u_{\lambda_n}}(\varphi_{\lambda_n}, t) = J_U(t). \tag{5.2}$$

The conclusion follows from (5.1) and (5.2). □

Lemma 5.2. *Let $u \in Lip_{loc}(1, 1/2)$ in a domain $\mathcal{D} \subset \mathbb{R}^{m+1}$ be a global L-subsolution, such that $Lu = 0$ in $\{u > 0\} \cup \{u \le 0\}^\circ$. If u^- is nondegenerate at $(x_0, t_0) \in \mathcal{D} \cap \partial\{u > 0\}$, then u^+ is also nondegenerate at (x_0, t_0).*

Proof. Arguing by contradiction, assume that there exists a sequence $\lambda_n \to 0$ such that

$$\lim_{n\to\infty} \frac{1}{{\lambda_n}^{m+3}} \iint_{Q^-_{\lambda_n}(x_0,t_0)} u^+ = 0.$$

Define $u_{\lambda_n} = \frac{1}{\lambda_n} u(x_0 + \lambda_n x, t_0 + {\lambda_n}^2 t)$. Since $u \in Lip_{loc}(1, 1/2)$ in $\mathcal{D}$ and $u(x_0, t_0) = 0$, there exist a subsequence, still denoted by λ_n, and $U \in Lip_{loc}(1, 1/2)$ in $\mathbb{R}^{m+1}$ such that $u_{\lambda_n} \to U$ uniformly on compact sets of $\mathbb{R}^{m+1}$. Hence

$$\iint_{Q_1^-} U^+ = \lim_{n\to\infty} \iint_{Q_1^-} u^+_{\lambda_n} = \lim_{n\to\infty} \frac{1}{{\lambda_n}^{m+3}} \iint_{Q^-_{\lambda_n}(x_0,t_0)} u^+ = 0.$$

This implies that $U^+ \equiv 0$ in Q_1^-, or $U \le 0$ in Q_1^-. Since U is globally subcaloric and vanishes at the origin, necessarily $U \equiv 0$ in Q_1^-. But this contradicts the fact that u^- is nondegenerate at (x_0, t_0), because

$$\lim_{n\to\infty} \iint_{Q_1^-} u^-_{\lambda_n} = \lim_{n\to\infty} \frac{1}{{\lambda_n}^{m+3}} \iint_{Q^-_{\lambda_n}(x_0,t_0)} u^- \ge c > 0.$$

This completes the proof. □

We are now ready to prove the asymptotic development for limit solutions when they are allowed to change sign.

Proof of Theorem 1.2. Without loss of generality, assume $(x_0, t_0) = (0, 0)$ and $\eta = \mathbf{e_1}$. Define $u_\lambda(x,t) = \frac{1}{\lambda} u(\lambda x, \lambda^2 t)$ and let $\rho > 0$ be such that $Q_\rho \subset\subset \mathcal{D}$. Since $u_\lambda \in Lip(1, 1/2)$ in $Q_{\rho/\lambda}$ uniformly in λ, and $u_\lambda(0,0) = 0$, there exist a subsequence $\lambda_n \to 0$ and a function $U \in Lip(1, 1/2)$ in $\mathbb{R}^{m+1}$ such that $u_{\lambda_n} \to U$ uniformly on compact subsets of $\mathbb{R}^{m+1}$. Using Lemmas 3.1 and 3.3, it is not difficult to see that U is caloric in $\{U > 0\} \cup \{U < 0\}$. Moreover, rescaling (1.3) we see that, for every $k > 0$

$$|\{u_\lambda > 0\} \cap \{x_1 < 0\} \cap Q_k| \to 0, \qquad |\{u_\lambda < 0\} \cap \{x_1 > 0\} \cap Q_k| \to 0$$

as $\lambda \to 0$. We deduce that U is nonnegative in $\{x_1 > 0\}$, nonpositive in $\{x_1 < 0\}$, and caloric in $\{U > 0\} \cup \{U < 0\}$. In particular, U is supercaloric in $\{x_1 < 0\}$. On the other hand, since $Lu^{\varepsilon_j} \geq 0$ in $\mathcal{D}$, also $Lu \geq 0$. Rescaling and passing to the limit, we find that U is subcaloric in $\mathbb{R}^{m+1}$ and so it is caloric in $\{x_1 < 0\}$. Moreover, since $U \in Lip(1, 1/2)$ in $\mathbb{R}^{m+1}$,

$$|U(x,t) - U(y,s)| \leq L(|x-y| + |t-s|^{1/2}) \qquad \text{for any } (x,t), (y,s) \in \{x_1 < 0\},$$

and $U\mid_{\{x_1=0\}} = 0$. From these facts we infer that necessarily $U(x,t) = \gamma x_1$ in $\{x_1 < 0\}$. Since $U \leq 0$ in $\{x_1 < 0\}$, we must have $\gamma \geq 0$. Finally, the nondegeneracy of u^- at the origin implies $\gamma > 0$. We now want to show that $U = \alpha x_1$ in $\{x_1 > 0\}$ for some positive constant α such that $\alpha^2 - \gamma^2 = 2M$. Consider, for $t > 0$,

$$J_U(t) = \frac{1}{t^2} \int_{-t}^0 \int_{\mathbb{R}^m} |\nabla U^+|^2 G(x,-s)\, dx\, ds \int_{-t}^0 \int_{\mathbb{R}^m} |\nabla U^-|^2 G(x,-s)\, dx\, ds,$$

where $G(x,t) = \frac{1}{(4\pi t)^{m/2}} e^{-\frac{|x|^2}{4t}}$. By Lemma 5.1 there exists $\delta \geq 0$ independent of the sequence λ_n such that $J_U(t) = \delta$ for all $t > 0$. Let us see that δ must be positive. In fact, $\gamma > 0$ forces $\int_{-t}^0 \int_{\mathbb{R}^m} |\nabla U^-|^2 G(x,-s)\, dx\, ds > 0$ for all $t > 0$. If $\int_{-t_0}^0 \int_{\mathbb{R}^m} |\nabla U^+|^2 G(x,-s)\, dx\, ds = 0$ for some $t_0 > 0$, then $U^+ = 0$ in $\mathbb{R}^m \times (-t_0, 0)$ and so, for any $r > 0$ such that $r^2 < t_0$,

$$0 = \lim_{n\to\infty} \frac{1}{r^{m+3}} \iint_{Q_r^-} u_{\lambda_n}^+ = \lim_{n\to\infty} \frac{1}{(\lambda_n r)^{m+3}} \iint_{Q_{\lambda_n r}^-} u^+,$$

which is absurd since, by Lemma 5.2, u^+ is nondegenerate at (x_0, t_0). Clearly, $J_U' \equiv 0$ and thus

$$\frac{2}{t} = \frac{I_1'}{I_1} + \frac{I_2'}{I_2},$$

where

$$I_1 = \int_{-t}^0 \int_{\mathbb{R}^m} |\nabla U^+|^2 G(x,-s)\, dx\, ds, \qquad I_2 = \int_{-t}^0 \int_{\mathbb{R}^m} |\nabla U^-|^2 G(x,-s)\, dx\, ds.$$

Integrating by parts we find that

$$\frac{1}{t} \geq \frac{\int_{\mathbb{R}^m} |\nabla U^+(x,-t)|^2 G(x,t)\, dx}{\int_{\mathbb{R}^m} |U^+(x,-t)|^2 G(x,t)\, dx} + \frac{\int_{\mathbb{R}^m} |\nabla U^-(x,-t)|^2 G(x,t)\, dx}{\int_{\mathbb{R}^m} |U^-(x,-t)|^2 G(x,t)\, dx}.$$

We now make the change of variable $x = t^{1/2}y$ and set $v_1^t(y) = U^+(t^{1/2}y, -t)$, $v_2^t(y) = U^-(t^{1/2}y, -t)$. We obtain

$$1 \geq \frac{\int_{\mathbb{R}^m} |\nabla v_1^t|^2 d\mu}{\int_{\mathbb{R}^m} |v_1^t|^2 d\mu} + \frac{\int_{\mathbb{R}^m} |\nabla v_2^t|^2 d\mu}{\int_{\mathbb{R}^m} |v_2^t|^2 d\mu}, \tag{5.3}$$

with $d\mu = \frac{1}{(4\pi)^{m/2}} e^{-|y|^2/4} dy$. Setting

$$\lambda_S = \inf \frac{\int_S |\nabla v|^2 d\mu}{\int_S |v|^2 d\mu},$$

where the infimum is taken over all $v \in Lip(1, 1/2)$ in $\mathbb{R}^m$, $v \equiv 0$ in S^c, we can rewrite (5.3) as

$$1 \geq \lambda_{\{v_1^t > 0\}} + \lambda_{\{v_2^t > 0\}}. \tag{5.4}$$

At this point we recall an important result of Beckner, Kenig and Pipher [BKP], see also [CK], which states that

$$\lambda_{S_1} + \lambda_{S_2} \geq 1$$

if S_1, S_2 are disjoint open sets in $\mathbb{R}^m$, and equality holds if, and only if, $S_2 = S_1^c$, with $S_1 = \{x_m > 0\}$, or a rotate of it, modulo sets of measure zero. This result, together with the fact that (5.4) holds, implies that $\{v_1^t > 0\} = \{\langle x, \eta\rangle > 0\}$ for some unit vector $\eta = \eta(t)$, $\{v_2^t > 0\} = \{\langle x, \eta\rangle < 0\}$ and

$$v_1^t(y) = \alpha(t)\langle y, \eta(t)\rangle^+, \qquad v_2^t(y) = \gamma(t)\langle y, \eta(t)\rangle^-$$

for some $\alpha(t), \gamma(t) > 0$. Hence

$$U^+(t^{1/2}y, -t) = \alpha(t)\langle y, \eta(t)\rangle^+, \qquad U^-(t^{1/2}y, -t) = \gamma(t)\langle y, \eta(t)\rangle^-,$$

and therefore

$$U^+(x, t) = \alpha(|t|)|t|^{-1/2}\langle x, \eta(|t|)\rangle^+, \qquad U^-(x, t) = \gamma(|t|)|t|^{-1/2}\langle x, \eta(|t|)\rangle^-,$$

for any $t < 0$. At this point we recall that $U(x, t) = -\gamma x_1^-$ in $\{x_1 < 0\}$ and so, in particular, $\eta(|t|) = \mathbf{e_1}$ for any $t < 0$. It follows that $U^+(x, t) = \tilde{\alpha}(t)x_1^+$ for any $t < 0$. But U is caloric where positive and so necessarily $\tilde{\alpha}(t) = \alpha$ for some positive constant α or $\tilde{\alpha}(t) = 0$. On the other hand, since U is continuous, also $\tilde{\alpha}$ is continuous in t and therefore, keeping in mind that U^+ cannot vanish because of the nondegeneracy of u^+ at the origin, we may conclude that

$$U^+(x, t) = \alpha x_1^+ \qquad \text{for any } t < 0.$$

We now need to show that $U^+(x, t) = \alpha x_1^+$ also for any $t \geq 0$. By Corollary A.1 in [CLW1], for any point $(0, \overline{x}', \overline{t})$, $\overline{x}' \in \mathbb{R}^{m-1}$, $\overline{t} \in \mathbb{R}$ there exists $\hat{\alpha} \geq 0$ such that

$$U(x, t) = \hat{\alpha} x_1^+ + o(|(x_1, x') - (0, \overline{x}')| + |t - \overline{t}|^{1/2}) \qquad \text{in } \{x_1 > 0\} \cap \{t \leq \overline{t}\}. \tag{5.5}$$

By virtue of Lemma 3.3, there exists a sequence $j_n \to \infty$ such that $\delta_n = \frac{\varepsilon_{j_n}}{\lambda_n} \to 0$ and

$$u^{\delta_n} = (u^{\varepsilon_{j_n}})_{\lambda_n} \to U$$

uniformly on compact sets of $\mathbb{R}^{m+1}$ as $n \to \infty$. Here $(u^{\varepsilon_{j_n}})_{\lambda_n}(x,t) = \frac{1}{\lambda_n} u^{\varepsilon_{j_n}}(\lambda_n x, {\lambda_n}^2 t)$. It is readily seen that u^{δ_n} is a solution to

$$L_n u^{\delta_n} = \beta_{\delta_n}(u^{\delta_n}),$$

where L_n is as in Lemma 3.2, with $(x_n, t_n) = (0,0)$. Next, define $U_\tau(x,t) = \frac{1}{\tau} U(\tau x_1, \tau x' + \overline{x}', \tau^2 t + \overline{t})$. Then $U_\tau \to \hat{\alpha} x_1^+$ uniformly on compact subsets of $\{x_1 > 0\} \cap \{t \leq 0\}$ as $\tau \to 0$, and $U_\tau = -\gamma x_1^-$ in $\{x_1 < 0\}$. We may apply Lemma 3.3 again (see also Remark 3.1), to conclude that there exist a sequence $\sigma_n = \frac{\delta_{j_n}}{\tau_n}$ and solutions u^{σ_n} to

$$L_{\sigma_n} u^{\sigma_n} = div\, A^{\sigma_n} \nabla u^{\sigma_n} - \partial_t u^{\sigma_n} + \mathbf{b}^{\sigma_n} \cdot \nabla u^{\sigma_n} + c^{\sigma_n} u^{\sigma_n} = \beta_{\sigma_n}(u^{\sigma_n})$$

such that $u^{\sigma_n} \to \hat{\alpha} x_1^+ - \gamma x_1^-$ uniformly on compact subsets of $\{t \leq 0\}$. Here $A^{\sigma_n}(x,t) = A^{j_n}(\tau_n x_1, \tau_n x' + \overline{x}', \tau_n^2 t + \overline{t})$, $\mathbf{b}^{\sigma_n}(x,t) = \tau_n \mathbf{b}^{j_n}(\tau_n x_1, \tau_n x' + \overline{x}', \tau_n^2 t + \overline{t})$, $c^{\sigma_n}(x,t) = \tau_n^2 c^{j_n}(\tau_n x_1, \tau_n x' + \overline{x}', \tau_n^2 t + \overline{t})$. By Proposition 3.5, we may conclude $\hat{\alpha}^2 - \gamma^2 = 2M$. In particular, α is independent of the point $(0, \overline{x}', \overline{t})$ and $\hat{\alpha} = \alpha$.

At this point, we only need to observe that the function $v(x,t) = U(x,t) - \alpha x_1^+$ is subcaloric in $E = \{x_1 > 0\} \cap \{t > -1\}$, and $v = 0$ on $\partial_p E$. We infer from the maximum principle that $U \leq \alpha x_1^+$ in E. Applying Hopf's maximum principle to the function $v(x,t) = U(x,t) - \alpha x_1$ in E and keeping (5.5) in mind, we see that necessarily

$$U = \alpha x_1^+ \qquad \text{in } \{x_1 > 0\}.$$

Finally, by Lemma 5.1 there exists $\delta > 0$, independent of λ_n, such that $\delta = J_U(t) = \frac{\alpha^2 \gamma^2}{4}$. Since $\alpha^2 - \gamma^2 = 2M$, α and γ do not depend on the sequence λ_n. In conclusion, we have proved that $u_\lambda(x,t) \to \alpha x_1^+ - \gamma x_1^-$ uniformly on compact subsets of $\mathbb{R}^{m+1}$, with $\alpha^2 - \gamma^2 = 2M$. The conclusion of the theorem readily follows. □

Our last main result shows that is actually possible to relax the nondegeneracy conditions (H3), (H4), and (H4') which appear in Theorems 1.1 and 1.2, provided that the free boundary is given by a differentiable surface, with non-vertical inward normal η. Without loss of generality we may assume

$$\eta = (\cos\theta, 0, \ldots, 0, \sin\theta), \qquad \text{with } 0 \leq \theta < \pi/2. \tag{5.6}$$

Theorem 5.1. *Let $\{u^{\varepsilon_j}\}$ be solutions to (SPP) in a domain $\mathcal{D} \subset \mathbb{R}^{m+1}$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of $\mathcal{D}$ as $\varepsilon_j \to 0$. Let $(x_0, t_0) \in \mathcal{D} \cap \partial\{u > 0\}$ be such that (H2) and (H5) holds. Suppose in addition that at (x_0, t_0) the free boundary $\partial\{u > 0\}$ is given by a* *differentiable surface, with inward normal* η *as in* (5.6). *Under these assumptions, there exist* $\alpha > 0$ *and* $\gamma \geq 0$ *such that*

$$u(x,t) = \alpha[(x-x_0)_1 + (\tan\theta)(t-t_0)]^+ - \gamma[(x-x_0)_1 + (\tan\theta)(t-t_0)]^- + o(|x-x_0| + |t-t_0|^{1/2}),$$

with $\alpha^2 - \gamma^2 = 2M$.

In order to prove Theorem 5.1 we will need the following results.

Proposition 5.1. *Let* u^{ε_j} *be solutions to (SPP) in a domain* $\mathcal{D} \subset \mathbb{R}^{m+1}$ *such that* $u^{\varepsilon_j} \to u$ *uniformly on compact subsets of* $\mathcal{D}$ *as* $\varepsilon_j \to 0$. *Let* $(x_0,t_0) \in \mathcal{D} \cap \partial\{u > 0\}$ *and assume that there exists an unit vector* $\eta \in \mathbb{R}^m$ *such that*

$$\liminf_{r\to 0^+} \frac{|\{u>0\} \cap \{\langle x-x_0,\eta\rangle > 0\} \cap Q_r^-(x_0,t_0)|}{|Q_r^-(x_0,t_0)|} = \alpha_1$$

and

$$\liminf_{r\to 0^+} \frac{|\{u<0\} \cap \{\langle x-x_0,\eta\rangle < 0\} \cap Q_r^-(x_0,t_0)|}{|Q_r^-(x_0,t_0)|} = \alpha_2,$$

with $\alpha_1 + \alpha_2 > 1/2$. *There exists a positive constant* $C = C(\alpha_1,\alpha_2,m,\nu,\Lambda)$ *such that for every* $r > 0$ *small*

$$\sup_{\partial_p Q_r^-(x_0,t_0)} u \geq Cr.$$

Since the proof of Proposition 5.1 is analogous to the one of Theorem 6.3 in [CLW1], we omit it and refer the reader to that source. The next result is a consequence of Lemma 5.1.

Lemma 5.3. *Let* $\{u^{\varepsilon_j}\}$ *be a family of solutions to (SPP) in a domain* $\mathcal{D} \subset \mathbb{R}^{m+1}$ *such that* $u^{\varepsilon_j} \to u$ *uniformly on compact subsets of* $\mathcal{D}$ *as* $\varepsilon_j \to 0$. *Let* $(x_0,t_0) \in \mathcal{D} \cap \partial\{u > 0\}$ *and define* $u_\lambda(x) = \frac{1}{\lambda}(x_0 + \lambda x, t_0 + \lambda^2 t)$. *Let* $\lambda_n, \tilde\lambda_n$ *be two sequences tending to zero and such that*

$$u_{\lambda_n} \to U = \alpha x_1^+ - \gamma x_1^- + o(|x| + |t|^{1/2}), \qquad u_{\tilde\lambda_n} \to \tilde U = \tilde\alpha x_1^+ + o(|x| + |t|^{1/2}),$$

uniformly on compact subsets of $\mathbb{R}^{m+1}$, *with* $\alpha > 0$ *and* $\tilde\alpha, \gamma \geq 0$. *Under these assumptions,* $\gamma = 0$.

Proof. Without loss of generality, we may assume $(x_0,t_0) = (0,0)$ and $A(0,0) = I$. By Lemma 5.1, there exists a constant δ (independent of the sequence λ_n) such that

$$\delta = J_U(t) = \frac{1}{t^2}\int_{-t}^0\int_{\mathbb{R}^m} |\nabla U^+|^2 G(x,-s)\,dx\,ds \int_{-t}^0\int_{\mathbb{R}^m} |\nabla U^-|^2 G(x,-s)\,dx\,ds \tag{5.7}$$

for every $t > 0$. Let $U_\lambda(x,t) = \frac{1}{\lambda}U(\lambda x, \lambda^2 t)$. Then $U_\lambda \to U_0 = \alpha x_1^+ - \gamma x_1^-$ uniformly on compact subsets of $\mathbb{R}^{m+1}$ as $\lambda \to 0$. Rescaling (5.7) we obtain

$$\delta = \frac{1}{t^2}\int_{-t}^0\int_{\mathbb{R}^m} |\nabla U_{\lambda_n}^+|^2 G(x,-s)\,dx\,ds \int_{-t}^0\int_{\mathbb{R}^m} |\nabla U_{\lambda_n}^-|^2 G(x,-s)\,dx\,ds. \tag{5.8}$$

We want to pass to the limit in this identity. By Lemma 3.3, there exists a sequence d_n such that $u^{d_n} \to U$ uniformly on compact subsets of $\mathbb{R}^{m+1}$, with u^{d_n} solutions to

$$L_n u^{d_n} = \beta_{d_n}(u^{d_n}),$$

where L_n is as in Lemma 3.2, with $(x_n, t_n) = (0, 0)$. We infer from Lemma 3.3 that $\nabla U_{\lambda_n} \to \nabla U_0$ in $L^2_{loc}(\mathbb{R}^{m+1})$, and on a subsequence still denoted by λ_n, $\nabla U_{\lambda_n} \to \nabla U_0$ a.e. in $\mathbb{R}^{m+1}$. Moreover, $|\nabla U_{\lambda_n}|$ is uniformly bounded in $\mathbb{R}^{m+1}$ because $U \in Lip(1, 1/2)$ in $\mathbb{R}^{m+1}$. Hence we may apply Lebesgue dominated convergence theorem in (5.8) to get $\delta = \alpha^2\gamma^2/4$. But the same argument for the sequence $\tilde{\lambda}_n$ shows $\delta = 0$. Since $\alpha > 0$, necessarily $\gamma = 0$. □

We can now present the proof of Theorem 5.1.

Proof of Theorem 5.1. For simplicity, we assume $(x_0, t_0) = (0, 0)$ and $\theta = 0$. Define, for $\lambda > 0$,

$$u_\lambda(x, t) = \frac{1}{\lambda} u(\lambda x, \lambda^2 t). \tag{5.9}$$

As proved in Theorem 1.2, on a subsequence $\lambda_n \to 0$, u_{λ_n} converges to a function $U \in Lip_{loc}(1, 1/2)$ in $\mathbb{R}^{m+1}$ uniformly on compact sets of $\mathbb{R}^{m+1}$. Moreover, U is a solution of the heat equation in $\{U > 0\} \cup \{U \le 0\}^\circ$. Consider now a linear scaling of u around the origin given by $v_\lambda(x, t) = \frac{1}{\lambda} u(\lambda x, \lambda t)$. If we start with u restricted to a cylinder $\Omega_R = \{|x| < R, |t| < R\}$, then v_λ is defined at least in a box $\Omega_{R/\lambda}$. Let $D_\lambda = \{(x, t) \in \Omega_{R/\lambda} \mid v_\lambda(x, t) > 0\}$, and let $\Pi = \{x_1 = 0\}$. Since Π is a tangent plane, for every $R > 0$ and $\sigma > 0$ the boundary of $D_\lambda \cap \Omega_R$ can be confined in the region $\Pi_\sigma = \{(x, t) \in \Omega_R \mid |x_1| \le \sigma\}$ if λ is sufficiently small. Thus, the region D_λ converges as $\lambda \to 0$ to the half space $\{x_1 > 0\}$, and the convergence is uniform in boxes Ω_R. Since $u_\lambda(x, t) = v_\lambda(x, \lambda t)$, the domain $\{u_\lambda > 0\}$ tends as $\lambda \to 0$ to the half space $\{x_1 > 0\}$. We conclude that U is a nonnegative solution of the heat equation in $\{x_1 > 0\}$, with boundary value 0 on Π, and satisfying $|U(x, t) - U(y, s)| \le L(|x - y| + |t - s|^{1/2})$ for any $(x, t), (y, s) \in \{x_1 > 0\}$. Hence, necessarily $U(x, t) = \alpha x_1^+$ in $\{x_1 > 0\}$ for some $\alpha \ge 0$. Arguing as in the proof of Theorem 1.2, it is not difficult to recognize that

$$U(x, t) = \alpha x_1^+ - \gamma x_1^-.$$

By Lemma 3.3, there exists a sequence u_{d_n} of solutions to (SPP) such that $u_{d_n} \to \alpha x_1^+ - \gamma x_1^-$ uniformly on compact subsets of $\mathbb{R}^{m+1}$. We now consider two cases.

<u>Case I</u> $\lim_{r \to 0} \frac{|\{u \equiv 0\} \cap Q_r|}{|Q_r|} = 0$.

Assume first $\gamma > 0$. We may thus apply Proposition 3.5 to conclude that $\alpha^2 - \gamma^2 = 2M$. Moreover, Lemma 5.1 ensures that α and γ be independent of the choice of the subsequence λ_n, and from this the conclusion of the theorem readily follows.

Suppose instead $\gamma = 0$. Since $0 \le B_{\varepsilon_n}(u^{\varepsilon_n}) \le M$, there exists $M(x,t) \in L^\infty(\mathcal{D})$, $0 \le M(x,t) \le M$, such that on a subsequence (still denoted by $B_{\varepsilon_n}(u^{\varepsilon_n})$) $B_{\varepsilon_n}(u^{\varepsilon_n}) \to M(x,t)$ $*$-weakly in $L^\infty(\mathcal{D})$. As in the proof of Proposition 3.5, $M(x,t) \equiv M$ in $\mathcal{D} \cap \{x_1 > 0\}$. Let $\psi \in C_0^\infty(\mathcal{D})$ and take $\partial_{x_1} u^\varepsilon \psi$ as test function in the weak formulation of (SPP). Proceeding as in the proof of Proposition 3.5 and passing to the limit as $\varepsilon_n \to 0$ we obtain

$$\begin{aligned}\iint \partial_t u\, \partial_{x_1} u\, \psi = \frac{1}{2}\iint \partial_{x_1} A\nabla u \cdot \nabla u\, \psi + \frac{1}{2}\iint A\nabla u \cdot \nabla u\, \partial_{x_1}\psi \\ - \iint A\nabla u \cdot \nabla\psi\, \partial_{x_1} u + \iint \mathbf{b}\cdot\nabla u\, \partial_{x_1} u\, \psi \\ + \iint c\, u\, \partial_{x_1} u\, \psi + M\iint_{\{u>0\}} \partial_{x_1}\psi + \iint_{\{u\equiv 0\}} M(x,t)\partial_{x_1}\psi.\end{aligned} \tag{5.10}$$

Now let $\psi_{\lambda_n}(x,t) = \lambda_n \psi(x/\lambda_n, t/\lambda_n{}^2)$, with $\psi \in C_0^\infty(\mathbb{R}^{m+1})$ and $supp\,\psi \subset Q_r$ for some $r > 0$. If λ_n is small enough, $supp\,\psi_{\lambda_n} \subset \mathcal{D}$ and therefore we may replace ψ with ψ_{λ_n} in (5.10). A change of variable then yields

(5.11)
$$\begin{aligned}\iint \partial_t u_{\lambda_n} \partial_{x_1} u_{\lambda_n} \psi = \frac{1}{2}\lambda_n \iint \partial_{x_1} A(\lambda_n x, \lambda_n{}^2 t)\nabla u_{\lambda_n} \cdot \nabla u_{\lambda_n}\, \psi + \frac{1}{2}\iint A^n \nabla u_{\lambda_n} \cdot \nabla u_{\lambda_n}\, \partial_{x_1}\psi \\ - \iint A^n \nabla u_{\lambda_n} \cdot \nabla\psi\, \partial_{x_1} u_{\lambda_n} + \iint \mathbf{b^n}\cdot\nabla u_{\lambda_n} \partial_{x_1} u_{\lambda_n}\, \psi \\ + \iint c^n u_{\lambda_n} \partial_{x_1} u_{\lambda_n} \psi + M\iint_{\{u_{\lambda_n}>0\}} \partial_{x_1}\psi + \iint_{\{u_{\lambda_n}\equiv 0\}} M(\lambda_n x, \lambda_n{}^2 t)\partial_{x_1}\psi.\end{aligned}$$

Here, A^n, $\mathbf{b^n}$, and c^n are as in Lemma 3.2, with $(x_n, t_n) = (0,0)$. Using Lemma 3.3 again we find $\nabla u_{\lambda_n} \to \alpha\chi_{\{x_1>0\}}\mathbf{e_1}$ in $L^2_{loc}(\mathbb{R}^{m+1})$, and $\partial_t u_{\lambda_n} \rightharpoonup 0$ weakly in $L^2_{loc}(\mathbb{R}^{m+1})$. Keeping in mind that A is continuously differentiable, A^n converges to the identity uniformly on compact sets of $\mathbb{R}^{m+1}$, $\|\mathbf{b}\|_\infty + \|c\|_\infty \le \Lambda$, $\|\nabla u_{\lambda_n}\|_\infty \le L$, and that $|\{u_{\lambda_n} \equiv 0\} \cap Q_r| \to 0$ as $\lambda \to 0$ for any $r > 0$ by the current assumption, we may pass to the limit in (5.11) obtaining

$$\frac{\alpha^2}{2}\iint_{\{x_1>0\}} \partial_{x_1}\psi = M\iint_{\{x_1>0\}} \partial_{x_1}\psi.$$

The latter identity, in turn, implies via integration by parts

$$M\iint_{\{x_1=0\}} \psi\, dx'\, dt = \frac{\alpha^2}{2}\iint_{\{x_1=0\}} \psi\, dx'\, dt.$$

Since $\psi \in C_0^\infty$ is arbitrary, we conclude that $\alpha^2 = 2M$, and $u_{\lambda_n}(x,t) \to \sqrt{2M}x_1^+$ as $n \to \infty$. The conclusion of the theorem is easily inferred from this.

<u>Case II</u> $\limsup_{r\to 0} \frac{|\{u\equiv 0\}\cap Q_r|}{|Q_r|} > 0$.

We prove that in this case $\gamma = 0$. Because of the current hypothesis, there exists a sequence $\tilde\lambda_n \to 0$ such that

$$\lim_{n\to\infty} \frac{|\{u \equiv 0\} \cap Q_{\tilde\lambda_n}|}{|Q_{\tilde\lambda_n}|} = 2c > 0,$$

and so

$$\lim_{n\to\infty} \frac{|\{u_{\tilde\lambda_n} \equiv 0\} \cap Q_1|}{|Q_1|} \geq c \tag{5.12}$$

for n large. On the other hand, on a subsequence (still denoted by $\tilde\lambda_n$), $u_{\tilde\lambda_n} \to \tilde U$ uniformly on compact subsets of $\mathbb{R}^{m+1}$, and $\tilde U(x,t) = \tilde\alpha x_1^+ - \tilde\gamma x_1^-$, for some $\tilde\alpha, \tilde\gamma \geq 0$. As shown in the proof of Case I, the existence of a classical normal to the free boundary $\mathcal{D} \cap \partial\{u > 0\}$ at the origin guarantees that $u > 0$ in $\{x_1 > 0\} \cap Q_r$ for all $r > 0$ sufficiently small, and therefore $u_{\tilde\lambda_n} > 0$ in $Q_1 \cap \{x_1 > 0\}$, if n is large enough. This implies, together with (5.12),

$$c \leq \frac{|\{\tilde U \equiv 0\} \cap Q_1|}{|Q_1|} = \frac{|\{\tilde U \equiv 0\} \cap \{x_1 < 0\} \cap Q_1|}{|Q_1|}.$$

Since $\tilde U(x,t) = -\tilde\gamma x_1^-$ in $\{x_1 < 0\}$, necessarily $\tilde\gamma = 0$. But then also $\gamma = 0$ by Lemma 5.3. We next show that $\alpha > 0$. If $\liminf_{r\to 0} \frac{|\{u<0\}\cap Q_r|}{|Q_r|} = 0$, then u^+ is nondegenerate at the origin by assumption. Hence, for every $r > 0$ and n sufficiently large

$$\frac{1}{r^{m+2}} \iint_{Q_r^-} u_{\lambda_n}^+ = \frac{1}{\lambda_n^{m+3} r^{m+2}} \iint_{Q_{\lambda_n r}^-} u^+ \geq cr,$$

which implies

$$\frac{1}{r^{m+2}} \iint_{Q_r^-} U^+ \geq cr$$

for all $r > 0$. This forces $\alpha > 0$. If instead $\liminf_{r\to 0} \frac{|\{u<0\}\cap Q_r|}{|Q_r|} > 0$, we may apply Proposition 5.1 to infer the existence of a positive constant C such that $\sup_{\partial_p Q_r^-} u \geq Cr$ for any $r > 0$ small enough. Rescaling and passing to the limit we find that, for $r > 0$, $\sup_{\partial_p Q_r^-} U \geq Cr$, and so necessarily again $\alpha > 0$. At this point we can proceed as in the proof of Theorem 1.1 to show $\alpha = \sqrt{2M}$. This proves that $U(x,t) = \sqrt{2M} x_1^+$ in $\mathbb{R}^{m+1}$, and the conclusion of the theorem follows also in this case. The proof is now complete. □

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