PROBLEMS WITH FREE INTERFACES AND FREE DISCONTINUITIES

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1. Introduction

In several models coming from very different applications, one needs to describe physical phenomena where the state function may present some regions of discontinuity. We may think for instance to problems arising in fracture mechanics, where the function which describes the displacement of the body has a jump along the fracture, phase transitions, or also to problems of image reconstruction, where the function which describes a picture (the intensity of black, for instance, in white and black pictures) has naturally some discontinuities along the profiles of the objects.

The analysis in Sobolev spaces is then no longer appropriate for this kind of problems, since Sobolev functions cannot have jump discontinuities along hypersurfaces, as on the contrary is required by the models above. For a rigorous presentation of variational problems involving functions with discontinuities, the essential tool is the space BV of functions with bounded variation. The first ideas about this space have been developed by De Giorgi in the fifties, in order to provide a variational framework to study the problems of minimal surfaces, and several monographs are nowadays available on the subject. We quote for instance the classical volumes of Evans and Gariepy [8], Federer [9], Giusti [13], Massari and Miranda [14], Ziemer [16], and the recent book by Ambrosio, Fusco and Pallara [2], where a systematic presentation is given, also in view of the applications mentioned above.

2. The space BV

Consider a generic open subset Ω of $\mathbb{R}^N$, that for simplicity we take bounded and with a Lipschitz boundary. In the following we denote by $\mathcal{L}^N(E)$ or simply by $|E|$ the Lebesgue measure of E in $\mathbb{R}^N$, while $\mathcal{H}^k$ denotes the k-dimensional *Hausdorff measure*.

Definition 2.1. *We say that a function $u \in L^1(\Omega)$ is afunction of bounded variation in Ω if its distributional gradient Du is a $\mathbb{R}^N$-valued finite Borel measure on Ω. In other words, we have*

$$\int_\Omega uD_i\phi\,dx = -\int_\Omega \phi\,dD_iu \qquad \forall\phi \in \mathcal{C}_c^\infty(\Omega), \quad \forall i = 1, \dots, N \tag{2.1}$$

where D_iu are finite Borel measures. The space of all functions of bounded variation in Ω is denoted by $BV(\Omega)$.

The space $BV(\Omega)$ is clearly a vector space and, with the norm

$$\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + |Du|(\Omega) \tag{2.2}$$

it becomes a Banach space. The total variation $|Du|(\Omega)$ appearing above is intended as

$$\begin{aligned} |Du|(\Omega) \quad &= \sup\Big\{\sum_{i=1}^N \int_\Omega \phi_i\,dD_iu \;:\; \phi \in \mathcal{C}_c^\infty(\Omega;\mathbb{R}^N),\ |\phi| \le 1\Big\} \\ &= \sup\Big\{-\int_\Omega u\,\mathrm{div}\,\phi\,dx \;:\; \phi \in \mathcal{C}_c^\infty(\Omega;\mathbb{R}^N),\ |\phi| \le 1\Big\} \end{aligned}$$

and is sometimes indicated by $\int_\Omega |Du|$. The space $BV_{loc}(\Omega)$ is defined in a similar way, requiring that $u \in BV(\Omega')$ for every $\Omega' \subset\subset \Omega$.

From the functional analysis point of view, the space $BV(\Omega)$ does not verify the nice properties of Sobolev spaces. In particular:

- the Banach space $BV(\Omega)$ is not separable;
- the Banach space $BV(\Omega)$ is not reflexive;
- the class of smooth functions is not dense in $BV(\Omega)$ for the norm (2.2).

The issues above motivate why the norm (2.2) is not very helpful in the study of variational problems involving the space $BV(\Omega)$. On the contrary, the weak* convergence defined below is much more suitable to treat minimization problems for integral functionals.

Definition 2.2. *We say that a sequence (u_n)* weakly* converges *in $BV(\Omega)$ to a function $u \in BV(\Omega)$ if $u_n \to u$ strongly in $L^1(\Omega)$ and $Du_n \to Du$ in the weak* convergence of measures.*

The weak* convergence on $BV(\Omega)$ satisfies the following properties:

- *(compactness)* every bounded sequence in $BV(\Omega)$ for the norm (2.2) admits a weakly* convergent subsequence;

- *(lower semicontinuity)* the norm (2.2) is sequentially lower semicontinuous with respect to the weak* convergence;
- *(density)* every function $u \in BV(\Omega)$ can be approximated, in the weak* convergence, by a sequence (u_n) of smooth functions.

The density property above can be actually made stronger: in fact, the approximation of (u_n) to u holds in the sense that

$$\begin{cases} u_n \to u \text{ strongly in } L^1(\Omega) \\ Du_n \to Du \text{ weakly* as measures} \\ |Du_n|(\Omega) \to |Du|(\Omega). \end{cases}$$

Further properties of the space $BV(\Omega)$ concern the embeddings into Lebesgue spaces, traces, and Poincaré-type inequalities. More precisely we have:

- *(embeddings)* the space $BV(\Omega)$ is embedded continuously into $L^{N/(N-1)}(\Omega)$ and compactly into $L^p(\Omega)$ for every $p < N/(N-1)$;
- *(traces)* every function $u \in BV(\Omega)$ has a boundary trace which belongs to $L^1(\partial\Omega)$, and the trace operator from $BV(\Omega)$ into $L^1(\partial\Omega)$ is continuous;
- *(Poincaré inequalities)* there exist suitable constants c_1 and c_2 such that for every $u \in BV(\Omega)$

$$\int_\Omega |u|\,dx \le c_1\Big[|Du|(\Omega) + \int_{\partial\Omega} |u|\,d\mathcal{H}^{N-1}\Big]$$
$$\int_\Omega |u - u_\Omega|\,dx \le c_2|Du|(\Omega) \qquad (\text{where } u_\Omega = \frac{1}{|\Omega|}\int_\Omega u\,dx).$$

3. Sets of finite perimeter

An important class of functions with bounded variation are those that can be written as 1_E, the *characteristic function* of a set E, taking the value 1 on E and 0 elsewhere. This is the natural class where many phase-transition problems with sharp interfaces may be framed.

Definition 3.1. *For a measurable set $E \subset \mathbb{R}^N$ the* perimeter *of E in Ω is defined as*

$$\mathrm{Per}(E,\Omega) = |D1_E|(\Omega).$$

The equality above is intended as $\mathrm{Per}(E,\Omega) = +\infty$ whenever $1_E \notin BV(\Omega)$. If $\mathrm{Per}(E,\Omega) < +\infty$ then the set E is called a set of finite perimeter *in Ω.*

Note that by the compactness property above for BV functions, a family of characteristic functions of sets with finite perimeter in a bounded open set Ω with equibounded perimeter is weakly*-precompact, and its limit is of the same form.

For a set E of finite perimeter in Ω we may define the inner normal versor and the reduced boundary as follows.

Definition 3.2. *Let E be a set of finite perimeter in Ω. We call* reduced boundary $\partial^* E$ *the set of all points $x \in \Omega \cap \operatorname{spt} |D1_E|$ such that the limit*

$$\nu_E(x) = \lim_{r \to 0} \frac{D1_E\big(B_r(x)\big)}{|D1_E|\big(B_r(x)\big)}$$

exists and satisfies $|\nu_E(x)| = 1$. The vector $\nu_E(x)$ is called the generalized inner normal versor *to E.*

In order to link the measure-theoretical objects introduced above with some structure property of sets of finite perimeter, we introduce, for every $t \in [0,1]$ and every measurable set $E \subset \mathbb{R}^N$, the set E^t defined by

$$E^t = \Big\{x \in \mathbb{R}^N \ : \ \lim_{r \to 0} \frac{|E \cap B_r(x)|}{|B_r(x)|} = t\Big\}. \tag{3.1}$$

For instance, if E is a smooth domain of $\mathbb{R}^N$, E^1 is the interior part of E, E^0 is its exterior part, while $E^{1/2}$ is the boundary ∂E.

The main properties of the reduced boundary and of the generalized inner normal versor are stated in the following result.

Theorem 3.3. *Let E be a set of finite perimeter in Ω. Then its reduced boundary $\partial^* E$ coincides $\mathcal{H}^{N-1}$-a.e. with the set $E^{1/2}$ introduced in Definition 3.1, and we have the equality*

$$\operatorname{Per}(E, \Omega) = \mathcal{H}^{N-1}(\Omega \cap \partial^* E) = \mathcal{H}^{N-1}(\Omega \cap E^{1/2}).$$

Moreover, the generalized inner normal versor $\nu_E(x)$ exists for $\mathcal{H}^{N-1}$-a.e. $x \in \partial^* E$, and we have

$$D1_E = \nu_E(x) \mathcal{H}^{N-1} \llcorner \partial^* E\ .$$

Note that the lower semicontinuity of $|D1_E|(\Omega)$ entails the lower semicontinuity of $E \mapsto \mathcal{H}^{N-1}(\Omega \cap \partial^* E)$ with respect to the weak*-convergence of 1_E. As a consequence we may apply the direct methods of the calculus of variations to obtain for example existence of minimizers of

$$\min\Big\{\operatorname{Per}(E, \mathbb{R}^N) - \int_E g\, dx\Big\},$$

that are *sets with prescribed mean curvature g*. This lower semicontinuity property can be further generalized, e.g. as in the following result for *anisotropic perimeters*.

Theorem 3.4. *Let $\varphi : S^{N-1} \to \mathbb{R}$ be a Borel function. The energy*

$$\int_{\Omega \cap \partial^* E} \varphi(\nu_E)\, d\mathcal{H}^{N-1}$$

is lower semicontinuous with respect to the weak*-convergence of 1_E in $BV(\Omega)$ if and only if the positively one-homogeneous extension of φ from S^{N-1} to $\mathbb{R}^N$ is convex.

This result immediately implies the existence of solutions of *isovolumetric problems* of the form

$$\min\Big\{\int_{\partial^* E}\varphi(\nu_E)\,d\mathcal{H}^{N-1}\ :\ |E|=c\Big\},$$

whose solutions are obtained by suitably scaling the *Wulff shape* of φ.

4. The structure of BV functions

The simplest situation occurs when $N=1$ and so Ω is an interval of the real line. In this case, decomposing the derivative u' into positive and negative parts, and taking their primitives, we obtain that $u\in BV(\Omega)$ if and only if u is the sum of two bounded monotone functions (one increasing and one decreasing). Therefore, in the one-dimensional case, the BV functions share all the properties of monotone functions.

The situation is more delicate when $N>1$, for which we need the notion of approximate limit.

Definition 4.1. *Let $u\in BV(\Omega)$. We say that u has the* approximate limit *z at x if*

$$\lim_{r\to 0}\frac{1}{|B_r(x)|}\int_{B_r(x)}|u(y)-z|\,dy=0\ .$$

The set where no approximate limit exists is called the approximate discontinuity set*, and is denoted by S_u. In a similar way, when $x\in S_u$ we may define the* approximate values *z^+ and z^-, by requiring that*

$$\lim_{r\to 0}\frac{1}{|B_r^+(x,\nu)|}\int_{B_r^+(x,\nu)}|u(y)-z^+|\,dy=0$$
$$\lim_{r\to 0}\frac{1}{|B_r^-(x,\nu)|}\int_{B_r^-(x,\nu)}|u(y)-z^-|\,dy=0$$

where

$$B_r^+(x,\nu)=\big\{y\in B_r(x)\ :\ (y-x)\cdot\nu>0\big\}$$
$$B_r^-(x,\nu)=\big\{y\in B_r(x)\ :\ (y-x)\cdot\nu<0\big\}$$

Analogous definitions can be given in the vector valued case, when $u\in BV(\Omega;\mathbb{R}^m)$.

The triplet (z^+,z^-,ν) in Definition 4.1 is unique up to interchanging z^+ with z^- and changing sign to ν, and is denoted by $\big(u^+(x),u^-(x),\nu_u(x)\big)$.

We are now in a position to describe the structure of the measure Du when $u\in BV(\Omega)$, or more generally $u\in BV(\Omega;\mathbb{R}^m)$. We first apply the Radon-Nikodym theorem to Du and we decompose it into absolutely continuous and singular parts: $Du=(Du)^a+(Du)^s$. We denote by ∇u the density of the absolutely continuous part, so that we have

$$Du=\nabla u\cdot\mathcal{L}^N+(Du)^s.$$

The singular part $(Du)^s$ can be further decomposed into a $(N-1)$ dimensional part, concentrated on the approximate discontinuity set S_u, and the remaining part, which vanishes on all sets with finite $\mathcal{H}^{N-1}$ measure. More precisely, if $u \in BV(\Omega;\mathbb{R}^m)$ we have

$$Du = \nabla u \cdot \mathcal{L}^N + \big(u^+(x) - u^-(x)\big) \otimes \nu_u(x) \cdot \mathcal{H}^{N-1} \llcorner S_u + (Du)^c\ ; \qquad (4.1)$$

the three terms on the right hand side are mutually singular and are respectively called the *absolutely continuous part*, the *jump part*, and the *Cantor part* of the gradient measure Du.

In the vector valued case Du is a $m \times N$ matrix of finite Borel measures, ∇u is a $m \times N$ matrix of functions in $L^1(\Omega)$, and the jump term in (4.1) is a $N-1$ dimensional measure of rank one. The structure of the Cantor part $(Du)^c$ is described by the Alberti's rank one theorem (see [1]).

Theorem 4.2. *For every $u \in BV(\Omega;\mathbb{R}^m)$ the Cantor part $(Du)^c$ is a measure with values in the $m \times N$ matrices of rank one.*

5. Convex functionals on BV

Many problems of the calculus of variations deal with the minimization of energies of the form

$$F(u) = \int_\Omega f(x, u, Du)\, dx\ ; \qquad (5.1)$$

the direct methods require, to obtain the existence of at least a minimizer, some coercivity hypotheses on F, as well as its lower semicontinuity. This last issue, already rather delicate when working in Sobolev spaces (see for instance the books [4] and [5]), presents additional difficulties when the unknown function u varies in the space $BV(\Omega)$, due to the fact that Du is a measure, and the precise meaning of the integral in (5.1) has to be clarified.

In this section we limit ourselves to consider the simpler situation of convex functionals, and we also assume that the integrand $f(x, u, Du)$ depends only on x and Du. It is then convenient to study the problem in the framework of functionals defined on the space of finite Borel vector measures $\mathcal{M}(\Omega;\mathbb{R}^k)$. Let $f : \mathbb{R}^N \times \mathbb{R}^k \to [0, +\infty]$ be a Borel function such that

- f is lower semicontinuous;
- $f(x, \cdot)$ is convex for every $x \in \mathbb{R}^N$.

We denote by $f^\infty(x, z)$ the recession function associated to f, given by

$$f^\infty(x, z) = \lim_{t \to +\infty} \frac{f(x, z_0 + tz)}{t}$$

where z_0 is any point in $\mathbb{R}^k$ such that $f(x, z_0) < +\infty$ (in fact, the definition above is independent of the choice of z_0). Then we may consider the functional

$$F(\lambda) = \int_\Omega f\big(x, \lambda^a(x)\big)\, dx + \int_\Omega f^\infty\Big(x, \frac{d\lambda^s}{d|\lambda^s|}\Big)\, d|\lambda^s| \qquad (5.2)$$

where $\lambda = \lambda^a \cdot dx + \lambda^s$ is the Lebesgue-Nikodym decomposition of λ into absolutely continuous and singular parts, and the notation $d\lambda^s/d|\lambda^s|$ stands for the density of λ^s with respect to its total variation $|\lambda^s|$. For simplicity, the last term in the right hand side of (5.2) is often denoted by $\int_\Omega f^\infty(x,\lambda^s)$.

For the functional F the following lower semicontinuity result holds (see for instance [4]).

Theorem 5.1. *Under the assumptions above the functional (5.2) is sequentially lower semicontinuous for the weak* convergence on $\mathcal{M}(\Omega;\mathbb{R}^k)$. Moreover, if*

$$f(x,z) \geq c_0|z| - a(x) \qquad \text{with } c_0 > 0 \text{ and } a \in L^1(\Omega), \tag{5.3}$$

then the functional F turns out to be coercive for the same topology.

From Theorem 5.1 we deduce immediately a lower semicontinuity result for functionals defined on $BV(\Omega;\mathbb{R}^m)$.

Corollary 5.2. *Under the assumptions above on the integrand f (with $k = mN$) the functional defined on $BV(\Omega;\mathbb{R}^m)$ by*

$$F(u) = \int_\Omega f\big(x,(Du)^a\big)\,dx + \int_\Omega f^\infty\Big(x,\frac{d(Du)^s}{d|Du|^s}\Big)\,d|Du|^s \tag{5.4}$$

is sequentially lower semicontinuous for the weak convergence. Moreover, under the assumption (5.3) the functional F is coercive with respect to the same topology.*

For some extensions of the result above to the case when $f(x,\cdot)$ is quasiconvex (in the vector valued situation $m > 1$) we refer the interested reader to [10] and to references therein.

Fixing boundary data is another difference between variational problems on Sobolev spaces and on BV spaces. Due to the fact that the class $\{u \in BV(\Omega)\ :\ u = u_0 \text{ on } \partial\Omega\}$ is not weakly* closed, to set in a correct way a minimum problem of Dirichlet type on $BV(\Omega)$ with datum $u_0 \in BV(\mathbb{R}^N)$ it is convenient to consider a larger domain $\Omega' \supset\supset \Omega$ and for every $u \in BV(\Omega)$ the extended function

$$\tilde{u} = \begin{cases} u & \text{on } \Omega \\ u_0 & \text{on } \Omega' \setminus \Omega \end{cases}$$

whose distributional gradient is

$$D\tilde{u} = Du \llcorner \Omega + Du_0 \llcorner \Omega' \setminus \overline{\Omega} + (u_0 - u)\nu_\Omega \mathcal{H}^{N-1} \llcorner \partial\Omega$$

being ν_Ω the exterior normal versor to Ω. We have then the functional on $BV(\Omega')$

$$\begin{aligned} \tilde{F}(\tilde{u}) &= \int_{\Omega'} f\big(x,(D\tilde{u})^a\big)\,dx + \int_{\Omega'} f^\infty\big(x,(D\tilde{u})^s\big) \\ &= \int_\Omega f\big(x,(Du)^a\big)\,dx + \int_{\Omega'\setminus\Omega} f\big(x,(Du_0)^a\big)\,dx + \int_\Omega f^\infty\big(x,(Du)^s\big) \\ &\quad + \int_{\Omega'\setminus\overline{\Omega}} f^\infty\big(x,(Du_0)^s\big) + \int_{\partial\Omega} f^\infty\big(x,(u_0-u)\nu_\Omega\big)\,d\mathcal{H}^{N-1}. \end{aligned}$$

If we drop the constant term $\int_{\Omega'\setminus\Omega} f\big(x,(Du_0)^a\big)\,dx + \int_{\Omega'\setminus\overline{\Omega}} f^\infty\big(x,(Du_0)^s\big)$, irrelevant for the minimization, we end up with the functional

$$F_{u_0}(u) = F(u) + \int_{\partial\Omega} f^\infty\big(x,(u_0-u)\nu_\Omega\big)\,d\mathcal{H}^{N-1}$$

where F is as in (5.4). The Dirichlet problem we consider is then

$$\min\Big\{F(u) + \int_{\partial\Omega} f^\infty\big(x,(u_0-u)\nu_\Omega\big)\,d\mathcal{H}^{N-1} \;:\; u\in BV(\Omega)\Big\}. \tag{5.5}$$

For instance, if $f(z)=|z|$, problem (5.5) becomes

$$\min\Big\{\int_\Omega |Du| + \int_{\partial\Omega} |u-u_0|\,d\mathcal{H}^{N-1} \;:\; u\in BV(\Omega)\Big\}.$$

Under the assumptions considered, the problem above admits a solution $u\in BV(\Omega)$, but in general we do not have $u=u_0$ on $\partial\Omega$ in the sense of BV traces.

6. Nonconvex functionals on BV

In order to introduce the class of nonconvex functionals on $BV(\Omega)$ let us denote $v = Du$ so that every functional $\Phi(v)$ provides an energy $F(u)$. If we work in the setting of Sobolev spaces, we have $u\in W^{1,p}(\Omega)$ $(p\ge 1)$ which implies $v\in L^p(\Omega;\mathbb{R}^N)$; now, it happens that in this case all "interesting" functionals Φ are convex. More precisely, it can be proved that a functional $\Phi : L^p(\Omega;\mathbb{R}^N)\to[0,+\infty]$ which is

- sequentially lower semicontinuous for the weak convergence of $L^p(\Omega;\mathbb{R}^N)$,
- local on $L^p(\Omega;\mathbb{R}^N)$ in the sense that $\Phi(v+w)=\Phi(v)+\Phi(w)$ whenever $v\cdot w\equiv 0$ in Ω,

has to be necessarily convex, and of the form

$$\Phi(v) = \int_\Omega \phi\big(x,v(x)\big)\,dx$$

for a suitable integrand ϕ such that $\phi(x,\cdot)$ is convex. Then the energies $F(u)$ defined on Sobolev spaces and obtained by a functional $\Phi(v)$ through the identification $v = Du$ are necessarily convex. This is no longer true if Φ is defined on the space $\mathcal{M}(\Omega;\mathbb{R}^N)$ of measures, and hence F is defined on $BV(\Omega)$. The first example of a nonconvex functional Φ on $\mathcal{M}(\Omega;\mathbb{R}^N)$ in the literature comes from the so called Mumford-Shah model for computer vision (see below) and is given by

$$\Phi(\lambda) = \int_\Omega |\lambda^a(x)|^2\,dx + \#(A_\lambda)$$

where λ^a is the absolutely continuous part of λ, A_λ is the set of atoms of λ, and $\#$ is the counting measure. The functional Φ is set equal to $+\infty$ on all measures λ whose singular part λ^s is nonatomic. A general representation result (see [3] and references therein) establishes that a functional $\Phi : \mathcal{M}(\Omega;\mathbb{R}^N)\to[0,+\infty]$ which is

- sequentially lower semicontinuous for the weak* convergence of $\mathcal{M}(\Omega;\mathbb{R}^N)$,
- local on $\mathcal{M}(\Omega;\mathbb{R}^N)$ in the sense that $\Phi(\lambda+\nu)=\Phi(\lambda)+\Phi(\nu)$ whenever λ and ν are mutually singular in Ω,

has to be of the form

$$\Phi(\lambda)=\int_\Omega \phi(x,\lambda^a)\,d\mu+\int_\Omega \phi^\infty(x,\lambda^c)+\int_\Omega \psi\big(x,\lambda^\#(x)\big)\,d\#$$

where μ is a nonnegative measure, $\lambda=\lambda^a\cdot dx+\lambda^c+\lambda^\#$ is the decomposition of λ into absolutely continuous, Cantor and atomic parts, $\phi(x,v)$ is an integrand convex in v and ϕ^∞ is its recession function. The novelty is now represented by the integrand $\psi(x,v)$ which has to be *subadditive* in v and satisfying the compatibility condition

$$\lim_{t\to+\infty}\frac{\phi(x,tv)}{t}=\lim_{t\to 0^+}\frac{\psi(x,tv)}{t}\ .$$

When ϕ has a superlinear growth the condition above gives that the slope of $\psi(x,\cdot)$ at the origin has to be infinite. For instance, in the Mumford-Shah case we have

$$\phi(x,v)=|v|^2 \qquad \psi(x,v)=\begin{cases}1 & \text{if } v\neq 0\\ 0 & \text{if } v=0.\end{cases} \tag{6.1}$$

Coming back to the case $u\in BV(\Omega)$, we have the decomposition (4.1)

$$Du=\nabla u\cdot\mathcal{L}^N+(Du)^c+[u]\nu_u(x)\cdot\mathcal{H}^{N-1}\llcorner S_u$$

where we considered for simplicity only the scalar case $m=1$ and denoted by $[u]$ the jump u^+-u^-. We have then the functional

$$F(u)=\int_\Omega \phi\big(x,\nabla u\big)\,dx+\int_\Omega \phi^\infty\big(x,(Du)^c\big)+\int_{S_u}\psi\big(x,[u]\nu_u\big)\,d\mathcal{H}^{N-1}.$$

For instance, in the homogeneous-isotropic case, when $\phi(x,v)$ and $\psi(x,v)$ are independent of x and only depend on $|v|$, the formula above reduces to

$$F(u)=\int_\Omega \phi\big(|\nabla u|\big)\,dx+\beta|Du|^c(\Omega)+\int_{S_u}\psi\big(|[u]|\big)\,d\mathcal{H}^{N-1}, \tag{6.2}$$

where β,ϕ,ψ satisfy the compatibility condition

$$\beta=\phi^\infty(1)=\lim_{t\to 0^+}\frac{\psi(t)}{t}\ . \tag{6.3}$$

In the original Mumford-Shah model for computer vision, Ω is a rectangle of the plane, $u_0:\Omega\to[0,1]$ represents the grey level of a picture, c_1 and c_2 are positive scale and contrast parameters, and the variational problem under consideration is

$$\min\Big\{\int_\Omega |\nabla u|^2\,dx+c_1\int_\Omega |u-u_0|^2\,dx+c_2\mathcal{H}^{N-1}(S_u)\ :\ (Du)^c\equiv 0\Big\}. \tag{6.4}$$

The solution u then represents the reconstructed image, whose contours are given by the jump set S_u. We refer to [7] and to the book [15] for further details about this model.

Analogously, in the case of the study of fractures of an elastic membrane, a problem similar to (6.4) provides the vertical displacement u of the membrane, together with its fracture set S_u. We refer to some recent papers (see [6], [12] and references therein) for a more detailed description of fracture mechanics problems, even in the more delicate vectorial setting of elasticity.

Using the functional F in (6.2) we have the generalized Mumford-Shah problem

$$\min\Big\{F(u)+c_1\int_\Omega|u-u_0|^2\,dx\ :\ u\in BV(\Omega)\Big\}$$

where ϕ is convex, ψ is subadditive, and the compatibility condition (6.3) is fulfilled.

If we set $K=S_u$ and assume it is closed, the Mumford-Shah problem can be rewritten as

$$\min\Big\{\int_{\Omega\setminus K}|\nabla u|^2\,dx+c_1\int_{\Omega\setminus K}|u-u_0|^2\,dx+c_2\mathcal{H}^{N-1}(K\cap\Omega)\ :\ K\subset\overline{\Omega}\text{ closed},\ u\in H^1(\Omega\setminus K)\Big\}.$$

and this justifies the name of *"free discontinuity problems"* that is often used in this setting.

The regularity properties of optimal pairs (u,K) are far from being fully understood; some partial results are available but the Mumford-Shah conjecture:

- *in the case $N=2$ for an optimal pair (u,K) the set K is locally the finite union of $C^{1,1}$ arcs*

remains still open. We refer to [2] for a list of the regularity results on the problem above that are known until now.

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