Manuscript submitted to
AIMS' Journals
Volume **00**, Number **0**, Xxxx **XXXX**

Website: http://AIMsciences.org

pp. **000–000**

QUASISTATIC EVOLUTION FOR PLASTICITY WITH SOFTENING: THE SPATIALLY HOMOGENEOUS CASE

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Abstract. The spatially uniform case of the problem of quasistatic evolution in small strain associative elastoplasticity with softening is studied. Through the introdution of a viscous approximation, the problem reduces to determine the limit behaviour of the solutions of a singularly perturbed system of ODE's in a finite dimensional Banach space. We see that the limit dynamics presents, for a generic choice of the initial data, the alternation of three possible regimes (elastic regime, slow dynamics, fast dynamics), which is determined by the sign of two scalar indicators, whose explicit expression is given.

1. **Introduction.** In plasticity theory the term softening refers to the reduction of the yield stress as plastic deformation proceeds. Classically this is described by a family of yield surfaces depending on a parameter ζ . The evolution laws are formulated in such a way that the yield surface shrinks when the time derivative of the plastic deformation is not zero.

In [1], this problem is investigated in the *quasistatic* case, in the framework of *small strain associative elastoplasticity* in a bounded and Lipschitz domain $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$. In this paper, we restrict our attention to the spatially homogeneous case in dimension N, with no volume forces and *prescribed boundary displacements* on the whole boundary of Ω. The system is driven by a time-dependent affine boundary condition $w(t,x)$, whose symmetrized spatial gradient $Ew(t,x)$ is independent of the space variable x and is denoted by $\xi(t)$. In this situation, the displacement $u(t,x)$ coincides with $w(t,x)$ and the unknowns are the elastic part $e(t)$ and the plastic part $p(t)$ appearing in the additive decomposition of the strain $Eu(t,x) = e(t) + p(t)$, as well as a scalar internal variable $z(t)$, which describes the time evolving yield surface. The stress $\sigma(t)$ is determined by the elastic part of the strain through the usual relation $\sigma(t) = \mathbb{C}e(t)$, where $\mathbb{C}$ is the tensor of elastic moduli. We introduce a parameter ζ, which is related to the internal variable z by the equation $\zeta = -V'(z)$, where $V : \mathbb{R} \to \mathbb{R}$ is a *concave* function of class C^3, called the softening potential; to simplify the mathematics of the problem, we will assume that the image of $-V'$ has a strictly positive distance from 0. For every value of the parameter ζ , the elastic domain – the set of admissible stresses enclosed by the yield surface – has the form $\{\sigma \in \mathbb{M}^{N\times N}_{sym} : \sigma_D \in K(\zeta)\}$ where $\mathbb{M}^{N\times N}_{sym}$ is the space of symmetric $N \times N$ matrices, σ_D denotes the deviatoric part of σ, and $K(\zeta)$ is a subset of the subspace $\mathbb{M}^{N\times N}_D$ of trace-free symmetric matrices.

We consider a slight variant of the model studied in [1], assuming that the set

$$K := \{(\sigma,\zeta) \in \mathbb{M}^{N\times N}_D \times [0,+\infty) : \sigma \in K(\zeta))\}$$

is a closed convex cone of the form

$$K := \{(\sigma,\zeta) \in \mathbb{M}^{N\times N}_D \times [0,+\infty) : \sigma \in \zeta K(1)\} \tag{1.1}$$

2000 *Mathematics Subject Classification.* 74C05 (34D15).

Key words and phrases. quasistatic evolution, rate independent processes, plasticity with softening.

where $K(1)$ is a compact convex set with C^2 boundary in $\mathbb{M}^{N\times N}_D$ containing 0 as an interior point, so that $\{0\}\times\mathbb{R}_+\subset K$.

The other ingredients of the model are the evolution laws for $p(t)$ and $z(t)$, resulting in the system

$$\begin{cases} e(t)+p(t)=\xi(t)\,, \\ \sigma(t)=\mathbb{C}e(t)\in K(\zeta(t))\,, \qquad \zeta(t)=-V'(z(t))\,, \\ (\dot p(t),\dot z(t))\in N_K(\sigma(t)_D,\zeta(t))\,, \end{cases} \tag{1.2}$$

where $N_K(\sigma_D,\zeta)$ denotes the normal cone to K at (σ_D,ζ). In view of the hypotheses on K, we have the monotonicity condition $\zeta_1<\zeta_2\Rightarrow K(\zeta_1)\subset K(\zeta_2)$. Moreover, we have that $\dot z(t)\le 0$; as $-V'$ is increasing, whenever $\dot z(t)\neq 0$, the monotonicity condition implies that the set $K(\zeta(t))$ shrinks leading to a softening response. Our hypotheses on K also imply that the internal variable may evolve only if the plastic part does.

To deal with the instabilities of the softening regime (see [1] for more details), it is useful to introduce a viscosity approximation to (1.2). Denoting the minimal distance projection of (σ_D,ζ) onto K by $\pi_K(\sigma_D,\zeta)$, for every $\varepsilon>0$ we consider the unconstrained system

$$\begin{cases} e_\varepsilon(t)+p_\varepsilon(t)=\xi(t)\,, \\ \sigma_\varepsilon(t)=\mathbb{C}e_\varepsilon(t)\,, \qquad \zeta_\varepsilon(t)=-V'(z_\varepsilon(t))\,, \\ (\dot p_\varepsilon(t),\dot z_\varepsilon(t))=N^\varepsilon_K(\sigma_\varepsilon(t)_D,\zeta_\varepsilon(t))\,, \end{cases} \tag{1.3}$$

where $N^\varepsilon_K(\sigma_D,\zeta):=\frac{1}{\varepsilon}((\sigma,\zeta)-\pi_K(\sigma,\zeta))$ is the usual approximation of the normal to K. A viscosity solution $(e(t),p(t),\sigma(t),z(t))$ to (1.2) is defined as a left continuous map which, for almost every time t, is the pointwise limit of a sequence $(e_\varepsilon(t),p_\varepsilon(t),\sigma_\varepsilon(t),z_\varepsilon(t))$ of solutions of (1.3). As the constraint acts only on the deviatoric part of the stress and $\mathbb{C}$ is assumed to map the line through the identity matrix into itself, as well as $\mathbb{M}^{N\times N}_D$ into $\mathbb{M}^{N\times N}_D$, only the projection of (1.3) on the finite dimensional Banach space $\mathbb{M}^{N\times N}_D\times\mathbb{R}$ will be considered; indeed, for every $\varepsilon>0$ we trivially have that, due to the hypotheses on $\mathbb{C}$

$$\begin{aligned} e_\varepsilon(t)&=(e_\varepsilon(t))_D+\tfrac{1}{N}[\mathrm{tr}(\xi(t)+e_\varepsilon(0)-\xi(0))]\,,\\ \sigma_\varepsilon(t)&=(\sigma_\varepsilon(t))_D+\tfrac{\kappa}{N}[\mathrm{tr}(\xi(t)+e_\varepsilon(0)-\xi(0))]\,, \end{aligned}$$

where $\kappa>0$ is the *modulus of compression* ; as the equation for the internal variable only depends on the deviatoric part of the stress, we only have to solve the projection of (1.3) onto $\mathbb{M}^{N\times N}_D\times\mathbb{R}$. Therefore, only this part of the system will be considered and the subscript D will be from now on omitted.

In this paper we study in detail the limit behavior as ε goes to 0 of the solutions of (1.3). We will see that the limit dynamics presents, for a generic choice of the initial data – some degenerate cases have indeed to be excluded – the alternation of three possible regimes:

a) **Elastic regime** . This situation occurs when in a time interval $[t_1,t_2]$, the plastic part, and thus the internal variable, do not evolve, while the stress is completely determined by the prescribed boundary displacement through the relation $\sigma(t)=\mathbb{C}(\xi(t)-\xi(t_1))$, for every $t\in[t_1,t_2]$; a necessary condition for this behavior to occur is clearly $(\mathbb{C}(\xi(t)-\xi(t_1)),\zeta(t_1))\in K$ for every $t\in[t_1,t_2]$.
b) **Slow dynamics** . In this situation the solution exhibits a plastic behavior and softening occurs, without singularities; the evolution can be studied using the standard time t, and the limit system in this case is given by (3.18). It is a differential system on ∂K which is called the system of the slow dynamics.
c) **Fast dynamics** . This is the situation where, in the softening regime , singular behavior occurs; this requires the use of a *fast time* $s:=\frac{1}{\varepsilon}t$. The corresponding limit system (4.1) is called the system of the fast dynamics. We will see that, at a jump time t_1, the right limit $\sigma(t_1+),\zeta(t_1+)$ of the solution is given by the asymptotic value

for $s \to +\infty$ of the solution of the system of the fast dynamics (4.1) issuing from the point $\sigma(t_1-), \zeta(t_1-)$ at $s = -\infty$

The alternation of these three regimes is determined by the sign of two scalar indicators; the first one, depending explicitly on time and on the state of the system, will be called the *elastic-inelastic indicator*. Its explicit expression is given by

$$\Phi(t, \sigma, \zeta) := n_\sigma(\sigma, \zeta) \cdot \mathbb{C}\dot{\xi}(t) \qquad \text{for every } (t, \sigma, \zeta) \in [0, +\infty] \times \partial K$$

where n_σ denotes the σ-component of the outward unit normal to K at (σ, ζ). The second one, only depending on the state of the system, will be called the *slow-fast indicator*; its explicit expression will be given by

$$\Psi(\sigma, \zeta) := [\frac{1}{g(\zeta)} n_\zeta^2(\sigma, \zeta) - n_\sigma(\sigma, \zeta) \cdot \mathbb{C} n_\sigma(\sigma, \zeta)] \qquad \text{for every } (\sigma, \zeta) \in \partial K,$$

where n_ζ denotes the ζ-component of the outward unit normal to K at (σ, ζ), and $g(\zeta)$ is the first derivative of the inverse function of $-V'$. For mathematical reasons, both the indicators will be suitably extended to the whole space, but they need to be evaluated only on the yield surface.

We now briefly describe how the two indicators determine the limit dynamics. We take an initial condition $(\sigma_0, \zeta_0) \in \overset{\circ}{K}$; then initially the solution is following the elastic regime, till it reaches the yield surface, at a time t_1, at a certain point (σ_1, ζ_1). Here the elastic-inelastic indicator must be nonnegative. In a generic situation it will be strictly positive, and this determines the appearance of a plastic behavior after the time t_1. The choice between the slow and the fast dynamics depends on the sign of the slow-fast indicator.

a) If $\Psi(\sigma_1, \zeta_1) < 0$ the solution has no singularity and is obtained by solving the system of the slow dynamics

$$\begin{cases} \mathbb{C}\dot{\xi}(t) - \dot{\sigma}_{sl}(t) = -\frac{\Phi(t, \sigma_{sl}(t), \zeta_{sl}(t))}{\Psi(\sigma_{sl}(t), \zeta_{sl}(t))}\, \mathbb{C}\, n_\sigma(\sigma_{sl}(t), \zeta_{sl}(t)), \\ g(\zeta_{sl}(t))\dot{\zeta}_{sl}(t) = -\frac{\Phi(t, \sigma_{sl}(t), \zeta_{sl}(t))}{\Psi(\sigma_{sl}(t), \zeta_{sl}(t))}\, n_\zeta(\sigma_{sl}(t), \zeta_{sl}(t)), \end{cases} \tag{1.4}$$

defined on ∂K, with Cauchy data (σ_1, ζ_1) at time t_1; this situation is studied in Section 3. This behavior persists as long as one of the two indicators does not vanish along the motion.

If at a time $\bar{t}$, we have that $\Phi(\bar{t}, \sigma_{sl}(\bar{t}), \zeta_{sl}(\bar{t}) = 0$ while Ψ remains strictly negative, elastic behavior may reappear, starting from the point $(\sigma_{sl}(\bar{t}), \zeta_{sl}(\bar{t}))$, in presence of some suitable higher order conditions, implying a change of sign of Φ along the motion; this situation is studied in Section 3.1.

If Φ remains stricly positive, the solution follows the system of the slow dynamics for all its maximal interval of existence, that is to say as long as Ψ does not vanish; if this happens in finite time, in presence of some suitable higher order conditions, implying a change of sign of Ψ along the motion, a transition from the slow to the fast dynamics occurs, and the solution jumps along the trajectory of the fast dynamics. This situation is studied in Section 5.

b) If $\Psi(\sigma_1, \zeta_1) > 0$ the solution is singular at time t_1 and jumps to the asymptotic value as $s \to +\infty$ of the solution of the system

$$\begin{cases} -\dot{\sigma}(s) = \mathbb{C}[\sigma(s) - \pi_\sigma(\sigma(s), \zeta(s))], \\ g(\zeta)\dot{\zeta}(s) = \zeta(s) - \pi_\zeta(\sigma(s), \zeta(s)), \\ \lim\limits_{s \to -\infty} (\sigma(s), \zeta(s)) = (\sigma_1, \zeta_1), \end{cases} \tag{1.5}$$

which is formally obtained by rescaling time in (1.3) according to $s = \frac{t}{\varepsilon}$, and neglecting the small perturbation $\varepsilon \chi_\varepsilon(s)$, where $\chi_\varepsilon(s) := \dot{\xi}(a_\varepsilon^1 + \varepsilon s)$ and a_ε^1 is a suitable sequence converging to t_1. This situation is studied in Section 4. In a generic situation, at the end of the jump the slow-fast indicator has negative sign, thus in a right neighborhood of t_1 the behavior of the system can be elastic or follow the slow dynamics equation, depending on the sign of the elastic-inelastic indicator.

By iterating these arguments at each critical time, we can completely describe the solution, except for some degenerate cases. A simple explicit example is studied in section 6.

Extensions to nonassociative elastoplasticity in the spatially uniform case, and extensions to non spatially uniform solutions will be considered in other forthcoming papers (see for instance [2], for examples of spatialy homogeneous solutions in the context of Cam-Clay plasticity).

2. **Notation and preliminaries.**

In this paper, we study spatially homogeneous solutions of a variant of the quasistatic evolution model for plasticity with softening, whose well-posedness has been investigated in a more general setting in [1]; we refer to that for the details of the model, and we limit ourselves only to recall the main points about the stress constraint and the softening potential. We shall suppose that the stress constraint is given by the closed convex cone:

$$K := \{(\sigma, \zeta) \in \mathbb{M}^{N\times N}_D \times [0, +\infty) : \sigma \in \zeta K(1)\} \tag{2.1}$$

where $K(1)$ is a compact convex set with C^2 boundary in $\mathbb{M}^{N\times N}_D$ containing 0 as an interior point, so that $\{0\} \times \mathbb{R}_+ \subset K$. The minimal distance projection onto the convex set K is denoted by $\pi := (\pi_\sigma, \pi_\zeta)$, while the outward unit normal to the set K is denoted by $n : (n_\sigma, n_\zeta)$. In our hypothesis it is easily seen that there exists two constants $0 < c_1 < c_2 < 1$ such that:

$$-1 < -c_2 \le n_\zeta(\sigma, \zeta) \le -c_1 < 0 \qquad \text{for every } (\sigma, \zeta) \in \partial K \setminus \{(0, 0)\}. \tag{2.2}$$

It follows that

$$-1 < -c_2 \le n_\zeta(\pi(\sigma, \zeta)) \le -c_1 < 0 \qquad \text{for every } (\sigma, \zeta) \in (\mathbb{M}^{N\times N}_D \times (0, +\infty)) \setminus \mathring{K}; \tag{2.3}$$

it is easy to deduce that

$$\pi_\zeta(\sigma, \zeta) \ge \zeta \qquad \text{for every } (\sigma, \zeta) \in \mathbb{M}^{N\times N}_D \times \mathbb{R} \tag{2.4}$$

The softening potential will be a function $V : \mathbb{R} \to \mathbb{R}$ of class C^3 with the following two properties:

$$V''(z) < 0 \qquad \text{for every } z \in \mathbb{R} \tag{2.5}$$

$$-V'(-\infty) = \alpha > 0 \tag{2.6}$$

where $V'(-\infty)$ denotes the limit of V' at $-\infty$; we shall also suppose for simplicity that also V'' has a limit at $-\infty$, which will obviously be 0 by (2.6). As shown in [1], the vanishing viscosity method for quasistatic evolution in plasticity with softening leads, in the spatially homogeneous case, to studying the asymptotic behavior, as $\varepsilon \to 0^+$ of the following singularly perturbed system of ODE's in the finite dimensional Banach space $\mathbb{M}^{N\times N}_D \times \mathbb{R}$:

$$\begin{cases} \dot\xi(t) - \dot e_\varepsilon(t) = \frac{1}{\varepsilon}[\sigma_\varepsilon(t) - \pi_\sigma(\sigma_\varepsilon(t), \zeta_\varepsilon(t))], \\ \dot z_\varepsilon(t) = \frac{1}{\varepsilon}[\zeta_\varepsilon(t) - \pi_\zeta(\sigma_\varepsilon(t), \zeta_\varepsilon(t))]. \end{cases} \tag{2.7}$$

Here and henceforth $\dot\xi(t)$ is the deviatoric part of the strain of the prescribed boundary condition, $e_\varepsilon(t)$ is the deviatoric part of the elastic strain of the solution, $\sigma_\varepsilon(t)$ is the deviatoric part of the stress, $z_\varepsilon(t)$ is an internal variable, and $\zeta_\varepsilon(t)$ is the dual internal variable. The constitutive relations between $\sigma_\varepsilon(t)$ and $e_\varepsilon(t)$ is given by $\sigma_\varepsilon(t) := \mathbb{C}e_\varepsilon(t)$, where $\mathbb{C} : \mathbb{M}^{N\times N}_{sym} \to \mathbb{M}^{N\times N}_{sym}$ is the elasticity tensor, while $z_\varepsilon(t)$ and $\zeta_\varepsilon(t)$ are related by $\zeta_\varepsilon(t) := -V'(z_\varepsilon(t))$ where V is the softening potential. We assume that $\mathbb{C}$ is positive definite and maps the line through the identity matrix into itself, as well as $\mathbb{M}^{N\times N}_D$ into $\mathbb{M}^{N\times N}_D$.

We suppose for simplicity that $\xi \in C^1([0,+\infty); \mathbb{M}^{N\times N}_D \times \mathbb{R})$. Using the constitutive relations, system (2.7) can be regarded as a first order system in normal form in the unknowns $e_\varepsilon(t)$ and $\zeta_\varepsilon(t)$. Local existence and uniqueness are then trivial, while global existence can be proved as in [1], Proposition 4.5.

In the dual variables $\sigma_\varepsilon(t)$ and $\zeta_\varepsilon(t)$ the system becomes

$$\begin{cases} \mathbb{C}\dot{\xi}(t) - \dot{\sigma}_\varepsilon(t) = \frac{1}{\varepsilon}\mathbb{C}[\sigma_\varepsilon(t) - \pi_\sigma(\sigma_\varepsilon(t), \zeta_\varepsilon(t))], \\ g(\zeta_\varepsilon(t))\dot{\zeta}_\varepsilon(t) = \frac{1}{\varepsilon}[\zeta_\varepsilon(t) - \pi_\zeta(\sigma_\varepsilon(t), \zeta_\varepsilon(t))]. \end{cases} \tag{2.8}$$

It obviously inherits the properties of global existence and uniqueness of the previous one. Here $g(\zeta)$ is the first derivative of the inverse function of $-V'$: in our hypotheses it is easily seen that g maps $(\alpha, -V'(+\infty))$ into $(0,+\infty)$ (here α is the positive constant given by (2.6), and $V'(+\infty)$ denotes the limit of V' at $+\infty$), and that

$$\lim_{\zeta\to\alpha} g(\zeta) = +\infty. \tag{2.9}$$

Moreover a simple change of variables yields that for every $\beta > \alpha$

$$\lim_{\varepsilon\to 0^+} \int_{\alpha+\varepsilon}^{\beta} g(\zeta)\, d\zeta = +\infty. \tag{2.10}$$

Since we want to consider a system which is initially in the elastic regime, we suppose that the initial conditions satisfy

$$(\sigma_0, \zeta_0) \in \mathring{K};$$

from (2.9) it follows that, for every $\varepsilon > 0$ the solution of (2.8) with the prescribed initial data satisfies

$$\zeta_\varepsilon(t) > \alpha \qquad \text{for every } t \in (0,+\infty).$$

Moreover (2.4) yields that the solution of (2.8) always satisfies

$$\zeta_\varepsilon(t) \le \zeta_\varepsilon(s) \le \zeta_0 \qquad \text{for every } 0 \le s \le t; \tag{2.11}$$

so, taking into account (2.9), we have that there exists $c_0 > 0$, depending only on g and ζ_0 such that, for every $\varepsilon > 0$:

$$g(\zeta_\varepsilon(t)) \ge \tfrac{1}{c_0} \qquad \text{for every } \varepsilon > 0 \text{ and every } t \in [0,+\infty). \tag{2.12}$$

We introduce the distance function from the convex set K

$$\varrho(\sigma,\zeta) := |(\sigma,\zeta) - \pi(\sigma,\zeta)|; \tag{2.13}$$

for every t such that $\varrho(\sigma_\varepsilon(t), \zeta_\varepsilon(t)) > 0$ equations (2.8) become

$$\begin{cases} \mathbb{C}\dot{\xi}(t) - \dot{\sigma}_\varepsilon(t) = \frac{1}{\varepsilon}\varrho(\sigma_\varepsilon(t), \zeta_\varepsilon(t))\, \mathbb{C}\, n_\sigma(\sigma_\varepsilon(t), \zeta_\varepsilon(t)), \\ g(\zeta_\varepsilon(t))\dot{\zeta}_\varepsilon(t) = \frac{1}{\varepsilon}\varrho(\sigma_\varepsilon(t), \zeta_\varepsilon(t))\, n_\zeta(\sigma_\varepsilon(t), \zeta_\varepsilon(t)). \end{cases} \tag{2.14}$$

Given the solution of (2.8) with the prescribed initial data we define

$$\varrho_\varepsilon(t) := \varrho(\sigma_\varepsilon(t), \zeta_\varepsilon(t)); \tag{2.15}$$

notice that $\varrho_\varepsilon(t)$ is Lipschitz continuous thus differentiable for almost every t; in particular it is differentiable for every t such that $\varrho_\varepsilon(t) > 0$ and we have

$$\frac{d}{dt}\varrho_\varepsilon(t) = n_\sigma(\pi(\sigma_\varepsilon(t), \zeta_e(t))) \cdot \dot{\sigma}_\varepsilon(t) + n_\zeta(\pi(\sigma_\varepsilon(t), \zeta_e(t)))\dot{\zeta}_\varepsilon(t)) \tag{2.16}$$

$$= -n_\sigma(\pi(\sigma_\varepsilon(t), \zeta_e(t))) \cdot (\mathbb{C}\dot{\xi}(t) - \dot{\sigma}_\varepsilon(t)) + \tag{2.17}$$

$$+ n_\zeta(\pi(\sigma_\varepsilon(t), \zeta_\varepsilon(t)))\dot{\zeta}_\varepsilon(t) + n_\sigma(\pi(\sigma_\varepsilon(t), \zeta_\varepsilon(t))) \cdot \mathbb{C}\dot{\xi}(t) \tag{2.18}$$

and so, by (2.14)

$$\frac{d}{dt}\varrho_\varepsilon(t) = \Phi(t, \sigma_\varepsilon(t), \zeta_\varepsilon(t)) + \frac{\varrho_\varepsilon(t)}{\varepsilon}\Psi(\sigma_\varepsilon(t), \zeta_\varepsilon(t)) \quad \text{whenever } \varrho_\varepsilon(t) > 0, \tag{2.19}$$

where

$$\Phi(t,\sigma,\zeta) := n_\sigma(\pi(\sigma,\zeta)) \cdot \mathbb{C}\dot{\xi}(t), \tag{2.20}$$

$$\Psi(\sigma,\zeta) := [\tfrac{1}{g(\zeta)} n_\zeta^2(\pi(\sigma,\zeta)) - n_\sigma(\pi(\sigma,\zeta)) \cdot \mathbb{C} n_\sigma(\pi(\sigma,\zeta))]. \tag{2.21}$$

The function Φ is defined on $[0,+\infty)\times[(\mathbb{M}^{N\times N}_D \times (\alpha,+\infty))\setminus \overset{\circ}{K})]$ and is continuous, while Ψ is defined on $(\mathbb{M}^{N\times N}_D \times(\alpha,+\infty))\setminus\overset{\circ}{K}$ and is of class C^1. In what follows, it is often convenient to consider extensions of Φ and Ψ to $[0,+\infty)\times\mathbb{M}^{N\times N}_D \times (\alpha,+\infty)$ and $\mathbb{M}^{N\times N}_D \times(\alpha,+\infty)$ of class C^0 and C^1, respectively. Notice that the partial derivatives of Ψ at each point of ∂K do not depend on the extension.

Recalling the assumptions on $\mathbb{C}$, with the use of (2.3) and (2.9), we may assume that there exists $\tilde{\alpha} > \alpha$ and a positive constant $\theta > 0$ such that

$$\Psi(\sigma,\zeta) \le -\theta \qquad \text{for every } (\sigma,\zeta) \in \mathbb{M}^{N\times N}_D \times(\alpha,\tilde{\alpha}). \tag{2.22}$$

We may assume also that there exists a positive constant λ such that:

$$\Psi(\sigma,\zeta) \ge -\lambda \qquad \text{for every } (\sigma,\zeta) \in \mathbb{M}^{N\times N}_D \times(\alpha,+\infty). \tag{2.23}$$

We will see in the next sections that the sign of Φ governs the transition from elastic to inelastic regime at times when the stress meets the yield surface, while in case of inelastic regime the sign of Ψ determines whether the quasistatic evolution follows the equation of the slow dynamics (softening without discontinuities) or jumps along the trajectory of the fast dynamics. For these reasons, Φ will be called *elastic-inelastic indicator*, while Ψ will be called *slow-fast indicator*.

3. **From the elastic regime to the slow dynamics.**

We start to study the asympotic behavior, as $\varepsilon \to 0^+$ of the solutions of (2.8). As we have supposed that the initial conditions are taken in such a way that

$$(\sigma_0,\zeta_0) \in \overset{\circ}{K},$$

the solutions at small times are trivially given for every ε, by

$$(\sigma(t),\zeta(t)) = (\sigma_0 + \mathbb{C}(\xi(t) - \xi(0)), \zeta_0); \tag{3.1}$$

this formula gives us the solution in the time interval $[0,t_1]$ where we put:

$$t_1 = \inf\{t > 0 : \ (\sigma(t),\zeta_0) \in \partial K\}. \tag{3.2}$$

We now consider the case where $t_1 < +\infty$, and we set

$$(\sigma_1,\zeta_1) := (\sigma_0 + \mathbb{C}(\xi(t_1) - \xi(0)), \zeta_0). \tag{3.3}$$

The following lemma gives the first elementary consequence of the positivity of the elastic-inelastic indicator Φ introduced in (2.20).

Lemma 3.1. *Let t_1, σ_1, ζ_1 be as in (3.2), (3.3), and suppose $t_1 < +\infty$ and suppose that $\Phi(t_1,\sigma_1,\zeta_1) > 0$. Let $\varepsilon > 0$, let $\varrho_\varepsilon(t)$ be as in (2.15) . Then, for every $t^* > t_1$ the set*

$$\{\varrho_\varepsilon(t) > 0\} \cap [t_1,t^*]$$

has strictly positive Lebesgue measure.

Proof. Otherwise, in the time interval $[t_1,t^*]$ the solution of (2.8) is given by the formula

$$(\sigma_\varepsilon(t),\zeta_\varepsilon(t)) = (\sigma_1 + \mathbb{C}(\xi(t) - \xi(0)), \zeta_1) \tag{3.4}$$

and $(\sigma_\varepsilon(t),\zeta_1) \in K$ for every t; as $(\sigma_1,\zeta_1) \in \partial K$, this implies $(\dot{\sigma}_\varepsilon(t_1), 0) \cdot n((\sigma_1,\zeta_1)) \le 0$. Using (3.4) $\mathbb{C}\dot{\xi}(t_1) \cdot n_\sigma(\sigma_1,\zeta_1) \le 0$; by the definition of Φ, we deduce $\Phi(t_1,\sigma_1,\zeta_1) \le 0$, which contradicts the hypothesis. ☐

Remark 3.2. Since the solutions, for $t < t_1$ are given by (3.4), we have $\Phi(t_1, \sigma_1, \zeta_1) \geq 0$: nothing can be said in general about the case $\Phi(t_1, \sigma_1, \zeta_1) = 0$ unless more hypotheses on the local behavior of the function Φ are at our disposal.

Supposing $\Phi(t_1, \sigma_1, \zeta_1) > 0$, we now may fix an open neighborhood $U_\delta := (t_1 - \delta, t_1 + \delta) \times B_\delta(\sigma_1, \zeta_1)$, where $B_\delta(\sigma_1, \zeta_1)$ denotes the open ball of radius $\delta > 0$ centered at (σ_1, ζ_1), in a way that there exists a positive constant $\gamma_2 > 0$ such that

$$\Phi(t, \sigma, \zeta) \geq \gamma_2 > 0 \qquad \text{for every } (t, \sigma, \zeta) \in U_\delta. \tag{3.5}$$

We define:

$$a_\varepsilon := \inf \{t \in (t_1, t_1 + \delta) : \ (\sigma_\varepsilon(t), \zeta_\varepsilon(t)) \in \partial B_\delta(\sigma_1, \zeta_1)\}. \tag{3.6}$$

The following lemma shows that, thanks to (3.5), the function $\frac{1}{\varepsilon}\varrho_\varepsilon(t)$ becomes greater than a fixed positive constant after a time t_ε converging to t_1 as $\varepsilon \to 0$, while the motion is still in $B_\delta(\sigma_1, \zeta_1)$; we shall see that this implies a transition to the inelastic regime.

Lemma 3.3. *Let t_1, σ_1, ζ_1, λ and Φ be as in (3.2), (3.3), (2.23), and (2.20), respectively. Suppose $t_1 < +\infty$ and $\Phi(t_1, \sigma_1, \zeta_1) > 0$, and let δ, a_ε, and γ_2, be as in (3.5) and (3.6). Let $\varepsilon > 0$ and $\varrho_\varepsilon(t)$ be as in (2.15). Define*

$$t_\varepsilon := \inf \left\{t \in (t_1, t_1 + \delta) : \ \tfrac{1}{\varepsilon}\varrho_\varepsilon(t) \geq \tfrac{\gamma_2}{2\lambda}\right\}. \tag{3.7}$$

Then:

a) $t_\varepsilon - t_1 \to 0$ *as* $\varepsilon \to 0^+$;
b) $t_1 < t_\varepsilon < a_\varepsilon$ *for* ε *sufficiently small;*
c) $\frac{1}{\varepsilon}\varrho_\varepsilon(t) \geq \frac{\gamma_2}{2\lambda}$ *for every* $t \in [t_\varepsilon, a_\varepsilon]$.

Proof. Let $s_\varepsilon := t_\varepsilon \wedge a_\varepsilon$. We first show that in (t_1, s_ε) one has

$$\varrho_\varepsilon(t) > 0. \tag{3.8}$$

Indeed $\dot\varrho_\varepsilon(t) = 0$ almost everywhere in the set $\{\varrho_\varepsilon(t) = 0\} \cap [t_1, s_\varepsilon]$, while in the set $\{\varrho_\varepsilon(t) > 0\} \cap [t_1, s_\varepsilon]$, one has by (2.19), (3.5), (2.23) and the hypothesis, that

$$\dot\varrho_\varepsilon(t) \geq \tfrac{\gamma_2}{2}.$$

Then by the fundamental theorem of calculus and by Lemma 3.1,we get

$$\varrho_\varepsilon(\tau) = \int_{\{\varrho_\varepsilon(t)>0\}\cap[t_1,\tau]} \dot\varrho_\varepsilon(t)\,dt \geq \tfrac{\gamma_2}{2}\mathcal{L}^1(\{\varrho_\varepsilon(t) > 0\} \cap [t_1, \tau]) > 0$$

for every $\tau \in [t_1, s_\varepsilon]$, which proves (3.8). Therefore $\{\varrho_\varepsilon(t) > 0\} \cap (t_1, s_\varepsilon] = (t_1, s_\varepsilon]$ so that the previous estimate and the definition of s_ε yield

$$\varepsilon\tfrac{\gamma_2}{2\lambda} \geq \varrho_\varepsilon(s_\varepsilon) \geq \tfrac{\gamma_2}{2}(s_\varepsilon - t_1), \tag{3.9}$$

which implies

$$s_\varepsilon - t_1 \to 0 \quad \text{as } \varepsilon \to 0^+. \tag{3.10}$$

Now suppose by contradiction that $s_\varepsilon = a_\varepsilon$ as $\varepsilon \to 0$ along a suitable sequence. Then $a_\varepsilon - t_1 \to 0$ as $\varepsilon \to 0^+$ and

$$\sup_{t\in[t_1,a_\varepsilon]} \tfrac{1}{\varepsilon}\varrho_\varepsilon(t) \leq \tfrac{\gamma_2}{2\lambda};$$

by the definition of a_ε, (2.12), and (2.14), this implies

$$
\begin{aligned}
\delta &= |(\sigma_\varepsilon(a_\varepsilon),\zeta_\varepsilon(a_\varepsilon))-(\sigma_1,\zeta_1)| \\
&\le |(\sigma_\varepsilon(a_\varepsilon)-\sigma_1,0)|+|(0,\zeta_\varepsilon(a_\varepsilon)-\zeta_1)| \\
&= |\int_{t_1}^{a_\varepsilon}\dot\sigma_\varepsilon(s)\,ds|+|\int_{t_1}^{a_\varepsilon}\dot\zeta_\varepsilon(s)\,ds| \\
&\le \int_{t_1}^{a_\varepsilon}|\mathbb{C}\dot\xi(s)-\dot\sigma_\varepsilon(s)|\,ds+|\mathbb{C}|\int_{t_1}^{a_\varepsilon}|\dot\xi(s)|\,ds+c_0\int_{t_1}^{a_\varepsilon}g(\zeta_\varepsilon(s))|\dot\zeta_\varepsilon(s)|\,ds \\
&\le (|\mathbb{C}|+c_0)(\sup_{t\in[t_1,a_\varepsilon]}\frac{1}{\varepsilon}\varrho_\varepsilon(t))(a_\varepsilon-t_1)+|\mathbb{C}|\int_{t_1}^{a_\varepsilon}|\dot\xi(s)|\,ds \\
&\le (|\mathbb{C}|+c_0)\tfrac{\gamma_2}{2\lambda}(a_\varepsilon-t_1)+|\mathbb{C}|\int_{t_1}^{a_\varepsilon}|\dot\xi(s)|\,ds,
\end{aligned}
\tag{3.11}
$$

a contradiction, since the right-hand side is infinitesimal. This proves part a) and part b) of the statement.

To prove part c), we observe that $\dot\varrho_\varepsilon(t_\varepsilon)\ge-\frac{\gamma_2}{2}+\gamma_2>0$, by (2.19) and (3.5). If c) is false, let t^1_ε be the first time in $(t_\varepsilon,a_\varepsilon)$ such that $\varrho_\varepsilon(t^1_\varepsilon)=\frac{\gamma_2}{2\lambda}$; then $\dot\varrho_\varepsilon(t^1_\varepsilon)\le 0$. On the other hand (2.19) and (3.5) yield $\dot\varrho_\varepsilon(t^1_\varepsilon)\ge-\frac{\gamma_2}{2}+\gamma_2>0$, a contradiction. ☐

We now focus on the case where the slow-fast indicator has negative sign at (σ_1,ζ_1). We will see that in this case the asymptotic behavior of the solutions of (2.14) is governed by the slow dynamics equation. As Theorem 3.8 will show, in a neighborhood of t_1 the function $\frac{1}{\varepsilon}\varrho_\varepsilon(t)$ remains uniformly bounded, so no discontinuities appear in the limit, while Lemma 3.3 assures that the limit equation is nontrivial, differently from the case of elastic regime. For a suitable choice of δ in the definition of the neighborhood U_δ satisfying (3.5), we may assume that there exists a positive constant γ_1 such that

$$\Psi(\sigma,\zeta)\le-\gamma_1 \qquad \text{for every } (\sigma,\zeta)\in B_\delta(\sigma_1,\zeta_1). \tag{3.12}$$

The proof of the main result of this section requires the use of the following general result about continuous dependence on a parameter, whose proof can be found in [4] (see also [3]); we also state and prove an elementary corollary which will be useful later.

Theorem 3.4. *Let f_ε and f_0 be Carathéodory functions defined on $[a,b]\times\mathbb{R}^m$ with values in $\mathbb{R}^m$, let t_ε, $t_0\in[a,b]$, and let x_ε, $x_0\in\mathbb{R}^m$. Assume that there exist two constants $L>0$ and $M>0$ such that*

$$|f_\varepsilon(t,x_2)-f_\varepsilon(t,x_1)|\le L\,|x_2-x_1|\,,$$
$$|f_\varepsilon(t,x)|\le M\,,$$

for every $\varepsilon>0$, every $t\in[a,b]$, and every x, x_1, $x_2\in\mathbb{R}^m$. Let $y_\varepsilon(t)$ and $y_0(t)$ be the solutions of the Cauchy problems

$$\begin{cases}\dot y_\varepsilon(t)=f_\varepsilon(t,y(t))\,,\\ y_\varepsilon(t_\varepsilon)=x_\varepsilon\,,\end{cases}\qquad\qquad \begin{cases}\dot y_0(t)=f_0(t,y(t))\,,\\ y_\varepsilon(t_0)=x_0\,.\end{cases}$$

If $t_\varepsilon\to t_0$, $x_\varepsilon\to x_0$, and for every $x\in\mathbb{R}^m$

$$\int_a^t f_\varepsilon(s,x)\,ds\to\int_a^t f(s,x)\,ds \qquad \textit{uniformly for } t\in[a,b]\,,$$

then $y_\varepsilon(t)\to y_0(t)$ uniformly for $t\in[a,b]$.

Corollary 3.5. *Let f_ε and f_0 be Carathéodory functions defined on $[a,b]\times\mathbb{R}^m$ with values in $\mathbb{R}^m$, let $t_\varepsilon \to a$, and let x_ε, $x_0 \in \mathbb{R}^m$. Assume that there exist two constants $L > 0$ and $M > 0$ such that*

$$|f_\varepsilon(t,x_2) - f_\varepsilon(t,x_1)| \le L\,|x_2 - x_1|\,,$$
$$|f_\varepsilon(t,x)| \le M\,,$$

for every $\varepsilon > 0$, every $t \in [t_\varepsilon, b]$, and every x, x_1, $x_2 \in \mathbb{R}^m$. Let $y_\varepsilon(t)$ and $y_0(t)$ be the solutions of the Cauchy problems

$$\begin{cases} \dot y_\varepsilon(t) = f_\varepsilon(t,y(t))\,, \\ y_\varepsilon(t_\varepsilon) = x_\varepsilon\,, \end{cases} \qquad \begin{cases} \dot y_0(t) = f_0(t,y(t))\,, \\ y_\varepsilon(t_0) = x_0\,. \end{cases}$$

If $x_\varepsilon \to x_0$, and for every $x \in \mathbb{R}^m$, and for every $\eta > 0$

$$\int_{a+\eta}^t f_\varepsilon(s,x)\,ds \to \int_{a+\eta}^t f(s,x)\,ds \qquad \textit{uniformly for } t \in [a+\eta, b]\,,$$

then $y_\varepsilon(t) \to y_0(t)$ uniformly on compact subintervals of $(a,b]$.

Proof. Define

$$g_\varepsilon(t,x) = \begin{cases} f_\varepsilon(t,x) \text{ if } t \ge t_\varepsilon \\ f_\varepsilon(t_\varepsilon,x) \text{ otherwise} \end{cases}$$

and let $z_\varepsilon(t)$ the solutions of the Cauchy problems

$$\begin{cases} \dot z_\varepsilon(t) = g_\varepsilon(t,z(t))\,, \\ z_\varepsilon(t_\varepsilon) = x_\varepsilon\,. \end{cases}$$

It is not difficult to see that previous theorem may be applied with $g_\varepsilon(t,x)$ in place of f_ε; then $z_\varepsilon(t) \to y_0(t)$ uniformly for $t \in [a,b]$; conclusion follows as, for every $\eta > 0$, when ε sufficiently small, $z_\varepsilon(t) = y_\varepsilon(t)$ in $[a+\eta, b]$ by the uniqueness of solutions to Cauchy problems. □

We also need the following auxiliary Lemma.

Lemma 3.6. *Let $\hat t > 0$, $(\hat\sigma, \hat\zeta) \in \partial K$, and $\hat t_\varepsilon$ a sequence such that*

$$\hat t_\varepsilon \to \hat t \textit{ as } \varepsilon \to 0^+;$$
$$(\sigma_\varepsilon(\hat t_\varepsilon), \zeta_\varepsilon(\hat t_\varepsilon)) \to (\hat\sigma, \hat\zeta) \textit{ as } \varepsilon \to 0^+.$$

Suppose that there exist two constants $\eta > 0, \gamma > 0$ such that, for every (σ,ζ) satisfying $|(\sigma,\zeta) - (\hat\sigma,\hat\zeta)| < \eta$, one has

$$\Psi(\sigma,\zeta) < -\gamma.$$

Let

$$b_\varepsilon^\eta := \inf\Big\{ t \in (\hat t_\varepsilon, \hat t + \eta) :\ (\sigma_\varepsilon(t), \zeta_\varepsilon(t)) \in \partial B_\eta(\hat\sigma,\hat\zeta) \Big\}.$$

Then there exist $L > 0$, $C(\eta,\gamma) > 0$ and a sequence $\hat s_\varepsilon$, which may be taken equal to $\hat t_\varepsilon$ whenever $\limsup\limits_{\varepsilon\to 0} \frac{\varrho_\varepsilon(\hat t_\varepsilon)}{\varepsilon} < +\infty$, such that

a) $\hat s_\varepsilon \to \hat t$ *as* $\varepsilon \to 0^+$*;*
b) $\liminf\limits_{\varepsilon\to 0} b_\varepsilon^\eta \ge \hat t + C(\eta,\gamma)$*;*
c) $(\sigma_\varepsilon(\hat s_\varepsilon), \zeta_\varepsilon(\hat s_\varepsilon)) \to (\hat\sigma,\hat\zeta)$ *as* $\varepsilon \to 0^+$*;*
d) $\frac{\varrho_\varepsilon(t)}{\varepsilon} \le \frac{L}{\gamma}$ *for every* $t \in [\hat s_\varepsilon, b_\varepsilon^\eta]$.

Proof. Choose L such that $|\mathbb{C}\dot\xi(t)| < L$ for every $t \in [\hat t-\eta, \hat t+\eta]$. Observe that, from (2.19) and the hypotheses, we get

$$\dot\varrho_\varepsilon(t) < -\gamma\tfrac{\varrho_\varepsilon(t)}{\varepsilon} + L \qquad \text{for a.e. } t \in [\hat t_\varepsilon, b^\eta_\varepsilon]; \tag{3.13}$$

indeed the inequality holds true also in the set $\{\varrho_\varepsilon(t) = 0\}$, as $\dot\varrho_\varepsilon(t) = 0$ almost everywhere in this set. Notice also that it is everywhere satisfied when $\varrho_\varepsilon(t) > 0$.

Let $M = \limsup_{\varepsilon\to 0} \frac{\varrho_\varepsilon(\hat t_\varepsilon)}{\varepsilon}$; we may assume, up to a subsequence, that this limsup is actually a limit. If $M < +\infty$, we may always assume, suitably enlarging the constant L, that $M < \frac{L}{\gamma}$. If $M = +\infty$, fix $\vartheta > 0$ and define $s^\eta_\varepsilon := \inf\{t \in [\hat t_\varepsilon, b^\eta_\varepsilon] | \frac{\varrho_\varepsilon(t)}{\varepsilon} \le \frac{L+\vartheta}{\gamma}\}$; then (3.13) yields

$$\dot\varrho_\varepsilon(t) < -\vartheta \qquad \text{for every } t \in [\hat t_\varepsilon, s^\eta_\varepsilon]; \tag{3.14}$$

integrating, we get

$$(s^\eta_\varepsilon - \hat t_\varepsilon)\vartheta < \varrho_\varepsilon(\hat t_\varepsilon) - \varrho_\varepsilon(s^\eta_\varepsilon). \tag{3.15}$$

As $\varrho_\varepsilon(\hat t_\varepsilon) \to 0$, we conclude that $s^\eta_\varepsilon \to \hat t$ as $\varepsilon \to 0^+$. From this fact and (3.15), we also get that $\lim_{\varepsilon\to 0} \varrho_\varepsilon(s^\eta_\varepsilon) = 0$, hence, integrating (3.13), we obtain

$$\lim_{\varepsilon\to 0} \int_{\hat t_\varepsilon}^{s^\eta_\varepsilon} \frac{\varrho_\varepsilon(s)}{\varepsilon}\, ds = 0. \tag{3.16}$$

We can then argue as in (3.11), and for every $t \in [\hat t_\varepsilon, s^\eta_\varepsilon]$ we have

$$|\sigma_\varepsilon(t) - \sigma_\varepsilon(\hat t_\varepsilon)| + |\zeta_\varepsilon(t) - \zeta_\varepsilon(\hat t_\varepsilon)| \le (|\mathbb{C}| + c_0) \int_{\hat t_\varepsilon}^{s^\eta_\varepsilon} \frac{\varrho_\varepsilon}{\varepsilon}(s)\, ds + |\mathbb{C}| \int_{\hat t_\varepsilon}^{s^\eta_\varepsilon} |\dot\xi(s)|\, ds,$$

where c_0 is the constant given by (2.12), and this gives, thanks to (3.16),

$$\lim_{\varepsilon\to 0} |\sigma_\varepsilon(t) - \sigma_\varepsilon(\hat t_\varepsilon)| + |\zeta_\varepsilon(t) - \zeta_\varepsilon(\hat t_\varepsilon)| = 0.$$

In particular we have $s^\eta_\varepsilon < b^\eta_\varepsilon$, when ε is sufficiently small.

So we put $\hat s_\varepsilon := s^\eta_\varepsilon$ when $M = +\infty$, while we put $\hat s_\varepsilon := \hat t_\varepsilon$ otherwise; up to redefining the constant L, we have, for every ε, $\frac{\varrho_\varepsilon(\hat s_\varepsilon)}{\varepsilon} \le \frac{L}{\gamma}$ and $\dot\varrho_\varepsilon(\hat s_\varepsilon) < 0$. Now, if d) is false, let s^1_ε be the first time in $(\hat s_\varepsilon, b^\eta_\varepsilon)$ such that $\varrho_\varepsilon(s^1_\varepsilon) = \frac{L}{\gamma}$; then $\dot\varrho_\varepsilon(t^1_\varepsilon) \ge 0$. On the other hand (3.13) yields $\dot\varrho_\varepsilon(t^1_\varepsilon) < -L + L = 0$, a contradiction.

It remains to prove only part b) of the statement. If we suppose $b^\eta_\varepsilon < \hat t + \eta$, otherwise the result is trivial, arguing as in (3.11) we have the estimate

$$\eta = |(\sigma_\varepsilon(b^\eta_\varepsilon) - \sigma_\varepsilon(\hat s_\varepsilon), \zeta_\varepsilon(t) - \zeta_\varepsilon(\hat s_\varepsilon)| \le (|\mathbb{C}| + c_0) \int_{\hat s_\varepsilon}^{b^\eta_\varepsilon} \frac{\varrho_\varepsilon}{\varepsilon}(s)\, ds + |\mathbb{C}| \int_{\hat s_\varepsilon}^{b^\eta_\varepsilon} |\dot\xi(s)|\, ds,$$

which implies, by part d) of the statement,

$$\eta < [(1+\gamma)|\mathbb{C}| + c_0]\tfrac{L}{\gamma}(b^\eta_\varepsilon - \hat s_\varepsilon); \tag{3.17}$$

since $\hat s_\varepsilon \to \hat t$ as $\varepsilon \to 0^+$, we obtain the conclusion with $C(\eta, \gamma) := \min\{\eta, \frac{\eta\gamma}{L[(1+\gamma)|\mathbb{C}|+c_0]}\}$. □

Consider now the differential equation on the open submanifold $K_0 = \partial K \cap \{\Psi(\sigma, \zeta) \ne 0\}$

$$\begin{cases} \mathbb{C}\dot\xi(t) - \dot\sigma_{sl}(t) = -\frac{\Phi(t,\sigma_{sl}(t),\zeta_{sl}(t))}{\Psi(\sigma_{sl}(t),\zeta_{sl}(t))}\, \mathbb{C}\, n_\sigma(\sigma_{sl}(t), \zeta_{sl}(t)), \\ g(\zeta_{sl}(t))\dot\zeta_{sl}(t) = -\frac{\Phi(t,\sigma_{sl}(t),\zeta_{sl}(t))}{\Psi(\sigma_{sl}(t),\zeta_{sl}(t))}\, n_\zeta(\sigma_{sl}(t), \zeta_{sl}(t)); \end{cases} \tag{3.18}$$

This will be called the equation of the slow dynamics: observe that this is a well-defined equation, since, for every $t \in [0, +\infty)$, the vector field

$$\chi_t(\sigma, \zeta) = (\mathbb{C}\dot\xi(t) + \tfrac{\Phi(t,\sigma,\zeta)}{\Psi(\sigma,\zeta)}\, \mathbb{C}\, n_\sigma(\sigma, \zeta), \tfrac{-\Phi(t,\sigma,\zeta)}{g(\zeta)\Psi(\sigma,\zeta)} n_\zeta(\sigma, \zeta))$$

is a tangent vector field to K_0, as a direct computation shows.

We may thus apply all standard results about local existence and uniqueness and the existence of a maximal interval where solutions to (3.18) are defined. So, let (t_1, t_2) the maximal interval of existence for the Cauchy problem associated to (3.18) with datum (σ_1,ζ_1); we easily have that

$$\limsup_{s\to t_2^-}\Psi(\sigma_{sl}(s),\zeta_{sl}(s))=0. \tag{3.19}$$

Indeed, as long as $\Psi(\sigma,\zeta)$ does not vanish along the solution, the right-hand side of (3.18) is locally bounded, and so is the solution.

Remark 3.7. Let $(\sigma(t),\zeta(t))$ a solution of (3.18) and define $e(t)$, $p(t)$, $z(t)$ through the constitutive relations in (1.2); we will have, by a direct computation, that $(\dot p(t),\dot z(t))=-\frac{\Phi(t,\sigma(t),\zeta(t))}{\Psi(\sigma(t),\zeta(t))}n(\sigma(t),\zeta(t))$, thus the flow rule in (1.2) is satisfied as long as $-\frac{\Phi(t,\sigma(t),\zeta(t))}{\Psi(\sigma(t),\zeta(t))}\geq 0$, that is as long as Φ does not become negative along the trajectory, as we are supposing that the slow-fast indicator Ψ is negative.

Viceversa, let $(\sigma(t),\zeta(t))$ a C^1 function with values on ∂K satisfying (1.2) in a certain interval of time; if we suppose $\Psi(\sigma(t),\zeta(t))\neq 0$, the flow rule and the condition

$$0=n_\sigma((\sigma(t),\zeta(t)))\cdot\dot\sigma(t)+n_\zeta((\sigma(t),\zeta(t)))\dot\zeta(t)$$

easily imply that $(\sigma(t),\zeta(t))$ satisfies (3.18) and that $-\frac{\Phi(t,\sigma(t),\zeta(t))}{\Psi(\sigma(t),\zeta(t))}\geq 0$.

We are now ready to prove the main result of this section.

Theorem 3.8. *Let t_1, σ_1, ζ_1, Φ, Ψ, and t_2 be as in (3.2), (3.3), (2.23), (2.20), (2.21), and (3.19) respectively. Suppose $t_1<+\infty$, $\Phi(t_1,\sigma_1,\zeta_1)>0$, and $\Psi(\sigma_1,\zeta_1)<0$. Let $(\sigma_{sl}(s),\zeta_{sl}(s))$ be the unique solution to the Cauchy problem associated to the equation of the slow dynamics (3.18) with datum (σ_1,ζ_1). Let $\bar t<t_2$ and suppose that there exists a constant $\gamma_3>0$ such that*

$$\Phi(s,\sigma_{sl}(s),\zeta_{sl}(s))\geq\gamma_3 \qquad \textit{for every } s\in[t_1,\bar t]. \tag{3.20}$$

Then $(\sigma_\varepsilon,\zeta_\varepsilon)$ converge uniformly to (σ_{sl},ζ_{sl}) as $\varepsilon\to 0^+$ in $[t_1,\bar t]$.

Proof. Let δ, γ_2, γ_1, and a_ε be given by (3.5), (3.12), and (3.6), respectively. We put $t^*=\liminf_{\varepsilon\to 0^+}a_\varepsilon$, and we apply Lemma 3.6 with $\hat t=\hat t_\varepsilon=t_1$, and $b_\varepsilon^\eta=a_\varepsilon$; we have that $t^*>t_1$, and, by part d) of the Lemma, we may assume that $\frac{\varrho_\varepsilon(t)}{\varepsilon}$ w^*-converge in $L^\infty((t_1,t^*))$ to some nonnegative function $\omega(t)$.

We write equation (2.8) in the form

$$\begin{cases}\mathbb{C}\dot\xi(t)-\dot\sigma(t)=\omega_1^\varepsilon(t,\sigma(t),\zeta(t))\\ g(\zeta(t))\dot\zeta(t)=\omega_2^\varepsilon(t,\sigma(t),\zeta(t));\end{cases}$$

where

$$\omega_1^\varepsilon(t,\sigma(t),\zeta(t)):=\tfrac{\varrho_\varepsilon(t)}{\varepsilon}h_\sigma(\sigma(t),\zeta(t))$$
$$\omega_2^\varepsilon(t,\sigma(t),\zeta(t)):=\tfrac{\varrho_\varepsilon(t)}{\varepsilon}h_\zeta(\sigma(t),\zeta(t)),$$

and h_σ, h_ζ are C^1 extensions to the whole space of $\mathbb{C}n_\sigma\circ\pi$ and $n_\zeta\circ\pi$, respectively, which we may assume to be globally Lipschitzian, as $\mathbb{C}n_\sigma\circ\pi$ and $n_\zeta\circ\pi$ are. Theorem 3.4 now provides the uniform convergence, along the sequence ε_k of the solutions of (2.8) to the solution of the problem

$$\begin{cases}\mathbb{C}\dot\xi(t)-\dot\sigma(t)=\omega(t)h_\sigma(\sigma(t),\zeta(t))\\ g(\zeta(t))\dot\zeta(t)=\omega(t)h_\zeta(\sigma(t),\zeta(t)),\end{cases} \tag{3.21}$$

with the same Cauchy data, in the interval $[t_1,t^*]$.

Now, Lemma 3.6, part d), implies that $(\sigma(t),\zeta(t))\in K$ for every $t\in[t_1,t^*]$, while Lemma 3.3 implies that, for every $t\in(t_1,t^*)$, the points $(\sigma_{\varepsilon_k}(t),\zeta_{\varepsilon_k}(t))$ do not belong to K when k is sufficiently large, so that $(\sigma(t),\zeta(t))\in\partial K$ for every $t\in[t_1,t^*]$. Thus, for every

$t \in [t_1, t^*]$, the functions $h_\sigma(\sigma(t), \zeta(t))$ and $h_\zeta(\sigma(t), \zeta(t))$ coincide with $\mathbb{C}n_\sigma(\sigma(t), \zeta(t))$ and $n_\zeta(\sigma(t), \zeta(t))$, respectively. Since $(\sigma(t), \zeta(t)) \in \partial K$, we must have, for every $t \in [t_1, t^*]$

$$\begin{aligned} 0 &= n_\sigma((\sigma(t), \zeta(t))) \cdot \dot\sigma(t) + n_\zeta((\sigma(t), \zeta(t)))\dot\zeta(t) \\ &= -n_\sigma((\sigma(t), \zeta(t))) \cdot (\mathbb{C}\dot\xi(t) - \dot\sigma(t)) + n_\zeta((\sigma(t), \zeta(t)))\dot\zeta(t) + \\ &\quad + n_\sigma((\sigma(t), \zeta(t))) \cdot \mathbb{C}\dot\xi(t); \end{aligned} \tag{3.22}$$

this in turn, recalling (3.21), (2.20) and (2.21) implies

$$0 = \omega(t)\Psi(\sigma(t), \zeta(t)) + \Phi(t, \sigma(t), \zeta(t)). \tag{3.23}$$

Notice that this is nothing more than formally passing (2.19) in the limit.

We then get that the solutions of (2.8) converge uniformly to the solution of the problem

$$\begin{cases} \mathbb{C}\dot\xi(t) - \dot\sigma(t) = -\frac{\Phi(t,\sigma(t),\zeta(t))}{\Psi(t,\sigma(t),\zeta(t))} n_\sigma(\sigma(t), \zeta(t)) \\ g(\zeta(t))\dot\zeta(t) = -\frac{\Phi(t,\sigma(t),\zeta(t))}{\Psi(t,\sigma(t),\zeta(t))} n_\zeta(\sigma(t), \zeta(t)), \end{cases} \tag{3.24}$$

with Cauchy data (σ_1, ζ_1), in the interval $[t_1, t^*]$, and by uniqueness, the limit is exactly $(\sigma_{sl}(t), \zeta_{sl}(t))$, which does not depend on the subsequence ε_k.

So, let $\tilde t$ the maximal time such that $(\sigma_\varepsilon, \zeta_\varepsilon)$ converge uniformly to $(\sigma_{sl}, \zeta_{sl})$ as $\varepsilon \to 0^+$ on compact subintervals of $[t_1, \tilde t)$; we have to show that $\tilde t > \bar t$. Let us argue by contradiction, supposing $\tilde t \le \bar t$. Define $(\tilde\sigma, \tilde\zeta) := (\sigma_{sl}(\tilde t), \zeta_{sl}(\tilde t))$ and observe that, by the hypotheses, there exist two constants $\eta > 0, \gamma > 0$ such that, for every $(t, \sigma, \zeta) \in [\tilde t - \eta, \tilde t + \eta] \times B_\eta(\tilde\sigma, \tilde\zeta)$, one has

$$\begin{aligned} \Psi(\sigma, \zeta) &< -\gamma \\ \Phi(t, \sigma, \zeta) &> \gamma. \end{aligned}$$

We may now fix $\tilde t - \frac{\eta}{2} < \tilde t_1 < \tilde t_2 < \tilde t < \tilde t_3 < \tilde t_1 + C(\frac{\eta}{2}, \gamma)$, where $C(\frac{\eta}{2}, \gamma)$ is the constant given by Lemma 3.6, in a way that $(\sigma_{sl}(\tilde t_1), \zeta_{sl}(\tilde t_1)) \in B_{\frac{\eta}{2}}(\tilde\sigma, \tilde\zeta)$ and we shall have that for every $(t, \sigma, \zeta) \in [\tilde t_1 - \frac{\eta}{2}, \tilde t + \frac{\eta}{2}] \times B_{\frac{\eta}{2}}(\sigma_{sl}(\tilde t_1), \zeta_{sl}(\tilde t_1))$,

$$\begin{aligned} \Psi(\sigma, \zeta) &< -\gamma \\ \Phi(t, \sigma, \zeta) &> \gamma \end{aligned} \tag{3.25}$$

By Lemma 3.6, applied with $\hat t = \hat t_\varepsilon = \tilde t_1$, we have that there exists $L > 0$ such that for ε sufficiently small $\frac{\varrho_\varepsilon(t)}{\varepsilon} \le \frac{L}{\gamma}$ for every $t \in [\tilde t_2, \tilde t_3]$. We now show that for ε sufficiently small

$$\frac{\varrho_\varepsilon(t)}{\varepsilon} \ge \frac{\gamma}{2\lambda} \qquad \text{for every } t \in [\tilde t_2, \tilde t_3], \tag{3.26}$$

where λ is the constant given by (2.23). To do that, we first prove that, for every $\varepsilon > 0$, and every $\theta > 0$, the set $\{\varrho_\varepsilon(t) > 0\} \cap [\tilde t_1, \tilde t_1 + \theta]$ has strictly positive Lebesgue measure; indeed, if not, applying Theorem 3.4, and taking into account that $(\sigma_\varepsilon, \zeta_\varepsilon)$ converge uniformly to $(\sigma_{sl}, \zeta_{sl})$ as $\varepsilon \to 0^+$ in $[\tilde t_1, \tilde t]$, we contradict (3.20). Then, the same argument of Lemma 3.3, with $\tilde t_1$ in place of t_1 and the time $b_\varepsilon^{\frac{\eta}{2}}$, introduced in Lemma 3.6, in place of a_ε, gives (3.26). Now we are in position to repeat the arguments of the previous step of the proof, and we get that the solutions of (2.8) uniformly converge in the interval $[\tilde t_2, \tilde t_3]$ to the solution of the problem

$$\begin{cases} \mathbb{C}\dot\xi(t) - \dot\sigma(t) = -\frac{\Phi(t,\sigma(t),\zeta(t))}{\Psi(t,\sigma(t),\zeta(t))} n_\sigma(\sigma(t), \zeta(t)) \\ g(\zeta(t))\dot\zeta(t) = -\frac{\Phi(t,\sigma(t),\zeta(t))}{\Psi(t,\sigma(t),\zeta(t))} n_\zeta(\sigma(t), \zeta(t)), \end{cases} \tag{3.27}$$

with Cauchy data $(\sigma(\tilde t_2), \zeta(\tilde t_2)) = (\sigma_{sl}(\tilde t_2), \zeta_{sl}(\tilde t_2))$, that is, by uniqueness, to $(\sigma_{sl}(t), \zeta_{sl}(t))$ and this contradicts the maximality of $\tilde t$. □

In general, $[t_1, t_2)$ may not be the maximal interval where solutions of (2.8) locally uniformly converge to the solution of (3.18), since condition (3.5) may fail before of t_2. The previous theorem shows that this is true if one has

$$\Phi(t, \sigma_{sl}(t), \zeta_{sl}(t)) > 0 \qquad \text{for every } t < t_2. \tag{3.28}$$

Assume instead that there exists $\bar{t} < t_2$ such that

$$\Phi(\bar{t}, \sigma_{sl}(\bar{t}), \zeta_{sl}(\bar{t})) = 0. \tag{3.29}$$

This is the case where, as we are going to discuss in the next subsection, elastic behavior may re-appear starting from the point $(\bar{\sigma}, \bar{\zeta}) := (\sigma_{sl}(\bar{t}), \zeta_{sl}(\bar{t})) \in \partial K$. In Section 5, on the contrary, we will assume (3.28): if moreover $t_2 < +\infty$, (3.30) below holds, and (5.2) is satisfied, we wiil show that a transition from the slow to the fast dynamics regime occurs. For this purpose, it will be useful the following proposition.

Proposition 3.9. *Let t_1, t_2, α be as in (3.2), (3.19) and (2.6) respectively, and let $(\sigma_{sl}(t), \zeta_{sl}(t))$ be as in (3.18). Assume (3.28), $t_2 < +\infty$ and that*

$$\liminf_{s \to t_2^-} \Phi(s, \sigma_{sl}(s), \zeta_{sl}(s)) > 0. \tag{3.30}$$

Then there exists

$$\lim_{s \to t_2^-} (\sigma_{sl}(s), \zeta_{sl}(s)) := (\sigma_2, \zeta_2) \in \partial K. \tag{3.31}$$

Moreover, we have

$$\zeta_2 > \alpha. \tag{3.32}$$

Proof. The existence of the limit for ζ is obvious as it is strictly decreasing in the considered interval, while (3.32) follows at once by (2.22). As $(\sigma_{sl}(s), \zeta_{sl}(s)) \in \partial K$ for every $s \in (t_1, t_2)$, $\sigma_{sl}(s)$ is continuous and bounded in the time interval (t_1, t_2). As the variable ζ is invertible, with inverse $t(\zeta)$, we can express σ in function of ζ; by (3.18), we then get that

$$-\frac{d}{d\zeta}\sigma_{sl}(\zeta) = \frac{g(\zeta)}{N_\zeta(\sigma_{sl}(\zeta), \zeta)}[\mathbb{C}\, N_\sigma(\sigma_{sl}(\zeta), \zeta) - \chi(\zeta)\frac{\Psi(\sigma_{sl}(\zeta), \zeta)}{\Phi(t(\zeta), \sigma_{sl}(\zeta), \zeta)}]$$

for every $\zeta \in (\zeta_2, \zeta_1)$; here we have put: $\chi(\zeta) := \dot{\xi}(t(\zeta))$. So, by (3.30), (3.32) and (2.2) $|\frac{d}{d\zeta}\sigma_{sl}(\zeta)|$ remains uniformly bounded in this interval. The conclusion follows. □

Remark 3.10. As a by-product, the limsup in (3.19) and the liminf in (3.30) are actually limits.

3.1. **Return to the elastic regime.**

In this subsection we assume (3.29) and we give some conditions which imply the return of the system to an elastic behaviour after the time $\bar{t}$, defined by (3.29). As $\bar{t} < t_2$, and since, by Theorem 3.8, $(\sigma_\varepsilon, \zeta_\varepsilon)$ converge uniformly to $(\sigma_{sl}, \zeta_{sl})$ as $\varepsilon \to 0^+$ on compact subintervals of $[t_1, \bar{t})$, with the help of Lemma 3.6 and Theorem 3.4 we can show, as in the proof of Theorem 3.8, that there exists $\vartheta > 0$ such that, in the time interval $[t_1, \bar{t} + \vartheta]$ the solutions of (2.8) converge, up to a subsequence, to the solution of the problem

$$\begin{cases} \mathbb{C}\dot{\xi}(t) - \dot{\sigma}(t) = \omega(t) h_\sigma(\sigma(t), \zeta(t)) \\ g(\zeta(t))\dot{\zeta}(t) = \omega(t) h_\zeta(\sigma(t), \zeta(t)), \end{cases} \tag{3.33}$$

with the Cauchy data $(\sigma_1, \zeta)_1$, for some suitable nonnegative function $\omega(t)$, and we have $(\sigma(t), \zeta(t)) = (\sigma_{sl}(t), \zeta_{sl}(t))$ in $(t_1, \bar{t})$. Thus $(\bar{\sigma}, \bar{\zeta}) := (\sigma_{sl}(\bar{t}), \zeta_{sl}(\bar{t})) = (\sigma(\bar{t}), \zeta(\bar{t}))$; recall that $\Phi(\bar{t}, \bar{\sigma}, \bar{\zeta}) = 0$. We give here two conditions assuring that, in a right neighborhood of $\bar{t}$, we have $\omega(t) \equiv 0$, that is, in the limit the system follows the equation of the elastic regime. Assume that there exists a sequence $t_n \to \bar{t}$ such that

$$\Phi(t_n, \sigma_{sl}(t_n), \zeta_{sl}(t_n)) < 0 \tag{3.34}$$

and that there exists $\eta > 0$ such that, for every $(t,s,\sigma,\zeta) \in (\bar{t},\bar{t}+\eta)\times(0,\eta)\times(B_\eta(\bar{\sigma},\bar{\zeta}))\cap\partial K$ satisfying $\Phi(t,\sigma,\zeta) \le 0$,

$$(\sigma + \mathbb{C}(\xi(t+s) - \xi(t)), \zeta) \in \overset{\circ}{K}. \tag{3.35}$$

We then have the following theorem.

Theorem 3.11. *Let $\bar{t}$ as in (3.29), $(\bar{\sigma},\bar{\zeta}) := (\sigma_{sl}(\bar{t}),\zeta_{sl}(\bar{t}))$, and assume that (3.34) and (3.35) hold. Let $(\sigma_{el}(t),\zeta_{el}(t) := (\bar{\sigma} + \mathbb{C}(\xi(t) - \xi(\bar{t})),\bar{\zeta})$ and*

$$\tau := \sup\{t > \bar{t}\,|(\sigma_{el}(s),\zeta_{el}(s)) \in \overset{\circ}{K} \text{ for every } s \in (\bar{t},t)\}.$$

Then $(\sigma_\varepsilon,\zeta_\varepsilon)$ converge uniformly to (σ_{el},ζ_{el}) as $\varepsilon \to 0^+$ on compact subsets of $[\bar{t},\tau)$.

Proof. Observe that, if (3.35) holds, τ is strictly larger than t and $\tau - \bar{t} \ge \eta$, where η is given by (3.35). Moreover, as the discussion at the beginning of this subsection shows, there exists $\vartheta > 0$ such that, in the time interval $[\bar{t},\bar{t}+\vartheta]$ the solutions of (2.8) converge, up to a subsequence, to the solution of the problem

$$\begin{cases} \mathbb{C}\dot{\xi}(t) - \dot{\sigma}(t) = \omega(t)h_\sigma(\sigma(t),\zeta(t)) \\ g(\zeta(t))\dot{\zeta}(t) = \omega(t)h_\zeta(\sigma(t),\zeta(t)), \end{cases} \tag{3.36}$$

with the Cauchy data $(\bar{\sigma},\bar{\zeta}) := (\sigma_{sl}(\bar{t}),\zeta_{sl}(\bar{t}))$, for some suitable nonnegative bounded function $\omega(t)$; thus we may fix $\delta < \eta$ such that $(\sigma(t),\zeta(t)) \in B_\eta(\bar{\sigma},\bar{\zeta})$ for every $t \in [\bar{t},\bar{t}+\delta]$.

Now, we first prove that the open set $A_{int} := \{t \in [\bar{t},\bar{t}+\delta)|(\sigma(t),\zeta(t)) \in \overset{\circ}{K}\}$ must be nonempty; indeed, if not, proceeding as in (3.22), we obtain that (3.23) is satisfied for every $t \in [\bar{t},\bar{t}+\delta]$; by uniqueness, this implies $(\sigma(t),\zeta(t)) = (\sigma_{sl}(t),\zeta_{sl}(t))$, but then (3.34) contradicts the nonnegativeness of $\omega(t)$. It is easily seen, as $(\sigma_\varepsilon,\zeta_\varepsilon)$ converge uniformly to (σ,ζ) in $[\bar{t},\bar{t}+\delta]$ that

$$\omega(t) \equiv 0 \qquad \text{for every } t \in A_{int}. \tag{3.37}$$

We now show that A_{int} is connected. Indeed, let $\hat{t} \in A_{int}$ and let $(\hat{t}_1,\hat{t}_2)$ the connected component containing $\hat{t}$. In $(\hat{t}_1,\hat{t}_2)$, we have, by (3.37), that $(\sigma(t),\zeta(t)) = (\sigma(\bar{t}_1) + \mathbb{C}(\xi(t) - \xi(\bar{t}_1),\zeta(\bar{t}_1))$. Notice that, as $(\sigma(\bar{t}_1),\zeta(\bar{t}_1)) \in \partial K$ by maximality, we have that $\Phi(\bar{t}_1,\sigma(\bar{t}_1),\zeta(\bar{t}_1)) \le 0$, if not the trajectory goes outside of K. Then, (3.35) implies that $\hat{t}_2 = \bar{t}+\delta$, thus proving that A_{int} is connected, that is $A_{int} = (\hat{t}_1,\bar{t}+\delta)$. Now, if $\hat{t}_1 > \bar{t}$, for every $t \in [\bar{t},\bar{t}_1]$ we must have $(\sigma(t),\zeta(t)) \in \partial K$, and again,by (3.22), (3.23) and (3.34), we get a contradiction. Thus the statement of the theorem is proved in $[\bar{t},\bar{t}+\delta]$; as, for every $t < \tau$, $(\sigma_{el}(t),\zeta_{el}(t)) \in \overset{\circ}{K}$, it is easily seen that the maximal interval such that the theorem holds is $[\bar{t},\tau)$. □

Remark 3.12. When ξ is at least C^2 regular, a simple sufficient condition implying both (3.34) and (3.35) is the following:

$$\mathbb{C}\ddot{\xi}(\bar{t}) \cdot n_\sigma(\bar{\sigma},\bar{\zeta}) + (\nabla_\sigma n_\sigma(\bar{\sigma},\bar{\zeta})\,\mathbb{C}\dot{\xi}(\bar{t})) \cdot \mathbb{C}\dot{\xi}(\bar{t}) < 0; \tag{3.38}$$

notice that $(\nabla_\sigma n_\sigma(\bar{\sigma},\bar{\zeta})\,\mathbb{C}\dot{\xi}(\bar{t})) \cdot \mathbb{C}\dot{\xi}(\bar{t})$ is exactly, except for a change of sign, the second fundamental form of ∂K at $(\bar{\sigma},\bar{\zeta})$ applied to the vector $(\mathbb{C}\dot{\xi}(\bar{t}),0)$, which is, by (3.29), tangent to ∂K at $(\bar{\sigma},\bar{\zeta})$.

Indeed, for what concerns (3.35), observe that, if (3.38) holds, by uniform continuity of the involved functions, we can find $\eta > 0$ such that, for every $(t,s,\sigma,\zeta) \in (\bar{t},\bar{t}+\eta)\times(0,\eta)\times(B_\eta(\bar{\sigma},\bar{\zeta}))\cap\partial K$

$$\frac{d}{ds}\Phi(t+s,\sigma + \mathbb{C}(\xi(t+s) - \xi(t)),\zeta) < 0, \tag{3.39}$$

since, by a direct computation, this derivative coincides with

$$\mathbb{C}\ddot{\xi}(t+s) \cdot n_\sigma(\sigma + \mathbb{C}(\xi(t+s) - \xi(t)),\zeta) \tag{3.40}$$

$$+(\nabla_\sigma n_\sigma(\sigma + \mathbb{C}(\xi(t+s) - \xi(t)),\zeta)\,\mathbb{C}\dot{\xi}(t+s)) \cdot \mathbb{C}\dot{\xi}(t+s); \tag{3.41}$$

thus, if $\Phi(t,\sigma,\zeta)\le 0$, for every $s\in(0,\eta)$,

$$\Phi(t+s,\sigma+\mathbb{C}(\xi(t+s)-\xi(t)),\zeta)<0. \tag{3.42}$$

Now, fix $(t,\sigma,\zeta)\in(\bar t,\bar t+\eta)\times(B_\eta(\bar\sigma,\bar\zeta))\cap\partial K$ with $\Phi(t,\sigma,\zeta)\le 0$, and define $\gamma(s):=(\sigma+\mathbb{C}(\xi(t+s)-\xi(t)),\zeta)$. Suppose, by contradiction, that there exists $s_1\in(0,\eta)$ such that $\gamma(s_1)\notin\overset{\circ}{K}$. We may assume $\gamma(s_1)\notin K$, since, if $\gamma(s_1)\in\partial K$, (3.42), which is equivalent to $\dot\gamma(s_1)\cdot n(\gamma(s_1))<0$, assures that in a left neighborhood of s_1, we have $\gamma(s)\notin K$. By the fundamental theorem of calculus we then get

$$0<\varrho(\gamma(s_1))=\int_0^{s_1}\tfrac{d}{ds}\varrho(\gamma(s))\,ds,$$

that is, explicitly calculating the derivative and recalling (2.20),

$$0<\varrho(\gamma(s_1))=\int_{\{\varrho(\gamma(s))>0\}\cap(0,s_1)}\Phi(t+s,\sigma+\mathbb{C}(\xi(t+s)-\xi(t)),\zeta)\,ds<0,$$

again by (3.42), which is a contradiction; thus (3.29) is proven.

For what concerns (3.34), observe that

$$\begin{aligned}\tfrac{d}{dt}\Phi(t,\sigma_{sl}(t),\zeta_{sl}(t)) \quad &= \quad \mathbb{C}\ddot\xi(t)\cdot n_\sigma(\sigma_{sl}(t),\zeta_{sl}(t))\\ &\quad +[\nabla n_\sigma(\sigma_{sl}(t),\zeta_{sl}(t))(\dot\sigma_{sl}(t),\dot\zeta_{sl}(t))]\cdot\mathbb{C}\dot\xi(t)\end{aligned}$$

coincides, by (3.29), at time $t=\bar t$ with the left-hand side of (3.38), thus is strictly negative if this one holds. Notice that, since for every $t<\bar t$, we have $\Phi(t,\sigma_{sl}(t),\zeta_{sl}(t))>0$, we get $\frac{d}{dt}\Phi(\bar t,\sigma_{sl}(\bar t),\zeta_{sl}(\bar t))\le 0$, thus condition (3.38) is actually a nondegeneracy condition which is satisfied in most cases.

4. **Fast dynamics.**

4.1. **The equation of the fast dynamics.**

We start this section by a qualitative study of the equation:

$$\begin{cases}-\dot\sigma(t)=\mathbb{C}[\sigma(t)-\pi_\sigma(\sigma(t),\zeta(t))],\\ g(\zeta)\dot\zeta(t))=\zeta(t)-\pi_\zeta(\sigma(t),\zeta(t));\end{cases} \tag{4.1}$$

we will see that, when condition (3.12) fails, in the limit the solutions of (2.8) present a jump which is governed by equation (4.1). Precisely, a rescaled version of the solutions of (2.8) converges to the heteroclinic orbit of (4.1) issuing from the point of ∂K reached at the jump time, whose existence and uniqueness, together with some other properties, are going to be proven in the following theorem.

Theorem 4.1. *Let $(\bar\sigma,\bar\zeta)\in\partial K$ and suppose that*

$$\Psi(\bar\sigma,\bar\zeta)>0 \tag{4.2}$$

or

$$\Psi(\bar\sigma,\bar\zeta)=0 \textit{ and } \nabla\Psi(\bar\sigma,\bar\zeta)\cdot(-g(\bar\zeta)\frac{\mathbb{C}n_\sigma(\bar\sigma,\bar\zeta)}{n_\zeta(\bar\sigma,\bar\zeta)},1)<0. \tag{4.3}$$

Then the problem

$$\begin{cases}-\dot\sigma(t)=\mathbb{C}[\sigma(t)-\pi_\sigma(\sigma(t),\zeta(t))],\\ g(\zeta)\dot\zeta(t)=\zeta(t)-\pi_\zeta(\sigma(t),\zeta(t)),\\ \lim\limits_{t\to-\infty}(\sigma(t),\zeta(t))=(\bar\sigma,\bar\zeta)\end{cases} \tag{4.4}$$

has a unique solution (up to time-translations); if we denote it with $(\bar\sigma(t),\bar\zeta(t))$, we have that the limit

$$(\sigma_\infty,\zeta_\infty):=\lim_{t\to+\infty}(\bar\sigma(t),\bar\zeta(t)) \tag{4.5}$$

exists and satisfies

$$(\sigma_\infty, \zeta_\infty) \in \partial K,\ \Psi(\sigma_\infty, \zeta_\infty) \le 0,\ \zeta_\infty > \alpha. \tag{4.6}$$

Proof. We first consider the equation:

$$\begin{cases} -\dot\sigma(t) = \varrho(\sigma(t), \zeta(t))\, \mathbb{C}\, n_\sigma(\pi_1(\sigma(t), \zeta(t))), \\ g(\zeta)\dot\zeta(t) = \varrho(\sigma(t), \zeta(t))\, n_\zeta(\pi_1(\sigma(t), \zeta(t))); \end{cases} \tag{4.7}$$

Here our regularity assumptions on K allow us to define a minimal distance projection π_1 to ∂K on a whole ball centered at $(\bar\sigma, \bar\zeta)$, which obviously coincide with π in the exterior of set K. Thus the functions $n \circ \pi$ and Ψ, too, are extended to a C^1 function defined on a whole ball centered at $(\bar\sigma, \bar\zeta)$, and clearly condition (2.2) is preserved. It is easily seen that the right-hand sides of (4.1) and (4.7) coincide in this ball.

The variable ζ is strictly decreasing along any motion starting in this fixed ball, thus we can express the time t in function of ζ and get the equation of the trajectories

$$-\sigma'(\zeta) = \tfrac{g(\zeta)}{n_\zeta(\pi_1(\sigma(\zeta),\zeta))}\, \mathbb{C}\, n_\sigma(\pi_1(\sigma(\zeta), \zeta)). \tag{4.8}$$

Conversely, every solution of (4.8) with $(\sigma(\zeta), \zeta) \notin K$ produces a solution of (4.7). By the local existence and uniqueness theorem for (4.8), there exists only an integral curve of (4.7) passing through the point $(\bar\sigma, \bar\zeta)$. The latter is a critical point for (4.1), so the first part of the statement will be proven once we show that this integral curve, which shall be denoted by $(\bar\sigma(\zeta), \zeta)$, points outwards the set K at $(\bar\sigma, \bar\zeta)$ in the direction of the motion. Precisely, introducing the oriented distance r from ∂K, since ζ is decreasing, we have to show that in a left open neighborhood of $\bar\zeta$ one has

$$r(\bar\sigma(\zeta), \zeta) > 0;$$

as $(\bar\sigma, \bar\zeta) \in \partial K$, it will suffice to show that in a left open neighborhood of $\bar\zeta$ one has

$$\tfrac{d}{d\zeta} r(\bar\sigma(\zeta), \zeta) < 0. \tag{4.9}$$

It is well known, indeed, that, provided we are in a sufficiently small neighborhood of $(\bar\sigma, \bar\zeta)$, the function r is smooth; moreover its gradient is given by $n \circ \pi_1$.

By a direct computation, similar to that done in order to get (2.19), and taking into account the equation of the trajectories, we have:

$$\tfrac{d}{d\zeta} r(\bar\sigma(\zeta), \zeta) = (\bar\sigma'(\zeta), 1) \cdot n(\pi_1(\bar\sigma(\zeta), \zeta)) = \tfrac{g(\zeta)}{n_\zeta(\pi_1(\bar\sigma(\zeta),\zeta))} \Psi(\bar\sigma(\zeta), \zeta) \tag{4.10}$$

hence to prove (4.9), by (2.2) and (2.12), we have only to show that there exist $\beta < \bar\zeta$ such that:

$$\Psi(\bar\sigma(\zeta), \zeta) > 0\ \forall\, \zeta \in (\beta, \bar\zeta). \tag{4.11}$$

This fact immediately follows by (4.2) or (4.3): in this latter case we have only to take one derivative along the trajectory to get that $\Psi(\bar\sigma(\zeta), \zeta)$ is strictly decreasing in $\bar\zeta$.

We now prove (4.5) for a generic orbit $(\sigma(t), \zeta(t))$ of (4.1) starting outside of the set K: trivially it cannot meet the set K at finite times. The existence of the limit

$$\zeta_\infty := \lim_{t \to +\infty} \zeta(t) \tag{4.12}$$

is a direct consequence of the strict monotonicity of $\zeta(t)$. Assume by contradiction that $\zeta_\infty = \alpha$; as usual, let us consider the corresponding trajectory $(\sigma(\zeta), \zeta)$. Along this trajectory, the oriented distance r from ∂K obviously conicide with the usual distance ϱ from K; we put $\rho(\zeta) := \varrho(\sigma(\zeta), \zeta)$ and clearly, for any $\zeta > \alpha$ we shall have $\rho(\zeta) > 0$; recall that, by (4.10), $\rho'(\zeta) = \frac{g(\zeta)}{n_\zeta(\pi(\sigma(\zeta),\zeta))} \Psi(\sigma(\zeta), \zeta)$.

Now let us fix $\varepsilon > 0$: by (2.22), and (2.2) we may fix $\beta > \alpha$ such that in $(\alpha + \varepsilon, \beta)$ one has $\frac{\Psi(\sigma(\zeta),\zeta)}{n_\zeta(\pi(\sigma(\zeta),\zeta))} \geq \theta$, so that by integration we get

$$\rho(\beta) - \rho(\alpha + \varepsilon) \geq \theta \int_{\alpha+\varepsilon}^{\beta} g(\zeta)\, d\zeta.$$

By (2.10), we conclude that:

$$\lim_{\varepsilon \to 0^+} \rho(\alpha + \varepsilon) = -\infty,$$

a contradiction as the function ρ is nonnegative.

We have then shown that $\zeta_\infty > \alpha$; it follows that g is bounded on any orbit of the system (4.1) and we get (4.5) applying to $\sigma(\zeta))$ the argument used in Proposition 3.9. As $(\sigma_\infty, \zeta_\infty)$ must be a critical point of (4.1), it is easy to show that $(\sigma_\infty, \zeta_\infty) \in \partial K$. Moreover differentiating the function $\varrho(\sigma(t), \zeta(t))$ with respect to the time, we get

$$\frac{d}{dt}\varrho(\sigma(t), \zeta(t)) = \Psi(\sigma(t), \zeta(t))\varrho(\sigma(t), \zeta(t));$$

now, as $(\sigma_\infty, \zeta_\infty) \in \partial K$, we conclude

$$\lim_{t \to +\infty} \varrho(\sigma(t), \zeta(t)) = 0$$

which implies, by differential inequalities

$$\liminf_{t \to +\infty} \Psi(\sigma(t), \zeta(t)) \leq 0;$$

and this finally implies $\Psi(\sigma_\infty, \zeta_\infty) \leq 0$. □

Remark 4.2. Indeed, we have also shown that every orbit of the system (4.1) has a unique ω- limit point (there is nothing to prove for orbits starting in the set K as it is made of fixed points); similarly we may show that the α- limit set has at most one point.

4.2. **From the elastic regime to the fast dynamics.**

We can now investigate what happens at time t_1, where t_1 is given by (3.2), when instead of condition (3.12), the slow-fast indicator has positive sign at (σ_1, ζ_1). Here (σ_1, ζ_1) are defined by (3.3). We shall see that, if also the elastic-inelastic indicator is positive, the limit of the solutions of (2.8) presents a discontinuity at time t_1.

We then suppose $\Phi(t_1, \sigma_1, \zeta_1) > 0$, and we now fix an open neighborhood $U_{\delta_1} := (t_1 - \delta_1, t_1 + \delta_1) \times B_{\delta_1}$, where $B_{\delta_1} := B_\delta(\sigma_1, \zeta_1)$, in a way that (3.5) holds. We define the first exit time from B_{δ_1} as

$$a^1_\varepsilon := \inf \{t \in (t_1 - \delta_1, t_1 + \delta_1) : \ (\sigma_\varepsilon(t), \zeta_\varepsilon(t)) \in \partial B_{\delta_1}(\sigma_1, \zeta_1)\}\,; \tag{4.13}$$

we also remember that in this situation, Lemma 3.3 holds. We also fix, starting from δ_1 a positive decreasing sequence $\delta_k \searrow 0^+$ and consequently we define, for every $k \in \mathbb{N}$, the exit times

$$a^k_\varepsilon := \inf \{t \in (t_1 - \delta_1, t_1 + \delta_1) : \ (\sigma_\varepsilon(t), \zeta_\varepsilon(t)) \in \partial B_{\delta_k}(\sigma_1, \zeta_1)\}\,. \tag{4.14}$$

We may suppose, possibly taking a smaller δ_1 in the definition of the neighborhood U_{δ_1} that there exists a positive constant γ_1 such that

$$\Psi(\sigma, \zeta) \geq \gamma_1 \qquad \text{for every } (\sigma, \zeta) \in B_{\delta_1}(\sigma_1, \zeta_1). \tag{4.15}$$

Next Lemma, which will be crucial in the remainder of the section, shows that the exit times a^k_ε tend to t_1 when ε goes to 0 and that the difference $a^1_\varepsilon - a^k_\varepsilon$ is of order ε for fixed k.

Lemma 4.3. *Let t_1, σ_1, ζ_1, Φ and Ψ be as in (3.2), (3.3), (2.20), and (2.21), respectively. Suppose $t_1 < +\infty$, $\Phi(t_1, \sigma_1, \zeta_1) > 0$, and assume (4.15). Let δ_1 be as in (3.5), fix a positive decreasing sequence $\delta_k \searrow 0^+$, starting from δ_1, and let a^k_ε be given for every $k \in \mathbb{N}$ by (4.14). Then, for every $k \in \mathbb{N}$:*

a) $a^k_\varepsilon \to t_1$ *as* $\varepsilon \to 0^+$*;*

b) $\sup_{\varepsilon>0} \frac{a_\varepsilon^1 - a_\varepsilon^k}{\varepsilon} \leq c_k < +\infty$,

where c_k *is a constant depending on* k*.*

Proof. Concerning part a) of the statement, it clearly suffices to show this is true for a_ε^1. We define t_ε as in Lemma 3.3; therefore we have that $t_\varepsilon < a_\varepsilon^1$ for small ε, and it suffices to show that

$$\limsup_{\varepsilon\to 0^+} (a_\varepsilon^1 - t_\varepsilon) = 0. \tag{4.16}$$

The same lemma shows that $\varrho_\varepsilon(t) > 0$ for every $t \in (t_1, a_\varepsilon^1)$, hence (2.19) holds. Moreover (3.5) and (4.15) imply $\dot\varrho_\varepsilon(t) \geq \gamma_1 \frac{1}{\varepsilon} \varrho_\varepsilon(t)$; dividing by $\varrho_\varepsilon(t)$, we get

$$\frac{\dot\varrho_\varepsilon(t)}{\varrho_\varepsilon(t)} \geq \frac{\gamma_1}{\varepsilon} \qquad \text{for every } t \in (t_1, a_\varepsilon^1). \tag{4.17}$$

Recall that by the definition of a_ε^1 clearly one has $\varrho_\varepsilon(a_\varepsilon^1) \leq \delta_1$, and that $\varrho_\varepsilon(t_\varepsilon) = \varepsilon \frac{\gamma_2}{2\lambda}$; thus, integrating (4.17) between t_ε and a_ε^1, we finally get the inequality

$$a_\varepsilon^1 - t_\varepsilon \leq \frac{\varepsilon}{\gamma_1} \log(\frac{2\lambda\delta_1}{\varepsilon\gamma_2})$$

which implies (4.16).

Concerning part b), we fix $k \in \mathbb{N}$; since $a_\varepsilon^k < t_1 + \delta_1$ for ε small enough, applying (4.17) we get

$$\begin{aligned}
\delta_k &= |(\sigma_\varepsilon(a_\varepsilon^k), \zeta_\varepsilon(a_\varepsilon^k)) - (\sigma_1, \zeta_1)| \\
&\leq |(\sigma_\varepsilon(a_\varepsilon^k) - \sigma_1, 0)| + |(0, \zeta_\varepsilon(a_\varepsilon^k) - \zeta_1)| \\
&= |\int_{t_1}^{a_\varepsilon^k} \dot\sigma_\varepsilon(s)\, ds| + |\int_{t_1}^{a_\varepsilon^k} \dot\zeta_\varepsilon(s)\, ds| \\
&\leq \int_{t_1}^{a_\varepsilon^k} |\mathbb{C}\dot\xi(s) - \dot\sigma_\varepsilon(s)|\, ds + |\mathbb{C}| \int_{t_1}^{a_\varepsilon^k} |\dot\xi(s)|\, ds + c_0 \int_{t_1}^{a_\varepsilon^k} g(\zeta_\varepsilon(s))|\dot\zeta_\varepsilon(s)|\, ds \qquad (4.18)\\
&\leq (|\mathbb{C}| + c_0) \int_{t_1}^{a_\varepsilon^k} \tfrac{\varrho_\varepsilon}{\varepsilon}(s)\, ds + |\mathbb{C}| \int_{t_1}^{a_\varepsilon^k} |\dot\xi(s)|\, ds \\
&\leq \tfrac{1}{\gamma_1}(|\mathbb{C}| + c_0) \int_{t_1}^{a_\varepsilon^k} \dot\varrho_\varepsilon(s)\, ds + |\mathbb{C}| \int_{t_1}^{a_\varepsilon^k} |\dot\xi(s)|\, ds \\
&\leq \tfrac{1}{\gamma_1}(|\mathbb{C}| + c_0)\varrho_\varepsilon(a_\varepsilon^k) + |\mathbb{C}| \int_{t_1}^{a_\varepsilon^k} |\dot\xi(s)|\, ds;
\end{aligned}$$

it follows that there exists a positive constant m_k such that for ε small enough:

$$\varrho_\varepsilon(a_\varepsilon^k) \geq m_k. \tag{4.19}$$

From this, integrating (4.17) between a_ε^k and a_ε^1 we get that for ε small enough

$$\tfrac{a_\varepsilon^1 - a_\varepsilon^k}{\varepsilon} \leq \tfrac{1}{\gamma_1} \log(\tfrac{\varrho_\varepsilon(a_\varepsilon^1)}{\varrho_\varepsilon(a_\varepsilon^k)}) \leq \tfrac{1}{\gamma_1} \log(\tfrac{\delta_1}{m_k}),$$

and conclusion then follows. □

We are now ready to prove the main result of this section.

Theorem 4.4. *Let* t_1, σ_1 *and* ζ_1*, be as in (3.2), and (3.3), respectively. Suppose* $t_1 < +\infty$*,* $\Phi(t_1, \sigma_1, \zeta_1) > 0$*, and assume (4.15). Let* δ_1 *be as in (3.5) and let* a_ε^1 *be given by (4.13).For* *every* $s \in \mathbb{R}$, *let* $(\sigma_\varepsilon^1(s), \zeta_\varepsilon^1(s)) := (\sigma_\varepsilon(a_\varepsilon^1 + \varepsilon s), \zeta_\varepsilon(a_\varepsilon^1 + \varepsilon s))$. *Then* $(\sigma_\varepsilon^1(s), \zeta_\varepsilon^1(s))$ *converges uniformly on compact subsets of* $\mathbb{R}$ *to a solution of the problem:*

$$\begin{cases} -\dot\sigma(s) = \mathbb{C}[\sigma(s) - \pi_\sigma(\sigma(s), \zeta(s))], \\ g(\zeta)\dot\zeta(s) = \zeta(s) - \pi_\zeta(\sigma(s), \zeta(s)), \\ \lim\limits_{s\to-\infty} (\sigma(s), \zeta(s)) = (\sigma_1, \zeta_1) \end{cases} \tag{4.20}$$

whose existence and uniqueness up to time translations is guaranteed by Theorem 4.1.

Proof. This proof is modelled on that of Lemma 4.3 of [5]. First of all, we observe that it suffices to prove the statement along a subsequence ε_k tending to 0. Indeed, the only difficulty is that the solutions of (4.20) may differ by a time translation, thus the limit could depend on the chosen subsequence. We are able to exclude this fact applying Lemma 4.4 in [5], with the same arguments as in the proof of Theorem 3.5 of the same paper. In view of that, we shall extract from now on subsequences without relabelling. We also define $\chi_\varepsilon(s) := \dot\xi(a_\varepsilon^1 + \varepsilon s)$.

We start by observing that the function $(\sigma_\varepsilon^1(s), \zeta_\varepsilon^1(s))$ solves the problem

$$\begin{cases} -\dot\sigma_\varepsilon^1(s) = \mathbb{C}[\sigma_\varepsilon^1(s) - \pi_\sigma(\sigma_\varepsilon^1(s), \zeta_\varepsilon^1(s))] - \varepsilon\mathbb{C}\,\chi_\varepsilon(s), \\ g(\zeta_\varepsilon^1(s))\dot\zeta_\varepsilon^1(s) = \zeta_\varepsilon^1(s) - \pi_\zeta(\sigma_\varepsilon^1(s), \zeta_\varepsilon^1(s)), \\ (\sigma_\varepsilon^1(0), \zeta_\varepsilon^1(0)) = (\sigma_\varepsilon(a_\varepsilon^1), \zeta_\varepsilon(a_\varepsilon^1)), \end{cases} \tag{4.21}$$

in the interval $[-\frac{a_\varepsilon^1}{\varepsilon}, \frac{t_1+\delta_1-a_\varepsilon^1}{\varepsilon}]$. As $(\sigma_\varepsilon(a_\varepsilon^1), \zeta_\varepsilon(a_\varepsilon^1))$ belongs to the compact set ∂B_{δ_1} we may assume, possibly passing to a subsequence that $(\sigma_\varepsilon(a_\varepsilon^1), \zeta_\varepsilon(a_\varepsilon^1)) \to \kappa_1 \in \partial B_{\delta_1}$ as $\varepsilon \to 0$. Notice that κ_1 has a strictly positive distance from K as a consequence of (4.19). Therefore, Lemma 4.3 and the Continuous Dependence Theorem imply that $(\sigma_\varepsilon^1(s), \zeta_\varepsilon^1(s))$ converges uniformly on compact subsets of $\mathbb{R}$, as $\varepsilon \to 0$, to the solution $(\sigma^1(s), \zeta^1(s))$ of the problem

$$\begin{cases} -\dot\sigma^1(s) = \mathbb{C}[\sigma^1(s) - \pi_\sigma(\sigma^1(s), \zeta^1(s))], \\ g(\zeta^1)\dot\zeta^1(s) = \zeta^1(s) - \pi_\zeta(\sigma^1(s), \zeta^1(s)), \\ (\sigma^1(0), \zeta^1(0)) = \kappa_1. \end{cases} \tag{4.22}$$

To conclude the proof we have to show that:

$$\lim_{s\to-\infty} (\sigma^1(s), \zeta^1(s)) = (\sigma_1, \zeta_1). \tag{4.23}$$

Actually, recalling Remark 4.2, it suffices to show that there exist $s_k \to +\infty$ such that:

$$\lim_{k\to+\infty} (\sigma^1(-s_k), \zeta^1(-s_k)) = (\sigma_1, \zeta_1). \tag{4.24}$$

To do that, we take δ_k and a_ε^k as in Lemma 4.3, and we define $S_\varepsilon^{1,k} := \frac{a_\varepsilon^1 - a_\varepsilon^k}{\varepsilon}$; by Lemma 4.3 and a diagunal argument, we may suppose, passing to a subsequence, that for every $k \in \mathbb{N}$ there exists

$$\lim_{\varepsilon\to0} S_\varepsilon^{1,k} := s_k \in \mathbb{R}_+.$$

Now we define $(\sigma_\varepsilon^k(s), \zeta_\varepsilon^k(s)) := (\sigma_\varepsilon(a_\varepsilon^k + \varepsilon s), \zeta_\varepsilon(a_\varepsilon^k + \varepsilon s))$; by repeating the above arguments we may suppose that for every $k \in \mathbb{N}$ there exists $\kappa_k \in \partial B_{\delta_k} \setminus K$ such that $(\sigma_\varepsilon^k(s), \zeta_\varepsilon^k(s))$ converges, as $\varepsilon \to 0$, uniformly on compact subsets of $\mathbb{R}$, to the solution $(\sigma^k(s), \zeta^k(s))$ of the problem

$$\begin{cases} -\dot\sigma^k(s) = \mathbb{C}[\sigma^k(s) - \pi_\sigma(\sigma^k(s), \zeta^k(s))], \\ g(\zeta^k)\dot\zeta^k(s) = \zeta^k(s) - \pi_\zeta(\sigma^k(s), \zeta^k(s)), \\ (\sigma^k(0), \zeta^k(0)) = \kappa_k. \end{cases} \tag{4.25}$$

Moreover, equality $(\sigma_\varepsilon^k(S_\varepsilon^{1,k}), \zeta_\varepsilon^k(S_\varepsilon^{1,k})) = (\sigma_\varepsilon(a_\varepsilon^1), \zeta_\varepsilon(a_\varepsilon^1))$ implies that $(\sigma^k(s_k), \zeta^k(s_k)) = \kappa_1$, hence by the uniqueness of solutions for Cauchy problems we get

$$(\sigma^k(s), \zeta^k(s)) = (\sigma^1(s - s_k), \zeta^1(s - s_k)); \tag{4.26}$$

this in turn implies that

$$(\sigma^1(-s_k), \zeta^1(-s_k)) = \kappa_k. \tag{4.27}$$

It follows that

$$\lim_{k\to+\infty}(\sigma^1(-s_k), \zeta^1(-s_k)) = (\sigma_1, \zeta_1); \tag{4.28}$$

as (σ_1, ζ_1) is an equilibrium point, necessarily $s_k \to +\infty$ as $k \to +\infty$; so, (4.24) is proven and conclusion follows. □

Remark 4.5. Let $(\sigma_\infty, \zeta_\infty)$ the unique ω-limit point of the solution of (4.20); by Theorem 4.1 $\Psi(\sigma_\infty, \zeta_\infty) \le 0$ and in a generic situation we may assume that strict inequality holds. By the previous theorem we may fix a sequence $t_{1,\varepsilon}$ converging to t_1 as $\varepsilon \to 0^+$ such that $(\sigma_\varepsilon(t_{1,\varepsilon}), \zeta_\varepsilon(t_{1,\varepsilon})) \to (\sigma_\infty, \zeta_\infty)$.

Now, if $\Phi(t_1, \sigma_\infty, \zeta_\infty) < 0$ it is easily seen that at time t_1 the elastic regime restarts from the point $(\sigma_\infty, \zeta_\infty)$; if instead $\Phi(t_1, \sigma_\infty, \zeta_\infty) > 0$, we can repeat the arguments of Section 3, with $t_{1,\varepsilon}$ in place of t_1, and Corollary 3.5 in place of Theorem 3.4, showing that in a right neighborhood of t_1 the solutions of (2.8) uniformly converge, on compact subintervals, to the solution of the slow dynamics equation given by (3.18) with initial datum $(\sigma_\infty, \zeta_\infty)$.

5. **From the slow dynamics to the fast dynamics.**

In this section of the paper we investigate the behavior of our system in a neighborhood of time t_2, introduced in (3.19): we are supposing from now on $t_2 < +\infty$. At smaller times, as we have seen in Section 3, the system is following the slow dynamics equation. We are assuming (3.30), so that the elastic regime does not restart at time t_2; Proposition 3.9 shows then that the function $(\sigma_{sl}(t), \zeta_{sl}(t))$ defined by (3.18), has a left limit at time t_2 that we shall denote with (σ_2, ζ_2); moreover (3.31), (3.32) hold together with:

$$\Psi(\sigma_2, \zeta_2) = 0 \text{ and } \Phi(t_2, \sigma_2, \zeta_2) > 0. \tag{5.1}$$

So we need an higher order condition on the slow-fast indicator Ψ to establish the behavior of the system at t_2: the simplest one, that we have already encountered in Theorem 4.1 and that we are going to assume in the remainder of the section, is

$$\nabla\Psi(\sigma_2, \zeta_2) \cdot (-g(\zeta_2)\tfrac{\mathbb{C}n_\sigma(\sigma_2,\zeta_2)}{n_\zeta(\sigma_2,\zeta_2)}, 1) < 0. \tag{5.2}$$

We will see that this condition implies a change of sign in the slow-fast indicator at point (σ_2, ζ_2) along the trajectories, and will determine the transition to the slow dynamics regime to the fast dynamics. To do that, we want to follow the scheme of Theorem 4.4, but, as the slow-fast indicator has no more a definite sign locally, while it had in the previous section, a little more difficulties have to be overcome. Precisely we need a refinement of Lemma 4.3.

As usual, we start by fixing an open neighborhood $U_{\delta_1} := (t_2 - \delta_1, t_2 + \delta_1) \times B_{\delta_1}(\sigma_2, \zeta_2)$ of (t_2, σ_2, ζ_2), in a way that (3.5) holds. If (5.2) holds we also may assume for a suitable choice of δ_1 there exists a positive constant M such that

$$\nabla\Psi(\sigma, \zeta) \cdot (-g(\zeta)\tfrac{\mathbb{C}n_\sigma(\pi(\sigma,\zeta))}{n_\zeta(\pi(\sigma,\zeta))}, 1) \le -M \qquad \text{for every } (\sigma, \zeta) \in B_{\delta_1}(\sigma_2, \zeta_2) \setminus \overset{\circ}{K}. \tag{5.3}$$

We also fix, thanks to Theorem 3.8, a sequence $t_{2,\varepsilon} \nearrow t_2$ such that

$$(\sigma_\varepsilon(t_{2,\varepsilon}), \zeta_\varepsilon(t_{2,\varepsilon})) \to (\sigma_2, \zeta_2). \tag{5.4}$$

Notice that we have (see Section 3) that, for every $\varepsilon > 0$,

$$\varrho(\sigma_\varepsilon(t_{2,\varepsilon}), \zeta_\varepsilon(t_{2,\varepsilon})) > 0. \tag{5.5}$$

We now define the exit time from $B_{\delta_1}(\sigma_2, \zeta_2)$

$$b^1_\varepsilon := \inf\{t \in (t_{2,\varepsilon}, t_2 + \delta_1) : \ (\sigma_\varepsilon(t), \zeta_\varepsilon(t)) \in \partial B_{\delta_1}(\sigma_2, \zeta_2)\}; \tag{5.6}$$

by the previous assumptions for small ε we will trivially have $t_{2,\varepsilon} < b^1_\varepsilon$. As in the previous section we fix a positive decreasing sequence $\delta_k \searrow 0^+$, starting from δ_1, and consequently we define, for every $k \in \mathbb{N}$,

$$b^k_\varepsilon := \sup\{t \in (t_{2,\varepsilon}, b^1_\varepsilon):\ (\sigma_\varepsilon(t), \zeta_\varepsilon(t)) \in \partial B_{\delta_k}(\sigma_2, \zeta_2)\}. \tag{5.7}$$

Notice the difference between the definition of the b^k_ε's and definition (4.14) of the a^k_ε's, which were defined as infima. In the present case, indeed, monotonocity of the function $\varrho_\varepsilon(t)$ is not a priori clear due to the different assumption on Ψ.

We are now ready to prove the following key lemma, whose statement is the analogous of Lemma 4.3.

Lemma 5.1. *Let t_2, σ_2, ζ_2, Φ and Ψ be as in (3.19), (3.31), (2.20), and (2.21), respectively. Suppose $t_2 < +\infty$, and assume (5.1) and (5.2). Let δ_1 be as in (3.5), fix a positive decreasing sequence $\delta_k \searrow 0^+$, starting from δ_1, and let b^k_ε be given for every $k \in \mathbb{N}$ by (5.7). Then, for every $k \in \mathbb{N}$:*

a) $b^k_\varepsilon \to t_2$ *as* $\varepsilon \to 0^+$;

b) $\sup_{\varepsilon>0} \frac{b^1_\varepsilon - b^k_\varepsilon}{\varepsilon} \le c_k < +\infty$,

where c_k is a constant depending on k.

Proof. Let $t_{2,\varepsilon}$ as in (5.4). Concerning part a) of the statement, it clearly suffices to show this is true for b^1_ε. By the definition of $t_{2,\varepsilon}$ this will be proved once we get:

$$\limsup_{\varepsilon \to 0^+} (b^1_\varepsilon - t_{2,\varepsilon}) = 0. \tag{5.8}$$

First of all, observe that by (5.5), (3.5) proceeding in the same way as in Lemma 3.3, we shall have $\varrho_\varepsilon(t) > 0$ for every $t \in (t_{2,\varepsilon}, b^1_\varepsilon)$; hence in this interval the trajectories of (2.8) and those of (2.14) coincide, and (2.19) holds. As $\zeta_2 > \alpha$ by Proposition 3.9, we may assume that $g(\zeta_\varepsilon(t))$ is uniformly bounded from above in $(t_{2,\varepsilon}, b^1_\varepsilon)$, provided we have chosen δ_1 suitably small ; by this and (2.2) we get the existence of a positive constant C such that

$$\dot\zeta_\varepsilon(t) \le -C\frac{\varrho_\varepsilon(t)}{\varepsilon} \qquad \text{for every } t \in (t_{2,\varepsilon}, b^1_\varepsilon). \tag{5.9}$$

In particular, for fixed $\varepsilon > 0$, the function $\dot\zeta_\varepsilon(t)$ never vanishes in the prescribed interval. We also immediately get, as $\zeta_\varepsilon(t)$ is uniformly bounded, that there exists a positive constant $\tilde R$ not depending on ε such that:

$$\int_{t_{2,\varepsilon}}^{b^1_\varepsilon} \frac{\varrho_\varepsilon(t)}{\varepsilon}\, dt \le \tilde R. \tag{5.10}$$

Now we differentiate the function Ψ along the trajectories and we get

$$\begin{aligned}
\frac{d}{dt}\Psi(\sigma_\varepsilon(t), \zeta_\varepsilon(t)) &= \nabla\Psi(\sigma_\varepsilon(t), \zeta_\varepsilon(t)) \cdot (\dot\sigma_\varepsilon(t), \dot\zeta_\varepsilon(t)) \\
&= \nabla\Psi(\sigma_\varepsilon(t), \zeta_\varepsilon(t)) \cdot (\mathbb{C}\dot\xi(t), 0) + \\
&\quad + \dot\zeta_\varepsilon(t)\nabla\Psi(\sigma_\varepsilon(t), \zeta_\varepsilon(t)) \cdot (-\tfrac{\mathbb{C}(\dot\xi(t) - \dot\sigma_\varepsilon(t))}{\dot\zeta_\varepsilon(t)}, 1) \\
&= \nabla\Psi(\sigma_\varepsilon(t), \zeta_\varepsilon(t)) \cdot (\mathbb{C}\dot\xi(t), 0) + \\
&\quad + \dot\zeta_\varepsilon(t)\nabla\Psi(\sigma_\varepsilon(t), \zeta_\varepsilon(t)) \cdot (-\tfrac{g(\zeta_\varepsilon(t))\mathbb{C}n_\sigma(\pi(\sigma_\varepsilon(t), \zeta_\varepsilon(t)))}{n_\zeta(\pi(\sigma_\varepsilon(t), \zeta_\varepsilon(t)))}, 1);
\end{aligned}$$

this equality, together with (5.9) and (5.3), implies that there exist two positive constants L and R such that

$$\frac{d}{dt}\Psi(\sigma_\varepsilon(t), \zeta_\varepsilon(t)) \ge R\frac{\varrho_\varepsilon(t)}{\varepsilon} - L|\mathbb{C}||\dot\xi(t)| \qquad \text{for every } t \in (t_{2,\varepsilon}, b^1_\varepsilon); \tag{5.11}$$

this will be crucial in the remainder of the proof.

We denote with M_ξ the supremum of $|\dot\xi(t)|$ in $(t_2-\delta_1, t_2+\delta_1))$, and we fix $0<\eta<\frac{R\gamma_2}{4L|\mathbb{C}|M_\xi}$, where γ_2 is the constant given by (3.5). For ε small enough, by the definition of $t_{2,\varepsilon}$, we shall have

$$\Psi(\sigma_\varepsilon(t_{2,\varepsilon}), \zeta_\varepsilon(t_{2,\varepsilon})) \geq -\eta.$$

We then define:

$$t^1_{2,\varepsilon} := \inf\{\in (t_{2,\varepsilon}, b^1_\varepsilon) :\ \frac{\varrho_\varepsilon(t)}{\varepsilon} \geq \frac{\gamma_2}{4\eta}\}$$
$$t^2_{2,\varepsilon} := \inf\{t \in (t_{2,\varepsilon}, b^1_\varepsilon) :\ \Psi(\sigma_\varepsilon(t), \zeta_\varepsilon(t)) \leq -2\eta\}.$$

Now, let $\tilde t_{2,\varepsilon} := t^1_{2,\varepsilon} \wedge t^2_{2,\varepsilon} \wedge b^1_\varepsilon$; exploiting (2.19), the same argument used to prove (3.10) shows that $\tilde t_{2,\varepsilon} \to t_2$ when ε goes to 0. First, we prove that $\tilde t_{2,\varepsilon} < b^1_\varepsilon$; to get this, we show that

$$(\sigma_\varepsilon(t), \zeta_\varepsilon(t)) \to (\sigma_2, \zeta_2) \tag{5.12}$$

uniformly for every $t \in [t_{2,\varepsilon}, \tilde t_{2,\varepsilon}]$. To do this, it suffices to proceed as in the proof of (3.11); as $\frac{\varrho_\varepsilon(t)}{\varepsilon}$ is equibounded in the time interval $[t_{2,\varepsilon}, \tilde t_{2,\varepsilon}]$, we may prove that

$$\lim_{\varepsilon\to 0} |(\sigma_\varepsilon(t), \zeta_\varepsilon(t)) - (\sigma_\varepsilon(t_{2,\varepsilon}), \zeta_\varepsilon(t_{2,\varepsilon}))| = 0$$

and (5.12) follows. Next we show that for small ε one has $\tilde t_{2,\varepsilon} < t^2_{2,\varepsilon}$, which in his turn implies $\tilde t_{2,\varepsilon} = t^1_{2,\varepsilon}$, so $t^1_{2,\varepsilon} \to t_2$ when ε goes to 0 and

$$\Psi(\sigma_\varepsilon(t^1_{2,\varepsilon}), \zeta_\varepsilon(t^1_{2,\varepsilon})) > -2\eta. \tag{5.13}$$

In fact if along some infinitesimal subsequence $\tilde t_{2,\varepsilon} = t^2_{2,\varepsilon}$, that is to say $\Psi(\sigma_\varepsilon(\tilde t_{2,\varepsilon}), \zeta_\varepsilon(\tilde t_{2,\varepsilon})) = -2\eta$, then, integrating (5.11), since the function $\varrho_\varepsilon(t)$ is positive we get

$$-\eta = \Psi(\sigma_\varepsilon(t^2_{2,\varepsilon}), \zeta_\varepsilon(t^2_{2,\varepsilon})) - \Psi(\sigma_\varepsilon(t_{2,\varepsilon}), \zeta_\varepsilon(t_{2,\varepsilon})) \geq -L|\mathbb{C}| \int_{t_{2,\varepsilon}}^{t^2_{2,\varepsilon}} |\dot\xi(t)|\, dt$$

which in the limit gives, by absolute continuity of the integral, that $-\eta \geq 0$, a contradiction.

So (5.13) holds, and $\frac{\varrho_\varepsilon(t^1_{2,\varepsilon})}{\varepsilon} = \frac{\gamma_2}{4\eta}$. Actually, we have

$$\frac{\varrho_\varepsilon(t)}{\varepsilon} > \frac{\gamma_2}{4\eta} \qquad \text{for every } t \in (t^1_{2,\varepsilon}, b^1_\varepsilon). \tag{5.14}$$

We prove this by contradiction. We observe that $\dot\varrho_\varepsilon(t^1_{2,\varepsilon}) \geq -\frac{\gamma_2}{2} + \gamma_2 > 0$, by (2.19) and (3.5). If (5.14) is false, let $t^3_{2,\varepsilon}$ be the first time in $(t^1_{2,\varepsilon}, b^1_\varepsilon)$ such that $\varrho_\varepsilon(t^3_{2,\varepsilon}) = \frac{\gamma_2}{4\eta}$; then $\dot\varrho_\varepsilon(t^3_{2,\varepsilon}) \leq 0$. But, by (5.11), for every $t \in (t^1_{2,\varepsilon}, t^3_{2,\varepsilon})$ we shall have $\frac{d}{dt}\Psi(\sigma_\varepsilon(t), \zeta_\varepsilon(t)) \geq 0$, hence, by (5.13),

$$\Psi(\sigma_\varepsilon(t), \zeta_\varepsilon(t) > -2\eta \qquad \text{for every } t \in (t^1_{2,\varepsilon}, t^3_{2,\varepsilon});$$

by this, (3.5), and (2.19) we infer that $\dot\varrho_\varepsilon(t^3_{2,\varepsilon}) \geq \gamma_2 - \frac{\gamma_2}{4} > 0$, which is a contradiction. Then, by (5.14) and (5.10), for ε sufficiently small we conclude that

$$\gamma_2(b^1_\varepsilon - t^1_{2,\varepsilon}) \leq 4\eta\tilde R; \tag{5.15}$$

as $t^1_{2,\varepsilon} - t_{2,\varepsilon} \to 0$, we get that $\limsup_{\varepsilon\to 0} \gamma_2(b^1_\varepsilon - t_{2,\varepsilon}) \leq 4\eta\tilde R$, and by the arbitrariness of η, (5.8) follows, so part a) of the statement is proved.

Concerning part b), we fix $k \in \mathbb{N}$, $k>1$. By the definition of b^k_ε and $t_{2,\varepsilon}$ we shall have, for any $t \in [b^k_\varepsilon, b^1_\varepsilon]$, that

$$\delta_k + o(1) = |(\sigma_\varepsilon(t), \zeta_\varepsilon(t) - (\sigma_\varepsilon(t_{2,\varepsilon}), \zeta_\varepsilon(t_{2,\varepsilon}))|;$$

it follows, proceeding as in (4.18), that there exists a positive constant W such that

$$\delta_k + o(1) \leq W(\int_{t_{2,\varepsilon}}^t \frac{\varrho_\varepsilon(s)}{\varepsilon}\, ds + \int_{t_{2,\varepsilon}}^t |\dot\xi(s)|\, ds). \tag{5.16}$$

This in turn implies, by (5.11) and the fundamental theorem of calculus that, up to redefining the constant W,

$$\delta_k + o(1) \leq W[\Psi(\sigma_\varepsilon(t), \zeta_\varepsilon(t)) - \Psi(\sigma_\varepsilon(t_{2,\varepsilon}), \zeta_\varepsilon(t_{2,\varepsilon})) + \int_{t_{2,\varepsilon}}^{t} |\dot{\xi}(s)|\, ds]. \tag{5.17}$$

By the definition of $t_{2,\varepsilon}$, $\Psi(\sigma_\varepsilon(t_{2,\varepsilon}), \zeta_\varepsilon(t_{2,\varepsilon})) = o(1)$; the absolute continuity of the integral and part a) of the statement now yield that, for ε small enough,

$$\Psi(\sigma_\varepsilon(t), \zeta_\varepsilon(t)) \geq \frac{\delta_k}{2W} \qquad \text{for every } t \in [b_\varepsilon^k, b_\varepsilon^1].$$

Substituting in (2.19), this gives

$$\dot{\varrho}_\varepsilon(t) \geq \frac{\delta_k}{2W}\frac{\varrho_\varepsilon(t)}{\varepsilon} \qquad \text{for every } t \in [b_\varepsilon^k, b_\varepsilon^1], \tag{5.18}$$

and we conclude that, for ε small enough

$$\frac{b_\varepsilon^1 - b_\varepsilon^k}{\varepsilon} \leq \frac{2W}{\delta_k}\log(\frac{\varrho_\varepsilon(b_\varepsilon^1)}{\varrho_\varepsilon(b_\varepsilon^k)}) \leq \frac{2W}{\delta_k}\log(\frac{\delta_1}{\varrho_\varepsilon(b_\varepsilon^k)}). \tag{5.19}$$

To get a lower bound for $\varrho_\varepsilon(b_\varepsilon^k)$ we observe that a fortiori (5.18) holds, with δ_{k+1} in place of δ_k, for any $t \in [b_\varepsilon^{k+1}, b_\varepsilon^k]$. Since clearly

$$\delta_k - \delta_{k+1} \leq |(\sigma_\varepsilon(b_\varepsilon^k), \zeta_\varepsilon(b_\varepsilon^k) - (\sigma_\varepsilon(b_\varepsilon^{k+1}), \zeta_\varepsilon(b_\varepsilon^{k+1})|,$$

proceeding as in (5.16), we obtain that there exists a positive constant $\tilde{W}$ such that

$$\delta_k - \delta_{k+1} \leq \tilde{W}(\int_{b_\varepsilon^{k+1}}^{b_\varepsilon^k} \frac{\varrho_\varepsilon(s)}{\varepsilon}\, ds + \int_{b_\varepsilon^{k+1}}^{b_\varepsilon^k} |\dot{\xi}(s)|\, ds). \tag{5.20}$$

Applying (5.18), with δ_{k+1} in place of δ_k, and the fundamental theorem of calculus, and neglecting the negative term $-\varrho_\varepsilon(b_\varepsilon^{k+1})$, we get, up to redefining the constant $\tilde{W}$, that for ε small enough

$$\varrho_\varepsilon(b_\varepsilon^k) \geq (\delta_k - \delta_{k+1})\frac{\delta_{k+1}}{2\tilde{W}} := m_k,$$

and conclusion then follows. □

We may now repeat the same argument of Theorem 4.4, which yields the main result of this section.

Theorem 5.2. *Let t_2, σ_2 and ζ_2, be as in (3.19), and (3.31), respectively. Suppose $t_2 < +\infty$ and assume (5.1) and (5.2). Let δ_1 be as in (3.5) and let b_ε^1 be given by (5.6). For every $s \in \mathbb{R}$, let $(\sigma_\varepsilon^1(s), \zeta_\varepsilon^1(s)) := (\sigma_\varepsilon(b_\varepsilon^1 + \varepsilon s), \zeta_\varepsilon(b_\varepsilon^1 + \varepsilon s))$. Then $(\sigma_\varepsilon^1(s), \zeta_\varepsilon^1(s))$ converges uniformly on compact subsets of $\mathbb{R}$ to a solution of the problem:*

$$\begin{cases} -\dot{\sigma}(s) = \mathbb{C}[\sigma(s) - \pi_\sigma(\sigma(s), \zeta(s))], \\ g(\zeta)\dot{\zeta}(s) = \zeta(s) - \pi_\zeta(\sigma(s), \zeta(s)), \\ \lim\limits_{s\to-\infty}(\sigma(s), \zeta(s)) = (\sigma_2, \zeta_2) \end{cases} \tag{5.21}$$

whose existence and uniqueness, up to time translations, is guaranteed by Theorem 4.1.

Remark 5.3. In a right neighborhood of t_2 the behavior of the system can be elastic or follow the slow dynamics equation exactly as discussed in Remark 4.5.

6. **An example.**

We conclude the paper by applying our results to a concrete example, where our choice of the data makes the expressions of the two indicators Φ and Ψ particularly simple; this allows us to determine a priori that the soluton has only one jump and to characterize the endpoint of the jump; also the slow-dynamics equation (3.18) takes in this case a simpler form.

Example 6.1. We consider a round cone K of the form

$$K := \{(\sigma,\zeta) \in \mathbb{M}^{N\times N}_D \times [0,+\infty) : |\sigma|^2 \leq \zeta^2\} \tag{6.1}$$

whose exterior normal is given, for every $(\sigma,\zeta) \in \partial K \setminus \{0,0\}$ by

$$n_K(\sigma,\zeta) = \tfrac{1}{\sqrt{2}}(\tfrac{\sigma}{\zeta}, -1). \tag{6.2}$$

We take $g(\zeta) = \frac{1}{(\zeta-1)f(\zeta)}$, where $f : [1,4) \to \mathbb{R}_+$ satisfies

$$f(\zeta) > 0 \text{ for every } \zeta \in [1,4) \tag{6.3}$$
$$f(2) = 1 \text{ and } f'(2) > -1; \tag{6.4}$$
$$f(3) = \tfrac{1}{2} \text{ and } f'(3) < -\tfrac{1}{4}; \tag{6.5}$$
$$f(\zeta) \neq \tfrac{1}{\zeta-1} \text{ for every } \zeta \in (1,4)\setminus\{2,3\}. \tag{6.6}$$

Such an f may be, for instance , the function $\frac{4-\zeta}{2}$.

We take the elasticity tensor equal to the identity tensor, and we consider a boundary datum of the form $\xi(t) = t\xi_0$, where $\xi_0 \neq 0 \in \mathbb{M}^{N\times N}_D$ and $|\xi_0| = 1$. We identify $\mathbb{M}^{N\times N}_D$ with $\mathbb{R}^m$, where $m = \frac{N(N+1)}{2} - 1$ in a way that ξ_0 is identified with the first vector of the canonical basis of $\mathbb{R}^m$, so $\xi(t) = (t,0,\ldots,0)$ and $\sigma \in \mathbb{M}^{N\times N}_D$ is represented by the m-uple $(\sigma_1,\ldots,\sigma_m)$.

As $\mathbb{C}$ is the identity, we have that $\Psi(\sigma,\zeta) = (1+\frac{1}{g(\zeta)})n^2_\zeta(\pi(\sigma,\zeta)) - 1$; in our particular case, this yields that the slow-fast indicator only depends on ζ and is of the form

$$\Psi(\zeta) = \tfrac{1}{2g(\zeta)} - \frac{1}{2}. \tag{6.7}$$

It follows from (6.4), (6.5), and (6.6) that $\Psi(\zeta) = 0$ if and only if $\zeta = 2$ or $\zeta = 3$, and that we have $\Psi'(2) > 0$, and $\Psi'(3) < 0$.

The elastic-inelastic indicator does not depend explicitly on t, as $\xi(t)$ is linear, and it is given by the formula

$$\Phi(\sigma,\zeta) = \tfrac{1}{\sqrt{2}}\tfrac{\sigma_1}{\zeta}. \tag{6.8}$$

Now we fix the initial condition $\sigma(0) = 0$, $\zeta(0) = \zeta_0 \in (3,4)$. At small times the evolution will obviously follow the elastic regime, thus being of the form $(\sigma(t),\zeta(t)) = (t,0,\ldots,0,\zeta_0)$, till at time $t_0 := \zeta_0$ it reaches ∂K at the point $(\zeta_0,0,\ldots,0,\zeta_0)$. At this time the elastic-inelastic indicator is strictly positive, while the slow-fast indicator has negative sign. By the results of Section 3, the solution then starts to follow the slow dynamics equation (3.18) with initial datum $(\sigma(t_0),\zeta(t_0)) = (\zeta_0,0,\ldots,0,\zeta_0)$. Moreover, as we have observed in Section 3, since the boundary datum is linear, we a priori know that the solution remains in the slow dynamics regime for all the maximal interval of definition of (3.18), which we shall denote by $[t_0,t_1)$. Equation (3.18) in our case takes the form

$$\begin{cases} 1 - \dot{\sigma_1}(t) = -\frac{\sigma_1(t)}{\zeta(t)}\frac{\sigma_1(t)}{2\zeta(t)\Psi(\zeta(t))}, \\ -\dot{\sigma_i}(t) = -\frac{\sigma_1(t)}{\zeta(t)}\frac{\sigma_i(t)}{2\zeta(t)\Psi(\zeta(t))}, & \text{for } i = 2,\ldots,m \\ \dot{\zeta}(t) = \frac{1}{g(\zeta(t))}\frac{\sigma_1(t)}{2\zeta(t)\Psi(\zeta(t))}; \end{cases} \tag{6.9}$$

by the equality $\frac{1}{g(\zeta)} = 2\Psi(\zeta)+1$, we easily see that the solution to (6.9) with the prescribed initial datum is given by $(\sigma(t),\zeta(t)) = (\zeta(t),0,\ldots,0,\zeta(t))$ where $\zeta(t)$ is the unique solution of the Cauchy problem

$$\begin{cases} \dot\zeta(t) = \frac{(\zeta(t)-1)f(\zeta(t))}{(\zeta(t)-1)f(\zeta(t))-1} \\ \zeta(t_0) = \zeta_0; \end{cases} \tag{6.10}$$

here we have inserted in place of g and Ψ their explicit expressions.

Observe now that, due to the choice of ζ_0 and the fact that $(\zeta-1)f(\zeta)-1<0$ when $\zeta\in(3,4)$, we have that $3<\zeta(t)<4$, and $\dot\zeta(t)<0$, for every $t\in[t_0,t_1)$; by the theory of autonomous equations this implies that $t_1<+\infty$ and that $\lim_{t\to t_1^-}\zeta(t)=3$, as $(\zeta-1)f(\zeta)-1$ vanishes for $\zeta=3$. Now we want to show that a jump occurs at time t_1. By the results of Section 5, we only have to show that (5.2) holds. Indeed, in our case, as the slow-fast indicator only depends on the ζ coordinate, (5.2) reduces to: $\Psi'(3)<0$, which is, as we already noticed, satisfied in our case.

To determine the endpoint of the jump, we consider the equation of the trajectories

$$-\sigma'(\zeta) = \frac{g(\zeta)}{n_\zeta(\pi_1(\sigma(\zeta),\zeta))}\, n_\sigma(\pi_1(\sigma(\zeta),\zeta)), \tag{6.11}$$

with initial datum $\sigma(3)=(3,0,\dots,0)$; here, as in Section 4.1, π_1 is the minimal distance projection to ∂K. Let $(\tilde\zeta,3)$ be the left maximal interval of definition of (6.11); clearly $\tilde\zeta\geq 1$. Recall that every solution of (6.11) with $(\sigma(\zeta),\zeta)\notin K$ produces a solution of (4.7), and that condition (5.2), as seen in Theorem 4.1, implies that for ζ sufficiently close to 3, $(\sigma(\zeta),\zeta)\notin K$. This yields that the endpoint of the jump, which we shall denote with $(\sigma_\infty,\zeta_\infty)$ as in Theorem 4.1 may be carachterized as follows:

$$\zeta_\infty := \sup\{\zeta\in(\tilde\zeta,3) | (\sigma(\zeta),\zeta)\in\partial K\} \text{ and } \sigma_\infty=\sigma(\zeta_\infty). \tag{6.12}$$

But, as $n\circ\pi_1$ leaves the plane $\{\sigma_2=\sigma_3=\dots=\sigma_m=0\}$ invariant, the solution to (6.11) is given, in our case, by $\sigma(\zeta)=(\sigma_1(\zeta),0,\dots,0)$ where $\sigma_1(\zeta)$ solves the Cauchy problem

$$\begin{cases} -\sigma_1'(\zeta) = \frac{g(\zeta)}{n_\zeta(\pi_1(\sigma_1(\zeta),0,\dots,0,\zeta))}\, n_1(\pi_1(\sigma_1(\zeta),0,\dots,0,\zeta)), \\ \sigma_1(3)=3, \end{cases}$$

where n_1 denotes the first component of the normal; this equation in our case, by a direct computation, simply reduces to

$$\begin{cases} \sigma_1'(\zeta) = g(\zeta) \\ \sigma_1(3)=3. \end{cases} \tag{6.13}$$

Now, as only the first component of σ is nonzero along the trajectory, (6.12) becomes

$$\zeta_\infty := \sup\{\zeta\in(\tilde\zeta,3) | |\sigma_1(\zeta)|=\zeta\}. \tag{6.14}$$

Moreover, condition $\Psi'(3)<0$ implies that for ζ sufficiently close to 3, $\sigma_1(\zeta)>\zeta$. Indeed, let $\theta(\zeta):=\sigma_1(\zeta)-\zeta$; by (6.13), recalling that $g(3)=1$, we have that $\theta(3)=\theta'(3)=0$; taking one more derivative, by the equality $\frac{1}{g(\zeta)}=2\Psi(\zeta)+1$, we get that $\theta''(3)$ has the opposite sign of $\Psi'(3)$, thus is strictly positive. Therefore $\theta(\zeta)$ has a minimum in $\zeta=3$. From this and (6.14), we get that

$$\zeta_\infty := \sup\{\zeta\in(\tilde\zeta,3) | \sigma_1(\zeta)=\zeta\}; \tag{6.15}$$

by (6.13), we conclude that

$$\zeta_\infty := \sup\{\zeta\in(1,3) | \int_\zeta^3 g(\eta)\,d\eta = 3-\zeta\}. \tag{6.16}$$

By (4.5) we also know that $\Psi(\zeta_\infty)\leq 0$; our hypotheses on Ψ then imply that $\zeta_\infty\leq 2$: actually, the strict inequality holds. Indeed, let $\theta(\zeta):=\sigma_1(\zeta)-\zeta$; if we suppose $\zeta_\infty=2$, by (6.13), recalling that $g(2)=1$, we have that $\theta(2)=\theta'(2)=0$; taking one more derivative, by the equality $\frac{1}{g(\zeta)}=2\Psi(\zeta)+1$, we get that $\theta''(2)$ has the opposite sign of $\Psi'(2)$, thus is strictly negative. Therefore $\theta(\zeta)$ has a maximum in $\zeta=2$, thus is negative in a right neighborhood of 2. But, as $\theta(\zeta)$ is positive in a left neighborhood of 3, we get a contradiction with the definition of ζ_∞ as a supremum.

Finally, as Ψ is strictly negative in $[1,\zeta_\infty]$, after the jump the solution follows the slow dynamics at all times, and repeating the same arguments used in the first part of this example, we get that for every $t\in(t_1,+\infty)$ the solution is given by $(\sigma(t),\zeta(t))=(\zeta(t),0,\dots,0,\zeta(t))$ where $\zeta(t)$ is the unique solution of the Cauchy problem

$$\begin{cases}\dot\zeta(t)=\frac{(\zeta(t)-1)f(\zeta(t))}{(\zeta(t)-1)f(\zeta(t))-1}\\ \zeta(t_1)=\zeta_\infty.\end{cases}$$

We summarize all these results in the next Proposition.

Proposition 6.2. *Let K as in* (6.1)*, let $g(\zeta)=\frac{1}{(\zeta-1)f(\zeta)}$, where $f:[1,4)\to\mathbb{R}_+$ satisfies* (6.3)-(6.6)*. Suppose that the elasticity tensor $\mathbb{C}$ is equal to the identity tensor, and consider a boundary datum of the form $\xi(t)=t\xi_0$, where $\xi_0\neq 0\in\mathbb{M}^{N\times N}_D$ and $|\xi_0|=1$. Fix the initial condition $\sigma(0)=0$, $\zeta(0)=\zeta_0\in(3,4)$. Let $t_0=\zeta_0$, let ζ_∞ as in* (6.16)*, and let $\zeta_1(t)$, and $\zeta_2(t)$ be the solutions of the Cauchy problems*

$$\begin{cases}\dot\zeta_1(t)=\frac{(\zeta_1(t)-1)f(\zeta_1(t))}{(\zeta_1(t)-1)f(\zeta_1(t))-1}\\ \zeta_1(t_0)=\zeta_0,\end{cases}\tag{6.17}$$

and

$$\begin{cases}\dot\zeta_2(t)=\frac{(\zeta_2(t)-1)f(\zeta_2(t))}{(\zeta_2(t)-1)f(\zeta_2(t))-1}\\ \zeta_2(t_1)=\zeta_\infty,\end{cases}\tag{6.18}$$

respectively; here $[t_0,t_1)$ is the maximal interval of existence for the solution of (6.17)*. Define*

$$\sigma(t)=\begin{cases}t\xi_0 \text{ for } t\in[0,t_0),\\ \zeta_1(t)\xi_0 \text{ for } t\in[t_0,t_1),\\ \zeta_2(t)\xi_0 \text{ for } t\in(t_1,+\infty)\end{cases}\qquad \zeta(t)=\begin{cases}\zeta_0(t) \text{ for } t\in[0,t_0),\\ \zeta_1(t) \text{ for } t\in[t_0,t_1),\\ \zeta_2(t) \text{ for } t\in(t_1,+\infty).\end{cases}$$

Then the solutions of (2.8) *with the prescribed initial condition converge to $(\sigma(t),\zeta(t)$ uniformly on compact subsets of $[0,t_1)\cup(t_1,+\infty)$. Moreover, $\zeta_\infty<2$.*

Remark 6.3. Notice that, according to Remark 3.7, the function $(\sigma(t),\zeta(t))$ defined in the previous proposition satifies (1.2) as long $\Psi(\sigma(t),\zeta(t))<0$; this makes a jump at time t_1 necessary to forbid that the internal variable ζ enters the interval $(2,3)$ where Ψ is positive.

Acknowledgments. The author wishes to thank prof G. Dal Maso for suggesting the problem and for useful advice.

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