Asymptotics and quantization for a mean-field equation of higher order

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Abstract

Given a regular bounded domain $\Omega \subset \mathbb{R}^{2m}$, we describe the limiting behavior of sequences of solutions to the mean field equation of order $2m$, $m \geq 1$,

$$(-\Delta)^m u = \rho \frac{e^{2mu}}{\int_\Omega e^{2mu} dx} \quad \text{in } \Omega,$$

under the Dirichlet boundary condition and the bound $0 < \rho \leq C$. We emphasize the relationship to the problem of prescribing the Q-curvature.

1 Introduction

Let $\Omega \subset \mathbb{R}^{2m}$ be a bounded domain with smooth boundary. Given a sequence of numbers $\rho_k > 0$, we consider solutions to the mean-field equation of higher order

$$(-\Delta)^m u_k = \rho_k \frac{e^{2mu_k}}{\int_\Omega e^{2mu_k} dx} \tag{1}$$

subject to the Dirichlet boundary condition

$$u_k = \partial_\nu u_k = \ldots = \partial_\nu^{m-1} u_k = 0 \quad \text{on } \partial\Omega. \tag{2}$$

As shown in Corollary 8 of [Mar1], every u_k is smooth. In this paper we study the limiting behavior of the sequence (u_k). We show that concentration-compactness phenomena together with geometric quantization occur. We particularly emphasize the interesting relationship with the thriving problem of prescribing the Q-curvature.

For any $\xi \in \overline{\Omega}$, let $G_\xi(x)$ denote the Green function of the operator $(-\Delta)^m$ on Ω with Dirichlet boundary condition (see e.g. [ACL]), i.e

$$\begin{cases} (-\Delta)^m G_\xi = \delta_\xi & \text{in } \Omega \\ G_\xi = \partial_\nu G_\xi = \ldots = \partial_\nu^{m-1} G_\xi = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

Also fix any $\alpha \in [0,1)$. We then have

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Theorem 1 *Let u_k be a sequence of solutions to* (1), (2) *and assume that*

$$0 < \rho_k \leq C.$$

Then one of the following is true:

(i) Up to a subsequence $u_k \to u_0$ in $C^{2m-1,\alpha}(\overline{\Omega})$ for some $u_0 \in C^\infty(\overline{\Omega})$.

(ii) Up to a subsequence, $\lim_{k\to\infty} \max_\Omega u_k = \infty$ and there is a positive integer N such that

$$\lim_{k\to\infty} \rho_k = N\Lambda_1, \quad \Lambda_1 = (2m-1)!|S^{2m}|. \tag{4}$$

Moreover there exists a non-empty finite set $S = \{x^{(1)}, \ldots, x^{(N)}\} \subset \Omega$ such that

$$u_k \to \Lambda_1 \sum_{i=1}^N G_{x^{(i)}} \quad \text{in } C^{2m-1,\alpha}_{\mathrm{loc}}(\overline{\Omega}\backslash S). \tag{5}$$

The mean field equation in dimensions 2 and 4 has been object of intensive study in the recent years. We refer e.g. to [NS], [Wei], [RW] and the references therein. In particular in [RW] the 4-dimensional analogous of our Theorem 1 was proven, and many of the ideas developed there are used in our treatment.

The geometric constant Λ_1 showing up in (4) and (5) is the total Q-curvature[1] of the round $2m$-dimensional sphere. It is worth explaining how this relation with Riemannian geometry arises. It will be shown in Lemma 6 below that one can blow up the u_k's at suitably chosen *concentration points*, and get in the limit a solution u_0 to the Liouville equation

$$(-\Delta)^m u_0 = (2m-1)! e^{2mu_0} \quad \text{in } \mathbb{R}^{2m} \tag{6}$$

with the bound

$$\int_{\mathbb{R}^{2m}} e^{2mu_0} dx < \infty. \tag{7}$$

Geometrically, if u_0 solves (6)-(7), then the conformal metric $e^{2u_0} g_{\mathbb{R}^{2m}}$ on $\mathbb{R}^{2m}$, where $g_{\mathbb{R}^{2m}}$ is the Euclidean metric, has constant Q-curvature equal to $(2m-1)!$ and finite volume. As shown in [CC], there are many such conformal metrics on $\mathbb{R}^{2m}$, and the crucial step in Lemma 6 below is to show that

$$u_0(x) = \eta_0(x) =: \log\left(\frac{2}{1+|x|^2}\right). \tag{8}$$

The function η_0 has the property that $e^{2\eta_0} g_{\mathbb{R}^{2m}} = (\pi^{-1})^* g_{S^{2m}}$, where $g_{S^{2m}}$ is the round metric on S^{2m}, and $\pi : S^{2m} \to \mathbb{R}^{2m}$ is the stereographic projection. This is the basic reason why the constant Λ_1 appears in Theorem 1. In particular

$$\int_{\mathbb{R}^{2m}} e^{2m\eta_0} dx = |S^{2m}|. \tag{9}$$

In order to show that (8) holds, we use the classification result of [Mar1] and a technique of [RS], which allows us to rule out all the solutions of (6) which are "non-spherical", hence whose total Q-curvature might be different from Λ_1.

[1] For the definition of Q-curvature we refer to [Cha], or to the introduction of [Mar1] and the references therein.

We can further exploit the connection with conformal geometry by referring to Theorem 1 in [Mar2], about the concentration-compactness phenomena for sequences of conformal metrics on $\mathbb{R}^{2m}$ with prescribed Q-curvature (compare [BM], [ARS] and [Rob] for 2 and 4-dimensional analogous results). We state a simplified version of this theorem in the appendix, since we shall use it several times.

The last crucial ingredient in the proof of Theorem 1 is a Pohozaev-type inequality which we discuss in the Appendix, and which we use in Lemma 11 and in Lemma 12 below.

One can also state Theorem 1 as an eigenvalue problem, as in [Wei]. In this case one replaces $\frac{\rho_k}{\int_\Omega e^{2mu_k}}$ by $\lambda_k > 0$ in (1) to get

$$(-\Delta)^m u_k = \lambda_k e^{2mu_k}. \tag{10}$$

The assumption $0 < \rho_k \leq C$ gets replaced by

$$\Sigma_k := \int_\Omega \lambda_k e^{2mu_k} dx \leq C, \tag{11}$$

and the boundary condition (2) still holds. Then Theorem 1 implies that either

(i) up to a subsequence $u_k \to u_0$ in $C^{2m-1,\alpha}_{\mathrm{loc}}(\overline{\Omega})$, or

(ii) up to a subsequence $\Sigma_k \to N\Lambda_1$ and (u_k) satisfies (5), with the same notation of Theorem 1.

Several times we use standard elliptic estimates. For the interior estimates one can safely rely on [GT] or [GM]. For the estimates up to the boundary, one can refer to [ADN]. Throughout the paper the letter C denotes a large universal constant which does not depend on k and can change from line to line, or even within the same line.

2 Proof of Theorem 1

The proof will be organized as follows. We shall see in Corollary 3, that if $\sup_\Omega u_k \leq C$, then u_k is bounded in $C^{2m-1,\alpha}(\overline{\Omega})$ and case (i) of Theorem 1 occurs. Then, after Corollary 3 we shall assume that

$$\lim_{k\to\infty} \sup_\Omega u_k = \infty, \tag{12}$$

and prove that case (ii) of Theorem 1 occurs. Let

$$\alpha_k := \frac{1}{2m} \log\left(\frac{(2m-1)! \int_\Omega e^{2mu_k} dx}{\rho_k}\right), \quad \hat{u}_k := u_k - \alpha_k. \tag{13}$$

Lemma 2 *Up to selecting a subsequence, we have $\alpha_k \geq -C$.*

Proof. Indeed

$$(-\Delta)^m \hat{u}_k = (2m-1)! e^{2m\hat{u}_k} \quad \text{in } \Omega \tag{14}$$

and

$$\hat{u}_k = -\alpha_k, \quad \partial_\nu \hat{u}_k = \ldots = \partial_\nu^{m-1} \hat{u}_k = 0 \quad \text{on } \partial\Omega.$$

Moreover

$$\int_\Omega e^{2m\hat{u}_k} dx = \frac{\rho_k}{(2m-1)!} \leq C. \tag{15}$$

Using the Green's representation formula, we infer

$$\hat{u}_k(x) = (2m-1)! \int_\Omega G_x(y) e^{2m\hat{u}_k(y)} dy - \alpha_k. \tag{16}$$

Then, integrating (16), using (15), the fact that $\|G_y\|_{L^1(\Omega)} \leq C$, with C independent of y, and the symmetry of G, i.e. $G_x(y) = G_y(x)$, we get

$$\int_\Omega |\hat{u}_k + \alpha_k| dx \leq C. \tag{17}$$

Now, according to Theorem 13 in the Appendix, we have that one of the following is true:

(i) $\hat{u}_k \to \hat{u}_0$ in $C^{2m-1,\alpha}_{\text{loc}}(\Omega)$ for some function u_0.

(ii) $\hat{u}_k \to -\infty$ locally uniformly in $\Omega \backslash \Omega_0$, for some closed nowhere dense (possibly empty) set Ω_0 of Hausdorff dimension at most $2m-1$.

In both cases the claim of the lemma easily follows from (17). □

Corollary 3 *The following facts are equivalent:*

(i) Up to selecting subsequences, $u_k \leq C$.

(ii) Up to selecting subsequences, $\hat{u}_k \leq C$.

(iii) Up to selecting subsequences, $u_k \to u_0$ in $C^{2m-1,\alpha}(\overline{\Omega})$ for some smooth function u_0.

Proof. (i) ⇒ (ii) follows at once from Lemma 2.

(ii) ⇒ (iii) follows by elliptic estimates, observing that

$$|(-\Delta)^m u_k| = |(-\Delta)^m \hat{u}_k| = \left|(2m-1)! e^{2m\hat{u}_k}\right| \leq C$$

and using (2).

(iii)⇒ (i) is obvious. □

Lemma 4 *For all $\ell \in \{1, \ldots, 2m-1\}$ and for $p \in [1, \frac{2m}{\ell})$, there exists $C = C(\ell, p)$ such that*

$$\int_{B_R(x_0)} |\nabla^\ell \hat{u}_k|^p dx \leq C R^{2m-ip}, \tag{18}$$

for any $B_R(x_0) \subset \Omega$.

Proof. We prove the claim by duality. Let $\varphi \in C_c^\infty(\Omega)$ and $q = \frac{p}{p-1}$. Differentiating (16), using Fubini's theorem, the relation $G_x(y) = G_y(x)$ and the estimate (see [DAS])

$$|\nabla^\ell G_y(x)| \le \frac{C}{|x-y|^\ell}, \tag{19}$$

we get

$$\begin{aligned}
\int_{B_R(x_0)} |\nabla^\ell \hat{u}_k| \varphi dx &\le C \int_{B_R(x_0)} \left(\int_\Omega \left| \nabla^\ell G_y(x) \right| e^{2m\hat{u}_k(y)} dy \right) |\varphi(x)|\, dx \\
&\le C \int_\Omega e^{2m\hat{u}_k(y)} \left(\int_{B_R(x_0)} |x-y|^{-\ell}\, |\varphi(x)|\, dx \right) dy \\
&\le C \|\varphi\|_{L^q(\Omega)} \int_\Omega e^{2m\hat{u}_k(y)} \left(\int_{B_R(x_0)} \frac{dx}{|x-y|^{\ell p}} \right)^{\frac{1}{p}} dy \\
&\le C \|\varphi\|_{L^q(\Omega)} R^{\frac{2m}{p}-\ell},
\end{aligned}$$

where in the last inequality we used $p < \frac{2m}{\ell}$, (15), and the simple estimate

$$\int_{B_R(x_0)} \frac{dx}{|x-y|^{\ell p}} \le \int_{B_R(y)} \frac{dx}{|x-y|^{\ell p}} \le C R^{2m-\ell p}.$$

The lemma follows at once. $\square$

Lemma 5 *Let $x_k \in \Omega$ be such that*

$$u_k(x_k) = \max_\Omega u_k \to \infty. \tag{20}$$

Let $\mu_k := 2e^{-\hat{u}_k(x_k)}$. Then $\frac{\mathrm{dist}(x_k, \partial\Omega)}{\mu_k} \to +\infty$.

Proof. Suppose that the conclusion of the lemma is false. Then the rescaled sets

$$\Omega_k := \tfrac{1}{\mu_k}(\Omega - x_k)$$

converge, up to rotation, to $(-\infty, t_0) \times \mathbb{R}^{2m-1}$ for some $t_0 \ge 0$. Define

$$\tilde{u}_k(x) := \hat{u}_k(x_k + \mu_k x) + \log(\mu_k), \quad x \in \Omega_k. \tag{21}$$

By (20) and Corollary 3 we have $\mu_k \to 0$. Fix $R > 0$ such that $B_R(0) \cap \partial\Omega_k \ne \emptyset$, and let $x \in B_R(0) \cap \Omega_k$. Then, for $1 \le \ell \le 2m-1$, using (16) and (19), we get

$$\begin{aligned}
\left| \nabla^\ell \tilde{u}_k(x) \right| &\le C\mu_k^\ell \int_\Omega |\nabla^\ell G_{x_k+\mu_k x}(y)| e^{2m\hat{u}_k(y)} dy \\
&\le C\mu_k^\ell \left(\int_{\Omega \setminus B_{2R\mu_k}(x_k)} \frac{1}{|x_k + \mu_k x - y|^\ell} e^{2m\hat{u}_k(y)} dy \right. \\
&\quad \left. + \int_{B_{2R\mu_k}(x_k)} \frac{1}{|x_k + \mu_k x - y|^\ell} e^{2m\hat{u}_k(y)} dy \right) \\
&\le CR^{-\ell} \int_\Omega e^{2m\hat{u}_k} dy + C\mu_k^{\ell-2m} \int_{B_{2R\mu_k}(x_k)} \frac{dy}{|x_k + \mu_k x - y|^\ell} \\
&\le C(R),
\end{aligned}$$

where we used that for $y \in \Omega \setminus B_{2R\mu_k}(x_k)$ and $x \in B_R(0) \cap \Omega_k$ we have $R\mu_k \leq |x_k + \mu_k x - y|$ and, for any $y \in \Omega$ we have $e^{2m\hat{u}_k(y)} \leq 2^{2m}\mu_k^{-2m}$. This implies

$$|\tilde{u}_k(x) - \tilde{u}_k(0)| \leq C(R)|x| \quad \text{for } |x| \leq R.$$

Choosing $x \in B_R(0) \cap \partial\Omega_k$ we get $|u_k(x_k)| = |\hat{u}_k(x_k) + \alpha_k| \leq C(R)$, contradicting (20). □

Remark. In the choice of the scales μ_k we are free to some extent. Our particular choice is made in order to give a cleaner form to the blow-up limit described in Lemma 6 and to make the connection with the problem of prescribing the Q-curvature more transparent. •

From now on we shall assume that (12) holds.

Lemma 6 *Let $\tilde{u}_k$ be defined as in* (21)*. Then, up to selecting a subsequence, we have*

$$\lim_{k\to+\infty} \tilde{u}_k(x) = \log\left(\frac{2}{1+|x|^2}\right) \quad \textit{in } C^{2m-1,\alpha}_{\mathrm{loc}}(\mathbb{R}^{2m}). \tag{22}$$

Proof. We give the proof in two steps.

Step 1. We first claim that up to a subsequence, $\tilde{u}_k \to \tilde{u}_0$ in $C^{2m-1,\alpha}_{\mathrm{loc}}(\mathbb{R}^{2m})$, for some smooth function $\tilde{u}_0$ satisfying

$$(-\Delta)^m \tilde{u}_0 = (2m-1)!e^{2m\tilde{u}_0}. \tag{23}$$

Let us first assume $m > 1$. We apply Theorem 13 on $\mathbb{R}^{2m}$ to the sequence $(\tilde{u}_k)$, where it is understood that one has to invade $\mathbb{R}^{2m}$ with bounded sets and extract a diagonal subsequence in order to get the local convergence on all of $\mathbb{R}^{2m}$. Since $\tilde{u}_k \leq \log 2$, we have $S_1 = \emptyset$, in the notation of Theorem 13. Then one of the following is true:

(i) $\tilde{u}_k \to \tilde{u}_0$ in $C^{2m-1,\alpha}_{\mathrm{loc}}(\mathbb{R}^{2m})$ for some function $\tilde{u}_0 \in C^{2m-1,\alpha}_{\mathrm{loc}}(\mathbb{R}^{2m})$, or

(ii-a) $\tilde{u}_k \to -\infty$ locally uniformly in $\mathbb{R}^{2m}$ (case $S_0 = \emptyset$), or

(ii-b) there exists a closed nowhere dense set $S_0 \neq \emptyset$ of Hausdorff dimension at most $2m-1$ and numbers $\beta_k \to \infty$ such that

$$\frac{\tilde{u}_k}{\beta_k} \to \varphi \text{ in } C^{2m-1,\alpha}_{\mathrm{loc}}(\mathbb{R}^{2m} \setminus S_0),$$

where

$$\Delta^m \varphi \equiv 0, \quad \varphi \leq 0, \quad \varphi \not\equiv 0 \text{ on } \mathbb{R}^{2m}, \quad \varphi \equiv 0 \text{ on } S_0. \tag{24}$$

Since $\tilde{u}_k(0) = \log 2$, (ii-a) can be ruled out. Assume now that (ii-b) occurs. From Liouville's theorem and (24), we get $\Delta\varphi \not\equiv 0$, hence for some $R > 0$ we have $\int_{B_R(0)} |\Delta\varphi| dx > 0$ and

$$\lim_{k\to\infty} \int_{B_R} |\Delta \tilde{u}_k| dx = \lim_{k\to\infty} \beta_k \int_{B_R(0)} |\Delta\varphi| dx = +\infty. \tag{25}$$

By (18), and using the change of variables $y = x_k + \mu_k x$, we get, for $1 \leq j \leq m-1$,

$$\begin{aligned}\int_{B_R(0)} |\Delta^j \tilde{u}_k| dx &= \mu_k^{-2m+2j} \int_{B_{R\mu_k}(x_k)} |\Delta^j \hat{u}_k| dy \\ &\leq C\mu_k^{-2m+2j} (R\mu_k)^{2m-2j} \leq CR^{2m-2j},\end{aligned} \tag{26}$$

which contradicts (25) for $j = 1$ and any fixed $R > 0$. Hence (i) occurs. Clearly $\tilde{u}_0$ satisfies (23) and our claim is proved.

For the case $m = 1$, we infer from Theorem 3 in [BM] that either case (i) or (ii-a) above occur, and case (ii-a) is ruled out as above.

Step 2. We now want to prove that $\tilde{u}_0 = \log \frac{2}{1+|x|^2}$. From Fatou's lemma and (15) we infer

$$\begin{aligned}\int_{\mathbb{R}^{2m}} e^{2m\tilde{u}_0} dx &= \lim_{R\to\infty} \int_{B_R(0)} e^{2m\tilde{u}_0} dx \leq \lim_{R\to\infty} \liminf_{k\to\infty} \int_{B_R(0)} e^{2m\tilde{u}_k} dx \\ &= \lim_{R\to\infty} \liminf_{k\to\infty} \int_{B_{R\mu_k}(x_k)} e^{2m\hat{u}_k} dx \leq \int_\Omega e^{2m\hat{u}_k} dx \leq C.\end{aligned}$$

If $m = 1$, then our claim follows directly from [CL]. Assume now $m > 1$. From Theorem 2 in [Mar1] we get that either

$$\tilde{u}_0 = \log \frac{2\lambda}{1 + \lambda^2 |x - x_0|^2} \tag{27}$$

for some $\lambda > 0$ and $x_0 \in \mathbb{R}^{2m}$, or there exists $j \in \{1, \ldots, m-1\}$ such that

$$\Delta^j \tilde{u}_0(x) \to a \text{ as } |x| \to +\infty, \tag{28}$$

for some constant $a < 0$. On the other hand, (28) implies that for every $R > 0$ large enough there is $k(R) \in \mathbb{N}$ such that

$$\int_{B_R(0)} |\Delta^j \tilde{u}_k| dx \geq \frac{|a|}{2} |B_R(0)| \geq \frac{R^{2m}}{C}, \quad \text{for } k \geq k(R).$$

This contradicts (26) in the limit as $R \to 0$, whence (27) has to hold. Since $\tilde{u}_k(0) = \max_{\Omega_k} \tilde{u}_k = \log 2$, the same facts hold for $\tilde{u}_0$. Therefore $x_0 = 0$ and $\lambda = 1$ in (27). This proves our second claim, hence the lemma. $\square$

Lemma 7 *There are $N > 0$ converging sequences $x_{k,i} \to x^{(i)}$, $1 \leq i \leq N$, with $\lim_{k\to\infty} u_k(x_{k,i}) = \infty$ such that, setting*

$$\tilde{u}_{k,i}(x) := \hat{u}_k(x_{k,i} + \mu_{k,i} x) + \log \mu_{k,i}, \quad \mu_{k,i} := 2e^{-\hat{u}_k(x_{k,i})}, \tag{29}$$

we have

(A_1) $\lim_{k\to\infty} \frac{|x_{k,i} - x_{k,j}|}{\mu_{k,i}} + \infty$ *for* $1 \leq i \neq j \leq N$,

(A_2) $\lim_{k\to\infty} \frac{\mathrm{dist}(x_{k,i}, \partial\Omega)}{\mu_{k,i}} = +\infty$, *for* $1 \leq i \leq N$

(A_3) $\tilde{u}_{k,i} \to \eta_0$ *in* $C^{2m-1,\alpha}_{\mathrm{loc}}(\mathbb{R}^{2m})$, *for* $1 \leq i \leq N$, *where* $\eta_0(x) = \log \left(\frac{2}{1+|x|^2} \right)$.

(A_4) For $1 \le i \le N$

$$\lim_{R\to\infty}\lim_{k\to\infty}\int_{B_{R\mu_{k,i}}(x_{k,i})} e^{2m\hat{u}_k}dx = |S^{2m}|. \tag{30}$$

(A_5) $\inf_{1\le i\le N}|x-x^{(i)}|^{2m}e^{2m\hat{u}_k(x)} \le C$ *for every* $x \in \Omega$.

Proof. We proceed inductively.

Step 1. For $N = 1$, choose $x_{k,1}$ such that $u_k(x_{k,1}) = \sup_\Omega u_k$. Then Lemma 5 and Lemma 6 imply that $(x_{k,1})$ satisfies (A_2) and (A_3). Moreover (A_1) is empty and (A_4) follows at once from (A_3) (9). If also (A_5) is satisfied, we are done. Otherwise we construct a new sequence, as in the inductive step below.

Step 2. Assume that ℓ sequences $\{(x_{k,i}) \to x^{(i)} : 1 \le i \le \ell\}$, have been constructed so that they satisfy (A_1), (A_2), (A_3) and (A_4), but not (A_5). Set

$$w_k(x) := \inf_{1\le i\le \ell}|x-x_{k,i}|^{2m}e^{2m\hat{u}_k(x)},$$

so that $\lim_{k\to\infty}\sup_\Omega w_k = \infty$, and choose $y_k \in \Omega$ such that $w_k(y_k) = \sup_\Omega w_k$. Then $y_k \to y$ up to a subsequence. Also set

$$\gamma_k = 2e^{-\hat{u}_k(y_k)}, \qquad v_k(x) = \hat{u}_k(y_k+\gamma_k x) + \log\gamma_k. \tag{31}$$

We claim that (A_1), (A_2), (A_3) and (A_4) hold for the $\ell+1$ sequences

$$\{(x_{k,i}) \to x^{(i)} : 1 \le i \le \ell+1\},$$

if we set

$$\begin{cases} x_{k,\ell+1} := y_k \\ x^{(\ell+1)} := y \\ \tilde{u}_{k,\ell+1} := v_k \\ \mu_{k,\ell+1} := \gamma_k \end{cases}$$

Since $w_k(y_k) \to +\infty$ we get

$$\lim_{k\to\infty}\frac{|y_k-x_{k,i}|}{\gamma_k} \ge \lim_{k\to\infty}\frac{w_k(y_k)^{\frac{1}{2m}}}{2} = +\infty \quad \text{for } 1 \le i \le \ell.$$

We claim that we also have

$$\lim_{k\to\infty}\frac{|y_k-x_{k,i}|}{\mu_{k,i}} = +\infty \quad \text{for } 1 \le i \le \ell.$$

Indeed, setting $\theta_{k,i} := \frac{y_k-x_{k,i}}{\mu_{k,i}}$, we have

$$|y_k-x_{k,i}|^{2m}e^{2m\hat{u}_k(y_k)} = |\theta_{k,i}|^{2m}\exp(2m[\hat{u}_k(x_{k,i}+\mu_{k,i}\theta_{k,i})+\log\mu_{k,i}]).$$

If our claim were false, then the right-hand side would be bounded thanks to (A_3), but then we would have $w_k(y_k) \le C$, against our assumption. This proves (A_1). Fix now $\varepsilon, R > 0$. Since $\max w_k$ is attained at y_k, and using (31), we have

$$e^{2mv_k(x)} \le 2^{2m}\frac{\inf_{1\le i\le\ell}|y_k-x_{k,i}|^{2m}}{\inf_{1\le i\le\ell}|y_k+\gamma_k x-x_{k,i}|^{2m}}. \tag{32}$$

Choose $k(\varepsilon, R)$ such that $|y_k - x_{k,i}| \geq \frac{R}{\varepsilon}\gamma_k$ for $k \geq k(\varepsilon, R)$ and $1 \leq i \leq \ell$. Then

$$\frac{|y_k - x_{k,i}|}{|y_k - x_{k,i} + \gamma_k x|} \leq \frac{1}{1-\varepsilon} \quad \text{for } x \in B_R(x),\ k \geq k(\varepsilon, R),\ 1 \leq i \leq \ell,$$

hence

$$e^{2mv_k(x)} \leq \frac{2^{2m}}{(1-\varepsilon)^{2m}} \quad \text{for } x \in B_R(0),\ k \geq k(\varepsilon, R).$$

With this information, we can apply the proofs of Lemma 5 and Lemma 6 to get (A_2) and (A_3) for $i = \ell + 1$. Finally, (A_4) follows from (A_3).

Step 3. The procedure has to stop, i.e. (A_5) has to be satisfied after a finite number of inductive steps. Indeed at the ℓ-th steps we get

$$\begin{aligned} \lim_{k\to\infty} \int_\Omega e^{2m\hat{u}_k} dx &\geq \lim_{R\to\infty} \lim_{k\to\infty} \sum_{i=1}^{\ell} \int_{B_{R\mu_{k,i}}(x_{k,i})} e^{2m\hat{u}_k(y)} dy \\ &= \lim_{R\to\infty} \lim_{k\to\infty} \sum_{i=1}^{\ell} \int_{B_R(0)} e^{2m\tilde{u}_{k,i}(y)} dy \\ &= \ell \int_{\mathbb{R}^{2m}} e^{2m\eta_0} dx = \ell |S^{2m}|, \end{aligned}$$

which, together with (15), gives an upper bound for ℓ. Setting N to be the ℓ at which our inductive procedure stops, we conclude. $\square$

From now on, the N converging sequences

$$\{x_{k,i} \to x^{(i)} : 1 \leq i \leq N\}$$

produced with Lemma 7 will be fixed and we shall set

$$S := \{x^{(i)} : 1 \leq i \leq N\}. \tag{33}$$

Lemma 8 *For $\ell \in \{1, \ldots, 2m-1\}$ there exists $C > 0$ such that*

$$\inf_{1\leq i\leq \ell} |x - x_{k,i}|^\ell \left|\nabla^\ell \hat{u}_k(x)\right| \leq C, \ \text{for } x \in \Omega. \tag{34}$$

Proof. As already noticed, we can use (16), (19) and the symmetry of G to get

$$|\nabla^\ell \hat{u}_k(x)| \leq C \int_\Omega \frac{e^{2m\hat{u}_k(y)}}{|x-y|^\ell} dy. \tag{35}$$

Let $\Omega_{k,i} := \{x \in \Omega :\ \mathrm{dist}(x, \{x_{k,1}, \ldots, x_{k,N}\}) = |x - x_{k,i}|\}$, fix $x \in \Omega_{k,i}$, and write

$$\int_{\Omega_{k,i}} \frac{e^{2m\hat{u}_k(y)}}{|x-y|^\ell} dy = \int_{\Omega_{k,i}\cap B_{k,i}} \frac{e^{2m\hat{u}_k(y)}}{|x-y|^\ell} dy + \int_{\Omega_{k,i}\setminus B_{k,i}} \frac{e^{2m\hat{u}_k(y)}}{|x-y|^\ell} dy, \tag{36}$$

where $B_{k,i} := B_{\frac{|x-x_{k,i}|}{2}}(x_{k,i})$. By Property (A_5) we get

$$e^{2m\hat{u}_k(y)} \leq C\,|y - x_{k,i}|^{-2m} \quad \text{for } y \in \Omega_{k,i} \setminus B_{k,i} \tag{37}$$

$$|x-y| \geq \frac{1}{2}|x - x_{k,i}| \quad \text{for } y \in \Omega_{k,i} \cap B_{k,i}. \tag{38}$$

Then, using (15) and (37), we get

$$\int_{\Omega_{k,i}\cap B_{k,i}} \frac{e^{2m\hat{u}_k(y)}}{|x-y|^\ell} dy \le \frac{C}{|x-x_{k,i}|^\ell}. \tag{39}$$

As for the last integral in (36), we write $\Omega_{k,i} \setminus B_{k,i} = \Omega_{k,i}^{(1)} \cup \Omega_{k,i}^{(2)}$, where

$$\Omega_{k,i}^{(1)} = (\Omega_{k,i}\backslash B_{k,i}) \cap B_{2|x-x_{k,i}|}(x), \quad \Omega_{k,i}^{(2)} = (\Omega_{k,i}\backslash B_{k,i})\backslash B_{2|x-x_{k,i}|}(x).$$

Then straightforward computations and (38) imply

$$\begin{aligned}
\int_{\Omega_{k,i}\backslash B_{k,i}} \frac{e^{2m\hat{u}_k(y)}dy}{|x-y|^\ell} &\le C\int_{\Omega_{k,i}^{(1)}} \frac{dy}{|y-x_{k,i}|^{2m}|x-y|^\ell} \\
&\quad + C\int_{\Omega_{k,i}^{(2)}} \frac{dy}{|y-x_{k,i}|^{2m}|x-y|^\ell} \\
&\le \frac{C}{|x-x_{k,i}|^{2m}} \int_{\Omega_{k,i}^{(1)}} \frac{dy}{|x-y|^\ell} + C\int_{\Omega_{k,i}^{(2)}} \frac{dy}{|y-x_{k,i}|^{2m+\ell}} \\
&\le \frac{C}{|x-x_{k,i}|^\ell}.
\end{aligned}$$

Summing up with (35), (36) and (39), the proof is complete. □

Lemma 9 *Up to a subsequence, we have*

$$\lim_{k\to\infty} \alpha_k = +\infty.$$

Proof. We argue by contradiction. Suppose $\lim_{k\to\infty} \alpha_k = \alpha_0 \in \mathbb{R}$.

Step 1. We claim that $S \subset \partial\Omega$, where S is as in (33), and there is a function $u_0 \in C^{2m-1,\alpha}(\overline{\Omega})$ such that

$$u_k \to u_0 \quad \text{in } C^{2m-1,\alpha}_{\text{loc}}(\overline{\Omega}\backslash S).$$

Moreover u_0 satisfies

$$\begin{cases} (-\Delta)^m u_0 = (2m-1)! e^{-2m\alpha_0} e^{2mu_0} \text{ in } \Omega \\ u_0 = \partial_\nu u_0 = \ldots = \partial_\nu^{m-1} u_0 = 0 \text{ in } \partial\Omega \end{cases} \tag{40}$$

Indeed (17) and the assumption that $\alpha_k \to \alpha_0$ imply that

$$\|\hat{u}_k\|_{L^1(\Omega)} \le C. \tag{41}$$

Since $\hat{u}_k$ satisfies (14) and (15), we can apply Theorem 13 from the appendix. This implies that one of the following is true

(i) Up to a subsequence, $\hat{u}_k \to \hat{u}_0$ in $C^{2m-1,\alpha}_{\text{loc}}(\Omega)$.

(ii) Up to a subsequence $\hat{u}_k \to -\infty$ locally uniformly in $\Omega\backslash\Omega_0$ for a set Ω_0 of Hausdorff dimension at most $2m-1$.

Clearly case (ii) contradicts (41), hence case (i) occurs and $S \subset \partial\Omega$. Using the boundary condition, Lemma 8, and elliptic estimates, we actually infer that $\hat{u}_k \to \hat{u}_0$ in $C^{2m-1,\alpha}_{\mathrm{loc}}(\overline{\Omega}\backslash S)$. Then clearly $u_k \to u_0 := \hat{u}_0 + \alpha_0$ in $C^{2m-1,\alpha}_{\mathrm{loc}}(\overline{\Omega}\backslash S)$ and u_0 satisfies (40).

We finally want to prove that u_0 is continuous in $\overline{\Omega}$, hence smooth. In the limit as $k \to \infty$, Lemma 8 implies

$$\inf_{1\le i\le N} |x - x^{(i)}||\nabla u_0(x)| \le C \quad \text{for } x \in \Omega \setminus S.$$

Fix $x^{(i)} \in S$ and $\delta > 0$ such that

$$|x - x^{(i)}|\,|\nabla u_0(x)| \le C \quad \text{for } x \in \Omega \cap B_\delta(x^{(i)}) \setminus \{x^{(i)}\}.$$

Then there is a constant $C > 0$ such that

$$|u(x) - u(y)| \le C \quad \text{for } x, y \in \Omega \cap B_\delta(x^{(i)}) \setminus \{x^{(i)}\},\ |x - x^{(i)}| = |y - x^{(i)}|.$$

By taking $y \in \partial\Omega$ and using (2), we obtain that u is bounded near $x^{(i)}$. Then (40) and elliptic regularity imply that $u_0 \in C^\infty(\overline{\Omega})$.

Step 2. If $S = \emptyset$, then Step 1 yields $u_k \to u_0$ in $C^{2m-1,\alpha}_{\mathrm{loc}}(\overline{\Omega})$, which contradicts the assumption $\sup_\Omega u_k \to +\infty$. Then let $x_0 \in S \subset \partial\Omega$. Take $\delta > 0$ such that $S \cap B_\delta(x_0) = \{x_0\}$, and set for $0 < r \le \delta$

$$\rho_{k,r} = \frac{\int_{\partial\Omega\cap B_r(x_0)} (x - x_0)\cdot\nu(x)|\Delta^{\frac{m}{2}} u_k|^2 d\sigma(x)}{\int_{\partial\Omega\cap B_r(x_0)} \nu(x_0)\cdot\nu(x)|\Delta^{\frac{m}{2}} u_k|^2 d\sigma(x)}, \tag{42}$$

where $\Delta^{\frac{m}{2}} u_k$ is defined as in (58) below, and $\nu(x)$ denotes the exterior normal to $\partial\Omega$ at x. Set also

$$y_{k,r} := x_0 + \rho_{k,r}\nu(x_0). \tag{43}$$

Up to taking δ even smaller, we may assume that

$$\frac{1}{2} \le \nu(x_0)\cdot\nu(x) \le 1 \quad \text{for } x \in \partial\Omega \cap \overline{B}_r(x_0),\ r \le \delta,$$

hence $|\rho_{k,r}| \le 2r$. Applying Lemma 15 to u_k on the domain $\Omega' := \Omega \cap B_r(x_0)$, with

$$Q = (2m-1)!e^{-2m\alpha_k}, \quad y = y_{k,r},$$

and by the property (A_4), we get

$$\begin{aligned}
\Lambda_1 \;&\le\; \lim_{k\to\infty} (2m-1)! \int_{\Omega'} e^{2m\hat{u}_k} dx \\
&=\; \lim_{k\to\infty} \frac{(2m-1)!}{2m} \int_{\partial\Omega'} (x - y_{k,r})\cdot\nu_{\Omega'} e^{2m\hat{u}_k} d\sigma \\
&\quad - \lim_{k\to\infty} \frac{1}{2}\int_{\partial\Omega'} (x - y_{k,r})\cdot\nu_{\Omega'} |\Delta^{\frac{m}{2}} u_k|^2 d\sigma + \lim_{k\to\infty}\int_{\partial\Omega'} f_k d\sigma,
\end{aligned} \tag{44}$$

where f_k is definded on $\partial\Omega'$ by

$$f_k(x) = \sum_{j=0}^{m-1} (-1)^{m+j+1}\nu_{\Omega'}\cdot\Big(\Delta^{\frac{j}{2}}((x - y_{k,r})\cdot\nabla u_k(x))\Delta^{\frac{2m-1-j}{2}} u_k(x)\Big). \tag{45}$$

Notice that (2) implies that $\nabla^\ell u_k = 0$ on $\partial\Omega$ for $0 \le \ell \le m-1$. Since each monomial of f_k contains a factor of the form $\partial^\gamma u_k$ for some multi-index γ with $|\gamma| \le m-1$, we get

$$\int_{\partial\Omega\cap B_r(x_0)} f_k d\sigma = 0.$$

Moreover

$$\frac{1}{2}\int_{\partial\Omega\cap B_r(x_0)} (x-y_{k,r})\cdot\nu_{\Omega'}|\Delta^{\frac{m}{2}}u_k|^2 d\sigma = 0$$

by (42) and (43). By (2) and Lemma 2, we also have

$$\left|\frac{(2m-1)!}{2m}\int_{\partial\Omega\cap B_r(x_0)} (x-y_{k,r})\cdot\nu_{\Omega'}e^{2m\hat{u}_k}\right| \le C\int_{\partial\Omega\cap B_r(x_0)} re^{-2m\alpha_k} \le Cr^{2m}.$$

All the other terms on the right-hand side of (44), namely the integrals over $\Omega\cap\partial B_r(x_0)$, are bounded by Cr^{2m-1} for $0 < r \le \delta$ and $k \ge k(r)$ large enough, since by Step 1 we have

$$\lim_{k\to\infty}\sup_{\partial B_r(x_0)\cap\Omega}|\nabla^\ell u_k - \nabla^\ell u_0| = 0, \quad |\nabla^\ell u_0| \le C, \quad 0 \le \ell \le 2m-1.$$

Therefore, taking the limit as $k \to 0$ first and $r \to 0$ then, we infer

$$\Lambda_1 \le Cr^{2m-1}.$$

This gives a contradiction as $r \to 0$, hence completing the proof. $\square$

Lemma 10 *Up to selecting a subsequence,*

$$\hat{u}_k \to -\infty \quad \textit{locally uniformly on } \overline{\Omega}\setminus S, \tag{46}$$

where S is as in (33). *Moreover*

$$\lim_{k\to+\infty} u_k = \sum_{i=1}^N \beta_i G_{x^{(i)}} \ \textit{in } C^{2m-1,\alpha}_{\mathrm{loc}}(\bar{\Omega}\setminus S), \tag{47}$$

with

$$\beta_i := (2m-1)!\lim_{\delta\to 0}\lim_{k\to\infty}\int_{B_\delta(x^{(i)})\cap\Omega} e^{2m\hat{u}_k}dy, \tag{48}$$

and $\beta_i \ge \Lambda_1$, for $1 \le i \le N$.

Proof. Step 1. We claim that $\hat{u}_k \to -\infty$ locally uniformly on $\overline{\Omega}\setminus S$. Indeed take $\delta > 0$ such that $\Omega_\delta := \Omega\setminus\cup_{i=1}^N \overline{B}_\delta(x_i)$ is connected and $\partial\Omega_\delta\cap\partial\Omega \ne \emptyset$. Lemma 8 implies that $\hat{u}_k$ is Lipschitz on Ω_δ, and we also have $\hat{u}_k = -\alpha_k$ on $\partial\Omega_\delta\cap\partial\Omega$, hence

$$|u_k| = |\hat{u}_k + \alpha_k| \le C_\delta \text{ in } \overline{\Omega}_\delta. \tag{49}$$

Since $\alpha_k \to +\infty$, we have $\hat{u}_k \to -\infty$ uniformly on $\overline{\Omega}_\delta$, hence the claim is proved.

Step 2. By (2) and Lemma 8, the u_k's are bounded in $C^0_{\mathrm{loc}}(\overline{\Omega}\setminus S)$. Since

$$(-\Delta)^m u_k = (2m-1)!e^{-2m\alpha_k}e^{2mu_k},$$

where the right-hand side is bounded $C^0_{\mathrm{loc}}(\overline{\Omega}\backslash S)$, by elliptic regularity we have that, up to a subsequence,

$$u_k \to \psi \quad \text{in } C^{2m-1,\alpha}_{\mathrm{loc}}(\overline{\Omega}\backslash S),$$

for some $\psi \in C^{2m-1,\alpha}_{\mathrm{loc}}(\overline{\Omega} \setminus S)$. Up to taking $\delta > 0$ smaller, we may assume that $\overline{B_\delta(x^{(i)})} \cap \overline{B_\delta(x^{(j)})} = \emptyset$ for $i \neq j$. Since $\hat{u}_k \to -\infty$ uniformly on the compact $\overline{\Omega}_\delta$, we have by (16)

$$\begin{aligned} \lim_{k\to\infty} u_k(x) &= (2m-1)! \lim_{k\to\infty} \int_\Omega G_x(y) e^{2m\hat{u}_k(y)} dy \\ &= (2m-1)! \lim_{k\to\infty} \sum_{i=1}^N \int_{B_\delta(x^{(i)})\cap\Omega} G_x(y) e^{2m\hat{u}_k(y)} dy. \end{aligned} \tag{50}$$

Now we want an explicit expression for ψ. Fix $x \in \overline{\Omega}\backslash S$. We observe that $G(x,\cdot)$ is smooth away from x; in particular it is continuous on $B_\delta(x^{(i)})$ for all i (up to decreasing δ). By (15), up to a subsequence we have

$$e^{2m\hat{u}_k}(y)dy \rightharpoonup \nu \quad \text{in } \overline{\Omega}$$

weakly in the sense of measures, for some positive Radon measure ν. On the other hand, since (46) implies that the support of ν is contained in S, we get

$$\nu = \sum_{i=i}^N \beta_i \delta_{x^{(i)}},$$

for some constants $\beta_i \geq 0$. Then (50) implies

$$\lim_{k\to\infty} u_k(x) = \sum_{i=1}^N \beta_i G_{x^{(i)}}(x) \quad \forall x \in \Omega \setminus S,$$

where β_i is as in (48). Now we fix a point $x^{(i)} \in S$ and we set $\mu_{k,i}$ and $x_{k,i}$ as in Lemma 6. By (A_4)

$$\lim_{k\to\infty} \int_{B_\delta(x^{(i)})\cap\Omega} e^{2m\hat{u}_k(x)}dx \geq \lim_{R\to\infty}\lim_{k\to\infty} \int_{B_{R\mu_k}(x_{k,i})} e^{2m\hat{u}_k(x)}dx = |S^{2m}|.$$

Taking the limit as $\delta \to 0$ we get $\beta_i \geq \Lambda_1$, as claimed. □

Lemma 11 *For any $x_0 \in \partial\Omega$ we have*

$$\lim_{r\to 0}\lim_{k\to+\infty} \int_{B_r(x_0)\cap\Omega} e^{2m\hat{u}_k}dx = 0. \tag{51}$$

In particular $S \cap \partial\Omega = \emptyset$.

Proof. Fix $x_0 \in \partial\Omega$. If $x_0 \notin S$, then (51) follows at once from Lemma 10. Then we can assume $x_0 = x^{(j)} \in \partial\Omega \cap S$ for some $1 \leq j \leq N$, and proceed by contradiction. Take $\delta > 0$ such that $S \cap B_\delta(x_0) = \{x_0\}$. Let $\nu : \partial\Omega \to S^{2m-1}$

be the outward pointing normal to $\partial\Omega$. Set $\rho_{k,r}$ and $y_{k,r}$ as in (42) and (43). Take $r>0$ so small that

$$\frac{1}{2}\leq \nu(x_0)\cdot\nu(x)\leq 1 \quad \text{for } x\in\partial\Omega\cap\overline{B}_r(x_0),$$

so that $|\rho_{k,r}|\leq 2r$. Applying Lemma 15 to u_k on the domain $\Omega':=\Omega\cap B_r(x_0)$, with

$$Q=(2m-1)!e^{-2m\alpha_k},\quad y=y_{k,r},$$

we obtain

$$\begin{aligned}(2m-1)!\int_{\Omega'}e^{2m\hat{u}_k}dx &= \frac{(2m-1)!}{2m}\int_{\partial\Omega'}(x-y_{k,r})\cdot\nu_{\Omega'}e^{2m\hat{u}_k}d\sigma \\ &\quad -\frac{1}{2}\int_{\partial\Omega'}(x-y_{k,r})\cdot\nu_{\Omega'}|\Delta^{\frac{m}{2}}u_k|^2d\sigma+\int_{\partial\Omega'}f_kd\sigma,\end{aligned}\tag{52}$$

where $f_k(x)$ is as in (45). Since each monomial of f contains a factor of the form $\partial^\gamma u_k$ with $|\gamma|\leq m-1$, we get

$$\int_{\partial\Omega\cap B_r(x_0)}f_kd\sigma=0.$$

Moreover, since $G_{x_0}\equiv 0$, and the derivatives of $G_{x^{(i)}}$ are bounded in $\overline{B_r(x_0)}$ for $x^{(i)}\neq x_0$, (47) implies

$$\lim_{k\to+\infty}\int_{\Omega\cap\partial B_r(x_0)}f_kd\sigma\leq Cr^{2m-1},$$

and

$$\lim_{k\to\infty}\frac{1}{2}\int_{\Omega\cap\partial B_r(x_0)}(x-y_{k,r})\cdot\nu|\Delta^{\frac{m}{2}}u_k|^2d\sigma\leq Cr^{2m}.$$

By the choice of $y_{k,r}$ we get again

$$\frac{1}{2}\int_{\partial\Omega\cap B_r(x_0)}(x-y_{k,r})\cdot\nu|\Delta^{\frac{m}{2}}u_k|^2d\sigma=0.$$

As for the first term on the right-hand side of (52), (2) and Lemma 2 imply

$$\int_{\partial\Omega'}(x-y_{k,r})\cdot\nu_{\Omega'}e^{-2m\alpha_k}e^{2mu_k}d\sigma\leq Cr^{2m}.$$

Summing up all the contributions, we get (51). $\square$

Lemma 12 *In* (47) *and* (48) *we have* $\beta_i=\Lambda_1$ *for all* $1\leq i\leq N$.

Proof. Since $S\cap\partial\Omega=\emptyset$, there exists $\delta>0$ such that $B_\delta(x^{(i)})\subset\Omega$, and $S\cap B_\delta(x^{(i)})=\{x^{(i)}\}$ for all $1\leq i\leq N$. Fix i with $1\leq i\leq N$ and suppose, up to a translation, that $x^{(i)}=0$. Recall that

$$\beta_i=(2m-1)!\lim_{\delta\to 0}\lim_{k\to\infty}\int_{B_\delta(0)}e^{2m\hat{u}_k}dx$$

By the Pohozaev identity of Lemma 15, applied to u_k on the domain $B_\delta := B_\delta(0)$ with $y = 0$ and $Q = (2m-1)!e^{-2m\alpha_k}$, we get

$$(2m-1)! \int_{B_\delta} e^{2m\hat{u}_k} dx = I_\delta(u_k) + II_\delta(u_k) + III_\delta(u_k), \tag{53}$$

where

$$\begin{aligned} I_\delta(u_k) &= \frac{\delta(2m-1)!}{2m} \int_{\partial B_\delta} e^{2m\hat{u}_k} d\sigma \\ II_\delta(u_k) &= -\frac{\delta}{2} \int_{\partial B_\delta} |\Delta^{\frac{m}{2}} u_k|^2 d\sigma \\ III_\delta(u_k) &= \sum_{j=0}^{m-1} (-1)^{m+j+1} \int_{\partial B_\delta} \nu \cdot \left(\Delta^{\frac{j}{2}} (x \cdot \nabla u_k) \Delta^{\frac{2m-1-j}{2}} u_k \right) d\sigma \end{aligned}$$

From Lemma 10 we infer

$$\begin{aligned} \lim_{k\to\infty} II_\delta(u_k) &= II_\delta(\beta_i G_0) = \beta_i^2 II_\delta(G_0) \\ \lim_{k\to\infty} III_\delta(u_k) &= III_\delta(\beta_i G_0) = \beta_i^2 III_\delta(G_0). \end{aligned}$$

Since the functions $e^{2m\hat{u}_k} \to 0$ in $C^0(\partial B_\delta)$, we have

$$\lim_{k\to\infty} I_\delta(u_k) = 0.$$

The Green function G_0 can be decomposed in the sum of a fundamental solution for the operator $(-\Delta)^m$ on $\mathbb{R}^{2m}$ and a so-called regular part R, which is smooth: Let us write

$$G_0 = g + R \quad \text{in } \overline{\Omega}$$

where

$$g(x) := \frac{1}{\gamma_{2m}} \log \frac{1}{|x|}, \quad \gamma_{2m} := \frac{\Lambda_1}{2}$$

satisfies $(-\Delta)^m g = \delta_0$ (see e.g. Proposition 22 in [Mar1]), and $R := G_0 - g \in C^\infty(\overline{\Omega})$. Since

$$|\nabla^j R| \le C, \quad |\nabla^j g| \le \frac{C}{\delta^j} \quad \text{on } \partial B_\delta, \tag{54}$$

we get

$$II_\delta(R+g) - II_\delta(g) \le C\delta \int_{\partial B_\delta} C \left(|\Delta^{\frac{m}{2}} g| + C \right) d\sigma \le C\delta^m.$$

For the terms in $III_\delta(R+g)$, (54) implies

$$\begin{aligned}
III_\delta^{(j)}(g+R) &:= \int_{\partial B_\delta} \nu \cdot \Big(\Delta^{\frac{j}{2}} \left(x \cdot \nabla (R+g)\right) \Delta^{\frac{2m-1-j}{2}} (R+g) \Big)\, d\sigma \\
&= \int_{\partial B_\delta} \nu \cdot \Big(\Delta^{\frac{j}{2}} \left(x \cdot \nabla g\right) \Delta^{\frac{2m-1-j}{2}} g \Big)\, d\sigma \\
&\quad + \int_{\partial B_\delta} \nu \cdot \Big(\Delta^{\frac{j}{2}} \left(x \cdot \nabla R\right) \Delta^{\frac{2m-1-j}{2}} g \Big)\, d\sigma \\
&\quad + \int_{\partial B_\delta} \nu \cdot \Big(\Delta^{\frac{j}{2}} \left(x \cdot \nabla g\right) \Delta^{\frac{2m-1-j}{2}} R \Big)\, d\sigma \\
&\quad + \int_{\partial B_\delta} \nu \cdot \Big(\Delta^{\frac{j}{2}} \left(x \cdot \nabla R\right) \Delta^{\frac{2m-1-j}{2}} R \Big)\, d\sigma \\
&= III_\delta^{(j)}(g) + O(\delta) \quad \text{as } \delta \to 0,
\end{aligned}$$

where $|O(\delta)| \le C\delta$ as $\delta \to 0$. Summing up all what we proved until now, we obtain

$$\beta_i = \beta_i^2 \lim_{\delta\to 0}\lim_{k\to\infty} \big[I_\delta(u_k) + II_\delta(u_k) + III_\delta(u_k) \big] = \beta_i^2 \lim_{\delta \to 0} \big[II_\delta(g) + III_\delta(g) \big].$$

On the other hand, since $II_\delta(g)$ and $III_\delta(g)$ do not depend on δ, it is enough to compute

$$\beta_i = II_\delta(g) + III_\delta(g) \tag{55}$$

for an arbitrary $\delta > 0$. Using the formula

$$\gamma_{2m}\Delta^k g = (-1)^k (2k-2)!! \frac{(2m-2)!!}{(2m-2k-2)!!} r^{-2k},$$

we find

$$II_\delta(g) = -\frac{\delta}{2}\int_{\partial B_\delta} \left[\frac{(2m-2)!!}{\gamma_{2m}} r^{-m} \right]^2 d\sigma = -|S^{2m-1}| \frac{[(2m-2)!!]^2}{2\gamma_{2m}^2}.$$

Observing that

$$\begin{aligned}
\Delta^k (x\cdot\nabla g) &= 2k\Delta^k g + r\partial_r \Delta^k g = 0, \\
\partial_r (x \cdot \nabla g) &= -r^{-1} - x\cdot\nabla(r^{-1}) = 0, \\
x\cdot\nabla g &= r\partial_r g = -\frac{1}{\gamma_{2m}}, \\
\gamma_{2m}\partial_r\Delta^k g &= (-1)^{k+1}(2k)!! \frac{(2m-2)!!}{(2m-2k-2)!!} r^{-2k-1}
\end{aligned}$$

we see that $III_\delta^{(j)}(g) = 0$ for $1 \le j \le m-1$, and

$$\begin{aligned}
III_\delta(g) &= III_\delta^{(0)}(g) = (-1)^{m+1} \int_{\partial B_\delta} (x\cdot\nabla g)\partial_r \Delta^{m-1} g\, d\sigma \\
&= |S^{2m-1}| \frac{[(2m-2)!!]^2}{\gamma_{2m}^2}.
\end{aligned}$$

From (55) we get

$$\frac{1}{\beta_i} = |S^{2m-1}| \frac{[(2m-2)!!]^2}{2\gamma_{2m}^2} = \frac{1}{(2m-1)!|S^{2m}|},$$

whence $\beta_i = \Lambda_1$. □

Proof of Theorem 1. By Corollary 3, it suffices to prove that, under the assumption (12), case (ii) of the theorem occurs. This follows at once putting together Lemmas 7, 10, 11 and 12. □

Appendix

A useful theorem

Several times we used the following theorem from [Mar2] (compare also [BM] and [ARS]).

Theorem 13 *Let Ω be a domain in $\mathbb{R}^{2m}$, $m > 1$, and let $(u_k)_{k\in\mathbb{N}}$ be a sequence of functions satisfying*

$$(-\Delta)^m u_k = (2m-1)! e^{2mu_k}. \tag{56}$$

Assume that

$$\int_\Omega e^{2mu_k} dx \leq C, \tag{57}$$

for all k and define the finite (possibly empty) set

$$S_1 := \left\{ x \in \Omega : \lim_{r\to 0^+} \lim_{k\to\infty} \int_{B_r(x)} (2m-1)! e^{2mu_k} dy \geq \frac{\Lambda_1}{2} \right\}.$$

Then one of the following is true.

(i) A subsequence converges in $C^{2m-1,\alpha}_{\mathrm{loc}}(\Omega)$ and $S_1 = \emptyset$.

(ii) There exist a subsequence, still denoted by (u_k), a closed nowhere dense set S_0 of Hausdorff dimension at most $2m-1$ such that, letting $\Omega_0 = S_0 \cup S_1$, we have $u_k \to -\infty$ locally uniformly in $\Omega \backslash \Omega_0$ as $k \to \infty$. Moreover there is a sequence of numbers $\beta_k \to \infty$ such that

$$\frac{u_k}{\beta_k} \to \varphi \text{ in } C^{2m-1,\alpha}_{\mathrm{loc}}(\Omega\backslash\Omega_0),$$

where $\varphi \in C^\infty(\Omega\backslash S_1)$, $S_0 = \{x \in \Omega : \varphi(x) = 0\}$, and

$$(-\Delta)^m \varphi \equiv 0, \quad \varphi \leq 0, \quad \varphi \not\equiv 0 \quad \text{in } \Omega\backslash S_1.$$

Pohozaev's identity

We now discuss a generalization of the celebrated Pohozaev identity to higher dimension, Lemma 15 below. A similar identity can be also found in [Xu]. We shall use the following notation:

$$\Delta^{\frac{m}{2}}u := \begin{cases} \Delta^n u \in \mathbb{R} & \text{if } m = 2n \text{ is even} \\ \nabla\Delta^n u \in \mathbb{R}^{2m} & \text{if } m = 2n+1 \text{ is odd,} \end{cases} \tag{58}$$

and we define $\Delta^j u \cdot \Delta^\ell u$ using the inner product of $\mathbb{R}^{2m}$, or the multiplication by a scalar or the product of $\mathbb{R}$ according to whether j and ℓ are integer or half-integer.

Preliminary to the proof of Pohozaev's identity, we need the following lemma.

Lemma 14 *Let $u \in C^{m+1}(\Omega)$, where $\Omega \subset \mathbb{R}^{2m}$ is open, and let $y \in \mathbb{R}^{2m}$ be fixed. We have*

$$\frac{1}{2}\operatorname{div}((x-y)|\Delta^{\frac{m}{2}}u|^2) = \Delta^{\frac{m}{2}}((x-y)\cdot\nabla u)\cdot\Delta^{\frac{m}{2}}u$$

Proof. By a simple translation we can assume $y = 0$. Let us first assume m even. Then

$$\begin{aligned} \frac{1}{2}\operatorname{div}(x|\Delta^{\frac{m}{2}}u|^2) &= m|\Delta^{\frac{m}{2}}u|^2 + \left[(x\cdot\nabla)\Delta^{\frac{m}{2}}u)\right]\cdot\Delta^{\frac{m}{2}}u \\ &= m(\Delta^{\frac{m}{2}}u + (x\cdot\nabla)\Delta^{\frac{m}{2}}u)\cdot\Delta^{\frac{m}{2}}u. \end{aligned} \tag{59}$$

Observing that $D^2 x = 0$ and use the Leibniz's rule, we also get

$$\begin{aligned} (x\cdot\nabla)\Delta^{\frac{m}{2}}u + m\Delta^{\frac{m}{2}}u &= (x\cdot\nabla)\Delta^{\frac{m}{2}}u + m\sum_{i,j=1}^{2m}\partial_{x^j}x^i\Delta^{\frac{m}{2}-1}\partial_{x_j}\partial_{x_i}u \\ &= \Delta^{\frac{m}{2}}(x\cdot\nabla u) \end{aligned} \tag{60}$$

Inserting (60) into (59) we conclude. ☐

Lemma 15 *Let $u \in C^{m+1}(\overline{\Omega})$, $Q \in \mathbb{R}$ satisfy*

$$(-\Delta)^m u = Qe^{2mu}$$

in $\Omega \subset \mathbb{R}^{2m}$. Let $y \in \mathbb{R}^{2m}$ be fixed. Then

$$\begin{aligned} \int_\Omega Qe^{2mu}dx &= \frac{1}{2m}\int_{\partial\Omega}(x-y)\cdot\nu Qe^{2mu}d\sigma - \frac{1}{2}\int_{\partial\Omega}(x-y)\cdot\nu|\Delta^{\frac{m}{2}}u|^2 d\sigma \\ &\quad + \sum_{j=0}^{m-1}(-1)^{m+j+1}\int_{\partial\Omega}\nu\cdot\Big(\Delta^{\frac{j}{2}}((x-y)\cdot\nabla u)\Delta^{\frac{2m-1-j}{2}}u\Big)d\sigma. \end{aligned}$$

Proof. The proof is a pretty straightforward application of integration by parts. We have

$$\int_{\partial\Omega}(x-y)\cdot\nu Qe^{2mu}d\sigma = \int_\Omega 2me^{2mu}Qdx + \int_\Omega 2m((x-y)\cdot\nabla u)e^{2mu}Qdx,$$

since both sides are equal to $\int_\Omega \operatorname{div}((x-y)e^{2mu})Qdx$. Then we use

$$\begin{aligned}\int_\Omega (x-y)\cdot\nabla u e^{2mu}Qdx &= (-1)^m\int_\Omega (x-y)\cdot\nabla u\Delta^m u dx\\ &= \int_\Omega \underbrace{\Delta^{\frac{m}{2}}((x-y)\cdot\nabla u)\Delta^m 2u}_{=\frac{1}{2}\operatorname{div}((x-y)|\Delta^{\frac{m}{2}}u|^2)}dx + \int_{\partial\Omega} f d\sigma,\end{aligned}$$

where

$$f(x) := \sum_{j=0}^{m-1}(-1)^{m+j}\nu\cdot\left(\Delta^{\frac{j}{2}}((x-y)\cdot\nabla u(x))\,\Delta^{\frac{2m-1-j}{2}}u(x)\right),\quad x\in\partial\Omega.$$

Moreover

$$\frac{1}{2}\int_\Omega \operatorname{div}((x-y)|\Delta^{\frac{m}{2}}u|^2)dx = \frac{1}{2}\int_{\partial\Omega}(x-y)\cdot\nu|\Delta^{\frac{m}{2}}u|^2 d\sigma.$$

Summing together we conclude. □

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