ON A HEAT PROBLEM INVOLVING PERTURBED HARDY-SOBOLEV OPERATOR

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Abstract

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$ and $0 \in \Omega$. It is known that the heat problem $\partial u/\partial t + L_{\lambda^*} u = 0 \quad \text{in } \Omega \times (0,\infty)$, $u(x,0) = u_0 \geq 0$, where $L_{\lambda^*} := -\Delta - \dfrac{\lambda^*}{|x|^2}$, $\lambda^* := \dfrac{(n-2)^2}{4}$, does not admit any non-trivial H_0^1 solutions, even locally in time. In this note we consider the perturbation operator $L_{\lambda^* q} := -\Delta - \dfrac{\lambda^* q(x)}{|x|^2}$, for some suitable bounded positive weight function q and determine the border line between the existence and non-existence of positive H_0^1 solutions for the above heat problem with the operator $L_{\lambda^* q}$. In the dimension $n = 2$, we have similar phenomena for the critical Hardy-Sobolev operator $L^* := -\Delta - \dfrac{1}{4|x|^2}\left(\log R/|x|\right)^{-2}$, for sufficiently large R.

§1 Introduction

Let Ω be bounded open subset in $\mathbb{R}^n$, $n \geq 3$ and $0 \in \Omega$. Baras and Goldstein in their classical paper [3], considered the following heat problem

$$\left\{\begin{array}{rcll} u_t - \Delta u &=& V(x)u & \text{in} \quad \Omega \times (0,\infty) \\ u(x,0) &=& u_0(x) & \text{in} \quad \Omega \\ u(x,t) &=& 0 & \text{on} \quad \partial\Omega \times (0,\infty). \end{array}\right. \tag{1.1}$$

For the critical potential $V(x) = \dfrac{\mu}{|x|^2}$, they have proved that the Cauchy-Dirichlet problem (1.1) has a global solution (*in the sense of distributions*) if $\mu \leq \lambda^* := \dfrac{(n-2)^2}{4}$ and no solution, even locally in time, if $\mu > \lambda^*$. It is well understood that, existence of solutions for the problem (1.1) is deeply connected to the Hardy-Sobolev inequality

$$\int_\Omega |\nabla u|^2 \, dx \geq \frac{(n-2)^2}{4} \int_\Omega \frac{|u|^2}{|x|^2} \, dx \tag{1.2}$$

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and the associated spectral problem $-\Delta u - \dfrac{\mu}{|x|^2}u = \lambda u$. By the inequality (1.2), it is easy to show that the problem (1.1) admits solution in the function space $C\left([0,\infty); L^2(\Omega)\right) \cap L^2\left(0,\infty; H_0^1(\Omega)\right)$, if $\mu < \lambda^*$. But a strong change of behaviour takes place at the transition value $\mu = \lambda^*$. In this case the behaviour of the solution is intimately related to the improved form of the classical Hardy inequality (1.2). Adimurthi, Chaudhuri and Ramaswamy [1] have improved the inequality (1.2), which states that for $n \geq 3$ and for any $R \geq e \sup_{\Omega} |x|$, there exist positive constants C_1 and C_2 such that,

$$\int_\Omega |\nabla u|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{|u|^2}{|x|^2}\, dx + C_1 \int_\Omega \frac{|u|^2}{|x|^2} \left(\log \frac{R}{|x|}\right)^{-2} dx + C_2 \int_\Omega |u|^2\, dx \quad (1.3)$$

holds for all $u \in H_0^1(\Omega)$.
But when $n = 2$ and $R \geq e^{e^e} \sup_{\Omega} |x|$, there exists a constant $C > 0$ such that, the following 2-dimensional Hardy-Sobolev inequality

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4} \int_\Omega \frac{|u|^2}{|x|^2} \left(\log \frac{R}{|x|}\right)^{-2} \left[1 + \left(\log \log \frac{R}{|x|}\right)^{-2}\right] dx + C \int_\Omega |u|^2\, dx\ , \quad (1.4)$$

holds for all $u \in H_0^1(\Omega)$.

For the case $n \geq 3$ and $\mu = \lambda^*$, Vazquez and Zuazua [5] proved that the solution $u(t)$ of the problem (1.1) with $u_0 \in L^2(\Omega)$ is in the Hilbert space $H(\Omega)$, for all time $t > 0$ but not in $H_0^1(\Omega)$, even locally in time, where $H(\Omega)$ is the completion of $H_0^1(\Omega)$ with respect to the *energy norm* $\|u\|_H := \left(\int_\Omega \left\{|\nabla u|^2 - \lambda^* \dfrac{u^2}{|x|^2}\right\} dx\right)^{1/2}$. It is interesting to notice that $H_0^1(\Omega) \subset H(\Omega) \subset \bigcap_{q<2} W_0^{1,q}(\Omega)$ and the inclusions are proper. So naturally a question arises: Is it possible to perturb the *Hardy-Sobolev operator* $L_{\lambda^*} := -\Delta - \dfrac{\lambda^*}{|x|^2}$, in such a way that the heat equation admits a non-negative solution $u(t) \in H_0^1(\Omega)$ for all $t > 0$ with $u_0 \geq 0$? Or what should be perturbation of the stationary operator L_{λ^*} which determines the border line between existence and non-existence of H_0^1 solutions? For the stationary operator L_{λ^*}, these questions was answered by Adimurthi and Sandeep in [2] and we state these results in Lemma 3.1 and Lemma 4.1. In this note we answer these questions for the heat problem.

Throughout in the text we assume $0 \leq q \leq 1$ and $\eta \geq 0$ be measurable functions on Ω such that $\eta \in L^\infty_{\rm loc}(\Omega \setminus \{0\})$, and it satisfies

$$\begin{cases} \lim\limits_{x\to 0} |x|^2 \left(\log \frac{1}{|x|}\right)^2 \eta(x) &= 0, \quad n \geq 3\,, \\ \lim\limits_{x\to 0} |x|^2 \left(\log \frac{1}{|x|}\right)^2 \left(\log\log \frac{1}{|x|}\right)^2 \eta(x) &= 0, \quad n = 2. \end{cases} \quad (1.5)$$

Now, let us define the following constant,

$$C(q,\eta) := \begin{cases} \inf\limits_{H_0^1(\Omega)} \left\{ \int\limits_\Omega \left[|\nabla u|^2 - \lambda^* \dfrac{|u|^2 q}{|x|^2} \right] dx \;\middle|\; \int\limits_\Omega |u|^2 \eta dx = 1 \right\}, & \text{for } n \geq 3 \\ \inf\limits_{H_0^1(\Omega)} \left\{ \int\limits_\Omega \left[|\nabla u|^2 - \dfrac{1}{4} \dfrac{|u|^2 q(x)}{(|x| \log R/|x|)^2} \right] dx \;\middle|\; \int\limits_\Omega |u|^2 \eta \, dx = 1 \right\} & \text{for } n = 2. \end{cases}$$

From the inequalities (1.3) and (1.4), it follows that $C(q,\eta) > 0$.

(1.1) Theorem. *Let $n \geq 3$, $u_0 \in L^\infty(\Omega)$, non-negative and $u_0 \not\equiv 0$, then the problem*

$$\begin{cases} u_t - \Delta u - \lambda^* \dfrac{q(x)}{|x|^2} u &= \lambda \eta u \quad \text{in} \quad \Omega \times (0,\infty) \\ u(x,0) &= u_0(x) \quad \text{in} \quad \Omega \\ u(x,t) &= 0 \quad \text{on} \quad \partial\Omega \times (0,\infty), \end{cases} \tag{1.6}$$

admits a unique global non-negative solution $u \in C\left([0,\infty); L^2(\Omega)\right) \cap L^2\left(0,\infty; H_0^1(\Omega)\right)$ with $u' \in L^2\left(0,\infty; H^{-1}(\Omega)\right)$, $u \not\equiv 0$; for all λ, $0 \leq \lambda \leq C(q,\eta)$, provided q satisfies

$$\liminf_{x \to 0} \left(\log \frac{1}{|x|} \right)^2 (1 - q(x)) > \frac{3}{(n-2)^2}. \tag{1.7}$$

But for any $u_0 \in L^2(\Omega)$, $u_0 \geq 0$, $u_0 \not\equiv 0$ and for any $\lambda \geq 0$ with q satisfying

$$\sup_{|x|<R} \left(\log \frac{R}{|x|} \right)^2 (1 - q(x)) \leq \frac{3}{(n-2)^2}, \quad R > 0 \tag{1.8}$$

the problem(1.6) does not admit any non-negative H^1 solution, even locally in time t.

(1.2) Theorem. *For $n = 2$, $u_0 \in L^\infty(\Omega)$, non-negative and $u_0 \not\equiv 0$, the problem*

$$\begin{cases} u_t - \Delta u - \dfrac{1}{4} \dfrac{q(x)}{(|x| \, \log \, R/|x|)^2} u &= \lambda \eta u \quad \text{in} \quad \Omega \times (0,\infty) \\ u(x,0) &= u_0(x) \quad \text{in} \quad \Omega \\ u(x,t) &= 0 \quad \text{on} \quad \partial\Omega \times (0,\infty), \end{cases} \tag{1.9}$$

admits a unique global non-negative solution $u \in C\left([0,\infty); L^2(\Omega)\right) \cap L^2\left(0,\infty; H_0^1(\Omega)\right)$ with $u' \in L^2\left(0,\infty; H^{-1}(\Omega)\right)$, $u \not\equiv 0$; for all λ, $0 \leq \lambda \leq C(q,\eta)$, provided q satisfies

$$\liminf_{x \to 0} \left(\log\log \frac{1}{|x|} \right)^2 (1 - q(x)) > 3\,. \tag{1.10}$$

But for any $u_0 \in L^2(\Omega)$, $u_0 \geq 0$, $u_0 \not\equiv 0$, and q satisfying

$$\sup_{|x|<R} \left(\log\log \frac{R}{|x|} \right)^2 (1 - q(x)) \leq 3, \quad R > 0 \tag{1.11}$$

the problem (1.9) does not admit any non-negative H^1 solution, even locally in time t, for any $\lambda \geq 0$.

(1.3) Remark. *For $n \geq 3$, by taking $q = 1 - \dfrac{C}{(\log R/|x|)^{\alpha}}$, and $\eta \equiv 1$, we can see that the problem (1.6) has a unique global solution in $H_0^1(\Omega)$, if and only if $0 < \alpha < 2$, or $\alpha = 2$ and $C > 3/(n-2)^2$.*

We denote by $W = W\left(0,\infty; H_0^1(\Omega), H^{-1}(\Omega)\right)$, the Hilbert space:

$$W\left(0,\infty; H_0^1(\Omega), H^{-1}(\Omega)\right) := \left\{ u \;\middle|\; u \in L^2\left(0,\infty; H_0^1(\Omega)\right),\, u' \in L^2\left(0,\infty; H^{-1}(\Omega)\right) \right\} ,$$

equipped with the norm $\|u\|_W = \left(\displaystyle\int_0^{\infty} \left(\|\nabla u(t)\|^2 + \|u'(t)\|_{H^{-1}}^2 \right) dt \right)^{1/2}$.

Definition. A function $u \in W\left(0,\infty; H_0^1(\Omega), H^{-1}(\Omega)\right)$, is said to be a solution to the problem (1.6) (similar for (1.9)), if for all $v \in H_0^1(\Omega)$

$$\frac{d}{dt} \int_{\Omega} u(t) v \, dx + \langle L_{\lambda^* q} u(t), v \rangle_{H^1} = 0 \tag{1.12}$$

in the sense of $D'(0,\infty)$ and $u(0) = u_0$. Here, $L_{\lambda^* q} := -\Delta - \lambda^* \dfrac{q(x)}{|x|^2} - \lambda\eta$, and $\langle L_{\lambda^* q}\, .\, ,\, .\, \rangle_{H^1} : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$, is the bilinear functional defined by

$$\langle L_{\lambda^* q} u, v \rangle_{H^1} := \int_{\Omega} \langle \nabla u, \nabla v \rangle \, dx - \lambda^* \int_{\Omega} \frac{u\, q(x)}{|x|^2} v \, dx - \lambda \int_{\Omega} uv\, \eta(x)\, dx \,, \quad \forall\ u,\, v \in H_0^1(\Omega) \ .$$

§2 Preliminaries

To prove the non-existence of local solutions, we need a Theorem of Baras and Goldstein [Theorem 2.2 and Remark 7.1, 4] and the following two Lemmas.

(2.1) Baras and Goldstein Theorem. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, $0 \in \Omega$ and $\phi > 0$, such that $V(x) := -\dfrac{\Delta\phi}{\phi} \in L^1(\Omega)$ and $\phi\Delta\phi \in L^1(\Omega)$. Assume that u solution (in the sense of distributions) to the heat problem (1.1) with $0 \leq u_0 \in L^2(\Omega)$ and $u_0 \not\equiv 0$. Moreover, if ϕ be such that $\displaystyle\int_{\Omega} u_0 \phi \, dx < \infty$ and for $B_r := B(0,r)$, some $r > 0$, the following weighted Sobolev inequality*

$$\left(\int_{B_r} |v|^p |\phi|^2 dx \right)^{2/p} \leq C(p,r,R) \int_{B_r} \left(|\nabla v|^2 + |v|^2 \right) |\phi|^2 dx \tag{2.13}$$

holds for all radial functions $v \in H^1(B_r)$ for some $p > 2$, then for any $\Omega' \subset\subset \Omega$ and $0 < t_1 < t_2 < \infty$ there exists $C(t_1, t_2, \Omega') > 0$ such that

$$u(x,t) \geq C(t_1, t_2, \Omega')\phi(x) \quad \text{in} \quad \Omega' \times [t_1, t_2] \,. \tag{2.14}$$

(2.2) Lemma. *Let $B_r := B(0,r) \subset \mathbb{R}^n$, $n \geq 3$, be the ball of radius $r > 0$. Let $R > 2r$, then there exists constant $C(p,r,R) > 0$ such that the following weighted Sobolev inequality*

$$\left(\int\limits_{B_r} |u|^p \frac{(\log R/|x|)^{-1}}{|x|^{n-2}} dx \right)^{2/p} \leq C(p,r,R) \int\limits_{B_r} \left(|\nabla u|^2 + |u|^2 \right) \frac{(\log R/|x|)^{-1}}{|x|^{n-2}} dx \quad (2.15)$$

holds for all radial functions $u \in H^1(B_r)$ and for every $p > 2$.

Proof. We first prove the inequality (2.15) for any $u \in C^1[0,r]$. Let $u \in C^1[0,2r]$ such that $u(2r) = 0$, $R > 2r$ and call $\phi(x) := |x|^{-\frac{n-2}{2}} (\log R/|x|)^{-1/2}$. Now observe that

$$\begin{aligned}
|u(s)| &= \left| -\int_s^{2r} u'(t) dt \right| \\
&\leq \int_s^{2r} |u'(t)| \phi t^{\frac{n-1}{2}} \left(\phi^{-1} t^{-\frac{n-1}{2}} \right) dt \\
&\leq \left(\int_s^{2r} |u'(t)|^2 \phi^2 t^{n-1} dt \right)^{1/2} \left(\int_s^{2r} (\log R/t) \frac{dt}{t} \right)^{1/2} \\
&= \frac{1}{2} \left(\int_s^{2r} |u'(t)|^2 \phi^2 t^{n-1} dt \right)^{1/2} \left[(\log R/s)^2 - (\log R/2r)^2 \right]^{1/2} \\
&\leq \frac{1}{2} \left(\int_0^{2r} |u'(t)|^2 \phi^2 t^{n-1} dt \right)^{1/2} (\log R/s) \, .
\end{aligned}$$

Therefore for any $p > 2$, we have

$$\begin{aligned}
\int_0^{2r} |u|^p \phi^2 t^{n-1} dt &\leq \left(\frac{1}{2} \right)^p \left(\int_0^{2r} |u'(t)|^2 \phi^2 t^{n-1} dt \right)^{p/2} \int_0^{2r} (\log R/t)^p \, \phi^2 t^{n-1} dt \\
&= \left(\frac{1}{2} \right)^p \left(\int_0^{2r} |u'(t)|^2 \phi^2 t^{n-1} dt \right)^{p/2} \int_0^{2r} (\log R/t)^{p-1} \, t \, dt \\
&\leq R^2 \left(\frac{1}{2} \right)^p \left(\int_0^{2r} |u'(t)|^2 \phi^2 t^{n-1} dt \right)^{p/2} \int_0^{\infty} t^{p-1} e^{-2t} \, dt \\
&= \frac{\Gamma(p) R^2}{2^{2p}} \left(\int_0^{2r} |u'(t)|^2 \phi^2 t^{n-1} dt \right)^{p/2} \qquad (2.16)
\end{aligned}$$

Now, let $\xi \in C^1[r,2r]$ such that $0 \leq \xi \leq 1$, $\xi = 1$ on $\left[r, \frac{5}{4} r \right]$ and $\xi = 0$ on $\left[\frac{3}{2} r, 2r \right]$. For $u \in C^1[0,r]$, define the extension $\tilde{u}$ on $[0,2r]$, by

$$\tilde{u}(s) := \begin{cases} u(s), & \text{for } 0 \leq s \leq r \, , \\ u(2r-s)\xi(s), & \text{for } r \leq s \leq 2r \, . \end{cases}$$

Therefore, by the inequality (2.16), we obtain

$$\begin{aligned}\left(\int_0^r |u|^p\phi^2 t^{n-1}dt\right)^{2/p} &\leq \left(\int_0^{2r} |\tilde{u}|^p\phi^2 t^{n-1}dt\right)^{2/p}\\ &\leq \frac{\left(R^2\Gamma(p)\right)^{2/p}}{8}\int_0^{2r}|\tilde{u}'|^2\phi^2 t^{n-1}dt\\ &\leq C\int_0^r |u'|^2\phi^2 t^{n-1}dt + 2C\int_r^{2r}|u'(2r-t)|^2\xi^2\phi^2 t^{n-1}dt\\ &+ 2C\int_r^{2r}|u(2r-t)|^2|\xi'|^2\phi^2 t^{n-1}dt\,. \qquad (2.17)\end{aligned}$$

Now, we notice that

$$\begin{aligned}\int_r^{2r}|u'(2r-t)|^2\xi^2\phi^2 t^{n-1}dt &= \int_r^{2r}|u'(2r-t)|^2\xi^2\left(\log R/t\right)^{-1}t\,dt\\ &\leq \int_r^{\frac{3r}{2}}|u'(2r-t)|^2\left(\log R/t\right)^{-1}t\,dt\\ &= \int_{\frac{r}{2}}^r |u'|^2\frac{2r-t}{t\,\log\left[R/(2r-t)\right]}\,dt\\ &= \int_{\frac{r}{2}}^r |u'|^2\frac{t}{\log R/t}\frac{(2r-t)\log R/t}{\log\left[R/(2r-t)\right]}\,dt\\ &\leq 3\left(\frac{\log 2R/r}{\log 2R/3r}\right)\int_0^r|u'|^2\frac{t}{\log R/t}dt\\ &= 3\left(\frac{\log 2R/r}{\log 2R/3r}\right)\int_0^r|u'|^2\phi^2\,t^{n-1}dt \qquad (2.18)\end{aligned}$$

and

$$\begin{aligned}\int_r^{2r}|u(2r-t)|^2|\xi'|^2\phi^2\,t^{n-1}dt &\leq \frac{C}{r^2}\int_r^{2r}|u(2r-t)|^2\phi^2\,t^{n-1}dt\\ &\leq \frac{C}{r^2}\left(\frac{\log 2R/r}{\log 2R/3r}\right)\int_0^r|u|^2\phi^2\,t^{n-1}dt\,. \qquad (2.19)\end{aligned}$$

Now the inequality (2.15) follows by substituting (2.18) and (2.19) in (2.17). ∎

(2.3) Lemma. *Let B_r, be the ball of radius $r>0$ in the plane $\mathbb{R}^2$. Assume $R\geq 2er$ and $\phi(x):=\left(\log R/|x|\right)^{1/2}\left(\log\log R/|x|\right)^{-1/2}$, then there exists constant $C(p,r,R)>0$ such that the following weighted Sobolev inequality*

$$\left(\int\limits_{B_r}|u|^p|\phi|^2dx\right)^{2/p}\leq C(p,r,R)\int\limits_{B_r}\left(|\nabla u|^2+|u|^2\right)|\phi|^2dx \qquad (2.20)$$

holds for any radial functions $u\in H^1(B_r)$ and for every $p>2$.

Proof. By following the proof of Lemma 2.2, we conclude the proof.

§3 Proof of Theorem 1.1

Existence Part : We observe that, $L_s^{-1} : L^2(\Omega) \to L^2(\Omega)$, for any $0 < s < 1$, is a compact, self-adjoint and positive definite operator. By the standard semigroup theory [4], there exists a unique function $u_s \in W\left(0,\infty; H_0^1(\Omega), H^{-1}(\Omega)\right)$, $u_s \geq 0$, $u_s \not\equiv 0$ satisfying the following heat equation

$$(u_s)_t + L_s u_s = 0 \qquad \text{in} \quad \Omega \times (0,\infty) \tag{3.21}$$

with the initial data

$$u_s(x,0) = u_0(x) \qquad \text{in } \Omega\,.$$

Now by taking $v = u_s(t)$ in (1.12), we have

$$\frac{1}{2}\frac{d}{dt}\int\limits_\Omega |u_s(t)|^2 dx = -\int\limits_\Omega |\nabla u_s(t)|^2 dx + \lambda^* s \int\limits_\Omega \frac{q|u_s(t)|^2}{|x|^2} dx + \lambda s \int\limits_\Omega \eta |u_s(t)|^2 dx\ . \tag{3.22}$$

Now our aim to show that the sequence $(u_s)_{0<s<1}$ is bounded in the function space $W\left(0,\infty; H_0^1(\Omega), H^{-1}(\Omega)\right)$. But this does not follow immediately from (3.21). So we plan to find a suitable test function ϕ such that u_s is pointwise bounded above by the function ϕ. To obtain such ϕ, we need the following Lemma. This Lemma is contained in the recent work of Adimurthi-Sandeep [2].

(3.1) Lemma. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$ with $0 \in \Omega$ and take q, η, $C(q,\eta)$ as defined in the introduction, with the extra assumption (1.7) on q. Then for any $0 \leq \lambda \leq C(q,\eta)$, there exists $u \in H_0^1(\Omega)$, $u > 0$, satisfying the following equation*

$$-\Delta u - \lambda^* \frac{q(x)}{|x|^2}\, u - \lambda \eta u = \lambda^1(\Omega, q, \eta) u \qquad \text{in} \quad \Omega \tag{3.23}$$

for

$$\lambda^1(\Omega,q,\eta) = \inf_{H_0^1(\Omega)} \left\{ \int\limits_\Omega |\nabla u|^2 dx - \lambda^* \int\limits_\Omega \frac{|u|^2 q}{|x|^2} dx - \lambda \int\limits_\Omega |u|^2 \eta dx \;\Big|\; \int\limits_\Omega |u|^2 dx = 1 \right\}. \tag{3.24}$$

Let Ω_1, be a bounded open subset of $\mathbb{R}^n$ such that $\bar{\Omega} \subset \Omega_1$. Take $v \in H_0^1(\Omega_1)$, $v > 0$ satisfying the equation (3.23) in the domain Ω_1, where $\lambda^1(\Omega_1, q, \eta)$ as in (3.24) and we simply denote, $\lambda^1(\Omega_1, q, \eta)$ by $\lambda^1(\Omega_1)$. Define the function ϕ on $\Omega_1 \times (0,\infty)$, by

$$\phi(x,t) := e^{-\lambda^1(\Omega_1)\, t} v(x)\,,$$

then $\phi \in W\left(0,\infty; H_0^1(\Omega_1), H^{-1}(\Omega_1)\right)$, $\phi > 0$ in $\Omega \times (0,\infty)$ and

$$\phi_t + L_{\lambda^* q}\phi = e^{-\lambda^1(\Omega_1)\, t}\left(-\lambda^1(\Omega_1) v + L_{\lambda^* q} v\right) = 0 \qquad \text{in} \quad \Omega_1 \times (0,\infty),$$

where $L_{\lambda^* q} := -\Delta - \lambda^* \dfrac{q(x)}{|x|^2} - \lambda\eta$. Hence ϕ is a solution to the problem

$$\left\{\begin{array}{rcll} \phi_t + L_{\lambda^* q}\phi &=& 0 & \text{in} \quad \Omega_1 \times (0,\infty) \\ \phi(x,0) &=& v(x) & \text{in} \quad \Omega_1 \\ \phi(x,t) &=& 0 & \text{on} \quad \partial\Omega_1 \times (0,\infty). \end{array}\right. \tag{3.25}$$

Now we define

$$w_s(x,t) := M\phi(x,t) - u_s(x,t) \qquad \text{on } \Omega \times (0,\infty)\,.$$

Since $u_s(x,0) = u_0(x) \in L^\infty(\Omega)$, we can choose some $M > 0$ such that

$$w_s(x,0) := w_s(0) = Mv(x) - u_0(x) \geq 0 \qquad \text{in } \Omega\,,$$

(we have taken $\Omega \subset\subset \Omega_1$ and $v \in H_0^1(\Omega_1)$, just to ensure that $v > 0$, on $\partial\Omega$, so that $w_s(0) \geq 0$ in Ω). From (3.21) and (3.25), we have $(w_s)_t + L_{\lambda^* q} w_s \geq 0$ in $\Omega \times (0,\infty)$ and hence $w_s \geq 0$ in $\Omega \times (0,\infty)$. In other words, we have

$$u_s \leq Me^{-\lambda^1(\Omega_1)\,t} v(x) \qquad \text{in} \quad \Omega \times (0,\infty). \tag{3.26}$$

To show that u_s is bounded in $L^2\left(0,\infty; H_0^1(\Omega)\right)$, we observe that, by using (3.26) in (3.22) and integrating from 0 to t

$$\begin{aligned} \frac{1}{2}\int\limits_\Omega |u_s(t)|^2 dx + \int_0^t \int\limits_\Omega |\nabla u_s(t)|^2 dx\, dt &= \lambda^* s \int_0^t \int\limits_\Omega \frac{q|u_s(t)|^2}{|x|^2} dx\, dt \\ &\quad + \lambda\, s \int_0^t \int\limits_\Omega \eta |u_s(t)|^2 dx\, dt + \frac{1}{2}\int\limits_\Omega |u_0(x)|^2 dt \\ &\leq \frac{M^2}{\lambda^1(\Omega_1)} \left(\lambda^* \int\limits_\Omega \frac{|v|^2}{|x|^2} dx + \lambda \int\limits_\Omega \eta(x)|v|^2 dx\right) \\ &\quad + \frac{1}{2}\int\limits_\Omega |u_0(x)|^2 dx \\ &\leq \frac{M^2}{\lambda^1(\Omega_1)} \int\limits_\Omega |\nabla v(t)|^2 dx + \frac{1}{2}\int\limits_\Omega |u_0(x)|^2 dx \end{aligned}$$

and hence, u_s is bounded in $L^2\left(0,\infty; H_0^1(\Omega)\right)$. Now from (3.21), we have for all $w \in H_0^1(\Omega)$ with $\|\nabla w\|_2 \leq 1$, by Hölder and Hardy inequality, we have

$$\begin{aligned} |\langle u_s'(t), w\rangle_{H^1}| &\leq \left(\int\limits_\Omega |\nabla u_s(t)|^2 dx\right)^{1/2} + \lambda^* \left(\int\limits_\Omega \frac{|u_s(t)|^2}{|x|^2} dx\right)^{1/2} \left(\int\limits_\Omega \frac{|w|^2}{|x|^2} dx\right)^{1/2} \\ &\quad + \lambda \left(\int\limits_\Omega \eta |u_s(t)|^2 dx\right)^{1/2} \left(\int\limits_\Omega \eta |w|^2 dx\right)^{1/2} \\ &\leq C \left(\int\limits_\Omega |\nabla u_s(t)|^2 dx\right)^{1/2} \end{aligned}$$

for some constant $C > 0$ and hence $u_s'(t)$ is bounded in $L^2\left(0,\infty; H^{-1}(\Omega)\right)$.

Therefore, in the limit $s \to 1$ (passing through a subsequence if necessary), we have

$$\begin{cases} u_s \rightharpoonup u & \text{weakly in} \quad L^2\left(0,\infty; H_0^1(\Omega)\right) \\ u_s' \rightharpoonup u' & \text{weakly in} \quad L^2\left(0,\infty; H^{-1}(\Omega)\right). \end{cases} \tag{3.27}$$

As $u \in W\left(0,\infty; H_0^1(\Omega), H^{-1}(\Omega)\right)$, $u(x,0) = u(0) \in L^2(\Omega)$. Since u_s satisfies the equation (3.21), we have for all $w \in H_0^1(\Omega)$ and for all $\zeta \in C_c^\infty(0,\infty)$,

$$-\int_0^\infty \int_\Omega u_s(t)\, w\, \zeta'(t)\, dxdt + \int_0^\infty \langle L_s u_s(t), w\rangle_{H^1}\, \zeta(t)\, dt = 0\ , \tag{3.28}$$

and hence by (3.27), we have as $s \to 1$

$$-\int_0^\infty \int_\Omega u(t)\, w\, \zeta'(t)\, dxdt + \int_0^\infty \langle L_{\lambda^* q} u, w\, \zeta(t)\rangle_{H^1}\, dt = 0 \tag{3.29}$$

for all $w \in H_0^1(\Omega)$ and for all $\zeta \in C_c^\infty(0,\infty)$. Therefore u satisfies the equation

$$u_t + L_{\lambda^* q} u = 0 \qquad \text{in } \Omega \times (0,\infty)\ ,$$

with the initial data $u(x,0) = u(0)$ in Ω and it is easy to show that $u(0) = u_0$. Since $u_0 \geq 0$ it follows that $u \geq 0$. Now we observe that for any $t > 0$, the solution u satisfies

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u(t)|^2 dx = -\int_\Omega |\nabla u(t)|^2 dx + \lambda^* \int_\Omega \frac{q|u(t)|^2}{|x|^2} dx + \lambda \int_\Omega \eta |u(t)|^2 dx\ .$$

By the definition of $\lambda^1(\Omega, q, \eta)$ in (3.24) and invoking Gronwall's Lemma, we have

$$\int_\Omega |u(t)|^2 dx \leq e^{-2\lambda^1(\Omega,q,\eta)\, t} \int_\Omega |u_0(x)|^2 dx \tag{3.30}$$

Hence, the uniqueness follows from (3.30). ∎

Non-existence part Let us assume that q satisfies the condition (1.7) and u be a solution of (1.6). From the condition (1.7), there exists $T > 0$ such that $B_T := B(0,T) \subset \Omega$ and

$$q(x) \geq V_0(x) := \left(1 - \frac{3}{(n-2)^2}\left(\log R/|x|\right)^{-2}\right) \quad \text{in } B_T\,. \tag{3.31}$$

Let us assume that, v satisfies

$$\begin{cases} v_t - \Delta v &= \lambda^* \dfrac{V_0(x)}{|x|^2}\, v & \text{in} & B_T \times (0,\infty) \\ u(x,0) &= u_0(x) & \text{in} & B_T \\ u(x,t) &= 0 & \text{on} & \partial B_T \times (0,\infty), \end{cases} \tag{3.32}$$

By using the inequality (1.3), it is easy to show that $u \geq v$ in $B_T \times (0,\infty)$. Hence without loss of generality we can assume $q(x) = V_0(x)$ in Ω and $\eta = 0$. Our aim is prove that, for any $0 < t_1 < t_2 < \infty$ and $\Omega' \subset\subset \Omega$, there exists a constant $C(t_1, t_2, \Omega') > 0$ such that

$$u(x,t) \geq C(t_1,t_2,\Omega')|x|^{-\frac{n-2}{2}} \left(\log R/|x|\right)^{-1/2} \quad \text{in } \Omega' \times [t_1,t_2]\,. \tag{3.33}$$

To prove the estimate (3.33) we follow the idea of Baras and Goldstein. In order to use the Theorem 2.1, we need to choose the test function ϕ such that $\phi\Delta\phi \in L^1(\Omega)$ and the weighted Sobolev inequality (2.13) holds with the weight ϕ. By Lemma 2.2, we notice that the function $\phi(x) := |x|^{-\frac{n-2}{2}} \left(\log R/|x|\right)^{-1/2}$, satisfies (2.13) but $\phi\Delta\phi \notin L^1(\Omega)$. So we need to regularize the function ϕ such that all the conditions in Theorem 2.1 holds. We notice that, for any $k > 1/2$, the function $\phi_k(x) := |x|^{-\frac{n-2}{2}} \left(\log R/|x|\right)^{-k}$ satisfies all the conditions of Theorem 2.1. Moreover, for any $k > 1/2$ it satisfies the following equation:

$$\phi_k\Delta\phi_k = \frac{\lambda^*\phi_k^2}{|x|^2}\left(1 - \frac{4k(k+1)}{(n-2)^2}\left(\log R/|x|\right)^{-2}\right) \quad \text{in } \Omega \tag{3.34}$$

Let $q_k := \left(1 - \frac{4k(k+1)}{(n-2)^2}\left(\log R/|x|\right)^{-2}\right)$ in Ω and u_k be a solution to the problem

$$\begin{cases} \partial u_k/\partial t - \Delta u_k &= \lambda^* \dfrac{q_k(x)}{|x|^2}\, u_k \quad \text{in} \quad \Omega \times (0,\infty) \\ u_k(x,0) &= u_0(x) \qquad\quad \text{in} \quad \Omega \\ u_k(x,t) &= 0 \qquad\quad \text{on} \quad \partial\Omega \times (0,\infty)\,. \end{cases}$$

Therefore by Theorem 2.1, there exists a constant $C(t_1, t_2, \Omega') > 0$, independent of k such that

$$u_k(x,t) \geq C(t_1,t_2,\Omega')|x|^{-\frac{n-2}{2}} \left(\log R/|x|\right)^{k} \quad \text{in } \Omega' \times [t_1,t_2]\,. \tag{3.35}$$

Since $V_0(x) \geq q_k(x)$, we obtain $u(x,t) \geq u_k(x,t)$ in $\Omega \times (0,\infty)$ for all $k > 1/2$. By considering the limit $k \to 1/2$, from (3.35), we obtain the estimate (3.33) and it contradicts the fact that $u(t) \in H_0^1(\Omega)$, because $\int\limits_\Omega \dfrac{|\phi|^2}{|x|^2} dx = \infty$. ∎

§4 Proof of Theorem 1.2

Existence : In the proof of Theorem 1.1, we have chosen the test function ϕ through the Lemma 3.1. Here also we have a similar Lemma, which will enable us to carry out the proof and this Lemma is again contained in [2].

(4.1) Lemma. *Let Ω be a bounded domain in $\mathbb{R}^2$, with $0 \in \Omega$ and $R \geq e^{e^e} \sup_{\Omega} |x|$. Take q, η, $C(q,\eta)$ as defined in section 5.1 with the extra assumption (1.10) on q. Then for any $0 \leq \lambda \leq C(q,\eta)$, there exists $u \in H_0^1(\Omega)$, $u > 0$, satisfying the following equation*

$$-\Delta u - \frac{1}{4} \frac{q(x)}{|x|^2 \left(\log R/|x|\right)^2} u - \lambda \eta u = \mu^1(\Omega, q, \eta) u \qquad \text{in} \quad \Omega \tag{4.36}$$

where,

$$\mu^1(\Omega, q, \eta) := \inf_{H_0^1(\Omega)} \left\{ \int_\Omega |\nabla u|^2 - \frac{1}{4} \int_\Omega \frac{q|u|^2}{|x|^2((\log R/|x|)^2} - \lambda \int_\Omega |u|^2 \eta \;\Big|\; \int_\Omega |u|^2 = 1 \right\} \tag{4.37}$$

Now, is easy to complete the proof by following proof of Theorem 1.1. Here, the only differences are, we have to use Lemma 4.1 instead of Lemma 3.1 and inequality (1.4) instead of (1.3) in the respective places . ∎

Proof of Non-Existence Here by taking $\phi_k := (\log R/|x|)^{1/2} (\log\log R/|x|)^{-k}$, $k > 1/2$ and using the Lemma 2.3 and Theorem 2.1, the non-existence follows.

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