THE NONLINEAR BENDING-TORSION THEORY FOR CURVED RODS AS Γ-LIMIT OF THREE-DIMENSIONAL ELASTICITY

LUCIA SCARDIA

Abstract. The problem of the rigorous derivation of one-dimensional models for nonlinearly elastic curved beams is studied in a variational setting. Considering different scalings of the three-dimensional energy and passing to the limit as the diameter of the beam goes to zero, a nonlinear model for strings and a bending-torsion theory for rods are deduced.

Keywords: dimension reduction, curved beams, nonlinear elasticity

2000 Mathematics Subject Classification: 74K10, 49J45

1. Introduction

This paper is part of a series of recent works concerning the rigorous derivation of lower dimensional models for thin domains from nonlinear three-dimensional elasticity, by means of $\Gamma-$ convergence.

The first result in this direction is due to E. Acerbi, G. Buttazzo and D. Percivale (see [1]), who deduced a nonlinear model for elastic strings by means of a 3D-1D dimension reduction. The two-dimensional analogue was studied by H. Le Dret and A. Raoult in [7], where they derived a nonlinear model for elastic membranes. The more delicate case of plates was justified more recently by G. Friesecke, R.D. James and S. Müller in [4] (see also [6] for a complete survey on plate theories). The case of shells was considered in [8] and [5].

As for one-dimensional models, nonlinear theories for elastic rods have been deduced by M.G. Mora, S. Müller (see [9], [10]) and, independently, by O. Pantz (see [11]). In all these results, as in [1], the beam is assumed to be straight in the unstressed configuration.

In this paper we study the case of a heterogeneous curved beam made of a hyperelastic material. Let $\Omega := (0, L) \times D$, where $L > 0$ and D is a bounded domain in $\mathbb{R}^2$, and let $h > 0$. We consider a beam, whose reference configuration is given by

$$\widetilde{\Omega}_h := \{\gamma(s) + h\,\xi\,\nu_2(s) + h\,\zeta\,\nu_3(s) : (s, \xi, \zeta) \in \Omega\},$$

where $\gamma : (0, L) \to \mathbb{R}^3$ is a smooth simple curve describing the mid-fiber of the beam, and $\nu_2, \nu_3 : (0, L) \to \mathbb{R}^3$ are two smooth vectors such that (γ', ν_2, ν_3) provide an orthonormal frame along the curve. In particular, the cross section of the beam is constant along γ and is given by the set hD. It is natural to parametrize $\widetilde{\Omega}_h$ through the map

$$\Psi^{(h)} : \Omega \to \widetilde{\Omega}_h, \quad (s, \xi, \zeta) \mapsto \gamma(s) + h\,\xi\,\nu_2(s) + h\,\zeta\,\nu_3(s),$$

which is one-to-one for h small enough.

The starting point of our approach is the elastic energy per unit volume

$$\tilde{I}^{(h)}(\tilde{v}) := \frac{1}{h^2} \int_{\widetilde{\Omega}_h} W\big(\big(\Psi^{(h)}\big)^{-1}(x), \nabla\tilde{v}(x)\big)dx$$

of a deformation $\tilde{v} \in W^{1,2}(\widetilde{\Omega}_h; \mathbb{R}^3)$. The stored energy density $W : \Omega \times \mathbb{M}^{3\times 3} \to [0, +\infty]$ has to satisfy some natural conditions; i.e.,

- W is frame indifferent: $W(z, RF) = W(z, F)$ for a.e. $z \in \Omega$, every $F \in \mathbb{M}^{3\times 3}$, and every $R \in SO(3)$;
- $W(z, F) \geq C\,\mathrm{dist}^2(F, SO(3))$ for a.e. $z \in \Omega$ and every $F \in \mathbb{M}^{3\times 3}$;
- $W(z, R) = 0$ for a.e. $z \in \Omega$ and every $R \in SO(3)$.

For the complete list of assumptions on W we refer to Section 2.

The aim of this work is to study the asymptotic behaviour of different scalings of the energy $\tilde{I}^{(h)}$, as $h \to 0$, by means of Γ-convergence (see [2] for a comprehensive introduction to Γ-convergence). Heuristic arguments suggest that, as in the case of straight beams, energies of order 1 correspond to stretching and shearing deformations, leading to a *string theory*, while energies of order h^2 correspond to bending flexures and torsions keeping the mid-fiber unextended, leading to a *rod theory*.

The main results of the paper are contained in Section 3, where we identify the Γ-limit of the sequence of functionals $\big(\tilde{I}^{(h)}/h^2\big)$. We first show a compactness result for sequences of deformations having equibounded energies (Theorem 3.3). More precisely, given a sequence $\big(\tilde{v}^{(h)}\big) \subset W^{1,2}(\widetilde{\Omega}_h;\mathbb{R}^3)$ with $\tilde{I}^{(h)}(\tilde{v}^{(h)})/h^2 \leq C$, we prove that there exist a subsequence (not relabelled) and some constants $c^{(h)} \in \mathbb{R}^3$ such that

$$\begin{aligned}
&\tilde{v}^{(h)} \circ \Psi^{(h)} - c^{(h)} \to v \quad \text{strongly in } W^{1,2}(\Omega;\mathbb{R}^3),\\
&\frac{1}{h}\,\partial_\xi\big(\tilde{v}^{(h)} \circ \Psi^{(h)}\big) \to d_2 \quad \text{strongly in } L^2(\Omega;\mathbb{R}^3),\\
&\frac{1}{h}\,\partial_\zeta\big(\tilde{v}^{(h)} \circ \Psi^{(h)}\big) \to d_3 \quad \text{strongly in } L^2(\Omega;\mathbb{R}^3),
\end{aligned}$$

where (v, d_2, d_3) belongs to the class

$$\begin{aligned}
\mathcal{A} := \{(v, d_2, d_3) \in W^{2,2}((0,L);\mathbb{R}^3) \times W^{1,2}((0,L);\mathbb{R}^3) \times W^{1,2}((0,L);\mathbb{R}^3) :\\
(v'(s) \,|\, d_2(s) \,|\, d_3(s)) \in SO(3) \text{ for a.e. } s \text{ in } (0,L)\}.
\end{aligned}$$

The key ingredient in the proof is a geometric rigidity theorem proved by G. Friesecke, R.D. James and S. Müller in [4]. In Theorems 3.5 and 3.6 we show that the Γ-limit of the sequence $\big(\tilde{I}^{(h)}/h^2\big)$ is given by

$$I(v, d_2, d_3) := \begin{cases} \dfrac{1}{2}\displaystyle\int_0^L Q_2\big(s, \big(R^T(s)R'(s) - R_0^T(s)R_0'(s)\big)\big)ds & \text{if } (v, d_2, d_3) \in \mathcal{A},\\ +\infty & \text{otherwise,}\end{cases} \tag{1.1}$$

where $R := (v' \,|\, d_2 \,|\, d_3)$, $R_0 := (\gamma' \,|\, \nu_2 \,|\, \nu_3)$, and Q_2 is a quadratic form arising from a minimization procedure involving the quadratic form of linearized elasticity (see (3.19)). We point out that in Theorems 3.5 and 3.6 we do not require any growth condition from above on the energy density W.

We notice that in the limit problem the behaviour of the rod is described by a triple (v, d_2, d_3). The function v represents the deformation of the mid-fiber, which satisfies $|v'| = 1$ a.e., because of the constraint $(v' \,|\, d_2 \,|\, d_3) \in SO(3)$ a.e.. Therefore, the admissible deformations are only those leaving the mid-fiber unextended. Moreover, the triple (v, d_2, d_3) provides an orthonormal frame along the deformed curve; in particular, d_2 and d_3 belong to the normal plane to the deformed curve and describe the rotation undergone by the cross section.

Since $R = (v' \,|\, d_2 \,|\, d_3)$ is a rotation a.e., the matrix $R^T R'$ is skew-symmetric a.e. and its entries are given by

$$(R^T R')_{1k} = -(R^T R')_{k1} = v' \cdot d_k' \quad \text{for } k = 2, 3, \quad (R^T R')_{23} = -(R^T R')_{32} = d_2 \cdot d_3'.$$

It is easy to see that the scalar products $v' \cdot d_k'$ are related to curvature and therefore, to bending effects, while $d_2 \cdot d_3'$ is related to torsion and twist. We remark also that the energy depends explicitly on the reference state of the beam through the quantity $R_0^T R_0'$, which encodes informations about the bending and torsion of the beam in the initial configuration.

We notice that, specifying $R_0 = Id$ in (1.1), we recover the result for straight rods obtained in [9] and [11].

The last section of the paper is devoted to the study of lower scalings of the energy. Assuming that the energy density W satisfies a growth condition from above, we prove the Γ-convergence of the sequence $\big(\tilde{I}^{(h)}\big)$ to a functional corresponding to a string model. Finally we show that intermediate scalings of the energy between 1 and h^2 lead to a trivial Γ-limit.

2. Notations and formulation of the problem

In this section we describe the geometry of the unstressed curved beam. Let $\gamma : [0, L] \to \mathbb{R}^3$ be a simple regular curve of class C^2 parametrized by the arc-length and let $\tau = \dot{\gamma}$ be its unitary tangent vector. We assume that there exists an orthonormal frame of class C^1 along the curve. More precisely, we assume that there exists $R_0 \in C^1([0, L]; \mathbb{M}^{3\times 3})$ such that $R_0(s) \in SO(3)$ for every $s \in [0, L]$ and $R_0(s)\, e_1 = \tau(s)$ for every $s \in [0, L]$, where e_i, for $i = 1, 2, 3$, denotes the i-th vector of the canonical basis of $\mathbb{R}^3$ and $SO(3) = \big\{R \in \mathbb{M}^{3\times 3} : R^T R = Id,\ \det R = 1\big\}$. We set $\nu_k(s) := R_0(s)\, e_k$ for $k = 2, 3$.

Let $D \subset \mathbb{R}^2$ be a bounded open connected set with Lipschitz boundary such that

$$\int_D \xi\, \zeta\, d\xi\, d\zeta = 0 \tag{2.1}$$

and

$$\int_D \xi\, d\xi\, d\zeta = \int_D \zeta\, d\xi\, d\zeta = 0, \tag{2.2}$$

where (ξ, ζ) stands for the coordinates of a generic point of D. Without loss of generality, we can assume $\mathcal{L}^2(D) = 1$. We set $\Omega := (0, L) \times D$.
The reference configuration of the thin beam is given by

$$\widetilde{\Omega}_h := \{\gamma(s) + h\, \xi\, \nu_2(s) + h\, \zeta\, \nu_3(s) : (s, \xi, \zeta) \in \Omega\},$$

where h is a small positive parameter. Clearly the curve γ and the set D represent the middle fiber and the cross section of the beam, respectively. The set $\widetilde{\Omega}_h$ is parametrized by the map

$$\Psi^{(h)} : \Omega \to \widetilde{\Omega}_h : \quad (s, \xi, \zeta) \mapsto \gamma(s) + h\, \xi\, \nu_2(s) + h\, \zeta\, \nu_3(s),$$

which is one-to-one for h small enough and of class C^1.

We assume that the thin beam is made of a hyperelastic material whose stored energy density $W : \Omega \times \mathbb{M}^{3\times 3} \to [0, +\infty]$ is a Carathéodory function satisfying the following hypotheses:

(i) there exists $\delta > 0$ such that the function $F \mapsto W(z, F)$ is of class C^2 on the set $\big\{F \in \mathbb{M}^{3\times 3} : \mathrm{dist}(F, SO(3)) < \delta\big\}$ for a.e. $z \in \Omega$;

(ii) the second derivative $\partial^2 W/\partial F^2$ is a Carathéodory function on the set

$$\Omega \times \{F \in \mathbb{M}^{3\times 3} :\ \mathrm{dist}(F, SO(3)) < \delta\} \tag{2.3}$$

and there exists a constant $C_1 > 0$ such that

$$\left|\frac{\partial^2 W}{\partial F^2}(z, F)[G, G]\right| \le C_1 |\, G\,|^2 \quad \text{for a.e. } z \in \Omega, \text{ every} F \text{with} \mathrm{dist}(F, SO(3)) < \delta$$

and every $G \in \mathbb{M}^{3\times 3}_{sym}$;

(iii) W is frame indifferent, i.e., $W(z, RF) = W(z, F)$ for a.e. $z \in \Omega$, every $F \in \mathbb{M}^{3\times 3}$ and every $R \in SO(3)$;

(iv) $W(z, R) = 0$ for every $R \in SO(3)$;

(v) $\exists\ C_2 > 0$ independent of z such that $W(z, F) \ge C_2\, \mathrm{dist}^2(F, SO(3))$ for a.e. $z \in \Omega$ and every $F \in \mathbb{M}^{3\times 3}$.

Notice that, since we do not require any growth condition from above, W is allowed to assume the value $+\infty$ outside the set (2.3). Therefore our treatment covers the physically relevant case in which $W = +\infty$ for $\det F < 0$, $W \to +\infty$ as $\det F \to 0^+$.

Let $\tilde{v} \in W^{1,2}(\widetilde{\Omega}_h; \mathbb{R}^3)$ be a deformation of $\widetilde{\Omega}_h$. The elastic energy per unit volume associated to $\tilde{v}$ is defined by

$$\tilde{I}^{(h)}(\tilde{v}) := \frac{1}{h^2} \int_{\widetilde{\Omega}_h} W\big(\big(\Psi^{(h)}\big)^{-1}(x), \nabla \tilde{v}(x)\big) dx. \tag{2.4}$$

The main part of this work is devoted to the study of the asymptotic behaviour as $h \to 0$ of the sequence of functionals $\tilde{I}^{(h)}/h^2$. In the final part we will also discuss the scaling $\tilde{I}^{(h)}/h^\alpha$ for $0 \le \alpha < 2$.

We conclude this section by analysing some properties of the map $\Psi^{(h)}$, which will be useful in the sequel. We will use the following notation: for any function $z \in W^{1,2}(\Omega;\mathbb{R}^3)$ we set

$$\nabla_h z := \left(\partial_s z \,\Big|\, \frac{1}{h}\,\partial_\xi z \,\Big|\, \frac{1}{h}\,\partial_\zeta z\right).$$

We observe that $\nabla_h \Psi^{(h)}$ can be written as the sum of the rotation R_0 and a perturbation of order h, that is,

$$\nabla_h \Psi^{(h)}(s,\xi,\zeta) = R_0(s) + h\,\left(\xi\,\nu_2'(s) + \zeta\,\nu_3'(s)\right) \otimes e_1.$$

From this fact it follows that, as $h \to 0$,

$$\nabla_h \Psi^{(h)}(s,\xi,\zeta) \to R_0(s) \quad \text{and} \quad \det\left(\nabla_h \Psi^{(h)}\right) \to \det R_0 = 1 \ \text{ uniformly.} \tag{2.5}$$

This implies that for h small enough $\nabla_h \Psi^{(h)}$ is invertible at each point of Ω. Since the inverse of $\nabla_h \Psi^{(h)}$ can be written as

$$\left(\nabla_h \Psi^{(h)}\right)^{-1}(s,\xi,\zeta) = R_0^T(s) - h\,R_0^T(s)\left[(\xi\,\nu_2'(s) + \zeta\,\nu_3'(s)) \otimes e_1\right] R_0^T(s) + O(h^2) \tag{2.6}$$

with $O(h^2)/h^2$ uniformly bounded, we have also that $\left(\nabla_h \Psi^{(h)}\right)^{-1}$ converges to R_0^T uniformly.

3. Derivation of the bending-torsion theory for curved rods

The aim of this section is the study of the asymptotic behaviour of the sequence of functionals

$$\frac{1}{h^2}\,\tilde{I}^{(h)}(\tilde{v}) = \frac{1}{h^4}\int_{\widetilde{\Omega}_h} W\big(\left(\Psi^{(h)}\right)^{-1}(x), \nabla\tilde{v}(x)\big)dx$$

under the assumptions (i)-(v) of Section 2.

3.1. Compactness.

We will show a compactness result for sequences of deformations having equibounded energy $\tilde{I}^{(h)}/h^2$. A key ingredient in the proof is the following rigidity result, proved by G. Friesecke, R.D. James and S. Müller in [4].

Theorem 3.1. *Let U be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. Then there exists a constant $C(U)$ with the following property: for every $u \in W^{1,2}(U;\mathbb{R}^n)$ there is an associated rotation $R \in SO(n)$ such that*

$$\|\nabla u - R\|_{L^2(U)} \leq C(U)\,\|\mathrm{dist}(\nabla u, SO(n))\|_{L^2(U)}\,.$$

Remark 3.2. The constant $C(U)$ can be chosen independent of U for a family of sets that are Bilipschitz images of a cube (with uniform Lipschitz constants), as remarked in [5].

We introduce the class of limiting admissible deformations

$$\begin{aligned}\mathcal{A} := \{(v,d_2,d_3) \in W^{2,2}((0,L);\mathbb{R}^3) \times W^{1,2}((0,L);\mathbb{R}^3) \times W^{1,2}((0,L);\mathbb{R}^3) :\\ (v'(s)\,|\,d_2(s)\,|\,d_3(s)) \in SO(3) \text{ for a.e. s in } (0,L)\}.\end{aligned} \tag{3.1}$$

Now we are ready to state and prove the main result of this subsection.

Theorem 3.3. *Let $\left(\tilde{v}^{(h)}\right)$ be a sequence in $W^{1,2}\big(\widetilde{\Omega}_h;\mathbb{R}^3\big)$ such that*

$$\frac{1}{h^2}\,\tilde{I}^{(h)}(\tilde{v}^{(h)}) \leq c < +\infty. \tag{3.2}$$

Then there exist a triple $(v,d_2,d_3) \in \mathcal{A}$, a map $\overline{R} \in W^{1,2}((0,L);\mathbb{M}^{3\times 3})$ with $\overline{R}(s) \in SO(3)$ for a.e. $s \in [0,L]$, and some constants $c^{(h)} \in \mathbb{R}^3$ such that, up to subsequences,

$$\tilde{v}^{(h)} \circ \Psi^{(h)} - c^{(h)} \to v \quad \textit{strongly in } W^{1,2}(\Omega;\mathbb{R}^3), \tag{3.3}$$

$$\frac{1}{h}\,\partial_\xi\big(\tilde{v}^{(h)} \circ \Psi^{(h)}\big) \to d_2 \quad \textit{strongly in } L^2(\Omega;\mathbb{R}^3), \tag{3.4}$$

$$\frac{1}{h}\,\partial_\zeta\big(\tilde{v}^{(h)} \circ \Psi^{(h)}\big) \to d_3 \quad \textit{strongly in } L^2(\Omega;\mathbb{R}^3), \tag{3.5}$$

$$\nabla\tilde{v}^{(h)} \circ \Psi^{(h)} \to \overline{R} \quad \textit{strongly in } L^2(\Omega;\mathbb{M}^{3\times 3}). \tag{3.6}$$

Moreover, for a.e. $s \in [0,L]$, we have $(v'(s)\,|\,d_2(s)\,|\,d_3(s)) = \overline{R}(s)\,R_0(s)$, where $R_0 = (\tau\,|\,\nu_2\,|\,\nu_3)$.

Proof. Let $\big(\tilde{v}^{(h)}\big)$ be a sequence in $W^{1,2}(\widetilde{\Omega}_h;\mathbb{R}^3)$ satisfying (3.2). The assumption (v) on W implies that

$$\int_{\widetilde{\Omega}_h} \mathrm{dist}^2\big(\nabla\tilde{v}^{(h)}(x), SO(3)\big)dx < C\,h^4$$

for a suitable constant C. Using the change of variables $\Psi^{(h)}$, we have

$$\int_{\Omega} \mathrm{dist}^2\big(\nabla\tilde{v}^{(h)}\circ\Psi^{(h)}, SO(3)\big)\,\det\big(\nabla_h\Psi^{(h)}\big)ds\,d\xi\,d\zeta \le c\,h^2. \tag{3.7}$$

From (2.5) and the estimate

$$\mathrm{dist}^2(F, SO(3)) \ge \frac{1}{2}\,|\,F\,|^2 - 3,$$

we get the bound

$$\int_{\Omega} \big|\,\nabla\tilde{v}^{(h)}\circ\Psi^{(h)}\big|^2 ds\,d\xi\,d\zeta \le c. \tag{3.8}$$

Define the sequence $F^{(h)} := \nabla\tilde{v}^{(h)}\circ\Psi^{(h)}$; from (3.8) it follows that there exists a function $F\in L^2(\Omega;\mathbb{M}^{3\times3})$ such that, up to subsequences,

$$F^{(h)} \rightharpoonup F \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times3}). \tag{3.9}$$

Using Theorem 3.1, we will show that this convergence is in fact strong in L^2 and that the limit function F is a rotation a.e. depending only on the variable along the mid-fiber and belonging to $W^{1,2}((0,L);\mathbb{M}^{3\times3})$. The idea is to divide the domain $\widetilde{\Omega}_h$ in small curved cylinders, which are images of homotetic straight cylinders through the same Bilipschitz function. Then, we can apply the rigidity theorem to each small curved cylinder with the same constant. In this way we construct a piecewise constant rotation, which is close to the deformation gradient $\nabla\tilde{v}^{(h)}$ in the L^2 norm. For every small enough $h>0$, let $K_h\in\mathbb{N}$ satisfy

$$h \le \frac{L}{K_h} < 2\,h.$$

For every $a\in[0,L)\cap\dfrac{L}{K_h}\,\mathbb{N}$, define the segments

$$S_{a,K_h} := \begin{cases} (a, a+2\,h) & \text{if } \ a < L - \dfrac{L}{K_h}, \\ (L-2\,h, L) & \text{otherwise.} \end{cases}$$

Now consider the cylinders $C_{a,h} := S_{a,K_h}\times D$ and the subsets of $\widetilde{\Omega}_h$ defined by $\widetilde{C}_{a,h} := \Psi^{(h)}(C_{a,h})$. Remark that $\widetilde{C}_{a,h}$ is a Bilipschitz image of a cube of size h, that is $(a,0,0)+h\,\big((0,2)\times D\big)$, through the map Ψ defined as

$$\Psi : [0,L]\times\mathbb{R}^2 \to \mathbb{R}^3, \quad (s, y_2, y_3) \mapsto \gamma(s) + y_2\,\nu_2(s) + y_3\,\nu_3(s).$$

By Theorem 3.1 we obtain that there exists a constant rotation $\widetilde{R}_a^{(h)}$ such that

$$\int_{\widetilde{C}_{a,h}} \big|\,\nabla\tilde{v}^{(h)} - \widetilde{R}_a^{(h)}\big|^2 dx \le c\int_{\widetilde{C}_{a,h}} \mathrm{dist}^2(\nabla\tilde{v}^{(h)}, SO(3))dx. \tag{3.10}$$

The subscript a in $\widetilde{R}_a^{(h)}$ is used to remember that the rotation depends on the cylinder $\widetilde{C}_{a,h}$. In particular, since $\Psi^{(h)}\big(\big(a, a+\frac{L}{K_h}\big)\times D\big)\subset\widetilde{C}_{a,h}$, we get

$$\int_{\Psi^{(h)}\left(\left(a,a+\frac{L}{K_h}\right)\times D\right)} \big|\,\nabla\tilde{v}^{(h)} - \widetilde{R}_a^{(h)}\big|^2 dx \le c\int_{\widetilde{C}_{a,h}} \mathrm{dist}^2(\nabla\tilde{v}^{(h)}, SO(3))dx. \tag{3.11}$$

Changing variables in the integral on the left-hand side, inequality (3.11) becomes

$$\begin{aligned}\int_{\left(a,a+\frac{L}{K_h}\right)\times D} \big|\nabla\tilde v^{(h)}\circ\Psi^{(h)}-\widetilde R_a^{(h)}\big|^2 \det\big(\nabla\Psi^{(h)}\big)ds\,d\xi\,d\zeta \\ \le c\int_{\widetilde C_{a,h}} \mathrm{dist}^2\big(\nabla\tilde v^{(h)},SO(3)\big)dx \\ \le c\int_{\widetilde C_{a,h}} W\big(\big(\Psi^{(h)}\big)^{-1}(x),\nabla\tilde v^{(h)}(x)\big)dx.\end{aligned}$$

Notice that $\det\big(\nabla\Psi^{(h)}\big)=h^2\det\big(\nabla_h\Psi^{(h)}\big)$ and, since $\det\big(\nabla_h\Psi^{(h)}\big)\to 1$ uniformly,

$$\int_{\left(a,a+\frac{L}{K_h}\right)\times D} \big|\nabla\tilde v^{(h)}\circ\Psi^{(h)}-\widetilde R_a^{(h)}\big|^2 ds\,d\xi\,d\zeta \le \frac{c}{h^2}\int_{\widetilde C_{a,h}} W\big(\big(\Psi^{(h)}\big)^{-1}(x),\nabla\tilde v^{(h)}(x)\big)dx. \tag{3.12}$$

Now define the map $R^{(h)}:[0,L)\to SO(3)$ given by

$$R^{(h)}(s):=\widetilde R_a^{(h)}\quad \text{for } s\in\Big[a,a+\frac{L}{K_h}\Big),\ a\in[0,L)\cap\frac{L}{K_h}\,\mathbb{N}.$$

Summing (3.12) over $a\in[0,L)\cap\frac{L}{K_h}\,\mathbb{N}$ leads to

$$\int_\Omega \big|\nabla\tilde v^{(h)}\circ\Psi^{(h)}-R^{(h)}\big|^2 ds\,d\xi\,d\zeta \le \frac{c}{h^2}\int_{\widetilde\Omega_h} W\big(\big(\Psi^{(h)}\big)^{-1}(x),\nabla\tilde v^{(h)}(x)\big)dx$$

for a suitable constant independent of h. By (3.2) we obtain

$$\int_\Omega \big|\nabla\tilde v^{(h)}\circ\Psi^{(h)}-R^{(h)}\big|^2 ds\,d\xi\,d\zeta \le c\,h^2. \tag{3.13}$$

Now, applying iteratively estimate (3.12) in neighbouring cubes, one can prove the following difference quotient estimate for $R^{(h)}$: for every $I'\subset\subset[0,L]$ and every $\delta\in\mathbb{R}$ with $|\,\delta|\le\mathrm{dist}(I',\{0,L\})$

$$\int_{I'} \big|\,R^{(h)}(s+\delta)-R^{(h)}(s)\,\big|^2 ds \le c\,(|\,\delta\,|+h)^2, \tag{3.14}$$

with c independent of I' and δ (see [9], proof of Theorem 2.1). Using the Fréchet-Kolmogorov criterion, we deduce that, for every sequence $(h_j)\to 0$, there exists a subsequence of $R^{(h_j)}$ which converges strongly in $L^2(I';\mathbb{M}^{3\times3})$ to some $\overline R\in L^2(I';\mathbb{M}^{3\times3})$, with $\overline R(s)\in SO(3)$ for a.e. $s\in I'$. From (3.9) and (3.13) it follows that $F=\overline R$ a.e.. Moreover (2.5) and (3.7) imply the convergence of the L^2 norm of $F^{(h)}$ to the L^2 norm of $\overline R$, hence

$$F^{(h)}\to\overline R\quad \text{strongly in } L^2(\Omega;\mathbb{M}^{3\times3}).$$

This proves (3.6), once the regularity of the function $\overline R$ is shown. To this aim, divide both sides of the inequality (3.14) by $(|\delta|+h)^2$ and let $h\to 0$; then

$$\int_{I'} \frac{\big|\overline R(s+\delta)-\overline R(s)\big|^2}{|\delta|^2}\,ds \le c \tag{3.15}$$

and so $\overline R\in W^{1,2}(I';\mathbb{M}^{3\times3})$. But this holds for every $I'\subset\subset[0,L]$ with a constant independent of the subset I', hence $\overline R\in W^{1,2}((0,L);\mathbb{M}^{3\times3})$.
Now notice that

$$\nabla_h\big(\tilde v^{(h)}\circ\Psi^{(h)}\big)=\big(\nabla\tilde v^{(h)}\circ\Psi^{(h)}\big)\nabla_h\Psi^{(h)}=F^{(h)}\nabla_h\Psi^{(h)}; \tag{3.16}$$

by (2.5) and (3.6) we deduce that

$$\nabla_h\big(\tilde v^{(h)}\circ\Psi^{(h)}\big)\longrightarrow\overline R\,R_0\quad \text{strongly in } L^2(\Omega;\mathbb{M}^{3\times3}). \tag{3.17}$$

In particular, we have

$$\nabla\big(\tilde v^{(h)}\circ\Psi^{(h)}\big)\longrightarrow\big(\overline R\,R_0e_1\big)\otimes e_1\quad \text{strongly in } L^2(\Omega;\mathbb{M}^{3\times3}). \tag{3.18}$$

By Poincaré inequality there exist some constants $c^{(h)}\in\mathbb{R}^3$ and a function v in $W^{1,2}(\Omega;\mathbb{R}^3)$ such that (3.3) is satisfied. Moreover (3.18) entails that the function v depends only on the variable s

in $[0,L]$ and satisfies $v' = \overline{R}\,R_0 e_1$. Setting $d_k := \overline{R}\,R_0 e_k$ for $k=2,3$, we have that $(v,d_2,d_3)\in\mathcal{A}$ and (3.4), (3.5) are satisfied by (3.17). □

3.2. **Bound from below.** Let $Q_3 : \Omega\times\mathbb{M}^{3\times3} \longrightarrow [0,+\infty)$ be twice the quadratic form of linearized elasticity; i.e.,

$$Q_3(z,G) := \frac{\partial^2 W}{\partial F^2}(z,Id)[G,G]$$

for a.e. $z\in\Omega$ and every $G\in\mathbb{M}^{3\times3}$. We introduce the quadratic form $Q_2 : [0,L]\times\mathbb{M}^{3\times3}_{\rm skew}\to[0,+\infty)$ defined by

$$Q_2(s,P) := \inf_{\substack{\hat\alpha\in W^{1,2}(D;\mathbb{R}^3)\\ \hat g\in\mathbb{R}^3}} \left\{ \int_D Q_3\Big(s,\xi,\zeta,R_0(s)\Big(P\begin{pmatrix}0\\ \xi\\ \zeta\end{pmatrix} + \hat g \,\Big|\, \partial_\xi\hat\alpha \,\Big|\, \partial_\zeta\hat\alpha\Big)R_0^T(s)\Big)d\xi\,d\zeta\right\}. \tag{3.19}$$

Remark 3.4. It is easy to check that the minimum in (3.19) is attained; moreover the minimizers depend linearly on P, hence Q_2 is a quadratic form of P. Notice also that if $P\in L^2((0,L);\mathbb{M}^{3\times3})$, then $\hat\alpha\in L^2(\Omega;\mathbb{R}^3)$ with $\partial_\xi\hat\alpha,\partial_\zeta\hat\alpha\in L^2(\Omega;\mathbb{R}^3)$, and $\hat g\in L^2((0,L);\mathbb{R}^3)$ (see [10, Remarks 4.1 - 4.3]).

In the following theorem we prove a lower bound for the energies $\tilde I^{(h)}/h^2$ in terms of the functional

$$I(v,d_2,d_3) := \begin{cases} \dfrac{1}{2}\displaystyle\int_0^L Q_2\big(s,\big(R^T(s)R'(s) - R_0^T(s)R_0'(s)\big)\big)ds & \text{if } (v,d_2,d_3)\in\mathcal{A},\\ +\infty & \text{otherwise},\end{cases} \tag{3.20}$$

where $R\in W^{1,2}((0,L);\mathbb{M}^{3\times3})$ denotes the matrix $R := (v'\,|\,d_2\,|\,d_3)$ and $\mathcal{A}$ is the class defined in (3.1).

Theorem 3.5. *Let $v\in W^{1,2}(\Omega;\mathbb{R}^3)$ and let $d_2,d_3\in L^2(\Omega;\mathbb{R}^3)$. Then, for every positive sequence (h_j) converging to zero and every sequence $\big(\tilde v^{(h_j)}\big)\subset W^{1,2}(\widetilde\Omega_{h_j};\mathbb{R}^3)$ such that*

$$\tilde v^{(h_j)}\circ\Psi^{(h_j)} \to v \quad \textit{strongly in } W^{1,2}(\Omega;\mathbb{R}^3), \tag{3.21}$$

$$\frac{1}{h_j}\,\partial_\xi\big(\tilde v^{(h_j)}\circ\Psi^{(h_j)}\big) \to d_2 \quad \textit{strongly in } L^2(\Omega;\mathbb{R}^3), \tag{3.22}$$

$$\frac{1}{h_j}\,\partial_\zeta\big(\tilde v^{(h_j)}\circ\Psi^{(h_j)}\big) \to d_3 \quad \textit{strongly in } L^2(\Omega;\mathbb{R}^3), \tag{3.23}$$

it turns out that

$$I(v,d_2,d_3) \le \liminf_{j\to\infty}\frac{1}{h_j^4}\int_{\widetilde\Omega_{h_j}} W\big(\big(\Psi^{(h_j)}\big)^{-1}(x),\nabla\tilde v^{(h_j)}(x)\big)dx. \tag{3.24}$$

Proof. In the following, for notational brevity, we will write simply h instead of h_j. Let $\big(\tilde v^{(h)}\big)$ be a sequence satisfying (3.21), (3.22) and (3.23). We can assume that

$$\liminf_{j\to\infty}\frac{1}{h_j^4}\int_{\widetilde\Omega_{h_j}} W\big(\big(\Psi^{(h_j)}\big)^{-1}(x),\nabla\tilde v^{(h_j)}(x)\big)dx \le C < +\infty,$$

otherwise (3.24) is trivial. Therefore, up to subsequences, (3.2) is satisfied. By Theorem 3.3 we deduce that $(v,d_2,d_3)\in\mathcal{A}$,

$$F^{(h)} := \nabla\tilde v^{(h)}\circ\Psi^{(h)} \longrightarrow \overline{R} \quad \text{strongly in } L^2(\Omega;\mathbb{M}^{3\times3}) \tag{3.25}$$

with $\overline{R}\in W^{1,2}((0,L);\mathbb{M}^{3\times3})$, $\overline{R}\in SO(3)$ a.e., and

$$R := (v'\,|\,d_2\,|\,d_3) = \overline{R}\,R_0. \tag{3.26}$$

Moreover, as in the proof of Theorem 3.3, we can construct a piecewise constant approximation $R^{(h)} : [0,L]\to SO(3)$ such that

$$\int_\Omega \big|F^{(h)} - R^{(h)}\big|^2 ds\,d\xi\,d\zeta \le c\,h^2 \tag{3.27}$$

and $R^{(h)} \to \overline{R}$ strongly in $L^2(I';\mathbb{M}^3)$ for every $I' \subset\subset [0,L]$. Define the functions $G^{(h)} : \Omega \to \mathbb{M}^{3\times 3}$ as

$$G^{(h)} := \frac{1}{h}\Big((R^{(h)})^T F^{(h)} - Id\Big) = \frac{1}{h}\Big((R^{(h)})^T \nabla_h v^{(h)} \big(\nabla_h \Psi^{(h)}\big)^{-1} - Id\Big). \tag{3.28}$$

By (3.27) they are bounded in $L^2(\Omega;\mathbb{M}^{3\times 3})$, so there exists $G \in L^2(\Omega;\mathbb{M}^{3\times 3})$ such that $G^{(h)} \rightharpoonup G$ weakly in $L^2(\Omega;\mathbb{M}^{3\times 3})$. We claim that

$$\liminf_{h\to 0} \frac{1}{h^4}\int_{\widetilde{\Omega}_h} W\Big(\big(\Psi^{(h)}\big)^{-1}(x), \nabla \tilde{v}^{(h)}(x)\Big)dx \geq \frac{1}{2}\int_\Omega Q_3(s,\xi,\zeta,G)ds\, d\xi\, d\zeta. \tag{3.29}$$

Performing the change of variables $\Psi^{(h)}$, we have

$$\begin{aligned}\frac{1}{h^4}\int_{\widetilde{\Omega}_h} W\Big(\big(\Psi^{(h)}\big)^{-1}(x), \nabla \tilde{v}^{(h)}(x)\Big)dx &= \frac{1}{h^2}\int_\Omega W\big(s,\xi,\zeta,F^{(h)}\big)\det\big(\nabla_h\Psi^{(h)}\big)ds\, d\xi\, d\zeta \\ &= \frac{1}{h^2}\int_\Omega W\big(s,\xi,\zeta,\big(R^{(h)}\big)^T F^{(h)}\big)\det\big(\nabla_h\Psi^{(h)}\big)ds\, d\xi\, d\zeta\end{aligned} \tag{3.30}$$

where the last equality follows from the frame indifference of W. Define the family of functions

$$\chi^{(h)}(s,\xi,\zeta) := \begin{cases} 1 & \text{in } \ \Omega \cap \big\{(s,\xi,\zeta) : \big|G^{(h)}(s,\xi,\zeta)\big| \leq h^{-\frac{1}{2}}\big\}, \\ 0 & \text{otherwise.}\end{cases}$$

From the boundedness of $G^{(h)}$ in $L^2(\Omega;\mathbb{M}^{3\times 3})$ we get that $\chi^{(h)} \to 1$ boundedly in measure, so that

$$\chi^{(h)} G^{(h)} \rightharpoonup G \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times 3}). \tag{3.31}$$

By expanding W around the identity, we obtain that for every $(s,\xi,\zeta) \in \Omega$ and $A \in \mathbb{M}^{3\times 3}$

$$W\big(s,\xi,\zeta,Id+A\big) = \frac{1}{2}\frac{\partial^2 W}{\partial F^2}(s,\xi,\zeta,Id+t\,A)[A,A]$$

where $0<t<1$ depends on the point (s,ξ,ζ) and on A. By (3.30) and by the definition of $G^{(h)}$

$$\begin{aligned}\frac{1}{h^2}\tilde{I}^{(h)}\big(\tilde{v}^{(h)}\big) &= \frac{1}{h^2}\int_\Omega W\big(s,\xi,\zeta,Id+h\,G^{(h)}\big)\det\big(\nabla_h\Psi^{(h)}\big)ds\, d\xi\, d\zeta \\ &\geq \frac{1}{h^2}\int_\Omega \chi^{(h)} W\big(s,\xi,\zeta,Id+h\,G^{(h)}\big)\det\big(\nabla_h\Psi^{(h)}\big)ds\, d\xi\, d\zeta \\ &= \frac{1}{2}\int_\Omega \chi^{(h)}\left(\frac{\partial^2 W}{\partial F^2}\big(s,\xi,\zeta,Id+h\,t(h)\,G^{(h)}\big)\big[G^{(h)},G^{(h)}\big]\right)\det\big(\nabla_h\Psi^{(h)}\big)ds\, d\xi\, d\zeta\end{aligned} \tag{3.32}$$

where $0<t(h)<1$ depends on (s,ξ,ζ) and on $G^{(h)}$. The last integral in the previous formula can be written as

$$\begin{aligned}&\frac{1}{2}\int_\Omega \chi^{(h)}\left(\frac{\partial^2 W}{\partial F^2}\big(s,\xi,\zeta,Id+h\,t(h)\,G^{(h)}\big)\big[G^{(h)},G^{(h)}\big]\right)\det\big(\nabla_h\Psi^{(h)}\big)ds\, d\xi\, d\zeta \\ &= \frac{1}{2}\int_\Omega \left(\chi^{(h)}\left(\frac{\partial^2 W}{\partial F^2}\big(s,\xi,\zeta,Id+h\,t(h)\,G^{(h)}\big)\big[G^{(h)},G^{(h)}\big] - Q_3\big(s,\xi,\zeta,G^{(h)}\big)\right)\right)\det\big(\nabla_h\Psi^{(h)}\big)ds\, d\xi\, d\zeta \\ &\qquad + \frac{1}{2}\int_\Omega Q_3\big(s,\xi,\zeta,\chi^{(h)}\,G^{(h)}\big)\det\big(\nabla_h\Psi^{(h)}\big)ds\, d\xi\, d\zeta.\end{aligned} \tag{3.33}$$

By Scorza-Dragoni theorem there exists a compact set $K \subset \Omega$ such that the function $\partial^2 W/\partial F^2$ restricted to $K \times \overline{B_\delta(Id)}$ is continuous, hence uniformly continuous. Since $h\,t(h)\,\chi^{(h)}\,G^{(h)}$ is uniformly small for h small enough, for every $\varepsilon > 0$ we have

$$\begin{aligned}&\frac{1}{2}\int_\Omega \chi^{(h)}\left(\frac{\partial^2 W}{\partial F^2}\big(s,\xi,\zeta,Id+h\,t(h)\,G^{(h)}\big)\big[G^{(h)},G^{(h)}\big] - Q_3\big(s,\xi,\zeta,G^{(h)}\big)\right)\det\big(\nabla_h\Psi^{(h)}\big)ds\, d\xi\, d\zeta \\ &\qquad \geq -\frac{\varepsilon}{2}\int_K \chi^{(h)}\big|\,G^{(h)}\big|^2\det\big(\nabla_h\Psi^{(h)}\big)ds\, d\xi\, d\zeta \geq -C\,\varepsilon\end{aligned}$$

for h small enough. As for the second integral in (3.33), by (2.5) and (3.30) we get

$$\liminf_{h\to 0}\frac{1}{2}\int_\Omega Q_3\big(s,\xi,\zeta,\chi^{(h)}\,G^{(h)}\big)\det\big(\nabla_h\Psi^{(h)}\big)ds\,d\xi\,d\zeta \geq \frac{1}{2}\int_\Omega Q_3\big(s,\xi,\zeta,G\big)ds\,d\xi\,d\zeta \tag{3.34}$$

since Q_3 is a nonnegative quadratic form. Combining (3.32), (3.33) and (3.34) we have

$$\liminf_{h\to 0}\frac{1}{h^2}\,\tilde{I}^{(h)}(\tilde{v}^{(h)}) \geq \frac{1}{2}\int_\Omega Q_3(s,\xi,\zeta,G)ds\,d\xi\,d\zeta\;-\;C\,\varepsilon$$

and, since ε is arbitrary, (3.29) is proved. It remains to identify G.

Fix $(\xi_0,\zeta_0)\in D$; let $\delta_0=\delta_0(\xi_0,\zeta_0)>0$ be such that $B_{2\,\delta_0}(\xi_0,\zeta_0)\subset D$ and let $U_0:=(0,L)\times B_{\delta_0}(\xi_0,\zeta_0)$. Fix $t\in\mathbb{R}-\{0\}$, $|\,t\,|<\delta_0$. For every $(s,\xi,\zeta)\in U_0$ we can define the difference quotients of the functions $G^{(h)}$ with respect to the variables ξ and ζ along the direction τ, given by

$$\begin{cases} H_t^{(h)}(s,\xi,\zeta):= & \dfrac{1}{t}\Big(G^{(h)}(s,\xi+t,\zeta)-G^{(h)}(s,\xi,\zeta)\Big)\,\tau(s),\\[2ex] K_t^{(h)}(s,\xi,\zeta):= & \dfrac{1}{t}\Big(G^{(h)}(s,\xi,\zeta+t)-G^{(h)}(s,\xi,\zeta)\Big)\,\tau(s),\end{cases}$$

and the corresponding difference quotients of the limit function G

$$\begin{cases} H_t(s,\xi,\zeta):= & \dfrac{1}{t}\Big(G(s,\xi+t,\zeta)-G(s,\xi,\zeta)\Big)\,\tau(s),\\[2ex] K_t(s,\xi,\zeta):= & \dfrac{1}{t}\Big(G(s,\xi,\zeta+t)-G(s,\xi,\zeta)\Big)\,\tau(s).\end{cases}$$

Since $G^{(h)}\rightharpoonup G$ in $L^2(\Omega;\mathbb{M}^{3\times 3})$ and $R^{(h)}\longrightarrow\overline{R}$ boundedly in measure, we have

$$\begin{aligned} H_t^{(h)}&\rightharpoonup H_t \quad \text{weakly in}\, L^2(U_0;\mathbb{R}^3)\ \text{and}\\ R^{(h)}\,H_t^{(h)}&\rightharpoonup \overline{R}\,H_t \quad \text{weakly in}\, L^2(U_0;\mathbb{R}^3).\end{aligned} \tag{3.35}$$

In terms of $F^{(h)}$ the left-hand side of (3.35) reads as

$$R^{(h)}(s)H_t^{(h)}(s,\xi,\zeta)=\frac{1}{h\,t}\Big(F^{(h)}(s,\xi+t,\zeta)-F^{(h)}(s,\xi,\zeta)\Big)\,\tau(s). \tag{3.36}$$

Now recall that, if we set $v^{(h)}:=\tilde{v}^{(h)}\circ\Psi^{(h)}$, we have

$$\nabla v^{(h)}=F^{(h)}\,\nabla\Psi^{(h)}; \tag{3.37}$$

in particular, taking the first column of the two matrices, we obtain

$$F^{(h)}(s,\xi,\zeta)\,\tau(s)=\partial_s v^{(h)}(s,\xi,\zeta)-h\,F^{(h)}(s,\xi,\zeta)\,(\xi\,\nu_2'(s)+\zeta\,\nu_3'(s)).$$

By the last equality and (3.36) we get

$$\begin{aligned} R^{(h)}(s)H_t^{(h)}(s,\xi,\zeta)=&\frac{1}{h\,t}\Big(\partial_s v^{(h)}(s,\xi+t,\zeta)-\partial_s v^{(h)}(s,\xi,\zeta)\Big)\\ &-\frac{1}{t}\Big((\xi+t)\,F^{(h)}(s,\xi+t,\zeta)-\xi\,F^{(h)}(s,\xi,\zeta)\Big)\,\nu_2'(s)\\ &-\frac{1}{t}\Big(\zeta\,F^{(h)}(s,\xi+t,\zeta)-\zeta\,F^{(h)}(s,\xi,\zeta)\Big)\,\nu_3'(s).\end{aligned} \tag{3.38}$$

For the first term we have

$$\begin{aligned}\frac{1}{h\,t}\,\partial_s\Big(v^{(h)}(s,\xi+t,\zeta)-v^{(h)}(s,\xi,\zeta)\Big)&=\frac{1}{h\,t}\,\partial_s\bigg(\int_\xi^{\xi+t}\partial_\xi v^{(h)}(s,\vartheta,\zeta)\,d\vartheta\bigg)\\ &=\partial_s\bigg(\frac{1}{t}\int_0^t\frac{1}{h}\,\partial_\xi v^{(h)}(s,\xi+\vartheta,\zeta)\,d\vartheta\bigg),\end{aligned}$$

so by (3.23) and (3.26)

$$\frac{1}{h\,t}\,\partial_s\Big(v^{(h)}(s,\xi+t,\zeta)-v^{(h)}(s,\xi,\zeta)\Big)\rightharpoonup d_2'(s)=\partial_s(\overline{R}(s)\,\nu_2(s))\quad\text{weakly in}\,W^{-1,2}(U_0;\mathbb{R}^3). \tag{3.39}$$

By (3.25) the second term in (3.38) converges to

$$\frac{1}{t}\Big((\xi+t)\,\overline{R}(s)-\xi\,\overline{R}(s)\Big)\,\nu_2'(s)=\overline{R}(s)\,\nu_2'(s)\quad\text{strongly in}\,L^2(U_0;\mathbb{R}^3)\tag{3.40}$$

and the last term to

$$\frac{1}{t}\Big(\zeta\,\overline{R}(s)-\zeta\,\overline{R}(s)\Big)\,\nu_3'(s)=0\quad\text{strongly in}\,L^2(U_0;\mathbb{R}^3).\tag{3.41}$$

Putting together (3.39), (3.40), (3.41) and (3.35)

$$\overline{R}(s)\,H_t(s,\xi,\zeta)=\partial_s(\overline{R}(s)\,\nu_2(s))-\overline{R}(s)\,\nu_2'(s)\ \text{a.e. in}\ U_0$$

and so

$$H_t(s,\xi,\zeta)=(\overline{R}(s))^T\,\overline{R}'(s)\,\nu_2(s)\ \text{a.e. in}\ U_0.\tag{3.42}$$

Repeating the same argument for $K_t^{(h)}$ we get

$$K_t(s,\xi,\zeta)=(\overline{R}(s))^T\,\overline{R}'(s)\,\nu_3(s)\ \text{a.e. in}\ U_0.\tag{3.43}$$

From the last two equalities we deduce that the functions H_t and K_t depend only on the variable s. Moreover, letting t go to 0 both in (3.42) and in (3.43), we get that the gradient of $G\,\tau$ w.r.to the variables (ξ,ζ) depends only on s, i.e.,

$$\nabla_{(\xi,\zeta)}\big(G(s,\xi,\zeta)\,\tau(s)\big)=(\overline{R}(s))^T\,\overline{R}'(s)\,(\nu_2(s)\,|\,\nu_3(s))\ \text{a.e. in}\ U_0.\tag{3.44}$$

Being this equality valid in $U_0=(0,L)\times B_{\delta_0}(\xi_0,\zeta_0)$, for an arbitrary $(\xi_0,\zeta_0)\in D$, we can conclude that it holds a.e. in the whole Ω. Since D is connected, we obtain that for a.e. $(s,\xi,\zeta)\in\Omega$

$$G(s,\xi,\zeta)\,\tau(s)=(\overline{R}(s))^T\,\overline{R}'(s)\,(\xi\,\nu_2(s)+\zeta\,\nu_3(s))+g(s)$$

with $g:[0,L]\to\mathbb{R}^3$. Remark that from the previous formula $g\in L^2((0,L);\mathbb{R}^3)$.
It remains to identify the components $G(s,\xi,\zeta)\,\nu_2(s)$ and $G(s,\xi,\zeta)\,\nu_3(s)$. By (3.37) we have

$$\begin{aligned}G^{(h)}(s,\xi,\zeta)\,\nu_2(s)&=\frac{1}{h}\Big((R^{(h)}(s))^TF^{(h)}(s,\xi,\zeta)\,\nu_2(s)-\nu_2(s)\Big)\\&=\frac{1}{h}\Big(h^{-1}(R^{(h)}(s))^T\partial_\xi v^{(h)}(s,\xi,\zeta)-\nu_2(s)\Big)\end{aligned}$$

and

$$\begin{aligned}G^{(h)}(s,\xi,\zeta)\,\nu_3(s)&=\frac{1}{h}\Big((R^{(h)}(s))^TF^{(h)}(s,\xi,\zeta)\,\nu_3(s)-\nu_3(s)\Big)\\&=\frac{1}{h}\Big(h^{-1}(R^{(h)}(s))^T\partial_\zeta v^{(h)}(s,\xi,\zeta)-\nu_3(s)\Big),\end{aligned}$$

so, if we define

$$\alpha^{(h)}(s,\xi,\zeta):=\frac{1}{h}\Big(h^{-1}(R^{(h)})^Tv^{(h)}(s,\xi,\zeta)-\xi\,\nu_2(s)-\zeta\,\nu_3(s)\Big)$$

it turns out that

$$\partial_\xi\alpha^{(h)}(s,\xi,\zeta)=G^{(h)}(s,\xi,\zeta)\,\nu_2(s)\quad\text{and}\quad\partial_\zeta\alpha^{(h)}(s,\xi,\zeta)=G^{(h)}(s,\xi,\zeta)\,\nu_3(s).\tag{3.45}$$

Applying the Poincaré inequality to the functions $\alpha^{(h)}$ for fixed s we obtain that for a.e. $s\in[0,L]$

$$\int_D\big|\,\alpha^{(h)}(s,\xi,\zeta)-\alpha_0^{(h)}(s)\,\big|^2\,d\xi\,d\zeta\le c\int_D\Big(\big|\,\partial_\xi\alpha^{(h)}(s,\xi,\zeta)\big|^2+\big|\,\partial_\zeta\alpha^{(h)}(s,\xi,\zeta)\big|^2\Big)\,d\xi\,d\zeta,$$

where $\alpha_0^{(h)}(s):=\int_D\alpha^{(h)}(s,\xi,\zeta)\,d\xi\,d\zeta$. Integrating over $[0,L]$, we have

$$\big|\big|\alpha^{(h)}-\alpha_0^{(h)}\big|\big|^2_{L^2(\Omega)}\le c\,\Big(\big|\big|\partial_\xi\alpha^{(h)}\big|\big|^2_{L^2(\Omega)}+\big|\big|\partial_\zeta\alpha^{(h)}\big|\big|^2_{L^2(\Omega)}\Big).$$

Since the right-hand side is bounded by (3.45), there exists a function $\alpha\in L^2(\Omega;\mathbb{R}^3)$ such that, up to subsequences,

$$\alpha^{(h)}-\alpha_0^{(h)}\rightharpoonup\alpha\quad\text{weakly in}\,L^2(\Omega;\mathbb{R}^3).$$

Moreover, from (3.45) we conclude that

$$\partial_\xi\alpha(s,\xi,\zeta)=G(s,\xi,\zeta)\,\nu_2(s)\quad\text{and}\quad\partial_\zeta\alpha(s,\xi,\zeta)=G(s,\xi,\zeta)\,\nu_3(s),\tag{3.46}$$

therefore $\partial_\xi\alpha, \partial_\zeta\alpha \in L^2(\Omega;\mathbb{R}^3)$. Now, define the functions $\hat\alpha(s,\xi,\zeta) := R_0^T(s)\,\alpha(s,\xi,\zeta)$ and $\hat g(s) := R_0^T(s)\,g(s)$. Thanks to these definitions and to (3.26), G can be written as

$$\begin{aligned} G &= \left(\left(R\,R_0^T\right)^T \left(R\,R_0^T\right)' R_0 \begin{pmatrix} 0 \\ \xi \\ \zeta \end{pmatrix} + g \,\middle|\, \partial_\xi\alpha \,\middle|\, \partial_\zeta\alpha \right) R_0^T \\ &= R_0 \left(\left(R^T R' + (R_0^T)' R_0\right) \begin{pmatrix} 0 \\ \xi \\ \zeta \end{pmatrix} + \hat g \,\middle|\, \partial_\xi\hat\alpha \,\middle|\, \partial_\zeta\hat\alpha \right) R_0^T \\ &= R_0 \left(\left(R^T R' - R_0^T R_0'\right) \begin{pmatrix} 0 \\ \xi \\ \zeta \end{pmatrix} + \hat g \,\middle|\, \partial_\xi\hat\alpha \,\middle|\, \partial_\zeta\hat\alpha \right) R_0^T, \end{aligned} \tag{3.47}$$

where the last equality follows from the identity $\left(R_0^T\right)' R_0 + R_0^T R_0' = 0$. Combining (3.29) and (3.47), we obtain

$$\liminf_{h\to 0} \frac{1}{h^2}\,\tilde I^{(h)}(\tilde v^{(h)}) \geq \frac{1}{2}\int_\Omega Q_3\left(s,\xi,\zeta, R_0(s)\left(P(s)\begin{pmatrix} 0 \\ \xi \\ \zeta \end{pmatrix} + \hat g \,\middle|\, \partial_\xi\hat\alpha \,\middle|\, \partial_\zeta\hat\alpha\right) R_0^T(s)\right) ds\, d\xi\, d\zeta,$$

with $P(s) := R^T(s)R'(s) - R_0^T(s)R_0'(s)$. By the definition of the quadratic form Q_2 in (3.19) we clearly have $\int_D Q_3(s,\xi,\zeta,G)d\xi\, d\zeta \geq Q_2(s,P(s))$, and so

$$\liminf_{h\to 0} \frac{1}{h^4}\int_{\widetilde\Omega_h} W\big(\big(\Psi^{(h)}\big)^{-1}(x), \nabla\tilde v^{(h)}(x)\big)dx \geq \frac{1}{2}\int_0^L Q_2\big(s, \big(R^T(s)R'(s) - R_0^T(s)R_0'(s)\big)\big)ds.$$

□

3.3. **Bound from above.** In this subsection we show that the lower bound proved in Theorem 3.5 is optimal.

Theorem 3.6. *For every sequence of positive (h_j) converging to 0 and for every $(v,d_2,d_3) \in \mathcal{A}$ there exists a sequence $\big(\tilde v^{(h_j)}\big) \subset W^{1,2}\big(\widetilde\Omega_{h_j};\mathbb{R}^3\big)$ such that*

$$\tilde v^{(h_j)} \circ \Psi^{(h_j)} \to v \quad \textit{strongly in } W^{1,2}(\Omega;\mathbb{R}^3), \tag{3.48}$$

$$\frac{1}{h_j}\,\partial_\xi\big(\tilde v^{(h_j)} \circ \Psi^{(h_j)}\big) \to d_2 \quad \textit{strongly in } L^2(\Omega;\mathbb{R}^3), \tag{3.49}$$

$$\frac{1}{h_j}\,\partial_\zeta\big(\tilde v^{(h_j)} \circ \Psi^{(h_j)}\big) \to d_3 \quad \textit{strongly in } L^2(\Omega;\mathbb{R}^3), \tag{3.50}$$

and

$$I(v,d_2,d_3) = \lim_{j\to\infty} \frac{1}{h_j^4}\int_{\widetilde\Omega_{h_j}} W\big(\big(\Psi^{(h_j)}\big)^{-1}(x), \nabla\tilde v^{(h_j)}(x)\big)dx, \tag{3.51}$$

where the class $\mathcal{A}$ and the functional I are defined in (3.1) and (3.20), respectively.

Proof. Let $(v,d_2,d_3) \in \mathcal{A}$. Assume in addition that $v \in C^2([0,L];\mathbb{R}^3)$ and $d_2, d_3 \in C^1([0,L];\mathbb{R}^3)$. Consider the functions $v^{(h)} : \Omega \to \mathbb{R}^3$ defined by

$$v^{(h)}(s,\xi,\zeta) := v(s) + h\,\xi\, d_2(s) + h\,\zeta\, d_3(s) + h\, q(s) + h^2\,\beta(s,\xi,\zeta),$$

with $q \in C^1([0,L];\mathbb{R}^3)$ and $\beta \in C^1(\overline\Omega;\mathbb{R}^3)$. We define $\tilde v^{(h)} := v^{(h)} \circ \big(\Psi^{(h)}\big)^{-1}$; these functions clearly satisfy (3.48). Moreover, since

$$\nabla_h\big(\tilde v^{(h)} \circ \Psi^{(h)}\big) = \nabla_h v^{(h)} = (v' \,|\, d_2 \,|\, d_3) + h\,\big(\xi\, d_2' + \zeta\, d_3' + q' \,|\, \partial_\xi\beta \,|\, \partial_\zeta\beta\big) + h^2\partial_s\beta \otimes e_1, \tag{3.52}$$

also (3.49) and (3.50) follow easily. In order to prove (3.51), we first observe that, performing the change of variables $(s,\xi,\zeta)=\big(\Psi^{(h)}\big)^{-1}(x)$, we obtain

$$\begin{aligned}\frac{1}{h^2}\,\tilde I^{(h)}\big(\tilde v^{(h)}\big) =&\frac{1}{h^2}\int_\Omega W\big(s,\xi,\zeta,\nabla\tilde v^{(h)}\circ\Psi^{(h)}\big)\det\big(\nabla_h\Psi^{(h)}\big)ds\,d\xi\,d\zeta\\ =&\frac{1}{h^2}\int_\Omega W\big(s,\xi,\zeta,\nabla_h\big(\tilde v^{(h)}\circ\Psi^{(h)}\big)\,\big(\nabla_h\Psi^{(h)}\big)^{-1}\big)\det\big(\nabla_h\Psi^{(h)}\big)ds\,d\xi\,d\zeta,\end{aligned}\tag{3.53}$$

where the last equality is justified observing that

$$\nabla_h\big(\tilde v^{(h)}\circ\Psi^{(h)}\big)=\big(\nabla\tilde v^{(h)}\circ\Psi^{(h)}\big)\,\big(\nabla_h\Psi^{(h)}\big).$$

Then, by the definition of $\tilde v^{(h)}$,

$$\frac{1}{h^2}\,\tilde I^{(h)}\big(\tilde v^{(h)}\big)=\frac{1}{h^2}\int_\Omega W\big(s,\xi,\zeta,\big(\nabla_h v^{(h)}\big)\,\big(\nabla_h\Psi^{(h)}\big)^{-1}\big)\det\big(\nabla_h\Psi^{(h)}\big)ds\,d\xi\,d\zeta.\tag{3.54}$$

Using (2.6) and (3.52) we get

$$\begin{aligned}\nabla_h v^{(h)}\,\big(\nabla_h\Psi^{(h)}\big)^{-1}=&\,R\,R_0^T+h\,(\xi\,d'_2+\zeta\,d'_3+q'\,|\,\partial_\xi\beta\,|\,\partial_\zeta\beta)\,R_0^T\\ &-h\,R\,R_0^T\big[(\xi\,\nu'_2+\zeta\,\nu'_3)\otimes e_1\big]\,R_0^T+O(h^2),\end{aligned}$$

where $R=(v'|d_2|d_3)$ and $O(h^2)/h^2$ is uniformly bounded. Now consider the rotation $\overline{R}(s)=R(s)\,R_0^T(s)$. Then

$$\overline{R}^T\nabla_h v^{(h)}\,\big(\nabla_h\Psi^{(h)}\big)^{-1}=\,Id+h\,\overline{R}^T(\xi\,d'_2+\zeta\,d'_3+q'\,|\,\partial_\xi\beta\,|\,\partial_\zeta\beta)\,R_0^T-h\,\big[(\xi\,\nu'_2+\zeta\,\nu'_3)\otimes e_1\big]\,R_0^T+O(h^2).$$

If we define the functions

$$B^{(h)}(s,\xi,\zeta):=\frac{1}{h}\Big(\overline{R}^T\,\nabla_h v^{(h)}\,\big(\nabla_h\Psi^{(h)}\big)^{-1}-Id\Big),$$

it turns out that

$$\begin{aligned}B^{(h)}=&\,(R_0\,R^T)(\xi\,d'_2+\zeta\,d'_3+q'\,|\,\partial_\xi\beta\,|\,\partial_\zeta\beta)R_0^T-\big[(\xi\,\nu'_2+\zeta\,\nu'_3)\otimes e_1\big]\,R_0^T+O(h)\\ =&\,R_0\,R^T\left(R'\begin{pmatrix}0\\ \xi\\ \zeta\end{pmatrix}+q'\,\Big|\,\partial_\xi\beta\,\Big|\,\partial_\zeta\beta\right)R_0^T-\left[\left(R'_0\begin{pmatrix}0\\ \xi\\ \zeta\end{pmatrix}\right)\otimes e_1\right]R_0^T+O(h)\\ =&\,R_0\left(\big(R^TR'-R_0^TR'_0\big)\begin{pmatrix}0\\ \xi\\ \zeta\end{pmatrix}+R^Tq\,\Big|\,R^T\partial_\xi\beta\,\Big|\,R^T\partial_\zeta\beta\right)R_0^T+O(h)\\ =:&\,G_{q,\beta}+O(h)\end{aligned}\tag{3.55}$$

where $O(h)/h$ is uniformly bounded. By frame indifference and the definition of $B^{(h)}$, we have

$$\begin{aligned}\frac{1}{h^2}\,W\big(s,\xi,\zeta,\nabla_h v^{(h)}\big(\nabla_h\Psi^{(h)}\big)^{-1}\big)=&\,\frac{1}{h^2}\,W\big(s,\xi,\zeta,\overline{R}^T\nabla_h v^{(h)}\big(\nabla_h\Psi^{(h)}\big)^{-1}\big)\\ =&\,\frac{1}{h^2}\,W\big(s,\xi,\zeta,Id+h\,B^{(h)}\big).\end{aligned}$$

Using (3.55) and the expansion of W around the identity, we obtain

$$\frac{1}{h^2}\,W\big(s,\xi,\zeta,\nabla_h v^{(h)}\big(\nabla_h\Psi^{(h)}\big)^{-1}\big)\to\frac{1}{2}\,Q_3(s,\xi,\zeta,G_{q,\beta})\quad\text{a.e..}$$

Moreover, the assumption (ii) gives the uniform bound

$$\frac{1}{h^2}\,W\big(s,\xi,\zeta,\nabla_h v^{(h)}\big(\nabla_h\Psi^{(h)}\big)^{-1}\big)\le\frac{1}{2}\,C_1\,|\,G_{q,\beta}\,|^2+C\in\,L^1(\Omega),$$

so, by the dominated convergence theorem and by (3.54) we conclude that

$$\lim_{h\to0}\frac{1}{h^4}\int_{\widetilde\Omega_h}W\Big(\big(\Psi^{(h)}\big)^{-1}(x),\nabla\tilde v^{(h)}(x)\Big)dx=\frac{1}{2}\int_\Omega Q_3(s,\xi,\zeta,G_{q,\beta})\,ds\,d\xi\,d\zeta.\tag{3.56}$$

This holds for every $q\in C^1([0,L];\mathbb{R}^3)$ and for every $\beta\in C^1(\overline{\Omega};\mathbb{R}^3)$.

Consider now the general case. Let $(v, d_2, d_3) \in \mathcal{A}$, and let $\hat{\alpha}(s,\cdot) \in W^{1,2}(D;\mathbb{R}^3)$, $\hat{g}(s)$ be a solution to the minimum problem (3.19) for $P = R^T R' - R_0^T R_0'$. By Remark 3.4, $\hat{\alpha} \in L^2(\Omega;\mathbb{R}^3)$ with $\partial_\xi \hat{\alpha}, \partial_\zeta \hat{\alpha} \in L^2(\Omega;\mathbb{R}^3)$ and $\hat{g} \in L^2((0,L);\mathbb{R}^3)$. In order to conclude the proof it is enough to construct a sequence of smooth deformations converging to (v, d_2, d_3), on which the energy $\tilde{I}^{(h)}/h^2$ converges to the right-hand side of (3.56) with q and β replaced by $R^T\hat{g}$ and $R^T\hat{\alpha}$, respectively. This can be done by repeating the same construction as in [9]. □

Remark 3.7 (Homogeneous rods). If the rod is made of a homogeneous material, i.e., $W(z,F) = W(F)$, for a.e. z in Ω and every $F \in \mathbb{M}^{3\times 3}$, then the limiting energy density Q_2 is given by the simpler formula

$$Q_2(s,P) = \inf_{\hat{\alpha}\in W^{1,2}(D;\mathbb{R}^3)} \left\{ \int_D Q_3\left(R_0(s)\left(P\begin{pmatrix} 0 \\ \xi \\ \zeta \end{pmatrix} \middle| \partial_\xi\hat{\alpha} \middle| \partial_\zeta\hat{\alpha} \right) R_0^T(s)\right) d\xi\, d\zeta \right\}. \tag{3.57}$$

In other words the optimal choice for $\hat{g}$ in (3.19) is $\hat{g} = 0$.
In order to show this, let $\hat{\alpha} \in W^{1,2}(D;\mathbb{R}^3)$ and let $\hat{g} \in \mathbb{R}^3$. We introduce the function

$$\tilde{\alpha}(s,\xi,\zeta) := \hat{\alpha}(s,\xi,\zeta) - \xi \int_D \partial_\xi\hat{\alpha}\, d\xi\, d\zeta - \zeta \int_D \partial_\zeta\hat{\alpha}\, d\xi\, d\zeta. \tag{3.58}$$

Then,

$$\begin{aligned} R_0\left(P\begin{pmatrix} 0 \\ \xi \\ \zeta \end{pmatrix} + \hat{g} \middle| \partial_\xi\hat{\alpha} \middle| \partial_\zeta\hat{\alpha} \right) R_0^T &= R_0\left(P\begin{pmatrix} 0 \\ \xi \\ \zeta \end{pmatrix} \middle| \partial_\xi\tilde{\alpha} \middle| \partial_\zeta\tilde{\alpha} \right) R_0^T \\ &\quad + R_0\left(\hat{g} \middle| \int_D \partial_\xi\hat{\alpha}\, d\xi\, d\zeta \middle| \int_D \partial_\zeta\hat{\alpha}\, d\xi\, d\zeta \right) R_0^T \\ &=: \tilde{G} + Z. \end{aligned}$$

By expanding the quadratic form Q_3, we have

$$\int_D Q_3(G) d\xi\, d\zeta = \int_D Q_3(\tilde{G}) d\xi\, d\zeta + \int_D Q_3(Z) d\xi\, d\zeta \geq \int_D Q_3(\tilde{G}) d\xi\, d\zeta, \tag{3.59}$$

where we used (2.2), the fact that $\partial_\xi\tilde{\alpha}$ and $\partial_\zeta\tilde{\alpha}$ have zero average on D and the non negativity of Q_3. From this inequality the thesis follows immediately.
Notice that, due to the nontrivial geometry of the body, the limit energy depends on the position over the curve γ even for a homogeneous material.

Remark 3.8 (Homogeneous and isotropic rods). Assume the density W is homogeneous and isotropic, that is,

$$W(F) = W(FR) \quad \text{for every } R \in SO(3).$$

Then the quadratic form Q_3 is given by

$$Q_3(G) = 2\,\mu \left| \frac{G + G^T}{2} \right|^2 + \lambda\,(\operatorname{tr} G)^2$$

for some constants $\lambda, \mu \in \mathbb{R}$. It is easy to show that for all $G \in \mathbb{M}^{3\times 3}$ and $R \in SO(3)$

$$Q_3(R\,G\,R^T) = Q_3(G),$$

and so, formula (3.57) reduces to

$$\begin{aligned} Q_2(P) &= \inf_{\hat{\alpha}\in W^{1,2}(D;\mathbb{R}^3)} \left\{ \int_D Q_3\left(P\begin{pmatrix} 0 \\ \xi \\ \zeta \end{pmatrix} \middle| \partial_\xi\hat{\alpha} \middle| \partial_\zeta\hat{\alpha} \right) d\xi\, d\zeta \right\} \\ &= \frac{1}{2\,\pi}\, \frac{\mu(3\,\lambda + 2\,\mu)}{\lambda + \mu}\, (p_{12}^2 + p_{13}^2) + \frac{\mu}{2\,\pi}\, p_{23}^2, \end{aligned}$$

where the last equality follows from [9, Remark 3.5]. This means that in the case of a homogeneous and isotropic material the quadratic form Q_2 is exactly the same as in the case of a straight rod treated in [9].

Remark 3.9 (Homogeneous rods with a circular cross section). Assume that the cross section D is a circle of radius $\frac{1}{\sqrt{\pi}}$ centred at the origin. In this case, the quadratic form Q_2 can be computed by a pointwise minimization. More precisely, for every s and for every P,

$$Q_2(s,P) = \frac{1}{4\pi}\min_{u,v,w}\left\{Q_3\left(R_0(s)\left(\begin{array}{c} p_{12} \\ 0 \\ -p_{23}\end{array}\middle|\, u \,\middle|\, v\right)R_0^T(s)\right) + Q_3\left(R_0(s)\left(\begin{array}{c} p_{13} \\ p_{23} \\ 0\end{array}\middle|\, v \,\middle|\, w\right)R_0^T(s)\right)\right\}.$$

The proof is completely analogous to [9, Remark 3.6].

4. Lower scalings of the energy

The content of this section is the study of the asymptotic behaviour of the functionals $\tilde{I}^{(h)}/h^\alpha$ for $0 \le \alpha < 2$, as $h \to 0$. In addition to conditions (i)-(v) of Section 2 we assume also that $W(z,F) = W(z_1,F)$ for every $z = (z_1,z_2,z_3) \in \mathbb{R}^3$ and every $F \in \mathbb{M}^{3\times 3}$, and that

(vi) $\exists\, C_3 > 0$ independent of z_1 such that $W(z_1,F) \le C_3\,\mathrm{dist}^2(F, SO(3))$ for a.e. z_1 and every $F \in \mathbb{M}^{3\times 3}$.

It is convenient to write the functionals $\tilde{I}^{(h)}$ as integrals over the fixed domain $\Omega = \left(\Psi^{(h)}\right)^{-1}\left(\tilde{\Omega}_h\right)$. Changing variables as in (3.53) and setting $v := \tilde{v} \circ \Psi^{(h)}$, we have

$$\tilde{I}^{(h)}(\tilde{v}) = \int_\Omega W\left(s, \left(\nabla_h v\right)\left(\nabla_h \Psi^{(h)}\right)^{-1}\right)\det\left(\nabla_h \Psi^{(h)}\right) ds\, d\xi\, d\zeta =: \tilde{J}^{(h)}(v).$$

We extend the functional to the space $L^2(\Omega;\mathbb{R}^3)$, setting

$$J^{(h)}(v) = \begin{cases} \tilde{J}^{(h)}(v) & \text{if } v \in W^{1,2}(\Omega;\mathbb{R}^3), \\ +\infty & \text{otherwise in } L^2(\Omega;\mathbb{R}^3). \end{cases}$$

The aim of this section is to determine the Γ-limit of $J^{(h)}/h^\alpha$, for $0 \le \alpha < 2$, as $h \to 0$, with respect to the strong topology of L^2.

4.1. **Derivation of the nonlinear theory for curved strings.** For this first part we specify $\alpha = 0$, so we are interested in the asymptotic behaviour of the functionals representing the energy per unit volume associated to a deformation of the reference configuration.

Theorem 4.1 (Compactness). *For every sequence $\left(v^{(h)}\right)$ in $L^2(\Omega;\mathbb{R}^3)$ such that*

$$J^{(h)}\left(v^{(h)}\right) \le c < +\infty \tag{4.1}$$

there exist a function $v \in W^{1,2}((0,L);\mathbb{R}^3)$ and some constants $c^{(h)} \in \mathbb{R}^3$ such that, up to subsequences,

$$v^{(h)} - c^{(h)} \rightharpoonup v \quad \textit{weakly in } W^{1,2}(\Omega;\mathbb{R}^3).$$

Proof. Let $\left(v^{(h)}\right)$ be a sequence in $L^2(\Omega;\mathbb{R}^3)$ satisfying (4.1). From the definition of the functional we have immediately that $v^{(h)} \in W^{1,2}(\Omega;\mathbb{R}^3)$. The assumptions on W and the uniform boundedness of $\left(\nabla_h \Psi^{(h)}\right)^{-1}$ and of $\det\left(\nabla_h \Psi^{(h)}\right)$ give the boundedness in $L^2(\Omega;\mathbb{M}^{3\times 3})$ of $\left(\nabla_h v^{(h)}\right)$ and hence of $\left(\nabla v^{(h)}\right)$. Therefore, using the Poincaré inequality

$$\left|\left|v^{(h)} - c^{(h)}\right|\right|_{L^2(\Omega;\mathbb{R}^3)} \le \left|\left|\nabla v^{(h)}\right|\right|_{L^2(\Omega;\mathbb{M}^{3\times 3})},$$

where $c^{(h)} \in \mathbb{R}^3$ is the mean value of $v^{(h)}$ over Ω, it turns out that the sequence $v^{(h)} - c^{(h)}$ is bounded in $W^{1,2}(\Omega;\mathbb{R}^3)$; hence there exists a function $v \in W^{1,2}(\Omega;\mathbb{R}^3)$ such that, up to subsequences,

$$v^{(h)} - c^{(h)} \rightharpoonup v \quad \text{weakly in } W^{1,2}(\Omega;\mathbb{R}^3).$$

Moreover since $\left(\nabla_h v^{(h)}\right)$ is bounded in $L^2(\Omega;\mathbb{M}^{3\times 3})$, we have

$$\partial_\xi v^{(h)} \to 0 \quad \text{and} \quad \partial_\zeta v^{(h)} \to 0 \quad \text{strongly in } L^2(\Omega;\mathbb{R}^3).$$

Therefore the limit function v depends only on the first variable. □

Theorem 4.2 (Γ-convergence)**.** *Let I be the functional defined as*

$$I(v) = \begin{cases} \displaystyle\int_0^L W_0^{**}(s, v'(s))\, ds & \text{if } v \in W^{1,2}((0,L);\mathbb{R}^3), \\ +\infty & \text{otherwise in } L^2(\Omega;\mathbb{R}^3), \end{cases} \tag{4.2}$$

where W_0^{**} is given by the convex envelope of the function $W_0 : [0,L] \times \mathbb{R}^3 \to \mathbb{R}$ defined as

$$W_0(s,z) := \inf\left\{W\left(s, (z \,|\, y_2 \,|\, y_3) R_0^T(s)\right) : y_2, y_3 \in \mathbb{R}^3\right\}.$$

Then

$$\Gamma - \lim_{h\to 0} J^{(h)} = I,$$

i.e., the following conditions are satisfied:
(i)(liminf inequality) for every $v \in L^2(\Omega;\mathbb{R}^3)$ and every sequence $\left(v^{(h)}\right) \subset L^2(\Omega;\mathbb{R}^3)$ such that $v^{(h)} \to v$ strongly in $L^2(\Omega;\mathbb{R}^3)$, it turns out that

$$I(v) \le \liminf_{h\to 0} J^{(h)}\left(v^{(h)}\right); \tag{4.3}$$

(ii)(limsup inequality) for every $v \in L^2(\Omega;\mathbb{R}^3)$ there exists a sequence $\left(v^{(h)}\right) \subset L^2(\Omega;\mathbb{R}^3)$ converging strongly to v in $L^2(\Omega;\mathbb{R}^3)$ such that

$$\limsup_{h\to 0} J^{(h)}\left(v^{(h)}\right) \le I(v). \tag{4.4}$$

Remark 4.3. Notice that, if $A := (z \,|\, y_2 \,|\, y_3)\, R_0^T$, then $A\,\tau = z$ and $A\,\nu_k = y_k$ for $k = 2, 3$. In other words, in the definition of W_0, the minimization is done with respect to the normal components of the matrix in the argument of W, keeping equal to z the tangential component.

Remark 4.4. Observe that conditions (iv) and (v) imply that for a.e. $s \in [0,L]$,

$$W_0^{**}(s,z) = 0 \quad \text{if and only if} \quad |z| \le 1, \tag{4.5}$$

(see [1]).

Proof. (of Theorem 4.2) (i) Let v and $v^{(h)}$ be as in the statement. We can assume that

$$\liminf_{h\to 0} J^{(h)}\left(v^{(h)}\right) < +\infty,$$

otherwise (4.3) is trivial. Therefore, up to subsequences, (4.1) is satisfied. From Theorem 4.1 we deduce that $v \in W^{1,2}((0,L);\mathbb{R}^3)$ and that the convergence is indeed weak in $W^{1,2}(\Omega;\mathbb{R}^3)$.
Now define the function $W_0 : [0,L] \times \mathbb{R}^3 \to \mathbb{R}$ as

$$W_0(s,z) := \inf\left\{W\left(s, (z \,|\, y_2 \,|\, y_3) R_0^T(s)\right) : y_2, y_3 \in \mathbb{R}^3\right\}.$$

Due to the coercivity assumptions this function is finite.

Notice that, since $R_0\, R_0^T = Id$, we can write

$$W\left(s, \nabla_h v^{(h)} \left(\nabla_h \Psi^{(h)}\right)^{-1}\right) = W\Big(s, \nabla_h v^{(h)} \left(\nabla_h \Psi^{(h)}\right)^{-1} R_0 R_0^T\Big)$$

and using the explicit expression of $\left(\nabla_h \Psi^{(h)}\right)^{-1}$ given in (2.6), i.e.,

$$\left(\nabla_h \Psi^{(h)}\right)^{-1}(s,\xi,\zeta) = R_0^T(s) - h\, R_0^T(s)\left[(\xi\, \nu_2'(s) + \zeta\, \nu_3'(s)) \otimes e_1\right] R_0^T(s) + O(h^2),$$

we have

$$\nabla_h v^{(h)} \left(\nabla_h \Psi^{(h)}\right)^{-1} R_0 e_1 \rightharpoonup v' \quad \text{weakly in } L^2(\Omega;\mathbb{R}^3). \tag{4.6}$$

So, from the definition of W_0

$$\begin{aligned} J^{(h)}\left(v^{(h)}\right) &\ge \int_\Omega W_0\left(s, \nabla_h v^{(h)} \left(\nabla_h \Psi^{(h)}\right)^{-1} R_0 e_1\right) \det\left(\nabla_h \Psi^{(h)}\right) ds\, d\xi\, d\zeta \\ &\ge \int_\Omega W_0^{**}\left(s, \nabla_h v^{(h)} \left(\nabla_h \Psi^{(h)}\right)^{-1} R_0 e_1\right) \det\left(\nabla_h \Psi^{(h)}\right) ds\, d\xi\, d\zeta. \end{aligned}$$

Now we pass to the lim inf in both sides of the previous inequality, using the uniform convergence of the determinant remarked in (2.5), and we get

$$\begin{aligned}\liminf_{h\to 0} J^{(h)}\big(v^{(h)}\big) &\geq \liminf_{h\to 0}\int_\Omega W_0^{**}\big(s,\big(\nabla_h v^{(h)}\big(\nabla_h\Psi^{(h)}\big)^{-1}R_0\big)e_1\big)\,\det\big(\nabla_h\Psi^{(h)}\big)\,ds\,d\xi\,d\zeta\\ &= \liminf_{h\to 0}\int_\Omega W_0^{**}\big(s,\big(\nabla_h v^{(h)}\big(\nabla_h\Psi^{(h)}\big)^{-1}R_0\big)e_1\big)\,ds\,d\xi\,d\zeta.\end{aligned}$$

Since the functional

$$G(u) := \int_\Omega W_0^{**}(s,u)\,ds\,d\xi\,d\zeta$$

is convex, it is sequentially weakly lower semicontinuous in $L^2(\Omega;\mathbb{R}^3)$; so, by (4.6) we can conclude that

$$\liminf_{h\to 0} J^{(h)}\big(v^{(h)}\big) \geq \int_0^L W_0^{**}(s,v'(s))\,ds. \tag{4.7}$$

(ii) Let v be a function in $W^{1,2}((0,L);\mathbb{R}^3)$, otherwise the bound in (4.4) is trivial. Let $w_2, w_3 \in W^{1,2}((0,L);\mathbb{R}^3)$ be arbitrary functions and consider the functions $v^{(h)}:\Omega\to\mathbb{R}^3$ defined by

$$v^{(h)}(s,\xi,\zeta) := v(s) + h\,\xi\,w_2(s) + h\,\zeta\,w_3(s).$$

Clearly, as $\nabla v^{(h)} = v'\otimes e_1 + h\,\big(\xi\,w_2' + \zeta\,w_3' \,|\, w_2 \,|\, w_3\big)$, we have that

$$v^{(h)} \to v \quad \text{strongly in } W^{1,2}(\Omega;\mathbb{R}^3). \tag{4.8}$$

Now we want to study the behaviour of the sequence

$$J^{(h)}\big(v^{(h)}\big) = \int_\Omega W\big(s,(\nabla_h v^{(h)})\big(\nabla_h\Psi^{(h)}\big)^{-1}\big)\det\big(\nabla_h\Psi^{(h)}\big)ds\,d\xi\,d\zeta$$

when $h\to 0$. Notice that the scaled gradient of $v^{(h)}$ satisfies

$$\nabla_h v^{(h)} = (v' \,|\, w_2 \,|\, w_3) + h\,(\xi\,w_2' + \zeta\,w_3')\otimes e_1 \to (v' \,|\, w_2 \,|\, w_3) \text{ a.e..} \tag{4.9}$$

So, by (2.5) and (vi), using the dominated convergence theorem we get

$$\begin{aligned}\lim_{h\to 0} J^{(h)}\big(v^{(h)}\big) &= \lim_{h\to 0}\int_\Omega W\big(s,(\partial_s v^{(h)} \,|\, w_2 \,|\, w_3)\big(\nabla_h\Psi^{(h)}\big)^{-1}\big)\det\big(\nabla_h\Psi^{(h)}\big)\,ds\,d\xi\,d\zeta\\ &= \int_0^L W\big(s,(v' \,|\, w_2 \,|\, w_3)\,R_0^T\big)\,ds.\end{aligned}$$

Up to now we have shown that for every choice of $w_2, w_3 \in W^{1,2}((0,L);\mathbb{R}^3)$, there exists a sequence $\big(v^{(h)}\big)$ such that (4.8) is satisfied and

$$\lim_{h\to 0} J^{(h)}\big(v^{(h)}\big) = \int_0^L W\big(s,(v' \,|\, w_2 \,|\, w_3)\,R_0^T\big)ds.$$

Therefore,

$$\begin{aligned}\Gamma-\limsup_{h\to 0} J^{(h)}(v) &:= \inf\left\{\limsup_{h\to 0} J^{(h)}\big(u^{(h)}\big) : u^{(h)}\to v \text{ strongly in } L^2(\Omega;\mathbb{R}^3)\right\}\\ &\leq \inf\left\{\int_0^L W\big(s,(v' \,|\, w_2 \,|\, w_3)\,R_0^T\big)\,ds : w_2,w_3\in W^{1,2}((0,L);\mathbb{R}^3)\right\}\\ &= \inf\left\{\int_0^L W\big(s,(v' \,|\, w_2 \,|\, w_3)\,R_0^T\big)\,ds : w_2,w_3\in L^2((0,L);\mathbb{R}^3)\right\},\end{aligned} \tag{4.10}$$

where the last equality is a consequence of the dominated convergence theorem and of the density of $W^{1,2}((0,L);\mathbb{R}^3)$ in $L^2((0,L);\mathbb{R}^3)$.

By the measurable selection lemma (see for example [3]) applied to the Carathéodory function

$$g:[0,L]\times\mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R},\quad (s,y_2,y_3)\mapsto g(s,y_2,y_3) := W\big(s,(v'(s) \,|\, y_2 \,|\, y_3)R_0^T(s)\big)$$

we obtain the existence of two measurable functions $w_2^0, w_3^0 : [0,L] \to \mathbb{R}^3$ satisfying

$$W\big(s, (v'(s) \,|\, w_2^0(s) \,|\, w_3^0(s))R_0^T(s)\big) = \inf_{y_2,y_3\in\mathbb{R}^3} W\big(s, (v'(s) \,|\, y_2 \,|\, y_3)R_0^T(s)\big) = W_0(s, v'(s)).$$

Moreover, from the coerciveness of W it follows that w_2^0, w_3^0 belong indeed to $L^2((0,L);\mathbb{R}^3)$ and so they are in competition for the infimum in (4.10). Hence, for every $v \in W^{1,2}((0,L);\mathbb{R}^3)$ we have

$$\Gamma - \limsup_{h\to 0} J^{(h)}(v) \le \int_0^L W_0(s, v'(s))\, ds =: \tilde{J}(v).$$

Now define the functional

$$J(v) = \begin{cases} \tilde{J}(v) & \text{if } v \in W^{1,2}((0,L);\mathbb{R}^3), \\ +\infty & \text{otherwise in } L^2(\Omega;\mathbb{R}^3); \end{cases} \tag{4.11}$$

clearly it turns out that

$$\Gamma - \limsup_{h\to 0} J^{(h)}(v) \le J(v) \quad \text{for every } v \in L^2(\Omega;\mathbb{R}^3). \tag{4.12}$$

As the lower semicontinuous envelope of J with respect to the strong topology of $L^2(\Omega;\mathbb{R}^3)$ is given by the functional I (see [2] and [7, Lemma 5]), the thesis follows immediately from (4.12). □

4.2. **Intermediate scaling.** In this subsection we show that scalings of the energy of order h^α, with $\alpha \in (0,2)$, lead to a trivial Γ-limit.

Theorem 4.5 (Compactness and Γ- convergence)**.** *Let $\mathcal{W}_1$ be the class of functions defined as*

$$\mathcal{W}_1 := \{v \in W^{1,2}((0,L);\mathbb{R}^3) : |v'(s)| \le 1 \text{ a.e.}\}. \tag{4.13}$$

For every sequence $\big(v^{(h)}\big)$ in $L^2(\Omega;\mathbb{R}^3)$ such that

$$\frac{1}{h^\alpha} J^{(h)}\big(v^{(h)}\big) \le c < +\infty \tag{4.14}$$

there exist a function $v \in \mathcal{W}_1$ and some constants $c^{(h)} \in \mathbb{R}$ such that, up to subsequences,

$$v^{(h)} - c^{(h)} \rightharpoonup v \quad \textit{weakly in } W^{1,2}(\Omega;\mathbb{R}^3).$$

Moreover,

$$\Gamma - \lim_{h\to 0} \frac{1}{h^\alpha} J^{(h)} = \begin{cases} 0 & \text{in } \mathcal{W}_1, \\ +\infty & \text{otherwise in } L^2(\Omega;\mathbb{R}^3). \end{cases} \tag{4.15}$$

Proof. Let $\big(v^{(h)}\big)$ be such that (4.14) is satisfied. Then

$$J^{(h)}\big(v^{(h)}\big) < c\, h^\alpha. \tag{4.16}$$

By Theorem 4.1 this implies that there exist $v \in W^{1,2}((0,L);\mathbb{R}^3)$ and some constants $c^{(h)} \in \mathbb{R}$ such that the sequence $v^{(h)} - c^{(h)}$ converges to v weakly in $W^{1,2}(\Omega;\mathbb{R}^3)$. Moreover by Theorem 4.2 and by (4.16)

$$0 = \liminf_{h\to 0} J^{(h)}\big(v^{(h)}\big) \ge \int_0^L W_0^{**}(s, v'(s))ds,$$

and this gives the additional condition that $|v'(s)| \le 1$ for almost every $s \in [0,L]$, thanks to Remark 4.4. Therefore $v \in \mathcal{W}_1$.

Let us prove (4.15). The liminf inequality follows directly from the fact that the energy density W is nonnegative and from the compactness. As for the limsup inequality we first notice that we can restrict our analysis to functions $v \in \mathcal{W}_1$, being the other case trivial. Since $|v'(s)| \le 1$ for a.e. $s \in [0,L]$, there exist two measurable functions $d_2, d_3 : [0,L] \to \mathbb{R}^3$ such that

$$(v'(s) \,|\, d_2(s) \,|\, d_3(s)) \in Co(SO(3)) \quad \text{for a.e. } s \in [0,L],$$

where $Co(SO(3))$ denotes the convex hull of $SO(3)$. As first step, we assume in addition that $(v' \,|\, d_2 \,|\, d_3)$ is a piecewise constant rotation; for simplicity we can limit ourselves to the case

$$(v'(s) \,|\, d_2(s) \,|\, d_3(s)) = \begin{cases} R_1 & \text{if } s \in [0, s_0[, \\ R_2 & \text{if } s \in [s_0, L] \end{cases}$$

with $R_1, R_2 \in SO(3)$. Now, let $\omega(h)$ be a sequence converging to zero, as $h \to 0$, and let P be a smooth function $P : [0,1] \longrightarrow SO(3)$, such that $P(0) = R_1$ and $P(1) = R_2$. Now consider a reparametrization of P, denoted by $P^{(h)}$ and given by

$$P^{(h)}(s) := P\left(\frac{s - s_0}{\omega(h)}\right).$$

Define the sequence $v^{(h)} : \Omega \to \mathbb{R}^3$ as

$$v^{(h)}(s,\xi,\zeta) := \begin{cases} R_1 \begin{pmatrix} s \\ h\,\xi \\ h\,\zeta \end{pmatrix} & \text{on } s \in [0, s_0[\times D, \\ \displaystyle\int_{s_0}^{s} \big(P^{(h)}\big)(\sigma) e_1 \, d\sigma + P^{(h)}(s) \begin{pmatrix} 0 \\ h\,\xi \\ h\,\zeta \end{pmatrix} + b^{(h)} & \text{on } \big[s_0, s_0 + \omega(h)\big] \times D, \\ R_2 \begin{pmatrix} s \\ h\,\xi \\ h\,\zeta \end{pmatrix} + d^{(h)} & \text{on } \big]s_0 + \omega(h), L\big] \times D, \end{cases}$$

where the constants $b^{(h)}$ and $d^{(h)}$ are chosen in order to make $v^{(h)}$ continuous. It turns out that the scaled gradient has the following expression:

$$\nabla_h v^{(h)} = \begin{cases} R_1 & \text{on } [0, s_0[\times D, \\ P^{(h)}(s) + \left(\big(P^{(h)}\big)'(s) \begin{pmatrix} 0 \\ h\,\xi \\ h\,\zeta \end{pmatrix} \right) \otimes e_1 & \text{on } \big[s_0, s_0 + \omega(h)\big] \times D, \\ R_2 & \text{on } \big]s_0 + \omega(h), L\big] \times D; \end{cases} \tag{4.17}$$

moreover $\nabla_h v^{(h)} \to (v' \,|\, d_2 \,|\, d_3)$ strongly in $L^2(\Omega; \mathbb{R}^3)$. In order to evaluate the functional on this sequence we use the fact that, by (v) and (2.5),

$$\frac{1}{h^\alpha} J^{(h)}\big(v^{(h)}\big) \le \frac{c}{h^\alpha} \int_\Omega \operatorname{dist}^2\big(\nabla_h v^{(h)} \left(\nabla_h \Psi^{(h)}\right)^{-1}, SO(3)\big) \, ds \, d\xi \, d\zeta. \tag{4.18}$$

From (4.17) the integral on the right-hand side of the previous expression can be written as

$$\int_0^{s_0} \int_D \operatorname{dist}^2\big(R_1 \left(\nabla_h \Psi^{(h)}\right)^{-1}, SO(3)\big) \, ds \, d\xi \, d\zeta + \int_{s_0+\omega(h)}^{L} \int_D \operatorname{dist}^2\big(R_2 \left(\nabla_h \Psi^{(h)}\right)^{-1}, SO(3)\big) \, ds \, d\xi \, d\zeta$$
$$+ \int_{s_0}^{s_0+\omega(h)} \int_D \operatorname{dist}^2\big(\nabla_h v^{(h)} \left(\nabla_h \Psi^{(h)}\right)^{-1}, SO(3)\big) \, ds \, d\xi \, d\zeta. \tag{4.19}$$

The first two terms in (4.19) give a contribution of order h^2 since, by (2.6), for $i = 1, 2$,

$$\begin{aligned} \operatorname{dist}^2\big(R_i \left(\nabla_h \Psi^{(h)}\right)^{-1}, SO(3)\big) &\le h^2 \operatorname{dist}^2\big(R_i \, R_0^T \left[(\xi\,\nu_2' + \zeta\,\nu_3') \otimes e_1\right] R_0^T, SO(3)\big) \\ &\le C\, h^2 \operatorname{dist}^2\big(\left[(\xi\,\nu_2' + \zeta\,\nu_3') \otimes e_1\right], SO(3)\big), \end{aligned}$$

so they can be neglected in the computation of the limit of (4.18). The only term we have to analyse is the last integral in (4.19). Set

$$A^{(h)}(s,\xi,\zeta) := \left(\big(P^{(h)}\big)' \begin{pmatrix} 0 \\ h\,\xi \\ h\,\zeta \end{pmatrix} \right) \otimes e_1.$$

Using again (2.6) we have that

$$\operatorname{dist}^2\big(\nabla_h v^{(h)} \left(\nabla_h \Psi^{(h)}\right)^{-1}, SO(3)\big) \le \operatorname{dist}^2\big(A^{(h)} \left(\nabla_h \Psi^{(h)}\right)^{-1}, SO(3)\big) \le C\, h^2 \left(\xi^2 + \zeta^2\right) \big|\, \big(P^{(h)}\big)' \,\big|^2,$$

so we get the following estimate:

$$\int_{s_0}^{s_0+\omega(h)} \int_D \operatorname{dist}^2\big(\nabla_h v^{(h)} \left(\nabla_h \Psi^{(h)}\right)^{-1}, SO(3)\big)\, ds\, d\xi\, d\zeta \le C\, h^2 \int_{s_0}^{s_0+\omega(h)} \left|\left(P^{(h)}\right)'\right|^2 ds$$
$$= C\, \frac{h^2}{\omega(h)} \int_0^1 \left|P'\right|^2 ds.$$

Notice that, if we choose $\omega(h) \sim h^\beta$, with $0 < \beta < 2-\alpha$, also this term can be neglected in (4.18), hence

$$\lim_{h\to 0} \frac{1}{h^\alpha}\, J^{(h)}\big(v^{(h)}\big) = 0$$

and this concludes the proof in the case $(v' \,|\, d_2 \,|\, d_3)$ is a piecewise constant rotation.

Consider now the general case. Since $(v' \,|\, d_2 \,|\, d_3) \in Co(SO(3))$ a.e., there exists a sequence of piecewise constant rotations $R_j : [0, L] \longrightarrow SO(3)$ such that $R_j \to (v' \,|\, d_2 \,|\, d_3)$ strongly in $L^2((0, L); \mathbb{M}^{3\times 3})$. For each element R_j of the sequence we can repeat the same construction done in the previous case and find a sequence $v_j^{(h)}$ whose scaled gradients $\nabla_h v_j^{(h)}$ converge to R_j as $h \to 0$ and such that for every j

$$\lim_{h\to 0} \frac{1}{h^\alpha} \int_\Omega W(s, \nabla_h v_j^{(h)} \left(\nabla_h \Psi^{(h)}\right)^{-1}) \det\big(\nabla_h \Psi^{(h)}\big) ds\, d\xi\, d\zeta = 0. \tag{4.20}$$

Now we can choose, for every j, an element of the sequence $v_j^{(h)}$, say $v_j^{(h_j)}$, in such a way that

$$\left\|\nabla_{h_j} v_j^{(h_j)} - R_j\right\|_{L^2(\Omega;\mathbb{M}^{3\times 3})} < \frac{1}{j} \tag{4.21}$$

and

$$\frac{1}{h_j^\alpha} \int_\Omega W(s, \nabla_{h_j} v_j^{(h_j)} \left(\nabla_{h_j} \Psi^{(h_j)}\right)^{-1}) \det\big(\nabla_{h_j} \Psi^{(h_j)}\big) ds\, d\xi\, d\zeta < \frac{1}{j}. \tag{4.22}$$

These estimates show that the sequence $v_j^{(h_j)}$ converges to $(v' \,|\, d_2 \,|\, d_3)$ strongly in $L^2((0, L); \mathbb{M}^{3\times 3})$ and that

$$\lim_{j\to\infty} \frac{1}{h_j^\alpha} \int_\Omega W(s, \nabla_{h_j} v^{(h_j)} \left(\nabla_{h_j} \Psi^{(h_j)}\right)^{-1}) \det\big(\nabla_{h_j} \Psi^{(h_j)}\big) ds\, d\xi\, d\zeta = 0. \tag{4.23}$$

This concludes the proof. □

Acknowledgments

I would like to thank Maria Giovanna Mora for having proposed to me the study of this problem and for many helpful and interesting suggestions. I would like also to thank Gianni Dal Maso for several stimulating discussions on the subject of this paper.
This work is part of the project "Calculus of Variations" 2004, supported by the Italian Ministry of Education, University, and Research.

References

[1] Acerbi E., Buttazzo G., Percivale D.: A variational definition of the strain energy for an elastic string. J. Elasticity **25**, 137-148 (1991)
[2] Dal Maso G.: An introduction to Γ-convergence. Birkhäuser, Boston, 1993
[3] Ekeland I., Temam R.: Convex analysis and variational problems. North-Holland, Amsterdam, 1976
[4] Friesecke G., James R.D., Müller S.: A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity. Comm. Pure Appl. Math. **55**, 1461-1506 (2002)
[5] Friesecke G., James R.D., Mora M.G., Müller S.: Derivation of nonlinear bending theory for shells from three-dimensional nonlinear elasticity by Γ-convergence. C. R. Math. Acad. Sci. Paris **336**, 697-702 (2003)
[6] Friesecke G., James R.D., Müller S.: A hierarchy of plate models derived from nonlinear elasticity by Γ-convergence. MPI-MIS Preprint **7** (2005)
[7] Le Dret H., Raoult A.: The nonlinear membrane model as variational limit of nonlinear three-dimensional elasticity. J. Math. Pures Appl. (9) **74**, 549-578 (1995)

[8] Le Dret H., Raoult A.: The membrane shell model in nonlinear elasticity: a variational asymptotic derivation. In Mechanics: from theory to computation, 59-84 Springer, New York, 2000

[9] Mora M.G., Müller S.: Derivation of the nonlinear bending-torsion theory for inextensible rods by Γ-convergence. Calc. Var. Partial Differential Equations **18**, 287-305 (2003)

[10] Mora M.G., Müller S.: A nonlinear model for inextensible rods as a low energy Γ-limit of three-dimensional nonlinear elasticity. Ann. Inst. H. Poincaré Anal. Non Linéaire **21**, 271-293 (2004)

[11] Pantz O.: Le modèle de poutre inextentionnelle comme limite de l' élasticité non-linéaire tridimensionnelle. Preprint (2002)

S.I.S.S.A., Via Beirut 2-4, 34014, Trieste, Italy
E-mail address: scardia@sissa.it