Global L^p estimates for degenerate Ornstein-Uhlenbeck operators*

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Abstract

We consider a class of degenerate Ornstein-Uhlenbeck operators in $\mathbb{R}^N$, of the kind

$$\mathcal{A} \equiv \sum_{i,j=1}^{p_0} a_{ij} \partial^2_{x_i x_j} + \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j}$$

where $(a_{ij}), (b_{ij})$ are constant matrices, (a_{ij}) is symmetric positive definite on $\mathbb{R}^{p_0}$ ($p_0 \leq N$), and (b_{ij}) is such that $\mathcal{A}$ is hypoelliptic. For this class of operators we prove global L^p estimates ($1 < p < \infty$) of the kind:

$$\left\| \partial^2_{x_i x_j} u \right\|_{L^p(\mathbb{R}^N)} \leq c \left\{ \| \mathcal{A} u \|_{L^p(\mathbb{R}^N)} + \| u \|_{L^p(\mathbb{R}^N)} \right\} \text{ for } i, j = 1, 2, ..., p_0$$

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and corresponding weak type (1,1) estimates. This result seems to be the first case of global estimates, in Lebesgue L^p spaces, for complete Hörmander's operators

$$\sum X_i^2 + X_0,$$

proved in absence of a structure of homogeneous group. We obtain the previous estimates as a byproduct of the following one, which is of interest in its own:

$$\left\| \partial^2_{x_i x_j} u \right\|_{L^p(S)} \leq c \left\| Lu \right\|_{L^p(S)}$$

for any $u \in C_0^\infty(S)$, where S is the strip $\mathbb{R}^N \times [-1, 1]$ and L is the Kolmogorov-Fokker-Planck operator $\mathcal{A} - \partial_t$. To get this estimate we use in a crucial way the left invariance of L with respect to a Lie group structure in $\mathbb{R}^{N+1}$ and some results on singular integrals on nonhomogeneous spaces recently proved in [2].

1 Introduction

Problem and main result

Let us consider the class of degenerate Ornstein-Uhlenbeck operators in $\mathbb{R}^N$:

$$\mathcal{A} = div\,(A\nabla) + \langle x, B\nabla \rangle = \sum_{i,j=1}^{N} a_{ij} \partial^2_{x_i x_j} + \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j},$$

where A and B are constant $N \times N$ matrices, A is symmetric and positive semidefinite. If we define the matrix:

$$C(t) = \int_0^t E(s)\, A E^T(s)\, ds, \text{ where } E(s) = \exp\left(-sB^T\right) \tag{1}$$

then it can be proved (see [18]) the equivalence between the three conditions:
- the operator $\mathcal{A}$ is hypoelliptic;
- $C(t) > 0$ for every $t > 0$;
- the following Hörmander's condition holds:

$$\text{rank}\mathcal{L}\left(X_1, X_2, ..., X_N, Y_0\right)(x) = N, \quad \text{for all } x \in \mathbb{R}^N,$$

where

$$Y_0 = \langle x, B\nabla \rangle \quad \text{and}$$

$$X_i = \sum_{j=1}^{N} a_{ij} \partial_{x_j} \quad i = 1, 2, ..., N.$$

Under one of these conditions it is proved in [18] that, for some basis of $\mathbb{R}^N$, the matrices A, B take the following form:

$$A = \begin{bmatrix} A_0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{2}$$

with $A_0 = (a_{ij})_{i,j=1}^{p_0}$ $p_0 \times p_0$ constant matrix ($p_0 \le N$), symmetric and positive definite:

$$\nu |\xi|^2 \le \sum_{i,j=1}^{p_0} a_{ij}\xi_i\xi_j \le \frac{1}{\nu} |\xi|^2 \tag{3}$$

for all $\xi \in \mathbb{R}^{p_0}$, some positive constant ν;

$$B = \begin{bmatrix} * & B_1 & 0 & \dots & 0 \\ * & * & B_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \dots & B_r \\ * & * & * & \dots & * \end{bmatrix} \tag{4}$$

where B_j is a $p_{j-1} \times p_j$ block with rank p_j, $j = 1, 2, ..., r$, $p_0 \ge p_1 \ge ... \ge p_r \ge 1$ and $p_0 + p_1 + ... + p_r = N$.

In this paper we consider hypoelliptic degenerate Ornstein-Uhlenbeck operators, with the matrices A, B already written as (2) and (4). For this class of operators, we shall prove the following global L^p estimates:

Theorem 1 *For every $p \in (1, \infty)$ there exists a constant $c > 0$, depending on p, N, p_0, the matrix B and the number ν in (3) such that for every $u \in C_0^\infty(\mathbb{R}^N)$ one has:*

$$\left\| \partial^2_{x_i x_j} u \right\|_{L^p(\mathbb{R}^N)} \le c \left\{ \|\mathcal{A}u\|_{L^p(\mathbb{R}^N)} + \|u\|_{L^p(\mathbb{R}^N)} \right\} \text{ for } i, j = 1, 2, ..., p_0 \tag{5}$$

$$\|Y_0 u\|_{L^p(\mathbb{R}^N)} \le c \left\{ \|\mathcal{A}u\|_{L^p(\mathbb{R}^N)} + \|u\|_{L^p(\mathbb{R}^N)} \right\}. \tag{6}$$

Moreover, the following weak type $(1,1)$ estimates hold:

$$\left| \left\{ x \in \mathbb{R}^N : \left| \partial^2_{x_i x_j} u(x) \right| > \alpha \right\} \right| \le \frac{c_1}{\alpha} \left\{ \|\mathcal{A}u\|_{L^1(\mathbb{R}^N)} + \|u\|_{L^1(\mathbb{R}^N)} \right\} \tag{7}$$

$$\left| \left\{ x \in \mathbb{R}^N : |Y_0 u(x)| > \alpha \right\} \right| \le \frac{c_1}{\alpha} \left\{ \|\mathcal{A}u\|_{L^1(\mathbb{R}^N)} + \|u\|_{L^1(\mathbb{R}^N)} \right\} \tag{8}$$

for every $\alpha > 0$, some constant c_1 depending on N, p_0, B and ν.

Global estimates in Hölder spaces analogous to (5)-(6) have been proved by Da Prato and Lunardi [7] in the nondegenerate case $p_0 = N$ (corresponding to the classical Ornstein-Uhlenbeck operator) and by Lunardi [20] in the degenerate case; L^p estimates in the nondegenerate case $p_0 = N$ have been proved by Metafune, Prüss, Rhandi and Schnaubelt [22] by a semigroup approach. Note that, even in the nondegenerate case, global estimates in L^p or Hölder spaces are not straightforward, due to the unboundedness of the first order coefficients. Under this regard, our weak (1,1) estimate seems to be new even in the nondegenerate case. L^2 estimates with respect to an invariant Gaussian measure have been proved by Lunardi [21] in the nondegenerate case, and by Farkas and Lunardi [11] in the degenerate case.

The operator $\mathcal{A}$ can be seen as the infinitesimal generator of the Ornstein-Uhlenbeck semigroup. This is the Markov semigroup associated to the stochastic differential equation:

$$d\xi(t) = B^T \xi(t)dt + \sqrt{2}\, A_0^{1/2} dW(t), \quad t > 0, \quad \xi(0) = x, \tag{9}$$

where $W(t)$ is a standard Brownian motion taking values in $\mathbb{R}^{p_0}$. This equation can describe the random motion of a particle in a fluid (see [28]). Several interpretations in physics and finance for the operator $\mathcal{A}$ or its evolutionary counterpart L (see below) are explained in the survey by Pascucci [24]. Nonlocal Ornstein-Uhlenbeck operators are studied by Priola and Zabczyk [25]. In infinite dimension, Ornstein-Uhlenbeck type operators arise naturally in the study of stochastic P.D.E.s (see [8], [9], [4] and the references therein).

Remark 2 *To make easier a comparison of our setting with that considered in several papers we have quoted so far, we point out the fact that the condition* $C(t) > 0$ *is equivalent to the condition*

$$Q_t \equiv \int_0^t \exp\left(sB^T\right) A \exp(sB)\, ds = \exp\left(tB^T\right) C(t) \exp(tB) > 0.$$

The operator Q_t *has also control theoretic meaning and is considered in [7], [8], [9], [11], [22], [25]. Also, note that it is enough to require that* $C(t)$ *or* Q_t *is positive definite for some* $t_0 > 0$ *in order to get that it is positive definite for all* $t > 0$.

Relation with the evolution operator

The evolution operator corresponding to $\mathcal{A}$,

$$L = \mathcal{A} - \partial_t,$$

is a Kolmogorov-Fokker-Planck ultraparabolic operator, which has been extensively studied in the last fifteen years. The largest part of the related literature is devoted to the case where an underlying structure of homogeneous group is present. In absence of this structure (that is, in the general situation we are interested in), this operator has been studied for instance by Lanconelli and Polidoro [18], Di Francesco and Polidoro [10], Cinti, Pascucci and Polidoro [5] (see also the survey [19], and references therein).

In particular, it is proved in [18] that the operator L is left-invariant with respect to the Lie group $\mathcal{K}$ whose underlying manifold is $\mathbb{R}^{N+1}$, endowed with the composition law

$$(x,t) \circ (\xi, \tau) = (\xi + E(\tau)\, x, t + \tau),$$

where $E(\tau) = \exp\left(-\tau B^T\right)$. Note that

$$(\xi, \tau)^{-1} = (-E(-\tau)\, \xi, -\tau).$$

It is straightforward to check the the left Haar measure on $\mathcal{K}$ is the Lebesgue measure on $\mathbb{R}^{N+1}$ and that the modular function is $\delta(x,t) = e^{t\mathrm{Tr}(B)}$. This means that the Jacobian of the inversion $\zeta = w^{-1}$ is given by

$$d\zeta = e^{\tau \mathrm{Tr} B} dw \tag{10}$$

so that

$$\int_{\mathbb{R}^{N+1}} f\left((x,t)^{-1}\right) dxdt = \int_{\mathbb{R}^{N+1}} f(x,t)\, e^{t\mathrm{Tr}(B)} dxdt. \tag{11}$$

We shall deduce global estimates (5) from an analogous estimate for L on the strip

$$S \equiv \mathbb{R}^N \times [-1,1],$$

which can be of independent interest:

Theorem 3 *For every $p \in (1,\infty)$ there exists a constant $c > 0$ such that*

$$\left\| \partial^2_{x_i x_j} u \right\|_{L^p(S)} \leq c \left\| Lu \right\|_{L^p(S)} \quad \text{for } i,j = 1,2,...,p_0, \tag{12}$$

for every $u \in C_0^\infty(S)$. The constant c depends on the same parameters than the c in Theorem 1.

To get the above L^p estimates, we have to set the problem in the suitable geometric framework, which for this specific class of operators has been studied in detail in [18], [10], while for general Hörmander's operators, with or without an underlying structure of homogeneous group, has been investigated by Folland [13], Rothschild and Stein [26], respectively.

In particular, L^p estimates for the second order derivatives have been proved in [13] on the whole space, but assuming the existence of a homogeneous group, and in [26] in the general case, but only locally. Therefore our results cannot be deduced by the existing theories.

Actually, Theorem 1 seems to be the first case of global estimates, in Lebesgue L^p spaces, for hypoelliptic degenerate Ornstein-Uhlenbeck operators, and more generally for complete Hörmander's operators

$$\sum X_i^2 + X_0,$$

in absence of an underlying structure of homogeneous group. We also want to stress that the group $\mathcal{K} = \left(\mathbb{R}^{N+1}, \circ\right)$ is not in general of polynomial growth (see (35)). Hence, in view of the results in [27], one cannot expect a global L^p estimate like (12) to be true on the whole $\mathbb{R}^{N+1}$ (instead that on a strip).

Our result can also be seen as a first step to study existence and uniqueness for the Cauchy problem related to L in L^p spaces, as well as to characterize the domain of the generator of the Ornstein-Uhlenbeck semigroup in L^p spaces. We plan to address these problems in the next future.

Strategy of the proof

Let us start noting that Theorem 3 easily implies Theorem 1, apart from the weak estimates (7), (8), which will be proved separately. Namely, let

$$\psi \in C_0^\infty(\mathbb{R})$$

be a cutoff function fixed once and for all, $\operatorname{sprt}\psi \subset [-1,1]$, $\int_{-1}^1 \psi(t)\,dt > 0$. If $u : \mathbb{R}^N \to \mathbb{R}$ is a C_0^∞ solution to the equation

$$\mathcal{A}u = f \text{ in } \mathbb{R}^N,$$

for some $f \in L^p\left(\mathbb{R}^N\right)$, let

$$U(x,t) = u(x)\,\psi(t);$$

then

$$LU(x,t) = f(x)\,\psi(t) - u(x)\,\psi'(t) \equiv F(x,t).$$

Therefore Theorem 3 applied to U gives

$$\left\|\partial^2_{x_i x_j} U\right\|_{L^p(S)} \leq c\,\|F\|_{L^p(S)} \quad \text{for } i,j = 1,2,...,p_0 \tag{13}$$

hence

$$\left\|\partial^2_{x_i x_j} u\right\|_{L^p(\mathbb{R}^N)} \leq c\left\{\|f\|_{L^p(\mathbb{R}^N)} + \|u\|_{L^p(\mathbb{R}^N)}\right\}$$

with c also depending on ψ. Note that (6) follows from (5).

We would like to describe now the general strategy of the proof of Theorem 3, as well as the main difficulties encountered. A basic idea is that of linking the properties of L to those of another operator of the same kind, which not only is left invariant with respect to a suitable Lie group of translations, but is also homogeneous of degree 2 with respect to a family of dilations (which are group automorphisms). Such an operator L_0 (see (16)) always exists under our assumptions, by [18], and has been called "the principal part" of L. Note that the operator L_0 fits the assumptions of Folland's theory [13]. However, to get the desired conclusion on L, this is not enough. Instead, we exploit the fact that, by results proved by [10], the operator L possesses a fundamental solution Γ with some good properties. First of all, Γ is translation invariant and has a fast decay at infinity, in space; this allows to reduce the desired L^p estimates to estimates of a singular integral operator whose kernel vanishes far off the pole. Second, this singular kernel, which has the form $\eta \cdot \partial^2_{x_i x_j}\Gamma$ where η is a radial cutoff function, satisfies "standard estimates" (in the language of singular integrals theory) with respect to a suitable "local quasisymmetric quasidistance" d, which is a key geometrical object in our study. Namely,

$$d(z,\zeta) = \left\|\zeta^{-1} \circ z\right\| \tag{14}$$

where $\zeta^{-1} \circ z$ is the Lie group operation related to the operator L, while $\|\cdot\|$ is a homogeneous norm related to the principal part operator L_0 (recall that L does not have an associated family of dilations, and therefore does not have a natural homogeneous norm). This "hybrid" quasidistance is not (and seemingly is not equivalent to) the control distance of any family of vector fields; even worse, it does not fulfill enough good properties in order to apply the standard theory of "singular integrals in spaces of homogeneous type" (in the sense of Coifman-Weiss [6]). More precisely, the problem is twofold:

(i) First, the function $d(z, \zeta)$ in (14) satisfies the quasisymmetric and quasitriangle inequalities only for $d(z, \zeta)$ bounded; this happens for instance on a fixed d-ball $B(z_0, R)$ (see Proposition 7).

(ii) On the other hand, in view of the rather involved geometry of the Lie group $\mathcal{K}$, it is not clear at all whether a doubling condition holds in the quasimetric space $(B(z_0, R), d)$. This means that we don't know whether an inequality of the kind

$$|B(z, 2r) \cap B(z_0, R)| \leq c\, |B(z, r) \cap B(z_0, R)| \quad \forall z \in B(z_0, R), r \leq R_0$$

actually holds: our strategy can only rely on those properties of the measure of balls proved in Proposition 9.

In other words, we have the following dilemma: if we choose as our space a compact set endowed with d and the Lebesgue measure, then d is a quasidistance but the doubling condition is not granted; and if we choose as our space the whole strip S, then we gain the doubling condition, but d is no longer a quasidistance.

This is why we cannot use the theory of "spaces of homogeneous type". Instead, we are forced to set the problem in a weaker abstract context ("bounded nonhomogeneous spaces"), and apply an ad hoc theory of singular integrals to get the desired L^p bound. The alluded ad hoc result has been proved by one of us in [2], in the spirit of the theory of singular integrals in nonhomogeneous spaces, which has been developed, since the late 1990's, by Nazarov-Treil-Volberg and other authors. With this machinery at hand, we can prove the desired L^p estimate for the singular integral with kernel $\eta \cdot \partial^2_{x_i x_j} \Gamma$ *on a ball.* To get the desired estimate on the whole strip $\mathbb{R}^N \times [-1, 1]$, still another nontrivial argument is needed, based on a covering lemma and exploiting both the existence of a group of translations, and the relevant properties of the quasidistance d.

Convention about constants. We have generally adopted the usual convention of denoting with the same letter c a constant which can vary from line to line. Sometimes, however, for the sake of clarity we have numbered different constants $c_1, c_2, ...$ through different inequalities.

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2 Background and known results

The principal part operator

Let us consider our operator L, with the matrices A, B written in the form (2), (4). We denote by B_0 the matrix obtained by annihilating every $*$ block in (4):

$$B_0 = \begin{bmatrix} 0 & B_1 & 0 & \dots & 0 \\ 0 & 0 & B_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & B_r \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \tag{15}$$

with B_j as in (4). By *principal part* of L we mean the operator

$$L_0 = div\,(A\nabla) + \langle x, B_0\nabla\rangle - \partial_t. \tag{16}$$

For every $\lambda > 0$, let us define the matrix of *dilations on* $\mathbb{R}^N$,

$$D\,(\lambda) = diag\left(\lambda I_{p_0}, \lambda^3 I_{p_1}, ..., \lambda^{2r+1} I_{p_r}\right)$$

where I_{p_j} denotes the $p_j \times p_j$ identity matrix, and the matrix of *dilations on* $\mathbb{R}^{N+1}$,

$$\delta\,(\lambda) = diag\left(\lambda I_{p_0}, \lambda^3 I_{p_1}, ..., \lambda^{2r+1} I_{p_r}, \lambda^2\right).$$

Note that

$$\det\left(\delta\,(\lambda)\right) = \lambda^{Q+2}$$

where

$$Q + 2 = p_0 + 3p_1 + ... + (2r+1)\,p_r + 2$$

is called the *homogeneous dimension of* $\mathbb{R}^{N+1}$. Analogously,

$$\det\left(D\,(\lambda)\right) = \lambda^Q$$

and Q is called the *homogeneous dimension of* $\mathbb{R}^N$. A remarkable fact proved in [18] is that the operator L_0 is homogeneous of degree two with respect to the dilations $\delta\,(\lambda)$, which by definition means that

$$L_0\left(u\left(\delta\,(\lambda)\,z\right)\right) = \lambda^2\left(L_0 u\right)\left(\delta\,(\lambda)\,z\right)$$

for every $u \in C_0^\infty\left(\mathbb{R}^{N+1}\right), z \in \mathbb{R}^{N+1}, \lambda > 0$.

If we define

$$C_0\,(t) = \int_0^t E_0\,(s)\,A E_0^T\,(s)\,ds, \text{ where } E_0\,(s) = \exp\left(-sB_0^T\right) \tag{17}$$

then the operator L_0 turns out to be left invariant with respect to the associated translations:

$$(x, t) \odot (\xi, \tau) = (\xi + E_0\,(\tau)\,x, t + \tau)\,;$$
$$(\xi, \tau)^{-1} = (-E_0\,(-\tau)\,\xi, -\tau)\,.$$

Moreover, the dilations $z \mapsto \delta(\lambda) z$ are automorphisms of the group $\left(\mathbb{R}^{N+1}, \odot\right)$.

There is a natural homogeneous norm in $\mathbb{R}^{N+1}$, induced by these dilations:

$$\|(x,t)\| = \sum_{j=1}^{N} |x_j|^{1/q_j} + |t|^{1/2}$$

where q_j are positive integers such that $D(\lambda) = diag\left(\lambda^{q_1}, ..., \lambda^{q_N}\right)$. Clearly, we have

$$\|\delta(\lambda) z\| = \lambda \|z\| \quad \text{for every } \lambda > 0, z \in \mathbb{R}^{N+1}.$$

Other properties of $\|\cdot\|$ will be stated later.

Fundamental solution

The following theorem collects some important known results about the fundamental solution of L:

Theorem 4 *Under the assumptions stated in the Introduction, the operator L possesses a fundamental solution*

$$\Gamma(z,\zeta) = \gamma\left(\zeta^{-1} \circ z\right) \text{ for } z, \zeta \in \mathbb{R}^{N+1},$$

with

$$\gamma(z) = \begin{cases} 0 & \text{for } t \leq 0 \\ \frac{(4\pi)^{-N/2}}{\sqrt{\det C(t)}} \exp\left(-\frac{1}{4}\left\langle C^{-1}(t)x, x\right\rangle - t\,TrB\right) & \text{for } t > 0 \end{cases}$$

where $z = (x,t)$ and $C(t)$ is as in (1). Recall that $C(t)$ is positive definite for all $t > 0$; hence $\gamma \in C^{\infty}\left(\mathbb{R}^{N+1} \setminus \{0\}\right)$. The following representation formulas hold:

$$u(z) = -(Lu * \gamma)(z) = -\int_{\mathbb{R}^{N+1}} \gamma\left(\zeta^{-1} \circ z\right) Lu(\zeta)\, d\zeta; \tag{18}$$

$$\partial^2_{x_i x_j} u(z) = -PV\left(Lu * \partial^2_{x_i x_j}\gamma\right)(z) + c_{ij} Lu(z) \tag{19}$$

for every $u \in C_0^{\infty}\left(\mathbb{R}^{N+1}\right)$, $i, j = 1, 2, ..., p_0$, for suitable constants c_{ij} which we do not need to specify. The "principal value" in (19) must be understood as

$$PV\left(Lu * \partial^2_{x_i x_j}\gamma\right)(z) \equiv \lim_{\varepsilon \to 0} \int_{\|\zeta^{-1} \circ z\| > \varepsilon} \left(\partial^2_{x_i x_j}\gamma\right)\left(\zeta^{-1} \circ z\right) Lu(\zeta)\, d\zeta.$$

The above theorem is proved in [15] (see also [18]), apart from (19) which is proved in [10, Proposition 2.11].

The fundamental solution $\Gamma_0(z,\zeta) = \gamma_0\left(\zeta^{-1} \circ z\right)$ of the principal part operator L_0 enjoys special properties; namely, for $t > 0$

$$\gamma_0(x,t) = \frac{(4\pi)^{-N/2}}{\sqrt{\det C_0(t)}} \exp\left(-\frac{1}{4}\left\langle C_0^{-1}(t)x, x\right\rangle\right) \tag{20}$$

with $C_0(t)$ as in (17); moreover (see [18, p.42]),

$$C_0\left(\lambda^2 t\right) = D(\lambda)\, C_0(t)\, D(\lambda) \qquad \forall \lambda, t > 0 \tag{21}$$

from which we can see that γ_0 is homogeneous of degree $-Q$:

$$\gamma_0(\delta(\lambda)(x,t)) = \lambda^{-Q}\gamma_0(x,t) \quad \forall \lambda > 0, (x,t) \in \mathbb{R}^{N+1} \setminus \{(0,0)\}.$$

Furthermore, the following relation links L to L_0 (see [18, Lemma 3.3]):

$$\langle C(t)\,x, x\rangle = \langle C_0(t)\,x, x\rangle (1 + O(t)) \text{ for } t \to 0; \tag{22}$$

$$\left\langle C^{-1}(t)\,x, x\right\rangle = \left\langle C_0^{-1}(t)\,x, x\right\rangle (1 + O(t)) \text{ for } t \to 0; \tag{23}$$

and (see [18, eqt. (3.14)]):

$$\det C(t) = \det C_0(t)\,(1 + O(t)) \text{ for } t \to 0. \tag{24}$$

3 Estimate on the nonsingular part of the integral

We now localize the singular kernel appearing in (19) introducing a cutoff function

$$\begin{aligned} &\eta \in C_0^\infty\left(\mathbb{R}^{N+1}\right) \text{ such that} \\ &\eta(z) = 1 \text{ for } \|z\| \le \rho_0/2; \\ &\eta(z) = 0 \text{ for } \|z\| \ge \rho_0, \end{aligned}$$

where $\rho_0 \le 1$ will be fixed later.

Let us rewrite (19) as:

$$\begin{aligned} \partial^2_{x_i x_j} u &= -PV\left(Lu * \left(\eta \partial^2_{x_i x_j}\gamma\right)\right) - \left(Lu * (1-\eta)\,\partial^2_{x_i x_j}\gamma\right) + c_{ij} Lu \\ &\equiv -PV\,(Lu * k_0) - (Lu * k_\infty) + c_{ij} Lu \end{aligned} \tag{25}$$

having set:

$$\begin{aligned} k_0 &= \eta \partial^2_{x_i x_j}\gamma \\ k_\infty &= (1-\eta)\,\partial^2_{x_i x_j}\gamma \end{aligned} \tag{26}$$

for $i, j = 1, 2, ..., p_0$ (we shall leave implicit the dependence of the kernels k_0, k_∞ on these indices i, j, as well as on the number ρ_0 appearing in the definition of the cutoff function η).

Since in k_∞ the singularity of $\partial^2_{x_i x_j}\gamma$ has been removed and $\partial^2_{x_i x_j}\gamma$ has a fast decay as $x \to \infty$, we can prove the following:

Proposition 5 *Let* $2S = \mathbb{R}^N \times [-2, 2]$. *For every* $\rho_0 > 0$ *there exists* $c = c(\rho_0) > 0$ *such that*

$$\int_{2S} |k_\infty(\zeta)|\, d\zeta \le c. \tag{27}$$

Note that this proposition easily implies the following:

Corollary 6 *For every $p \in [1,\infty]$ there exists a constant $c > 0$ only depending on p, N, p_0, ν and the matrix B such that:*

$$\left\| -\left(Lu * k_\infty\right) + c_{ij} Lu \right\|_{L^p(S)} \le c \left\| Lu \right\|_{L^p(S)} \text{ for every } u \in C_0^\infty\left(S\right), \tag{28}$$

$i, j = 1, ..., p_0$.

Proof of the Corollary. Since the modular function is bounded on $2S$ and $2S$ is invariant under inversion, (27) also implies

$$\int_{2S} \left| k_\infty\left(\zeta^{-1}\right)\right| d\zeta \le c. \tag{29}$$

In turn, (27) and (29) imply that for every $z \in S$,

$$\int_S \left| k_\infty\left(\zeta^{-1} \circ z\right)\right| d\zeta \le c \tag{30}$$

$$\int_S \left| k_\infty\left(z^{-1} \circ \zeta\right)\right| d\zeta \le c. \tag{31}$$

Now, as soon as we know that the kernel G of an integral operator

$$T : f \longmapsto \int G\left(x,y\right) f\left(y\right) dy$$

satisfies uniform bounds

$$\sup_x \int \left|G\left(x,y\right)\right| dy + \sup_y \int \left|G\left(x,y\right)\right| dx \le c < \infty,$$

this implies the continuity of T on L^p for every $p \in [1,\infty]$ (see e.g. Theorem 6.18 p.193 in [12]). Therefore (30) and (31) imply the Corollary. ■

Proof of Proposition 5. Since we are not interested in the exact dependence of the constant c on ρ_0, for the sake of simplicity we shall prove the Proposition for $\rho_0 = 1$. An analogous proof can be done for every ρ_0, finding a constant c which depends on ρ_0.

Recalling that, for $t > 0$, we have

$$\gamma\left(x,t\right) = \frac{\left(4\pi\right)^{-N/2}}{\sqrt{\det C\left(t\right)}} \exp\left(-\frac{1}{4}\left\langle C^{-1}\left(t\right)x, x\right\rangle - t\mathrm{Tr}B\right),$$

let us compute:

$$\left(\partial_{x_i}\gamma\right)\left(x,t\right) = -\frac{1}{2}\gamma\left(x,t\right)\left\langle C^{-1}\left(t\right)x, e_i\right\rangle$$

$$\left(\partial^2_{x_i x_j}\gamma\right)\left(x,t\right) = \frac{1}{2}\gamma\left(x,t\right)\left\{\frac{1}{2}\left\langle C^{-1}\left(t\right)x, e_j\right\rangle\left\langle C^{-1}\left(t\right)x, e_i\right\rangle - \left\langle C^{-1}\left(t\right)e_j, e_i\right\rangle\right\}$$

(where we have denoted by e_i the i-th unit vector in $\mathbb{R}^N$). Since the matrix $C^{-1}(t)$ is symmetric and positive definite, we can bound

$$\left|\left\langle C^{-1}(t)\,x, e_j\right\rangle\right| \leq \left\langle C^{-1}(t)\,x, x\right\rangle^{1/2}\left\langle C^{-1}(t)\,e_j, e_j\right\rangle^{1/2}.$$

By (23) and (21) we have:

$$\left\langle C^{-1}(t)\,e_j, e_j\right\rangle = \left\langle C_0^{-1}(t)\,e_j, e_j\right\rangle(1+O(t)) \quad \text{for } t \to 0 \tag{32}$$

and

$$\left\langle C_0^{-1}(t)\,e_j, e_j\right\rangle = \left\langle C_0^{-1}(1)\,D\left(\frac{1}{\sqrt{t}}\right)e_j, D\left(\frac{1}{\sqrt{t}}\right)e_j\right\rangle \leq c\left|D\left(\frac{1}{\sqrt{t}}\right)e_j\right|^2 =$$

$$(\text{since } j \in \{1,2,...,p_0\}\,) = c\left|\frac{1}{\sqrt{t}}e_j\right|^2 = \frac{c}{t}.$$

This shows that

$$\left\langle C^{-1}(t)\,e_j, e_j\right\rangle \leq \frac{c}{t}(1+O(t)), \text{ and}$$

$$\left|\left\langle C^{-1}(t)\,e_j, e_i\right\rangle\right| \leq \left\langle C^{-1}(t)\,e_j, e_j\right\rangle^{1/2}\left\langle C^{-1}(t)\,e_i, e_i\right\rangle^{1/2} \leq \frac{c}{t}(1+O(t)),$$

for $t \to 0$. Therefore

$$\left|\partial^2_{x_i x_j}\gamma(x,t)\right| \leq \frac{1}{2}\gamma(x,t)\left\{\frac{c}{t}\left\langle C^{-1}(t)\,x,x\right\rangle + \frac{c}{t}\right\}(1+O(t))$$

$$= \frac{c}{\sqrt{\det C(t)}}\exp\left(-\frac{1}{4}\left\langle C^{-1}(t)\,x,x\right\rangle - t\mathrm{Tr}B\right)\left\{\frac{1}{t}\left\langle C^{-1}(t)\,x,x\right\rangle + \frac{1}{t}\right\}(1+O(t))$$

$$\leq \frac{c}{t\sqrt{\det C(t)}}\exp\left(-\frac{(1-\delta)}{4}\left\langle C^{-1}(t)\,x,x\right\rangle - t\mathrm{Tr}B\right)$$

for some $\delta > 0$, all $t \in [-1,1]$.

Let us rewrite the last inequality as

$$\left|\partial^2_{x_i x_j}\gamma(x,t)\right| \leq \frac{c}{t}\gamma_\delta(x,t), \tag{33}$$

$$\text{with } \gamma_\delta(x,t) = \frac{(4\pi)^{-N/2}}{\sqrt{\det C(t)}}\exp\left(-\frac{(1-\delta)}{4}\left\langle C^{-1}(t)\,x,x\right\rangle - t\mathrm{Tr}B\right).$$

With this bound in hand, we can now bound the following integral:

$$\begin{aligned}\int_{2S}|k_\infty(\zeta)|\,d\zeta &= \int_{2S}\left|\left((1-\eta)\,\partial^2_{x_i x_j}\gamma\right)(\zeta)\right|d\zeta \\ &\leq \int_{\mathbb{R}^N\times(-2,2),\|\zeta\|\geq 1/2}\left|\left(\partial^2_{x_i x_j}\gamma\right)(\zeta)\right|d\zeta \\ &= \int_{\mathbb{R}^N\times(-2,2),\|\zeta\|\geq 1/2,\|(x,0)\|\leq 1/4}\left|\left(\partial^2_{x_i x_j}\gamma\right)(x,t)\right|dxdt+ \\ &+ \int_{\mathbb{R}^N\times(-2,2),\|\zeta\|\geq 1/2,\|(x,0)\|>1/4}\left|\left(\partial^2_{x_i x_j}\gamma\right)(x,t)\right|dxdt \\ &\equiv I + II.\end{aligned}$$

Now,

$$
\begin{aligned}
I &\leq c \int_{1/16 \leq |t| \leq 2, \|(x,0)\| \leq 1/4} \frac{(4\pi)^{-N/2}}{t\sqrt{\det C(t)}} \exp\left(-\frac{(1-\delta)}{4}\left\langle C^{-1}(t)\, x, x\right\rangle - t\operatorname{Tr} B\right) dxdt \\
&\leq c \int_{|x| \leq c_1} \exp\left(-c_2 |x|^2\right) dx \leq c
\end{aligned}
$$

where we have used (23) and the facts that

$$
\left\langle C_0^{-1}(t)\, x, x\right\rangle \geq c \left| D\left(\frac{1}{\sqrt{t}}\right) x\right|^2 \geq c\,|x|^2 \text{ since } |t| \leq 2
$$

while, by (24),

$$
t\sqrt{\det C(t)} \geq c_1 \sqrt{\det C_0(t)} = c_2 t^{Q/2} \geq c_3 \text{ since } |t| \geq 1/16.
$$

To handle II, we start noting that, if $\|(x,0)\| > 1/4$, by (23) we can write

$$
\begin{aligned}
&\exp\left(-c_1 \left\langle C^{-1}(t)\, x, x\right\rangle\right) \leq \exp\left(-c_2 \left\langle C_0^{-1}(t)\, x, x\right\rangle\right) \\
&\leq \exp\left(-c_3 \left| D\left(\frac{1}{\sqrt{t}}\right) x\right|^2\right) \leq \exp\left(-c_4 \frac{|x|^2}{t}\right) \\
&\leq \exp\left(-\frac{c_5}{t}\right) \leq c_6 t
\end{aligned}
$$

hence

$$
\begin{aligned}
II &\leq \int_{\mathbb{R}^N \times (-2,2), \|\zeta'\| \geq 1/2, \|(x,0)\| > 1/4} \frac{c}{\sqrt{\det C(t)}} \exp\left(-c_7 \left\langle C^{-1}(t)\, x, x\right\rangle\right) dxdt \\
&\text{(letting } C^{-1/2}(t)\, x = y) \\
&= \int_{\mathbb{R}^N \times (-2,2)} c \exp\left(-c_7 |y|^2\right) dydt = c.
\end{aligned}
$$

■

By Corollary 6 and (25), our final goal will be achieved as soon as we shall prove that

$$
\|PV(Lu * k_0)\|_{L^p(S)} \leq c\, \|Lu\|_{L^p(S)} \tag{34}
$$

for every $u \in C_0^\infty(S)$, $i, j = 1, ..., p_0, 1 < p < \infty$. The proof of (34) will be carried out in the following sections, and concluded with Theorem 22.

4 Estimates on the singular kernel

To prove the singular integral estimate (34), we have to introduce some more structure in our setting. Let:

$$
d(z, \zeta) = \left\|\zeta^{-1} \circ z\right\|.
$$

Recall that $\circ$ is the translation induced by the the operator L (or more precisely by the matrix B), and $\|\cdot\|$ the homogeneous norm induced by the dilations associated to the principal part operator L_0 (see §2). This object has been introduced and used in [10], and turns out to be the right geometric tool to describe the properties of the singular kernel γ_0. Namely, the following key properties have been proved in [10]:

Proposition 7 *(See Lemma 2.1 in [10]). For every compact set $K \subset \mathbb{R}^N$ there exists a constant $c_K \geq 1$ such that*

$$\begin{aligned}
&\left\|z^{-1}\right\| \leq c_K \|z\| \text{ for every } z \in K \times [-1,1] \\
&\|z \circ \zeta\| \leq c_K \{\|z\| + \|\zeta\|\} \text{ for every } \zeta \in S, z \in K \times [-1,1] .
\end{aligned}$$

Since the set $\{z : \|z\| \leq 1\}$ is compact, in terms of d the above inequalities imply that there exists a constant $c > 0$ such that:

$$\begin{aligned}
&d(z,\zeta) \leq c d(\zeta,z) \ \ \forall z,\zeta \in S \text{ with } d(\zeta,z) \leq 1 \\
&d(z,\zeta) \leq c\{d(z,w) + d(w,\zeta)\} \ \ \forall z,\zeta,w \in S \text{ with } d(z,w) \leq 1, d(\zeta,w) \leq 1.
\end{aligned}$$

Let us define the d-balls:

$$B(z,\rho) = \left\{\zeta \in \mathbb{R}^{N+1} : d(z,\zeta) < \rho\right\}.$$

Lemma 8 *The d-balls are open with respect to the Euclidean topology. Moreover, the family of d-open subsets of $\mathbb{R}^{N+1}$ (saying that a set Ω is open whenever for every $x \in \Omega$ there exists $\rho > 0$ such that $B(x,\rho) \subset \Omega$) coincides with the Euclidean topology.*

Proof. Since the map $w \mapsto z_0 \circ w$ is continuous with respect to the Euclidean topology, there exists $\varepsilon \in (0,1)$ such that if $|w| < \varepsilon$ then $|z_0 - z_0 \circ w| < \rho$. We claim that $B(z_0,\varepsilon) \subset B^E(z_0,\rho)$, where B^E denotes an Euclidean ball. Indeed, pick $z \in B(z_0,\varepsilon)$ and set $w = z_0^{-1} \circ z$. Then $\|w\| < \varepsilon < 1$. Therefore, it follows immediately from the definition of $\|w\|$ that $|w| \leq \|w\| < \varepsilon$. Thus $|z_0 - z| = |z_0 - z_0 \circ w| < \rho$ and the claim is proved.

Conversely, since the function $\zeta \mapsto d(z_0,\zeta)$ is continuous, $B(z_0,\rho)$ is open with respect to the Euclidean topology; in particular, $B(z_0,\rho)$ contains an Euclidean ball centered at z_0, so that the two topologies coincide. ■

The relevant information about the measure of d-balls are contained in the following:

Proposition 9 *There exists a constant $c > 0$ such that:*

(i) The following dimensional bound holds:

$$|B(z,\rho)| \leq c\rho^{Q+2} \text{ for every } z \in S, 0 < \rho < 1.$$

(ii) The following doubling condition holds in S:

$$|B(z,2\rho) \cap S| \leq c\,|B(z,\rho) \cap S| \quad \text{for every } z \in S, 0 < \rho < 1.$$

Proof. Let us compute the integral

$$|B(z,\rho)| = \int_{\|\zeta^{-1}\circ z\|<\rho} d\zeta.$$

Setting $\zeta^{-1}\circ z = w$ and applying (10) we have, if $z=(x,t)\,, w=(\xi,\tau)$:

$$|B(z,\rho)| = \int_{\|(\xi,\tau)\|<\rho} e^{\tau \operatorname{Tr} B} d\xi d\tau. \tag{35}$$

(By the way, (35) shows that the group $\mathcal{K}$ has not polynomial growth, generally). Since $z\in S$, in particular, $|t|\le 1$, $|t-\tau|\le\rho^2$, hence $|\tau|\le 2$ and the last integral is bounded by

$$e^{2\operatorname{Tr}B}\int_{\|w\|<\rho} dw = c\rho^{Q+2},$$

where we have used the fact that the norm $\|\cdot\|$ is homogeneous with respect to the dilations $\delta(\rho)$ and $\det(\delta(\rho)) = \rho^{Q+2}$. This proves (i).

To prove (ii), let $z=(x,t)$ be in S and assume, to fix ideas, that $t\ge 0$. Then

$$\begin{aligned}
|B(z,\rho)\cap S| &= \int_{\|(\xi,\tau)\|<\rho, |t-\tau|<1} e^{\tau \operatorname{Tr} B} d\xi d\tau \\
&\ge c\int_{\|(\xi,\tau)\|<\rho, 0\le\tau\le 1} d\xi d\tau \\
(\text{since } \rho<1) &= c\int_{\|w\|<\rho} dw = c\rho^{Q+2}\int_{\|w'\|<1} dw' \\
&= c\rho^{Q+2} \ge c\,|B(z,2\rho)| \ge c\,|B(z,2\rho)\cap S|
\end{aligned}$$

by (i). ■

We also need the following bounds of the fundamental solution Γ in terms of d:

Proposition 10 *(See Proposition 2.7 in [10]) The following "standard estimates" hold for Γ: there exist $c>0$ and $M>1$ such that*

(i) for all $z,\zeta\in S$,

$$\left|\partial^2_{x_i x_j}\Gamma(z,\zeta)\right| \le \frac{c}{d(z,\zeta)^{Q+2}};$$

(ii) for all $z,\zeta,w\in S$ with $Md(z,\zeta)\le d(\zeta,w)\le 1$,

$$\left|\partial^2_{x_i x_j}\Gamma(\zeta,w) - \partial^2_{x_i x_j}\Gamma(z,w)\right| \le c\frac{d(z,\zeta)}{d(\zeta,w)^{Q+3}}.$$

Moreover, estimate (ii) still holds for the kernel $\Gamma^*(\cdot,w) = \Gamma(w,\cdot)$.

Actually, estimate (ii) with respect to the exchanged variables is not explicitly proved in [10]; nevertheless, it can be proved with similar techniques. We leave the details to the interested reader.

An easy computation shows that the previous estimates extend to the kernel $k_0 = \eta \partial^2_{x_i x_j} \gamma$:

Proposition 11 *There exists $c > 0$ and $M > 1$ such that*

$$\left| k_0 \left(\zeta^{-1} \circ z \right) \right| \leq \frac{c}{d(z,\zeta)^{Q+2}} \quad \forall z, \zeta \in S$$

$$\left| k_0 \left(w^{-1} \circ \zeta \right) - k_0 \left(w^{-1} \circ z \right) \right| \leq c \frac{d(z,\zeta)}{d(\zeta,w)^{Q+3}} \quad \forall z, \zeta, w \in S, \; M d(z,\zeta) \leq d(\zeta,w) \leq 1.$$

Moreover, the same estimate holds for the kernel $k_0^*(z) = k_0 \left(z^{-1} \right)$.

Remark 12 *We can always assume that M is large enough, so that the conditions*

$$M d(z,\zeta) \leq d(\zeta,w) \leq 1$$

imply

$$c_1 d(z,w) \leq d(\zeta,w) \leq c_2 d(z,w)$$

for some absolute constants $c_1, c_2 > 0$.

We also need the following:

Lemma 13 *There exists $c > 0$ such that*

$$\left| \int_{r_1 < \|\zeta^{-1} \circ z\| < r_2} k_0 \left(\zeta^{-1} \circ z \right) d\zeta \right| \leq c$$

for all $z \in 2S$, all r_1, r_2 with $0 < r_1 < r_2$. Moreover, for every $z \in 2S$, the limit

$$\lim_{\varepsilon \to 0^+} \int_{\|\zeta^{-1} \circ z\| > \varepsilon} k_0 \left(\zeta^{-1} \circ z \right) d\zeta$$

exists, is finite, and independent of z. The same conclusions hold for the kernel $k_0^*(z) = k_0 \left(z^{-1} \right)$.

Proof. Let ρ_0 be the positive constant introduced at the beginning of Section 3. We may always assume that $r_2 \leq \rho$, because $k_0(w) = 0$ for $\|w\| > \rho_0$.

The change of variables $w = \zeta^{-1} \circ z$ (see (10)) shows that

$$\int_{r_1 < \|\zeta^{-1} \circ z\| < r_2} k_0 \left(\zeta^{-1} \circ z \right) d\zeta = \int_{r_1 < \|w\| < r_2} k_0(w) e^{\tau \operatorname{Tr} B} dw \tag{36}$$

with $w = (\xi, \tau)$

$$= \int_{r_1 < \|w\| < r_2} k_0(w) dw + \int_{r_1 < \|w\| < r_2} k_0(w) \left(e^{\tau \operatorname{Tr} B} - 1 \right) dw \equiv A(r_1, r_2) + B(r_1, r_2).$$

Then, since we can assume $r_2 \leq \rho_0$, we have

$$\left|B\left(r_1, r_2\right)\right| \leq \int_{\|w\|<r_2} c \frac{|\tau|}{\|w\|^{Q+2}} dw \leq c \int_{\|w\|<r_2} \frac{dw}{\|w\|^{Q}} \rightarrow 0 \text{ as } r_2 \rightarrow 0. \tag{37}$$

As to $A\left(r_1, r_2\right)$, assume first that $r_2 \leq \frac{\rho_0}{2}$, then

$$\begin{aligned} A\left(r_1, r_2\right) &= \int_{r_1<\|w\|<r_2} \partial_{x_i x_j}^2 \gamma(w)\, dw = \\ &= \int_{\|w\|=r_2} \partial_{x_i} \gamma(w)\, \nu_j d\sigma(w) - \int_{\|w\|=r_1} \partial_{x_i} \gamma(w)\, \nu_j d\sigma(w) \\ &\equiv I\left(r_2\right) - I\left(r_1\right) \end{aligned} \tag{38}$$

by the divergence theorem. It is shown in [10, Lemma 2.10] that

$$I(\rho) \rightarrow \int_{\|w\|=1} \partial_{x_i} \gamma_0(w)\, \nu_j d\sigma(w) \text{ as } \rho \rightarrow 0 \tag{39}$$

with γ_0 as in (20). Since, on the other hand, $I(\rho)$ is continuous for $\rho \in (0, 1/2]$, we conclude that $I(\rho)$ is bounded for $\rho \in \left[0, \frac{\rho_0}{2}\right]$. This implies the first statement in the Lemma if $r_2 \leq \rho_0/2$.

Next, if $\rho_0/2 \leq r_2 \leq \rho_0$, we can write

$$A\left(r_1, r_2\right) \leq \left|\int_{r_1<\|w\|<\rho_0/2} k_0(w)\, dw\right| + \left|\int_{\rho_0/2<\|w\|<r_2} k_0(w)\, dw\right|.$$

Now the first term can be bounded as above, while the second is bounded by

$$\int_{\rho_0/2 \leq \|w\| \leq \rho_0} c\, \|w\|^{-(2+Q)}\, dw = c.$$

The second statement in the Lemma follows by (37), (38) and (39).

To prove the same conclusions for the kernel $k_0^*(z) = k_0\left(z^{-1}\right)$, we can write

$$\int_{r_1<\|\zeta^{-1} \circ z\|<r_2} k_0\left(z^{-1} \circ \zeta\right) d\zeta = \int_{r_1<\|w^{-1}\|<r_2} k_0(w)\, dw.$$

Now, using the equivalence between $\left\|w^{-1}\right\|$ and $\|w\|$ and the bound $\left|k_0(w)\right| \leq c/\|w\|^{Q+2}$, it is quite standard to reduce the study of this integral to that of $\int_{r_1<\|w\|<r_2} k_0(w)\, dw$, so that the above arguments allow to conclude the proof. ■

5 L^p estimates of singular integrals on nonhomogeneous spaces

We now want to apply to our singular kernel an abstract result, proved in [2], which we are going to recall now.

Let X be a set. A function $d : X \times X \rightarrow \mathbb{R}$ is called a *quasisymmetric quasidistance* on X if there exists a constant $c_d \geqslant 1$ such that for every $x, y, z \in X$:

$$d(x,y) \geqslant 0 \text{ and } d(x,y) = 0 \Leftrightarrow x = y;$$

$$d(x,y) \leq c_d d(y,x); \tag{40}$$

$$d(x,y) \leqslant c_d (d(x,z) + d(z,y)). \tag{41}$$

If d is a quasisymmetric quasidistance, then

$$d^*(x,y) = d(x,y) + d(y,x)$$

is a quasidistance, equivalent to d, in the sense that, for suitable constants $c_1, c_2 > 0$,

$$c_1 d^*(x,y) \leq d(x,y) \leq c_2 d^*(x,y) \text{ for all } x, y \in X.$$

We will call d^* *the symmetrized quasidistance* of d.

Definition 14 *We will say that* (X, d, μ, k) *is a* nonhomogeneous space with Calderón-Zygmund kernel k *if:*

1. (X, d) *is a set endowed with a quasisymmetric quasidistance* d*, such that the* d*-balls are open with respect to the topology induced by* d*;*

2. μ *is a positive regular Borel measure on* X*, and there exist two positive constants* A, n *such that:*

$$\mu(B(x,\rho)) \leq A\rho^n \text{ for every } x \in X, \rho > 0; \tag{42}$$

3. $k(x,y)$ *is a real valued measurable kernel defined in* $X \times X$*, and there exists a positive constant* β *such that:*

$$|k(x,y)| \leq \frac{A}{d(x,y)^n} \text{ for every } x, y \in X; \tag{43}$$

$$|k(x,y) - k(x_0,y)| \leq A \frac{d(x_0,x)^\beta}{d(x_0,y)^{n+\beta}} \tag{44}$$

for every $x_0, x, y \in X$ *with* $d(x_0,y) \geq A d(x_0,x)$, *where* n, A *are as in (42).*

Theorem 15 *(See Theorem 3 in [2]). Let* (X, d, μ, k) *be a bounded and separable nonhomogeneous space with Calderón-Zygmund kernel* k*. Also, assume that*

(i) $k^*(x,y) \equiv k(y,x)$ *satisfies (44);*

(ii) there exists a constant $B > 0$ *such that*

$$\left| \int_{d(x,y)>\rho} k(x,y)\, d\mu(y) \right| + \left| \int_{d(x,y)>\rho} k^*(x,y)\, d\mu(y) \right| \leqslant B \tag{45}$$

for every $\rho > 0, x \in X$;

(iii) for a.e. $x \in X$, the limits

$$\lim_{\rho \to 0} \int_{d(x,y)>\rho} k(x,y)\, d\mu(y)\,; \qquad \lim_{\rho \to 0} \int_{d(x,y)>\rho} k^*(x,y)\, d\mu(y)$$

exist and are finite. Then the operator

$$\begin{aligned} Tf(x) &= PV \int_X k(x,y) f(y)\, d\mu(y) \\ &\equiv \lim_{\varepsilon \to 0} T_\varepsilon f(x) \equiv \lim_{\varepsilon \to 0} \int_{d(x,y)>\varepsilon} k(x,y) f(y)\, d\mu(y) \end{aligned}$$

is well defined for every $f \in L^1(X)$, and

$$\|Tf\|_{L^p(X)} \le c_p \|f\|_{L^p(X)} \text{ for every } p \in (1,\infty)\,;$$

moreover, T is weakly $(1,1)$ continuous. The constant c_p only depends on all the constants implicitly involved in the assumptions: $p, c_d, A, B, n, \beta, diam(X)$.

In the following we will write

$$PV \int k(x,y) f(y)\, d\mu(y) \equiv \lim_{\varepsilon \to 0} \int_{d(x,y)>\varepsilon} k(x,y) f(y)\, d\mu(y)\,.$$

We will also need the notion of Hölder space in this context:

Definition 16 (Hölder spaces) *We will say that $f \in C^\alpha(X)$, for some $\alpha > 0$, if*

$$\|f\|_\alpha \equiv \|f\|_\infty + |f|_\alpha \equiv \sup_{x \in X} |f(x)| + \sup_{x,y \in X, x \ne y} \frac{|f(x) - f(y)|}{d(x,y)^\alpha} < \infty.$$

We now come back to our original setting of $S \subset \mathbb{R}^{N+1}$ endowed with the local quasidistance d introduced in section 4. Our aim is to apply the previous abstract result to the singular integral T with kernel k_0 on a bounded domain, say a d-ball $B(z_0, R)$. More precisely, as we shall see later, what we need is an estimate of the kind

$$\|Tf\|_{L^p(B(z_0,R))} \le c \|f\|_{L^p(B(z_0,R))}$$

for $1 < p < \infty$, where R is a small radius fixed once and for all, z_0 is every point in the strip S, and the constant c is independent from z_0. Note that, by Proposition 7, our d is actually a quasisymmetric quasidistance in $X = B(z_0, R)$, as soon as R is small enough; moreover, by Proposition 9 the Lebesgue measure of a d-ball satisfies the required dimensional bound (42) with $n = Q+2$. Also, Proposition 11 and Lemma 13 suggest that the kernel k_0 satisfies the properties required by Theorem 15. However, there is a subtle problem with this last assertion, as explained in the following

Remark 17 *Saying that k_0 satisfies the cancellation property (45) in $B(z_0,R)$ means that*

$$\left|\int_{\zeta\in B(z_0,R):d(z,\zeta)>r} k_0\left(\zeta^{-1}\circ z\right)d\zeta\right|+\left|\int_{\zeta\in B(z_0,R):d(z,\zeta)>r} k_0\left(z^{-1}\circ \zeta\right)d\zeta\right|\leq c$$

whereas what we know (see Lemma 13) is that

$$\left|\int_{\zeta\in \mathbb{R}^{N+1}:d(z,\zeta)>r} k_0\left(\zeta^{-1}\circ z\right)d\zeta\right|+\left|\int_{\zeta\in \mathbb{R}^{N+1}:d(z,\zeta)>r} k_0\left(z^{-1}\circ \zeta\right)d\zeta\right|\leq c.$$

The point is that restricting the kernel k_0 to the domain $B(z_0,R)$ can destroy the cancellation property.

To overcome this problem, a more cautious choice consists in cutting the kernel smoothly, by a couple of Hölder continuous cutoff functions. Namely, we have the following

Proposition 18 *Let k_0 be the above kernel (see (26)). There exists a constant $R_0>0$ such that, for every $z_0\in S$, $R\leq R_0$, if a,b are two cutoff functions belonging to $C^{\alpha}\left(\mathbb{R}^{N+1}\right)$ for some $\alpha>0$, with $sprt\,a$, $sprt\,b\subset B(z_0,R)$, and we set*

$$k(x,y)=a(x)\,k_0\left(y^{-1}\circ x\right)b(y), \tag{46}$$

then:

(a) k satisfies (43),(44) and (45) in $B(z_0,R)$ (with possibly different constants). Explicitly, "(45) in $B(z_0,R)$" means

$$\left|\int_{y\in B(z_0,R):d(x,y)>r} k(x,y)\,d\mu(y)\right|\leq c \text{ for all } x\in B(z_0,R),r>0. \tag{47}$$

(b) for every $x\in B(z_0,R)$ there exists

$$h(x)\equiv\lim_{\varepsilon\to 0}\int_{y\in B(z_0,R):d(x,y)>\varepsilon} k(x,y)\,d\mu(y).$$

Finally, all the constants appearing in the above estimates about k depend on z_0,R and the cutoff functions a,b only through the C^{α} norms of a,b.

Remark 19 *Since in this Proposition and its proof the distinction between space and time variables is irrelevant, changing for a moment our notation we have denoted by $x,y,x_0...$ the variables in $\mathbb{R}^{N+1}$, and by $d\mu$ the Lebesgue measure $dxdt$ in $\mathbb{R}^{N+1}$.*

Proof. We choose R_0 small enough so that $x,y\in B(z_0,R_0)$ imply

$$d(x,y)+d(y,x)\leq 1.$$

Let $0 < R \leq R_0$.

We shall apply several times the properties of the kernel k_0 proved in Proposition 11 and Lemma 13. Also, we shall use twice the following simple fact:

$$\int_{d(x,y)<\rho} \frac{d\mu(y)}{d(x,y)^{Q+2-\alpha}} \leq c\rho^{\alpha} \text{ for all } x \in 2S,\ 0 < \rho \leq R_0 \tag{48}$$

which can be checked by a dilation argument and exploiting the fact that the modular function is bounded on the strip $2S$.

(a) The kernel k satisfies condition (43) in $B(z_0, R)$, because k_0 satisfies the analogous condition in S, by Proposition 11. As to (44), we can write

$$\begin{aligned} k(x,y) - k(x_0,y) &= [a(x) - a(x_0)]\, k_0\left(y^{-1} \circ x\right) b(y) + \\ &+ a(x_0)\left[k_0\left(y^{-1} \circ x\right) - k_0\left(y^{-1} \circ x_0\right)\right] b(y) = I + II. \end{aligned}$$

Now, for $d(x_0, y) > M d(x_0, x)$

$$|I| \leq |a|_{\alpha}\, d(x, x_0)^{\alpha} \cdot \frac{c}{d(x,y)^{Q+2}} \|b\|_{\infty} \leq c \frac{d(x,x_0)^{\alpha}}{d(x_0,y)^{Q+2+\alpha}}.$$

We have implicitly used the fact that the functions $d(x_0, y), d(x, y)$ are bounded by some absolute constant (since $x_0, x, y \in B(z_0, R)$), and the equivalence between $d(x_0, y)$ and $d(x, y)$, which holds under the assumption $d(x_0, y) > M d(x_0, x)$ (see Remark 12).

Moreover, still by Proposition 11, the kernel k_0 satisfies (44) with $n = Q+2$ and $\beta = 1$. Thus

$$|II| \leq c\, \|a\|_{\infty} \|b\|_{\infty} \frac{d(x, x_0)}{d(x_0, y)^{Q+3}} \leq c \frac{d(x, x_0)^{\alpha}}{d(x_0, y)^{Q+2+\alpha}},$$

hence (44) holds for k in $B(z_0, R)$, with $n = Q+2, \beta = \alpha$.

To check (47) let us start by noting that, since $\operatorname{sprt} b \subset B(z_0, R)$, we can write, for every $x \in B(z_0, R)$ and $r > 0$,

$$\begin{aligned} \int_{y \in B(z_0,R): d(x,y)>r} k(x,y)\, d\mu(y) &= a(x) \int_{y \in B(z_0,R): d(x,y)>r} k_0\left(y^{-1} \circ x\right) b(y)\, d\mu(y) \\ &= a(x) \int_{y \in \mathbb{R}^{N+1}: d(x,y)>r} k_0\left(y^{-1} \circ x\right) b(y)\, d\mu(y). \end{aligned}$$

Note that there exists some absolute constant $c > 0$ such that $b(y)$ vanishes if $x \in B(z_0, R)$ and $d(x, y) \geq cR$; hence the last integral equals

$$\begin{aligned} &a(x) \int_{y \in \mathbb{R}^{N+1}: r<d(x,y)<cR} k_0\left(y^{-1} \circ x\right) [b(y) - b(x)]\, d\mu(y) + \\ &+ a(x)\, b(x) \int_{y \in \mathbb{R}^{N+1}: r<d(x,y)<cR} k_0\left(y^{-1} \circ x\right) d\mu(y) \equiv I + II. \end{aligned}$$

Since $z_0 \in S$ and $x \in B(z_0, R)$, we can assume $x \in 2S$. Then, by (48)

$$|I| \leq c \|a\|_\infty |b|_\alpha \int_{d(x,y)<cR} \frac{d(x,y)^\alpha}{d(x,y)^{Q+2}} d\mu(y) = c \|a\|_\infty |b|_\alpha R^\alpha \leq c \|a\|_\infty |b|_\alpha R_0^\alpha$$

while, by Lemma 13,

$$|II| \leq \|a\|_\infty \|b\|_\infty \left| \int_{y \in \mathbb{R}^{N+1}: r_1 < d(x,y) < r_2} k_0 \left(y^{-1} \circ x\right) d\mu(y) \right| \leq c \|a\|_\infty \|b\|_\infty .$$

(b) To show the existence of $h(x)$ let us consider, for $0 < \varepsilon_1 < \varepsilon_2$ and a fixed $x \in B(z_0, R)$,

$$\begin{aligned}
&\int_{y \in B(z_0,R): d(x,y) > \varepsilon_1} k(x,y)\, d\mu(y) - \int_{y \in B(z_0,R): d(x,y) > \varepsilon_2} k(x,y)\, d\mu(y) \\
&= a(x) \int_{y \in B(z_0,R): \varepsilon_1 < d(x,y) < \varepsilon_2} k_0 \left(y^{-1} \circ x\right) b(y)\, d\mu(y) \\
&= a(x) \int_{y \in \mathbb{R}^{N+1}: \varepsilon_1 < d(x,y) < \varepsilon_2} k_0 \left(y^{-1} \circ x\right) b(y)\, d\mu(y) \\
&= a(x) \int_{y \in \mathbb{R}^{N+1}: \varepsilon_1 < d(x,y) < \varepsilon_2} k_0 \left(y^{-1} \circ x\right) [b(y) - b(x)]\, d\mu(y) + \\
&+ a(x)\, b(x) \int_{y \in \mathbb{R}^{N+1}: \varepsilon_1 < d(x,y) < \varepsilon_2} k_0 \left(y^{-1} \circ x\right) d\mu(y) \\
&\equiv II_1 + II_2.
\end{aligned}$$

Now,

$$\begin{aligned}
|II_1| &\leq \|a\|_\infty \int_{d(x,y)<\varepsilon_2} \left| k_0 \left(y^{-1} \circ x\right) [b(y) - b(x)] \right| d\mu(y) \\
&\leq c \|a\|_\infty |b|_\alpha \int_{d(x,y)<\varepsilon_2} \frac{d(x,y)^\alpha}{d(x,y)^{Q+2}} d\mu(y) \\
&\leq c \|a\|_\infty |b|_\alpha \varepsilon_2^\alpha
\end{aligned}$$

by (48), since $x \in 2S$. On the other hand,

$$|II_2| \leq \|a\|_\infty \|b\|_\infty \left| \int_{y \in \mathbb{R}^{N+1}: \varepsilon_1 < d(x,y) < \varepsilon_2} k_0 \left(y^{-1} \circ x\right) d\mu(y) \right|$$

which tends to zero as $\varepsilon_2 \to 0$, by Lemma 13. This proves the existence of the limit $h(x)$. ■

In view of Proposition 11 and Lemma 13, Proposition 18 can be applied also to the adjoint kernel $k_0^*(x) = k_0\left(x^{-1}\right)$. Therefore, from Theorem 15, Proposition 18 and the previous discussion, we immediately have the following:

Corollary 20 *For every fixed $z_0 \in S$, let*

$$Tf(z) = PV \int_{B(z_0,R)} k(z,\zeta) f(\zeta) \, d\zeta,$$

with k, R as in the previous Proposition. Then for every $p \in (1,\infty)$ there exists $c > 0$ such that

$$\|Tf\|_{L^p(B(z_0,R))} \leq c \|f\|_{L^p(B(z_0,R))}$$

for every $f \in L^p(B(z_0,R))$. The constant c depends on the cutoff functions a, b only through their C^α norms, and does not depend on z_0 and R.

We still need the following covering argument:

Lemma 21 *For every $r_0 > 0$ and $K > 1$ there exist $\rho \in (0, r_0)$, a positive integer M and a sequence of points $\{z_i\}_{i=1}^\infty \subset S$ such that:*

$$S \subset \bigcup_{i=1}^\infty B(z_i, \rho);$$

$$\sum_{i=1}^\infty \chi_{B(z_i,K\rho)}(z) \leq M \quad \forall z \in S.$$

The proof of this result uses arguments which are quite standard in doubling quasimetric settings or in locally compact groups (see [14, Lemma 8], [1]). Recall, however, that the set S is neither a group nor a doubling space.

Since this property is better proved in an abstract context, we postpone its proof to the next section, and proceed to conclude the proof of our main result:

Theorem 22 *For a suitable choice of the number ρ_0 appearing in the definition of the kernel k_0 (see §3), for every $p \in (1,\infty)$, there exists a positive constant c, depending on p, N, p_0, ν and the matrix B such that*

$$\|PV(f * k_0)\|_{L^p(S)} \leq c \|f\|_{L^p(S)}$$

for every $f \in L^p(S)$.

Proof. Pick a cutoff function

$$\begin{aligned} A &\in C_0^\alpha(S) \text{ such that:} \\ A(z) &= 1 \text{ for } \|z\| < \rho_0; \\ A(z) &= 0 \text{ for } \|z\| > 2\rho_0 \end{aligned}$$

where the number ρ_0, to be fixed later, is the same appearing in the definition of the cutoff function η and the kernel k_0 (see (26) in §3). Let

$$a_i(z) = A\left(z^{-1} \circ z_i\right) \text{ for } i = 1, 2, ...;$$

Since $k_0(\zeta^{-1} \circ z)$ vanishes for $d\,(z, \zeta) > \rho_0$, we have that

$$z \in B(z_i, \rho_0) \text{ and } k_0(\zeta^{-1} \circ z) \neq 0 \implies \zeta \in B(z_i, C\rho_0) \tag{49}$$

for some absolute constant C. Define a second cutoff function

$$\begin{gathered} B \in C_0^{\alpha}(S) \text{ such that:} \\ B(z) = 1 \text{ for } \|z\| < C\rho_0; \\ B(z) = 0 \text{ for } \|z\| > 2C\rho_0 \end{gathered}$$

where C is the constant appearing in (49). Let

$$b_i(z) = B\left(z^{-1} \circ z_i\right) \text{ for } i = 1, 2, ...$$

Note that:

$$\begin{aligned} \|a_i\|_{C^{\alpha}} &= \|A\|_{C^{\alpha}} \text{ for } i = 1, 2, ... \\ \|b_i\|_{C^{\alpha}} &= \|B\|_{C^{\alpha}} \text{ for } i = 1, 2, ... \end{aligned} \tag{50}$$

Set

$$k_i(z, \zeta) = k_0(\zeta^{-1} \circ z) a_i(z) b_i(\zeta).$$

Let now R_0 be as in Proposition 18; set $r_0 = R_0/2C$ and let us apply Lemma 21 for this r_0: there exists $\rho_0 < r_0$ such that

$$S \subset \bigcup_{i=1}^{\infty} B(z_i, \rho_0); \tag{51}$$

$$\sum_{i=1}^{\infty} \chi_{B(z_i, 2C\rho_0)}(z) \leq M \quad \forall z \in S. \tag{52}$$

We eventually chose this value for the constant ρ_0.

Recall that $Tf = PV(f * k_0)$. By (51) we can write

$$\|Tf\|_{L^p(S)} \leq \sum_{i=1}^{\infty} \|Tf\|_{L^p(B(z_i, \rho_0))}. \tag{53}$$

On the other side, by (49) for every $z \in B(z_i, \rho_0)$ we have

$$\begin{aligned} Tf(z) &= PV \int_{\mathbb{R}^{N+1}} k_0\left(\zeta^{-1} \circ z\right) f(\zeta) d\zeta \\ &= a_i(z) PV \int_{\mathbb{R}^{N+1}} k_0\left(\zeta^{-1} \circ z\right) b_i(\zeta) f(\zeta) d\zeta \\ &= PV \int_{B(z_i, 2C\rho_0)} k_i(z, \zeta) f(\zeta) d\zeta \equiv T_i f(z) \end{aligned}$$

hence

$$\sum_{i=1}^{\infty} \|Tf\|_{L^p(B(z_i, \rho_0))} = \sum_{i=1}^{\infty} \|T_i f\|_{L^p(B(z_i, \rho_0))}. \tag{54}$$

Since $2C\rho_0 \leq R_0$, the kernel k_i also satisfies the assumptions of Proposition 18. Hence by Corollary 20 we have

$$\|T_i f\|_{L^p(B(z_i,2C\rho_0))} \leq c\,\|f\|_{L^p(B(z_i,2C\rho_0))} \tag{55}$$

with c independent of i, by (50). By (52) to (55) and we conclude

$$\|Tf\|_{L^p(S)} \leq c\sum_{i=1}^{\infty}\|f\|_{L^p(B(z_i,2C\rho_0))} \leq cM\,\|f\|_{L^p(S)}$$

which ends the proof. ■

Conclusion of the proof of Theorems 1 and 3. Theorem 22 and Corollary 6 imply Theorem 3, by (25). As we have shown in §1, Theorem 3 in turn implies (5)-(6) in Theorem 1. To finish the proof of Theorem 1 we are left to prove the weak $(1,1)$-estimates (7)-(8). This will be done here.

Let $u \in C_0^{\infty}(S)$. By (25) in §3 we can write, for every $\alpha > 0$:

$$\begin{aligned}
&\left|\left\{z \in S : \left|\partial^2_{x_i x_j} u(z)\right| \geq \alpha\right\}\right| \leq \left|\left\{z \in S : |PV\,(Lu * k_0)(z)| \geq \frac{\alpha}{3}\right\}\right| + \\
&+ \left|\left\{z \in S : |(Lu * k_\infty)(z)| \geq \frac{\alpha}{3}\right\}\right| + \left|\left\{z \in S : |c_{ij} Lu(z)| \geq \frac{\alpha}{3}\right\}\right| \\
&\equiv A + B + C.
\end{aligned}$$

Now, by Corollary 6

$$B + C \leq \frac{3}{\alpha}\left\{\|Lu * k_\infty\|_{L^1(S)} + \|c_{ij} Lu\|_{L^1(S)}\right\} \leq \frac{c}{\alpha}\,\|Lu\|_{L^1(S)}.$$

To bound A, we revise as follows the proof of Theorem 22, writing (with the same meaning of symbols and letting $f \equiv Lu$):

$$\begin{aligned}
A &= \left|\left\{z \in S : |Tf(z)| \geq \frac{\alpha}{3}\right\}\right| \\
&\leq \sum_{i=1}^{\infty}\left|\left\{z \in B(z_i,\rho_0) : |Tf(z)| \geq \frac{\alpha}{3}\right\}\right| \\
&= \sum_{i=1}^{\infty}\left|\left\{z \in B(z_i,\rho_0) : |T_i f(z)| \geq \frac{\alpha}{3}\right\}\right| \\
&\leq \sum_{i=1}^{\infty}\left|\left\{z \in B(z_i,2C\rho_0) : |T_i f(z)| \geq \frac{\alpha}{3}\right\}\right| \\
&\leq \sum_{i=1}^{\infty}\frac{c}{\alpha}\,\|f\|_{L^1(B(z_i,2C\rho_0))} \leq \frac{cM}{\alpha}\,\|f\|_{L^1(S)}
\end{aligned}$$

where we used the fact that T_i is also weak $(1,1)$ continuous on $L^1(B(z_i,2C\rho_0))$, by Theorem 15. This proves the weak estimate on the strip:

$$\left|\left\{z \in S : \left|\partial^2_{x_i x_j} u(z)\right| \geq \alpha\right\}\right| \leq \frac{c}{\alpha}\,\|Lu\|_{L^1(S)}. \tag{56}$$

Next, we take a cutoff function $\psi \in C_0^\infty(-1,1)$ such that $\psi(t) \geq 1$ in $\left[-\frac{1}{2},\frac{1}{2}\right]$ and, for every $u \in C_0^\infty\left(\mathbb{R}^N\right)$, apply (56) to ψu, getting

$$\begin{aligned}
&\left|\left\{x \in \mathbb{R}^N : \left|\partial^2_{x_i x_j} u(x)\right| \geq \alpha\right\}\right| \\
&\leq \left|\left\{(x,t) \in \mathbb{R}^N \times \left[-\frac{1}{2},\frac{1}{2}\right] : \left|\psi(t)\, \partial^2_{x_i x_j} u(x)\right| \geq \alpha\right\}\right| \\
&\leq \left|\left\{z \in S : \left|\partial^2_{x_i x_j}(u\psi)(z)\right| \geq \alpha\right\}\right| \\
&\leq \frac{c}{\alpha} \left\|L(u\psi)\right\|_{L^1(S)} \\
&\leq \frac{c}{\alpha} \left\{\|\mathcal{A}u\|_{L^1(\mathbb{R}^N)} + \|u\|_{L^1(\mathbb{R}^N)}\right\}.
\end{aligned}$$

So we have proved (7); then (8) follows from (7) using the equation, and this ends the proof. ■

6 A covering lemma

To make our proof of Theorem 22 complete, we are left to prove Lemma 21. Here we will do this, by using a general abstract argument.

Definition 23 *We say that (X,d,μ) is a* locally invariant quasimetric space *if the following conditions hold:*

(i) $d : X \times X \to \mathbb{R}_+$ is a function such that for some constant $C > 0$

(i_1) For every $x, y \in X$, if $d(y,x) \leq 1$ then $d(x,y) \leq Cd(y,x)$

(i_2) For every $x,y,z \in X$, if $d(x,z) \leq 1$ and $d(y,z) \leq 1$ then

$$d(x,y) \leq C\left(d(x,z) + d(z,y)\right).$$

(ii) μ is a positive measure defined on a σ-algebra of subsets of X which contains the d-balls

$$B(x,\rho) = \{y \in X : d(y,x) < \rho\}, \quad x \in X, \rho > 0.$$

Moreover, every d-ball has positive and finite measure.

(iii) There exists $R > 0$ such that if $0 < R_1 < R_2 \leq R$ then there exists $C = C(R_1,R_2)$ such that

$$\mu(B(x,R_2)) \leq C\mu(B(x,R_1)) \text{ for any } x \in X. \tag{57}$$

Remark 24 *Note that $(S,d,dxdt)$ is a locally invariant quasimetric space. Namely, condition (i) follows from Proposition 7, (ii) follows from Lemma 8 and (iii) follows from Proposition 9. Hence the following theorem will imply Lemma 21, and therefore will conclude the proof of Theorem 3.*

Theorem 25 *Let (X, d, μ) be a locally invariant quasimetric space. Then for every $r_0 > 0$ and $K > 1$, there exist $\rho \in (0, r_0)$, a positive integer M and a countable set $\{x_i\}_{i\in A} \subset X$ such that:*

1.
$$\bigcup_{i\in A} B(x_i, \rho) = X;$$

2.
$$\sum_{i\in A} \chi_{B(x_i, K\rho)} \leq M^2.$$

Proof. First of all, we claim that for every $\rho > 0$, X admits a maximal countable family of disjoint balls of radius ρ. Namely: the existence of a maximal family (of arbitrary cardinality) of disjoint balls of radius ρ follows by Zorn's Lemma; let us show that this family $\{B(x_\alpha, \rho)\}_\alpha$ must be countable. Otherwise, since for every fixed $x_0 \in X$,

$$X = \bigcup_{n=1}^{\infty} B(x_0, n),$$

at least one ball $B(x_0, n)$ should contain an uncountable family of disjoint balls of radius ρ. By (ii) in Definition 23, every such ball has positive measure, and this would imply that $B(x_0, n)$ has infinite measure, which contradicts the same definition. This proves the claim.

Then, let $\left\{B\left(x_i, \frac{\rho}{C(C+1)}\right)\right\}_{i\in A}$ be a countable maximal family of disjoint balls.

Fix $x \in X$. There exists $i \in A$ such that

$$B\left(x, \frac{\rho}{C(C+1)}\right) \cap B\left(x_i, \frac{\rho}{C(C+1)}\right) \neq \emptyset.$$

To estimate $d(x_i, x)$, we consider $y \in B\left(x, \frac{\rho}{C(C+1)}\right) \cap B\left(x_i, \frac{\rho}{C(C+1)}\right)$, and we find

$$d(x_i, x) \leq C(d(x_i, y) + d(y, x)) < C\left(\frac{\rho}{C(C+1)} + C \cdot \frac{\rho}{C(C+1)}\right) = \rho$$

where, to apply (i_1)-(i_2) in Definition 23, we have assumed $\rho \leq 1$. This proves (1).

To prove (2), fix an arbitrary $i \in A$; we want to estimate how many $j \in A$ satisfy the property

$$B(x_i, K\rho) \cap B(x_j, K\rho) \neq \emptyset. \tag{58}$$

Fix x_i and x_j and suppose there exists $y \in B(x_i, K\rho) \cap B(x_j, K\rho)$. We assume $K\rho \leq 1$; hence

$$d(x_i, x_j) \leq C(d(x_i, y) + d(y, x_j)) \leq C(K\rho + CK\rho) = C(1 + C)K\rho,$$

and we assume $C(1+C)K\rho \leq 1$. Now suppose that for $j=1,2,...,N$ we have (58); we want to estimate N.

Take $z \in B(x_j, K\rho)$. Since $K\rho \leq 1$ and $d(x_i, x_j) \leq 1$ we have

$$\begin{aligned} d(x_i, z) &\leq C(d(x_i, x_j) + d(x_j, z)) \leq C(C(1+C)K\rho + K\rho) \\ &= K\rho\left(C\left(C^2 + C + 1\right)\right) \equiv K'\rho \end{aligned}$$

with $K' > 1$. The previous computation shows that

$$\bigcup_{j=1}^{N} B(x_j, K\rho) \subset B(x_i, K'\rho).$$

Since by construction the balls $B\left(x_i, \frac{\rho}{C(C+1)}\right)$ are pairwise disjoint, we have

$$\begin{aligned} \sum_{j=1}^{N} \mu\left(B\left(x_j, \frac{\rho}{C(C+1)}\right)\right) &= \mu\left(\bigcup_{j=1}^{N} B\left(x_j, \frac{\rho}{C(C+1)}\right)\right) \\ &\leq \mu\left(\bigcup_{j=1}^{N} B(x_j, K\rho)\right) \leq \mu(B(x_i, K'\rho)). \end{aligned}$$

Assuming $K'\rho \leq R$ (with R as in (iii) of Definition 23), we also have, for some constant M only depending on C and ρ,

$$\sum_{j=1}^{N} \mu\left(B\left(x_j, \frac{\rho}{C(C+1)}\right)\right) \leq M\mu\left(B\left(x_i, \frac{\rho}{C(C+1)}\right)\right).$$

Now fix any $j = 1,2,...,N$. Note that i satisfies (58); repeating the previous argument exchanging i with j we get

$$\mu\left(B\left(x_i, \frac{\rho}{C(C+1)}\right)\right) \leq \mu(B(x_j, K'\rho)) \leq M\mu\left(B\left(x_j, \frac{\rho}{C(C+1)}\right)\right).$$

We have found that for every $j = 1,2,...,N$

$$\sum_{k=1}^{N} \mu\left(B\left(x_k, \frac{\rho}{C(C+1)}\right)\right) \leq M^2\mu\left(B\left(x_j, \frac{\rho}{C(C+1)}\right)\right).$$

Letting

$$a = \min_{j=1,2,...,N} \mu\left(B\left(x_j, \frac{\rho}{C(C+1)}\right)\right)$$

we get

$$Na \leq \sum_{k=1}^{N} \mu\left(B\left(x_k, \frac{\rho}{C(C+1)}\right)\right) \leq M^2 a$$

and since, by (iii), $0 < a < \infty$, we infer $N \leq M^2$, which is (2), provided ρ satisfies all the conditions we have imposed so far:

$$\rho \leq 1;\ K\rho \leq 1; K'\rho \equiv K\rho\left(C\left(1 + C\left(1 + C\right)\right)\right) \leq R.$$

■

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