THE LIMIT OF $W^{1,1}$ HOMEOMORPHISMS WITH FINITE DISTORTION

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Abstract. We show that the limit f of a weakly convergent sequence of $W^{1,1}$ homeomorphisms f_j with finite distortion has finite distortion as well, provided that it is a homeomorphism. Moreover, the lower semicontinuity of the distortions is deduced both in case of outer and inner distortion.

1. Introduction

In this paper we study the convergence of a sequence of homeomorphisms $f_j : \Omega \mapsto \Omega'$ of Sobolev class $W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ with finite distortion, where Ω and Ω' are bounded open sets in $\mathbb{R}^n$, $n \geq 2$.

Recall that a mapping $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ is said to be of *finite distortion* if its Jacobian J_f is strictly positive almost everywhere on the set where $Df \neq 0$. For such a mapping the *distortion* K_f is defined as

$$K_f(x) = \begin{cases} \dfrac{|Df(x)|^n}{J_f(x)} & if \;\; J_f(x) > 0, \\ 1 & otherwise. \end{cases} \tag{1.1}$$

Notice that $K_f(x)$ is the smallest function greater than or equal to 1 and such that

$$|Df(x)|^n \leq K_f(x) J_f(x) \qquad \text{for a.e. } x \in \Omega. \tag{1.2}$$

Our first result deals with the convergence of the inverse mappings f_j^{-1} of a sequence f_j of homeomorphisms of finite distortion. In fact, a recent result proved in [11] (see also [9], [10] for the case $n = 2$ and [3], [12]) states that if $f \in W^{1,n-1}(\Omega, \Omega')$ is a homeomorphism of finite distortion and $|Df|$ belongs to the Lorentz space $L^{n-1,1}(\Omega)$, then the inverse map f^{-1} belongs to $W^{1,n-1}(\Omega'; \Omega)$ and has finite distortion too. Note that when $n = 2$ the Lorentz space $L^{1,1}$ coincides with L^1 and thus no additional assumption is required on f, besides $f \in W^{1,1}(\Omega, \Omega')$.

In particular our Theorem 3.2 shows that if f_j is a sequence of homeomorphisms of finite distortion, satisfying reasonable equi-boundedness assumptions, then the inverse mappings f_j^{-1} converge weakly in $W^{1,1}$.

In the literature the study of a sequence of mappings of finite distortion has been also considered from a different point of view, namely to find under which conditions weak limits are also maps of finite distortion. To this aim, we recall the following result, proved in [8], where the maps f_j are assumed to converge weakly in $W^{1,n}$ to f and the corresponding distortions K_{f_j} converge in the biting sense to some function K.

Theorem 1.1. *Suppose that $f_j : \Omega \mapsto \mathbb{R}^n$ is a sequence of mappings of finite distortion which converge weakly in $W^{1,n}(\Omega, \mathbb{R}^n)$ to f and suppose that the functions K_{f_j} converge in the biting* *sense to K. Then f has finite distortion and*

$$K_f(x) \le K(x) < \infty \qquad \text{for a.e. } x \in \Omega\,.$$

A more general version of this result has been proved in [13] in the context of Orlicz-Sobolev spaces.

An important tool in the proof of Theorem 1.1 is the continuity of the Jacobian operator

$$f \in W^{1,n}(\Omega, \mathbb{R}^n) \mapsto J_f \in L^1(\Omega)$$

with respect to weak convergence in $W^{1,n}$ of mappings of finite distortion and weak convergence in L^1 of Jacobians. Notice that such a continuity is not guaranteed, even in dimension $n = 2$, when we assume that mappings f_j belong only to $W^{1,1}$ and converge weakly in $W^{1,1}$. On the other hand this result pertains to mappings of finite distortion which are not necessarily one-to-one, though they are continuous, as a consequence of the required summability of their gradients.

In this paper we present a different kind of result. On one side, we assume more on the maps f_j and f by requiring that they are both homeomorphisms, on the other side, we weaken significantly the integrability assumptions on the gradients by requiring only that $Df_j, Df \in L^1$. Denoting by $\mathrm{Hom}(\Omega, \Omega')$ the set of all homeomorphisms between Ω and Ω', our main result reads as follows.

Theorem 1.2. *Let $f_j, f \in W^{1,1}(\Omega, \mathbb{R}^n) \cap \mathrm{Hom}(\Omega, \Omega')$, with $f_j \rightharpoonup f$ weakly in $W^{1,1}(\Omega, \mathbb{R}^n)$. Assume that*

$$|Df_j(x)|^n \leqslant K_j(x) J_{f_j}(x) \qquad \text{for a.e. } x \in \Omega\,, \tag{1.3}$$

where $K_j : \Omega \to [1, \infty)$ is a Borel function for all j and K_j converges in the biting sense to K. Then f is a map of finite distortion and $K_f(x) \le K(x)$ for a.e. $x \in \Omega$.

Finally, we observe that in Theorem 1.2 the finite distortion assumption (1.3) can be replaced by a similar one involving inner distortion (see Theorem 4.1).

2. Preliminary results

In the sequel it will be convenient to work with a pointwise definition of a gradient of a Sobolev map. To this aim let us consider a function $f \in L^1_{\rm loc}(\Omega, \mathbb{R}^N)$. We say that f is *approximately differentiable* at x if there exists a $N \times n$ matrix, denoted by $Df(x)$, such that

$$\lim_{r \to 0} \fint_{B_r(x)} \frac{|f(y) - f(x) - Df(x)(y - x)|}{r}\, dy = 0\,. \tag{2.1}$$

From (2.1) we have in particular that

$$\lim_{r \to 0} \fint_{B_r(x)} |f(y) - f(x)|\, dy = 0\,,$$

hence x is a Lebesgue point for f. The *approximate gradient* $Df(x)$ is uniquely determined by equality (2.1) and it can be easily checked that the set

$$\mathcal{D}_f = \{x \in \Omega :\ f \text{ is approximately differentiable at } x\}$$

is a Borel set and the map $Df : \mathcal{D}_f \mapsto \mathbb{R}^{nN}$ is a Borel map ([2, Proposition 3.71]).

In the sequel by Df we shall always denote the approximate gradient defined above. Note that if f is differentiable in the classical sense at x, the approximate gradient $Df(x)$ coincides with the usual gradient. Moreover, if $f \in W^{1,1}_{\rm loc}(\Omega, \mathbb{R}^N)$, then f is approximately differentiable almost everywhere in Ω and its approximate differential coincides almost everywhere with the distributional gradient ([2, Proposition 3.83]).

Another feature of the definition (2.1) is its local nature. In fact, if $f, g \in L^1_{\rm loc}(\Omega, \mathbb{R}^N)$, then ([2, Proposition 3.73])

$$Df(x) = Dg(x) \qquad \text{for a.e. } x \in \mathcal{D}_f \cap \mathcal{D}_g \cap \{f = g\}\,. \tag{2.2}$$

Finally, we remark that definition (2.1) of approximate gradient is slightly stronger than the one appearing in [7]. However, for a Sobolev map the two definitions agree, up to a set of measure zero.

Next lemma is a technical result that will be useful in the sequel.

Lemma 2.1. *Let $f : \Omega \mapsto \Omega'$ be a one-to-one map such such that $f \in W^{1,1}(\Omega, \Omega')$ and $f^{-1} \in W^{1,1}(\Omega'; \Omega)$. Set $E = \big\{y \in \mathcal{D}_{f^{-1}} : |J_{f^{-1}}(y)| > 0\big\}$. Then, there exists a Borel set $A \subset E$, with $|E \setminus A| = 0$ such that $f^{-1}(A) \subset \{x \in \mathcal{D}_f : |J_f(x)| > 0\}$, with the property that*

$$Df^{-1}(y) = \big[Df\big(f^{-1}(y)\big)\big]^{-1} \qquad \textit{for all } y \in A\,.$$

Proof. Fix $\varepsilon > 0$. By a well known approximation result there exist a Lipschitz map $h : \mathbb{R}^n \mapsto \mathbb{R}^n$ and a measurable set $F_\varepsilon \subset E$, with $|E \setminus F_\varepsilon| < \varepsilon$, such that $f^{-1}(y) = h(y)$ for all $y \in F_\varepsilon$. As a consequence, recalling (2.2), we have that $Df^{-1}(y) = Dh(y)$ for a.e. $y \in F_\varepsilon$, hence $|J_h(y)| > 0$ for a.e. $y \in F_\varepsilon$.
Thus, by the Lipschitz linearization lemma of Federer ([2, Lemma 2.74] or [7, Lemma 3.2.2]), F_ε can be decomposed, up to a set of zero measure, into the union of countably many, pairwise disjoint, compact sets H_i such that for all i, the map $h_{|H_i}$ is invertible, $\big(h_{|H_i}\big)^{-1}$ is Lipschitz, h is differentiable, $|J_h(y)| > 0$ and $Df^{-1}(y) = Dh(y)$ for all $y \in H_i$. Finally, let us denote by $g_i : \mathbb{R}^n \mapsto \mathbb{R}^n$ a Lipschitz function such $g_i(x) = \big(h_{|H_i}\big)^{-1}(x)$ for all $x \in h(H_i)$. Since $h(g_i(x)) = x$ for all $x \in h(H_i)$ and $g_i(h(y)) = y$ for all $y \in H_i$, using the a.e. differentiability of Lipschitz functions and (2.2) again we easily get that for all i

$$Dh(g_i(x)) = \big[Dg_i(x)\big]^{-1} \qquad \text{for a.e. } x \in h(H_i)\,.$$

Since $g_i(x) = f(x)$ for every $x \in h(H_i)$, from the equality above we deduce that for all i there exists a null Borel set $M_i \subset h(H_i) = f^{-1}(H_i)$ such that f is approximately differentiable at every point $x \in f^{-1}(H_i) \setminus M_i$, and

$$Dh(f(x)) = \big[Df(x)\big]^{-1} \qquad \text{for any } x \in f^{-1}(H_i) \setminus M_i\,,$$

i.e., $Dh(y) = \big[Df\big(f^{-1}(y)\big)\big]^{-1}$ for all $y \in H_i \setminus f(M_i)$. Notice that $f(M_i) = g_i(M_i)$ and thus, since g_i is a Lipschitz map, we may deduce that $f(M_i)$ is a Borel set of zero Lebesgue measure. In conclusion, recalling that $Df^{-1}(y) = Dh(y)$ for all $y \in \cup_i H_i$, we have proved that the approximate gradient $Df(x)$ exists for all $x \in \cup_i(f^{-1}(H_i) \setminus M_i)$ and

$$Df^{-1}(y) = \big[Df\big(f^{-1}(y)\big)\big]^{-1} \qquad \text{for all } y \in \cup_i(H_i \setminus f(M_i))\,,$$

where $\cup_i(H_i \setminus f(M_i))$ is a Borel subset of F_ε of full measure. From this equality, the assertion easily follows. $\square$

Remark 2.2. [Validity of the Area formula] In the sequel we are going to use the *area formula* for maps in $W^{1,1}_{loc}(\Omega, \mathbb{R}^n) \cap \mathrm{Hom}(\Omega, \Omega')$. To this aim, we recall that if f is such a map, and $\mathcal{D}_f$ is the set of points in Ω where f is approximately differentiable, then the area formula holds in $\mathcal{D}_f$, i.e.,

$$\int_{\mathcal{D}_f} \varphi(f(x))|J_f(x)|\,dx = \int_{f(\mathcal{D}_f)} \varphi(y)\,dy \tag{2.3}$$

for every nonnegative Borel function φ in $\mathbb{R}^n$. Equality (2.3) is proved by covering $\mathcal{D}_f$ with a countable family of measurable sets such that the restriction of f to each member of the family is a Lipschitz map ([7, Theorem 3.1.8]) and by applying the usual area formula for Lipschitz maps. In particular, denoting by $\mathcal{J}_f^0 \subset \mathcal{D}_f$, the set of points where J_f is zero, we have that $|f(\mathcal{J}_f^0)| = 0$. This result can be viewed as a *weak version of the classical Sard theorem.*
Notice that, as a consequence of (2.3), we have that for any Borel set $E \subset \Omega$ and any nonnegative Borel function φ in $\mathbb{R}^n$ the following inequality holds

$$\int_E \varphi(f(x))|J_f(x)|\,dx \le \int_{f(E)} \varphi(y)\,dy\,. \tag{2.4}$$

However, if f satisfies the (N) Lusin condition, inequality (2.4) clearly holds as an equality.

Next theorem is a slight variant of a result proved in [11], with the only difference that the (outer) distortion K_f defined in (1.1) is replaced by the inner distortion. To this aim, let us recall that a mapping $f \in W^{1,1}(\Omega, \mathbb{R}^n)$ is said to be of *finite inner distortion* if its Jacobian J_f is strictly positive almost everywhere on the set where $\operatorname{Adj} Df \neq 0$. Here, if A is a $n \times n$ matrix, $\operatorname{Adj} A$ denotes the transpose of the cofactor matrix of A. If f is a map of finite inner distortion, similarly to (1.2), we call *inner distortion* of f the smallest function $K_f^I \ge 1$ such that

$$|\operatorname{Adj} Df(x)|^n \le K_f^I(x) J_f(x)^{n-1} \qquad \text{for a.e. } x \in \Omega\,. \tag{2.5}$$

Notice that in (2.5) and in the rest of the paper, by $|A|$ we denote the operator norm of the $n \times n$ matrix A, i.e., $|A| = \sup\{|A\xi| : \xi \in \mathbb{R}^n,\ |\xi| = 1\}$.

Clearly, a map of finite (outer) distortion is also of finite inner distortion and in dimension $n = 2$ the two notions coincide. In general, as a consequence of the Hadamard inequality $|\operatorname{Adj} A| \le |A|^{n-1}$, we have immediately that if f has finite distortion, then $K_f^I(x) \le (K_f(x))^{n-1}$ for all x and the inequality can be strict if $n \ge 3$.

Theorem 2.3. *Let $f \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap \operatorname{Hom}(\Omega, \Omega')$ be a map such that $|Df| \in L^{n-1,1}(\Omega)$. Let us assume that there exists a Borel function $K : \Omega \to [1, \infty)$ such that*

$$|\operatorname{Adj} Df(x)|^n \le K(x) J_f(x)^{n-1} \qquad \text{a.e. in } \Omega\,. \tag{2.6}$$

Then, f^{-1} is a $W^{1,1}(\Omega'; \mathbb{R}^n)$ map of finite distortion. Moreover,

$$|Df^{-1}(y)|^n \le K(f^{-1}(y)) J_{f^{-1}}(y) \qquad \text{a.e. in } \Omega' \tag{2.7}$$

and

$$\int_{\Omega'} |Df^{-1}(y)|\,dy = \int_\Omega |\operatorname{Adj} Df(x)|\,dx\,. \tag{2.8}$$

Proof. The proof that f^{-1} is a $W^{1,1}(\Omega'; \mathbb{R}^n)$ map of finite distortion goes exactly as the proof of Theorem 1.2 in [11], where the finite distortion assumption on f was used only to derive Lemma 3.2. However, one can easily check that in the proof of this lemma only the weaker assumption (2.6) is actually needed. Thus, we are reduced to prove only (2.7) and (2.8).
Notice that, since f^{-1} is a map of finite distortion, in order to prove (2.7) it is enough to restrict ourselves to the points $y \in A$, where A is the Borel set provided by Lemma 2.1. To this aim, let

us denote by $F \subset \Omega$ a Borel set, with $|F| = 0$ such that (2.6) holds for all $x \in \Omega \setminus F$. Then, for any $y \in A \setminus f(F)$, from Lemma 2.1 and (2.6) we have

$$|Df^{-1}(y)|^n = \frac{\big|\mathrm{Adj}\, Df\big(f^{-1}(y)\big)\big|^n}{J_f(f^{-1}(y))^n} \leq \frac{K(f^{-1}(y))}{J_f(f^{-1}(y))} = K(f^{-1}(y))J_{f^{-1}}(y)\,. \tag{2.9}$$

Then, (2.7) follows, since from area formula (2.3) we get

$$\int_{A\cap f(F)} J_{f^{-1}}(y)\, dy = |f^{-1}(A) \cap F| = 0\,,$$

hence $|A \cap f(F)| = 0$.

Using Lemma 2.1 and recalling that f^{-1} is a map of finite distortion, from (2.9) and the area formula we have

$$\begin{aligned}\int_{\Omega'} |Df^{-1}(y)|\, dy &= \int_A |Df^{-1}(y)|\, dy = \int_A \frac{\big|\mathrm{Adj}\, Df\big(f^{-1}(y)\big)\big|}{J_f(f^{-1}(y))}\, dy \\ &= \int_A \big|\mathrm{Adj}\, Df\big(f^{-1}(y)\big)\big| J_{f^{-1}}(y)\, dy \leq \int_\Omega \big|\mathrm{Adj}\, Df(x)\big|\, dx\,.\end{aligned}$$

To show the opposite inequality, let us apply Lemma 2.1 again, thus getting a Borel set $\widetilde{A} \subset \widetilde{E} = \{x \in \mathcal{D}_f : J_f(x) > 0\}$, such that $|\widetilde{E} \setminus \widetilde{A}| = 0$ and $Df(x) = \big[Df^{-1}(f(x))\big]^{-1}$ for all $x \in \widetilde{A}$. Then, from the assumption (2.6) and the area formula we obtain

$$\int_\Omega \big|\mathrm{Adj}\, Df(x)\big|\, dx = \int_{\widetilde{A}} \big|\mathrm{Adj}\, Df(x)\big|\, dx = \int_{\widetilde{A}} \big|Df^{-1}(f(x))\big| J_f(x)\, dx \leq \int_{\Omega'} |Df^{-1}(y)|\, dy\,,$$

thus proving (2.8). □

3. Weak convergence of the inverse mappings

Let us start with the following

Lemma 3.1. *Let $f_j, f \in \mathrm{Hom}(\Omega, \Omega')$ be such that $f_j \to f$ uniformly in Ω. Then, $f_j^{-1} \to f^{-1}$ locally uniformly in Ω'.*

Proof. Fix a compact subset H of Ω'. We argue by contradiction. If f_j^{-1} does not converge uniformly to f^{-1}, we can find an increasing sequence $\{j_r\}$ and a corresponding sequence of points $y_{j_r} \in H$ such that $y_{j_r} \to y \in H$ and

$$\liminf_{r\to\infty} |f_{j_r}^{-1}(y_{j_r}) - f^{-1}(y_{j_r})| > 0\,. \tag{3.1}$$

On the other hand, by the uniform convergence of f_j to f we have that $f(f_{j_r}^{-1}(y_{j_r})) - y_{j_r} = f(f_{j_r}^{-1}(y_{j_r})) - f_{j_r}(f_{j_r}^{-1}(y_{j_r})) \to 0$ as $r \to \infty$. From this, recalling that $y_{j_r} \to y$, we deduce that $f(f_{j_r}^{-1}(y_{j_r})) \to y$, and in turn, by the continuity of f^{-1}, that

$$\lim_{r\to\infty} \big(f_{j_r}^{-1}(y_{j_r}) - f^{-1}(y_{j_r})\big) = 0$$

which contradicts (3.1). Hence, the result follows. □

Theorem 3.2. *Let $f_j, f \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap \mathrm{Hom}(\Omega, \Omega')$, with $f_j \rightharpoonup f$ weakly in $W^{1,n-1}(\Omega, \mathbb{R}^n)$ and $f_j \to f$ uniformly in Ω. Assume that*

$$\big|\mathrm{Adj}\, Df_j(x)\big|^n \leq K(x) J_{f_j}(x)^{n-1} \qquad \textit{for a.e. } x \in \Omega\,,$$

where $K(x) : \Omega \to [1, \infty)$ is a Borel function. Assume also that $|Df_j| \in L^{n-1,1}(\Omega)$ for all j and that the sequence $\mathrm{Adj}\, Df_j$ is equi-integrable in Ω.

Then the maps f_j^{-1} *converge weakly in* $W^{1,1}(\Omega';\mathbb{R}^n)$ *and locally uniformly in* Ω' *to* f^{-1}*.*

Remark 3.3. Notice that in Theorem 3.2, the additional assumptions

$$|Df_j| \in L^{n-1,1}(\Omega) \quad \text{for all } j, \qquad\qquad \operatorname{Adj} Df_j \quad \text{equi-integrable in } \Omega\,,$$

play a role only when the dimension n is greater than 2. Infact, if $n = 2$, they are a consequence of the identity $L^{1,1} = L^1$ and of the weak convergence in $W^{1,1}(\Omega, \mathbb{R}^2)$ of the maps f_j.

Proof of Theorem 3.2. The local uniform convergence of f_j^{-1} to f^{-1} follows from Lemma 3.1. Moreover from Theorem 2.3, we know that f_j^{-1} are all maps of finite distortion belonging to $W^{1,1}(\Omega';\mathbb{R}^n)$.
Therefore, we need only to show that the sequence Df_j^{-1} is equi-integrable. To this aim, let us set, for $h \in \mathbb{N}$, $F_h = \{x \in \Omega : K(x) > h\}$. For any Borel set $E \subset \Omega'$, we have

$$\int_E |Df_j^{-1}(y)|\,dy = \int_{E\setminus f_j(F_h)} |Df_j^{-1}(y)|\,dy + \int_{E\cap f_j(F_h)} |Df_j^{-1}(y)|\,dy = I_1 + I_2\,. \tag{3.2}$$

If $y \in E \setminus f_j(F_h)$, then $K\big(f_j^{-1}(y)\big) \le h$, hence, by applying (2.7) to each f_j and using Hölder inequality and inequality (2.4),

$$I_1 \le h^{1/n} \int_E J_{f_j^{-1}}(y)^{1/n}\,dy \le (h|\Omega|)^{1/n}|E|^{(n-1)/n}\,. \tag{3.3}$$

To estimate I_2, define for $j, h \in \mathbb{N}$

$$E_{jh} = E \cap f_j(F_h) \cap A_j$$

where, for all j, A_j is the set relative to f_j^{-1} provided by Lemma 2.1. Recalling that by Theorem 2.3 f_j^{-1} is a map of finite distortion, from coarea formula (2.3) and Lemma 2.1, we get

$$\begin{aligned} I_2 &= \int_{E_{jh}} \frac{|Df_j^{-1}(y)|}{J_{f_j^{-1}}(y)} J_{f_j^{-1}}(y)\,dy = \int_{f_j^{-1}(E_{jh})} \frac{|Df_j^{-1}(f_j(x))|}{J_{f_j^{-1}}(f_j(x))}\,dx \\ &= \int_{f_j^{-1}(E_{jh})} \big|\operatorname{Adj} Df_j(x)\big|\,dx \le \int_{F_h} \big|\operatorname{Adj} Df_j(x)\big|\,dx\,. \end{aligned}$$

From this inequality, (3.2) and (3.3), we conclude that for any measurable set $E \subset \Omega'$ and for any $j, h \in \mathbb{N}$

$$\int_E |Df_j^{-1}(y)|\,dy \le \int_{F_h} \big|\operatorname{Adj} Df_j(x)\big|\,dx + (h|\Omega|)^{1/n}|E|^{(n-1)/n}\,.$$

Recalling that $K(x) < \infty$ a.e. in Ω, we have that $|F_h| \to 0$ as $h \to \infty$. Therefore, from the equiintegrability of the sequence $\operatorname{Adj} Df_j$ we deduce that, given any $\varepsilon > 0$, there exists h_ε such that

$$\sup_{j\in\mathbb{N}} \int_{F_{h_\varepsilon}} \big|\operatorname{Adj} Df_j(x)\big|\,dx < \varepsilon\,.$$

Therefore, if $|E| < \dfrac{\varepsilon^{n/(n-1)}}{(h_\varepsilon|\Omega|)^{1/(n-1)}}$, we get that for all j

$$\int_E |Df_j^{-1}(y)|\,dy < 2\varepsilon\,,$$

thus proving the equi-integrability of the sequence Df_j^{-1}.

□

4. Lower semicontinuity of the distortion

In this section we establish the lower semicontinuity of the distortions of a sequence of homeomorphisms converging weakly in $W^{1,1}$ (see Corollary 4.2 below). This property is an immediate consequence of Theorem 1.2 whose proof is also given here.

To this aim, let us recall that a sequence of measurable functions $h_j : \Omega \to \mathbb{R}$ is said to converge in the *biting sense* in Ω to a measurable function $h : \Omega \to \mathbb{R}$ if there exists an increasing sequence of measurable sets $E_k \subset \Omega$, with $\cup_k E_k = \Omega$, such that $h_j, h \in L^1(E_k)$ for all j, k and $h_j \rightharpoonup h$ weakly in $L^1(E_k)$ for all k.

An important feature of this convergence is the property that if h_j is a sequence bounded in $L^1(\Omega)$, then there exists a subsequence h_{j_r} converging in the biting sense in Ω (see [4] or [1, Lemma 1.6]).

Proof of Theorem 1.2. From the weak convergence of f_j to f in $W^{1,1}(\Omega, \mathbb{R}^n)$ it follows that $f_j(x) \to f(x)$ a.e. in Ω. For any $\sigma > 0$ denote by $\Omega_\sigma \subset \Omega$ a measurable set such that $f_j \to f$ uniformly in Ω_σ, $K_j \rightharpoonup K$ weakly in $L^1(\Omega_\sigma)$ and $|\Omega \setminus \Omega_\sigma| < \sigma$.
For all $M > 1$ we set

$$L_M = \big\{x \in \mathcal{D}_f : \, K(x) + |Df(x)| \le M\big\} \setminus f^{-1}\big(\mathcal{J}^0_{f^{-1}}\big)\,,$$

where $\mathcal{D}_f$ is the set of points where f is approximately differentiable and $\mathcal{J}^0_{f^{-1}}$ is the set of points in $\mathcal{D}_{f^{-1}}$, where $J_{f^{-1}} = 0$. We are going to show that

$$\int_H |Df(x)|^n \, dx \le \int_H K(x) J_f(x)\, dx \qquad \text{for all compact sets } H \subset L_M \cap \Omega_\sigma\,. \tag{4.1}$$

In fact, once this inequality is proved, since $\mathcal{D}_f$ has full measure in Ω and by the weak Sard theorem $|f^{-1}\big(\mathcal{J}^0_{f^{-1}}\big)| = 0$, from the arbitrariety of H, M and σ we easily conclude that $|Df(x)|^n \le K(x)J_f(x)$ for a.e. $x \in \Omega$.
So, let us fix a compact subset H of $L_M \cap \Omega_\sigma$. Given a nonnegative function $\varphi \in C_0(\Omega)$, from the assumption (1.3) and the weak convergence of f_j to f in $W^{1,1}(\Omega, \mathbb{R}^n)$, we immediately get

$$\int_H |Df(x)|\varphi(x)\, dx \le \liminf_{j\to\infty} \int_H |Df_j(x)|\varphi(x)\, dx \le \liminf_{j\to\infty} \int_H \big(K_{f_j}(x) J_{f_j}(x)\big)^{1/n}\varphi(x)\, dx\,. \tag{4.2}$$

Let us now denote by ψ a bounded, strictly positive, continuous function in Ω. By applying Hölder inequality (once if $n = 2$ and twice if $n \ge 3$) and inequality (2.4) for f_j, we get

$$\begin{aligned} \int_H \big(K_j J_{f_j}\big)^{1/n}\varphi\, dx &\le \left(\int_H \big(K_j\psi\big)^{\frac{1}{n-1}} \varphi^{\frac{n(n-2)}{(n-1)^2}}\, dx\right)^{\frac{n-1}{n}} \left(\int_H \frac{\varphi^{\frac{n}{n-1}}(x) J_{f_j}(x)}{\psi(x)}\, dx\right)^{\frac{1}{n}} \\ &\le \left(\int_H K_j(x)\psi(x)\, dx\right)^{\frac{1}{n}} \left(\int_H \varphi^{\frac{n}{n-1}}(x)\, dx\right)^{\frac{n-2}{n}} \left(\int_H \frac{\varphi^{\frac{n}{n-1}}(x) J_{f_j}(x)}{\psi(x)}\, dx\right)^{\frac{1}{n}} \\ &\le \left(\int_H K_j(x)\psi(x)\, dx\right)^{\frac{1}{n}} \left(\int_H \varphi^{\frac{n}{n-1}}(x)\, dx\right)^{\frac{n-2}{n}} \left(\int_{\Omega'} \frac{\varphi^{\frac{n}{n-1}}(f_j^{-1}(y))}{\psi(f_j^{-1}(y))}\chi_H(f_j^{-1}(y))\, dy\right)^{\frac{1}{n}}. \end{aligned} \tag{4.3}$$

Fix $y \in \Omega'$. We claim that if there exists a subsequence f_{j_r} of f_j such that $f_{j_r}^{-1}(y) \in H$ for every r, then $f_{j_r}^{-1}(y) \to f^{-1}(y)$.
To this aim, notice that since H is compact and $f_{j_r}^{-1}(y) \in H$ for every r, then, up to a subsequence, $f_{j_r}^{-1}(y) \to x_0 \in H$. Thus, by the continuity of f, we have that $f(f_{j_r}^{-1}(y)) \to f(x_0)$.

Moreover, from the uniform convergence of f_j to f in H we have also

$$\lim_{j\to\infty}|f(f_{j_r}^{-1}(y))-y|=\lim_{j\to\infty}|f(f_{j_r}^{-1}(y))-f_{j_r}(f_{j_r}^{-1}(y))|=0$$

and so $f(x_0)=y$, i.e. $x_0=f^{-1}(y)\in H$, thus proving the claim.
As a consequence of the claim we have just proved, we have in particular that

$$\limsup_{j\to\infty}\frac{\varphi^{\frac{n}{n-1}}(f_j^{-1}(y))}{\psi(f_j^{-1}(y))}\chi_H(f_j^{-1}(y))\le\frac{\varphi^{\frac{n}{n-1}}(f^{-1}(y))}{\psi(f^{-1}(y))}\chi_H(f^{-1}(y))\qquad\text{for all } y\in\Omega'.$$

Thus, combining (4.2) and (4.3), and passing to the limit as $j\to\infty$, by Fatou Lemma and the weak convergence of K_j in H, we get

$$\begin{aligned}&\int_H|Df(x)|\varphi(x)\,dx\\ &\le\limsup_{j\to\infty}\left(\int_H K_j\psi\,dx\right)^{\frac1n}\left(\int_H\varphi^{\frac{n}{n-1}}(x)\,dx\right)^{\frac{n-2}{n}}\left(\int_{\Omega'}\frac{\varphi^{\frac{n}{n-1}}(f_j^{-1}(y))}{\psi(f_j^{-1}(y))}\chi_H(f_j^{-1}(y))\,dy\right)^{\frac1n}\\ &\le\left(\int_H K\psi\,dx\right)^{\frac1n}\left(\int_H\varphi^{\frac{n}{n-1}}(x)\,dx\right)^{\frac{n-2}{n}}\left(\int_{f(H)}\frac{\varphi^{\frac{n}{n-1}}(f^{-1}(y))}{\psi(f^{-1}(y))}\,dy\right)^{\frac1n}.\end{aligned}\tag{4.4}$$

Now, let us fix $m\in\mathbb{N}$ and set $E_m=\{x\in\mathcal{D}_f:\,1/m\le J_f(x)\le m\}$. Given $\varepsilon>0$, we denote by ψ_h a sequence of continuous, equibounded functions such that $\psi_h(x)\ge\varepsilon$ for all $x\in\Omega$ such that

$$\psi_h(x)\to J_f(x)\chi_{E_m}(x)+\varepsilon\qquad\text{for a.e. } x\in\Omega.$$

Recall that $H\subset\mathcal{D}_f$ and that , by (2.3), $f_{|\mathcal{D}_f}$ satisfies the Lusin (N) property. Thus,

$$\psi_h(f^{-1}(y))\to J_f(f^{-1}(y))\chi_{E_m}(f^{-1}(y))+\varepsilon\qquad\text{for a.e. } y\in f(H).$$

Thus, inserting ψ_h in place of ψ in (4.4) and passing to the limit, first as $h\to\infty$ and then as $m\to\infty$, we get

$$\int_H|Df|\varphi dx\le\left(\int_H K(J_f(x)+\varepsilon)dx\right)^{\frac1n}\left(\int_H\varphi^{\frac{n}{n-1}}dx\right)^{\frac{n-2}{n}}\left(\int_{f(H)}\frac{\varphi^{\frac{n}{n-1}}(f^{-1}(y))}{J_f(f^{-1}(y))\chi_E(f^{-1}(y))+\varepsilon}dy\right)^{\frac1n},$$

where $E=\{x\in\mathcal{D}_f:\,J_f(x)>0\}$.
Recalling that $|f(\mathcal{J}_f^0)|=0$, we have that $|f(H\setminus E)|=0$, hence $\chi_E(f^{-1}(y))=1$ for a.e. $y\in f(H)$. Thus, letting $\varepsilon\to0$ in the inequality above, we get

$$\int_H|Df(x)|\varphi(x)dx\le\left(\int_H K(x)J_f(x)dx\right)^{\frac1n}\left(\int_H\varphi^{\frac{n}{n-1}}(x)dx\right)^{\frac{n-2}{n}}\left(\int_{f(H\cap E)}\frac{\varphi^{\frac{n}{n-1}}(f^{-1}(y))}{J_f(f^{-1}(y))}dy\right)^{\frac1n}.$$

By the definition of L_M, it follows that $f(H\cap E)\cap\mathcal{J}_{f^{-1}}^0=\emptyset$. Therefore, since $\Omega'\setminus\mathcal{D}_{f^{-1}}$ is a null set, from Lemma 2.1 we have that $J_{f^{-1}}(y)=1/J_f(f^{-1}(y))$ for a.e. $y\in f(H\cap E)$ and thus, using (2.4), we get

$$\int_H |Df|\varphi\,dx \le \left(\int_H K(x)J_f(x)\,dx\right)^{\frac{1}{n}} \left(\int_H \varphi^{\frac{n}{n-1}}(x)\,dx\right)^{\frac{n-2}{n}} \left(\int_{f(H\cap E)} \varphi^{\frac{n}{n-1}}(f^{-1}(y))J_{f^{-1}}(y)\,dy\right)^{\frac{1}{n}}$$
$$\le \left(\int_H K(x)J_f(x)\,dx\right)^{\frac{1}{n}} \left(\int_H \varphi^{\frac{n}{n-1}}(x)\,dx\right)^{\frac{n-2}{n}} \left(\int_{H\cap E} \varphi^{\frac{n}{n-1}}(x)\,dx\right)^{\frac{1}{n}}$$
$$\le \left(\int_H K(x)J_f(x)\,dx\right)^{\frac{1}{n}} \left(\int_H \varphi^{\frac{n}{n-1}}(x)\,dx\right)^{\frac{n-1}{n}} .$$

Finally, let us replace φ in this inequality by φ_h, where $\varphi_h \in C_0(\Omega)$, $0 \le \varphi_h(x) \le M^{n-1}$ for all $h \in \mathbb{N}$ and any $x \in \Omega$ and

$$\varphi_h(x) \to |Df(x)|^{n-1} \qquad \text{for a.e. } x \in L_M .$$

Then, letting $h \to \infty$, we get

$$\int_H |Df(x)|^n\,dx \le \left(\int_H K(x)J_f(x)\,dx\right)^{\frac{1}{n}} \left(\int_H |Df(x)|^n\,dx\right)^{\frac{n-1}{n}} , \tag{4.5}$$

hence (4.1) follows. This concludes the proof. □

A slightly different result is obtained with a simple variant of the argument used in the proof of Theorem 1.2.

Theorem 4.1. *Let $f_j, f \in W^{1,n-1}(\Omega,\mathbb{R}^n) \cap \mathrm{Hom}(\Omega,\Omega')$ with $f_j \rightharpoonup f$ weakly in $W^{1,n-1}(\Omega,\mathbb{R}^n)$. Assume that*

$$\big|\mathrm{Adj}\,Df_j(x)\big|^n \le K_j(x)J_{f_j}^{n-1} \qquad \text{for a.e. } x \in \Omega , \tag{4.6}$$

where $K_j : \Omega \to [1,\infty)$ is a Borel function and K_j converges to K in the biting sense. Then f has finite inner distortion $K_f^I \le K(x)$ for a.e. $x \in \Omega$.

Proof. Assume $n \ge 3$, since for $n = 2$ the assertion reduces to Theorem 1.2.
As in the proof of Theorem 1.2, we start by observing that $f_j(x) \to f(x)$ a.e. in Ω and that for any $\sigma > 0$ there exists a measurable set $\Omega_\sigma \subset \Omega$ such that $f_j \to f$ uniformly in Ω_σ, $K_j \rightharpoonup K$ in $L^1(\Omega_\sigma)$, with $|\Omega \setminus \Omega_\sigma| < \sigma$.
For $M > 1$ we set

$$L_M \;=\; \{x \in \mathcal{D}_f :\; K(x) + \big|\mathrm{Adj}\,Df_j(x)\big| \le M\} \setminus f^{-1}\big(\mathcal{J}^0_{f^{-1}}\big) .$$

Our aim is to show that for every compact set $H \subset L_M \cap \Omega_\sigma$ we have

$$\int_H \big|\mathrm{Adj}\,Df_j(x)\big|^n\,dx \le \int_H K(x)J_f(x)^{n-1}\,dx . \tag{4.7}$$

Indeed, as before, establishing this inequality will conclude the proof.
Thus, let us fix a compact set H and a nonnegative function $\varphi \in C_0(\Omega)$. Setting for all $(x, A) \in \Omega \times \mathbb{R}^{n^2}$

$$F(x, A) = \chi_H(x)\varphi(x)\big|\mathrm{Adj}\,A\big| ,$$

F turns out to be a polyconvex integrand with growth $(n-1)$. Therefore, using the lower semicontinuity theorem by Acerbi-Fusco ([1]), from (4.6) we have

$$\int_H \big|\mathrm{Adj}\,Df(x)\big|\varphi(x)dx \le \liminf_{j\to\infty} \int_H \big|\mathrm{Adj}\,Df_j(x)\big|\varphi(x)dx \le \liminf_{j\to\infty} \int_H \big(K_j(x)J_{f_j}(x)^{n-1}\big)^{\frac{1}{n}}\varphi(x)dx .$$

Let us denote by ψ a strictly positive and bounded continuous function in Ω. By Hölder's inequality and (2.4) we get

$$\int_H \big(K_j(x)J_{f_j}(x)^{n-1}\big)^{\frac{1}{n}}\varphi(x)\,dx \le \left(\int_H K_j(x)\psi^{n-1}(x)\,dx\right)^{\frac{1}{n}}\left(\int_H \frac{J_{f_j}(x)\varphi^{\frac{n}{n-1}}(x)}{\psi(x)}\,dx\right)^{\frac{n-1}{n}}$$

$$\le \left(\int_H K_j(x)\psi^{n-1}(x)\,dx\right)^{\frac{1}{n}}\left(\int_{\Omega'} \frac{\varphi^{\frac{n}{n-1}}(f_j^{-1}(y))}{\psi(f_j^{-1}(y))}\chi_H(f_j^{-1}(y))\,dy\right)^{\frac{n-1}{n}}$$

and then, arguing exactly as in the proof of Theorem 1.2, we deduce first the inequality

$$\int_H \big|\mathrm{Adj}\,Df(x)\big|\varphi(x)\,dx \le \left(\int_H K(x)\psi^{n-1}(x)\,dx\right)^{\frac{1}{n}}\left(\int_{f(H)} \frac{\varphi^{\frac{n}{n-1}}(f^{-1}(y))}{\psi(f^{-1}(y))}\,dy\right)^{\frac{n-1}{n}}$$

and then

$$\int_H \big|\mathrm{Adj}\,Df(x)\big|\varphi(x)\,dx \le \left(\int_H K(x)J_f(x)^{n-1}\,dx\right)^{\frac{1}{n}}\left(\int_H \varphi^{\frac{n}{n-1}}(x)\,dx\right)^{\frac{n-1}{n}}.$$

Finally, replace in this inequality φ by $\varphi_h \in C_0(\Omega)$, $0 \le \varphi_h \le M^{n-1}$, such that

$$\varphi_h(x) \to \big|\mathrm{Adj}\,Df(x)\big|^{n-1} \qquad \text{for a.e. } x \in L_M$$

and let $h \to \infty$ to obtain

$$\int_H \big|\mathrm{Adj}\,Df(x)\big|^n\,dx \le \left(\int_H K(x)J_f^{n-1}\,dx\right)^{\frac{1}{n}}\left(\int_H \big|\mathrm{Adj}\,Df(x)\big|^n\,dx\right)^{\frac{n-1}{n}}.$$

From this inequality (4.7) follows, thus concluding the proof. □

Corollary 4.2. *Let $f_j, f : \Omega \to \Omega'$ be maps satisfying the assumptions of Theorem 1.2. Then f is a map with finite distortion and*

$$\int_\Omega K_f(x)\,dx \le \liminf_{j\to\infty} \int_\Omega K_{f_j}(x)\,dx\,. \tag{4.8}$$

Proof. In order to prove (4.8) we may assume without loss of generality that the lim inf on the right hand side is a limit and that is finite. If this is the case, there exists a subsequence $K_{f_{j_r}}$ converging in the biting sense to a measurable function $\widetilde{K}$. Thus, from Theorem 1.2, we have

$$\int_\Omega K_f(x)\,dx \le \int_\Omega \widetilde{K}(x)\,dx \le \lim_{r\to\infty}\int_\Omega K_{f_{j_r}}(x)\,dx$$

and the assertion follows. □

A similar result clearly holds for the inner distortions if in Corollary 4.2 we assume that the maps f_j, f satisfy the asssumptions of Theorem 4.1.

Example 4.3. Let $\varphi : \mathbb{R} \to [c, +\infty)$, $c > 0$, be a 1-periodic function, strictly increasing in $(0,1)$ and such that $\displaystyle\int_0^1 \varphi dt = 1$, but $\varphi \notin L^p((0,1))$ for all $p > 1$. Set, for all $j \in \mathbb{N}$, $(x,y) \in \Omega = (0,1)\times(0,1)$,

$$f_j(x,y) = \left(\int_0^x \varphi(jt)dt, \int_0^y \varphi(jt)dt\right).$$

Then, f_j is a sequence of homeomorphisms from Ω onto Ω weakly converging to the identity map f in $W^{1,1}(\Omega,\Omega)$. All maps f_j are of finite distortion and for a.e. $(x,y)\in\Omega$

$$K_{f_j}(x,y)=\frac{\max\{\varphi^2(jx),\varphi^2(jy)\}}{\varphi(jx)\varphi(jy)}\,.$$

Thus, the functions K_{f_j} converge weakly in $L^1(\Omega)$ to the constant function

$$K\equiv\int_0^1\int_0^1\frac{\max\{\varphi^2(s),\varphi^2(t)\}}{\varphi(s)\varphi(t)}\,dsdt\,.$$

Recalling that φ is strictly increasing in $(0,1)$, we easily get that

$$K\equiv 2\int_0^1\varphi(s)\,ds\int_0^s\frac{1}{\varphi(t)}\,dt>2\int_0^1\varphi(s)\frac{s}{\varphi(s)}\,ds=1\equiv K_f\,,$$

thus showing that the inequality $K_f\le K$ provided by Theorem 1.2 can be everywhere strict even in very simple situations. Notice also that since $f_j\notin W^{1,2}$ for all j, Theorem 1.1 does not apply to this example.

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