Surface measures and convergence of the Ornstein-Uhlenbeck semigroup in Wiener spaces

Luigi Ambrosio*, Alessio Figalli†

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Abstract

We study points of density 1/2 of sets of finite perimeter in infinite-dimensional Gaussian spaces and prove that, as in the finite-dimensional theory, the surface measure is concentrated on this class of points. Here density 1/2 is formulated in terms of the pointwise behaviour of the Ornstein-Uhlembeck semigroup.

Dans cet article nous étudions la structure de l'ensemble des points avec densité 1/2 pour les ensemble de périmètre fini dans un espace gaussien infini-dimensionnel. Nous démontrons que, comme dans le cas de dimension finie, la mesure de surface est concentrée sur cet ensemble de points. Ici, la définition de points avec densité 1/2 est donnée en utilisant le comportement ponctuel du semigroupe de Ornstein-Uhlembeck.

1 Introduction

The theory of sets of finite perimeter and BV functions in Wiener spaces, i.e., Banach spaces endowed with a Gaussian Borel probability measure γ, has been initiated by Fukushima and Hino in [14, 15, 16]. More recently, some basic questions of the theory have been investigated in [17] and in [3, 5] (see also [4] for a slightly different framework). One motivation for this theory is the development of Gauss-Green formulas in infinite-dimensional domains; as in the finite-dimensional theory, it turns out that for nonsmooth domains the surface measure might be supported in a set much smaller than the topological boundary (see also the precise analysis made in [22], in a particular class of infinite-dimensional domains).

*Scuola Normale Superiore, p.za dei Cavalieri 7, I-56126 Pisa, Italy, e–mail: l.ambrosio@sns.it
†The University of Texas at Austin, Department of Mathematics, 1 University Station C1200, Austin TX 78756, USA, e–mail: figalli@math.utexas.edu

The basic question we would like to consider is the research of infinite-dimensional analogues of the classical fine properties of BV functions and sets of finite perimeter in finite-dimensional (Gaussian) spaces.

For this reason we start first with a discussion of the finite-dimensional theory, referring to [11] and [2] for much more on this subject. Recall that a Borel set $E \subset \mathbb{R}^m$ is said to be of *finite perimeter* if there exists a vector valued measure $D\chi_E = (D_1\chi_E, \dots, D_m\chi_E)$ with finite total variation in $\mathbb{R}^m$ satisfying the integration by parts formula:

$$\int_E \frac{\partial \phi}{\partial x_i}\, dx = -\int_{\mathbb{R}^m} \phi \, dD_i\chi_E \qquad \forall i = 1, \dots, m, \ \forall \phi \in C^1_c(\mathbb{R}^m). \tag{1}$$

De Giorgi proved in [9] a deep result on the structure of $D\chi_E$. First of all he identified a set $\mathcal{F}E$, called by him *reduced boundary*, on which $|D\chi_E|$ is concentrated, and defined a pointwise inner normal $\nu_E(x) = (\nu_{E,1}(x), \dots, \nu_{E,m}(x))$ (see (49)); then, through a suitable blow-up procedure, he proved that $\mathcal{F}E$ is countably rectifiable (more precisely, it is contained in the union of countably many graphs of Lipschitz functions defined on hyperplanes of $\mathbb{R}^m$); finally, he proved the representation formula $D\chi_E = \nu_E \mathscr{S}^{m-1} \llcorner \mathcal{F}E$, where $\mathscr{S}^{m-1}$ is the $(m-1)$-dimensional spherical Hausdorff measure in $\mathbb{R}^m$. In light of these results, the integration by parts formula reads

$$\int_E \frac{\partial \phi}{\partial x_i}\, dx = -\int_{\mathcal{F}E} \phi \nu_{E,i} \, d\mathscr{S}^{m-1} \qquad \forall i = 1, \dots, m, \ \forall \phi \in C^1_c(\mathbb{R}^m).$$

A few years later, Federer proved in [10] that the same representation result of $D\chi_E$ holds for another concept of boundary, called *essential boundary*:

$$\partial^* E := \left\{ x \in \mathbb{R}^m : \ \limsup_{r \downarrow 0} \frac{\mathscr{L}^m(B_r(x) \cap E)}{\mathscr{L}^m(B_r(x))} > 0, \qquad \limsup_{r \downarrow 0} \frac{\mathscr{L}^m(B_r(x) \setminus E)}{\mathscr{L}^m(B_r(x))} > 0 \right\},$$

where $\mathscr{L}^m$ is the m–dimensional Lebesgue measure (this corresponds to points neither of density 0, nor of density 1). Indeed, a consequence of the De Giorgi's blow-up procedure is that $\mathcal{F}E \subset \partial^* E$ (because tangent sets to E at all points in the reduced boundary are halfspaces, whose density at the origin is 1/2), and in [10] it is shown that $\mathscr{S}^{m-1}(\partial^* E \setminus \mathcal{F}E) = 0$. Since the set $E^{1/2}$ of points of density 1/2

$$E^{1/2} := \left\{ x \in \mathbb{R}^m : \ \lim_{r \downarrow 0} \frac{\mathscr{L}^m(B_r(x) \cap E)}{\mathscr{L}^m(B_r(x))} = \frac{1}{2} \right\},$$

is in between the two, one can also use it as a good definition of boundary.

When looking for the counterpart of De Giorgi's and Federer's results in infinite-dimensional spaces, one can consider a suitable notion of "distributional derivative" along Cameron-Martin directions $D_\gamma \chi_E$ and surface measure $|D_\gamma \chi_E|$. But, several difficulties arise:

(i) The classical concept of Lebesgue approximate continuity, underlying also the definition of essential boundary, seems to fail or seems to be not reproducible in Gaussian spaces (X,γ). For instance, in [20] it is shown that in general the balls of X cannot be used, and in any case the norm of X is not natural from the point of view of the calculus in Wiener spaces, where no intrinsic metric structure exists and the "differentiable" structure is induced by H.

(ii) Suitable notions of codimension-1 Hausdorff measure, of rectifiability and of essential/reduced boundary have to be devised.

Nevertheless, some relevant progresses have been obtained by Feyel-De la Pradelle in [12], by Hino in [17] and, on the rectifiability issue, by the first author, Miranda and Pallara in [5]. In [12] a family of spherical Hausdorff pre-measures $\mathscr{S}^{\infty-1}_F$ has been introduced by looking at the factorization $X = \mathrm{Ker}(\Pi_F)\otimes F$, with F m-dimensional subspace of H, considering the measures $\mathscr{S}^{m-1}$ on the m-dimensional fibers of the decomposition. A crucial monotonicity property of these pre-measures with respect to F allows to define $\mathscr{S}^{\infty-1}_{FDP}$ (here, FDP stands for Feyel-De la Pradelle) as $\lim_F \mathscr{S}^{\infty-1}_F$, the limit being taken in the sense of directed sets. This Hausdorff measure, when restricted to the boundary of a "nice" set (in the sense of Malliavin calculus) is then shown to be consistent with the surface measure defined in [1]. In [17] this approach has been used to build a Borel set $\partial^*_{\mathcal{F}}E$, called cylindrical essential boundary, for which the representation formula

$$|D_\gamma\chi_E| = \mathscr{S}^{\infty-1}_{\mathcal{F}}\llcorner\partial^*_{\mathcal{F}}E \tag{2}$$

holds. Here $\mathcal{F}=\{F_n\}_{n\geq 1}$ is an nondecreasing family of finite-dimensional subspaces of $\tilde H$ (see (8) for the definition of $\tilde H$) whose union is dense in H and $\mathscr{S}^{\infty-1}_{\mathcal{F}} = \lim_n \mathscr{S}^{\infty-1}_{F_n}$. Notice that, while the left hand side in the representation formula is independent of the choice of $\mathcal{F}$, both the cylindrical essential boundary and $\mathscr{S}^{\infty-1}_{\mathcal{F}}$ a priori depend on $\mathcal{F}$ (see Remark 2.6 for a more detailed discussion). The problem of getting a representation formula in terms of a coordinate-free measure $\mathscr{S}^{\infty-1}$ is strongly related to the problem of finding coordinate-free definitions of reduced/essential boundary.

In this paper, answering in part to questions raised in [17] and in [5], we propose an infinite-dimensional counterpart of $E^{1/2}$ and use it to provide a coordinate-free version of (2).

In view of the quite general convergence results illustrated in [21] it is natural, in this context, to think of the Ornstein-Uhlenbeck semigroup $T_t\chi_E$ starting from χ_E, for small t, as an analog of the mean value of χ_E on small "balls". Also, it is already known starting from [8] (see also [15, 16, 3, 19]) that surfaces measures are intimately connected to the behavior of $T_t\chi_E$ for small t. Our first main result provides strong convergence of $T_t\chi_E$ as $t\downarrow 0$, if we take the surface measure as reference measure:

Theorem 1.1. *Let E be a Borel set of finite perimeter in (X,γ). Then*

$$\lim_{t\downarrow 0}\int_X |T_t\chi_E-\frac{1}{2}|^2\,d|D_\gamma\chi_E|=0.$$

Since $|D_\gamma\chi_E|$ is orthogonal w.r.t. γ, it is crucial for the validity of the result that $T_t\chi_E$ is not understood in a functional way (i.e., as an element of $L^\infty(X,\gamma)$), but really in a pointwise way through Mehler's formula (10). In this respect, the choice of a Borel representative is important, see also Proposition 2.2 and (14).

The proof of Theorem 1.1 is based on two results: first, by a soft argument based on the product rule for weak derivatives, we show the weak* convergence of $T_t\chi_E$ to $1/2$ in $L^\infty(X,|D_\gamma\chi_E|)$. Then, by a quite delicate finite-dimensional approximation and factorization of the OU semigroup, we show the apriori estimate

$$\limsup_{t\downarrow 0}\int_X |T_t\chi_E|^2\,d|D_\gamma\chi_E|^2\le\frac{1}{4}|D_\gamma\chi_E|(X).$$

Notice that in finite dimensions Theorem 1.1 is easy to show, using the fact that sets of finite perimeter are, for $|D_\gamma\chi_E|$-a.e. x, close to halfspaces on small balls centered at x (see the proof of Proposition 3.1 and also Remark 4.2).

Thanks to Theorem 1.1, we can choose an infinitesimal sequence $(t_i)\downarrow 0$ such that

$$\sum_i\int_X |T_{t_i}\chi_E-\frac{1}{2}|\,d|D_\gamma\chi_E|<\infty, \tag{3}$$

This choice of (t_i) ensures in particular the convergence of $T_{t_i}\chi_E$ to $1/2$ $|D_\gamma\chi_E|$-a.e. in X, and motivates the next definition:

Definition 1.2 (Points of density 1/2)**.** *Let $(t_i)\downarrow 0$ be such that $\sum_i\sqrt{t_i}<\infty$ and* (3) *holds. We denote by $E^{1/2}$ the set*

$$E^{1/2}:=\left\{x\in X:\ \lim_{i\to\infty}T_{t_i}\chi_E(x)=\frac{1}{2}\right\}. \tag{4}$$

Notice that $|D_\gamma\chi_E|$ is concentrated on $E^{1/2}$. With this definition, and defining $\mathscr{S}^{\infty-1}$ as the supremum of $\mathscr{S}^{\infty-1}_F$ among all finite-dimensional subspaces of $\tilde H$, we can prove our second main result:

Theorem 1.3. *Let $(t_i)\downarrow 0$ be such that $\sum_i\sqrt{t_i}<\infty$ and* (3) *holds. Then the set $E^{1/2}$ defined in* (4) *has finite $\mathscr{S}^{\infty-1}$-measure and*

$$|D_\gamma\chi_E|=\mathscr{S}^{\infty-1}\llcorner E^{1/2}. \tag{5}$$

As we said, an advantage of (5) is its coordinate-free character, see also Remark 2.6 for a more detailed comparison with Hino's cylindrical definition of essential boundary. A drawback is its dependence on (t_i); however, this dependence enters only in the definition of $E^{1/2}$, and not in the one of $\mathscr{S}^{\infty-1}$. Moreover, it readily follows from Theorem 1.3 that $E^{1/2}$ is uniquely determined up to $\mathscr{S}^{\infty-1}$-negligible sets (i.e., different sequences produce equivalent sets). We consider merely as a (quite) technical issue the replacement of $\mathscr{S}^{\infty-1}$ with the larger measure $\mathscr{S}^{\infty-1}_{FDP}$ (defined considering *all* finite-dimensional subspaces of H) in (5), for the reasons explained in Remark 2.4.

As an example of application of the structure result for $|D_\gamma \chi_E|$ provided by (5), we can provide a precise formula for the distributional derivative of the union of two disjoints sets of finite perimeter. Given a set E of finite perimeter, write $D_\gamma \chi_E = \nu_E |D_\gamma \chi_E|$, with $\nu_E : X \to H$ a Borel vectorfield satisfying $|\nu_E|_H = 1$ $|D_\gamma \chi_E|$-a.e. in X. With this notation we have:

Corollary 1.4. *Let E and F be sets of finite perimeter with $\gamma(E \cap F) = 0$. Then $E \cup F$ has finite perimeter,*

$$\nu_{E\cup F} \mathscr{S}^{\infty-1} \llcorner (E \cup F)^{1/2} = \nu_E \mathscr{S}^{\infty-1} \llcorner (E^{1/2} \setminus F^{1/2}) + \nu_F \mathscr{S}^{\infty-1} \llcorner (F^{1/2} \setminus E^{1/2}), \quad (6)$$

and $\nu_E(x) = -\nu_F(x)$ at $\mathscr{S}^{\infty-1}$-a.e. $x \in E^{1/2} \cap F^{1/2}$.

An important feature in the above result is that, since $(E \cup F)^{1/2}$, $E^{1/2}$, and $F^{1/2}$ are uniquely determined up to $\mathscr{S}^{\infty-1}$-negligible sets, one does not have to specify which sequences (t_i) one uses to define the sets (and the sequences could all be different). On the other hand, if one would try to deduce the analogous result stated in terms of cylindrical boundaries, it seems to us that one would be obliged to choose the same family $\mathcal{F} = \{F_n\}_{n\geq 1}$ for all the three sets (see Remark 2.6).

Let us conclude this introduction pointing out that our results can be considered as the analogous of Federer's result to an infinite dimensional setting. In [5, Section 7], the authors gave a list of some open problems related to the rectifiability result, and gave potential alternative definitions of essential and reduced boundary. As we will show in the appendix, the approach used in Proposition 4.3 to prove the weak* convergence of $T_t \chi_E$ to $1/2$ in $L^\infty(X, |D_\gamma \chi_E|)$ is flexible enough to give a "weak form" of the fact that $|D_\gamma \chi_E|$ is concentrated also on a kind of reduced boundary. Apart from this, many other natural questions remain open. In particular, the main open problem is still to find some analogous of De Giorgi's blow-up theorem (i.e., understanding in which sense, for $|D_\gamma \chi_E|$-a.e. $x \in X$, the blow-up of E around x converges to an half-space, see the proof of Proposition 3.1).

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2 Notation and preliminary results

We assume that $(X, \|\cdot\|)$ is a separable Banach space and γ is a Gaussian probability measure on the Borel σ-algebra of X. We shall always assume that γ is nondegenerate (i.e., all closed proper subspaces of X are γ-negligible) and centered (i.e., $\int_X x\,d\gamma = 0$). We denote by H the Cameron-Martin subspace of X, that is

$$H := \left\{ \int_X f(x)x\,d\gamma(x) : f \in L^2(X,\gamma) \right\},$$

and, for $h \in H$, we denote by $\hat{h}$ the corresponding element in $L^2(X,\gamma)$; it can be characterized as the Fomin derivative of γ along h, namely

$$\int_X \partial_h \phi\,d\gamma = -\int_X \hat{h}\phi\,d\gamma \tag{7}$$

for all $\phi \in C^1_b(X)$. Here and in the sequel $C^1_b(X)$ denotes the space of continuously differentiable cylindrical functions in X, bounded and with a bounded gradient. The space H can be endowed with an Hilbertian norm $|\cdot|_H$ that makes the map $h \mapsto \hat{h}$ an isometry; furthermore, the injection of $(H, |\cdot|_H)$ into $(X, \|\cdot\|)$ is compact.

We shall denote by $\tilde{H} \subset H$ the subset of vectors of the form

$$\int_X \langle x^*, x\rangle x\,d\gamma(x), \qquad x^* \in X^*. \tag{8}$$

This is a dense (even w.r.t. to the Hilbertian norm) subspace of H. Furthermore, for $h \in \tilde{H}$ the function $\hat{h}(x)$ is precisely $\langle x^*, x\rangle$ (and so, it is continuous).

Given a m-dimensional subspace $F \subset \tilde{H}$ we shall frequently consider an orthonormal basis $\{h_1, \ldots, h_m\}$ of F and the factorization $X = F \oplus Y$, where Y is the kernel of the continuous linear map

$$x \in X \mapsto \Pi_F(x) := \sum_{i=1}^m \hat{h}_i(x)h_i \in F. \tag{9}$$

The decomposition $x = \Pi_F(x) + (x - \Pi_F(x))$ is well defined, thanks to the fact that $\Pi_F \circ \Pi_F = \Pi_F$ and so $x - \Pi_F(x) \in Y$; in turn this follows by $\hat{h}_i(h_j) = \langle \hat{h}_i, \hat{h}_j\rangle_{L^2} = \delta_{ij}$.

Thanks to the fact that $|h_i|_H = 1$, this induces a factorization $\gamma = \gamma_F \otimes \gamma_Y$, with γ_F the standard Gaussian in F (endowed with the metric inherited from H) and γ_Y Gaussian in $(Y, \|\cdot\|)$. Furthermore, the orthogonal complement $F^\perp$ of F in H is the Cameron-Martin space of (Y, γ_Y).

2.1 BV functions and Sobolev spaces

Here we present the definitions of Sobolev and BV spaces. Since we will consider bounded functions only, we shall restrict to this class for ease of exposition.

Let $u : X \to \mathbb{R}$ be a bounded Borel function. Motivated by (7), we say that $u \in W^{1,1}(X, \gamma)$ if there exists a (unique) H-valued function, denoted by ∇u, with $|\nabla u|_H \in L^1(X, \gamma)$ and

$$\int_X u \partial_h \phi \, d\gamma = - \int_X \phi \langle \nabla u, h \rangle_H \, d\gamma + \int_X u \phi \hat{h} \, d\gamma$$

for all $\phi \in C^1_b(X)$ and $h \in H$.

Analogously, following [15, 16], we say that $u \in BV(X, \gamma)$ if there exists a (unique) H-valued Borel measure $D_\gamma u$ with finite total variation in X satisfying

$$\int_X u \partial_h \phi \, d\gamma = - \int_X \phi \, d\langle D_\gamma u, h \rangle_H + \int_X u \phi \hat{h} \, d\gamma$$

for all $\phi \in C^1_b(X)$ and $h \in H$.

In the sequel, shall mostly consider the case when $u = \chi_E : X \to \{0, 1\}$ is the characteristic function of a set E, although some statements are more natural in the general BV context. Notice the inclusion $W^{1,1}(X, \gamma) \subset BV(X, \gamma)$, given by the identity $D_\gamma u = \nabla u \gamma$.

2.2 The OU semigroup and Mehler's formula

In this paper, the Ornstein-Uhlenbeck semigroup $T_t f$ will always be understood as defined by the *pointwise* formula

$$T_t f(x) := \int_X f(e^{-t} x + \sqrt{1 - e^{-2t}} y) \, d\gamma(y) \tag{10}$$

which makes sense whenever f is bounded and Borel. This convention will be important when integrating $T_t f$ against potentially singular measures, see for instance (14).

We shall also use the dual OU semigroup T_t^*, mapping signed measures into signed measures, defined by the formula

$$\langle T_t^* \mu, \phi \rangle := \int_X T_t \phi \, d\mu \qquad \phi \text{ bounded Borel.} \tag{11}$$

In the next proposition we collect a few properties of the OU semigroup needed in the sequel (see for instance [7] for the Sobolev case and [5] for the BV case).

Proposition 2.1. *Let $u : X \to \mathbb{R}$ be bounded and Borel and $t > 0$. Then $T_t u \in W^{1,1}(X, \gamma)$ and:*

(a) *if $u \in W^{1,1}(X,\gamma)$ then, componentwise, it holds $\nabla T_t u = e^{-t} T_t \nabla u$;*

(b) *if $u \in BV(X,\gamma)$ then, componentwise, it holds $\nabla T_t u\gamma = e^{-t} T_t^*(D_\gamma u)$.*

The next result is basically contained in [7, Proposition 5.4.8], we state and prove it because we want to emphasize that the regular version of the restriction of $T_t f$ to $y+F$, $y \in Y$, provided by the Proposition, is for γ_Y-a.e. y precisely the one pointwise defined in Mehler's formula.

Proposition 2.2. *Let u be a bounded Borel function and $t>0$. With the above notation, for γ_Y-a.e. $y \in Y$ the map $z \mapsto T_t u(z,y)$ is smooth in F.*

Proof. Let us prove, for the sake of simplicity, Lipschitz continuity (in fact, the only property we shall need) for γ_Y-a.e. y, with a bound on the Lipschitz constant depending only on t and on the supremum of $|u|$. We use the formula

$$\partial_h T_t u(x) = \frac{e^{-t}}{\sqrt{1-e^{-2t}}} \int_X u(e^{-t}x + \sqrt{1-e^{-2t}}y)\hat{h}(y)\, d\gamma(y) \qquad h \in H$$

for the weak derivative and notice that, if u is cylindrical, this provides also the classical derivative. On the other hand, the formula provides also the uniform bound $\sup|\partial_h T_t u| \le c(t)|h|_H \sup|u|$. The uniform bound and Fubini's theorem ensure that the class of functions for which the stated property is true contains all cylindrical functions and it stable under pointwise equibounded limits. By the monotone class theorem, the stated property holds for all bounded Borel functions. □

The next lemma provides a rate of convergence of $T_t u$ to u when u belongs to $BV(X,\gamma)$; the proof follows the lines of the proof of Poincaré inequalities, see [7, Theorem 5.5.11].

Lemma 2.3. *Let $u \in BV(X,\gamma)$. Then*

$$\int_X |T_t u - u|\, d\gamma \le c_t |D_\gamma u|(X)$$

with $c_t := \sqrt{\frac{2}{\pi}} \int_0^t \frac{e^{-s}}{\sqrt{1-e^{-2s}}}\, ds$, $c_t \sim 2\sqrt{t/\pi}$ as $t \downarrow 0$.

Proof. It obviously suffices to bound with $c_t|D_\gamma u|(X)$ the expression

$$\int_X \int_X |u(x) - u(e^{-t}x + \sqrt{1-e^{-2t}}y)|\, d\gamma(x) d\gamma(y). \tag{12}$$

Standard cylindrical approximation arguments reduce the proof to the case when u is smooth, X is finite-dimensional and γ is the standard Gaussian. Since

$$\begin{aligned} u(e^{-t}x + \sqrt{1-e^{-2t}}y) - u(x) &= \int_0^1 \frac{d}{d\tau} u(e^{-t\tau}x + \sqrt{1-e^{-2t\tau}}y)\, d\tau \\ &= t\int_0^1 \nabla(e^{-t\tau}x + \sqrt{1-e^{-2t\tau}}y) \cdot \left(-e^{-t\tau}x + \frac{e^{-2t\tau}y}{\sqrt{1-e^{-2t\tau}}}\right) d\tau \end{aligned}$$

we can estimate the expression in (12) with

$$t\int_0^1 \frac{e^{-t\tau}}{\sqrt{1-e^{-2t\tau}}}\int_X\int_X |\nabla u(e^{-t\tau}x+\sqrt{1-e^{-2t\tau}}y)\cdot(-\sqrt{1-e^{-2t\tau}}x+e^{-t\tau}y)|\,d\gamma(x)d\gamma(y)d\tau.$$

Now, for τ fixed we can perform the "Gaussian rotation"

$$(x,y)\mapsto\big(e^{-t\tau}x+\sqrt{1-e^{-2t\tau}}y,-\sqrt{1-e^{-2t\tau}}x+e^{-t\tau}y\big)$$

to get

$$t\int_0^1 \frac{e^{-t\tau}}{\sqrt{1-e^{-2t\tau}}}\int_X\int_X |\nabla u(v)\cdot w|\,d\gamma(w)d\gamma(v)d\tau.$$

Eventually we use the fact that $\int_X |\xi\cdot w|\,d\gamma(w)=\sqrt{2/\pi}|\xi|$ to get

$$t\sqrt{\frac{2}{\pi}}\int_0^1 \frac{e^{-t\tau}}{\sqrt{1-e^{-2t\tau}}}\,d\tau\int_X |\nabla u|(v)\,d\gamma(v).$$

A change of variables leads to the desired expression of c_t. □

Notice that the proof of the lemma provides the slightly stronger information

$$\int_X\int_X |u(x)-u(e^{-t}x+\sqrt{1-e^{-2t}}y)|\,d\gamma(x)d\gamma(y)\le c_t|D_\gamma u|(X). \tag{13}$$

This more precise formulation will be crucial in the proof of Proposition 4.1.

2.3 Product rule

In the proof of Proposition 4.3 we shall use the product rule

$$D_\gamma(\chi_E v)=\chi_E\nabla v\gamma+vD_\gamma\chi_E$$

for $v\in W^{1,1}(X,\gamma)$ and E with finite perimeter. In general the proof of this property is delicate, even in finite-dimensional spaces, since a precise representative of v should be used to make sense of the product $vD_\gamma\chi_E$. However, in the special case when $v=T_tf$ with $t>0$ and f bounded Borel, the product rule, namely

$$D_\gamma(\chi_E T_tf)=\chi_E\nabla T_tf\gamma+T_tfD_\gamma\chi_E. \tag{14}$$

holds provided we understand T_tf as pointwise defined in Mehler's formula. The argument goes by pointwise approximation by better maps, very much as in Proposition 2.2, and we shall not repeat it.

2.4 Factorization of T_t and $D_\gamma u$

Let us consider the decomposition $X = F \oplus Y$, with $F \subset \tilde{H}$ finite-dimensional. Denoting by T_t^F and T_t^Y the OU semigroups in F and Y respectively, it is easy to check (for instance first on products of cylindrical functions on F and Y, and then by linearity and density) that also the action of T_t can be "factorized" in the coordinates $x = (z, y) \in F \times Y$ as follows:

$$T_t f(z, y) = T_t^Y \big(w \mapsto T_t^F f(\cdot, w)(z)\big)(y) \tag{15}$$

for any bounded Borel function f.

Let us discuss, now, the factorization properties of $D_\gamma u$. Let us write $D_\gamma u = \nu_u |D_\gamma u|$ with $\nu_u : X \to H$ Borel vectorfield with $|\nu_u|_H = 1$ $|D_\gamma u|$-a.e. Moreover, given a Borel set B, define

$$B_y := \{z \in F : \ (z, y) \in B\}, \qquad B_z := \{y \in Y : \ (z, y) \in B\}.$$

The identity

$$\int_B |\pi_F(\nu_u)| \, d|D_\gamma u| = \int_Y |D_{\gamma_F} u(\cdot, y)|(B_y) \, d\gamma_Y(y) \qquad B \text{ Borel} \tag{16}$$

is proved in [5, Theorem 44.2] (see also [3, 17] for analogous results), where $\pi_F : H \to F$ is the orthogonal projection. Along the similar lines, one can also show the identity

$$\int_B |\pi_{F^\perp}(\nu_u)| \, d|D_\gamma u| = \int_F |D_{\gamma_Y} u(z, \cdot)|(B_z) \, d\gamma_F(z) \qquad B \text{ Borel} \tag{17}$$

with $\pi_F + \pi_{F^\perp} = \mathrm{Id}$. In the particular case $u = \chi_E$, with the notation

$$E_y := \{z \in F : \ (z, y) \in E\}, \qquad E_z := \{y \in Y : \ (z, y) \in E\} \tag{18}$$

the identities (16) and (17) read respectively as

$$\int_B |\pi_F(\nu_E)| \, d|D_\gamma \chi_E| = \int_Y |D_{\gamma_F} \chi_{E_y}|(B_y) \, d\gamma_Y(y) \qquad B \text{ Borel,} \tag{19}$$

$$\int_B |\pi_{F^\perp}(\nu_E)| \, d|D_\gamma \chi_E| = \int_F |D_{\gamma_Y} \chi_{E_z}|(B_z) \, d\gamma_F(z) \qquad B \text{ Borel} \tag{20}$$

with $D_\gamma \chi_E = \nu_E |D_\gamma \chi_E|$.

Remark 2.4. Having in mind (19) and (20), it is tempting to think that the formula holds for any orthogonal decomposition of H (so, not only when $F \subset \tilde{H}$), or even when none of the parts if finite-dimensional. In order to avoid merely technical complications we shall not treat this issue here because, in this more general situation, the "projection maps" $x \mapsto y$ and $x \mapsto z$ are no longer continuous. The problem can be solved removing sets of small capacity, see for instance [12] for a more detailed discussion.

As a corollary of the above formulas, we can prove the following important semicontinuity result for open sets:

Proposition 2.5. *For any open set $A \subset X$ the map*

$$u \mapsto |D_\gamma u|(A)$$

is lower semicontinuous in $BV(X;\gamma)$ with respect to the $L^1(X,\gamma)$ convergence.

Proof. Let $u_k \to u$ in $L^1(X,\gamma)$. It suffices to prove the result under the additional assumption that

$$\sum_k \int_X |u_k - u| \, d\gamma < \infty. \tag{21}$$

Let $F \subset \tilde{H}$ be a finite dimensional subspace, let $X = F \times Y$ be the associated factorization, and use coordinates $x = (z, y) \in F \times Y$ as before.

Thanks to (21) and Fubini's theorem, $u_k(\cdot, y) \to u(\cdot, y)$ in $L^1(F, \gamma_F)$ for γ_Y-a.e. $y \in Y$. Hence, by the lower semicontinuity of the total variation in finite dimensional spaces (see for instance [2, Remark 3.5] for a proof when γ_F is replaced by the Lebesgue measure) we obtain

$$|D_{\gamma_F} u(\cdot, y)|(A_y) \leq \liminf_{k\to\infty} |D_{\gamma_F} u_k(\cdot, y)|(A_y) \qquad \text{for } \gamma_Y\text{-a.e. } y \in Y,$$

where $A_y := \{z \in F : (z, y) \in A\}$. Integrating with respect to γ_Y and using Fatou's lemma we get

$$\int_Y |D_{\gamma_F} u(\cdot, y)|(A_y) \, d\gamma_{Y_n} \leq \liminf_{k\to\infty} \int_Y |D_{\gamma_F} u_k(\cdot, y)|(A_y) \, d\gamma_Y,$$

which together with (16) gives

$$\int_A |\pi_F(\nu_u)| \, d|D_\gamma u| \leq \liminf_{k\to\infty} \int_A |\pi_F(\nu_u)| \, d|D_\gamma u_k| \leq \liminf_{k\to\infty} |D_\gamma u_k|(A)$$

(recall that $|\nu_u|_H = 1$). Since $|\pi_F(\nu_u)| \uparrow 1$ as F increases to a dense subspace of H, we conclude by the monotone convergence theorem. □

2.5 Finite-codimension Hausdorff measures

We start by introducing, following [12], pre-Hausdorff measures which, roughly speaking, play the same role of the pre-Hausdorff measures $\mathscr{S}^n_\delta$ in the finite-dimensional theory.

Let $F \subset \tilde{H}$ be finite-dimensional, $m \geq k \geq 0$ and, with the notation of the previous section, define

$$\mathscr{S}^{\infty-k}_F(B) := \int_Y \int_{B_y} G_m \, d\mathscr{S}^{m-k} \, d\gamma_Y(y) \qquad B \text{ Borel} \tag{22}$$

where $m = \dim(F)$ and G_m is the standard Gaussian density in F (so that $\mathscr{S}^{\infty-0}_F = \gamma$). It is proved in [12] that $y \mapsto \int_{B_y} G_m \, d\mathscr{S}^{m-k}$ is γ_Y-measurable whenever B is Suslin (so, in particular, when B is Borel), therefore the integral makes sense. The first key monotonicity property noticed in [12], based on [10, 2.10.27], is

$$\mathscr{S}^{\infty-k}_F(B) \leq \mathscr{S}^{\infty-k}_G(B) \qquad \text{whenever } F \subset G \subset \tilde{H}$$

provided $\mathscr{S}^{m-k}$ in (22) is understood as the *spherical* Hausdorff measure of dimension $m-k$ in F. This naturally leads to the definition

$$\mathscr{S}^{\infty-k}(B) := \sup_F \mathscr{S}^{\infty-k}_F(B) \qquad B \text{ Borel}, \tag{23}$$

where the supremum runs among all finite-dimensional subspaces F of $\tilde{H}$. Notice, however, that strictly speaking the measure defined in (23) does not coincide with the one in [12], since all finite-dimensional subspaces of H are considered therein. We make the restriction to finite-dimensional subspaces of $\tilde{H}$ for the reasons explained in Remark 2.4. However, still $\mathscr{S}^{\infty-k}$ is defined in a coordinate-free fashion.

These measures have been related for the first time to the perimeter measure $D_\gamma \chi_E$ in [17]. Hino defined the F-essential boundaries (obtained collecting the essential boundaries of the finite-dimensional sections $E_y \subset F \times \{y\}$)

$$\partial^*_F E := \{(z, y) : \ z \in \partial^* E_y\} \tag{24}$$

and noticed another key monotonicity property (see also [5, Theorem 5.2])

$$\mathscr{S}^{\infty-1}_F(\partial^*_F E \setminus \partial^*_G E) = 0 \qquad \text{whenever } F \subset G \subset \tilde{H}. \tag{25}$$

Then, choosing a sequence $\mathcal{F} = \{F_1, F_2, \ldots\}$ of finite-dimensional subspaces of $\tilde{H}$ whose union is dense he defined

$$\mathscr{S}^{\infty-1}_{\mathcal{F}} := \sup_n \mathscr{S}^{\infty-1}_{F_n}, \qquad \partial^*_{\mathcal{F}} E := \liminf_{n\to\infty} \partial^*_{F_n} E \tag{26}$$

and proved that

$$|D_\gamma \chi_E| = \mathscr{S}^{\infty-1}_{\mathcal{F}} \llcorner \partial^*_{\mathcal{F}} E. \tag{27}$$

Remark 2.6. If we compare (27) with (5), we see that both the measure and the set are defined in (5) in a coordinate-free fashion, using on one hand all finite-dimensional subspaces of $\tilde{H}$, on the other hand the OU semigroup. In this respect, it seems to us particularly difficult to compare null sets w.r.t. $\mathscr{S}^{\infty-1}_{\mathcal{F}}$ and $\mathscr{S}^{\infty-1}_{\mathcal{F}'}$ when $\mathcal{F} \neq \mathcal{F}'$; so, even though the left hand side in (27) is coordinate-free, it seems difficult to extract from this information a "universal" set. On the other hand, combining (5) and (27) we obtain that $E^{1/2}$ is equivalent to $\partial^*_{\mathcal{F}} E$, up to $\mathscr{S}^{\infty-1}_{\mathcal{F}}$-null sets (observe that, on the other hand, it is not even clear that $\partial^*_{\mathcal{F}} E$ has $\mathscr{S}^{\infty-1}$ finite measure). So, in some sense, $E^{1/2}$ is "minimal" against the "maximal" measure $\mathscr{S}^{\infty-1}$.

3 Finite-dimensional facts

Throughout this section we assume that (X, γ) is a finite-dimensional Gaussian space, with the associated OU semigroup T_t. We assume that the norm of X is equal to the Cameron-Martin norm, so that we can occasionally identify X with $\mathbb{R}^m$, $m = \dim X$, and identify γ with the product $G_m \mathscr{L}^m$ of m standard Gaussians. Give a Borel set E, we shall denote by E^1 (resp. E^0) the set of density points of E (resp. rarefaction points) with respect to the Lebesgue measure (it would be the same to consider γ, since this measure is locally comparable to $\mathscr{L}^m$).

In this finite dimensional setting, the first result is that the statement of Theorem 1.1 can be improved, getting pointwise convergence up to $|D_\gamma \chi_E|$-negligible sets:

Proposition 3.1. *Let $E \subset X$ be with finite γ-perimeter. Then, as $t \downarrow 0$, $T_t \chi_E \to 1/2$ $|D_\gamma \chi_E|$-a.e. in X.*

Proof. In this proof we identify X with $\mathbb{R}^m$. Since $|D_\gamma \chi_E| = G_m |D\chi_E|$, we know that E has locally finite Euclidean perimeter. Hence, the finite-dimensional theory ensures that $|D\chi_E|$-almost every point x the rescaled and translated sets $(E - x)/r$ locally converge in measure as $r \downarrow 0$ to an halfspace passing through the origin (see for instance [2, Theorem 3.59(a)]). We obtain that for $|D_\gamma \chi_E|$-almost every point x the sets

$$E_{t,x} := \frac{E - e^{-t}x}{\sqrt{1 - e^{-2t}}}$$

locally converge in measure as $t \downarrow 0$ to an halfspace (here we use the fact that translating by $e^{-t}x$ instead of x is asymptotically the same, since $1 - e^{-t} = o(\sqrt{1 - e^{-2t}})$ as $t \downarrow 0$). Hence, it suffices to show that $T_t \chi_E(x) \to 1/2$ at all points x where this convergence holds. We compute:

$$\begin{aligned} T_t \chi_E(x) &= (2\pi)^{-m/2} \int_{\mathbb{R}^m} \chi_E(e^{-t}x + \sqrt{1 - e^{-2t}}y) e^{-|y|^2/2} \, dy \\ &= (2\pi)^{-m/2} \int_{E_{t,x}} e^{-|y|^2/2} \, dy. \end{aligned}$$

Taking the limit as $t \downarrow 0$ yields $(2\pi)^{-m/2} \int_H e^{-|w|^2/2} \, dw$ for some halfspace H with $0 \in \partial H$. By rotation invariance the value of the limit equals $1/2$. $\square$

In the next proposition we carefully estimate the blow-up rate of the density of $T_t^* \mu$ as $t \downarrow 0$ when μ is a codimension one Hausdorff measure on a "nice" hypersurface.

Proposition 3.2. *Let $K \subset \mathbb{R}^m$ be a Borel set contained in the union of finitely many C^1 compact hypersurfaces. Then, for all $\varepsilon > 0$, there exist $K_\varepsilon \subset K$ and $t_\varepsilon > 0$ such that $\mathscr{S}^{m-1}(K \setminus K_\varepsilon) < \varepsilon$ and*

$$\sqrt{t} T_t^* \big(G_m \mathscr{S}^{m-1} \llcorner K_\varepsilon \big) \le \gamma \qquad \forall t \in (0, t_\varepsilon).$$

Proof. We can assume with no loss of generality that $1 + \varepsilon^2 < 2\pi$. For any $y \in K$, let $r_y > 0$ be a radius such that:

- $K \cap B_{r_y}(y)$ is contained inside a C^1 submanifold S_y;
- there exists an orthogonal transformation $Q_y : \mathbb{R}^m \to \mathbb{R}^m$ such that $Q_y(S_y)$ is contained inside the graph of a Lipschitz function $u_y : B^{m-1}_{r_y} \subset \mathbb{R}^{m-1} \to \mathbb{R}$;
- the Lipschitz constant of u_y is bounded by ε.

By compactness, there exists a finite set of points $y_1, \dots, y_N$ such that

$$K \subset \bigcup_{i=1}^{N} B_{r_{y_i}}(y_i).$$

Let us define the disjoints family of sets $A_1 = K \cap B_{r_{y_1}}(y_1)$, $A_i := K \cap B_{r_{y_i}}(y_i) \setminus \left(\cup_1^{i-1} A_j\right)$ for $i = 2, \dots, N$. For any given $\varepsilon > 0$, we can find compact sets $E_i \subset A_i$ such that

$$\sum_{i=1}^{N} \mathscr{S}^{m-1}(A_i \setminus E_i) < \varepsilon, \qquad \min_{1 \le i \ne j \le N} \operatorname{dist}(E_i, E_j) =: 2\delta > 0.$$

Let us set $K_\varepsilon := \cup_{i=1}^{N} E_i$, and let $R > 0$ be sufficiently large so that $K_\varepsilon \subset B_R$. Thanks to Lemma 3.3 below applied with $\Gamma = Q_{y_i}(E_i)$ for $i = 1 \dots, N$, since G_m is invariant under orthogonal transformations there exists $t_i > 0$ such that

$$\sqrt{t} T_t^* \left(G_m \mathscr{S}^{m-1} \llcorner E_i\right) \le \sqrt{\frac{1+\varepsilon^2}{2\pi}} \Omega_{m,R} \left(\operatorname{dist}(\cdot, E_i)/\sqrt{t}\right) \gamma \qquad \forall t \in (0, t_i).$$

This implies that, for $0 < t < \min_i t_i$,

$$\sqrt{t} T_t^* \left(G_m \mathscr{S}^{m-1} \llcorner K_\varepsilon\right) \le \sqrt{\frac{1+\varepsilon^2}{2\pi}} \sum_{i=1}^{N} \Omega_{m,R} \left(\operatorname{dist}(\cdot, E_i)/\sqrt{t}\right) \gamma.$$

Recalling that $\operatorname{dist}(E_i, E_j) \ge 2\delta > 0$ for $i \ne j$, for all $x \in \mathbb{R}^m$ it holds $\operatorname{dist}(x, E_i) > \delta$ for all i with at most one exception. Hence, since $\Omega_{m,R} \le 1$ and $\Omega_{m,R}(s) \to 0$ as $s \to +\infty$, we get

$$\sqrt{\frac{1+\varepsilon^2}{2\pi}} \sum_{i=1}^{N} \Omega_{m,R} \left(\operatorname{dist}(\cdot, E_i)/\sqrt{t}\right) \le \sqrt{\frac{1+\varepsilon^2}{2\pi}} \left(1 + (N-1)\Omega_{m,R}\left(\delta/\sqrt{t}\right)\right) \le 1$$

for t sufficiently small, which concludes the proof. $\square$

Lemma 3.3. *Let $A \subset \mathbb{R}^{m-1}$ be a bounded Borel set, let $u : A \mapsto \mathbb{R}$ be a Lipschitz function with Lipschitz constant ℓ, and let $\Gamma := \{(z, u(z)) :\ z \in A\}$ be the graph of u. Assume that $\Gamma \subset B_R$ for some $R > 0$. Then, there exist a continuous function $\Omega_{m,R} : [0, +\infty) \to [0, 1]$, depending only on m and R, and $\bar{t} > 0$, such that $\Omega_{m,R}(s) \to 0$ as $s \to +\infty$, and*

$$\sqrt{t}T_t^*\big(G_m \mathscr{S}^{m-1} \llcorner \Gamma\big) \le \sqrt{\frac{1+\ell^2}{2\pi}}\Omega_{m,R}\left(\mathrm{dist}(x,\Gamma)/\sqrt{t}\right)\gamma \qquad \forall t \in (0, \bar{t}).$$

Proof. Let us first observe that, given a test function $f : \mathbb{R}^m \to \mathbb{R}$, it holds

$$\begin{aligned}\int_{\mathbb{R}^m} f\, dT_t^*\big(G_m \mathscr{S}^{m-1} \llcorner \Gamma\big) &= \int_\Gamma T_t f(y) G_m(y)\, d\mathscr{S}^{m-1}(y) \\ &= \int_{\mathbb{R}^m} f(x) \int_\Gamma \frac{e^{-\frac{|e^{-t}x|^2 - 2e^{-t}x\cdot y + |e^{-t}y|^2}{2(1-e^{-2t})}}}{(1-e^{-2t})^{m/2}} G_m(y)\, d\mathscr{S}^{m-1}(y)\, d\gamma(x).\end{aligned}$$

Hence, we have to show that, for any $x = (x', x_m) \in \mathbb{R}^{m-1} \times \mathbb{R}$, the expression

$$\sqrt{t}\int_\Gamma \frac{e^{-\frac{|e^{-t}x|^2 - 2e^{-t}x\cdot y + |e^{-t}y|^2}{2(1-e^{-2t})}}}{(1-e^{-2t})^{m/2}} G_m(y)\, d\mathscr{S}^{m-1}(y) = \frac{\sqrt{t}}{(2\pi)^{m/2}(1-e^{-2t})^{m/2}} \int_\Gamma e^{-\frac{|e^{-t}x-y|^2}{2(1-e^{-2t})}}\, d\mathscr{S}^{m-1}(y)$$

is bounded by $\sqrt{\frac{1+\ell^2}{2\pi}}\Omega_{m,R}\left(\mathrm{dist}(x,\Gamma)/\sqrt{t}\right)$ for t sufficiently small (independent of x), with $\Omega_{m,R}$ as in the statement.

Thanks to the area formula and the bound on the Lipschitz constant, we can write

$$\begin{aligned}&\frac{\sqrt{t}}{(2\pi)^{m/2}(1-e^{-2t})^{m/2}} \int_\Gamma e^{-\frac{|e^{-t}x-y|^2}{2(1-e^{-2t})}}\, d\mathscr{S}^{m-1}(y) \\ = \;& \frac{\sqrt{t}}{(2\pi)^{m/2}(1-e^{-2t})^{m/2}} \int_A e^{-\frac{|e^{-t}x'-y'|^2}{2(1-e^{-2t})}} e^{-\frac{|e^{-t}x_m-u(y')|^2}{2(1-e^{-2t})}} \sqrt{1+|\nabla u(y')|^2}\, dy' \\ \le \;& \frac{\sqrt{1+\ell^2}\sqrt{t}}{(2\pi)^{m/2}(1-e^{-2t})^{m/2}} \int_A e^{-\frac{|e^{-t}x'-y'|^2}{2(1-e^{-2t})}} e^{-\frac{|e^{-t}x_m-u(y')|^2}{2(1-e^{-2t})}}\, dy'.\end{aligned}$$

Now, since $t \le 1 - e^{-2t}$ for t small, we can bound the above expression by

$$\sqrt{\frac{1+\ell^2}{2\pi}} \frac{1}{(2\pi)^{(m-1)/2}(1-e^{-2t})^{(m-1)/2}} \int_A e^{-\frac{|e^{-t}x'-y'|^2}{2(1-e^{-2t})}} e^{-\frac{|e^{-t}x_m-u(y')|^2}{2(1-e^{-2t})}}\, dy'. \tag{28}$$

First of all we observe that, since

$$\frac{1}{(2\pi)^{(m-1)/2}(1-e^{-2t})^{(m-1)/2}} \int_A e^{-\frac{|e^{-t}x'-y'|^2}{2(1-e^{-2t})}}\, dy' = T_t\chi_A(x') \le 1,$$

the quantity in (28) is trivially bounded by $(1+\ell^2)/(2\pi)$.

To show the existence of a function $\Omega_{m,R}$ as in the statement of the lemma, we split the integral over A into the one over $A\setminus B_{\mathrm{dist}(x,\Gamma)/2}(x')$, and the one over $A\cap B_{\mathrm{dist}(x,\Gamma)/2}(x')$.

To estimate the first integral, we bound $e^{-|e^{-t}x_m-u(y')|^2/[2(1-e^{-2t})]}$ by 1. Moreover, we observe that

$$
\begin{aligned}
T_t\chi_{A\setminus B_{\mathrm{dist}(x,\Gamma)/2}(x')}(x') &\le \frac{1}{(2\pi)^{(m-1)/2}(1-e^{-2t})^{(m-1)/2}}\int_{\mathbb{R}^{m-1}\setminus B_{\mathrm{dist}(x,\Gamma)/2}(x')} e^{-\frac{|e^{-t}x'-y'|^2}{2(1-e^{-2t})}}\,dy' \\
&= \frac{1}{(2\pi)^{(m-1)/2}}\int_{\mathbb{R}^{m-1}\setminus B_{\mathrm{dist}(x,\Gamma)/[2\sqrt{1-e^{-2t}}]}} e^{-\frac{|e^{-t}x'-x'-\sqrt{1-e^{-2t}}z'|^2}{2(1-e^{-2t})}}\,dz' \\
&= \frac{1}{(2\pi)^{(m-1)/2}}\int_{\mathbb{R}^{m-1}\setminus B_{\mathrm{dist}(x,\Gamma)/[2\sqrt{1-e^{-2t}}]}} e^{-\frac{\left|z'+\sqrt{\frac{1-e^{-t}}{1+e^{-t}}}x'\right|^2}{2}}\,dz'.
\end{aligned}
$$

We now remark that $-|a+b|^2\le -|a|^2/2+|b|^2$ for all $a,b\in\mathbb{R}^{m-1}$, $1-e^{-2t}\le 2t$, and $\frac{1-e^{-t}}{1+e^{-t}}\le t$ for t small. Hence, the above expression is bounded from above by

$$
\frac{1}{(2\pi)^{(m-1)/2}}\int_{\mathbb{R}^{m-1}\setminus B_{\mathrm{dist}(x,\Gamma)/(2\sqrt{2t})}} e^{-|z'|^2/4}e^{t|x'|^2/2}\,dz'.
$$

Since $\Gamma\subset B_R$ for some R, it holds $|x'|\le|x|\le R+\mathrm{dist}(x,\Gamma)$, and so the above quantity can be bounded from above by

$$
\begin{aligned}
&\frac{1}{(2\pi)^{(m-1)/2}}e^{tR^2}e^{t\mathrm{dist}(x,\Gamma)^2}\int_{\mathbb{R}^{m-1}\setminus B_{\mathrm{dist}(x,\Gamma)/(2\sqrt{2t})}} e^{-|z'|^2/4}\,dz' \\
\le\;& \frac{m\omega_m}{(2\pi)^{(m-1)/2}}e^{R^2}e^{\mathrm{dist}(x,\Gamma)^2/100t}\int_{\mathrm{dist}(x,\Gamma)/(4\sqrt{t})}^{\infty} e^{-\tau^2/4}\tau^{m-1}\,d\tau
\end{aligned}
$$

for t small (here ω_m denotes the measure of the unit ball in $\mathbb{R}^m$).

To control the second integral over $A\cap B_{\mathrm{dist}(x,\Gamma)/2}(x')$, we bound $T_t\chi_{A\cap B_{\mathrm{dist}(x,\Gamma)/2}(x')}(x')$ by 1 and we estimate from above, uniformly for $y'\in B_{\mathrm{dist}(x,\Gamma)/2}(x')$, the quantity

$$
e^{-\frac{|e^{-t}x_m-u(y')|^2}{2(1-e^{-2t})}}.
$$

We proceed as follows: for $y'\in B_{\mathrm{dist}(x,\Gamma)/2}(x')$, by the definition of $\mathrm{dist}(x,\Gamma)$, we have

$$
4|x'-y'|^2\le \mathrm{dist}(x,\Gamma)^2\le |x'-y'|^2+|x_m-u(y')|^2,
$$

which implies $3|x'-y'|^2\le|x_m-u(y')|^2$, and so $\mathrm{dist}(x,\Gamma)^2\le 4|x_m-u(y')|^2/3$. Thus, using again the estimate $-|a-b|^2\le-|a|^2/2+|b|^2$, for t small enough we obtain

$$
e^{-\frac{|e^{-t}x_m-u(y')|^2}{2(1-e^{-2t})}}\le e^{-\frac{|x_m-u(y')|^2}{4(1-e^{-2t}))}}e^{\frac{(1-e^{-t})^2|x_m|^2}{(1-e^{-2t})}}\le e^{-\mathrm{dist}(x,\Gamma)^2/(16t)}e^{t|x_m|^2}.
$$

Since $|x_m| \le |x| \le R + \mathrm{dist}(x,\Gamma)$, we conclude that

$$e^{-\frac{|e^{-t}x_m - u(y')|^2}{2(1-e^{-2t})}} \le e^{R^2} e^{-\mathrm{dist}(x,\Gamma)^2/(20t)} \qquad \forall y' \in B_{\mathrm{dist}(x,\Gamma)/2}(x')$$

for t small enough.

Hence, it suffices to define

$$\Omega_{m,R}(s) := \min\left\{1, \frac{m\omega_m}{(2\pi)^{(m-1)/2}} e^{R^2} e^{s^2/100} \int_{s/4}^{\infty} e^{-\tau^2/4}\tau^{m-1}\,d\tau + e^{R^2}e^{-s^2/20}\right\}$$

(recall that $\int_{s/4}^{\infty} e^{-\tau^2/4}\tau^{m-1}\,d\tau \sim c_m e^{-s^2/64}s^{m-2}$ as $s \to +\infty$) to conclude the proof. □

The next lemma is stated with outer integrals $\int_Y^*$; this suffices for our purposes and avoids the difficulty of proving that the measures σ_y we will dealing with have a measurable dependence w.r.t. y.

Lemma 3.4. *Let $(Y,\mathcal{F},\mu)$ be a probability space and, for $t>0$ and $y\in Y$, let $g_{t,y}: X \to [0,1]$ be Borel maps. Assume also that:*

(a) $\{\sigma_y\}_{y\in Y}$ are positive finite Borel measures in X, with $\int_Y^* \sigma_y(X)\,d\mu(y)$ finite;

(b) $\sigma_y = G_m\mathscr{S}^{m-1}\llcorner\Gamma_y$ for μ-a.e. y, with Γ_y countably $\mathscr{S}^{m-1}$-rectifiable.

Then

$$\limsup_{t\downarrow 0} \int_Y^* \int_X T_t g_{t,y}(x)\,d\sigma_y(x)d\mu(y) \le \limsup_{t\downarrow 0} \frac{1}{\sqrt{t}} \int_Y^* \int_X g_{t,y}(x)\,d\gamma(x)d\mu(y). \tag{29}$$

Proof. We prove first the lemma under the stronger assumption that, for μ-a.e. $y\in Y$, there exists $t_y > 0$ such that

$$T_t^*\sigma_y \le \frac{1}{\sqrt{t}}\gamma \qquad \forall t \in (0,t_y).$$

Fix $\varepsilon > 0$ small, and set $Y_\varepsilon := \{y \in Y:\ t_y > \delta\}$, where $\delta = \delta(\varepsilon) > 0$ is chosen sufficiently small in such a way that $\int_{Y_\varepsilon}^* \int_X T_t g_{t,y}\,d\sigma_y d\mu(y) + \varepsilon \ge \int_Y^* \int_X T_t g_{t,y}\,d\sigma_y d\mu(y)$ (this is possible, by the continuity properties of the upper integral). For $t \in (0,\delta)$ we estimate the integrals in (29) with Y_ε in place of Y:

$$\int_{Y_\varepsilon}^* \int_X T_t g_{t,y}\,d\sigma_y d\mu(y) = \int_{Y_\varepsilon}^* \int_X g_{t,y}\,dT_t^*\sigma_y d\mu(y) \le \frac{1}{\sqrt{t}} \int_Y^* \int_X g_{t,y}\,d\gamma d\mu(y).$$

Hence, letting $t \downarrow 0$ yields (29) with an extra summand ε in the right hand side. Since ε is arbitrary we conclude.

Finally, in the general case when Γ_y is countably $\mathscr{S}^{m-1}$-rectifiable we can find for any $\varepsilon > 0$ sets $\Gamma'_y \subset \Gamma_y$ contained in the union of finitely many hypersurfaces such that $\sigma_y(\Gamma_y \setminus \Gamma'_y) < \varepsilon/2$ and then, thanks to Proposition 3.2, sets $\Gamma''_y \subset \Gamma'_y$ with $\sigma_y(\Gamma'_y \setminus \Gamma''_y) < \varepsilon/2$ in such a way that the estimate (29) holds when σ_y is replaced by $G_m \mathscr{S}^{m-1} \llcorner \Gamma''_y$. Since $T_t g_t \leq 1$ we can let $\varepsilon \downarrow 0$ to obtain (29). □

In the proof of Theorem 1.3 we need a Poincaré inequality involving capacities. Recall that the 1-dimensional capacity $c_1(G)$ of a Borel set G can be defined as:

$$c_1(G) := \inf \left\{ |Du|(\mathbb{R}^m) : \ u \in L^{m/(m-1)}(\mathbb{R}^m), \ G \subset \operatorname{int}(\{u \geq 1\}) \right\}$$

(see [23, §5.12]; other equivalent definitions involve the Bessel capacity). The following result is known (see for instance [23, Theorem 5.13.3]) but we reproduce it for the reader's convenience in the simplified case when v is continuous.

Lemma 3.5. *Let $v \in W^{1,1}(B_r) \cap C(B_r)$ and let $G \subset B_r$ be a Borel set with $c_1(G) > 0$. Then, for some dimensional constant κ, it holds*

$$\frac{1}{\omega_m r^m} \int_{B_r} |v| \, dx \leq \frac{\kappa}{c_1(G)} \int_{B_r} |\nabla v| \, dx$$

whenever v vanishes c_1-a.e. on G.

Proof. By a scaling argument, suffices to consider the case $r = 1$. By a truncation argument (i.e., first considering the positive and negative parts and then replacing v by $\min\{v, n\}$ with $n \in \mathbb{N}$) we can also assume that v is nonnegative and bounded. By homogeneity of both sides, suffices to consider the case $0 \leq v \leq 1$. In this case the statement follows by applying the inequality

$$\mathscr{L}^m(B_1 \setminus E) \leq \frac{\kappa}{c_1(G)} |D\chi_E|(B_1) \qquad \text{whenever } E \text{ is open and } G \subset E \tag{30}$$

with $E = \{v < t\}$, $t \in (0, 1)$, and then integrating both sides with respect to t and using the coarea formula. Hence, we are led to the proof of (30). Now, if $\mathscr{L}^m(E) \geq \omega_m/2$ we can apply the relative isoperimetric inequality in B_1 to get

$$\mathscr{L}^m(B_1 \setminus E) \leq c_m |D\chi_E|(B_1) \leq \frac{\kappa}{c_1(G)} |D\chi_E|(B_1)$$

provided we choose κ so large that $\kappa \geq c_1(B_1) c_m$ (observe that $c_1(G) \leq c_1(B_1)$). On the other hand, if $\mathscr{L}^m(E) \leq \omega_m/2$ then we estimate $\mathscr{L}^m(B_1 \setminus E)$ from above with ω_m and it suffices to show that $|D\chi_E|(B_1) \geq c_1(G)\omega_m/\kappa$ for $\kappa = \kappa(m)$ large enough. In this case we can find a compactly supported BV function u coinciding with χ_E on B_1 with

$$|Du|(\mathbb{R}^m) \leq c'_m \big(|D\chi_E|(B_1) + \mathscr{L}^m(E \cap B_1) \big) \leq c'_m (1 + c_m) |D\chi_E|(B_1)$$

(see for instance [2, Proposition 3.21] for the existence of a continuous linear extension operator from $BV(B_1)$ to $BV(\mathbb{R}^m)$). It follows that $c_1(G) \le c'_m(1+c_m)|D\chi_E|(B_1)$, so suffices to take κ such that $\kappa/\omega_m \ge c'_m(1+c_m)$. $\square$

In the sequel we shall extensively use the following identity between null sets w.r.t. c_1 and null sets w.r.t. to codimension one Hausdorff measure, see for instance [23, Lemma 5.12.3]:

$$c_1(G) = 0 \qquad \Longleftrightarrow \qquad \mathscr{S}^{m-1}(G) = 0. \tag{31}$$

Lemma 3.6. *Let $G \subset \mathbb{R}^m$ be a Borel set. Then*

$$\limsup_{r\downarrow 0} \frac{c_1(G\cap B_r(x))}{r^{m-1}} > 0 \qquad \text{for } c_1\text{-a.e. } x \in G.$$

Proof. Let $L \subset G$ be the Borel set of points where the limsup is null and assume by contradiction that $c_1(L) > 0$. Then (31) yields $\mathscr{S}^{m-1}(L) > 0$ as well and we can find, thanks to [6], a compact subset L' with $0 < \mathscr{S}^{m-1}(L') < \infty$. We will prove that

$$\liminf_{r\downarrow 0} \frac{c_1(L'\cap \overline{B}_r(x))}{\mathscr{S}^{m-1}(L'\cap \overline{B}_r(x))} > 0 \qquad \text{for } \mathscr{S}^{m-1}\text{-a.e. } x \in L'. \tag{32}$$

Combining this information with the well-know fact (see for instance [2, (2.43)])

$$\limsup_{r\downarrow 0} \frac{\mathscr{S}^{m-1}(L'\cap \overline{B}_r(x))}{r^{m-1}} > 0 \qquad \text{for } \mathscr{S}^{m-1}\text{-a.e. } x \in L', \tag{33}$$

we obtain

$$\limsup_{r\downarrow 0} \frac{c_1(L'\cap \overline{B}_r(x))}{r^{m-1}} > 0 \qquad \text{for } \mathscr{S}^{m-1}\text{-a.e. } x \in L',$$

in contradiction with the inclusion $L' \subset L$ and the fact that $\mathscr{S}^{m-1}(L') > 0$.

To conclude the proof, we check (32). Let $L'' \subset L'$ be the Borel set of points where the liminf in (32) is null; for all $\varepsilon > 0$ we can find, thanks to Vitali covering theorem, a disjoint cover of $\mathscr{S}^{m-1}$-almost all of L'' by disjoint closed balls $\{\overline{B}_{r_i}(x_i)\}_{i\in I}$ satisfying $c_1(L'\cap\overline{B}_{r_i}(x_i)) \le \varepsilon\mathscr{S}^{m-1}(L'\cap\overline{B}_{r_i}(x_i))$. Thanks to (31) the balls cover also c_1-almost all of L'', so the countable subadditivity of capacity yields $c_1(L'') \le \varepsilon\mathscr{S}^{m-1}(L')$. Since ε is arbitrary we conclude that $c_1(L'') = 0$, whence $\mathscr{S}^{m-1}(L'') = 0$ by (31). $\square$

Proposition 3.7. *Let $(u_n) \subset W^{1,1}(X,\gamma)\cap C(X)$ be convergent in $L^1(X,\gamma)$ to χ_E, with E of finite perimeter, and satisfying*

$$\limsup_{n\to\infty} \int_X |\nabla u_n|\, d\gamma \le |D_\gamma \chi_E|(X). \tag{34}$$

Then

$$L := \left\{ x : \lim_{n\to\infty} u_n(x) = \frac{1}{2} \right\}$$

is contained, up to $\mathscr{S}^{m-1}$-negligible sets, in the essential boundary of E.

Proof. Possibly passing to the smaller sets

$$L \cap \left(\bigcap_{n=m}^{\infty} \left\{ x \in X : |u_n(x) - \frac{1}{2}| \le \frac{1}{4} \right\} \right)$$

which monotonically converge to L as $m \to \infty$, we can assume with no loss of generality that $|u_n - 1/2| \le 1/4$ on L.

Let us prove, first, that (34) yields the weak* convergence in the duality with $C_b(X)$ of $|\nabla u_n|\gamma$ to $|D_\gamma \chi_E|$. It suffices to apply the lower semicontinuity of the total variation in open sets (see Proposition 2.5) to get

$$\liminf_{n\to\infty} \int_A |\nabla u_n|\, d\gamma \ge |D_\gamma \chi_E|(A) \qquad \text{for all } A \subset X \text{ open}$$

and then to apply [2, Proposition 1.80].

Denoting by E^1 the set of density points of E, it suffices to show that $c_1(L \cap E^1) = 0$; indeed, the same property with the complement of E and $1 - u_n$ gives $c_1(L \cap E^0) = 0$, where E is the set of rarefaction points of E, and since $E^0 \cup E^1$ is the complement of the essential boundary of E we conclude thanks to (31).

We now assume by contradiction that $G := L \cap E^1$ has strictly positive capacity. Since $|D\chi_E|(B_r(y)) = o(r^{m-1})$ for $\mathscr{S}^{m-1}$-a.e. $y \in E^1$ and thanks to Lemma 3.6, we find a point $x \in G$ and radii $r_i \downarrow 0$ such that $\lim_i c_1(G \cap B_{r_i}(x))/r_i^{m-1} > 0$ and $|D\chi_E|(B_{r_i}(x)) = o(r_i^{m-1})$. Let $\phi : [0,1] \to [0,1]$ be the piecewise affine function identically equal to $1/2$ on $[1/4, 3/4]$ and with derivative equal to 2 on $(0, 1/4) \cup (3/4, 1)$. Since $\phi \circ u_n$ are identically equal to $1/2$ on $L \supset G$, we can apply Lemma 3.5 to $1/2 - \phi \circ u_n$ in the ball $B_{r_i}(x)$ to get

$$r_i^{-m} \int_{B_{r_i}(x)} |\phi \circ u_n - \frac{1}{2}|\, dy \le \frac{2\kappa\omega_m}{c_1(G \cap B_{r_i}(x_i))} \int_{B_{r_i}(x)} |\nabla u_n|\, dy.$$

Since $\phi(0) = 0$ and $\phi(1) = 1$, passing to the limit as $n \to \infty$ and using the weak* convergence of $|\nabla u_n|\gamma$ to $|D_\gamma \chi_E|$ yields

$$r_i^{-m} \int_{B_{r_i}(x)} |\chi_E - \frac{1}{2}|\, dy \le \frac{2\kappa\omega_m}{c_1(G \cap B_{r_i}(x_i))} \int_{B_{r_i}(x)} \frac{1}{G_m}\, d|D_\gamma \chi_E|.$$

Since $r_i^{m-1}/c_1(G \cap B_{r_i}(x_i))$ is uniformly bounded as $i \to \infty$ and $|D\chi_E|(B_{r_i}(x)) = o(r_i^{m-1})$ we conclude that

$$r_i^{-m} \int_{B_{r_i}(x)} |\chi_E - \frac{1}{2}|\, dy \to 0 \qquad \text{as } r_i \downarrow 0,$$

contradicting the fact that $x \in E^1$. □

4 Convergence of $T_t\chi_E$ to $1/2$

In this section we shall prove Theorem 1.1. By a well-known convergence criterion in L^2, the stated convergence will be a consequence of the weak* convergence of $T_t\chi_E$ to $1/2$ in $L^\infty(X,|D_\gamma\chi_E|)$, that we shall prove in Proposition 4.3, and the following apriori estimate (see also Remark 4.2):

Proposition 4.1. *For any set E with finite perimeter in (X,γ) it holds*

$$\limsup_{t\downarrow 0}\int_X |T_t\chi_E|^2\,d|D_\gamma\chi_E| \le \frac{1}{4}|D_\gamma\chi_E|(X). \tag{35}$$

Proof. In this proof we shall use the simpler notation

$$T_tf(x)=\int_F f(y)\rho_t^X(x,dy)$$

for the action of the OU semigroup. Comparing with Mehler's formula (10), we see that the measure $\rho_t^X(x,\cdot)$ is nothing but the law of $y\mapsto e^{-t}x+\sqrt{1-e^{-2t}}y$ under γ (not absolutely continuous w.r.t. γ if $t>0$ and X is infinite-dimensional).

Let $f_t=T_t\chi_E$ and write, as in (15),

$$f_t(z,y)=\int_Y\int_F \chi_{E_{y'}}(z')\rho_t^F(z,dz')\rho_t^Y(y,dy')$$

where $H=F\oplus F^\perp$ is an orthogonal decomposition of H, $F\subset\tilde H$ is finite-dimensional, $X=F\oplus Y$ and $\gamma=\gamma_F\otimes\gamma_Y$ are the corresponding decompositions of X and γ and $E_y=\{z\in F:(z,y)\in E\}$. Then Hölder's inequality yields

$$f_t^2(z,y)\le\int_Y\left(\int_{E_{y'}}\rho_t^F(z,dz')\right)^2\rho_t^Y(y,dy'), \tag{36}$$

so that it suffices to estimate from above the upper limits of the integrals

$$\int_X\left[\int_Y\left(\int_{E_{y'}}\rho_t^F(z,dz')\right)^2\rho_t^Y(y,dy')\right]d|D_\gamma\chi_E|(x) \tag{37}$$

as $t\downarrow 0$, with $|D_\gamma\chi_E|(X)/4$. First of all, we notice that the quantity in square parentheses is less than 1; hence, since (19) ensures that the measures in X

$$|D_{\gamma_F}\chi_{E_y}|(dz)\otimes\gamma_Y(dy)$$

monotonically converge to $|D_\gamma \chi_E|$ as $F \uparrow H$ (more precisely, as F increases to a vector space dense in H), it suffices to estimate with $|D_\gamma \chi_E|(X)/4$ the upper limit as $t \downarrow 0$ of the integrals

$$\int_Y \int_F \left[\int_Y \left(\int_{E_{y'}} \rho_t^F(z, dz') \right)^2 \rho_t^Y(y, dy') \right] d|D_{\gamma_F} \chi_{E_y}|(z) d\gamma_Y(y). \tag{38}$$

Now, if in (38) we replace the innermost integral on $E_{y'}$ with an integral on E_y, thanks to Fatou's lemma and Proposition 3.1 (observe that $\int_{E_y} \rho_t^F(z, dz') \leq 1$) we get immediately

$$\begin{aligned} & \limsup_{t \downarrow 0} \int_Y \int_F \left(\int_{E_y} \rho_t^F(z, dz') \right)^2 d|D_{\gamma_F} \chi_{E_y}|(z) d\gamma_Y(y) \\ \leq & \int_Y \int_F \limsup_{t \downarrow 0} \left(\int_{E_y} \rho_t^F(z, dz') \right)^2 d|D_{\gamma_F} \chi_{E_y}|(z) d\gamma_Y(y) \\ \leq & \frac{1}{4} \int_Y |D_{\gamma_F} \chi_{E_y}|(F) \, d\gamma_Y(y). \end{aligned}$$

Since the quantity above is less than $|D_\gamma \chi_E|(X)/4$, we are led to show that the lim sup as $t \downarrow 0$ of the expressions

$$\int_Y \int_F \int_Y \left| \left(\int_{E_{y'}} \rho_t^F(z, dz') \right)^2 - \left(\int_{E_y} \rho_t^F(z, dz') \right)^2 \right| \rho_t^Y(y, dy') \, d|D_{\gamma_F} \chi_{E_y}|(z) d\gamma_Y(y)$$

can be made arbitrarily small, choosing F large enough. To this aim, bounding the difference of the squared integrals with twice their difference, using again that $\int_{E_y} \rho_t^F(z, dz') \leq 1$ it suffices to estimate the simpler expressions

$$\int_Y \int_F \int_Y \left| \left(\int_{E_{y'}} \rho_t^F(z, dz') - \int_{E_y} \rho_t^F(z, dz') \right) \right| \rho_t^Y(y, dy') \, d|D_{\gamma_F} \chi_{E_y}|(z) d\gamma_Y(y). \tag{39}$$

We can now estimate (39) from above with

$$\int_Y \int_F T_t^F g_{t,y}(z) \, d|D_{\gamma_F} \chi_{E_y}|(z) d\gamma_Y(y),$$

where T_t^F denotes the OU semigroup in (F, γ_F) and

$$g_{t,y}(z) := \int_Y |\chi_{E_{y'}}(z) - \chi_{E_y}(z)| \rho_t^Y(y, dy').$$

Keeping y fixed, by applying Lemma 3.4 with $\sigma_y = |D_{\gamma_F} \chi_{E_y}|$ we get that the limsup as $t \downarrow 0$ of the expression in (39) is bounded above by

$$\limsup_{t \downarrow 0} \frac{1}{\sqrt{t}} \int_Y \int_X g_{t,y}(z) \, d\gamma_F(z) d\gamma_Y(y). \tag{40}$$

Since we can also write $g_{t,y}(z) = \int_Y \big|\chi_{E_z}(y) - \chi_{E_z}(y')\big| \rho_t^Y(y, dy')$, by (13) we get

$$\int_Y g_{t,y}(z)\, d\gamma_Y(y) = \int_Y \int_Y |\chi_{E_z}(y) - \chi_{E_z}(y')| \rho_t^Y(y, dy')\, d\gamma_Y(y) \le c_t |D_{\gamma_Y} \chi_{E_z}|(Y),$$

so that an integration w.r.t. z and Fubini's theorem give that the lim sup in (40) is bounded above by (taking also into account that $c_t \sim 2\sqrt{t/\pi}$)

$$\frac{2}{\sqrt{\pi}} \int_F |D_{\gamma_Y} \chi_{E_z}|(Y)\, d\gamma_F(z).$$

But, according to (20), we can represent the expression above as

$$\frac{2}{\sqrt{\pi}} \int_X |\pi_{F^\perp}(\nu_E)|\, d|D_\gamma \chi_E|.$$

Since $|\pi_{F^\perp}(\nu_E)| \downarrow 0$ as F increases to a dense subspace of H, we conclude. □

Remark 4.2. In the previous proof we used that the statement is true in finite dimensions, see Proposition 3.1. But actually Proposition 3.1 provides also a stronger information, and the proof above could be slightly modified to obtain directly Theorem 1.1 from this stronger information. However, we prefer to emphasize a softer and surely more elementary proof of the weak* convergence of T_t. Indeed, we believe that the softer argument below (based just on the product rule (14) and some elementary arguments) has an interest in his own. In particular, a variant of this argument allows to prove that $|D_\gamma \chi_E|$ is also concentrated on a kind of reduced boundary (see the Appendix).

Proposition 4.3. *As $t \downarrow 0$, $T_t \chi_E$ weak* converge to $1/2$ in $L^\infty(X, |D_\gamma \chi_E|)$.*

Proof. Let $t_i \downarrow 0$ be such that $f_i := T_{t_i} \chi_E$ weak* converge to some function f as $i \to \infty$. It suffices to show that $f \ge 1/2$ up to $|D_\gamma \chi_E|$-negligible sets. Indeed, the same property applied to $X \setminus E$ yields $1 - f \ge 1/2$ up to $|D_\gamma \chi_{X \setminus E}|$-negligible sets, and since the surface measures of E and $X \setminus E$ are the same we obtain that $f = 1/2$ in $L^\infty(X, |D_\gamma \chi_E|)$. Since $T_t \chi_E$ is uniformly bounded in $L^\infty(X, |D_\gamma \chi_E|)$, from the arbitrariness of (t_i) the stated convergence property as $t \downarrow 0$ follows.

By approximation, it suffices to show that

$$2 \int_A f\, d|D_\gamma \chi_E| \ge |D_\gamma \chi_E|(A) \tag{41}$$

for any open set $A \subset X$; by inner approximation with smaller open sets whose boundary is $|D_\gamma \chi_E|$-negligible, we can also assume in the proof of (41) that $|D_\gamma \chi_E|(\partial A) = 0$. We use the product rule (14) to obtain

$$D_\gamma(f_i \chi_E) = f_i D_\gamma \chi_E + \chi_E \nabla f_i \gamma.$$

Then, we use the relations $\nabla T_t v = e^{-t}T_t^* D_\gamma v$ (see Proposition 2.1(b)) and $|\nabla T_t v| \le e^{-t}T_t^*|D_\gamma v|$ with $v = \chi_E$ and $t = t_i$ to get

$$|D_\gamma(f_i\chi_E)| \le f_i|D_\gamma\chi_E| + T_{t_i}^*|D_\gamma\chi_E|.$$

Let us now evaluate both measures on A:

$$|D_\gamma(f_i\chi_E)|(A) \le \int_A f_i\, d|D_\gamma\chi_E| + \int_X T_{t_i}\chi_{A\cap E}\, d|D_\gamma\chi_E|.$$

Since $T_{t_i}\chi_{A\cap E} \le \min\{f_i, T_{t_i}\chi_A\}$ we can further estimate

$$|D_\gamma(f_i\chi_E)|(A) \le 2\int_A f_i\, d|D_\gamma\chi_E| + \int_{X\setminus A} T_{t_i}\chi_A\, d|D_\gamma\chi_E|.$$

Finally, since $f_i\chi_E \to \chi_E$ in $L^1(X,\gamma)$, it suffices to use the fact that $T_t\chi_A \to 0$ pointwise on $X\setminus\overline{A}$ and the lower semicontinuity of the total variation in open sets (see Proposition 2.5) to get (41). □

5 Representation of the perimeter measure

In this section we shall prove Theorem 1.3. We fix an orthogonal decomposition $X = F\oplus F^\perp$ of H, with $F \subset \tilde{H}$ finite-dimensional, and denote by $X = F\oplus Y$ the corresponding decomposition of X. We define E_y, $y\in Y$, as in (18) and, correspondingly, the essential boundary $\partial_F^* E$ as in (24).

Our main goal will be to show that the set $E^{1/2}$ (as defined in Definition 1.2), namely

$$\left\{x\in X:\ \lim_{i\to\infty} T_{t_i}\chi_E(x) = \frac{1}{2}\right\}$$

is contained in $\partial_F^* E$ up to $\mathscr{S}_F^{\infty-1}$-negligible sets, i.e.,

$$\mathscr{S}_F^{\infty-1}(E^{1/2}\setminus\partial_F^* E) = 0. \tag{42}$$

Proof of (42). Let $f_{i,y}(z) = T_{t_i}\chi_E(z,y)$. Since $\sum_i\sqrt{t_i} < \infty$ we can use the estimates

$$\int_Y \sum_i \int_F |f_{i,y} - \chi_{E_y}|\, d\gamma_F\, d\gamma_Y(y) = \sum_i \int_X |T_{t_i}\chi_E - \chi_E|\, d\gamma \le |D_\gamma\chi_E|(X)\sum_i c_{t_i},$$

with c_t as in Lemma 2.3, to obtain that $f_{i,y}\to\chi_{E_y}$ in $L^1(\gamma_F)$ for γ_Y-a.e. $y\in Y$. Our first task will be to show the existence of a subsequence $t_{i(j)}$ such that

$$\lim_{j\to\infty}\int_F |\nabla_F f_{i(j),y}|\, d\gamma_F = |D_{\gamma_F}\chi_{E_y}|(F) \qquad \text{for } \gamma_Y\text{-a.e. } y\in Y. \tag{43}$$

To this aim, we first show that

$$\int_Y \left(\int_F |\nabla_F f_{i,y}| \, d\gamma_F \right) d\gamma_Y \leq \int_Y |D_{\gamma_F} \chi_{E_y}|(F) \, d\gamma_Y. \tag{44}$$

In order to prove (44) we use Proposition 2.1(b) to get $|\nabla_F f_i|\gamma \leq T^*_{t_i}|\pi_F(D_\gamma \chi_E)|$, hence

$$\int_X |\nabla_F f_{i,y}| \, d\gamma \leq |\pi_F(D_\gamma \chi_E)|(X)$$

and using (19) we conclude that (44) holds.

Condition (43) now follows by the $L^1(Y, \gamma_Y)$ convergence of $\int_F |\nabla_F f_{i,y}| \, d\gamma_F$ to $|D_{\gamma_F} \chi_{E_y}|(F)$; in turn, applying a convergence criterion (see for instance [2, Exercise 1.19]) this follows by the lim inf inequality

$$\liminf_{i \to \infty} \int_F |\nabla_F f_{i,y}| \, d\gamma_F \geq |D_{\gamma_F} \chi_{E_y}|(F) \qquad \text{for } \gamma_Y\text{-a.e. } y \in Y.$$

(a consequence of the lower semicontinuity of total variation) together with convergence of the L^1 norms ensured by (44).

Now, we fix y such that all functions $f_{i,y}$ are continuous and both conditions

$$\lim_{i \to \infty} \int_F |f_{i,y} - \chi_{E_y}| \, d\gamma_F = 0, \qquad \lim_{j \to \infty} \int_F |\nabla_F f_{i(j),y}| \, d\gamma_F = |D_{\gamma_F} \chi_{E_y}|(F)$$

hold and apply Proposition 3.7 to obtain that the y section of $E^{1/2}$, contained in

$$\left\{ z \in F : \lim_{j \to \infty} f_{i(j),y}(z) = \frac{1}{2} \right\}$$

is also contained, up to $\mathscr{S}^{m-1}$-negligible sets, in $\partial^* E_y$. Since Proposition 2.2 and (43) ensure that the set of exceptional y's is γ_Y-negligible, the definition of $\mathscr{S}^{\infty-1}_F$ yields (42). □

Having achieved (42) we can now prove Theorem 1.3. To this aim, we fix a nondecreasing family $\mathcal{F} = \{F_1, F_2, \ldots\}$ of finite-dimensional subspaces of $\tilde{H}$ whose union is dense in H and, using (42) in conjunction with (25), for $n \leq m$ we get

$$\mathscr{S}^{\infty-1}_{F_n}(E^{1/2} \setminus \bigcap_{i=m}^{\infty} \partial^*_{F_i} E) = 0.$$

Letting $m \to \infty$ it follows that $\mathscr{S}^{\infty-1}_{F_n}(E^{1/2} \setminus \partial^*_{\mathcal{F}} E) = 0$, and since n is arbitrary this proves that

$$\mathscr{S}^{\infty-1}_{\mathcal{F}}(E^{1/2} \setminus \partial^*_{\mathcal{F}} E) = 0. \tag{45}$$

Now, we know that $|D_\gamma \chi_E| = \mathscr{S}^{\infty-1}_{\mathcal{F}} \llcorner \partial^*_{\mathcal{F}} E$, hence evaluating both measures on $\partial^*_{\mathcal{F}} E \setminus E^{1/2}$ and using the fact that $|D_\gamma \chi_E|$ is concentrated on $E^{1/2}$ we get

$$\mathscr{S}^{\infty-1}_{\mathcal{F}}(\partial^*_{\mathcal{F}} E \setminus E^{1/2}) = 0. \tag{46}$$

The combination of (45) and (46) gives

$$|D_\gamma \chi_E| = \mathscr{S}^{\infty-1}_{\mathcal{F}} \llcorner E^{1/2}.$$

But, since $\mathcal{F}$ is arbitrary, this yields that $E^{1/2}$ has finite $\mathscr{S}^{\infty-1}$-measure and (5), concluding the proof.

6 Derivative of the union of disjoint sets

In this section we prove Corollary 1.4. Let us remark that, although the result is standard in finite dimensions and could be proved in different ways (for instance, using De Giorgi's rectifiability theorem), the argument below is very elementary. Although the proof is more or less the same as the one in [13, Lemma 2.2] (where the authors are dealing with the classical notion of perimeter in $\mathbb{R}^m$), we believe it is worth to repeat the argument for reader's convenience, and for underlying the importance of the fact that in our representation formula (5) the measure $\mathscr{S}^{\infty-1}$ is universal.

Proof of Corollary 1.4. The fact that $E \cup F$ has finite perimeter follows immediately from the definition.

Since the sets $(E \cup F)^{1/2}$, $E^{1/2}$, $F^{1/2}$ are $\mathscr{S}^{\infty-1}$-uniquely determined, we can assume that they all have been defined using the same sequence (t_i).

As $\gamma(E \cap F) = 0$ we have $\chi_{E\cup F} = \chi_E + \chi_F$, so that by (5)

$$\begin{aligned} \nu_{E\cup F} \mathscr{S}^{\infty-1} \llcorner (E \cup F)^{1/2} &= D_\gamma \chi_{E\cup F} = D_\gamma \chi_E + D_\gamma \chi_F \\ &= \nu_E \mathscr{S}^{\infty-1} \llcorner E^{1/2} + \nu_F \mathscr{S}^{\infty-1} \llcorner F^{1/2}. \end{aligned} \tag{47}$$

Since $E^{1/2} \cap F^{1/2} \subset \{x \in X : \lim_{i\to\infty} T_{t_i} \chi_{E\cup F}(x) = 1\}$ we have

$$(E \cup F)^{1/2} \cap E^{1/2} \cap F^{1/2} = \emptyset, \tag{48}$$

so (6) follows from (47). Moreover, by (47) and (48), for every Borel set $C \subseteq E^{1/2} \cap F^{1/2}$ we have

$$\int_C \nu_E + \nu_F \, d\mathscr{S}^{\infty-1} = \int_{C\cap(E\cup F)^{1/2}} \nu_{E\cup F} \, d\mathscr{S}^{\infty-1} = 0,$$

which implies that $\nu_E = -\nu_F$ at $\mathscr{S}^{\infty-1}$-a.e. point in $E^{1/2} \cap F^{1/2}$, as desired. ∎

7 Appendix: The reduced boundary

The classical finite-dimensional definition of reduced boundary [9] is based on the requirements of existence of the limit

$$\nu_E(x) := \lim_{r\downarrow 0} \frac{D\chi_E(B_r(x))}{|D\chi_E|(B_r(x))} \tag{49}$$

and modulus of the limit $\nu_E(x)$ equal to 1. It is not hard to show that points in the reduced boundary are Lebesgue points for the vector field ν_E, relative to $|D\chi_E|$, hence the proof that $|D\chi_E|$-almost every point x is in the reduced boundary is based on Besicovitch covering theorem, a result not available in infinite dimensions.

In [5, Definition 7.2], the authors proposed the following definition of reduced boundary based on the OU semigroup:

Definition 7.1 (Gaussian Reduced Boundary)**.** *Let E be a Borel set of finite perimeter in (X,γ). We denote by $\mathcal{F}E$ the set of points $x \in X$ where the limit*

$$\nu_E(x) := \lim_{t\downarrow 0} T_t\left(\frac{T_t^* D_\gamma\chi_E}{T_t^*|D_\gamma\chi_E|}\right)(x) \tag{50}$$

exists and satisfies $|\nu_E(x)| = 1$.

As observed in [5, Section 7], a natural open problem is to prove that $|D_\gamma\chi_E|$ is concentrated on $\mathcal{F}E$. Here, we show how the soft argument used in the proof of Proposition 4.3 allows to prove easily the weaker result

$$\lim_{t\downarrow 0} T_t h_t = 1 \quad \text{in } L^1(X, |D_\gamma\chi_E|) \qquad \text{with} \qquad h_t := \frac{|T_t^* D_\gamma\chi_E|}{T_t^*|D_\gamma\chi_E|}. \tag{51}$$

In particular, we deduce that along any subsequence $(t_i) \downarrow 0$ such that

$$\sum_i \int_X |T_{t_i} h_{t_i} - 1|\, d|D_\gamma\chi_E| < \infty$$

it holds

$$\lim_{i\to\infty} T_{t_i}\left(\frac{|T_{t_i}^* D_\gamma\chi_E|}{T_{t_i}^*|D_\gamma\chi_E|}\right)(x) = 1 \qquad \text{for } |D_\gamma\chi_E|\text{-a.e. } x \in X.$$

Proof of (51). Set $f_t := T_t\chi_E$. Arguing as in the proof of Proposition 4.3, the product rule (14) yields

$$|D_\gamma(f_t\chi_E)|(X) \leq \int_X f_t\, d|D_\gamma\chi_E| + \int_X h_t\chi_E\, dT_t^*|D_\gamma\chi_E|.$$

Replacing E by $X \setminus E$ and and f_t by $1 - f_t$, we also have

$$|D_\gamma((1-f_t)\chi_{X\setminus E})|(X) \leq \int_X (1-f_t)\, d|D_\gamma \chi_E| + \int_X h_t(1-\chi_E)\, dT_t^*|D_\gamma \chi_E|.$$

Adding together the two inequalities above, we obtain

$$\begin{aligned}|D_\gamma(f_t\chi_E)|(X) + |D_\gamma((1-f_t)\chi_{X\setminus E})|(X) &\leq |D_\gamma \chi_E|(X) + \int_X h_t\, dT_t^*|D_\gamma \chi_E| \\ &= |D_\gamma \chi_E|(X) + \int_X T_t h_t\, d|D_\gamma \chi_E|.\end{aligned}$$

By lower semicontinuity of the total variation (see Proposition 2.5), letting $t \downarrow 0$ we get

$$\begin{aligned}2|D_\gamma \chi_E|(X) &\leq \liminf_{t\downarrow 0}\Big(|D_\gamma(f_t\chi_E)|(X) + |D_\gamma((1-f_t)\chi_{X\setminus E})|(X)\Big) \\ &\leq |D_\gamma \chi_E|(X) + \liminf_{t\downarrow 0}\int_X T_t h_t\, d|D_\gamma \chi_E|,\end{aligned}$$

so that

$$|D_\gamma \chi_E|(X) \leq \liminf_{t\downarrow 0}\int_X T_t h_t\, d|D_\gamma \chi_E|.$$

This, combined with the fact that $0 \leq T_t h_t \leq 1$ (as $0 \leq h_t \leq 1$) proves that

$$\int_X |T_t h_t - 1|\, d|D_\gamma \chi_E| = \int_X (1 - T_t h_t)\, d|D_\gamma \chi_E| \to 0 \qquad \text{as } t \downarrow 0,$$

as desired. □

References

[1] H. Airault and P. Malliavin, *Intégration géométrique sur l'espace de Wiener*, Bull. des Sciences Math., **112** (1988), 25–74.

[2] L. Ambrosio, N. Fusco and D. Pallara, "Functions of bounded variation and free discontinuity problems", Oxford Mathematical Monographs, 2000.

[3] L. Ambrosio, S. Maniglia, M. Miranda Jr. and D. Pallara, *BV functions in abstract Wiener spaces*, J. Funct. Anal., **258** (2010), 785–813.

[4] L. Ambrosio, G. Da Prato and D. Pallara, *BV functions in a Hilbert space with respect to a Gaussian measure*, Preprint, (2009), http://cvgmt.sns.it/cgi/get.cgi/papers/ambdappal/

[5] L. Ambrosio, M. Miranda and D. Pallara, *Sets with finite perimeter in Wiener spaces, perimeter measure and boundary rectifiability,* Discrete Contin. Dyn. Syst. Series A, **28** (2010), 591–606.

[6] A.P. Besicovitch, *On the existence of subsets of finite measure of sets of infinite measure*, Indag. Math., **14** (1952), 339–344.

[7] V.I. Bogachev, "Gaussian Measures", American Mathematical Society, 1998.

[8] E. De Giorgi, *Definizione ed espressione analitica del perimetro di un insieme*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Nat., **8** (1953), 390–393.

[9] E. De Giorgi, *Su una teoria generale della misura* $(r-1)$*-dimensionale in uno spazio ad* r *dimensioni*, Ann. Mat. Pura Appl., **4** (1954), 191–213.

[10] H. Federer, *A note on the Gauss–Green theorem*, Proc. Amer. Math. Soc., **9** (1958), 447–451.

[11] H. Federer, "Geometric measure theory", Springer, 1969.

[12] D. Feyel and A. De la Pradelle, *Hausdorff measures on the Wiener space*, Potential Anal., **1** (1992), 177–189.

[13] A. Figalli, F. Maggi and A. Pratelli, *A mass transportation approach to quantitative isoperimetric inequalities*, Invent. Math., **182** (2010), 167–211.

[14] M. Fukushima, *On semimartingale characterization of functionals of symmetric Markov processes*, Electron J. Probab., **4** (1999), 1–32.

[15] M. Fukushima, *BV functions and distorted Ornstein-Uhlenbeck processes over the abstract Wiener space*, J. Funct. Anal., **174** (2000), 227–249.

[16] M. Fukushima and M. Hino, *On the space of BV functions and a related stochastic calculus in infinite dimensions*, J. Funct. Anal., **183** (2001), 245–268.

[17] M. Hino, *Sets of finite perimeter and the Hausdorff–Gauss measure on the Wiener space*, J. Funct. Anal., **258** (2010), 1656–1681.

[18] M. Ledoux, *Isoperimetry and Gaussian analysis*, in "Lectures on Probability Theory and Statistics", Saint Flour, 1994 Lecture Notes in Mathematics, 1648, Springer, 1996.

[19] M. Ledoux, *Semigroup proof of the isoperimetric inequaliy in Euclidean and Gaussian spaces*, Bull. Sci. Math., **118** (1994), 485–510.

[20] D. Preiss, *Gaussian measures and the density theorem*, Comment. Math. Univ. Carolin., **22** (1981), 181–193.

[21] E.M. Stein, "Topics in Harmonic Analysis related to the Littlewood-Paley theory", Annals of Mathematics Studies **63**, Princeton University Press, 1970.

[22] L. Zambotti, *Integration by parts formulae on convex sets of paths and applications to SPDEs with reflection*, Probab. Theory Relat. Fields, **123** (2002), 579–600.

[23] W.P. Ziemer, "Weakly differentiable functions", Graduate Texts in Mathematics, Springer, 1989.