TOWARDS DIFFERENTIAL CALCULUS IN STRATIFIED GROUPS

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Contents

1. Introduction

The relationship between the geometry of stratified groups and several branches of Mathematics, as PDEs, Differential Geometry, Complex Analysis, Control Theory and Geometric Measure Theory has known an increasing interest in the last decade.

The initial motivation of the present work was the study of an implicit function theorem between stratified groups with respect to the corresponding notion differentiability. Then we realized that to formulate and prove this theorem a series of basic results were needed. These ones naturally lead us to the development of an Intrinsic Differential Calculus for group-valued mappings defined on stratified groups.

This project belongs to the general program of developing Analysis in metric spaces, as explained in several books [2], [6], [16], [27], [29], [61]. Other monographs are focused on Lie groups and sub-Riemannian manifolds, [5], [10], [18], [26], [49], [64]. Calculus and Analysis in stratified groups have been developed by several authors with a vast literature, here we mention a list of references for a rough overview that is certainly far from being complete, [1], [3], [4], [7], [8], [9], [11], [12], [13], [14], [15], [20], [21], [22], [24], [28], [33], [35], [36], [42], [44], [43], [46], [47], [48], [52], [56].

In the present work the notion of differentiability plays a central role and essentially enters every result. This notion in the setting of stratified groups has been introduced by P. Pansu to obtain a rigidity theorem for quasi-isometries of the quaternionic hyperbolic space, where he proved a Rademacher type theorem for Lipschitz mappings with respect to his notion of differentiability, [51].

Our *intrinsically C^1-smooth Calculus* starts from the following

Theorem 1.1 (Characterization of continuously P-differentiable mappings)**.** *Let $\Omega \subset \mathbb{G}$ be an open set and let $f : \Omega \longrightarrow \mathbb{M}$. Then the following statements are equivalent*

- *f is continuously P-differentiable,*
- *letting $f(x) = \exp \circ F(x) = \exp \sum_{j=1}^{\upsilon} F_j(x)$, the mappings $F_j : \Omega \longrightarrow W_j$, equal to $\pi_j \circ F$, are continuously P-differentiable and the formulae*

$$\pi_1 \circ df(x) = dF_1(x) \tag{1}$$

$$dF_j(x)(h) = \sum_{n=2}^{\upsilon} \frac{(-1)^n}{n!}\, \pi_j\left([F(x), dF(x)(h)]_{n-1}\right) . \tag{2}$$

  hold for every $j = 2, \ldots, \upsilon$ and every $h \in \mathbb{G}$.
- *f is continuously h-differentiable contact.*

The integer υ denotes the step of $\mathbb{M}$ and the symbol d denotes the P-differential read between Lie algebras.

We have denoted by $\mathbb{G}$ and $\mathbb{M}$ a stratified group and a graded group, respectively. The corresponding Lie algebras $\mathcal{G}$ of $\mathbb{G}$ and $\mathcal{M}$ of $\mathbb{M}$ are decomposed into the direct sums of layers V_i and W_i, respectively. The mappings π_j indicate the canonical projection onto the j-th layer of a graded algebra.

Notice that h-differentiability, introduced in Definition 2.4, is a much weaker notion than P-differentiability. According to Theorem 1.1, we have that continuous h-differentiability and contact property given by (2) yield continuous P-differentiability. The main point of Theorem 1.1 is that it allows for the study of P-differentiable mappings regardless of the notion of P-differentiability and using instead the following system of first order nonlinear PDEs

$$\left\{\begin{array}{l} X_1F_j = \sum_{n=2}^{\upsilon} \frac{(-1)^n}{n!}\, \pi_j\left([F, X_1F]_{n-1}\right) \\ X_2F_j = \sum_{n=2}^{\upsilon} \frac{(-1)^n}{n!}\, \pi_j\left([F, X_2F]_{n-1}\right) \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ X_mF_j = \sum_{n=2}^{\upsilon} \frac{(-1)^n}{n!}\, \pi_j\left([F, X_mF]_{n-1}\right) \end{array}\right. \tag{3}$$

for every $j = 2, \ldots, \upsilon$, where $(X_1, \ldots, X_m)$ is a basis of the first layer of $\mathcal{G}$. In fact, formulae (2) recursively define the P-differential dF_j for every $j = 2, \ldots, \upsilon$, starting from dF_1 and in particular yield the system (3). These $m \times (\upsilon - 1)$ equations exactly characterize the contact property of f and can be used to study existence of Lipschitz extensions for mappings between stratified groups, [45]. Another feature of these equations is that they allow to transmit the regularity of F_1 to the remaining components F_j of higher order. The same phenomenon occurs in the "algebraic regularity" proved in Theorem 6.1 of [9].

The crucial point in the proof of Theorem 1.1 is the quantitative estimate (52) for the P-difference quotient of a horizontal curve, that is obtained in Theorem 4.4. This estimate plays a key role also in the proof of the mean value inequality and it allows for the extension of Pansu result, [51], on a.e. P-differentiability of Lipschitz mappings to the case when the target is a graded group, see Corollary 4.5. On the other hand, this key estimate is obtained in a rather "simple" way, once the technical Lemma 3.3 is adopted, and it boils down to integrate recursively the differential equations

$$\dot\gamma_i(t) = \sum_{n=2}^{\upsilon} \frac{(-1)^n}{n!}\, \pi_i\left([\gamma(t), \dot\gamma(t)]_{n-1}\right), \qquad i = 2, \ldots, \upsilon, \tag{4}$$

that must be satisfied by every horizontal curve $\Gamma = \exp\gamma \in \mathbb{M}$, where $\gamma_i = \pi_i \circ \gamma$ and $i = 2, \ldots, \upsilon$, according to Proposition 4.1. Lemma 3.3 is one of the major technical results of this paper and will appear several times throughout the present work. It corresponds to the linearization of the homogeneous addends c_n appearing in the Baker-Campbell-Hausdorff formula (13).

If one assumes apriori C^1 smoothness of f, then the characterization of contact mappings through P-differentiabiltiy is already shown in [65]. The intriguing feature of Theorem 1.1, along with its technical difficulty, stems from the fact that we do not assume any "extrinsic" regularity of f, making this characterization sharp. Continuously h-differentiable mappings are continuous, since their components are locally Lipschitz with respect to the homogeneous distance of the domain. Furthermore, adding the contact property to the continuous h-differentiability, then Theorem 1.1 and Corollary 5.8 yield the local Lipschitz property of f with respect to homogeneous distances. However, let us point out that one can find continuously h-differentiable contact mappings that are not differentiable on sets of positive measure with respect to the Euclidean notion of differentiability, [41].

Our next result is the mean value inequality for P-differentiable mappings, that represents a crucial tool in the proof of our implicit function theorem. Even in this case, the proof relies on a preliminary study of horizontal curves. In Corollary 5.4 we characterize Lipschitz curves in graded groups with respect to a homogeneous distance as Lipschitz curves in the Euclidean sense that also satisfy the differential equations (4). In the case of stratified groups, where the Lie bracket condition is satisfied, this is a well known fact that can be found in Section 11.1 of [27] for the more general Carnot-Carathéodory spaces.

Theorem 1.2 (Mean Value Inequality)**.** *Let $\Omega\subset\mathbb{G}$ be an open subset and consider a continuously P-differentiable mapping $f:\Omega\longrightarrow\mathbb{M}$. Let $\Omega_1,\Omega_2\subset\mathbb{G}$ be open subsets such that*

$$\Big\{x\in\mathbb{G}\,\Big|\,d(x,\Omega_1)\le c(\mathbb{G},d)\,N\,\mathrm{diam}(\Omega_1)\Big\}\subset\Omega_2\,, \tag{5}$$

where Ω_2 is compactly contained in Ω. Here $c(\mathbb{G},d)$ and N are geometric constants only depending on the metric space $(\mathbb{G},d)$, see both Lemma 4.6 and Definition 4.4. Then there exist a constant C, only depending on $\mathbb{G}$, $\max_{x\in\overline{\Omega}_2}\|dF_1(x)\|$ and on the modulus of continuity $\omega_{\overline{\Omega}_2,dF_1}$ of $x\longrightarrow df(x)$, defined in (60), such that

$$\frac{\rho\Big(f(x)^{-1}f(y),Df(x)(x^{-1}y)\Big)}{d(x,y)}\le C\ \ \big[\omega_{\overline{\Omega}_2,df}\big(N\,c(\mathbb{G},d)\,d(x,y)\big)\big]^{1/\iota^2}\,. \tag{6}$$

for every $x,y\in\overline{\Omega}_1$, with $x\neq y$. The integer ι denotes the step of $\mathbb{G}$.

Several difficulties are hidden in this estimate. First, P-differentiable mappings in general are not C^1 smooth in the classical sense. Second, working by single components does not suffice, since this would lead to estimates on the Euclidean norm of the difference quotient, that does not fit into the notion of P-differentiability. Third, the family of horizontal curves along which one integrates the P-differential is not manageable. In fact, condition (5) has only the technical motivation to make sure that the special family of piecewise horizontal lines that connect points of Ω_1 and along which we apply (52) are all contained in Ω_2.

An immediate corollary of the mean value inequality is a short proof of the inverse mapping theorem in stratified groups. On the other hand, the difficulty in applying Theorem 1.2 to prove an intrinsic implicit function theorem arises from the algebraic problems related to a proper factorization of the group $\mathbb{G}$. In the commutative case, that is the classical viewpoint, the implicit mapping is defined by decomposing the space into a product of two linear subspaces, naturally given by the tangent space to the level set and one of its complementary subspaces, as for instance the orthogonal subspace, if a scalar product is fixed. Extending this argument to stratified groups does not work when we seek the second subspace. In fact, the first one is automatically defined as the kernel of the P-differential of the defining mapping, that is also a normal subgroup, but in general we cannot claim a group structure in the "complementary subspace". For this reason, according to our terminology, we consider the special class of *h-epimorphisms*, as the "regular surjective h-homomorphisms", that yield the natural splitting of $\mathbb{G}$, when they are the P-differential of the defining mapping. Notice that this splitting of the group is a necessary condition in order to state an implicit function theorem. In fact, the existence of the complementary subgroup H in Theorem 1.3 is a consequence of Proposition 7.14, since $Df(\overline{x})$ is an h-epimorphism. To emphasize this special class of surjective h-homomorphisms, in Example 2.4, we provide a surjective h-homomorphism that is not an h-epimorphism.

Notice that not all Lie subgroups are considered, but only the special class of homogeneous subgroups, that are closed under dilations. This turns out to be rather natural, in the perspective to interpret a subgroup as blow-up of a suitable intrinsically regular set, according to Theorem 1.6. Using the terminology of Section 7, two homogeneous subgroups P and H are *complementary* if they satisfy the conditions $\mathbb{G} = PH$ and $P \cap H = \{e\}$.

Now we are in the position to establish one of the central results of this paper.

Theorem 1.3 (Implicit Function Theorem)**.** *Let $\Omega \subset \mathbb{G}$ be an open set and let $f : \Omega \longrightarrow \mathbb{M}$ be continuously P-differentiable, where $\overline{x} \in \Omega$ and the P-differential $Df(\overline{x}) : \mathbb{G} \longrightarrow \mathbb{M}$ is an h-epimorphism. Let N be the kernel of $Df(\overline{x})$ and let H be a complementary subgroup. Then there exist $r, s > 0$, with $D^N_{\overline{n},r} D^H_{\overline{h},s} \subset \Omega$, along with a unique mapping $\varphi : D^N_{\overline{n},r} \longrightarrow D^H_{\overline{h},s}$, such that*

$$f^{-1}\big(f(\overline{x})\big) \cap D^N_{\overline{n},r} D^H_{\overline{h},s} = \{n\varphi(n) \mid n \in D^N_{\overline{n},r}\}. \tag{7}$$

Furthermore, there exists a constant $\kappa > 0$ such that the Lipschitz-type estimate

$$d\big(\varphi(n), \varphi(n')\big) \leq \kappa\, d\big(\varphi(n')^{-1} n^{-1} n' \varphi(n')\big) \tag{8}$$

holds. In particular, the mapping φ is $1/\iota$-Hölder continuous with respect to the metrics d in $D^H_{\overline{h},r}$ and $\|\cdot\|$ in $D^N_{\overline{n},s}$.

The induced splitting of $\mathbb{G}$ is precisely an inner semidirect product, that is in general not direct. This should explain why the classical contraction-mapping principle here seems to be not applicable. More precisely, to use this argument, in the case $\overline{x}$ equals the unit element $e \in \mathbb{G}$, one should consider the mapping $F(h) = L(h)^{-1} f(nh)$, where L is the restriction of the P-differential $Df(e)$ to the complementary subgroup H. In the classical case, one shows that the mapping F is a contraction for every n sufficiently close to e. In stratified groups, using the notion of P-differentiability, the analogous argument does not work, due to noncommutativity. To overcome this point, we show that the mapping $F_n(h) = f(nh)$ is uniformly biLipschitz with respect to n and has constant nonvanishing topological degree as n varies in a compact neighbourhood of the unit element of N. This gives existence and uniqueness of the implicit mapping. We also wish to point out how inequality (8) surprisingly fits into the intrinsic notion of Lipschitz mapping given in Definition 3.1 of [23], when the ambient space is an Heisenberg group.

It is now natural to investigate the natural counterpart of the implicit function theorem, namely, the rank theorem. To translate this theorem in terms of continuously P-differentiable mappings, we first single out the class injective h-homomorphisms that provide the splitting of the target group, according to Proposition 7.15.

Theorem 1.4 (Rank Theorem). *Let $f : \Omega \longrightarrow \mathbb{M}$ be a continuously P-differentiable mapping, where Ω is an open subset of $\mathbb{G}$. Let $\overline{x} \in \Omega$ and let $Df(\overline{x}) : \mathbb{G} \longrightarrow \mathbb{M}$ be the P-differential. Let us assume that $Df(\overline{x})$ is an h-monomorphism of image H and let N be a normal complementary subgroup. Let $p : \mathbb{M} \longrightarrow H$ be the associated canonical projection. Then there exist neighbourhoods $V \subset \Omega$ of $\overline{x}$ and $W \subset H$ of $p(f(\overline{x}))$ along with mappings $\varphi : W \longrightarrow N$, $J : H \longrightarrow \mathbb{G}$ and $\Psi : f(V) \longrightarrow \mathbb{M}$ such that we have*

$$f(V) = \{h\varphi(h) \mid h \in W\} \quad \text{and} \quad \Psi \circ f \circ J_{|V'} = I_{|V'}, \tag{9}$$

where $I : H \hookrightarrow \mathbb{M}$ is the restriction of the identity mapping $Id_{\mathbb{M}}$ to $H \subset \mathbb{M}$, the open subset $V' \subset H$ is equal to $J^{-1}(V)$ and $J = \big(p \circ Df(\overline{x})\big)^{-1}$. Furthermore, setting $F = \exp^{-1} \circ \varphi$, then there exists $C > 0$ such that for every $h, h' \in W$ we have

$$\|F(h) - F(h')\| \leq C\, d(h, h')\,. \tag{10}$$

The proof of this result uses completely different tools with respect to the implicit function theorem. Here the key observation is that the projection $p : \mathbb{M} \longrightarrow H$ is an h-epimorphism, hence it is Lipschitz. Then we apply our inverse mapping theorem.

Implicit function theorem and rank theorem are the standard tools to define differentiable manifolds. Analogously, in an obvious way Theorem 1.3 and Theorem 1.4 define subsets that are expected to possess some intrinsic regularity. These are the $(\mathbb{G}, \mathbb{M})$-regular sets, that we distinguish into those contained in $\mathbb{G}$, that are suitable level sets and those of $\mathbb{M}$, that are suitable images, see Section 10 for more details. By estimate (114), that immediately follows from (8), $(\mathbb{G}, \mathbb{M})$-regular sets of $\mathbb{G}$ can be locally parametrized by $1/\iota$-Hölder mappings. In the case of $(\mathbb{H}^1, \mathbb{R})$-regular sets of $\mathbb{H}^1$, hence $\iota = 2$, B. Kirchheim and F. Serra Cassano have proved that the embedding Hölder exponent $1/2$ cannot be improved in general. In the same work [33], the authors provide through a nontrivial construction an example of $(\mathbb{H}^1, \mathbb{R})$-regular set of $\mathbb{H}^1$ of Euclidean Hausdorff dimension $5/2$, that is clearly not rectifiable in the Federer sense, [17]. Understanding the intrinsic regularity of these sets is certainly hard and very far from being completely understood. For instance, a fine characterization of $(\mathbb{H}^n, \mathbb{R})$-regular sets as suitable 1-codimensional intrinsic graphs has been recently established in [1].

As an immediate consequence of both rank theorem and implicit function theorem, we have the following

Corollary 1.5. *Every $(\mathbb{G}, \mathbb{M})$-regular set is locally an intrinsic graph.*

Intrinsic graphs and $(\mathbb{G}, \mathbb{M})$-regular sets of $\mathbb{G}$ first appeared in the works of Franchi, Serapioni and Serra Cassano, [19], [20] in the case $\mathbb{M} = \mathbb{R}$ and more recently in Heisenberg groups with $\mathbb{M} = \mathbb{R}^k$, [22]. In general stratified groups, these sets have been considered in [42], where the problem of studying their metric and topological properties has been raised. For instance, Corollary 1.5 shows that the topological codimension of $(\mathbb{G},\mathbb{M})$-regular sets corresponds to the topological dimension of $\mathbb{M}$.

A detailed study of $(\mathbb{H}^n,\mathbb{R}^k)$-regular sets, $1\le k\le n$, has been accomplished in [22], where a corresponding implicit function theorem and an area-type formula have been established. In the same work, following the Rumin complex [58], the authors lay the foundations for an intrinsic theory of currents, where "rectifiable" sets correspond to both $(\mathbb{H}^n,\mathbb{R}^k)$-regular sets and $(\mathbb{R}^k,\mathbb{H}^n)$-regular sets. In the terminology of [22], these two class of sets correspond to *low codimensional* $\mathbb{H}$*-regular surfaces* and *low dimensional* $\mathbb{H}$*-regular surfaces*, respectively. As an example of another geometry to be considered, Theorem 12.6 provides the complete list of all possible intrinsically regular sets of the 6-dimensional complexified Heisenberg group $\mathbb{H}^1_2$. To obtain this classification we need both algebraic and analytical tools. In fact, we first find all possible factorizations of $\mathbb{H}^1_2$ into an inner semidirect product of two complementary subgroups, then we apply our implicit function theorem joined with Theorem 12.1.

Low dimensional $\mathbb{H}$-regular surfaces of $\mathbb{H}^n$ are nontrivial examples of $(\mathbb{G},\mathbb{M})$-regular sets of $\mathbb{M}$, where $\mathbb{G}=\mathbb{R}^k$ and $\mathbb{M}=\mathbb{H}^n$. More generally, Theorem 12.1 characterizes $(\mathbb{R}^k,\mathbb{M})$-submanifolds as k-dimensional C^1-Legendrian submanifolds of $\mathbb{M}$. This theorem is a straightforward consequence of Theorem 1.1. As another consequence of Theorem 12.1, Legendrian submanifolds are intrinsic graphs. If we consider general couples of stratified groups, then $(\mathbb{G},\mathbb{M})$-regular sets of $\mathbb{M}$ can be thought of as a smooth version of "N-rectifiable sets" studied in [52], where $\mathbb{G}=N$.

From the metric viewpoint, estimate (10) shows a stronger regularity of image sets with respect to level sets, where the "nonlinear estimate" (8) holds. However, in the direction of intrinsic regularity we have the following

Theorem 1.6 (Intrinsic Blow-up)**.** *Under hypotheses of Theorem 1.3, we consider the set* $S=f^{-1}\big(f(\overline{x})\big)$. *Then for every* $R>0$ *we have*

$$D_R\cap\delta_{1/\lambda}\big((\overline{x})^{-1}S\big)\longrightarrow D_R\cap N\quad as\quad \lambda\to 0^+$$

with respect to the Hausdorff convergence of sets. In particular, $\operatorname{Tan}(S,\overline{x})=N$.

Under hypotheses of Theorem 1.4, setting $S=f(V)$, *for every* $R>0$, *we have*

$$D_R\cap\delta_{1/\lambda}\big((\overline{x})^{-1}S\big)\longrightarrow D_R\cap H\quad as\quad \lambda\to 0^+$$

with respect to the Hausdorff convergence of sets. In particular, $\operatorname{Tan}(S,\overline{x})=H$.

As a consequence of this theorem, we notice another strong difference between level sets and image sets in general. In fact, all homogeneous tangent cones to a $(\mathbb{G},\mathbb{M})$-regular set of $\mathbb{M}$ are clearly h-isomophic to $\mathbb{M}$ and in particular have all the same Hausdorff dimension. Furthermore, by the area formula of [39], their Hausdorff dimension coincides with that of $\mathbb{G}$ and we have an integral formula for their Hausdorff measure. On the other hand, homogeneous tangent cones to a $(\mathbb{G},\mathbb{M})$-regular set of $\mathbb{G}$ are not necessarily h-isomorphic to each other, as Example 10.4 shows. By the way, Corollary 10.3 shows that all of the homogeneous tangent cones to a $(\mathbb{G},\mathbb{M})$-regular set of $\mathbb{G}$ have the same Hausdorff dimension and this one equals to

$$\mathcal{H}\text{-dim}(\mathbb{G}) - \mathcal{H}\text{-dim}(\mathbb{M}).$$

This suggests that the Hausdorff dimension of a $(\mathbb{G},\mathbb{M})$-regular set of $\mathbb{G}$ should coincide with that of its homogeneous tangent cones, but this problem still claims to be investigated.

From the previous results, it is clear that the richness of $(\mathbb{G},\mathbb{M})$-regular sets is connected to the richness of h-homomorphisms between $\mathbb{G}$ and $\mathbb{M}$ along with their factorizing properties. To advertize novel applications of both the rank theorem and the implicit function theorem, in Section 11 we consider some special couples of groups where all injective h-homomorphisms are h-monomorphisms and all surjective h-homomorphisms are h-epimorphisms. We believe that possibly many more "geometries" can be discovered through a deeper algebraic investigation.

Finally, we give a concise overview of the paper. In Section 2, we recall the main definitions that will be used throughout. Section 3 develops the technical machinery of the paper, that plays a key role in the proofs of our main results. In Section 4, we establish quantitative estimates on the P-difference quotient of horizontal curves and we apply them to characterize continuously P-differentiable mappings. In Section 5 we characterize absolutely continuous curves with respect to a homogeneous distance and through this characterization we prove the Lipschitz property of continuously P-differentiable mappings with values in a graded group. Section 6 contains the proof of the mean value inequality for continuously P-differentiable mappings. As an immediate application, in Subsection 6.1 we give a proof of the inverse mapping theorem in stratified groups. Section 7 recalls the notion of homogeneous subgroup, introduces complementary subgroups and gives simple characterizations of both h-epimorphisms and h-monomorphisms. In Section 8 we show that the quotient of a graded group by a normal homogeneous subgroup is still a graded group and the analogous statement holds for stratified groups. Section 9 is devoted to the proof of both the implicit function theorem and the rank theorem. In Section 10 we prove the everywhere existence of the homogeneous tangent cone to $(\mathbb{G},\mathbb{M})$-regular sets and we show by an example that they might not be h-isomorphic to each other in the case of $(\mathbb{G},\mathbb{M})$-regular sets of $\mathbb{G}$. In Section 11 we give the notions of h-quotients and of h-embeddings. We introduce the notion of factorizing group as a quotient and factorizing group as a subgroup, providing corresponding examples. In Section 12, we characterize all C^1 smooth Legendrian submanifolds in graded groups as $(\mathbb{R}^k,\mathbb{M})$-regular sets and we find all intrinsically regular sets of Heisenberg groups and of the complexified Heisenberg group.

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2. Preliminaries and definitions

All Lie groups we consider in this paper are real, connected, simply connected and finite dimensional. A *graded group* is a Lie group $\mathbb{G}$, whose Lie algebra $\mathcal{G}$ can be written as the direct sum of subspaces V_i, called layers, such that

$$[V_i, V_j] \subset V_{i+j} \tag{11}$$

and $\mathcal{G} = V_1 \oplus \cdots \oplus V_\iota$. The integer ι is the step of nilpotence of $\mathbb{G}$, [18]. A graded group $\mathbb{G}$ is *stratified* if its layers satisfy the stronger condition $[V_i, V_j] = V_{i+j}$.

The grading of $\mathcal{G}$ allows us to introduce a one-parameter group of Lie algebra automorphisms $\delta_r : \mathcal{G} \longrightarrow \mathcal{G}$, defined as $\delta_r(X) = r^i$ if $X \in V_i$, where $r > 0$. These mappings are called *dilations*. Taking into account that the exponential mapping $\exp : \mathcal{G} \longrightarrow \mathbb{G}$ is a diffeomorphism for simply connected nilpotent Lie groups, we can read dilations in the group $\mathbb{G}$ through the mapping exp and mantain the same notation. Recall from Theorem 2.14.3 of [63] that the differential of the exponential mapping is given by the followig formula

$$d\exp(X) = \mathrm{Id} - \sum_{n=2}^{\infty} \frac{(-1)^n}{n!}\, \mathrm{ad}(X)^{n-1}\,, \tag{12}$$

Example 2.1. Let $\mathbb{H}^1$ denote the Heisenberg group and let $\mathfrak{h}^1 = \mathrm{span}\{X, Y, Z\}$ be its Lie algebra with $[X, Y] = Z$. Then the "vertical" subgroup $\exp\big(\mathrm{span}\{X, Z\}\big)$ is a graded group, but it is not stratified.

The group operation can be read in the algebra as follows

$$X \circledcirc Y = \sum_{j=1}^{\iota} c_n(X, Y) \tag{13}$$

where $c_1(X, Y) = X + Y$ and the addends c_n are given by induction through the Baker-Campbell-Hausdorff formula

$$\begin{aligned}(n+1)\, c_{n+1}(X, Y) &= \frac{1}{2}\, [X - Y, c_n(X, Y)] \\ &+ \sum_{\substack{p\ge 1\\ 2p\le n}} K_{2p} \sum_{\substack{k_1,\dots,k_{2p}>0\\ k_1+\cdots k_{2p}=n}} [c_{k_1}(X, Y), [\cdots, [c_{k_{2p}}(X, Y), X + Y],], \dots,],\end{aligned} \tag{14}$$

see Lemma 2.15.3 of [63]. Analyzing (14), one easily notices that

$$c_n(\lambda X, \lambda Y) = \lambda^n\, c_n(X, Y) \tag{15}$$

for every $X, Y \in \mathcal{G}$ and $\lambda \in \mathbb{R}$. These formulae will be important in the next section.

The metric structure of a graded group is given by a continuous, left invariant distance $d : \mathbb{G} \times \mathbb{G} \longrightarrow \mathbb{R}$ such that $d(\delta_r x, \delta_r y) = r\, d(x, y)$ for every $x, y \in \mathbb{G}$ and $r > 0$. Every distance satisfying these properties is a *homogeneous distance*.

The Carnot-Carathéodory distance is an important example of homogeneous distance that can be defined in stratified groups, since they satisfy the Lie bracket generating condition, see for instance [25].

Notice that graded groups may not satisfy this condition. On the other hand, according to Example 2.1, when a graded group is contained in a stratified group, the restriction of the Carnot-Caratheodory distance to the graded subgorup provides an example of homogeneous distance. In general, it is possible to construct homogeneous distances in every graded group, [21], [55].

We denote by e the unit element and to simplify notation we set $d(x) = d(x, e)$. Notice that left invariance and symmetry of d imply the equality $d(x^{-1}) = d(x)$. An open ball of center x and radius r with respect to a homogeneous distance will be denoted by $B_{x,r}$. The corresponding closed ball will be denoted by $D_{x,r}$

2.1. **h-homomorphisms and notions of differentiability.**

Definition 2.1 (h-homomorphism)**.** Let $\mathbb{G}$ and $\mathbb{M}$ be graded groups with dilations $\delta_r^{\mathbb{G}}$ and $\delta_r^{\mathbb{M}}$, respectively. We say that a group homomorphism $L : \mathbb{G} \longrightarrow \mathbb{M}$ such that $L(\delta_r^{\mathbb{G}} x) = \delta_r^{\mathbb{M}} L(x)$ for every $x \in \mathbb{G}$ and $r > 0$ is a *homogeneous homomorphism*, in short *h-homomorphism*. Invertible h-homomorphisms will be called *h-isomorphisms*.

Remark 2.2. Analogous terminology will be used for the corresponding Lie algebra homomorphisms of graded algebras that commute with dilations.

Remark 2.3. Concerning injective and surjective h-homomorphisms, we will use the classical terminology of h-epimorphism and h-monomorphisms to indicate special classes of surjective and injective h-homomorphisms. In fact, a surjective linear mapping of vector spaces is characterized by the existence of a right inverse that is also linear. Analogously, injective linear mappings are characterized by the existence of a linear left inverse mapping. The analogous characterization for either surjective or injective algebra homomorphisms does not work, as we will see in the next example.

Example 2.4. Let $l : \mathfrak{h}^1 \longrightarrow \mathbb{R}^2$ be a surjective h-homomorphism. Clearly $l(\mathfrak{v}) = \mathbb{R}^2$ and $l(\mathfrak{z}) = \{0\}$, where $\mathfrak{h}^1 = \mathfrak{v} \oplus \mathfrak{z}$. We show that there is no right inverse that is also an h-homomorphism. Assume by contradiction that there exists an h-homomorphism $\tau : \mathbb{R}^2 \longrightarrow \mathfrak{h}^1$ that is a right inverse. Then the property $l \circ \tau = \mathrm{Id}_{\mathbb{R}^2}$ and the fact that τ is an h-homomorphism imply that $\tau(\mathbb{R}^2)$ is a 2-dimensional homogeneous subalgebra of $\mathfrak{h}^1$. Clearly $\tau(\mathbb{R}^2)$ cannot intersect $\ker l = \mathfrak{z}$, but this conflicts with Example 7.9, where we show that any 2-dimensional homogeneous subalgebra of $\mathfrak{h}^1$ contains $\mathfrak{z}$.

As a consequence of the previous example, in the category of graded algebras and h-homomorphisms, requiring the existence of a right inverse homomorphism is a stronger condition than surjectivity. This motivates the following

Definition 2.2. We say that an h-homomorphism is an *h-epimorphism* if it has a right inverse that is also an h-homomorphism. We say that an h-homomorphism is an *h-monomorphism* if it has a left inverse that is also an h-homomorphism.

In Subsection 7.2, we will see how either h-epimorphisms or h-monomorphisms can be characterized by their property of factorizing either the domain or the codomain.

Definition 2.3 (P-differentiability)**.** Let $\mathbb{G}$ and $\mathbb{M}$ be graded groups with homogeneous distances d and ρ, respectively. Let Ω be an open subset of $\mathbb{G}$ and consider $f : \Omega \longrightarrow \mathbb{M}$. We say that f is *P-differentiable* at $x \in \Omega$ if there exists an h-homomorphism $L : \mathbb{G} \longrightarrow \mathbb{M}$ such that

$$\frac{\rho\left(f(x)^{-1}f(xh), L(h)\right)}{d(h)} \longrightarrow 0 \quad \text{as} \quad h \to e\,.$$

The h-homomorphism L satisfying this limit is unique and it is called *P-differential* of f at x. We denote L by $Df(x)$. When we read the P-differential between Lie algebras, we will denote it by $df(x)$.

Definition 2.4 (Horizontal differentiability)**.** Let $\Omega \subset \mathbb{G}$ be an open set and let M be a smooth manifold. We consider a mapping $f : \Omega \longrightarrow M$ and $x \in \Omega$. We say that f is *horizontally differentiable* at x, in short *h-differentiable* at x, if there exists a neighbourhood U of the origin in the first layer V_1 such that the restriction

$$U \ni X \longrightarrow f(x \exp X) \in M$$

is differentiable at the origin. The differential $L : H_x\mathbb{G} \longrightarrow T_{f(x)}M$ will be denoted by $d_H f(x)$. We say that f is *continuously* h-differentiable in the case $x \longrightarrow d_H f(x)$ is a continuous mapping. If M is a graded group $\mathbb{M}$, then we also use the notation $D_H f(x) : \exp V_1 \longrightarrow \mathbb{M}$ to denote the h-differential between the corresponding groups.

Remark 2.5. We notice that P-differentiability of $\mathbb{R}^k$-valued mappings on an open subset $\Omega \subset \mathbb{G}$ implies h-differentiability. It suffices to restrict P-differentiability to horizontal directions and to observe that $d(\exp X)/\|X\|$ is bounded away from zero and from above with constants independent from X as it varies in V_1. This yields

$$D_H f(x)(\exp X) = Df(x)(\exp X) \qquad \text{for every} \qquad X \in V_1. \tag{16}$$

Then h-differentiability in general is a weaker notion. On the other hand, the little regularity of h-differentiable mappings suffices to introduce contact mappings.

Definition 2.5 (Contact mapping)**.** Let $f : \Omega \longrightarrow \mathbb{M}$ be an h-differentiable mapping. We say that f is a *contact mapping* if the inclusion $d_H f(x)(H_x\mathbb{G}) \subset H_{f(x)}\mathbb{M}$ holds for every $x \in \Omega$.

Remark 2.6. Notice that $\mathbb{R}^k$-valued continuously h-differentiable mappings are automatically contact. This family of mappings have been already considered in [20] as mappings of $\mathbf{C}^1_{\mathbb{G}}$.

Proposition 2.7 (Chain rule)**.** *Let $f : \Omega \longrightarrow \mathbb{U}$ be P-differentiable at $x \in \Omega$ and let $g : \Upsilon \longrightarrow \mathbb{M}$ be P-differentiable at $f(x) \in \Upsilon$, where $\Omega \subset \mathbb{G}$ and $\Upsilon \subset \mathbb{U}$ are open subsets and $f(\Omega) \subset \Upsilon$. Then $g \circ f : \Omega \longrightarrow \mathbb{M}$ is P-differentiable at x and $D(g \circ f)(x) = Dg\big(f(x)\big) \circ Df(x)$.*

The proof of this result is straightforward, see for instance Proposition 3.2.5 of [38].

3. Technical lemmata

Throughout this section, we denote by $\mathbb{G}$ a graded group, equipped with a homogeneous distance d. Its Lie algebra $\mathcal{G}$ of layers V_i has step ι. On $\mathcal{G}$, seen as a finite dimensional real vector space we fix a norm $\|\cdot\|$. Bilinearity of brackets yields a constant $\beta>0$, such that for every $X,Y\in\mathcal{G}$ we have

$$\|[X,Y]\|\leq\beta\,\|X\|\,\|Y\|. \tag{17}$$

Lemma 3.1. *Let $\nu>0$ and let $n=2,\dots,\iota$. Then there exists a constant $\alpha_n(\nu)$ only depending on n and ν such that*

$$\|c_n(X,Y)\|\leq\alpha_n(\nu)\;\|[X,Y]\| \tag{18}$$

whenever $\|X\|,\|Y\|\leq\nu$.

Proof. Our statement is trivial for $n=2$, being $c_2(X,Y)=[X,Y]/2$. Assume that it is true for every $j=2,\dots,n$, with $n\geq2$. It suffices to observe that $[c_{k_{2p}}(X,Y),X+Y]\neq0$ in (14) implies $k_{2p}>1$, then inductive hypothesis yields

$$\|c_{k_{2p}}(X,Y)\|\leq\alpha_{k_{2p}}(\nu)\;\|[X,Y]\|\,.$$

Using this estimate in (14) and observing that $\|c_{k_i}(X,Y)\|\leq2\nu$, whenever $k_i=1$, the thesis follows. $\square$

Lemma 3.2. *Let $c_n(X,Y)$ be as in* (13)*. Then for each $n=2,\dots,\iota$ there exists a set of real numbers $\left\{e_{n,\alpha}\mid\alpha\in\{1,2\}^{n-1}\right\}$ only depending on $\mathcal{G}$ such that for every $A_1,A_2\in\mathcal{G}$ we have*

$$c_n(A_1,A_2)=\sum_{\alpha\in\{1,2\}^{n-1}}e_{n,\alpha}\;L_n(A_\alpha,A_1+A_2)\,, \tag{19}$$

where $A_\alpha=(A_{\alpha_1},\dots,A_{\alpha_{n-1}})$, $L_1=Id_{\mathcal{G}}$ and for $n\geq2$ the n-linear mapping $L_n:\mathcal{G}^n\longrightarrow\mathcal{G}$ is defined by

$$L_n(X_1,X_2,\dots,X_n)=[X_1,[X_2,[\dots,[X_{n-1},X_n]]\dots]\,. \tag{20}$$

Proof. Let $L_1:\mathcal{G}\longrightarrow\mathcal{G}$ be the identity mapping and let $k_1,\dots,k_p$ be positive integers with $n=k_1+k_2\cdots+k_p$ where $p\in\mathbb{N}$. Iterating Jacoby identity of the Lie product, it is not difficult to check that

$$\begin{aligned}&[L_{k_1}(X^1_1,\dots,X^1_{k_1}),[L_{k_2},(X^2_1,\dots,X^2_{k_2}),[\cdots,\\&[L_{k_{p-1}}(X^{p-1}_1,\dots,X^{p-1}_{k_{p-1}}),L_{k_p},(X^p_1,\dots,X^p_{k_p})]]\dots]\\&=\sum_{\sigma_1\in S_{k_1},\dots,\sigma_{p-1}\in S_{k_{p-1}}}s_{\sigma_1,\dots,\sigma_{p-1}}\;\;L_n\big(X^1_{\sigma_1},X^2_{\sigma_2},\dots,X^{p-1}_{\sigma_1},X^p\big)\,,\end{aligned} \tag{21}$$

where S_j is the set of all permutations on j elements, $s_{\sigma_1,\dots,\sigma_{p-1}}\in\{-1,0,1\}$,

$$X^p=(X^p_1,\dots,X^p_{k_p})\quad\text{and}\quad X^j_{\sigma_j}=(X^j_{\sigma_j(1)},X^j_{\sigma_j(2)},\dots,X^j_{\sigma_j(k_j)})\,.$$

Our statement can be proved by induction. It is clearly true for $n=2$ taking $e_1=1/2$ and $e_2=0$, due to the formula $c_2(A_1,A_2)=[A_1,A_2]/2$. Let us assume that (19) holds for all $c_j(A_1,A_2)$ with $j\le n$. By recurrence equation (14), we have

$$
\begin{aligned}
(22)\,(n+1)\,c_{n+1}(A_1,A_2) &= \frac{1}{2}\sum_{\alpha\in\{1,2\}^{n-1}} e_{n,\alpha}\ [A_1-A_2, L_n(A_\alpha, A_1+A_2)] \\
&+ \sum_{\substack{p\ge1\\2p\le n}} K_{2p} \sum_{\substack{k_1,\dots,k_{2p}>0\\k_1+\cdots k_{2p}=n}} \sum_{\substack{\alpha_i\in\{1,2\}^{k_i-1}\\i=1,\dots,2p}} e_{k_1,\alpha_1}\, e_{k_2,\alpha_2}\ \cdots\ e_{k_{2p},\alpha_{2p}} \\
&[L_{k_1}(A_{\alpha_1},A_1+A_2),[L_{k_2}(A_{\alpha_2},A_1+A_2,),[\cdots,[L_{k_{2p}}(A_{\alpha_{2p}},A_1+A_2),A_1+A_2]]\dots]
\end{aligned}
$$

Then applying (21) we have proved that $c_{n+1}(A_1,A_2)$ can be represented as a linear combination of terms $L_{n+1}(A_\alpha, A_1+A_2)$ where $\alpha\in\{1,2\}^n$. This concludes our proof. □

Following the notation of [51], Section 4.5, in the next definition we introduce the iterated Lie bracket.

Definition 3.1. Let $X,Y\in\mathcal{G}$. The k-th bracket is defined by

$$
[X,Y]_k = \underbrace{[X,[X,[\cdots,[X}_{k\text{ times}},Y],],\dots,] \quad \text{and} \quad [X,Y]_0=Y\,. \tag{23}
$$

Lemma 3.3. *Let $c_n(X,Y)$ be as in (13), where $n=2,\dots,\iota$. Then we have*

$$
c_n(X,Y)=\frac{(-1)^{n-1}}{n!}\ \left[\frac{Y-X}{2},X+Y\right]_{n-1}+R_n(X,Y) \tag{24}
$$

and for every $\nu>0$ there exists a positive nondecreasing function $C(n,\cdot)$ such that

$$
\|R_n(X,Y)\|\le C(n,\nu)\,\|X+Y\|^3 \tag{25}
$$

whenever $\|X\|,\|Y\|\le\nu$.

Proof. Our statement is trivial for $n=2$, being $R_2(X,Y)$ vanishing for any $X,Y\in\mathcal{G}$. Let us consider $n\ge3$. We apply formula (19) to (14), getting

$$
c_n(X,Y)=-\frac{1}{n}\ \left[\frac{Y-X}{2},c_{n-1}(X,Y)\right]+E_n(X,Y) \tag{26}
$$

where for every $A_1,A_2\in\mathcal{G}$ we have

$$
\begin{aligned}
E_n(A_1,A_2) &= \sum_{\substack{p\ge1\\2p\le n-1}} K_{2p} \sum_{\substack{k_1,\dots,k_{2p}>0\\k_1+\cdots k_{2p}=n-1}} \sum_{\substack{\alpha_i\in\{1,2\}^{k_i-1}\\i=1,\dots,2p}} e_{k_1,\alpha_1}\, e_{k_2,\alpha_2}\ \cdots\ e_{k_{2p},\alpha_{2p}} \\
&[L_{k_1}(A_{\alpha_1},A_1+A_2),[L_{k_2}(A_{\alpha_2},A_1+A_2,),[\cdots,[L_{k_{2p}}(A_{\alpha_{2p}},A_1+A_2),A_1+A_2]]\dots]\,.
\end{aligned}
$$

As a consequence, there exist constants $\widetilde{C}_p > 0$ such that

$$\|E_n(X,Y)\| \le \sum_{1\le p\le (n-1)/2} \widetilde{C}_p \;\; \nu^{n-2p-1}\, \|X+Y\|^{2p+1} \tag{27}$$
$$\le \sum_{1\le p\le (n-1)/2} 4^p\, \widetilde{C}_p \;\; \nu^{n-3}\, \|X+Y\|^3 \,.$$

Iterating (26), we get

$$c_n(X,Y) = \frac{(-1)^2}{n(n-1)}\left[\frac{Y-X}{2}, c_{n-2}(X,Y)\right]_2 + E_n(X,Y) \tag{28}$$
$$-\frac{1}{n}\left[\frac{Y-X}{2}, E_{n-1}(X,Y)\right] = \frac{(-1)^{n-1}}{n!}\left[\frac{Y-X}{2}, c_1(X,Y)\right]_{n-1} + E_n(X,Y)$$
$$+\sum_{k=1}^{n-2}\frac{(-1)^k}{n(n-1)\cdots(n-k+1)}\left[\frac{Y-X}{2}, E_{n-k}(X,Y)\right]_k$$

therefore (27) concludes our proof. □

Remark 3.4. As an immediate consequence of Lemma 3.3, we achieve the estimate

$$\|(-\xi)\circledcirc\eta\| \le C(\nu)\|\xi-\eta\| \tag{29}$$

for every $\xi,\eta\in\mathcal{G}$ satisfying $\|\xi\|,\|\eta\|\le\nu$. To see this, it suffices to apply (24) and (25) to the Baker-Campbell-Hausdorff formula $(-\xi)\circledcirc\eta = -\xi+\eta+\sum_{n=2}^{\iota} c_n(-\xi,\eta)$.

As an immediate extension of inequality (+) at p.13 of [51], see also [50], we have the following lemma.

Lemma 3.5. *Let $\pi^i : \mathcal{G} \longrightarrow V_i\oplus\cdots\oplus V_\iota$ be the natural projection and let U be a bounded open neighbourhood of the unit element $e\in\mathbb{G}$. Then there exists a constant $K_U>0$, depending on U, such that*

$$\|\pi^i\left(\exp^{-1}(x)\right)\| \le K_U\; d(x)^i \tag{30}$$

holds for every $x\in U$.

Proof. We first define $\mathcal{S} = \{v\in\mathcal{G}\mid d(\exp v)=1\}$. Let us fix $v\in\mathcal{S}$ and $s>0$ satisfying the condition $s\le M$, with $M=\max_{x\in U} d(x)$. We have

$$\|\pi^i(\delta_s v)\| = s^i\left\|\sum_{j=i}^{\iota} s^{j-i}\,\pi_j(v)\right\| \le Cs^i\,,$$

where $C=\sum_{j=i}^{\iota} M^{j-i}\max_{u\in\mathcal{S}}\|\pi_j(u)\|$. This concludes the proof. □

Remark 3.6. From Proposition 1.5 of [18], it is easy to check that there exists a constant $\kappa(\nu)$ such that whenever $\|\xi\|\le\nu$ there holds

$$d(\exp\xi) \le \kappa(\nu)\,\|\xi\|^{1/\iota}\,. \tag{31}$$

Remark 3.7. As a byproduct of (29) and (31) we obtain the well known estimate

$$d\big(\exp\xi,\exp\eta\big)\le C(\nu)\,\|\xi-\eta\|^{1/\iota} \tag{32}$$

for every $\xi,\eta\in\mathcal{G}$ satisfying $\|\xi\|,\|\eta\|\le\nu$, see [57] for its proof in the more general setting of vector fields satisfying the Hörmander condition.

Remark 3.8. Notice that for every homogeneous distance d, there exists a constant $C_{d,\|\cdot\|}$ such that

$$C^{-1}_{d,\|\cdot\|}\,\sum_{j=1}^{\iota}\|\pi_j(\xi)\|^{1/j}\le d(\exp\xi)\le C_{d,\|\cdot\|}\,\sum_{j=1}^{\iota}\|\pi_j(\xi)\|^{1/j}\,.$$

In particular, we have

$$d(\exp\xi,\exp\eta)\ge C^{-1}_{d,\|\cdot\|}\,\|\pi_1(\xi-\eta)\|\,. \tag{33}$$

for every $\xi,\eta\in\mathcal{G}$.

Lemma 3.9. *Let $x,y\in\mathbb{G}$ and let $\nu>0$ be such that $d(x),d(y)\le\nu$. Then there exists a constant $C(\nu)$ only depending on $\mathbb{G}$ and on ν such that*

$$d(y^{-1}xy)\le C(\nu)\,\|\exp^{-1}(x)\|^{1/\iota}\,. \tag{34}$$

PROOF. Let us fix $x=\exp\xi$ and $y=\exp h$. By Lemma 3.5 we find a constant $k_\nu>0$ such that $\|\xi\|,\|h\|\le k_\nu$. The Baker-Campbell-Hausdorff fomuls yields

$$(-h)\circledcirc\big(\xi\circledcirc h\big)=\xi+\sum_{n=2}^{\iota}c_n(\xi,h)+\sum_{n=2}^{\iota}c_n(-h,\xi\circledcirc h). \tag{35}$$

The same formula along with (18), also gives the estimate

$$\|\xi\circledcirc h\|\le k_\nu\left(2+\sum_{n=2}^{\iota}\alpha_n(k_\nu)\,\beta\,k_\nu\right)=A(\nu).$$

The bilinear estimate (18) also yields

$$\|c_n(-h,\xi\circledcirc h)\|\le\alpha_n\big(A(\nu)\big)\,\|[h,\xi\circledcirc h]\|. \tag{36}$$

Observing that

$$[h,\xi\circledcirc h]=\big[h,\xi+\sum_{n=2}^{\iota}c_n(\xi,h)\big]$$

we achieve the estimate

$$\|[h,\xi\circledcirc h]\|\le\beta\,\|h\|\,\|\xi\|\,\big(1+\sum_{n=2}^{\iota}\alpha_n(k_\nu)\,\|h\|\big).$$

Joining this estimate with (36) we achieve

$$\|c_n(-h,\xi\circledcirc h)\|\le\beta\,\alpha_n\big(A(\nu)\big)\left(1+\sum_{n=2}^{\iota}\nu\,\alpha_n(\nu)\right)\|h\|\,\|\xi\| \tag{37}$$

Thus, formula (35), estimates (18) and (37) give a constant $B(\nu)>0$ such that

$$\|(-h)\circledcirc\xi\circledcirc h\|\le\|\xi\|\,\big(1+B(\nu)\|h\|\big). \tag{38}$$

The previous inequality yields

$$\|(-h)\circledcirc\xi\circledcirc h\|\le k_\nu\,\big(1+k_\nu\,B(\nu)\big),$$

then (31) gives a constant $B_1(\nu)>0$ such that

$$d(y^{-1}xy)\le B_1(\nu)\,\|(-h)\circledcirc\xi\circledcirc h\|^{1/\iota}$$

Thus, applying (38) we achieve a constant $B_2(\nu)>0$ such that

$$d(y^{-1}xy)\le B_2(\nu)\,\|\xi\|^{1/\iota}\,.$$

This finishes the proof. □

Remark 3.10. The previous lemma also provides another variant of its estimate. In fact, the condition $d(x)\le\nu$ implies that $\|\exp^{-1}(x)\|$ is bounded by $d(x)$ up to a factor only depending on ν, due to Lemma 3.5. As a result, the assumption $d(x),d(y)\le\nu$ gives a constant $C(\nu)$ such that the following estimate holds

$$d(y^{-1}xy)\le C(\nu)\;d(x)^{1/\iota}. \tag{39}$$

Lemma 3.11. *Let N be a positive integer and let $A_j,B_j\in\mathbb{G}$ with $j=1,\dots,N$. Let $\nu>0$ be such that $d(B_jB_{j+1}\cdots B_N)\le\nu$ and $d(A_j,B_j)\le\nu$ for every $j=1,\dots,N$. Then there exists $K_\nu>0$ such that*

$$d\big(A_1A_2\cdots A_N,B_1B_2\cdots B_N\big)\le K_\nu\,\sum_{j=1}^N d(A_j,B_j)^{1/\iota}. \tag{40}$$

Proof. We define

$$\hat B_j=B_jB_{j+1}\cdots B_N,\qquad \hat A_j=A_jA_{j+1}\cdots A_N$$

and use left invariance of d to obtain

$$d\big(\hat A_1,\hat B_1\big)\le d(A_N,B_N)+\sum_{j=1}^{N-1}d\big(\hat B_{j+1}A_j^{-1}B_j\hat B_{j+1}\big)\,.$$

By our hypothesis and (39) we get

$$d\big(\hat A_1,\hat B_1\big)\le d(A_N,B_N)+C(\nu)\,\sum_{j=1}^{N-1}d\big(A_j,B_j\big)^{1/\iota}\le d(A_N,B_N)^{1/\iota}\nu^{1-1/\iota}$$

$$+C(\nu)\,\sum_{j=1}^{N-1}d\big(A_j,B_j\big)^{1/\iota}\le\max\{C(\nu),\nu^{1-1/\iota}\}\sum_{j=1}^{N}d(A_j,B_j)^{1/\iota}.$$

This shows the validity of (40). □

Lemma 3.12. *Let $X, Y, D_1, D_2 \in \mathcal{G}$ and let $\nu \geq \max\{\|X\|, \|Y\|, \|D_1\|, \|D_2\|\}$. Then there exists a constant $\kappa_n > 0$ such that*

$$\|c_n(X + D_1, Y + D_2) - c_n(X, Y)\| \leq \gamma_n \, \nu^{n-1} \, \max\{\|D_1\|, \|D_2\|\} \tag{41}$$

for every $n = 2, \dots, \iota$.

PROOF. We set $X = A_1^1$, $D_1 = A_1^2$, $Y = A_2^1$ and $D_2 = A_2^2$ and apply (19). Taking into account that L_n is a multilinear mapping defined in (20), we obtain

$$\begin{aligned}
&c_n(A_1^1 + A_1^2, A_2^1 + A_2^2) \\
&= \sum_{\alpha\in\{1,2\}^{n-1}} e_{n,\alpha} \, L_n(A_{\alpha_1}^1 + A_{\alpha_1}^2, \dots, A_{\alpha_{n-1}}^1 + A_{\alpha_{n-1}}^2, A_1^1 + A_1^2 + A_2^1 + A_2^2) \\
&= \sum_{\alpha\in\{1,2\}^{n-1}} e_{n,\alpha} \sum_{\gamma\in\{1,2\}^{n-1}} L_n(A_{\alpha_1}^{\gamma_1}, \dots, A_{\alpha_{n-1}}^{\gamma_{n-1}}, A_1^1 + A_1^2 + A_2^1 + A_2^2) \\
&= \sum_{\alpha\in\{1,2\}^{n-1}} e_{n,\alpha} \sum_{\gamma\in\{1,2\}^{n-1}} L_n(A_{\alpha_1}^{\gamma_1}, \dots, A_{\alpha_{n-1}}^{\gamma_{n-1}}, A_1^1 + A_2^1) \\
&+ \sum_{i=1,2} \sum_{\alpha\in\{1,2\}^{n-1}} e_{n,\alpha} \sum_{\gamma\in\{1,2\}^{n-1}} L_n(A_{\alpha_1}^{\gamma_1}, \dots, A_{\alpha_{n-1}}^{\gamma_{n-1}}, A_i^2) \\
&= c_n(A_1^1, A_2^1) + \sum_{\alpha\in\{1,2\}^{n-1}} e_{n,\alpha} \sum_{\substack{\gamma\in\{1,2\}^{n-1} \\ \gamma\neq(1,1,\dots,1)}} L_n(A_{\alpha_1}^{\gamma_1}, \dots, A_{\alpha_{n-1}}^{\gamma_{n-1}}, A_1^1 + A_2^1) \\
&+ \sum_{i=1,2} \sum_{\alpha\in\{1,2\}^{n-1}} e_{n,\alpha} \sum_{\gamma\in\{1,2\}^{n-1}} L_n(A_{\alpha_1}^{\gamma_1}, \dots, A_{\alpha_{n-1}}^{\gamma_{n-1}}, A_i^2)
\end{aligned}$$

where the equality in the line before last we have applied again formula (19). Taking into account the multilinear estimate

$$\|L_n(X_1, X_2, \dots, X_n)\| \leq \beta^n \prod_{j=1}^{n} \|X_j\|,$$

where β is given in (17). As a consequence, a short calculation yields

$$\|c_n(A_1^1 + A_1^2, A_2^1 + A_2^2) - c_n(A_1^1, A_2^1)\| \leq 2^{n+1} \beta^n \nu^{n-1} \max\{\|A_1^2\|, \|A_2^2\|\} \sum_{\alpha\in\{1,2\}^{n-1}} |e_{n,\alpha}| \,,$$

leading us to the conclusion. □

4. P-DIFFERENTIABILITY OF CURVES IN GRADED GROUPS

In this section we study differentiability properties of curves $\Gamma : [a, b] \longrightarrow \mathbb{M}$, where $\mathbb{M}$ is a graded group. We denote by γ the corresponding curve $\exp^{-1} \circ \Gamma$ with values in the Lie algebra $\mathcal{M}$ of $\mathbb{M}$. The components of γ taking values in the layers W_i are denoted by γ_i, hence $\gamma = \sum_{i=1}^{\upsilon} \gamma_i$. We have $\gamma_i = \pi_i \circ \gamma$, where $\pi_i : \mathcal{M} \longrightarrow W_i$ is the canonical projection on layers of degree i and $\mathcal{M} = W_1 \oplus \cdots \oplus W_\upsilon$. We denote by $\|\cdot\|$ a fixed norm on $\mathcal{M}$. The *horizontal subspace* of $\mathbb{M}$ at $x \in \mathbb{M}$ is defined as follows

$$H_x\mathbb{M} = \{U(x) \mid U \in W_1\}.$$

Definition 4.1. We say that a curve $\Gamma : [a,b] \longrightarrow \mathbb{M}$ is *horizontal* if $\gamma : [a,b] \longrightarrow \mathcal{M}$ is absolutely continuous and for a.e. differentiability point $t \in [a,b]$ the inclusion $\dot\gamma(t) \in H_{\gamma(t)}\mathbb{M}$ holds.

Identifying any $T_x\mathbb{M}$ with $\mathcal{M}$ and applying formula (12) for the differential of the exponential mapping, we have

$$\dot\Gamma(t) = \dot\gamma(t) - \sum_{n=2}^{\upsilon} \frac{(-1)^n}{n!}\,\mathrm{ad}(\gamma(t))^{n-1}(\dot\gamma(t)) = \dot\gamma(t) - \sum_{n=2}^{\upsilon} \frac{(-1)^n}{n!}\,[\gamma(t),\dot\gamma(t)]_{n-1}\,. \tag{42}$$

Then $\dot\Gamma(t) \in W_1$ if and only if

$$\pi_i\Big(\dot\gamma(t) - \sum_{n=2}^{\upsilon} \frac{(-1)^n}{n!}\,[\gamma(t),\dot\gamma(t)]_{n-1}\Big) = 0$$

for all $i \geq 2$. This immediately proves the following

Proposition 4.1. *Let $\gamma : [a,b] \longrightarrow \mathcal{M}$ be an absolutely continuous curve. Then Γ is a horizontal curve if and only if the differential equation*

$$\dot\gamma_i(t) = \sum_{n=2}^{\upsilon} \frac{(-1)^n}{n!}\,\pi_i\left([\gamma(t),\dot\gamma(t)]_{n-1}\right) \tag{43}$$

a.e. holds for each $i = 2,\ldots,\upsilon$.

Definition 4.2. Let $\gamma : [a,b] \longrightarrow \mathcal{M}$ be a locally summable curve and let $\lambda \neq 0$. The *sup-average* of γ at t is defined as follows

$$\mathcal{A}_t^\lambda(\gamma) = \begin{cases} \sup\limits_{0\leq\tau\leq\lambda} \displaystyle\fint_t^{t+\tau} \|\gamma(l)\|\,dl & \text{if } \lambda > 0 \\ \sup\limits_{\lambda\leq\tau\leq 0} \displaystyle\fint_{t+\tau}^{t} \|\gamma(l)\|\,dl & \text{if } \lambda < 0 \end{cases}. \tag{44}$$

Notice that for some t the sup-average $\mathcal{A}_t^\lambda(\gamma)$ takes values in $[0,+\infty]$, as γ is not necessarily bounded in a neighbourhood of t.

Lemma 4.2. *There exist strictly increasing functions $\Upsilon_i : [0,+\infty[\longrightarrow [0,+\infty[$, with $i = 2,\ldots,\upsilon$, which only depend on $\mathcal{M}$, are infinitesimal at zero and satisfy the following properties. For any horizontal curve $\Gamma : [-\alpha,\alpha] \longrightarrow \mathbb{M}$ such that $\Gamma(0) = e$, the estimates*

$$\left|\int_0^\lambda \|\dot\gamma_i(t)\|\,dt\right| \leq \Upsilon_i(L)\,\mathcal{A}_0^\lambda(\dot\gamma_1 - X)\,|\lambda|^i \tag{45}$$

hold, where $L = \max\left\{\mathcal{A}_0^{-\alpha}(\dot\gamma_1), \mathcal{A}_0^{\alpha}(\dot\gamma_1)\right\}$, $X \in \mathcal{M}$, $\|X\| \leq L$ and the function $\mathcal{A}_0^\lambda(\cdot)$ has been introduced in Definition 4.2.

PROOF. First of all, we assume that $\mathcal{A}_0^\lambda(\dot\gamma_1)<+\infty$, otherwise the proof becomes trivial. Due to Proposition 4.1, the differential equations (43) hold. Applying (43) for $i=2$, we achieve

$$\dot\gamma_2(t)=\frac{t}{2}\left[\frac{\gamma_1(t)}{t},\dot\gamma_1(t)\right]=\frac{t}{2}\left(\left[\fint_0^t\dot\gamma_1(\tau)\,d\tau,X\right]+\left[\fint_0^t\dot\gamma_1(\tau)\,d\tau,\dot\gamma_1(t)-X\right]\right). \tag{46}$$

For every $0<s<\alpha$, it follows that

$$\int_0^s\|\dot\gamma_2(t)\|\,dt\le\frac{s}{2}\int_0^s\left\|\left[\fint_0^t\dot\gamma_1(\tau)\,d\tau,X\right]\right\|dt+\frac{s}{2}\int_0^s\left\|\left[\fint_0^t\dot\gamma_1(\tau)\,d\tau,\dot\gamma_1(t)-X\right]\right\|dt$$
$$\le\frac{\beta Ls}{2}\left(\int_0^s\mathcal{A}_0^t(\dot\gamma_1-X)\,dt+s\,\mathcal{A}_0^s(\dot\gamma_1-X)\right).$$

This shows that

$$\int_0^s\|\dot\gamma_2(t)\|\,dt\le\beta\,L\,s^2\,\mathcal{A}_0^s(\dot\gamma_1-X). \tag{47}$$

From the grading property (11), equations (43) can be stated as follows

$$\dot\gamma_j(t)=\sum_{n=2}^{\upsilon}\frac{(-1)^n}{n!}\sum_{\substack{1\le l_1,\dots,l_n\le\upsilon\\ l_1+\cdots+l_n=j}}[\gamma_{l_1}(t),[\gamma_{l_2}(t),[\cdots,[\gamma_{l_{n-1}}(t),\dot\gamma_{l_n}(t)],],\dots,] \tag{48}$$

for every $j=2,\dots,\upsilon$. We will proceed by induction, assuming that for every $j=2,\dots,i$ and every $\upsilon\ge i$, there exists a positive constant $\kappa_j(\beta)$ such that

$$\int_0^s\|\dot\gamma_j(t)\|\,dt\le\kappa_j(\beta)\,F_j(L)\,\mathcal{A}_0^s(\dot\gamma_1-X)\,s^j, \tag{49}$$

where $F_j:[0,+\infty[\longrightarrow[0,+\infty[$ is a strictly increasing function, that is infinitesimal at zero. We have already proved this statement for $i=2$. Using (48) for $j=i+1$ and applying the inductive hypothesis (49), we get

$$\int_0^s\|\dot\gamma_{i+1}(t)\|\,dt\le\sum_{n=2}^{\upsilon}\frac{\beta^n}{n!}\sum_{\substack{1\le l_1,\dots,l_n\le\upsilon\\ l_1+\cdots+l_n=i+1}}\int_0^s\|\gamma_{l_1}(t)\|\|\gamma_{l_2}(t)\|\cdots\|\gamma_{l_{n-1}}(t)\|\|\dot\gamma_{l_n}(t)\|\,dt$$
$$\le\sum_{n=2}^{\upsilon}\frac{\beta^n}{n!}\sum_{\substack{1\le l_1,\dots,l_n\le\upsilon\\ l_1+\cdots+l_n=i+1}}\int_0^s\|\dot\gamma_{l_1}(t)\|\,dt\int_0^s\|\dot\gamma_{l_2}(t)\|\,dt\cdots\int_0^s\|\dot\gamma_{l_{n-1}}(t)\|\,dt\int_0^s\|\dot\gamma_{l_n}(t)\|\,dt$$
$$\le\sum_{n=2}^{\upsilon}\frac{\beta^n}{n!}\sum_{\substack{1\le l_1,\dots,l_n\le\upsilon\\ l_1+\cdots+l_n=i+1}}\kappa_{l_1}\cdots\kappa_{l_n}\,F_{l_1}(L)\cdots F_{l_n}(L)\;\mathcal{A}_0^s(\dot\gamma_1-X)^n\,s^{i+1}$$
$$\le\sum_{n=2}^{\upsilon}\frac{\beta^n}{n!}\sum_{\substack{1\le l_1,\dots,l_n\le\upsilon\\ l_1+\cdots+l_n=i+1}}\kappa_{l_1}\cdots\kappa_{l_n}\,F_{l_1}(L)\cdots F_{l_n}(L)\;(2L)^{n-1}\,\mathcal{A}_0^s(\dot\gamma_1-X)\,s^{i+1}.$$

This proves estimates (49) for every $j = 2, \dots, \upsilon$. To complete the proof, it suffices to apply these estimates to the Lipschitz curve $\tilde\Gamma(t) = \Gamma(-t)$ and replace X by $-X$. $\square$

Corollary 4.3. *There exists a strictly increasing function $\Upsilon : [0, +\infty[\longrightarrow [0, +\infty[$, infinitesimal at zero and only depending on $\mathcal{M}$ that has the following property. Let $\Gamma : [a, b] \longrightarrow \mathbb{M}$ be a horizontal curve and assume that $t \in [a, b]$ is an approximate continuity points t of $\dot\gamma_1$. Then*

$$\left\| \pi_i \left(\exp^{-1} \left(\delta_{1/h} \big(\Gamma(t)^{-1} \Gamma(t+h) \big) \right) \right) \right\| \le \Upsilon(L_t) \, \mathcal{A}_t^h \big(\dot\gamma_1 - \dot\gamma_1(t) \big) \tag{50}$$

for every $i = 2, \dots, \upsilon$, where $L_t = \max \big\{ \mathcal{A}_t^{b-t}(\dot\gamma_1), \mathcal{A}_t^{a-t}(\dot\gamma_1) \big\}$.

Proof. Clearly, γ_1 is absolutely continuous. Let t be an approximate continuity point of $\dot\gamma_1$ and then also a differentiability point. It suffices to apply Lemma 4.2 to the curve $h \longrightarrow \Gamma(t)^{-1}\Gamma(t+h)$ with $X = \dot\gamma_1(t)$. In fact, we observe that

$$\pi_1 \left(\exp^{-1} \left(\Gamma(t)^{-1} \Gamma(t+h) \right) \right) = \gamma_1(t+h) - \gamma_1(t)$$

and that

$$\mathcal{A}_0^\lambda \big(\dot\gamma_1(t + \cdot) - \dot\gamma_1(t) \big) = \mathcal{A}_t^\lambda \big(\dot\gamma_1(\cdot) - \dot\gamma_1(t) \big) \, .$$

This implies that

$$\left\| \pi_i \Big(\exp^{-1} \left(\Gamma(t)^{-1} \Gamma(t+h) \right) \Big) \right\| \le \Upsilon_i(L_t) \, |h|^i \, \mathcal{A}_t^h \big(\dot\gamma_1 - \dot\gamma_1(t) \big) \tag{51}$$

where Υ_i are given in Lemma 4.2 and

$$L_t = \max \left\{ \mathcal{A}_t^{b-t}(\dot\gamma_1), \mathcal{A}_t^{a-t}(\dot\gamma_1) \right\} \, .$$

Thus, setting $\Upsilon = \max_{i=2,\dots,\upsilon} \Upsilon_i$, and using the definition of dilations, our claim follows. $\square$

Theorem 4.4. *There exists a nondecreasing function $\Upsilon : [0, +\infty[\longrightarrow [0, +\infty[$ only depending on $\mathcal{M}$ with the following property. If $\Gamma : [a, b] \longrightarrow \mathbb{M}$ is a horizontal curve, then for all approximate continuity points t of $\dot\gamma_1$ the following estimate*

$$\left\| \delta_{1/h} \Big(- h \, \dot\gamma_1(t) \circledcirc \big(- \gamma(t) \big) \circledcirc \gamma(t+h) \Big) \right\| \le \Upsilon(L_t) \, \mathcal{A}_t^h \big(\dot\gamma_1 - \dot\gamma_1(t) \big) \tag{52}$$

holds, where $L_t = \max \big\{ \mathcal{A}_t^{b-t}(\dot\gamma_1), \mathcal{A}_t^{a-t}(\dot\gamma_1) \big\}$. In particular, Γ is a.e. P-differentiable.

Proof. Let t be an approximate continuity point of $\dot\gamma_1$ and define

$$h \longrightarrow \Gamma(t)^{-1} \Gamma(t+h) = \exp \theta(h) = \exp \big(\theta_1(h) + \cdots + \theta_\upsilon(h) \big) \, ,$$

with $\theta_i(h) \in V_i$. Notice that in particular $\theta_1(h) = \gamma_1(t+h) - \gamma_1(t)$. In view of (50), there exists a constant $K > 0$ depending on $\mathcal{M}$ and a strictly increasing function $\Upsilon : [0, +\infty[\longrightarrow [0, +\infty[$ infinitesimal at zero such that

$$\left\| \frac{\theta_i(h)}{h^i} \right\| \le \Upsilon(L_t) \, \mathcal{A}_t^h \big(\dot\gamma_1 - \dot\gamma_1(t) \big). \tag{53}$$

for every $i = 2, \ldots, \upsilon$, where we have set

$$L_t = \max\left\{\mathcal{A}_t^{b-t}(\dot\gamma_1), \mathcal{A}_t^{a-t}(\dot\gamma_1)\right\}.$$

Thus, taking into account that t is a differentiability point of γ_1 and that 0 is an approximate continuity point of $\dot\theta_1$, we obtain that Γ is P-differentiable at t and that

$$D\Gamma(t)(\lambda) = \exp\big(\lambda\,\dot\gamma_1(t)\big).$$

Notice also that

$$\left\|\frac{\theta_1(h)}{h} - \dot\gamma_1(t)\right\| \le \mathcal{A}_t^h\big(\dot\gamma_1 - \dot\gamma_1(t)\big),$$

then (53) yields

$$\big\|\delta_{1/h}\theta(h) - \dot\gamma_1(t)\big\| \le \Big[1 + (\upsilon-1)\Upsilon(L_t)\Big]\ \mathcal{A}_t^h\big(\dot\gamma_1 - \dot\gamma_1(t)\big). \tag{54}$$

Applying (24) and (54) to

$$-\dot\gamma_1(t) \circledcirc \delta_{1/h}\theta(h) = \delta_{1/h}\theta(h) - \dot\gamma_1(t) + \sum_{n=2}^{\upsilon} c_n\big(-\dot\gamma_1(t), \delta_{1/h}\theta(h)\big),$$

we get the estimate

$$\begin{aligned}\big\|-\dot\gamma_1(t) \circledcirc \delta_{1/h}\theta(h)\big\| \le{} & \Big[1 + (\upsilon-1)\Upsilon(L_t)\Big]\ \mathcal{A}_t^h\big(\dot\gamma_1 - \dot\gamma_1(t)\big) \\ & + \sum_{n=2}^{\upsilon} \frac{\beta^{n-1}}{n!}\left\|\frac{\delta_{1/h}\theta(h) + \dot\gamma_1(t)}{2}\right\|^{n-1}\Big[1 + (\upsilon-1)\Upsilon(L_t)\Big]\mathcal{A}_t^h\big(\dot\gamma_1 - \dot\gamma_1(t)\big) \\ & + \sum_{n=2}^{\upsilon}\big\|R_n\left(-\dot\gamma_1(t), \delta_{1/h}\theta(h)\right)\big\|\,.\end{aligned} \tag{55}$$

Observing that $\|\dot\gamma_1(t)\| \le L_t$ and $\mathcal{A}_t^h\big(\dot\gamma_1 - \dot\gamma_1(t)\big) \le 2L_t$, inequality (54) implies that

$$\big\|\delta_{1/h}\theta(h)\big\| \le 3L_t\Big[1 + (\upsilon-1)\Upsilon(L_t)\Big] = \Upsilon_1(L_t),$$

then we apply (25) to the third line of (55), obtaining

$$\begin{aligned}\big\|-\dot\gamma_1(t) \circledcirc \delta_{1/h}\theta(h)\big\| \le{} & \left(1 + \sum_{n=2}^{\upsilon}\frac{\beta^{n-1}}{n!}\left(\frac{\Upsilon_1(L_t) + L_t}{2}\right)^{n-1}\right) \\ & \Big[1 + (\upsilon-1)\Upsilon(L_t)\Big]\mathcal{A}_t^h\big(\dot\gamma_1 - \dot\gamma_1(t)\big) \\ & + \left(\sum_{n=2}^{\upsilon} C\big(n, \Upsilon_1(L_t)\big)\right)\Big[1 + (\upsilon-1)\Upsilon(L_t)\Big]^3 4\,L_t^2\,\mathcal{A}_t^h\big(\dot\gamma_1 - \dot\gamma_1(t)\big).\end{aligned} \tag{56}$$

This concludes the proof. $\square$

Corollary 4.5 (Pansu). *Let A be a measurable set of $\mathbb{G}$ and let $f : A \longrightarrow \mathbb{M}$ be a Lipschitz mapping. Then f is a.e. P-differentiable.*

PROOF. By Theorem 4.4, rectifiable curves in graded groups are in particular a.e. P-differentiable. This allows us to extend Proposition 3.6 of [39] to the case of mappings with values in a graded group $\mathbb{M}$. Thus, repeating the same arguments of the proof of Theorem 3.9 of [39] the claim is achieved. $\Box$

4.1. **Characterization of P-differentiable mappings.** Here we denote by $\mathbb{G}$ and $\mathbb{M}$ two arbitrary graded groups, where $\mathbb{G}$ is stratified and $\Omega \subset \mathbb{G}$ is an open set. To obtain estimates on the P-difference quotient of P-differentiable mappings, we will use the following family of piecewise horizontal lines.

Definition 4.3. Let N be a positive integer, $X_1, \ldots, X_m$ be a basis of the first layer V_1, $i_1, \ldots, i_N \in \{1, \ldots, m\}$ be fixed integers and for every $a = (a_1, \ldots, a_N) \in \mathbb{R}^N$ define

$$P^s(a) = \begin{cases} e & \text{if } s = 0 \\ \delta_{a_1} h_{i_1} \delta_{a_2} h_{i_2} \cdots \delta_{a_s} h_{i_s} & \text{if } s = 1, \ldots, N \end{cases}, \tag{57}$$

where $h_1 = \exp X_1, \ldots, h_m = \exp X_m$ and we have assumed that $d(h_i) = 1$.

Lemma 1.40 of [18] yields the following

Lemma 4.6. *For every stratified group $\mathbb{G}$ there exists an integer N and a family of integers $\{i_1, \ldots, i_N\} \subset \{1, \ldots, m\}$ depending on $\mathbb{G}$, such that the mapping P^N of Definition 4.3 sends a neighbourhood of the origin in $\mathbb{R}^N$ onto a neighbourhood of the identity $e \in \mathbb{G}$.*

Definition 4.4. Under conditions of Lemma 4.6, we define the number

$$c(\mathbb{G}, d) = \max_{\substack{s=1,\ldots,N \\ a=(a_s)\in(P^N)^{-1}(D_1)}} |a_s|, \tag{58}$$

that only depends on the algebraic structure of $\mathbb{G}$ and on the homogeneous distance d used to define both P^N and the closed unit ball D_1.

Remark 4.7. Being the mapping P^N homogeneous with respect to dilations, namely, $P^N(ra) = \delta_r\big(P^N(a)\big)$, it follows that P^N is surjective onto $\mathbb{G}$. Homogeneity yields

$$\max_{\substack{s=1,\ldots,N \\ a=(a_s)\in(P^N)^{-1}(D_r)}} |a_s| = r\, c(\mathbb{G}, d) \tag{59}$$

for every $r > 0$ and implies that the preimage of compact sets of $\mathbb{G}$ is compact in $\mathbb{R}^N$. This shows that definition (58) is well posed.

Definition 4.5. Let $f : K \longrightarrow Y$ be a vector valued continuous function on a compact metric space (K, ρ). Then we define the *modulus of continuity* of f on K as

$$\omega_{K,f}(t) = \max_{\substack{x,y\in K \\ \rho(x,y)\le t}} \|f(x) - f(y)\|\,. \tag{60}$$

Theorem 4.8. *Let $f:\Omega\longrightarrow\mathbb{M}$ be P-differentiable at $x\in\Omega$, where $f=\exp\circ F$ and $F=F_1+\cdots+F_\upsilon$ with $F_j:\Omega\longrightarrow W_j$. Then every F_j is P-differentiable at x and we have the formulae*

$$\pi_1\circ df(x)=dF_1(x) \tag{61}$$

$$dF_i(x)(h)=\sum_{n=2}^{\upsilon}\frac{(-1)^n}{n!}\,\pi_i\left([F(x),dF(x)(h)]_{n-1}\right)\,. \tag{62}$$

for every $i=2,\dots,\upsilon$ and every $h\in\mathbb{G}$

Proof. We will use the notation $|h|=d(h)$. By definition of P-differentiability we have that

$$\delta_{1/|h|}\left(f(x)^{-1}f(xh)\right)$$

converges to $Df(x)(h)$ as $h\to e$, uniformly with respect to the parameter $\delta_{1/|h|}h$ varying in a compact set. Then the difference quotients

$$\pi_i\circ\exp^{-1}\left[\delta_{1/|h|}\left(f(x)^{-1}f(xh)\right)\right],$$

uniformly converge for every $i=1,\dots,\upsilon$. In other words, the difference quotient

$$\frac{F_1(xh)-F_1(x)}{|h|}$$

along with

$$\frac{F_i(xh)-F_i(x)+\sum_{n=2}^{\upsilon}\pi_i\left(c_n\big(-F(x),F(xh)\big)\right)}{|h|^i} \tag{63}$$

converge as $h\to 0$ whenever $i=2,\dots,\upsilon$. In particular, F_1 is P-differentiable at x and

$$\pi_1\circ\exp^{-1}\big(Df(x)(h)\big)=DF_1(x)(h)$$

that implies (61). Writing (63) for $i=2$, we notice that this expression can be written as

$$\frac{F_2(xh)-F_2(x)-\frac12[F_1(x),F_1(xh)]}{|h|^2}\,. \tag{64}$$

The convergence of the previous quotient and P-differentiability of F_1 at x imply that F_2 is also differentiable at x and there holds

$$DF_2(x)(h)=\frac{1}{2}\,[F_1(x),DF_1(x)(h)].$$

By induction, we assume that for $j\geq 2$, the vector-valued mapping F_i is differentiable at x and that

$$DF_i(x)(h)=\sum_{n=2}^{\upsilon}\frac{(-1)^n}{n!}\,\pi_i\left([\overline{F}_{i-1}(x),D\overline{F}_{i-1}(x)(h)]_{n-1}\right)$$

holds for every $i=1,\dots,j$, where we have set

$$\overline{F}_i=F_1+F_2+\cdots+F_i.$$

By P-differentiability (63), we have

$$\frac{F_{i+1}(xh) - F_{i+1}(x) + \sum_{n=2}^{\upsilon} \pi_{i+1}\left(c_n\big(- F(x), F(xh)\big)\right)}{|h|} \longrightarrow 0 \quad \text{as} \quad h \to 0\,. \tag{65}$$

Now we notice that

$$\pi_{i+1}\left(c_n\big(- F(t), F(t+h)\big)\right) = \pi_{i+1}\left(c_n\big(- \overline{F}_i(t), \overline{F}_i(t+h)\big)\right)\,. \tag{66}$$

Furthermore, inductive hypothesis and Lemma 3.3 ensure the existence of

$$\lim_{|h|\to 0} \frac{c_n(-\overline{F}_i(x), \overline{F}_i(x+h))}{|h|} = \frac{(-1)^{n-1}}{n!}\left[\overline{F}_i(x), D\overline{F}_i(x)(h)\right]_{n-1}. \tag{67}$$

Thus, joining (65), (66) and (67), it follows that there exists $DF_{i+1}(x)$ and

$$DF_{i+1}(x)(h) = \sum_{n=2}^{i+1} \frac{(-1)^n}{n!}\, \pi_{i+1}\left([\overline{F}_i(x), D\overline{F}_i(x)]_{n-1}\right)\,.$$

This shows that $F : \Omega \longrightarrow \mathcal{M}$ is P-differentiable at t, hence in the previous formula we can replace $\overline{F}_i$ by F, achieving (62). This ends the proof. □

Corollary 4.9. *Let $\Gamma : [a,b] \longrightarrow \mathbb{M}$ be P-differentiable at $t \in [a,b]$. Then γ is differentiable at t and satisfies equations* (43).

This corollary is an immediate consequence of Theorem 4.8. Now we can prove the first result stated in the introduction.

Proof of Theorem 1.1. If $f : \Omega \longrightarrow \mathbb{M}$ is continuously P-differentiable, then Theorem 4.8 implies the P-differentiability of all F_j's and the validity of both formulae (1) and (2). These equations in turn give the continuoity of the P-differential of every F_j. This shows that the first condition implies the second one.

Now, we assume that the second condition holds. As observed in Remark 2.5, P-differentiability of vector-valued mappings implies h-differentiability, hence all F_j's are in particular h-differentiable. In addition, formula (16) yields the continuity of the horizontal differential. By our assumption (1) and (2) hold, therefore combining these equations with Proposition 4.1, it follows that f also has the contact property. We have shown that f is continuously h-differentiable contact, that is the third condition.

Finally, we assume that f is a continuously h-differentiable contact mapping and consider

$$\Gamma_X(t) = \exp\Big(\sum_{j=1}^{\iota} \gamma_{X,j}(t)\Big) = f\big(c_X(t)\big) = \exp\Big(\sum_{j=1}^{\iota} F_j\big(c_X(t)\big)\Big)$$

where $c_X(t) = x\exp(tX)$ and $\exp^{-1}\circ\Gamma_X = \gamma_X = \sum_{j=1}^{\iota} \gamma_{X,j}$ with $t \longrightarrow \gamma_{X,j}(t) \in V_j$. By definition of h-differentiability we immediately observe that also F_j are continuously h-differentiable and we have

$$\dot\gamma_{X,j}(t) = XF_j(c_X(t))$$

By our assumptions, we know that Γ_X is a horizontal curve, then we apply the key estimate (52) to Γ_X at $t=0$, achieving

$$\left\|\delta_{1/h}\Big(-h\,XF_1(x)\circledcirc\big(-\gamma_X(0)\big)\circledcirc\gamma_X(h)\Big)\right\|\leq\Upsilon(L)\;\mathcal{A}_0^h\big((XF_1)\circ c_X-XF_1(x)\big)$$

where $\Upsilon:[0,+\infty[\longrightarrow[0,+\infty[$ is a nondecreasing function depending on $\mathbb{M}$, $\mathcal{A}_0^h$ is defined in (44) and

$$L=\max\left\{\mathcal{A}_0^{\varepsilon}(\dot\gamma_{X,1}),\mathcal{A}_0^{-\varepsilon}(\dot\gamma_{X,1})\right\}.$$

Here $\varepsilon>0$ is fixed such that $c_X([-\varepsilon,\varepsilon])\subset U$ for every $X\in V_1$ with $\|X\|=1$, where $U\subset\Omega$ is a fixed compact neighbourhood of x. For every $|h|\leq\varepsilon$, we have

$$\mathcal{A}_0^h\big((XF_1)\circ c_X-XF_1(x)\big)\leq\max_{\substack{y,z\in U\\ \|\exp^{-1}(y)-\exp^{-1}(z)\|\leq|h|}}\|XF_1(z)-XF_1(y)\|\,.$$

We notice that $\dot\Gamma_X(0)=XF_1(x)=d_Hf(x)(X)$, then setting

$$\omega_{d_Hf,U}(|h|)=\max_{\substack{y,z\in U\\ \|\exp^{-1}(y)-\exp^{-1}(z)\|\leq\varepsilon}}\|d_Hf(y)-d_Hf(z)\|$$

and observing that

$$L\leq\max_{y\in U}\|d_Hf(y)\|$$

$$\left\|\delta_{1/h}\Big(-h\,XF_1(x)\circledcirc\big(-\gamma_X(0)\big)\circledcirc\gamma_X(h)\Big)\right\|\leq\Upsilon\big(\max_{y\in U}\|d_Hf(y)\|\big)\;\omega_{d_Hf,U}(|h|)\,. \tag{68}$$

The right hand side is independent of x and tends to zero as $h\to0$, then taking the exponential of elements in the left hand side, we get the uniform convergence of

$$\delta_{1/h}\bigg(\exp\big(-hXF_1(y)\big)\Big(f(y)^{-1}f(y\exp(hX)\big)\Big)\bigg)$$

to the unit element e as $h\to0$, where y varies in the smaller neighbourhood

$$U_{-\varepsilon}=\{y\in U\mid \mathrm{dist}_{\|\cdot\|}(y,U^c)\geq\varepsilon\}\,.$$

of x. Thus, arguing as in Corollaire 3.3 of [51], we achieve the P-differentiability of f in a possibly smaller neighbourhood of x. This shows the everywhere P-differentiability of f. Observing that

$$d_Hf(x)(X)=df(x)(X) \tag{69}$$

for every $x\in\Omega$ and $X\in V_1$, we get that $x\longrightarrow Df(x)_{|\exp V_1}$ is continuous. The fact that $df(x)_{|V_j}$ polynomially depends on $df(x)_{|V_1}$ concludes the proof. □

5. Absolutely continuous curves in graded groups

This section is devoted to the characterization of absolutely continuous curves in graded groups. $\mathbb{M}$ and $\mathcal{M}$ denote a graded group along with its Lie algebra, equipped with a homogeneous distance ρ and a norm $\|\cdot\|$, respectively.

Definition 5.1. Let $\gamma : [0,s] \longrightarrow \mathcal{M}$ be an absolutely continuonus curve in $\mathcal{M}$ and let $\Sigma = (t_0, t_1, \ldots, t_N)$ be a partition of $[0,s]$, where $t_0 = 0$, $t_i < t_{i+1}$ and $t_N = s$ for some $N \in \mathbb{N}$. The associated "sum" with respect to the group operation in $\mathcal{M}$ is defined by

$$\sigma_\Sigma(\gamma)(s) = \sum_{k=0}^{N-1} \gamma(t_k)^{-1} \circledcirc \gamma(t_{k+1})\,. \tag{70}$$

We also set $\|\Sigma\| = \max_{1\le i\le N}(t_i - t_{i-1})$.

Lemma 5.1. *Let $\gamma : [0,s] \longrightarrow \mathcal{M}$ be an absolutely continuous curve. Then there exists*

$$\lim_{\|\Sigma\|\to 0^+} \sigma_\Sigma(s) = \gamma(s) - \gamma(0) + \sum_{n=2}^{\iota} \frac{(-1)^{n-1}}{n!} \int_0^s [\gamma(l), \dot\gamma(l)]_{n-1}\, dl\,. \tag{71}$$

Proof. Applying (13) and (70), we get

$$\sigma_\Sigma(s) = \sum_{k=0}^{N-1} \sum_{n=1}^{\iota} c_n\big(-\gamma(t_k), \gamma(t_{k+1})\big)\,,$$

then formula (24) yields

$$\sigma_\Sigma(s) = \gamma(s) - \gamma(0) + \sum_{k=0}^{N-1}\sum_{n=2}^{\iota} \frac{(-1)^{n-1}}{n!} \left[\frac{\gamma(t_k)+\gamma(t_{k+1})}{2}, \gamma(t_{k+1}) - \gamma(t_k)\right]_{n-1} \tag{72}$$
$$+ \sum_{k=0}^{N-1}\sum_{n=2}^{\iota} R_n\big(-\gamma(t_k), \gamma(t_{k+1})\big)\,.$$

We define $\mu_k = \big(\gamma(t_k) + \gamma(t_{k+1})\big)/2$, then the absolute continuity of γ yields

$$\sum_{k=0}^{N-1} [\mu_k, \gamma(t_{k+1}) - \gamma(t_k)]_{n-1} = \sum_{k=0}^{N-1} \left[\mu_k, \int_{t_k}^{t_{k+1}} \dot\gamma(l)\, dl\right]_{n-1} = \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} [\mu_k, \dot\gamma(l)]_{n-1}\, dl\,.$$

Now we notice that

$$\sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} [\mu_k, \dot\gamma(l)]_{n-1}\, dl = \int_0^s [\gamma(l), \dot\gamma(l)]_{n-1}\, dl$$
$$+ \sum_{p=0}^{n-2}\sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} [\mu_k, [\mu_k - \gamma(l), [\gamma(l), \dot\gamma(l)]_p]]_{n-p-2}\, dl,$$

hence the estimate

$$\left\| \sum_{p=0}^{n-2} \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} [\mu_k, [\mu_k - \gamma(l), [\gamma(l), \dot\gamma(l)]_p]]_{n-p-2}\, dl \right\|$$
$$\le (p-1)\, \beta^{n-2}\, \eta(\|\Sigma\|) \max_{t\in[0,s]} \|\gamma(t)\|^{n-2} \int_0^s \|\dot\gamma(l)\|\, dl$$

immediately proves the validity of the following

$$\lim_{\|\Sigma\|\to 0^+} \sum_{k=0}^{N-1} \left[\frac{\gamma(t_k)+\gamma(t_{k+1})}{2}, \gamma(t_{k+1}) - \gamma(t_k) \right]_{n-1} = \int_0^s [\gamma(l), \dot\gamma(l)]_{n-1}\, dl\,, \tag{73}$$

where we have set

$$\eta(\tau) = \sup_{|\beta-\alpha|\le\tau} \left| \int_\alpha^\beta \|\dot\gamma(l)\|\, dl \right|$$

which goes to zero as $\tau \to 0^+$, due to the absolute continuity of γ. Taking into account (25), (72) and (73), then limit (71) follows. □

Remark 5.2. Incidentally, the limit (73) exactly coincides with the "$(n-1)$-ième aire balayée" of Definition 4.6 in [51].

Definition 5.2. Let $\Gamma : [a,b] \longrightarrow \mathbb{M}$ be a continuous curve. Then we define

$$\mathrm{Var}_a^t \Gamma = \sup_{\substack{t_0=a<t_1<\cdots<t_N=t \\ N\in\mathbb{N}}} \sum_{k=1}^{N} \rho\left(\Gamma(t_k), \Gamma(t_{k-1})\right) .$$

Theorem 5.3. *Let $\Gamma : [a,b] \longrightarrow \mathbb{M}$ be a curve and define $\gamma = \exp^{-1} \circ \Gamma = \sum_{i=1}^{\upsilon} \gamma_i$, where γ_i takes values in V_i. Then the following statements are equivalent:*

(1) Γ *is absolutely continuous,*
(2) γ *is absolutely continuous and the differential equation*

$$\dot\gamma_i(t) = \sum_{n=2}^{\upsilon} \frac{(-1)^n}{n!}\, \pi_i\left([\gamma(t), \dot\gamma(t)]_{n-1}\right) \tag{74}$$

is satisfied a.e. for every $i \ge 2$.

If one of the previous conditions holds, then there exists a constant $C > 0$ only depending on ρ and $\|\cdot\|$, such that for any $\tau_1 < \tau_2$, we have

$$\rho\big(\Gamma(\tau_1), \Gamma(\tau_2)\big) \le C \int_{\tau_1}^{\tau_2} \|\dot\gamma_1(t)\|\, dt\,. \tag{75}$$

Proof. We first assume that Γ is absolutely continuous with respect to a homogeneous distance ρ fixed in $\mathbb{M}$. Observing that the image of Γ is bounded and applying Lemma 3.5 with $i = 1$, we immediately conclude that γ is also absolutely continuous with respect to the norm $\|\cdot\|$ fixed in $\mathcal{M}$; then γ is a.e. differentiable. Let $a < \tau < b$

be both an approximate continuity point of $\dot\gamma$ and a differentiability point of the total variation

$$[a,b]\ni l \longrightarrow \mathrm{Var}_a^l\Gamma \tag{76}$$

We fix $\varepsilon>0$ suitable small. We fix a bounded open neighbourhood U of e and choose $\delta>0$ such that $\exp\big(\gamma(\tau+t_{i-1})^{-1}\circledcirc\gamma(\tau+t_i)\big)\in U$ whenever $\Sigma=(t_0,t_1,\dots,t_N)$ is an arbitrary partition of $[0,s]$ with $\|\Sigma\|\le\delta$ and $0<s\le\varepsilon$. For every $i=1,\dots,\upsilon$, there exists a constant $k_i>0$ such that

$$\|\pi_i\big(\sigma_\Sigma(\gamma(\tau+\cdot))(s)\big)\|\le k_i\,\|\pi^i\big(\sigma_\Sigma(\gamma(\tau+\cdot))(s)\big)\| \tag{77}$$

$$\le k_i\,K_U\sum_{j=1}^N\rho\big(\Gamma(\tau+t_j),\Gamma(\tau+t_{j-1})\big)^i\le k_i\,K_U\left(\mathrm{Var}_\tau^{\tau+s}\Gamma\right)^i. \tag{78}$$

Taking the limit in the previous inequality as $\|\Sigma\|\longrightarrow0^+$ and applying Lemma 5.1, we obtain

$$\begin{aligned}&\left\|\gamma_i(\tau+s)-\gamma_i(\tau)+\sum_{n=2}^{\upsilon}\frac{(-1)^{n-1}}{n!}\int_0^s\pi_i\left([\gamma(\tau+l),\dot\gamma(\tau+l)]_{n-1}\right)dl\right\|\\ &\le\ k_i\,K_U\left(\mathrm{Var}_\tau^{\tau+s}\Gamma\right)^{i-1}\left(\mathrm{Var}_a^{\tau+s}\Gamma-\mathrm{Var}_a^{\tau}\Gamma\right)\end{aligned}$$

By our assumptions, τ is a differentiability point of both γ and of the total variation (76) and it is an approximate continuity point of $\dot\gamma$. Thus, dividing by s the last inequality and taking the limit as $s\to0^+$ for every $i\ge2$, equations (74) follow.

Now we assume the γ is absolutely continuous and that (74) a.e. holds for every $i\ge2$. Let us fix $a<\tau_1<\tau_2<b$ and consider

$$h\longrightarrow\Theta(h)=\Gamma(\tau_1)^{-1}\Gamma(\tau_1+h)=\exp\theta(h)=\exp\big(\theta_1(h)+\theta_2(h)+\cdots+\theta_\upsilon(h)\big),$$

where $\theta_i(h)\in V_i$. In view of Proposition 4.1 and observing that Θ, as left translated of Γ, is a.e. horizontal, we have that

$$\dot\theta_i(h)=\sum_{n=2}^{\upsilon}\frac{(-1)^n}{n!}\,\pi_i\left([\theta(h),\dot\theta(h)]_{n-1}\right) \tag{79}$$

a.e. holds and θ is absolutely continuous. Now, we wish to prove by induction that there exists a constant $C_i>0$, only depending on the norm constant β, such that

$$\int_0^{h_0}\|\dot\theta_i(t)\|\,dt\le C_i\left(\int_0^{h_0}\|\dot\theta_1\|\right)^i \tag{80}$$

for every $i\ge1$, with $h_0=\tau_2-\tau_1$. To prove (80), we first write (79) as follows

$$\dot\theta_i(h)=\sum_{n=2}^{\upsilon}\frac{(-1)^n}{n!}\sum_{\substack{1\le l_1,\dots,l_n\le\upsilon\\ l_1+\cdots+l_n=i}}[\theta_{l_1}(h),[\theta_{l_2}(h),[\cdots,[\theta_{l_{n-1}}(h),\dot\theta_{l_n}(h)],],\dots,],\tag{81}$$

since $\mathbb{M}$ is a graded group. For $i=1$, inequality (80) is trivial, choosing $C_1 \geq 1$. Let us assume that (80) holds for every $i \leq j$. Due to (81) for $i=j+1$ and our induction hypothesis, we have

$$\int_0^{h_0} \|\dot\theta_{j+1}(t)\|\, dt \leq \sum_{n=2}^{\upsilon} \frac{\beta^{n-1}}{n!} \sum_{\substack{1\leq l_1,\ldots,l_n\leq \upsilon\\ l_1+\cdots+l_n=j+1}} \int_0^{h_0} \|\theta_{l_1}(t)\|\,\|\theta_{l_2}(t)\|\cdots\|\theta_{l_{n-1}}(t)\|\,\|\dot\theta_{l_n}(t)\|\, dt$$

$$\leq \sum_{n=2}^{\upsilon} \frac{\beta^{n-1}}{n!} \sum_{\substack{1\leq l_1,\ldots,l_n\leq \upsilon\\ l_1+\cdots+l_n=j+1}} C_{l_1}\cdots C_{l_n}\Big(\int_0^{h_0}\|\dot\theta_1(t)\|\,dt\Big)^{j+1}$$

$$= \left(\int_0^{h_0}\|\dot\theta_1(t)\|\,dt\right)^{j+1}\Big(\sum_{n=2}^{\upsilon} \frac{\beta^{n-1}}{n!} \sum_{\substack{1\leq l_1,\ldots,l_n\leq \upsilon\\ l_1+\cdots+l_n=j+1}} C_{l_1}\cdots C_{l_n}\Big).$$

Thus, we have proved that

$$\|\theta_i(h_0)\| = \big\|\pi_i\big(-\gamma(\tau_1)\circledcirc\gamma(\tau_2)\big)\big\| \leq C_i \left(\int_{\tau_1}^{\tau_2}\|\dot\gamma_1\|\right)^i, \tag{82}$$

for every $i=1,\ldots,\upsilon$. These estimates show the validity of (75), whose proves in turn that Γ is absolutely continuous. $\square$

Corollary 5.4. *Let $\Gamma:[a,b]\longrightarrow\mathbb{M}$ be a curve and define $\gamma=\exp^{-1}\circ\Gamma=\sum_{i=1}^{\upsilon}\gamma_i$, where γ_i takes values in V_i. Then the following statements are equivalent:*

(1) *Γ is Lipschitz,*
(2) *γ is Lipschitz and the differential equation*

$$\dot\gamma_i(t) = \sum_{n=2}^{\upsilon}\frac{(-1)^n}{n!}\,\pi_i\left([\gamma(t),\dot\gamma(t)]_{n-1}\right) \tag{83}$$

is satisfied a.e. for every $i\geq 2$.

If one of the previous conditions holds, then there exists a constant $C>0$ only depending on ρ and $\|\cdot\|$, such that

$$C^{-1}\,\mathrm{Lip}(\gamma_1)\leq \mathrm{Lip}(\Gamma)\leq C\,\mathrm{Lip}(\gamma_1) \tag{84}$$

Proof. In view of Theorem 5.3, if Γ is Lipschitz, then γ is absolutely continuous and (83) a.e. hold. By the general estimate (33), γ_1 is Lipschitz and we have a constant $C_1>0$, only depending on ρ and $\|\cdot\|$, such that

$$C_1^{-1}\,\mathrm{Lip}(\gamma_1)\leq\mathrm{Lip}(\Gamma).$$

Thus, from (75) the estimate (84) follows. Applying recursively the equations (83), the Lipschitz property of γ_1 implies the Lipschitz property of all γ_j's. Conversely, if γ is Lipschitz and (83) a.e. hold, then Theorem 5.3 implies that Γ is absolutely continuous and the estimate (75) yields the Lipschitz property of Γ. $\square$

Corollary 5.5. *Let $\Gamma : [a,b] \longrightarrow \mathbb{M}$ be an absolutely continuous curve. Then the following formula holds*

$$\mathrm{Var}_a^b \Gamma = \int_a^b \rho\Big(\exp\left(\dot\gamma_1(t)\right)\Big)\, dt\,.$$

where $\mathrm{Var}_a^b\Gamma$ is computed with respect to the homogeneous distance ρ.

Proof. Our claim is a consequence of a.e. P-differentiability of Γ proved in Theorem 4.4 and the general formula

$$\mathrm{Var}_a^b \Gamma = \int_a^b |\dot\Gamma(t)| \tag{85}$$

that holds in metric spaces, where $|\dot\Gamma(t)|$ is the metric derivative, that in this case coincides with $\rho\big(\exp\left(\dot\gamma_1(t)\right)\big)$, by definition of P-differentiability. Formula (85) can be found in [2] for Lipschitz curves in metric spaces. Its extension to absolutely continuous curves is straightforard and can be proved in a similar manner, taking into account formula (75). □

5.1. **Lipschitz property of P-differentiable mappings.** In the sequel, $\mathbb{G}$ will denote a stratified group with Lie algebra $\mathcal{G}$ and $\Omega \subset \mathbb{G}$ will be an open set. The mapping $f : \Omega \longrightarrow \mathbb{M}$ will be represented in the algebra by $F = \exp^{-1} \circ f : \Omega \longrightarrow \mathcal{M}$ and $F_j = \pi_j \circ F$, where $\pi_j : \mathcal{M} \longrightarrow W_j$ are the canonical projections onto the layers.

Lemma 5.6. *Let V be a finite dimensional normed vector space and let $F : \Omega \longrightarrow V$ be continuously P-differentiable. There exist constants $\mu_0 > 1$ and $C_0 > 0$, only depending on $(\mathbb{G}, d)$, such that*

$$\|F(x) - F(y)\| \leq C_0 \max_{z \in D_{\xi,\mu_0 r}} \|dF(z)\| \; d(x,y)$$

for every $x, y \in D_{\xi,r}$, where $D_{\xi,\mu_0 r} \subset \Omega$.

Proof. We consider $c_0 = c(\mathbb{G}, d)$ as in (58) and fix $\mu_0 = 2(1 + c_0 N)$. If $D_{\xi,\mu_0 r} \subset \Omega$ for a suitable small $r > 0$, then one can check that

$$A_{\xi,r} = \{ y \in \mathbb{G} \mid \mathrm{dist}(y, B_{\xi,r}) \leq 2\, c_0\, N\, r\} \subset B_{\xi,\mu_0 r}\,.$$

Let $x, y \in B_{\xi,r}$ and observe that Remark 4.7 implies the existence of $a \in (P^N)^{-1}(D_1)$ such that $y = x\delta_{d(x,y)} P^N(a)$. We fix $\lambda = d(x,y)$, then taking into account that $x \in B_{\xi,r}$ one easily realizes that

$$[0, \lambda\, a_s] \ni l \longrightarrow x\delta_\lambda P^{s-1}(a) \delta_l h_{i_s} \in A_{\xi,r} \subset B_{\xi,\mu_0 r} \subset \Omega$$

for every $s = 1, \ldots, N$. Now, for every horizontal direction $h = \exp X \in \exp(V_1)$ and every $z \in \Omega$ such that

$$[0,1] \ni t \longrightarrow \Gamma(t) = z \exp tX = z\delta_t h \in \Omega$$

we have the integral formula

$$F(xh) - F(x) = \int_0^1 DF(x \exp tX)(h)\, dt\,,$$

since $t \longrightarrow F\circ\Gamma(t)$ is continuously differentiable with $\big(F\circ\Gamma\big)'(t) = DF(x\exp tX)(h)$. It follows that

$$\|F(xh) - F(x)\| \le \|X\| \int_0^1 \|dF(x\delta_t h)\|\, dt\,. \tag{86}$$

For every $s = 1, \dots, N$, we apply (86) to the curve

$$\Gamma(t) = x\delta_\lambda P^{s-1}(a)\delta_{t\lambda a_s} h_{i_s} \in D_{\xi,\mu_0 r} \quad \text{for every} \quad t \in [0,1]$$

obtaining the final estimates

$$\begin{aligned} \|F(y) - F(x)\| &\le \sum_{s=1}^N \|\exp^{-1}\big(\delta_{\lambda a_s} h_{i_s}\big)\| \int_0^1 \|dF(x\delta_\lambda P^{s-1}(a)\delta_{t\lambda a_s} h_{i_s})\|\, dt \\ &\le c_0\, d(x,y) \max_{1\le s\le N} \|\exp^{-1}\big(h_{i_s}\big)\| \max_{y\in D_{\xi,\mu_0 r}} \|dF(y)\| = C_0\, d(x,y) \max_{y\in D_{\xi,\mu_0 r}} \|dF(y)\|\,, \end{aligned} \tag{87}$$

that conclude the proof. □

Proposition 5.7. *Let $c : [a,b] \longrightarrow \Omega$ be a Lipschitz curve and let $f : \Omega \longrightarrow \mathbb{M}$ be a continuously P-differentiable mapping. Then there exists a constant $C > 0$, only depending on d, ρ and the norms on $\mathcal{G}$ and $\mathcal{M}$, such that $\Gamma = f \circ c$ is Lipschitz and the following formulae hold*

$$\rho\big(\Gamma(t), \Gamma(\tau)\big) \le C \,\mathrm{Lip}(c) \max_{x\in c([a,b])} \|dF_1(x)\|\, |t-\tau|\,, \tag{88}$$

$$\dot\gamma_1(t) = dF_1\big(c(t)\big)\circ dc(t) = dF_1\big(c(t)\big)\circ \dot\alpha_1(t)\,, \tag{89}$$

where we have set $\gamma_j = \pi_j \circ \exp^{-1}\circ\Gamma$, $\alpha_j = \pi_j\circ\exp^{-1}\circ c$,

$$\Gamma = \exp\big(\gamma_1 + \cdots + \gamma_\upsilon\big) \qquad \textit{and} \qquad c = \exp(\alpha_1 + \cdots + \alpha_\iota)\,.$$

Proof. Let $c : [a,b] \longrightarrow \Omega$ be a horizontal curve, where $f : \Omega \longrightarrow \mathbb{M}$ is continuously P-differentiable. Due to Theorem 4.8, F_j are continuously P-differentiable, hence Lemma 5.6 yields their local Lipschitz property. Thus, the composition

$$F \circ c : [a,b] \longrightarrow \mathcal{M}$$

is locally Lipschitz, hence it is absolutely continuous. By Corollary 5.4, c is horizontal, then Theorem 4.4 yields the a.e. P-differentiability of c. In view of the chain rule stated in Proposition 2.7, it follows that Γ, equal to $f\circ c$, is a.e. P-differentiable and

$$d\Gamma(t) = df\big(c(t)\big)\big(dc(t)\big)\,.$$

for a.e. $t \in [a,b]$. Applying Corollary 4.9 and the characterization of Proposition 4.1, it follows that Γ is horizontal. In view of (61), the formula above yields the a.e. validity of (89), since c is horizontal. Due to (89) and (84), it follows that Γ is Lipschitz and formula (88) follows. □

Corollary 5.8. *Every continuously P-differentiable mapping $f : \Omega \longrightarrow \mathbb{M}$ is locally Lipschitz.*

Proof. It suffices to observe that Ω is locally connected by geodesics with respect to a Carnot-Carathéodory distance fixed on $\mathbb{G}$ and then apply Proposition 5.7. □

6. Mean value inequality

Throughout this section, $\mathbb{G}$ and $\mathcal{G}$ denote a stratified group along with its Lie algebra, $\Omega \subset \mathbb{G}$ is a fixed open set, d is a homogeneous distance on $\mathbb{G}$. $\mathbb{M}$ is a graded group equipped with a homogeneous distance ρ. When the norm $\|\cdot\|$ is applied to a vector of either $T\mathbb{G}$ or $T\mathbb{M}$, it is understood that it represents the left invariant Finsler norm generated by the fixed norm in the Lie algebra of either $\mathbb{G}$ or $\mathbb{M}$. The mapping $f : \Omega \longrightarrow \mathbb{M}$ will be represented in the algebra by $F = \exp^{-1} \circ f : \Omega \longrightarrow \mathcal{M}$ and $F_j = \pi_j \circ F$, where $\pi_j : \mathcal{M} \longrightarrow W_j$ are the canonical projections onto the layers.

Theorem 6.1. *Let $f : \Omega \longrightarrow \mathbb{M}$ be a continuously P-differentiable mapping and let Ω' be an open subset compactly contained in Ω. Let $c : [a, b] \longrightarrow \Omega'$ be a Lipschitz curve with $\|\dot c(t)\| \le 1$ a.e. Then there exists a constant C, depending on the modulus of continuity of dF_1 on $\overline{\Omega'}$ and on $L = \max_{x\in\overline{\Omega'}} \|dF_1(x)\|$, such that at approximate continuity points t of $\dot c$ the following estimate holds*

$$\rho\Big(Df\big(c(t)\big)\big(\exp \dot c(t)\, h\big), f\big(c(t)\big)^{-1} f\big(c(t+h)\big)\Big) \tag{90}$$
$$\le C\left[L^{1/\iota} \sup_{s\in I_{t,t+h}} \left|\fint_t^{t+h} \|\dot c(l) - \dot c(t)\|\, dl\right|^{1/\iota} + \max_{s\in I_{t,t+h}} \|dF_1\big(c(s)\big) - dF_1\big(c(t)\big)\|^{1/\iota}\right]|h|,$$

where we have set $I_{t,t+h} = [\min\{t, t+h\}, \max\{t, t+h\}]$.

Proof. By Proposition 5.7, it follows that $\Gamma = f \circ c = \exp \circ \gamma$ is Lipschitz and

$$\dot\gamma_1(t) = dF_1(c(t))\big(\dot\alpha_1(t)\big)\,, \tag{91}$$

where $df = \exp^{-1} \circ Df \circ \exp$ and $c = \exp\big(\alpha_1 + \cdots + \alpha_\iota\big)$. Observing that $\|\dot\alpha_1\| \le 1$ a.e., we then apply Theorem 4.4 to Γ. As a consequence, taking into account (91), the estimate (52) applied to Γ yields

$$\left\|\delta_{1/h}\Big[-h\, dF_1\big(c(t)\big)\big(\dot\alpha_1(t)\big) \circledcirc \big(-F\big(c(t)\big) \circledcirc F\big(c(t+h)\big)\Big]\right\| \tag{92}$$
$$\le \Upsilon(L) \sup_{s\in I_{t,t+h}} \left|\fint_t^s \|dF_1\big(c(l)\big)\big(\dot\alpha_1(l)\big) - dF_1\big(c(t)\big)\big(\dot\alpha_1(t)\big)\| dl\right|$$
$$\le \Upsilon(L)\left[L \sup_{s\in I_{t,t+h}} \left|\fint_t^s \|\dot\alpha_1(l) - \dot\alpha_1(t)\|\, dl\right| + \max_{s\in I_{t,t+h}} \|dF_1\big(c(s)\big) - dF_1\big(c(t)\big)\|\right]$$

where Υ is the a nondecreasing function defined in Theorem 4.4. Now, applying (32) to (92), we obtain a constant $C > 0$ such that (90) holds. It is not difficult to notice that C only depends on the quantities stated in our claim. □

Corollary 6.2. *Let $f : \Omega \longrightarrow \mathbb{M}$ be a continuously P-differentiable mapping and let Ω_1, Ω_2 be open subsets such that Ω_2 is compactly contained in Ω and for every $x, y \in \Omega_1$ such that $x^{-1}y \in \exp(V_1)$ we have $x\delta_l(x^{-1}y) \in \overline{\Omega}_2$ whenever $0 \le l \le 1$. Then there exists a constant C, depending on the modulus of continuity of $\omega_{\overline{\Omega}_2,dF_1}$ and* *on* $\max_{x\in\overline{\Omega_2}}\|dF_1(x)\|$ *such that*

$$\rho\left(Df(x)(x^{-1}y),f(x)^{-1}f(y)\right)\le C\,\omega_{\overline{\Omega}_2,dF_1}\big(d(x,y)\big)^{1/\iota}\,d(x,y)\,. \tag{93}$$

whenever $x,y\in\Omega_1$ *and* $x^{-1}y\in\exp V_1$.

Proof. It suffices to apply Theorem 6.1 to the curve $c(l)=x\delta_l u$, with $[a,b]=[0,\rho(x,y)]$, $t=0$, $h=d(x,y)$ and $u=\delta_{1/\rho(x,y)}(x^{-1}y)$. Then

$$\rho\left(Df(x)(x^{-1}y),f(x)^{-1}f(y)\right)\le C\left[\max_{l\in I_{t,t+h}}\|dF_1(c(l))-dF_1(x)\|^{1/\iota}\right]d(x,y)\,.$$

Observing that $d(c(l),x)\le d(x,y)$, estimate (93) follows. □

Proof of Theorem 1.2. Let $x,y\in\Omega_1$ and let $a\in(P^N)^{-1}(D_1)$ be such that $P^N(a)=\delta_{1/d(x,y)}\big(x^{-1}y\big)$. Thus, due to Definition 4.3, we have

$$d(P^s(a))\le N\,c(\mathbb{G},d)$$

where $c(\mathbb{G},d)$ and N are defined in (58) and in Lemma 4.6, respectively. For the sequel, we set $c_0=c(\mathbb{G},d)$. Analogously, we have

$$d\big(\delta_t P^{s-1}(a)\delta_l h_{i_s}\big)\le c_0\,N\,\mathrm{diam}(\Omega_1)$$

if $l\in[0,ta_s]$ and $0\le t\le\mathrm{diam}(\Omega_1)$. We fix $t_0=d(x,y)\le\mathrm{diam}(\Omega_1)$, hence from our assumptions it follows that each curve

$$[0,t_0a_s]\ni l\longrightarrow x\delta_{t_0}P^{s-1}(a)\delta_l h_{i_s}$$

is contained in Ω_2. Applying (90) to this curve, there exists a constant C_1, only depending on both $\omega_{\overline{\Omega}_2,dF_1}$ and $\max_{x\in\overline{\Omega_2}}\|dF_1(x)\|$, such that

$$\begin{aligned}&\rho\Big(f(x\delta_{t_0}P^{s-1}(a))^{-1}f\big(x\delta_{t_0}P^s(a)\big),Df(x\delta_{t_0}P^{s-1}(a))\big(\delta_{a_st_0}h_{i_s}\big)\Big)\\&\le C_1\,|a_s\,t_0|\,\omega_{\overline{\Omega}_2,dF_1}\big(c_0\,t_0\big)^{1/\iota}.\end{aligned} \tag{94}$$

Then we obtain the key uniform estimate

$$\begin{aligned}&\rho\Big(\delta_{1/t_0}\Big(f(x\delta_{t_0}P^{s-1}(a))^{-1}f\big(x\delta_{t_0}P^s(a)\big)\Big),Df(x\delta_{t_0}P^{s-1}(a))\big(\delta_{a_s}h_{i_s}\big)\Big)\\&\le C_1\,c_0\,\omega_{\overline{\Omega}_2,dF_1}\big(c_0\,t_0\big)^{1/\iota}.\end{aligned} \tag{95}$$

We fix $\nu=\max_{x\in\overline{\Omega_2}}\|dF_1(x)\|$, so that (31) yields a constant $K_\nu>0$ such that

$$\rho\big(Df(x)\big)\le k_\nu\|dF_1(x)\|^{1/\iota}$$

where we have set $\rho(Df(x))=\max_{d(h)=1}\rho\big(Df(x)(h)\big)$. We have the estimate

$$\rho\big(Df(x)(\delta_{a_s}h_{i_s}\cdots\delta_{a_N}h_{i_N})\big)\le c_0\,k_\nu\,N\,\nu^{1/\iota} \tag{96}$$

In view of (95) and (96), we use (40), getting a constant C_2, only depending on $\mathbb{G}$, ν and $\omega_{\overline{\Omega},dF_1}$ such that

$$\rho\Big(\delta_{1/t_0}\big(f(x)^{-1}f\big(x\delta_{t_0}P^N(a)\big)\big), Df(x)\big(P^N(a)\big)\Big) \leq C_2\sum_{s=1}^{N}\rho\Big(\delta_{1/t_0}\big(f(x\delta_{t_0}P^{s-1}(a))^{-1}f\big(x\delta_{t_0}P^s(a)\big)\big), Df(x)\big(\delta_{a_s}h_{i_s}\big)\Big)^{1/\iota}, \tag{97}$$

where we have used the decomposition

$$\delta_{1/t_0}\big(f(x)^{-1}f\big(x\delta_{t_0}P^N(a)\big)\big) = \prod_{s=1}^{N}\delta_{1/t_0}\Big(f\big(x\delta_{t_0}P^{s-1}(a)\big)^{-1}f\big(x\delta_{t_0}P^s(a)\big)\Big). \tag{98}$$

By (95) we have

$$\rho\Big(\delta_{1/t_0}\Big(f\big(x\delta_{t_0}P^{s-1}(a)\big)^{-1}f\big(x\delta_{t_0}P^s(a)\big)\Big), Df(x)\big(\delta_{a_s}h_{i_s}\big)\Big)^{1/\iota} \leq \big(C_1\,c_0\big)^{1/\iota}\,\omega_{\overline{\Omega}_2,dF_1}\big(c_0\,t_0\big)^{1/\iota^2} + \rho\Big(Df(x\delta_{t_0}P^{s-1}(a)(\delta_{a_s}h_{i_s}), Df(x)\big(\delta_{a_s}h_{i_s}\big)\Big)^{1/\iota} \tag{99}$$

By (32) we have a constant $C_3>0$ depending on c and ν, such that

$$\rho\Big(Df(x\delta_{t_0}P^{s-1}(a))(\delta_{a_s}h_{i_s}), Df(x)\big(\delta_{a_s}h_{i_s}\big)\Big) \leq c_0\;C_3\;\omega_{\overline{\Omega_2},dF_1}(N\,c_0\,t_0)^{1/\iota} \tag{100}$$

Joining (97), (99) and (100), we achieve

$$\rho\Big(\delta_{1/t_0}\big(f(x)^{-1}f\big(x\delta_{t_0}P^N(a)\big)\big), Df(x)\big(P^N(a)\big)\Big) \leq N\,C_2\Big(c_0^{1/\iota}\,C_1^{1/\iota}\omega_{\overline{\Omega}_2,dF_1}(c_0t_0)^{1/\iota^2} + c_0^{1/\iota}\,C_3^{1/\iota}\omega_{\overline{\Omega}_2dF_1}(N\,c_0\,t_0)^{1/\iota^2}\Big).$$

Taking into account that $y = x\delta_{t_0}P^N(a)$, then the previous estimate can be precisely written as follows

$$\frac{\rho\Big(f(x)^{-1}f(y), Df(x)(x^{-1}y)\Big)}{d(x,y)} \leq N\,C_2\,c_0^{1/\iota}\Big(C_1^{1/\iota}\omega_{\overline{\Omega}_2,dF_1}(c_0\,d(x,y))^{1/\iota^2} + C_3^{1/\iota}\omega_{\overline{\Omega}_2dF_1}(N\,c_0\,d(x,y))^{1/\iota^2}\Big).$$

The estimate obtained in Ω_1 easily extends to its closure. This concludes the proof. $\square$

6.1. **Inverse mapping theorem.** As an immediate consequence of the mean value inequality we achieve the inverse mapping theorem for P-differentiable mappings.

Theorem 6.3. *Let Ω be an open subset of $\mathbb{G}$ and let $f:\Omega\longrightarrow\mathbb{G}$ be a continuously P-differentiable mapping and assume that $Df(x):\mathbb{G}\longrightarrow\mathbb{G}$ is invertible at some $\overline{x}\in\Omega$. Then there exists a neighbourhood U of $\overline{x}$ such that the restriction $f_{|U}$ has an inverse mapping g that is also P-differentiable and for every $y\in f(U)$ we have*

$$Dg(y) = Df(g(y))^{-1}.$$

PROOF. By continuity of $x \longrightarrow Df(x)$, there exists a compact neighbourhood U' of $\overline{x}$ and a number $\mu > 0$ such that

$$\min_{x \in U'} \min_{d(u)=1} \rho\big(Df(x)(u)\big) = \mu.$$

Then triangle inequality and estimates (6) give us a constant $\beta > 0$ and an open neighbourhood U contained in U' and depending on the modulus of continuity $\omega_{df,U'}$, such that

$$d\big(f(x), f(y)\big) \geq \beta\, d(x,y)$$

whenever $x, y \in U$. Then $f_{|U}$ is invertible onto its image $f(U)$ and by the domain invariance theorem, see for instance Theorem 3.3.2 of [37], $f_{|U}$ is an open mapping. Then the inverse mapping $g : f(U) \longrightarrow U$ is continuous and the chain rule of Proposition 2.7 concludes the proof. □

7. Homogeneous subgroups

Throughout this section, we assume that all subgroups are closed, connected and simply connected Lie subgroups, if not otherwise stated. Here $\mathbb{G}$ denotes an arbitrary graded group, that is not necessarily stratified. Its Lie algebra is denoted by $\mathcal{G}$. From 5.2.4 of [62], we recall the notion of homogeneous subalgebra and the corresponding notion of homogeneous subgroup.

Definition 7.1 (Homogeneous subalgebra)**.** Let $\mathfrak{p} \subset \mathcal{G}$ be a Lie subalgebra. We say that $\mathfrak{p}$ is a *homogeneous subalgebra* if $\delta_r \mathfrak{p} \subset \mathfrak{p}$ for every $r > 0$.

Remark 7.1. It is not difficult to find examples of subalgebras which are not homogeneous. It suffices to consider $\mathcal{L} = \mathrm{span}\{X + Z\}$, which is a subalgebra of the Heisenberg algebra $\mathfrak{h}^1$ of brackets $[X, Y] = Z$. However, $\delta_2(X + Z) = 2X + 4Z \notin \mathcal{L}$.

Proposition 7.2. *Let $\mathfrak{a}$ be a homogeneous subalgebra of $\mathcal{G}$, where $\mathcal{G}$ is decomposed into the direct sum $V_1 \oplus \cdots \oplus V_\iota$. Then we have*

$$\mathfrak{a} = (V_1 \cap \mathfrak{a}) \oplus (V_2 \cap \mathfrak{a}) \oplus \cdots \oplus (V_\iota \cap \mathfrak{a}).$$

PROOF. We have only to prove that each vector $\xi \in \mathfrak{a}$ with the unique decomposition $\xi = \sum_{j=1}^{\iota} \xi_j$, with $\xi_j \in V_j$, satisfies the property $\xi_j \in \mathfrak{a}$. By hypothesis, $\delta_r \xi \in \mathfrak{a}$ whenever $r > 0$, then the closedness of $\mathfrak{a}$ implies that

$$\lim_{r \to 0^+} \frac{1}{r}\, \delta_r \xi = \xi_1 \in \mathfrak{a}.$$

This implies that $\xi - \xi_1 \in \mathfrak{a}$, then

$$\lim_{r \to 0^+} \frac{1}{r^2}\, \delta_r(\xi - \xi_1) = \xi_2 \in \mathfrak{a}.$$

Iterating this process by induction, our claim follows. □

Corollary 7.3. *Every homogeneous subalgebra $\mathfrak{a} \subset \mathcal{G}$ is a graded algebra.*

Remark 7.4. Let $\mathfrak{a} = \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_\iota$ be a homogeneous subalgebra. In general, some factor $\mathfrak{a}_j$ might be the null space. Consider for instance the homogeneous subalgebra $\mathfrak{a} = V_2 \oplus V_4 \oplus \cdots \oplus V_{2[\iota/2]}$, where $\mathcal{G} = V_1 \oplus V_2 \oplus \cdots \oplus V_\iota$.

Definition 7.2 (Homogeneous subgroup)**.** Let $P \subset \mathbb{G}$ be a Lie subgroup. We say that P is a *homogeneous subgroup* if $\delta_r P \subset P$ for every $r > 0$.

It is clear that all properties of homogeneous subalgebras are exactly translated to homogeneous subgroups. Thus, in the sequel we will equivalently work with either homogeneous subalgebras or homogeneous subgroups.

7.1. **Complementary subgroups.** This subsection is devoted to the notion of complementary subgroup, that plays an important role in the algebraic side of this paper.

When A and B are subsets of an abstract group G we will use the notation

$$AB = \{ab \,|\, a \in A, b \in B\}.$$

Definition 7.3 (Complementary subgroup)**.** Let P be a homogeneous subgroup of $\mathbb{G}$. If there exists a homogeneous subgroup H of $\mathbb{G}$ satisfying the properties $PH = \mathbb{G}$ and $P \cap H = \{e\}$, then we say that H is a *complementary subgroup* to P.

Remark 7.5. Recall that for abstract subgroups A, B of an abstract group G, the subset AB is an abstract subgroup if and only if $AB = BA$, see for instance [30]. As a consequence, H is complementary to P if and only if P is complementary to H.

Due to the previous remark, being complementary is a symmetric relation and we can say that two subgroups are *complementary*.

Remark 7.6. Let H and P be complementary subgroups of $\mathbb{G}$ and let $g \in \mathbb{G}$. Then it is immediate to check that there exist unique elements $h, h' \in H$ and $p, p' \in P$ such that $g = hp = p'h'$.

The following proposition characterizes complementary subgroups through their Lie subalgebras.

Proposition 7.7. *Let $\mathfrak{p}$ and $\mathfrak{h}$ be homogeneous subalgebras of $\mathcal{G}$ and let P and H denote their corresponding homogeneous subgroups, respectively. Then the condition $\mathfrak{p} \oplus \mathfrak{h} = \mathcal{G}$ is equivalent to require that P and H are complementary subgroups. Furthermore, if one of these conditions hold, then the mapping*

$$\phi : \mathfrak{p} \times \mathfrak{h} \longrightarrow \mathbb{G}, \qquad \phi(W, Y) = \exp W \ \exp Y \tag{101}$$

is a diffeomophism.

Proof. Assume that $\mathfrak{p} \oplus \mathfrak{h} = \mathcal{G}$. By Theorem 2.10.1 of [63] the exist bounded neighbourhoods Ω_1 and Ω_2 of the origin in $\mathfrak{p}$ ad in $\mathfrak{h}$, respectively, such that the mapping restricted to $\Omega_1 \times \Omega_2$ is a diffeomorphism onto some neighbourhood U of e. Observing that

$$\delta_r\big(\phi(W, Y)\big) = \phi(\delta_r W, \delta_r Y)$$

for every $r > 0$, it follows that ϕ is invertible. This shows that P and H are complementary subgroups. Viceversa, suppose that P and H are complementary subgroups and assume by contradiction that there exists $X \in \mathfrak{p} \cap \mathfrak{h} \setminus \{0\}$. This implies that $\exp X \in H \cap P = \{e\}$, that is absurd. Now, we wish to show that $\mathfrak{p} \oplus \mathfrak{h} = \mathcal{G}$. By contradiction, we assume that $\dim\big(\mathfrak{p} \oplus \mathfrak{h}\big) < \dim \mathcal{G}$. We notice that the mapping ϕ defined in (101) has everywhere injective differential, since $\mathfrak{h} \cap \mathfrak{p} = \{0\}$ and both $\exp_{|\mathfrak{p}}$ and $\exp_{|\mathfrak{h}}$ have everywhere injective differential. Thus, $\phi(\mathfrak{p} \times \mathfrak{h}) \subset \mathbb{G}$ is an open subset, whose dimension is less than $\dim \mathcal{G}$. As a consequence, ϕ cannot be surjective and this conflicts with hypothesis $PH = \mathbb{G}$. □

Remark 7.8. Joining Proposition 7.2 and Proposition 7.7, we have the following property. If K and P are complementary subgroups, then

$$Q = \mathcal{H}\text{-dim}(K) + \mathcal{H}\text{-dim}(P), \tag{102}$$

where $\mathcal{H}$-dim denotes the Hausdorff dimension with respect to a fixed homogeneous distance. One can interpret this fact as a "proper splitting" of the ambient group even with respect to the metric point of view.

Through Proposition 7.7, it is easy to observe that not any subgroup admits a complementary subgroup.

Example 7.9. Let us consider the second layer $\mathfrak{n} = \text{span}\{Z\}$ of the Heisenberg algebra $\mathfrak{h}^1$, with bracket relations $[X, Y] = Z$. Then the normal subgroup $N = \exp(\mathfrak{n})$ does not possess any complementary subgroup, see also [22]. In fact, let $\mathfrak{a}$ be any 2-dimensional homogeneous subalgebra of $\mathfrak{h}^1$. Then Proposition 7.2 gives the decomposition $\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2$, where $\mathfrak{a}_j$ are contained in the j-th layer, $j = 1, 2$. If $\mathfrak{a}_2 = \{0\}$, then $\mathfrak{a} = \mathfrak{a}_1 = \mathfrak{v}$ is not a subalgebra of $\mathfrak{h}^1$. Thus, $\mathfrak{a}$ must contain $\mathfrak{n}$ and this conflicts with existence of a 2-dimensional subalgebra complementary to $\mathfrak{n}$.

Remark 7.10. More generally there do not exist 2-dimensional subalgebras $\mathfrak{a}$ of $\mathfrak{h}^1$ such that $\mathfrak{a} \oplus \mathfrak{n} = \mathfrak{h}^1$, even if we do not require the homogeneity of $\mathfrak{a}$. In fact, let $\mathfrak{a} = \text{span}\{U + \alpha Z, U' + \alpha' Z\}$ be a 2-dimensional subalgebra of $\mathfrak{h}^1$, where U, U' belong to the first layer $\mathfrak{v}$. If $[U, U'] = 0$ then they are proportional and $\mathfrak{a} \oplus \mathfrak{n}$ is 2-dimensional. It follows that $[U, U'] = \gamma Z$, with $\gamma \neq 0$. On the other hand, $\mathfrak{a}$ is a subalgebra, hence $Z \in \mathfrak{a}$ and this would imply $\mathfrak{a} = \mathfrak{h}^1$. This is also a contradiction, then such a complementary subalgebra cannot exist.

We also point out that a complementary subgroup, when it exists need not be unique, as we show in the next

Example 7.11. All the homogeneous subalgebras

$$\mathfrak{a}_\lambda = \text{span}\{X + \lambda Y\} \qquad \lambda \in \mathbb{R},$$

of $\mathbb{H}^1$ yield complementary subgroups $A_\lambda = \exp \mathfrak{a}_\lambda$ of $S = \exp\big(\text{span}\{Y, Z\}\big)$.

Next, we wish to see how the notion of complementary subgroup is translated for the corresponding subalgebras, when $\mathbb{G}$ is a graded group. First of all, we notice that a decomposition of $\mathcal{G}$ as a direct sum of two subspaces does not ensure a corresponding decomposition of $\mathbb{G}$ as the product of their images through the exponential mapping. This simple fact is shown in the next example.

Example 7.12. We consider the Heisenberg algebra $\mathfrak{h}^1$ with basis (X, Y, Z) and bracket relation $[X, Y] = Z$. We consider the following subspaces of $\mathfrak{h}^1$

$$\begin{cases} \mathfrak{u} = \operatorname{span}\{X, Y\} \\ \mathfrak{w} = \operatorname{span}\{Y + \frac{1}{2}Z\} \end{cases} .$$

It is immediate to check that $\mathfrak{u} \oplus \mathfrak{w} = \mathfrak{h}^1$, but $\exp \mathfrak{u}\ \exp \mathfrak{w} \neq \mathbb{H}^1$, since

$$\exp(-X + \lambda Z) \notin \exp \mathfrak{u}\ \exp \mathfrak{w}$$

for every $\lambda \neq 0$.

Even the decomposition of $\mathcal{G}$ as the direct sum of two subalgebras does not imply that their corresponding subgroups factorize $\mathbb{G}$, as we show in the next

Example 7.13. We consider the Heisenberg algebra $\mathfrak{h}^2$ with basis (X_1, Y_1, X_2, Y_2, Z) and bracket relations $[X_1, Y_1] = [X_2, Y_2] = Z$. We consider the following subalgebras of $\mathfrak{h}^2$

$$\begin{cases} \mathfrak{a} = \operatorname{span}\{X_1, X_2, Z + Y_1\} \\ \mathfrak{b} = \operatorname{span}\{Y_1, Y_2\} \end{cases} .$$

It is immediate to check that $\mathfrak{a} \oplus \mathfrak{b} = \mathfrak{h}^2$, but

$$\exp(2X_1 + Z) \notin AB,$$

where $A = \exp \mathfrak{a}$ and $B = \exp \mathfrak{b}$ are the corresponding Lie subgroups.

7.2. **h-epimorphisms and h-monomorphisms.** In this subsection we use complementary subgroups to characterize both h-epimorphisms and h-monomorphisms. These characterizations will be used in the proof of both the implicit function theorem and the rank theorem.

Proposition 7.14 (Characterization of h-epimorphisms)**.** *Let $L : \mathbb{G} \longrightarrow \mathbb{M}$ be a surjective h-homomorphism and let N be its kernel. The conditions*

(1) *there exists a subgroup H complementary to N*
(2) *L is an h-epimorphism*

are equivalent and if one of them holds, then the restriction $L_{|H}$ is an h-isomorphism.

Proof. We first assume the validity of the first condition. Then we consider the restriction $T = L_{|H} : H \longrightarrow \mathbb{M}$. Let $m \in \mathbb{M}$ and let $g \in \mathbb{G}$ such that $L(g) = m$. Then the property $NH = \mathbb{G}$ implies that $g = nh$, where $(n, h) \in N \times H$. As a consequence, we have

$$L(nh) = L(h) = T(h) = m,$$

hence T is surjective. If we have $T(h) = e$, then

$$h \in H \cap N = \{e\}.$$

We have shown that $L_{|H}$ is an h-homomorphism. Clearly, $T^{-1} : \mathbb{M} \longrightarrow H$ is an h-homomorphism and satisfies $L \circ T^{-1} = \mathrm{Id}_{\mathbb{M}}$, hence L is an h-epimorphism. Conversely, if L is an h-epimorphism, then there exists a right inverse $R : \mathbb{M} \longrightarrow \mathbb{G}$ that is also an h-homomorphism. We set $H = R(\mathbb{M})$ and easily observe that $H \cap N = \{e\}$. Let $g \in \mathbb{G}$ and consider

$$m = L(g) = L\big(R(m)\big),$$

therefore $g^{-1}R(m) \in N$ and this implies that $g \in HN$. It follows that $\mathbb{G} = HN = NH$ and this concludes the proof. □

Proposition 7.15 (Characterization of h-monomorphisms). *Let $T : \mathbb{G} \longrightarrow \mathbb{M}$ be an injective h-homomorphism and let H be its image. The conditions*

(1) *there exists a normal subgroup N complementary to H,*
(2) *there exists an h-epimorphism $p : \mathbb{M} \longrightarrow H$ such that $p_{|H} = Id_H$,*
(3) *T is an h-monomorphism*

are equivalent.

Proof. We show that (1) implies (2). We define the projection $p : \mathbb{M} \longrightarrow H$ that associates to any element $m = nh \in \mathbb{M}$, with $(n, h) \in N \times H$ the element $h \in H$. This definition is well posed, since N and H are complementary. The fact that N is normal and the uniqueness of representation of the product nh give

$$p\big((n_1h_1)(n_2h_2)\big) = p\big((n_1h_1n_2h_1^{-1})(h_1h_2)\big) = h_1h_2 = p(n_1h_1)\,p(n_2h_2)\,.$$

It is trivial to observe that p is homogeneous and that its restriction to H is exactly Id_H. Then p is a surjective h-homomorphism. Furthermore, H is a complementary subgroup of the kernel N, then Proposition 7.14 implies that p is an h-epimorphism.

To prove that (2) implies (3), we define the mapping $L = J \circ p$, where the injectivity of T allows us to define the h-isomorphism $J : H \longrightarrow \mathbb{G}$ such that $J\big(T(g)\big) = g$ for every $g \in \mathbb{G}$. Then L is an h-homomorphism as composition of h-homomorphisms. In addition, one can easily verify that $L \circ T = \mathrm{Id}_{\mathbb{G}}$.

We are left to show that (3) implies (1). Bt definition of h-monomorphism we have an h-homomorphism $L : \mathbb{M} \longrightarrow \mathbb{G}$ that is a left inverse of T. Let N be its kernel, that is a normal homogeneous subgroup of $\mathbb{M}$. We have to show that N and H are complementary subgroups. Let $m \in \mathbb{M}$ and consider $L(m) = g \in \mathbb{G}$. We have

$$L\big(m\,T(g^{-1})\big) = L(m)g^{-1} = e\,,$$

then $m\,T(g^{-1}) = n \in N$, namely, $m = n\,T(g) \in NH$. Let $x \in N \cap H$ and let $g \in \mathbb{G}$ such that $x = T(g)$. Thus, we get

$$e_{\mathbb{G}} = L(x) = L \circ T(g) = g\,,$$

that implies $g = e_{\mathbb{G}}$ and $x = T(e_{\mathbb{G}}) = e_{\mathbb{M}}$, namely, $N \cap H = \{e_{\mathbb{M}}\}$. □

8. Quotients of graded groups

In this section we show that the group quotient of graded groups is still graded with a natural left invariant and homogeneous distance. We first recall some elementary facts of Lie groups Theory in order to study the relationship between quotients of Lie algebras and quotients of Lie groups.

Let G be a real Lie group and let H be a Lie subgroup of G. Recall that the quotient G/H has a unique manifold structure that makes the projection $\pi : G \longrightarrow G/H$ a smooth mapping. G/H is called homogeneous manifold, see Theorem 3.58 of [66].

If we consider a normal Lie subgroup N, then G/N is in addition a Lie group, according to Theorem 3.64 of [66] and in this case

$$\pi : G \longrightarrow G/N$$

is clearly is a Lie group homomorphism. As a result, by Theorem 3.14 of [66]

$$d\pi : \mathcal{G} \longrightarrow \overline{\mathcal{G}}$$

is a Lie algebra homomorphism, where $\mathcal{G}$ and $\overline{\mathcal{G}}$ are the Lie algebras of G and G/N, respectively.

Remark 8.1. The mapping $d\pi$ is also surjective. In fact, Theorem 3.58 ensures the existence of a neighbourhood $W \subset G/N$ of the unit element $\overline{e}$ of G/N and a smooth mapping $\tau : W \longrightarrow G$ such that $\pi \circ \tau = \mathrm{Id}_W$. Let $\Gamma :]-\varepsilon, \varepsilon[\longrightarrow W$ be a smooth curve with $\dot{\Gamma}(0) = \overline{X}_{\overline{e}} \in T_{\overline{e}}\big(G/N\big)$. Then the curve $\gamma = \tau \circ \Gamma :]-\varepsilon, \varepsilon[\longrightarrow G$ satisfies $\pi \circ \gamma = \Gamma$ and $d\pi(X_e) = \overline{X}_{\overline{e}}$, where $X_e = \dot{\gamma}(0)$. This shows surjectivity of $d\pi$.

Proposition 8.2. *The Lie algebra $\overline{\mathcal{G}}$ of G/N is isomophic to the quotient algebra $\mathcal{G}/\mathcal{N}$, where $\mathcal{G}$ is the Lie algebra of G and $\mathcal{N}$ is the ideal corresponding to the normal subgroup N. Furthermore, we can represent the exponential mapping of G/N on $\mathcal{G}/\mathcal{N}$ by setting $\mathrm{Exp} : \mathcal{G}/\mathcal{N} \longrightarrow G/N$ and*

$$\mathrm{Exp}(X + \mathcal{N}) = \pi\big(\exp(X)\big)$$

where $\exp : \mathcal{G} \longrightarrow G$ is the exponential mapping of G, $\pi : G \longrightarrow G/N$ is the canonical projection and the diagram

$$\begin{tikzcd}
\mathcal{G}/\mathcal{N} \arrow[r, "\overline{d\pi}"] \arrow[dr, "\mathrm{Exp}"'] & \overline{\mathcal{G}} \arrow[d, "\overline{\exp}"] \\
& G/N
\end{tikzcd}$$

commutes, where $\overline{\exp} : \overline{\mathcal{G}} \longrightarrow G/N$ is the canonical exponential mapping of G/N.

Proof. We have seen above that $d\pi : \mathcal{G} \longrightarrow \overline{\mathcal{G}}$ is surjective, hence

$$\dim\big(\ker(d\pi)\big) = \dim(\mathcal{N}).$$

For every $U \in \mathcal{N}$ the curve $\pi\big(\exp(tX)\big)$ is constantly equal to the unit element $\overline{e}$ of G/N, then $d\pi(X) = 0$ and clearly $\mathcal{N} \subset \ker d\pi$. It follows that $\mathcal{N} = \ker d\pi$ and it is well defined the algebra isomorphism $\overline{d\pi} : \mathcal{G}/\mathcal{N} \longrightarrow \overline{\mathcal{G}}$, where $p : \mathcal{G} \longrightarrow \mathcal{G}/\mathcal{N}$ is the canonical projection and

$$[p(X), p(Y)] = p\big([X, Y]\big)$$

for every $X, Y \in \mathcal{G}$. If we consider the diagram

(103)
$$\begin{tikzcd}
\mathcal{G} \arrow[d, "p"'] \arrow[dr, "d\pi"] & \\
\mathcal{G}/\mathcal{N} \arrow[r, "\overline{d\pi}"] \arrow[dr, "\mathrm{Exp}"'] & \overline{\mathcal{G}} \arrow[d, "\overline{\exp}"] \\
& G/N
\end{tikzcd}$$

then we are left to show that its lower part commutes. In fact, by definition of $\overline{d\pi}$, for every $X \in \mathcal{G}$ we have

$$\overline{\exp} \circ \overline{d\pi}\big(p(X)\big) = \overline{\exp} \circ d\pi(X). \tag{104}$$

On the other hand, due to Theorem 3.32 of [66], the diagram

$$\begin{tikzcd}
G \arrow[r, "\pi"] & G/N \\
\mathcal{G} \arrow[u, "\exp"] \arrow[r, "d\pi"] & \overline{\mathcal{G}} \arrow[u, "\overline{\exp}"']
\end{tikzcd}$$

commutes, namely $\overline{\exp} \circ d\pi(X) = \pi \circ \exp(X)$ for every $X \in \mathcal{G}$. It follows that

$$\overline{\exp} \circ \overline{d\pi} = \pi \circ \exp = \mathrm{Exp}.$$

This concludes the proof. $\square$

Proposition 8.3. *Let $\mathbb{M}$ be a graded group of algebra $\mathcal{M} = W_1 \oplus \cdots \oplus W_\upsilon$ and let N be a normal homogeneous subgroup of corresponding ideal $\mathcal{N} = \mathcal{N}_1 \oplus \cdots \oplus \mathcal{N}_\upsilon$. Then the surjective Lie algebra homomorphism $d\pi : \mathcal{M} \longrightarrow \overline{\mathcal{M}}$ induces a grading*

$$\overline{W}_1 \oplus \cdots \oplus \overline{W}_\upsilon$$

on the Lie algebra $\overline{\mathcal{M}}$ of $\mathbb{M}/N$, that is h-isomorphic to

$$W_1/\mathcal{N}_1 \oplus \cdots \oplus W_\upsilon/\mathcal{N}_\upsilon\,,$$

where $d\pi(W_i) = \overline{W}_i$ is linearly isomorphic to $W_i/\mathcal{N}_i$. Furthermore, a one parameter group of dilations $\overline{\delta}_r$ can be defined on $\mathbb{M}/N$ such that $\pi : \mathbb{M} \longrightarrow \mathbb{M}/N$ is an h-homomorphism. Finally, if $\mathbb{M}$ is stratified then so is $\mathbb{M}/N$.

Proof. The decomposition of $\mathcal{N}$ into factors $\mathcal{N}_i$ follows from Corollary 7.3, where some of $\mathcal{N}_i$'s are possibly vanishing. By Proposition 8.2 the Lie algebra $\overline{\mathcal{M}}$ of the quotient $\mathbb{M}/N$ is isomorphic to $\mathcal{M}/\mathcal{N}$ and

$$p : \mathcal{M} \longrightarrow \mathcal{M}/\mathcal{N} \qquad \text{is equivalent to} \qquad d\pi : \mathcal{M} \longrightarrow \overline{\mathcal{M}}\,,$$

by commutativity of diagram (103), hence we consider the grading induced by p. Let us check that $p(W_i) \cap p(W_j) = \{0\}$ whenever $i \neq j$. By contradiction, let $\overline{X} \in p(W_i) \cap p(W_j)$ be nonvanishing. Then there exist $X_i \in W_i \setminus \mathcal{N}$ and $X_j \in W_j \setminus \mathcal{N}$ such that $X_i = X_j + Z$, where $Z \in \mathcal{N}$. Since $\mathcal{N}$ is homogeneous, we must have $Z = Z_i + Z_j$, where $Z_i \in W_i \cap \mathcal{N}$ and $Z_j \in W_j \cap \mathcal{N}$, hence $Z_i = X_i$ and $Z_j = -X_j$ this conflicts with our initial assumption on X_i and X_j. This shows that $\overline{\mathcal{M}} = \overline{W}_1 \oplus \cdots \oplus \overline{W}_v$ is a grading, where $\overline{W}_i = d\pi(W_i)$. This immediately implies that $\overline{\mathcal{M}}$ is stratified when so is $\mathcal{M}$. If we set $\overline{\delta}_r(\overline{X}_i) = r^i \overline{X}_i$ for every $\overline{X}_i \in \overline{W}_i$, then one can easily check that $d\pi \circ \delta_r = \overline{\delta}_r \circ d\pi$ and π is an h-homomorphism. Finally, we consider the following commutative diagram

$$\begin{tikzcd} \mathcal{M} \arrow[r, "p"] \arrow[d, "p_0"'] & \mathcal{M}/\mathcal{N} \\ W_1/\mathcal{N}_1 \oplus \cdots \oplus W_v/\mathcal{N}_v \arrow[ur, "J"] & \end{tikzcd}$$

where we have set

$$p_0(w_1 + \cdots + w_v) = \big(w_1 + \mathcal{N}_1, \ldots, w_v + \mathcal{N}_v\big)$$

and the following definition

$$J\big(w_1 + \mathcal{N}_1, \ldots, w_v + \mathcal{N}_v\big) = w_1 + \cdots + w_v + \mathcal{N}$$

is well posed. It is immediate to observe that $\ker J = \{0\}$, then J is a linear isomorphism. The commutativity of the diagram above implies that

$$p(W_i) = J\big(W_i/\mathcal{N}_i\big)\,.$$

This concludes the proof. $\square$

Remark 8.4. The induced grading on $\mathcal{M}/\mathcal{N}$ considered on Proposition 8.3 is given by the subspaces $\overline{W}_j = d\pi(W_j)$, that satisfy $\mathcal{M}/\mathcal{N} = \overline{W}_1 \oplus \cdots \oplus \overline{W}_v$, where some of these factors are possibly vanishing. On the other hand, when $\mathcal{N}$ is not homogeneous $\mathcal{M}/\mathcal{N}$ still can have a grading with dilations, but these ones do not commute with π, namely, π is no longer an h-homomorphism.

Example 8.5. Let $W_1 = \operatorname{span}\{X_1, X_2, X_3\}$ and $W_2 = \operatorname{span}\{Z\}$ where the only nontrivial bracket relation is $[X_1, X_2] = Z$. $\mathcal{M} = W_1 \oplus W_2$ is a 2-step stratified algebra and $\mathcal{N} = \operatorname{span}\{X_3 - Z\}$ is an ideal of $\mathcal{M}$ that is clearly not homogeneous. Elements of $\mathcal{M}/\mathcal{N}$ are $\overline{X}_i = X_i + \mathcal{N}$, where $i = 1, 2, 3$ and $\overline{X}_3 = Z + \mathcal{N}$. Clearly, the only nontrivial bracket relation is $[\overline{X}_1, \overline{X}_2] = \overline{X}_3$ and $\mathcal{M}/\mathcal{N}$ is isomorphic to the 3-dimensional Heisenberg algebra. On the other hand, there are no dilations $\overline{\delta}_r$ on $\mathcal{M}/\mathcal{N}$ such that $d\pi \circ \delta_r = \overline{\delta}_r \circ d\pi$, since

$$d\pi\big(\delta_r X_3\big) = r\, d\pi(X_3) = r\, \overline{X}_3 \qquad \text{and} \qquad d\pi\big(\delta_r Z\big) = r^2\, d\pi(Z) = r^2\, \overline{X}_3.$$

We also notice that in this case $d\pi$ cannot induce any grading on $\overline{\mathcal{M}}$, since $d\pi(W_1) \cap d\pi(W_2) = \{\overline{X}_3\}$.

9. Implicit Function Theorem and Rank Theorem

In this section we prove both the implicit function theorem and the rank theorem stated in the introduction. We will use the following notation $D^N_{\overline{n},r}=D_{\overline{n},r}\cap N$ and $D^H_{\overline{h},s}=D_{\overline{h},s}\cap H$.

Proof of Theorem 1.3. Our arguments are divided into two main steps.

Step 1: *Existence.*

Due to Proposition 7.14, the restriction $Df(\overline{x})(h):H\longrightarrow\mathbb{G}$ is invertible, then

$$\min_{\substack{d(h)=1\\ h\in H}}\rho\big(Df(\overline{x})(h)\big)>0.$$

By continuity of $\Omega\times H\ni(x,h)\longrightarrow\rho\big(Df(x)(h)\big)$, there exists $R>0$ such that

$$\mu=\min_{x\in D_{\overline{x},R}}\min_{\substack{d(h)=1\\ h\in H}}\rho\big(Df(x)(h)\big)>0\quad\text{and}\quad\overline{\mu}=\max_{x\in D_{\overline{x},R}}\max_{\substack{d(h)=1\\ h\in H}}\rho\big(Df(x)(h)\big)>0 \tag{105}$$

with $D_{\overline{x},R}\subset\Omega$. Now we set $R_0=R/(2+2\,c(\mathbb{G},d)\,N)$, where $c(\mathbb{G},d)$ and N are as (58) and Lemma 4.6, respectively. Then condition (5) is satisfied and Theorem 1.2 yields the estimate

$$\max_{x,y\in D_{\overline{x},R_0}}\rho\big(f(x)^{-1}f(y),Df(x)(x^{-1}y)\big)\le C\ \left[\omega_{D_{\overline{x},R},df}\big(N\,c\,d(x,y)\big)\right]^{1/\iota^2}d(x,y)\,. \tag{106}$$

Thus, by definition of $\overline{\mu}$ in (105) we have

$$\rho\big(f(x),f(y)\big)\le\Big(C\big[\omega_{D_{\overline{x},R},df}\big(N\,c\,d(x,y)\big)\big]^{1/\iota^2}+\overline{\mu}\Big)\ d(x,y) \tag{107}$$

for every $x,y\in D_{\overline{x},R_0}$. Let $(\overline{n},\overline{h})$ be the unique couple of $N\times H$ such that $\overline{n}\overline{h}=\overline{x}$ and let $r,s>0$ be such that $D^N_{\overline{n},r}D^H_{\overline{h},s}\subset D_{\overline{x},R_0}$. Then, using definition of μ in (105), for every $n\in D^N_{\overline{n},r}$ and every $h,h'\in D^H_{\overline{h},s}$ there holds

$$\rho\big(f(nh),f(nh')\big))\ge\Big(\mu-C\big[\omega_{D_{\overline{x},R},df}\big(N\,c\,d(h,h')\big)\big]^{1/\iota^2}\Big)\ d(h,h') \tag{108}$$

observing that $d(nh,nh')=d(h,h')$. Using (107) and (108) and possibly taking a smaller $s>0$ depending on $C,N,c>0$ we get a constant $\beta>0$ such that

$$\beta^{-1}\,d(h,h')\le\rho\big(F_n(h),F_n(h')\big)\le\beta\,d(h,h') \tag{109}$$

where for each $n\in D^N_{\overline{n},r}$ we have defined

$$F_n:D^H_{\overline{h},s}\longrightarrow\mathbb{M},\qquad F_n(h)=f(nh).$$

Observing that $F_{\overline{n}}(\overline{h})=f(\overline{x})$ and taking into account the biLipschitz estimate (109) we have

$$F_{\overline{n}}(h)\ne f(\overline{x})\quad\text{for every}\quad h\in\partial D^H_{\overline{h},s}\,.$$

Injectivity of $F_{\overline{n}}$ implies that

$$\deg\big(F_{\overline{n}},B^H_{\overline{h},s},f(\overline{x})\big)\in\{-1,1\},$$

see for instance Theorem 3.3.3 of [37]. By continuity of

$$n \longrightarrow \max_{h\in\partial D^H_{\overline{h},s}} \rho\big(f(nh), f(\overline{n}h)\big)\,,$$

up to choosing a smaller $r>0$, we can assume that

$$\max_{h\in\partial D^H_{\overline{h},s}} \rho\big(f(nh), f(\overline{n}h)\big) < \frac{s}{2\beta} \tag{110}$$

for every $n\in D^N_{\overline{n},r}$. As a consequence, applying (109), it follows that

$$\rho\big(F_n(h), f(\overline{x})\big) \ge \rho\big(F_{\overline{n}}(h), f(\overline{x})\big) - \rho\big(F_{\overline{n}}(h), F_n(h)\big) > \frac{s}{2\beta}$$

for every $h\in\partial D^H_{\overline{h},s}$ and every $n\in D^N_{\overline{n},r}$. For an arbitrary $n\in D^N_{\overline{n},r}\setminus\{\overline{n}\}$ one can consider the continuous curve $\gamma:[0,1]\longrightarrow D^N_{\overline{n},r}$, defined by $\gamma(t)=\overline{n}\delta_t\big((\overline{n})^{-1}n\big)$. Notice that γ has image in $D^N_{\overline{n},r}$, since N is a homogeneous subgroup of $\mathbb{G}$. By previous estimates, the mapping $\Phi:[0,1]\times D_{\overline{h},s}\longrightarrow\mathbb{M}$ defined by

$$\Phi(t,h) = f\big(\gamma(t)h\big)$$

is a homotopy between $F_{\overline{n}}$ and F_n such that $\Phi(t,h)\neq f(\overline{x})$ for every $t\in[0,1]$ and every $h\in\partial D^H_{\overline{h},s}$. Thus, homotopy invariance of topological degree, [53], implies

$$\deg\big(F_n, B^H_{\overline{h},s}, f(\overline{x})\big) = \deg\big(F_{\overline{n}}, B^H_{\overline{h},s}, f(\overline{x})\big) \neq 0,$$

hence there exists at least one element $h'\in D^H_{\overline{h},s}$, depending on n, such that $F_n(h')=f(\overline{x})$. Injectivity of F_n gives uniqueness of h', hence there exists $\varphi: D^N_{\overline{n},r}\longrightarrow D^H_{\overline{h},s}$, uniquely defined, such that

$$F_n\big(\varphi(n)\big) = f\big(n\,\varphi(n)\big) = f(\overline{x}).$$

This concludes the proof of the first step.

Step 2: *Regularity.*

We keep the same notation of the previous step. By definition of $\mu>0$ in (105) the restriction $L(x)=\big(Df(x)\big)_{|H}: H\longrightarrow\mathbb{M}$ is invertible for every $x\in D_{\overline{x},R_0}$ and setting $T(x)=L(x)^{-1}$, we have

$$d\big(T(x)(m)\big) \le \mu^{-1}\,\rho(m) \tag{111}$$

for every $m\in\mathbb{M}$. Due to (106), we can choose a possibly smaller $R>0$, hence a smaller $R_0=R/(2+2cN)$ such that

$$\max_{x,y\in D_{\overline{x},R_0}} \rho\big(f(x)^{-1}f(y), Df(x)(x^{-1}y)\big) \le \frac{\mu\, d(x,y)}{2}\,.$$

It follows that the remainder $E(n,n')$ in the expression

$$f\big(n\varphi(n)\big)^{-1} f(n\varphi(n')) = L\big(n\varphi(n)\big)\big(\varphi(n)^{-1}\varphi(n')\big)E(n,n')$$

satisfies the uniform estimate

$$\rho\big(E(n,n')\big) \le \frac{\mu}{2}\, d\big(\varphi(n),\varphi(n')\big) \tag{112}$$

for every $n, n' \in D^N_{\overline{n},r}$. The implicit mapping φ satisfies

$$f\big(n\varphi(n)\big)^{-1} f\big(n\varphi(n')\big) = \Big(f\big(n\varphi(n')\big)^{-1} f\big(n'\varphi(n')\big)\Big)^{-1},$$

therefore one easily gets

$$\varphi(n)^{-1}\varphi(n') = T\big(n\varphi(n)\big)\left(f\big(n\varphi(n')\big)^{-1} f\big(n'\varphi(n')\big)\right)^{-1} T\big(n\varphi(n)\big)\big(E(n,n')\big)^{-1}.$$

As a result, in view of (111) and (112), we obtain

$$d\big(\varphi(n),\varphi(n')\big) \le \frac{2}{\mu}\,\rho\left(f\big(n\varphi(n')\big), f\big(n'\varphi(n')\big)\right).$$

By (107), up to choosing a possibly smaller $R > 0$, we can suppose that

$$\rho\big(f(x), f(y)\big) \le 2\,\overline{\mu}\; d(x,y)$$

for every $x, y \in D_{\overline{x},R_0}$, hence

$$d\big(\varphi(n),\varphi(n')\big) \le \frac{4\,\overline{\mu}}{\mu}\, d\big(\varphi(n')^{-1}n^{-1}n'\varphi(n')\big). \tag{113}$$

Finally, formulae (34) and (29) lead us to the conclusion. □

Remark 9.1. Under hypotheses of Theorem 1.3, one immediately gets

$$d\big(n\varphi(n), n'\varphi(n')\big) \le (1+\kappa)\; d\big(\varphi(n')^{-1}n^{-1}n'\varphi(n')\big), \tag{114}$$

where κ is as in the statement of the above mentioned theorem.

Proof of Theorem 1.4. By Proposition 7.15 we have a projection $p : \mathbb{M} \longrightarrow H$ that is an h-epimorphism and $p_{|H} = \mathrm{Id}_H$. We also know that the kernel N of p is complementary to H. As a first consequence, the mapping $p \circ f : \Omega \longrightarrow H$ is continuously P-differentiable and $D(p \circ f)(\overline{x}) : \mathbb{G} \longrightarrow H$ is invertible. Due to Theorem 6.3, there exist open neighbourhoods $V \Subset \Omega$ and $W \Subset H$ of $\overline{x}$ and of $p\big(f(\overline{x})\big)$, respectively, such that the restriction

$$(p\circ f)_{|\overline{V}} : \overline{V} \longrightarrow \overline{W}$$

is invertible. We denote by $\psi : \overline{W} \longrightarrow \overline{V}$ its inverse function. By Remark 7.6, there exists a unique function $f_N : \Omega \longrightarrow N$ such that $f(x) = p\big(f(x)\big) f_N(x)$ for every $x \in \Omega$, hence we consider $g : \overline{W} \longrightarrow \mathbb{M}$ defined by

$$g(h) = f\big(\psi(h)\big) = h\, f_N\big(\psi(h)\big)$$

for every $h \in \overline{W}$. Uniqueness of factorization implies that

$$g(W') = g(W') \cap W'N,$$

whenever $W' \subset \overline{W}$. If W' is open, then so is $W'N$, since the mapping (101) is open. As a consequence, g is an open mapping on its image, hence so is the restriction $f_{|V}$. This implies that $f_{|V}$ is a topological embedding. If we set

$$\varphi = f_N \circ \psi \quad \text{and} \quad \Psi = I \circ J^{-1} \circ \psi \circ p_{|f(V)}$$

then $\varphi(h) = h^{-1}g(h)$ and (9) holds, where I and J are defined in the statement of Theorem 1.4. We notice that g is continuously P-differentiable on $\overline{W}$, then in particular it is Lipschitz on $\overline{W}$ up to a suitable choice of W, for instance a closed ball. Our next computations are performed in the Lie algebra $\mathcal{G}$ of $\mathbb{G}$. We set $\varphi = \exp\circ F$ and $g = \exp\circ G$, therefore Lemma 3.5 gives a constant $C > 0$, depending on both $\overline{W}$ and the Lipschitz constant of g such that

$$\|G(h) - G(h')\| \le C\, d(h,h') \quad \text{and} \quad \|\xi - \xi'\| \le C\, d(h,h'), \tag{115}$$

where $h = \exp\xi$ and $h' = \exp\xi'$, with $h, h' \in \overline{W}$. We have

$$\begin{aligned} F(h) - F(h') &= (-\xi) \circledcirc G(h) - \big((-\xi') \circledcirc G(h')\big) \\ &= \xi' - \xi + G(h) - G(h') + \sum_{n=2}^{\iota} \left[c_n\big(-\xi, G(h)\big) - c_n\big(-\xi', G(h')\big)\right], \end{aligned}$$

due to the formula (13). From (41), it follows that

$$\|F(h) - F(h')\| \le \|\xi' - \xi\| + \|G(h) - G(h')\| + \sum_{n=2}^{\iota} \gamma_n\, \nu^{n-1}\big(\|\xi - \xi'\| + \|G(h) - G(h')\|\big),$$

where we have set

$$\nu = 2\left(\max_{\eta\in\overline{W}} \|G(\eta)\| + \sup_{\eta\in\overline{W}} \|\eta\|\right).$$

By virtue of (115), we get a constant $C_1 > 0$, depending on $\overline{W}$, ν and the Lipschitz constant of g such that

$$\|F(h) - F(h')\| \le C_1\, d(h,h').$$

This concludes the proof. □

10. Intrinsic graphs and $(\mathbb{G},\mathbb{M})$-regular sets

The notion of complementary subgroup allows us to set properly the well established notion of intrinsic graph in those graded groups that admit a factorization by two complementary subgroups.

Definition 10.1. Let P and H be complementary subgroups of $\mathbb{G}$ and let $S \subset \mathbb{G}$. We say that S is an *intrinsic graph* with respect to (P, H) if there exists a subset $A \subset P$ and a mapping $\varphi : A \longrightarrow H$ such that

$$S = \{p\,\varphi(p) \mid p \in A\}.$$

Remark 10.1 (Translations of intrinsic graphs)**.** As already observed in [22], intrinsic graphs are preserved under left translations. If P and H are complementary subgroups of $\mathbb{G}$ and $S = \{p\,\varphi(p) \mid p \in A\}$ is an intrinsic graph with respect to (P, H), then the left translated $xp\,\varphi(p)$, for some $x \in \mathbb{G}$, is of the form $\tilde p\,\psi(\tilde p)$ where $\tilde p \in \tilde A \subset P$ and $\psi : \tilde A \longrightarrow H$. In fact, by unique decomposition of elements in $\mathbb{G}$ with respect to P

and H, see Remark 7.6, there exist functions $\psi_1 : A \longrightarrow P$ and $\psi_2 : A \longrightarrow H$ such that

$$xp\,\varphi(p) = \psi_1(p)\psi_2(p)\,.$$

To conclude, it suffices to show that ψ_1 is injective. If $\psi_1(p) = \psi_1(p')$, then

$$xp\,\varphi(p)\psi_2(p)^{-1} = xp'\varphi(p')\psi_2(p')^{-1},$$

where $\varphi(p)\psi_2(p)^{-1}, \varphi(p')\psi_2(p')^{-1} \in H$. Again, uniqueness yields $p = p'$.

Definition 10.2 (($\mathbb{G},\mathbb{M}$)-regular set of $\mathbb{G}$)**.** Let $\mathbb{G}$ be a stratified group and let $\mathbb{M}$ be a graded group such that $\mathbb{M}$ is an h-quotient of $\mathbb{G}$. We say that a subset $S \subset \mathbb{G}$ is $(\mathbb{G},\mathbb{M})$-*regular* if for every point $\overline{x} \in S$, there exists an open neighbourhood U of $\overline{x}$ and a continuously P-differentiable mapping $f : U \longrightarrow \mathbb{M}$ such that $S \cap U = f^{-1}(e)$ and $Df(x) : \mathbb{G} \longrightarrow \mathbb{M}$ is an h-epimorphism for every $x \in U$.

Definition 10.3 (($\mathbb{G},\mathbb{M}$)-regular set of $\mathbb{M}$)**.** Let $\mathbb{G}$ be a stratified group and let $\mathbb{M}$ be a graded group such that $\mathbb{G}$ h-embeds into $\mathbb{M}$. A subset $S \subset \mathbb{M}$ is $(\mathbb{G},\mathbb{M})$-*regular* if for every point $\overline{x} \in S$, there exist open neighbourhoods $U \subset \mathbb{M}$ of $\overline{x}$ and $V \subset \mathbb{G}$ of $e \in \mathbb{G}$, along with a continuously P-differentiable topological embedding $f : V \longrightarrow \mathbb{M}$, such that $S \cap U = f(V)$ and $Df(y) : \mathbb{G} \longrightarrow \mathbb{M}$ is an h-monomorphism for every $y \in V$.

Definition 10.4 (($\mathbb{G},\mathbb{M}$)-regular set)**.** When a subset S is either $(\mathbb{G},\mathbb{M})$-regular in $\mathbb{G}$ or $(\mathbb{G},\mathbb{M})$-regular in $\mathbb{M}$, we simply say that it is $(\mathbb{G},\mathbb{M})$-regular without further specification, or we can say that it is an *intrinsically regular set.*

10.1. **Tangent cone to** $(\mathbb{G},\mathbb{M})$**-regular sets.** The notion of "tangent cone" given in 3.1.21 of [17] can be easily extended to graded groups.

Definition 10.5. Let $\mathbb{G}$ be a graded group and let $S \subset \mathbb{G}$ with $a \in \mathbb{G}$. The *homogeneous tangent cone* of S at a is the homogeneous subset

$$\begin{aligned}\text{(116)}\qquad \mathrm{Tan}(S,a) = \Big\{ & v \in \mathbb{G} \mid v = \lim_{k\to\infty} \delta_{r_k}(a^{-1}s_k), \text{ for some sequences } (r_k) \subset \mathbb{R}^+, \\ & (s_k) \subset S, \text{ where } s_k \to a \Big\}.\end{aligned}$$

Remark 10.2. If $a \in \overline{S}$, then $\mathrm{Tan}(S,a) \neq \emptyset$.

Proof of Theorem 1.6. Replacing f with $f \circ l_x$, where $l_x(y) = xy$, it is not restrictive to assume that $\overline{x} = e$, since

$$(\overline{x})^{-1}S = (f \circ l_{\overline{x}})^{-1}(f \circ l_{\overline{x}})(e)$$

and $Df(\overline{x}) = D(f \circ l_{\overline{x}})(e)$. Now we use the notation of Theorem 1.3 applying it in the case $\overline{x} = e$. From P-differentiability of f at e and representation (7), we have

$$f(e) = f(n\varphi(n)) = f(e)Df(e)\big(\varphi(n)\big)o\big(n\varphi(n)\big).$$

Setting $L = Df(e)$, we obtain

$$d\big(\varphi(n)\big) = d\left(L_{|H}^{-1}\big(o(n\varphi(n))\big)\right) \le \varepsilon\big(d(n) + d(\varphi(n))\big) \tag{117}$$

for n suitable close to e and $\varepsilon > 0$ small, arbitrarily fixed. From this, we easily conclude that φ is P-differentiable at e_N and $D\varphi(e_N)$ is the null h-homomorphism. Now, we arbitrarily fix $R > 0$ and consider $\lambda > 0$ sufficiently small, such that

$$D_R \cap \delta_{1/\lambda} S = D_R \cap \{\delta_{1/\lambda} n \delta_{1/\lambda}\varphi(n) \mid n \in D_r^N\}.$$

Let (λ_k) be an infinitesimal sequence of positive numbers. Using Proposition 4.5.5 of [2], we have to prove the following conditions:

(i) if $x = \lim_{k\to\infty} x_k$ for some sequence (x_k) such that $x_k \in D_R \cap \delta_{1/\lambda_k} S$, then we have $x \in D_R \cap N$;

(ii) if $x \in D_R \cap N$, then there exists a sequence (x_k) such that $x_k \in D_R \cap \delta_{1/\lambda_k} S$ and $x_k \to x$.

To prove (i), we write $x_k = \big(\delta_{1/\lambda_k} n_k\big)\big(\delta_{1/\lambda_k}\varphi(n_k)\big)$ and use Proposition 7.7 with respect to our complementary subgroups N and H. Through the global diffeomorphism ϕ in (101), convergence of the sequence (x_k) implies convergence of both $(\delta_{1/\lambda_k} n_k)$ and $\big(\delta_{1/\lambda_k}\varphi(n_k)\big)$. We notice that the limit n_0 of $(\delta_{1/\lambda_k} n_k)$ belongs to N. P-differentibility of φ at e_N gives

$$\lim_{k\to\infty} \delta_{1/\lambda_k}\varphi\big(\delta_{\lambda_k} n_k'\big) = D\varphi(e_N)(n_0) = e_H$$

where we have set $\delta_{1/\lambda_k} n_k = n_k'$. Then $x_k \to x = n_0 \in D_R \cap N$ and our claim is achieved. Now we choose $x \in N \cap D_R$ and consider a sequence (x_l) contained in $N \cap B_R$ converging to x. We observe that for every fixed l, there exists a sufficiently large k_l such that $\delta_{\lambda_{k_l}} x_l \in D_r^N$ and

$$\delta_{1/\lambda_{k_l}}\big(\delta_{\lambda_{k_l}} x_l \varphi(\delta_{\lambda_{k_l}} x_l)\big) \in B_R \cap \delta_{1/\lambda_k} S,$$

since P-differentiability of φ at e_N implies that the sequence $\delta_{1/\lambda_k}\big(\delta_{\lambda_k} x_l \varphi(\delta_{\lambda_k} x_l)\big)$ converges to x_l as $k \to \infty$. The fact that $D\varphi(e_N)$ is the null h-homomorphism also implies that

$$\delta_{1/\lambda_{k_l}}\big(\delta_{\lambda_{k_l}} x_l \varphi(\delta_{\lambda_{k_l}} x_l)\big) \longrightarrow x D\varphi(e_N)(x) = x \quad \text{as} \quad l \to \infty.$$

Finally, we notice that conditions (i) and (ii) exactly prove that the two inclusions between N and $\mathrm{Tan}(S, e)$. This concludes the proof of the first part of Theorem 1.6.

Now, we keep the same notation used in the proof of Theorem 1.4 and observe that it is not restrictive assuming that $f(\overline{x}) = e_{\mathbb{M}}$. In fact, in the general case it suffices to replace f with the left translated $x \to f(\overline{x})^{-1} f(x)$, that holds the same assumptions. We know that the mapping $p \circ f_{|V}$ is invertible with P-differentiable inverse $\psi : W \longrightarrow V$ and $g(h) = f \circ \psi(h) = h\varphi(h)$ for every $h \in W$. By Proposition 2.7, we have that $Dg(e_H) : H \hookrightarrow \mathbb{M}$ identically embeds H into $\mathbb{M}$, then

$$g(h) = h\, o(h) = h\, \varphi(h)$$

so that $\varphi(h) = o(h)$. This implies that there exists $D\varphi(e_H)$ and it is constantly equal to the unit element $e_N \in N$. To show the remaining claims, one can argue exactly as in the first part of this proof, using the fact that φ is P-differentiable at e_H with null P-differential at this point and replacing N with H. $\square$

Corollary 10.3. *Let S be a $(\mathbb{G},\mathbb{M})$-regular set of $\mathbb{G}$. Then for every $x\in S$, we have*

$$\mathcal{H}\text{-dim}\big(\mathrm{Tan}(S,x)\big)=\mathcal{H}\text{-dim}\mathbb{G}-\mathcal{H}\text{-dim}\mathbb{M}\,.$$

PROOF. By Theorem 1.6, $\mathrm{Tan}(S,x)$ is h-isomorphic to $N=\ker Df(x)$, where f is the continuously P-differentiable mapping defining S around x as a level set. Due to Proposition 7.14, the factorizing property of h-epimorphisms yields a complementary subgroup H that is h-isomorphic to $\mathbb{M}$. Due to formula (102), our claim follows. □

Example 10.4. The fact that homogeneous tangent cones to a $(\mathbb{G},\mathbb{M})$-regular set of $\mathbb{G}$ have a fixed Hausdorff dimension does not mean that they are all algebraically h-isomorphic. Let us consider the mapping

$$f:\mathbb{H}^2\longrightarrow\mathbb{R}^2,\qquad (x_1,x_2,x_3,x_4,x_5)\longrightarrow\Big(\sqrt{x_2^2+x_3^2},x_4\Big)$$

where $p=\exp\big(\sum_{j=1}^5 x_jX_j\big)\in\mathbb{H}^2$ and $[X_1,X_2]=[X_3,X_4]=X_5$ are the nontrivial bracket relations of $\mathbb{H}^2$. The Pansu differential $Df(x)$ is represented by the matrix

$$\begin{pmatrix} 0 & \frac{x_2}{\sqrt{x_2^2+x_3^2}} & \frac{x_3}{\sqrt{x_2^2+x_3^2}} & 0 & 0 \\ 0 & 0 & 0 & x_4 & 0 \end{pmatrix}.$$

Then, defining the connected open set

$$\Omega=\big\{(x_1,x_2,x_3,x_4,x_5)\in\mathbb{H}^2\mid x_4>0,\ x_2^2+x_3^2>0\big\}$$

we notice that $S=\Omega\cap f^{-1}\big((1,1)\big)$ is $(\mathbb{H}^2,\mathbb{R}^2)$-regular in $\mathbb{H}^2$. Now we fix $\xi=(0,1,0,1,0)$ and $\eta=(0,0,1,1,0)$, observing that $\xi,\eta\in S$. Thus, it is easy to check that $\mathrm{Tan}(S,\xi)$ is commutative and $\mathrm{Tan}(S,\eta)$ is h-isomorphic to $\mathbb{H}^1$.

11. Factorizing groups

In this section we investigate the algebraic conditions under which either surjective or injective h-homomorphisms are h-monomorphisms or h-epimorphisms, respectively. Let $\mathbb{G}$ and $\mathbb{M}$ be arbitrary graded groups with algebras $\mathcal{G}$ and $\mathcal{M}$.

Definition 11.1 (h-quotients and h-embeddings)**.** We say that $\mathbb{M}$ is an *h-quotient* of $\mathbb{G}$ if there exists a normal homogeneous subgroup $N\subset\mathbb{G}$ such that $\mathbb{G}/N$ is h-isomorphic to $\mathbb{M}$. Analogously, $\mathbb{G}$ *h-embeds* into $\mathbb{M}$ if and only if there exists a homogeneous subgroup H of $\mathbb{M}$ which is h-isomorphic to $\mathbb{G}$.

Proposition 11.1. *There exists a normal subgroup N of $\mathbb{G}$ such that $\mathbb{G}/N$ is h-isomorphic to $\mathbb{M}$ if and only if there exists a surjective h-homomorphism $L:\mathbb{G}\longrightarrow\mathbb{M}$.*

PROOF. If the first condition holds, then we denote by $T:\mathbb{G}/N\longrightarrow\mathbb{M}$ the canonical h-isomorphism and notice that $L=T\circ\pi:\mathbb{G}\longrightarrow\mathbb{M}$ is a surjective h-homomrphism, where $\pi:\mathbb{G}\longrightarrow\mathbb{G}/N$ is the canonical projection, that is also an h-homomorphism, according to Proposition 8.3. The converse is trivial. □

Remark 11.2. The previous proposition shows that $\mathbb{M}$ is an *h-quotient* of $\mathbb{G}$ if and only if there exists a surjective h-homomorphism $L : \mathbb{G} \longrightarrow \mathbb{M}$. In view of Proposition 8.3, we also notice that $\mathbb{M}$ is stratified when so is $\mathbb{G}$. On the other side, it is easy to check that $\mathbb{G}$ h-embeds into $\mathbb{M}$ if and only if there exists an injective h-homomorphism $T : \mathbb{G} \longrightarrow \mathbb{M}$.

Lemma 11.3. *If m is the dimension of the first layer of $\mathcal{G}$, then $\mathbb{R}^k$ is an h-quotient of $\mathbb{G}$ if and only if $k \leq m$.*

Proof. Let $V_1 \oplus V_2 \oplus \cdots \oplus V_\iota$ be the direct decomposition of $\mathcal{G}$ into its layers. Let $\mathfrak{u}$ be an $(m-k)$-dimensional subspace of V_1. Then $\mathcal{N} = \mathfrak{u} \oplus V_2 \oplus \cdots \oplus V_\iota$ is an ideal and clearly $\mathcal{G}/\mathcal{N}$ is h-isomorphic to $\mathbb{R}^k$. By Proposition 8.2, $\mathbb{G}/N$ is also h-isomorphic to $\mathbb{R}^k$, where $N = \exp \mathcal{N}$. Conversely, assume that $\mathbb{R}^k$ is an h-quotient of some $\mathbb{G}$. From Proposition 8.3, it follows that there exists N of ideal $\mathcal{N} = \mathcal{N}_1 \oplus \cdots \oplus \mathcal{N}_\iota$ such that $\mathbb{R}^k$ is linearly isomorphic to $V_1/\mathcal{N}_1$, then $k \leq \dim V_1 = m$. □

Example 11.4. The Heisenberg group $\mathbb{H}^k$ is not an h-quotient of $\mathbb{H}^n$, whenever $n > k$. By contradiction, assume that there exists a normal subgroup $N = \exp \mathfrak{n}$ such that $\mathbb{H}^n/N$ is h-isomorphic to $\mathbb{H}^k$. In terms of algebras, we have a $2(n-k)$-dimensional ideal $\mathfrak{n}$ of $\mathfrak{h}^n$ such that $\mathfrak{h}^n/\mathfrak{n}$ is h-isomorphic to $\mathfrak{h}^k$. This implies that there exists $\mathfrak{n} + Z' \in \mathfrak{h}^n/\mathfrak{n}$ that corresponds to a nonvanishing element of the second layer of $\mathfrak{h}^k$. It follows that $\mathfrak{n} \subset \mathfrak{v}$ and $[\mathfrak{n}, \mathfrak{n}] = \{0\}$, since $\mathfrak{v}$ is the first layer of $\mathfrak{h}^n$. If we pick an element $X \in \mathfrak{v}$ such that $[X, \mathfrak{n}] \neq \{0\}$, then we meet a contradiction. In fact, $X \notin \mathfrak{n}$, since $\mathfrak{n}$ is commutative and this conflicts with the fact that $\mathfrak{n}$ is an ideal.

Remark 11.5. In view of Remark 11.2, the previous example shows the nonexistence of surjective h-homomorphisms from $\mathbb{H}^n$ to $\mathbb{H}^k$ whenever $n > k$, as it was proved in Theorem 2.8 of [40], by explicit representation of h-homomorphisms from $\mathbb{H}^n$ to $\mathbb{H}^k$.

Definition 11.2 (Factorizing group as a quotient)**.** We say that $\mathbb{M}$ *factorizes* $\mathbb{G}$ *as a quotient* if it is an h-quotient of $\mathbb{G}$ and every normal subgroup N of $\mathbb{G}$ such that $\mathbb{G}/N$ is h-isomorphic to $\mathbb{M}$ has a complementary subgroup H.

Definition 11.3 (Factorizing group as a subgroup)**.** We say that $\mathbb{G}$ *factorizes* $\mathbb{M}$ *as a subgroup* if it h-embeds into $\mathbb{M}$ and every subgroup H of $\mathbb{M}$ which is h-isomorphic to $\mathbb{G}$ has a complementary normal subgroup N.

As a corollary of both Proposition 7.14 and Proposition 7.15, we have the following

Proposition 11.6. $\mathbb{M}$ *factorizes* $\mathbb{G}$ *as a quotient if and only if every surjective h-homomorphism is an h-epimorphism and* $\mathbb{G}$ *factorizes* $\mathbb{M}$ *as a subgroup if and only if every injective h-homomorphism is an h-monomorphism.*

Proposition 11.7. *If $\mathbb{R}^k$ h-embeds into $\mathbb{G}$, then it factorizes $\mathbb{G}$ as a subgroup.*

Proof. By hypothesis, the class of k-dimensional subgroups $H = \exp \mathfrak{h}$ of $\mathbb{G}$ which are h-isomorphic to $\mathbb{R}^k$ is nonempty and it correspond to k-dimensional commutative subgroups contained in $\exp V_1$. Let $\mathcal{N}_1$ be a subspace of V_1 such that $\mathfrak{h} \oplus \mathcal{N}_1 = V_1$. Then $\mathcal{N} = \mathcal{N}_1 \oplus V_2 \oplus \cdots \oplus V_\iota$ is an ideal and $N = \exp \mathcal{N}$ is complementary to H. □

Remark 11.8. If $\mathbb{R}^k$ h-embeds into $\mathbb{M}$, then every injective h-homomorphism from $\mathbb{R}^k$ to $\mathbb{M}$ is an h-monomorphism. This immdiately follows joining Proposition 11.6 and Proposition 11.7.

Definition 11.4 (Factorizing group)**.** We say that $\mathbb{G}$ *factorizes* $\mathbb{M}$ if it factorizes $\mathbb{M}$ both as a subgroup and as a quotient.

Remark 11.9. It might be interesting from an algebraic viewpoint to investigate whether factorizing groups as quotient are also factorizing as a subgroup and viceversa.

Proposition 11.10. *$\mathbb{R}$ factorizes every stratified group.*

Proof. By Lemma 11.3, $\mathbb{R}$ is an h-quotient of an arbitrary stratified group $\mathbb{G}$. Let $N = \exp\big(\mathcal{N}_1 \oplus \cdots \oplus \mathcal{N}_\iota\big)$ be a homogeneous normal subgroup such that $\mathbb{G}/N$ is h-isomorphic to $\mathbb{R}$. By Proposition 8.3, we must have that $V_1/\mathcal{N}_1$ is one-dimensional. The Lie algebra $\mathcal{G}$ of $\mathbb{G}$ is the direct sum $V_1 \oplus \cdots \oplus V_\iota$. We choose $X \in V_1 \setminus \mathcal{N}_1$ and define

$$\mathfrak{h} = \{tX \mid t \in \mathbb{R}\}.$$

It follows that $\mathfrak{n}$ and $\mathfrak{h}$ are homogeneous subalgebras such that $\mathfrak{n} \oplus \mathfrak{h} = \mathcal{G}$. By virtue of Proposition 7.7, $\mathbb{R}$ factorizes $\mathbb{G}$ as a quotient. Finally, by Proposition 11.7 $\mathbb{R}$ factorizes $\mathbb{G}$ as a subgroup, hence our proof is complete. $\square$

Example 11.11. One can easily find $\mathbb{G}$ and $\mathbb{M}$ such that there exist h-epimorphisms $L : \mathbb{G} \longrightarrow \mathbb{M}$, although $\mathbb{M}$ does not factorize $\mathbb{G}$ as a quotient. In fact, let us consider the 2-step stratified group $\mathbb{G}$ with Lie algebra $\mathcal{G} = V_1 \oplus V_2$, where

$$V_1 = \mathrm{span}\{X_1, X_2, X_3, X_4\}, \qquad V_2 = \mathrm{span}\{Z_{23}, Z_{24}, Z_{34}\}$$

and the only nontrivial bracket relations are the following

$$[X_2, X_3] = Z_{23}, \quad [X_2, X_4] = Z_{24}, \quad \text{and} \quad [X_3, X_4] = Z_{34}\,.$$

Let $L_1 : \mathbb{G} \longrightarrow \mathbb{R}^2$ be the h-epimorphism defined by

$$L_1\Big(\exp\big(\sum_{i=1}^4 x_i X_i + z_{23} Z_{23} + z_{24} Z_{24} + z_{34} Z_{34} \big) \Big) = (x_1, x_2)$$

and let $\ker L_1 = \exp \mathfrak{n} = \exp(\mathfrak{n}_1 \oplus \mathfrak{n}_2) = N$. We have $\mathfrak{n}_2 = V_2$ and $\mathfrak{n}_1 = \mathrm{span}\{X_3, X_4\}$, then $H = \exp\{X_1, X_2\}$ is complementary to N, as it easily follows from Proposition 7.7. However, if we consider the h-epimorphism $L_2 : \mathbb{G} \longrightarrow \mathbb{R}^2$ defined by

$$L_2\Big(\exp\big(\sum_{i=1}^4 x_i X_i + z_{23} Z_{23} + z_{24} Z_{24} + z_{34} Z_{34} \big) \Big) = (x_3, x_4),$$

then $\ker L_2 = \exp \mathfrak{n}$, with $\mathfrak{n} = \mathrm{span}\{X_1, X_2\} \oplus V_2$ and we will check that $N = \exp \mathfrak{n}$ does not admit any complementary subgroup. By contradiction, if H is complementary to N, then Proposition 7.14 shows that the restriction $T : H \longrightarrow \mathbb{R}^2$ of L_2 is a group isomorphism. In particular, H is commutative. In addition, Proposition 7.7

shows that $\mathfrak{h}\oplus\mathfrak{n}=\mathcal{G}$, where $\mathfrak{h}$ denotes the Lie algebra of H. Then $\mathfrak{h}$ is a 2-dimensional commutative subalgebra of $\mathcal{G}$. We consider a basis (v,w) of $\mathfrak{h}$, given by

$$v=\sum_{j=1}^{4}\alpha_j X_j+Z, \quad \text{and} \quad w=\sum_{j=1}^{4}\beta_j X_j+T,$$

where $T,Z\in V_2$. The decomposition $\mathfrak{h}\oplus\mathfrak{n}=\mathcal{G}$ implies that $(v,w,X_1,X_2,Z_{23},Z_{24},Z_{34})$ is a basis of $\mathcal{G}$, hence we must have $\alpha_3\beta_4-\beta_3\alpha_4\neq 0$. As a consequence,

$$[v,w]=(\alpha_2\beta_3-\alpha_3\beta_2)Z_{23}+(\alpha_2\beta_4-\alpha_4\beta_2)Z_{24}+(\alpha_3\beta_4-\alpha_4\beta_3)Z_{34}\neq 0\,.$$

This conflicts with the fact that $\mathfrak{h}$ is commutative.

11.1. **Factorizations in some H-type groups.** Our aim here is to present some factorizing properties of $\mathbb{R}^k$ in some important H-type groups. These groups were introduced in [31].

Definition 11.5. Let $\mathfrak{g}$ be a Lie algebra equipped with an inner product $\langle\cdot,\cdot\rangle$ and let $\mathfrak{z}$ be a nontrivial subspace of $\mathfrak{g}$ such that $[\mathfrak{g},\mathfrak{z}]=0$ and $[\mathfrak{g},\mathfrak{g}]\subset\mathfrak{z}$. Let $\mathfrak{v}$ be the orthogonal space of $\mathfrak{z}$ and define $J:\mathfrak{z}\longrightarrow \mathrm{End}(\mathfrak{v})$ by the formula

$$\langle J_Z X,Y\rangle=\langle Z,[X,Y]\rangle \tag{118}$$

for every $Z\in\mathfrak{z}$ and $X,Y\in\mathfrak{v}$. If J satisfies the condition

$$|J_Z X|=|Z|\,|X|\,,$$

then we say that $\mathfrak{g}$ is an *H-type algebra.*

From the previous definition, it follows that

$$[X,J_Z X]=|X|^2\,Z \qquad \text{for every} \qquad (X,Z)\in\mathfrak{v}\times\mathfrak{z}\,. \tag{119}$$

Then $\mathfrak{g}=\mathfrak{v}\oplus\mathfrak{z}$ is a 2-step stratified algebra of first layer $\mathfrak{v}$ and second layer $\mathfrak{z}$. In the sequel, we will utilize another well known formula

$$J_Z J_W+J_W J_Z=-2\,\langle Z,W\rangle\,\mathrm{Id}_{\mathfrak{v}}. \tag{120}$$

Example 11.12. Let (X,Y,Z) be an orthonormal basis of the Heisenberg algebra $\mathfrak{h}^1$, where $[X,Y]=Z$. Then setting $\mathfrak{z}=\mathrm{span}\{Z\}$, $\mathfrak{v}=\mathrm{span}\{X,Y\}$ and

$$J_Z X=Y, \qquad J_Z Y=-X$$

extended by linearity, it follows that J makes $\mathfrak{h}^1$ an H-type group. The higher dimensional Heisenberg algebras $\mathfrak{h}^n$ can be seen as direct product of the irreducible Heisenberg algebras isomorphic to $\mathfrak{h}^1$ as follows

$$\mathfrak{h}^n=\mathfrak{v}_1\oplus\cdots\oplus\mathfrak{v}_n\oplus\mathfrak{z},$$

where (X_i,Y_i) is an orthonormal basis of $\mathfrak{v}_i$ and $J_Z X_i=Y_i$, $J_Z Y_i=-X_i$. Here we notice that setting $\omega(U_1,U_2)=\langle J_Z U_1,U_2\rangle$, we define a symplectic form on the $2n$-dimensional first layer $\mathfrak{v}=\mathfrak{v}_1\oplus\cdots\oplus\mathfrak{v}_n$.

Example 11.13. It is easy to check that $\mathbb{R}^2$ does not factorize $\mathfrak{h}^1$ as a quotient, although it is an h-quotient of $\mathfrak{h}^1$. In fact, the only 1-dimensional ideal $\mathfrak{n}$ of $\mathfrak{h}^1$ is the second layer $\mathfrak{z}$ and we have already shown in Example 7.9 that there do not exist 2-dimensional subalgebras complementary to $\mathfrak{z}$.

However, as first observed in [22], $\mathbb{R}^k$ factorizes the Heisenberg group $\mathbb{H}^n$ as a quotient, whenever $k \leq n$. In view of Proposition 7.14, the next statement translates Proposition 3.24 of [22] into our setting, with a different proof.

Proposition 11.14. *If $1 \leq k \leq n$, then $\mathbb{R}^k$ factorizes $\mathbb{H}^n$.*

Proof. By Lemma 11.3, $\mathbb{R}^k$ is an h-quotient $\mathbb{H}^n$. Let N be a normal homogeneous subgroup of $\mathbb{H}^n$ such that $\mathbb{H}^n/N$ is h-isomorphic to $\mathbb{R}^k$. Let $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$ be the Lie algebra of N, which is also an ideal of $\mathfrak{h}^n$. It is clear that $\mathfrak{n}_2 = \mathfrak{z}$, where $\mathfrak{v}$ and $\mathfrak{z}$ denote the first and the second layers of $\mathfrak{h}^n$, respectively. If we prove that there exists a k-dimensional commutative subalgebra $\mathfrak{s}$ contained in $\mathfrak{v}$ such that $\mathfrak{s} \oplus \mathfrak{n}_1 = \mathfrak{v}$, then Proposition 7.7 concludes the proof.

To prove this, we fix a unit vector $Z \in \mathfrak{z}$ and consider the mapping $J_Z : \mathfrak{v} \longrightarrow \mathfrak{v}$ that makes $\mathfrak{h}^n$ an H-type algebra, see Example 11.12. We set $\mathfrak{n}_1 \cap J_Z(\mathfrak{n}_1) = \mathfrak{w}$ and notice that $J_Z(\mathfrak{w}) = \mathfrak{w}$. If $\mathfrak{w} \neq \{0\}$, then $\mathfrak{w}$ is a symplectic subspace with respect to the symplectic form $\omega(X, Y) = \langle J_Z X, Y \rangle$. In fact, one can easily construct the following orthogonal basis $(e_1, \ldots, e_l, J_Z(e_1), \ldots, J_Z(e_l))$ of $\mathfrak{w}$. We proceed as follows, choosing a unit vector $e_1 \in \mathfrak{w}$ and then considering the orthogonal $J_Z(e_1) \in \mathfrak{w}$. The procedure continues selecting a unit vector $e_2 \in \mathfrak{w} \cap \operatorname{span}\{e_1, J_Z(e_1)\}^\perp$ and checking that $J_Z(e_2) \in \mathfrak{w} \cap \operatorname{span}\{e_1, J_Z(e_1)\}^\perp$ and it is repeated up to reaching the desired basis. Notice that this basis is orthonormal by construction. If $p = \dim(\mathfrak{n}_1) > 2l$, then we choose $u_1 \in \mathfrak{n}_1 \cap \mathfrak{w}^\perp$ and observe that $J_Z(u_1) \notin \mathfrak{n}_1$, otherwise $u_1, J_Z(u_1)$ would belong to $\mathfrak{w}$, that is a contradiction. One can iterate this argument up to reaching $u_1, \ldots, u_{p-2l}$, where for every $j = 1, \ldots, p-2l$ we have

$$J_Z(u_j) \notin \mathfrak{n}_1.$$

By construction $u_1, \ldots, u_{p-2l}, J_Z(u_1), \ldots, J_Z(u_{p-2l})$ are orthogonal vectors and

$$(e_1, \ldots, e_l, J_Z(e_1), \ldots, J_Z(e_l), u_1, \ldots, u_{p-2l})$$

is an orthonormal basis of $\mathfrak{n}_1$. If $(p - l) < n$, then repeating the previous argument we can complete

$$(e_1, \ldots, e_l, J_Z(e_1), \ldots, J_Z(e_l), u_1, \ldots, u_{p-2l}, J_Z(u_1), \ldots, J_Z(u_{p-2l})) \tag{121}$$

to the following orthonormal basis of $\mathfrak{v}$

$$\begin{aligned}&\big(e_1, \ldots, e_l, J_Z(e_1), \ldots, J_Z(e_l), u_1, \ldots, u_{p-2l}, J_Z(u_1), \ldots, J_Z(u_{p-2l}),\\ &w_1, \ldots, w_r, J_Z(w_1), \ldots, J_Z(w_r)\big).\end{aligned}$$

This is precisely a symplectic basis with respect to $\omega(\cdot, \cdot)$, since

$$\langle Z, [w_s, J_Z w_s] \rangle = \langle J_Z w_s, J_Z w_s \rangle = 1 = \langle Z, [e_i, J_Z e_i] \rangle = \langle Z, [u_j, J_Z u_j] \rangle$$

implies that $[w_s, J_Z w_s] = [e_i, J_Z e_i] = [u_j, J_Z u_j] = Z$ and in the same way one can check that these are the only nontrivial brackets. For instance, we have

$$\langle Z, [J_Z(u_i), w_j]\rangle = \langle J_Z^2(u_i), w_j\rangle = -\langle u_i, w_j\rangle = 0$$

that gives $[J_Z(u_i), w_j] = 0$. Finally, the dimensional condition $2l + 2(p-2l) + 2r = 2n$ gives $l - r = p - n \geq 0$, by our hypothesis on $p \geq n$. Thus, the basis

$$\Big(J_Z(u_1), \ldots, J_Z(u_{p-2l}), w_1 + e_1, \ldots, w_r + e_r, J_Z(w_1) - J_Z(e_1), \ldots, J_Z(w_r) - J_Z(e_r)\Big)$$

defines a k-dimensional subspace $\mathfrak{s}$ that is commutative and by construction satisfies $\mathfrak{s} \cap \mathfrak{n}_1 = \{0\}$. This shows that $\mathbb{R}^k$ factorizes $\mathbb{H}^n$ as a quotient. Due to Proposition 11.7, $\mathbb{R}^k$ also factorizes $\mathbb{H}^n$ as a subgroup. This concludes the proof. □

Remark 11.15. Joining Proposition 11.6 and Proposition 11.10, it follows that every surjective h-homomorphism $L : \mathbb{G} \longrightarrow \mathbb{R}$ and every injective h-homomorphism $L : \mathbb{R} \longrightarrow \mathbb{G}$ are an h-epimorphism and an h-monomoprhism, respectively. Joining Proposition 11.6 and Proposition 11.14, for every $1 \leq k \leq n$, it follows that every surjective h-homomorphism $L : \mathbb{H}^n \longrightarrow \mathbb{R}^k$ and every injective h-homomorphism $L : \mathbb{R}^k \longrightarrow \mathbb{H}^n$ are an h-epimorphism and an h-monomoprhism, respectively. On the other hand, Example 11.13 shows that $\mathbb{R}^k$ is not an h-quotient of $\mathbb{H}^n$ whenever $k > n$.

Another example of H-type group is the complexified Heisenberg group $\mathbb{H}_2^1$, where the center $\mathfrak{z}$ of its Lie algebra $\mathfrak{h}_2^1$ is 2-dimensional and the first layer $\mathfrak{v}$ has dimension four. More information on this group can be found in [54].

Lemma 11.16. *Let $\mathfrak{h}_2^1 = \mathfrak{v} \oplus \mathfrak{z}$ be the 6-dimensional real Lie algebra of the complexified Heisenberg group $\mathbb{H}_2^1$ and let $X \in \mathfrak{v}$ with $|X| = 1$. Let (Z_1, Z_2) be an orthonormal basis of $\mathfrak{z}$. Then the following vectors*

$$R_0 = X, \quad R_1 = J_{Z_1} X, \quad R_2 = J_{Z_2} X, \quad R_3 = J_{Z_1} J_{Z_2} X, \tag{122}$$

form an orthonormal basis of $\mathfrak{v}$ and the only nontrivial bracket relations are given by

$$[R_0, R_1] = [R_2, R_3] = Z_1, \quad [R_0, R_2] = -[R_1, R_3] = Z_2. \tag{123}$$

Proof. Using just the properties of J and formula $[Y, J_Z Y] = |Y|^2 Z$ for every $Y \in \mathfrak{v}$ and $Z \in \mathfrak{z}$ it is easy to check that R_i form an orthonormal basis of $\mathfrak{v}$. The previous formula also yields $[R_0, R_1] = Z_1$ and $[R_0, R_2] = Z_2$. In addition, we have

$$\langle Z_1, [R_1, R_2]\rangle = \langle J_{Z_1}^2 X, J_{Z_2} X\rangle = -\langle X, J_{Z_2} X\rangle = 0,$$
$$\langle Z_2, [R_1, R_2]\rangle = \langle J_{Z_2} J_{Z_1} X, J_{Z_2} X\rangle = \langle J_{Z_1} X, X\rangle = 0,$$

then $[R_1, R_2] = 0$. To prove that $[R_0, R_3] = 0$, $[R_1, R_3] = -Z_2$ and $[R_2, R_3] = Z_1$, one argues in the same way, using also formula (120). □

Proposition 11.17. $\mathbb{R}^2$ *factorizes the complexified Heisenberg group $\mathbb{H}_2^1$ as a quotient.*

Proof. Lemma 11.3 ensures that $\mathbb{R}^2$ is an h-quotient of $\mathbb{H}_2^1$. Let N be the kernel of an h-epimorphism $L : \mathbb{H}_2^1 \longrightarrow \mathbb{R}^2$ and set $\mathfrak{n} = \exp^{-1}(N) = \mathfrak{n}_1 \oplus \mathfrak{n}_2$, with $\mathfrak{n}_1 = \mathfrak{n} \cap V_1$ and $\mathfrak{n}_2 = \mathfrak{n} \cap V_2$, due to Proposition 7.2. Clearly $\mathfrak{n}_2 = \mathfrak{z}$ and $\dim(\mathfrak{n}_1) = 2$. We have either $\dim([\mathfrak{n}_1, \mathfrak{n}_1]) = 0$ or $\dim([\mathfrak{n}_1, \mathfrak{n}_1]) = 1$. In the first case we choose an orthonormal basis (X,Y) of $\mathfrak{n}_1$ and represent Y as a linear combination of R_i, according to (122). Then the fact that (X,Y) is an orthonormal basis and $[X,Y]=0$ imply that $Y=J_{Z_1}J_{Z_2}X$ for a fixed orthonormal basis (Z_1,Z_2) of the center $\mathfrak{z}$. Now we simply notice that the commutative subalgebra $\mathfrak{h}=\mathrm{span}\{J_{Z_1}X, J_{Z_2}X\}$ satisfies $\mathfrak{h}\oplus\mathfrak{n}=\mathfrak{h}_2^1$. Let us consider the remaining case and take an orthonormal basis (X,Y) of $\mathfrak{n}_1$. We have $[X,Y]=Z\neq 0$. Let (T_1,T_2) be an orthonormal basis of $\mathfrak{z}$ such that $T_1=Z/|Z|$. Replacing (Z_1,Z_2) in Lemma 11.16 with the orthonormal basis (T_1,T_2), we get $Y=\alpha_1 J_{T_1}X+\alpha_3 J_{T_1}J_{T_2}X$, with $|\alpha_1|=|Z|>0$. By direct computation, one can check that the commutative subalgebra $\mathfrak{h}=\mathrm{span}\{X-\lambda J_{T_2}X, \lambda J_{T_1}X+J_{T_1}J_{T_2}X\}$ satisfies the condition $\mathfrak{h}\oplus\mathfrak{n}=\mathfrak{h}_2^1$ if we fix $\lambda\neq 0$ and $\lambda^{-1}\neq\alpha_3/\alpha_1$. This concludes the proof. □

Proposition 11.18. *Let $\mathfrak{h}_2^1=\mathfrak{v}\oplus\mathfrak{z}$ be the complexified Heisenberg algebra. Then there do not exist commutative subalgebras of $\mathfrak{v}$ with dimension greater than two.*

Proof. By the general formula (119), for every $X\in\mathfrak{v}\setminus\{0\}$ the mapping

$$\mathrm{ad\ X}:\mathfrak{v}\longrightarrow\mathfrak{z}\,,\qquad Y\longrightarrow[X,Y]$$

is surjective, then its kernel is 2-dimensional. By contradiction, the existence of a commutative subalgebra of $\mathfrak{v}$ with dimension greater than two would conflict with the dimension of the kernel. □

Corollary 11.19. *For $k=3,4$ we have that $\mathbb{R}^k$ does not h-embeds into $\mathbb{H}_2^1$ and does not factorize $\mathbb{H}_2^1$ as a quotient.*

Proof. The first assertion immediately follows from both Definition 11.1 and Proposition 11.18. Concerning the proof of the second assertion, in view of Proposition 11.6 we consider a surjective h-homomorphism $L:\mathfrak{h}_2^1\longrightarrow\mathbb{R}^3$. By contradiction, if L is an h-epimorphism, then we get a 3-dimensional subalgebra of $\mathfrak{v}$ that is h-isomorphic to $\mathbb{R}^3$. This conflicts with Proposition 11.18 and concludes the proof. □

11.2. **Factorizations in some free stratified groups.** We denote by $\mathfrak{g}_{p,\upsilon}$ the free υ-step stratified algebra on p generators. The corresponding simply connected Lie group will be denoted by $\mathbb{G}_{p,\upsilon}$.

Remark 11.20. We notice that $\mathfrak{g}_{p,\upsilon}$ is an h-quotient of $\mathfrak{g}_{r,\upsilon}\oplus\mathfrak{a}$ for every $p\leq r$, where $\mathfrak{a}$ is a stratified algebra and $\oplus$ denotes the direct product of Lie algebras. It suffices to consider the basis

$$\begin{aligned}&\mathcal{B}_{r,\upsilon}=\big\{[X_{j_1},[X_{j_2},[\cdots,[X_{j_{s-1}},X_{j_s}],],\ldots,]\mid 1\leq s\leq\upsilon,\\&(j_1,\ldots,j_s)\in\mathcal{A}^s_{r,\upsilon}\subset\{1,\ldots,r\}^s\big\}\end{aligned}$$

of $\mathfrak{g}_{r,\upsilon}$ and the basis

$$\begin{aligned}&\mathcal{B}'_{p,\upsilon}=\big\{[X'_{j_1},[X'_{j_2},[\cdots,[X'_{j_{s-1}},X'_{j_s}],],\ldots,]\mid 1\leq s\leq\upsilon,\\&(j_1,\ldots,j_s)\in\mathcal{A}'^s_{p,\upsilon}\subset\{1,\ldots,p\}^s\big\}\end{aligned}$$

of $\mathfrak{g}_{p,\upsilon}$, setting

$$L\big([X_{j_1},[X_{j_2},[\cdots,[X_{j_{s-1}},X_{j_s}],],\ldots,]\big) := [X'_{j_1},[X'_{j_2},[\cdots,[X'_{j_{s-1}},X'_{j_s}],],\ldots,]\,,$$

for every $1 \leq s \leq \upsilon$ and every $(j_1,\ldots,j_s) \in \mathcal{A}'^s_{p,\upsilon} \subset \mathcal{A}^s_{r,\upsilon}$,

$$L\big([X_{j_1},[X_{j_2},[\cdots,[X_{j_{s-1}},X_{j_s}],],\ldots,]\big) := 0$$

for every $1 \leq s \leq \upsilon$ and every $(j_1,\ldots,j_s) \in \mathcal{A}^s_{r,\upsilon} \setminus \mathcal{A}'^s_{p,\upsilon}$ and $L(\mathfrak{a}) = \{0\}$. It is easy to check that $L : \mathfrak{g}_{r,\upsilon} \oplus \mathfrak{a} \longrightarrow \mathfrak{g}_{p,\upsilon}$ is a surjective h-homomorphism.

Proposition 11.21. *Let $\mathbb{P}$ be a stratified group such that $\mathbb{G}_{p,\upsilon}$ is an h-quotient of $\mathbb{P}$. Then $\mathbb{G}_{p,\upsilon}$ factorizes $\mathbb{P}$ as a quotient.*

Proof. We apply Proposition 11.6. By hypothesis, we have $L : \mathcal{P} \longrightarrow \mathfrak{g}_{p,\upsilon}$ that is a surjective h-homomorphism, where $\mathcal{P} = W_1 \oplus \cdots \oplus W_\iota$ is the stratified algebra of $\mathbb{P}$. Let $X_1,\ldots,X_p$ be generators of $\mathfrak{g}_{p,\upsilon}$. Then there exist $U_1,\ldots,U_p \in W_1$ such that $L(U_i) = X_i$. By hypothesis, we consider the basis

$$\mathcal{B} = \{[X_{j_1},[X_{j_2},[\cdots,[X_{j_{s-1}},X_{j_s}],],\ldots,] \mid 1 \leq s \leq \upsilon,\ (j_1,\ldots,j_s) \in \mathcal{A}^s \subset \{1,\ldots,p\}^s\}$$

of $\mathfrak{g}_{p,\upsilon}$. The homomorphism property of L implies that

$$L\big([U_{j_1},[U_{j_2},[\cdots,[U_{j_{s-1}},U_{j_s}],],\ldots,]\big) = [X_{j_1},[X_{j_2},[\cdots,[X_{j_{s-1}},X_{j_s}],],\ldots,]\,,$$

then the family

$$\mathcal{B}' = \{[U_{j_1},[U_{j_2},[\cdots,[U_{j_{s-1}},U_{j_s}],],\ldots,] \mid 1 \leq s \leq \upsilon,\ (j_1,\ldots,j_s) \in \mathcal{A}'^s \subset \{1,\ldots,p\}^s\}$$

is a basis of $\mathfrak{h} = \text{Lie-span}\{U_1,\ldots,U_p\}$, then L maps $\mathfrak{h}$ h-isomorphically onto $\mathfrak{g}_{p,\upsilon}$. □

Corollary 11.22. *$\mathbb{H}^1$ factorizes $\mathbb{G}_{r,2} \times \mathbb{G}'$ as a quotient for every $r \geq 2$.*

Proof. Observe that $\mathfrak{h}^1 = \mathfrak{g}_{2,2}$, take into account Remark 11.20 and apply Proposition 11.21. □

The following corollaries are straightforward.

Corollary 11.23. *$\mathbb{H}^1$ factorizes $\mathbb{G}_{r,2} \times \mathbb{G}_{r,2} \times \cdots \times \mathbb{G}_{r,2}$ as a quotient for every $r \geq 2$.*

Corollary 11.24. *$\mathbb{H}^1$ factorizes $\mathbb{H}^1 \times \mathbb{H}^1 \times \cdots \times \mathbb{H}^1$ as a quotient.*

Remark 11.25. Although $\mathbb{H}^1$ factorizes $\mathbb{H}^1 \times \mathbb{H}^1 \times \cdots \times \mathbb{H}^1$ as a quotient, one notice that there are a few nontrivial h-epimorphisms between these groups. In fact, they are all of the form

$$L : \overbrace{\mathbb{H}^1 \times \mathbb{H}^1 \times \cdots \times \mathbb{H}^1}^{n-times} \longrightarrow \mathbb{H}^1 \qquad L(a_1,\ldots,a_n) = J(a_k)$$

for some fixed $k \in \{1,\ldots,n\}$ and some h-isomorphism $J : \mathbb{H}^1 \longrightarrow \mathbb{H}^1$.

12. Examples of $(\mathbb{G},\mathbb{M})$-regular sets

Existence of different types of intrinsically regular sets in a given graded group $\mathbb{P}$ depends on the corresponding algebraic factorizations. In correspondence to the fact that $\mathbb{R}^k$ factorizes a graded group $\mathbb{M}$ as a subgroup, we have the following

Theorem 12.1. *Let $\mathbb{M}$ be a graded group and let n be the maximum over all dimensions of commutative subalgebras contained in the first layer. Then the family of k-dimensional Legendrian C^1 smooth submanifolds is nonempty if and only if $1 \leq k \leq n$ and it coincides with that of $(\mathbb{R}^k,\mathbb{M})$-regular sets of $\mathbb{M}$.*

Proof. Every k-dimensional commutative subalgebra of the first layer also represents a trivial example of k-dimensional Legendrian submanifold. This shows, by definition of n, that this family is nonempty for every $1 \leq k \leq n$. Recall that a k-dimensional Legendrian submanifold is locally parametrized by a C^1 contact mapping defined on an open subset of $\mathbb{R}^k$. In other words, it is locally parametrized by a continuously h-differentiable contact mapping with injective differential. Then Theorem 1.1 shows that this mapping is continuously P-differentiable. Furthermore, since in this case classical differentiability coincides with h-differentiability, then formula (69) shows that the classical differential coincides with the P-differential. Then the P-differential of the local parametrization is an injective h-homomorphism. As a result, by Remark 11.8 the P-differential is an h-monomorphism. Taking into account Definition 10.3, we have shown that every C^1 smooth Legendrian submanifold contained in $\mathbb{M}$ is an $(\mathbb{R}^k,\mathbb{M})$-regular set of $\mathbb{M}$. Clearly, we have also proved that there do not exist k-dimensional Legendrian submanifold when $k > n$, since the injective P-differential of the local parametrization would imply the existence of a commutative subalgebra in the first layer of $\mathcal{M}$ with dimension greater than k, that is a contradiction.

Now, let S be an $(\mathbb{R}^k,\mathbb{M})$-regular set, where clearly $1 \leq k \leq n$. Again Theorem 1.1 shows that continuously P-differentiable mappings on $\mathbb{R}^k$ exactly correspond to contact mappings of class C^1, then S is a C^1 smooth Legendrian submanifold. $\square$

It is well known that in the Heisenberg group $\mathbb{H}^n$ the only intrinsically regular sets are $(\mathbb{H}^n,\mathbb{R}^k)$-regular sets and $(\mathbb{R}^k,\mathbb{H}^n$-regular sets with $1 \leq k \leq n$, [22].

Definition 12.1. We say that a homogeneous subgroup of $\mathbb{H}^n$ is *horizontal* if its Lie algebra is contained in the first layer $\mathfrak{v}$ and *vertical* if it contains the second layer $\mathfrak{z}$.

Remark 12.2. Notice that horizontal subgroups are always commutative, whereas homogeneous normal subgroups exactly characterize vertical subgroups.

Proposition 12.3. *Every couple of complementary subgroups of $\mathbb{H}^n$ is formed by a horizontal and a vertical subgroup.*

Proof. Let $\mathfrak{a}$ and $\mathfrak{b}$ be the corresponding homogeneous subalgebras of A and B, respectively. From our hypothesis and Proposition 7.7, it follows that $\mathfrak{a} \oplus \mathfrak{b} = \mathfrak{h}^n$. In particular, $Z \in \mathfrak{a} \oplus \mathfrak{b}$ hence we can suppose that $\mathfrak{a}$ is the subalgebra containing a vector $W = W_0 + \lambda Z$ with $\lambda \neq 0$. Homogeneity of $\mathfrak{a}$ gives $W_0 + r\lambda Z \in \mathfrak{a}$ for every $r > 0$, therefore $W_0 \in \mathfrak{a}$ and $Z \in \mathfrak{a}$. This shows that $\mathfrak{a}$ is an ideal of $\mathfrak{h}^n$. Clearly, the same argument shows that $\mathfrak{b}$ cannot have vectors of the form $U_0 + \mu Z$ with $\mu \neq 0$, otherwise Z would belong to $\mathfrak{b}$, that conflicts with the decomposition $\mathfrak{a} \oplus \mathfrak{b} = \mathfrak{h}^n$. This shows that $\mathfrak{b}$ is contained in the first layer of $\mathfrak{h}^n$ and concludes the proof. □

The special factorization of $\mathbb{H}^n$ cannot be extended to all H-type groups.

Example 12.4. Let us consider the complexified Heisenberg group $\mathbb{H}^1_2$ along with the bases (R_0, R_1, R_2, R_3) and (Z_1, Z_2), defined in Lemma 11.16. We define the homogeneous commutative subalgebras $\mathfrak{a} = \operatorname{span}\{R_0, R_3, Z_1\}$ and $\mathfrak{b} = \operatorname{span}\{R_1, R_2, Z_2\}$. Clearly, $\mathfrak{a} \oplus \mathfrak{b} = \mathfrak{h}^1_2$, hence from Proposition 7.7 it follows that $A = \exp \mathfrak{a}$ and $B = \exp \mathfrak{b}$ are complementary subgroups of $\mathbb{H}^1_2$. On the other hand, $[R_0, R_2] = Z_2 \notin \mathfrak{a}$ and $[R_0, R_1] = Z_1 \notin \mathfrak{b}$ impliy that both $\mathfrak{a}$ and $\mathfrak{b}$ are not ideals of $\mathfrak{h}^1_2$.

On the other hand, we have the following

Proposition 12.5. *Let N and H be complementary subgroups of $\mathbb{H}^1_2$, where N is a normal. Then N contains the center of the group,* top-$\dim(N) \in \{4, 5\}$ *and H is commutative and horizontal, namely, $H \subset \exp \mathfrak{v}$.*

Proof Let $N = \exp \mathfrak{n}$ and $H = \exp \mathfrak{h}$. If $\dim(\mathfrak{n}) \geq 3$, then $\mathfrak{n} \supset \mathfrak{z}$ by formula (119). Then in this case $\dim(\mathfrak{h}) \leq 3$ and $\mathfrak{h} \subset \mathfrak{v}$. By Proposition 11.18, it follows that $\dim(\mathfrak{h}) \leq 2$. Then the case $\dim(\mathfrak{n}) = 3$ cannot occur. If $\dim(\mathfrak{n}) \leq 2$, then $\mathfrak{n} \subset \mathfrak{z}$, then Proposition 11.18 prevents the existence of $\mathfrak{h} \subset \mathfrak{v}$, that should be commutative. As a result, the only allowed possibilities are $\dim(\mathfrak{n}) = 4$ and $\dim(\mathfrak{n}) = 5$. □

Theorem 12.6. *The only intrinsically regular sets of $\mathbb{H}^1_2$ are contained in the following list*

1. $(\mathbb{H}^1_2, \mathbb{R})$*-regular sets,*
2. $(\mathbb{H}^1_2, \mathbb{R}^2)$*-regular sets,*
3. $(\mathbb{R}^2, \mathbb{H}^1_2)$*-regular sets,*
4. $(\mathbb{R}, \mathbb{H}^1_2)$*-regular sets.*

Proof. By Proposition 7.14 and Proposition 12.5, we are allowed to consider only level sets defined through continuously P-differentiable mappings with values in $\mathbb{R}^k$, $k = 1, 2$ and defined on an open set of $\mathbb{H}^1_2$. Every C^1 mapping $f : \Omega \subset \mathbb{H}^1_2 \longrightarrow \mathbb{R}^2$ with surjective differential defines a $(\mathbb{H}^1_2, \mathbb{R}^2)$-regular set, since it is continuously P-differential and its P-differential is an h-epimorphism due to Proposition 11.17. This allows us to apply Theorem 1.3. We argue in the same way for C^1 mappings $f : \Omega \subset \mathbb{H}^1_2 \longrightarrow \mathbb{R}$, due to Proposition 11.10.

Analogously, Proposition 7.15 and Proposition 12.5 allow us to consider only image sets from open subsets of $\mathbb{R}^k$, $k = 1, 2$ with values in $\mathbb{H}^1_2$. By Theorem 12.1, $(\mathbb{R}, \mathbb{H}^1_2)$-regular sets and $(\mathbb{R}^2, \mathbb{H}^1_2)$-regular sets of $\mathbb{H}^1_2$ exist and correspond to one dimensional and two dimensional C^1 smooth Legendrian submanifolds of $\mathbb{H}^1_2$. □

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