# Collisions and phase-vortex interactions in dissipative Ginzburg-Landau dynamics

F. Bethuel, G. Orlandi and D. Smets

**Abstract**

In this paper we describe a natural framework for the vortex dynamics in the parabolic complex Ginzburg-Landau equation in $\mathbb{R}^2$. This general setting does not rely on any assumption of well-preparedness and has the advantage to be valid even after collision times. We analyze carefully collisions leading to annihilation. A new phenomenon is identified, the phase-vortex interaction, related to persistence of low frequency oscillations, and leading to an unexpected drift in the motion of vortices.

*2000 Mathematics Subject Classification : 35B40, 35K55, 35Q40.*

## 1 Introduction

In this paper we continue our investigations initiated in [6] on the complex-valued parabolic Ginzburg-Landau equation

$$(\mathrm{PGL})_\varepsilon \qquad \begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon = \dfrac{1}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2) & \text{on } \mathbb{R}^N \times \mathbb{R}_*^+, \\ u_\varepsilon(x, 0) = u_\varepsilon^0(x) & \text{for } x \in \mathbb{R}^N, \end{cases}$$

where the initial datum $u_\varepsilon^0$ verifies the bound

$$(\mathrm{H}_0) \qquad \mathcal{E}_\varepsilon(u_\varepsilon^0) = \int_{\mathbb{R}^N} e_\varepsilon(u_\varepsilon^0) = \int_{\mathbb{R}^N} \frac{|\nabla u_\varepsilon^0|^2}{2} + \frac{1}{4\varepsilon^2}(1 - |u_\varepsilon^0|^2)^2 \leq M_0 |\log \varepsilon|$$

and $M_0$ is some fixed given constant. Our main focus in this sequel is on the specificities of the two-dimensional case $N = 2$. However, a part of the analysis is valid in arbitrary dimension and completes the one in [6] (where the emphasis was put on $N \geq 3$).

The evolution in time, and in particular its asymptotics as $\varepsilon \to 0$, has already attracted much attention. The picture in dimension two is somewhat different from the one in higher dimensions. In dimension $N \geq 3$, the original time scale is essentially the only appropriate one in order to describe the evolution. On the other hand, it is necessary to introduce an accelerated time scale in dimension $N = 2$ in order to describe some part of the dynamics.

Evidence for the last assertion was first provided on a formal level in [22, 23, 14], and then rigorously in the case of "well-prepared" data in [20, 17, 30, 27]. In particular, such well-prepared data have well defined vortices of degree $+1$ or $-1$, and the diverging part of the energy is entirely provided by those vortices. In this framework, it is shown that in the accelerated time $t = |\log \varepsilon| s$ vortices evolve according to a simple ordinary differential equation up to the first collision time.

Our purpose in this paper is to study similarly the asymptotics in dimension $N = 2$ relaxing completely the assumption on the well-preparedness. More precisely, the only assumption on the initial data is the natural energy bound ($\mathrm{H}_0$). The motivation comes from our previous investigation on the higher dimensional case [6], where important differences with the case of prepared data were pointed out[1].

A typical initial datum which we wish to handle[2] is given by

$$u_\varepsilon^0(z) = \exp(i\varphi_\varepsilon^0(z)) \prod_{i=1}^{l} f(\frac{|z-a_i|}{\varepsilon}) \left(\frac{z-a_i}{|z-a_i|}\right)^{d_i} \quad \text{on } \mathbb{R}^2, \tag{1}$$

where $f$ is a smooth non negative function on $\mathbb{R}^+$ such that $f(0) = 0$, $f \equiv 1$ outside of a compact set, $d_i \in \mathbb{Z}$ with $\sum_i d_i = 0$, and the phase $\varphi_\varepsilon^0$ verifies the bound

$$\|\nabla\varphi_\varepsilon^0\|_{L^2(\mathbb{R}^2)}^2 \leq C|\log\varepsilon|.$$

Our analysis shows that, in contrast with the higher dimensional case and with existing results on the two-dimensional case, the phase and the vortices[3] do actually interact in the accelerated time scale $t = |\log\varepsilon|s$. This phenomenon is related to persistence of low frequency oscillations in the phase, leading to an additional and somewhat unexpected drift term acting on vortices. This phenomenon would not be observed on a fixed bounded domain.[4]

The second point we wish to emphasize is that our analysis is **not** restricted by the occurrence of collisions. On the other hand, our results provide only a weak form of regularity for motion of vortices: in particular the motion of multiple degree vortices, with possible splittings and recombinations, remains a delicate open issue. A first step in this direction is provided by Theorem 3, where we describe the evolution of clusters of vortices of total degree zero. We show complete annihilation after a time proportional to the square of the confinement radius. In particular, vortices of degree zero are excluded except at a finite number of occurrences, which correspond to collisions. Even in the case of well-prepared data, this provides some new information, and also answers an open question raised by Jerrard and Soner ([17], Remark 2.2).

In the accelerated time, we set

$$\mathfrak{u}_\varepsilon(z,s) = u_\varepsilon(z, s|\log\varepsilon|).$$

Our first result establishes some compactness and rigidity for $\mathfrak{u}_\varepsilon$.

**Theorem 1.** *There exist a function $\vec{c} : \mathbb{R}_*^+ \to \mathbb{R}^2$, and for each $s>0$, a finite set $\{a_i(s)\}_{1\leq i\leq l(s)}$ of $\mathbb{R}^2$ and $l(s)$ integers $d_i(s) \in \mathbb{Z}$, such that, for a subsequence $\varepsilon_n \to 0$,*

$$\mathfrak{u}_{\varepsilon_n} \times \nabla\mathfrak{u}_{\varepsilon_n}(z,s) \to w_* \times \nabla w_*(z,s) + \vec{c}(s) \qquad \text{as } n \to +\infty, \tag{2}$$

*and $|\mathfrak{u}_{\varepsilon_n}| \to 1$, uniformly on every compact set $K \subset \mathbb{R}^2 \times \mathbb{R}_*^+ \setminus \Sigma_\mathfrak{v}$. Here, we have set*

$$w_*(z,s) = \prod_{i=1}^{l(s)} \left(\frac{z-a_i(s)}{|z-a_i(s)|}\right)^{d_i(s)}, \tag{3}$$

[1] The evolution in case of prepared data in dimension $N \geq 3$ has been studied in [23, 18, 21].

[2] Assumption $(H_0)$ obviously allows to handle a much larger class.

[3] Which we termed the linear and the topological modes respectively in [6].

[4] One may wonder if it is physically relevant to work on the whole of $\mathbb{R}^2$. For the related Gorkov-Eliashberg equation for superconductivity, the physical domain has to be rescaled by a factor diverging with $\varepsilon$, which allows the same long-range interaction phenomenon.

*and*

$$\Sigma_{\mathfrak{v}} = \cup_{s>0}\Sigma_{\mathfrak{v}}^s = \cup_{s>0}\cup_{i=1}^{l(s)}\{a_i(s)\}.$$

*Moreover, there exist constants $l_0$, $d_0$ and $c_0$ depending only on $M_0$ such that for every $s>0$,*

$$l(s) \leq l_0, \quad |d_i(s)| \leq d_0, \quad \textit{and} \quad |\vec{c}(s)| \leq \frac{c_0}{\sqrt{s}}.$$

We would like to draw the attention of the reader to the fact that degree zero vortices, i.e. points $a_i(s)$ such that $d_i(s) = 0$ **do not** enter explicitly in the expression (3) of $w_*(z,s)$. However, their possible presence plays an important role in the description of the set $\Sigma_\nu$ and in the convergence stated in (2), so that at this stage of the analysis they cannot be removed a priori (see Proposition 1 below for further results on this issue).

Theorem 1 should be compared with the higher dimensional counterpart obtained in [6]. In the original time scale, there is no compactness for the functions due to possible wild oscillations in the phase. After times of the order of $|\log\varepsilon|$, these oscillations have been damped to order one.

In the special case of well-prepared data, similar results have been established, up to collision time, in [17, 20]: in their case, however, the additional term $\vec{c}$ is not observed. This new term is related to possible divergence of energy in the phase, and more precisely to (extremely) low frequency terms. Here is an explicit example of initial datum giving rise to a non-zero term $\vec{c}$ : take $u_0^\varepsilon$ as in (1) and

$$\varphi_\varepsilon^0(z) = \sqrt{|\log\varepsilon|}e^{-\frac{|z-a(\varepsilon)|^2}{4|\log\varepsilon|}},$$

where $a(\varepsilon) = \sqrt{|\log\varepsilon|}\vec{e}_1$. Using the explicit evolution of Gaussians by the heat equation, an elementary computation leads to the formula[5] $\vec{c}(s) = \frac{1}{2(1+s)^2}\exp(-\frac{1}{4(1+s)})\vec{e}_1$.

Clearly, the set $\Sigma_{\mathfrak{v}}$ in Theorem 1 contains the trajectory of vortices (as far as they can be defined!). Our next result provides some regularity properties for $\Sigma_{\mathfrak{v}}$.

**Theorem 2.** *The set $\Sigma_{\mathfrak{v}}$ is closed in $\mathbb{R}^2\times\mathbb{R}_*^+$ and of locally finite two-dimensional parabolic Hausdorff measure. Moreover, there exists $\alpha>0$ depending only on $M_0$ such that for each $s>0$ there exists $s'>s$ such that*

$$\Sigma_{\mathfrak{v}} \cap \mathbb{R}^2\times[s,s') \subset \cup_{i=1}^{l(s)}\mathcal{P}(a_i(s),s), \tag{4}$$

*where, for $(z,s)\in\mathbb{R}^2\times\mathbb{R}_*^+$, $\mathcal{P}(z,s)$ denotes the parabolic cone defined by*

$$\mathcal{P}(z,s) = \{(z',s')\in\mathbb{R}^2\times\mathbb{R}^+ \ \textit{s.t.} \ \ s'-s\geq\alpha|z'-z|^2\}.$$

In the case of well-prepared initial data, with $d_i = \pm1$, it is known from [20, 17] that the points $a_i(s)$ evolve according to the motion law

$$\frac{d}{ds}a_i(s) = 2\nabla_{a_i}\Big(\sum_{j\neq i} d_id_j\log|a_i-a_j|\Big),$$

[5]In order to keep this paper of reasonable size we will not work out the details here.

up to the first collision time. For initial data of the form (1) and with $d_i = \pm 1$ for all $i$, the motion law for the vortices would be given, similarly, by

$$\frac{d}{ds} a_i(s) = 2\nabla_{a_i}\Big(\sum_{j\neq i} d_i d_j \log|a_i - a_j|\Big) + d_i \vec{c}(s)^{\perp}. \tag{5}$$

In particular, in this range, the set $\Sigma_{\mathfrak{v}}$ is a disjoint finite union of smooth curves. We therefore strongly believe that Theorem 2 is not optimal, and that in the general case $\Sigma_{\mathfrak{v}}$ is a finite union of smooth curves, with possible branching corresponding to collisions and splitting of vortices of multiple degree. As a consequence, such a set would be one-dimensional rectifiable, whereas we only obtained a bound on the two-dimensional parabolic Hausdorff measure. However, to improve Theorem 2 and go beyond the parabolic scaling, one will need some way to describe the evolution of the vortex cores.[6]

Our next theorem settles the question of annihilation.[7]

**Theorem 3.** *Let $s_0 > 0$, $R > 0$ and $a \in \mathbb{R}^2$. Assume that $\sum_{a_i(s_0)\in B(a,R)} d_i(s_0) = 0$ and that for some $0 < \kappa < 1$*

$$\Sigma_{\mathfrak{v}}^{s_0} \cap B(a,R) \subset B(a,\kappa R). \tag{6}$$

*There exists positive constants $\kappa_0$, $K_1$ and $K_2$ depending only on $M_0$ such that, if $\kappa \leq \kappa_0$ then*

$$\Sigma_{\mathfrak{v}}^{s} \cap B(a, \frac{R}{2}) = \emptyset,$$

*for every $s \in [s_0 + K_1\kappa^2 R^2, s_0 + K_2 R^2]$.*

Theorem 3 has several consequences, both of global and local nature. First, if at some time $s_0$ all vortices $a_i(s_0)$ are contained in a ball of radius $R$, and of total degree zero[8], then at a later time $s_0 + CR^2$ they have completely disappeared and $w_*$ is constant. A second one is the following:

**Proposition 1.** *The topological degrees $d_i(s)$ are non zero except for a finite number of times.*

**Remark 1.** The result described in Theorem 3 and Proposition 1 do not hold for the original time scale, or even intermediate time scales. In particular degree zero vortices may survive on a full time interval in the original time scale. A way to construct such limit vortices is to take a vortex-antivortex pair (for more details, see the "additional comments" in Section 3, after the proof of Theorem 3.1).

As previously mentioned, the above results allow to give an answer to Remark 2.2 in [17][9], concerning collision for a prepared datum with two vortices of degree $+1$ and $-1$, for instance

$$u_\varepsilon^0(z) = f(\frac{z-1}{\varepsilon}) f(\frac{z+1}{\varepsilon}) \frac{(z-1)}{|z-1|} \left(\frac{(z+1)}{|z+1|}\right)^{-1}.$$

[6] This can be done in some specific cases, for instance we believe that our method would allow us to handle the case $|d_i| \leq 1$, but that the general case presumably does not have a simple answer. Indeed, splitting of multiple degree vortices involves discussions related to stable and unstable manifolds, and the resulting behavior is therefore very sensitive to the initial datum.

[7] Related results are announced for [28] based on different type of arguments.

[8] This is not always the case under assumption $(H_0)$. Take as initial datum $u_\varepsilon^0$ with a $+1$ vortex at the origin and a -1 vortex at a distance of order $\varepsilon^{-1}$. Then $l(s) = 1$ for all $s$, $a_1(s) = 0$, $d_1(s) = 1$ and $w_*(z) = z/|z|$.

[9] The method described allows to treat collisions of total degree zero. However collisions with total non zero degree are not excluded, and are not treated here.

In view of [17], it is known that the solution has two vortices $a_i$, $i=-1,1$ given by $a_i(s) = (-1)^i\sqrt{1-2s}$. In particular, these two vortices will collide at time $S=\frac{1}{2}$. They disappear after this collision time, as a consequence of Theorem 3, and $w_*$ is constant afterward.

Although they did not appear explicitly in our previous statements, the Radon measures $\mathfrak{v}_\varepsilon^s$ defined for $s \geq 0$ on $\mathbb{R}^2 \times \{s\}$ by

$$\mathfrak{v}_\varepsilon^s(x) = \frac{e_\varepsilon(\mathfrak{u}_\varepsilon(x,s))}{|\log\varepsilon|}\,dx$$

are central in the proofs. These quantities possess remarkable properties inherited from the equation $(\mathrm{PGL})_\varepsilon$. As a matter of fact, the points $a_i$ will appear as concentration points of these measures. The following preliminary result insures first that their asymptotic limits actually do exist.

**Theorem 4.** *Assume $(H_0)$ holds. There exist a sequence $\varepsilon_n \to 0$ and, for each $s \geq 0$, a measure $\mathfrak{v}_*^s$ on $\mathbb{R}^2 \times \{s\}$ such that*

$$\mathfrak{v}_{\varepsilon_n}^s \rightharpoonup \mathfrak{v}_*^s \qquad \text{as } n \to \infty. \tag{7}$$

In view of assumption $(\mathrm{H}_0)$ and the energy inequality $\|\mathfrak{v}_\varepsilon^s\| \leq M_0$, $\forall s \geq 0$, for **fixed s** it is straightforward to find a sequence $\varepsilon_n \to 0$ such that $\mathfrak{v}_{\varepsilon_n}^s$ converges as $n \to +\infty$. The main difficulty in Theorem 4 is to find a sequence $\varepsilon_n$ for which the convergence holds for **all** positive times. Clearly, convergence in (7) requires some specific property for the family $(\mathfrak{v}_\varepsilon^s)_{0<\varepsilon<1}$, which may be interpreted as a regularity in time. In the original time scale, the result described in Theorem 4 is well-known, and its proof relies on the so-called semi-decreasing property (see [8]). In contrast, in the accelerated time scale, the proof is much less direct, and is obtained at a late stage of our PDE analysis.

Finally, our last result relates the points $a_i(s)$ with the measures $\mathfrak{v}_*^s$, and provides some further properties of $\mathfrak{v}_*^s$.

**Theorem 5.** *i) For every $s > 0$, we have*

$$\mathfrak{v}_*^s = \sum_{i=1}^{l(s)} \theta_i(s)\delta_{a_i(s)}$$

*for some non negative densities $\theta_i(s)$. Moreover, we have*

$$\theta_i(s) \geq \pi|d_i(s)| \qquad \forall\, i = 1, ..., l(s). \tag{8}$$

*ii) For every $s_0 > 0$ and every $\chi \in \mathcal{D}(\mathbb{R}^2, \mathbb{R}^+)$ such that $\mathrm{supp}(\nabla\chi) \cap \cup_{i=1}^{l(s)}\{a_i(s_0)\} = \emptyset$, the function $s \mapsto \mathfrak{v}_*^s(\chi)$ is non-increasing on some interval $[s_0, s_0']$ with $s_0' > s_0$. Moreover, the function $s \mapsto \|\mathfrak{v}_*^s\|$ is non-increasing on $\mathbb{R}_*^+$.*

*iii) There exists some universal constant $\eta_0 > 0$ such that if for some time $s_0 > 0$, and some $i \in \{1, ..., l(s_0)\}$,*

$$d_i(s_0) = 0,$$

*then*

$$\lim_{s \to s_0^-} ||\mathfrak{v}_*^s|| - \lim_{s \to s_0^+} ||\mathfrak{v}_*^s|| \geq \frac{\eta_0}{2}. \tag{9}$$

*In particular, for all but finitely many $s > 0$, $|d_i(s)| \geq 1$ and thus $\theta_i(s) \geq \pi$.*

The plan of the paper does not follow the order of the Theorems, which was chosen for expository purposes. As already mentioned, the guiding thread will be the concentration points of energy measures which in turn allow to define the vortices and their degrees. A preliminary step is to describe the asymptotic in the original time scale.[10] We then first prove Theorem 2, 4 and 5 i). Our analysis relies heavily on three distinct ingredients. The first one is the decomposition of $u_\varepsilon$ given in Proposition 4.1, which allows to identify and remove the oscillatory and non topological part of the energy. This technique was used extensively in [6], here we extend it to the long time range. It requires therefore specific parabolic and elliptic linear estimates for measure data unbounded in one direction.[11] The second ingredient, the Cylinders Lemma (Proposition 4.3), gives an upper bound on the speed of concentration sets. This kind of lemma have already a long history [10, 25, 20, 19], our arguments are however qualitatively different and do not rely on energy lower bounds nor on the precise description of vortex cores. Finally, concentration sets and vortices are related through a third ingredient, the Clearing-Out Lemma.[12]
In the last part of the paper, starting in Section 6, we prove some compactness properties for the functions $u_\varepsilon$ themselves, and obtain rigidity formulas leading to Theorem 1, 3, and 5 ii) and iii).

In order to conclude this introduction, we would like to emphasize once more that our work has left aside the difficult question of the precise dynamics in the general setting considered here. As mentioned, this would require a further understanding of high multiplicity vortices, and in particular the mechanism of their splittings and possible recombinations. The case $d_i = \pm 1$ is much simpler, we intend to establish rigorously the motion law (5) in a different place. The general case is still a challenge to us.

**Acknowledgments.** We wish to thank warmly the referee for his judicious remarks and his very careful reading, which we believe led to a substantial improvement of the manuscript.

# 2 Some properties of $(\mathrm{PGL})_\varepsilon$

In this section we collect some elements entering in the study of $(\mathrm{PGL})_\varepsilon$. Set

$$\mu_\varepsilon^t(x) = \frac{e_\varepsilon(u_\varepsilon(x,t))}{|\log \varepsilon|}\, dx.$$

We begin with

## 2.1 Classical identities for the evolution of $\mu_\varepsilon^t$

**Lemma 2.1.** *Let $u_\varepsilon$ be a solution of $(PGL)_\varepsilon$. Then, $\forall\, \chi \in \mathcal{D}(\mathbb{R}^N)$ and $\forall\, t \geq 0$ we have*

$$\frac{d}{dt}\int_{\mathbb{R}^N} \chi(x)\, d\mu_\varepsilon^t = -\int_{\mathbb{R}^N\times\{t\}} \chi(x)\frac{|\partial_t u_\varepsilon|^2}{|\log\varepsilon|}\, dx + \int_{\mathbb{R}^N\times\{t\}} \nabla\chi(x)\cdot \frac{-\partial_t u_\varepsilon \cdot \nabla u_\varepsilon}{|\log \varepsilon|}\, dx. \tag{2.1}$$

In most applications we will assume $\chi \geq 0$, so that the first term on the r.h.s. of (2.1) is non positive. In order to handle the second term, and to get rid of the time derivative $\partial_t u_\varepsilon$, it is often useful to invoke another identity involving the stress-energy tensor.

[10] In particular, we complete in Appendix C some arguments which were only briefly sketched in [6].
[11] These are developed in an Appendix.
[12] Another approach avoiding this type of argument is exposed in [27] and [28].

**Lemma 2.2.** *Let $\vec{X} \in \mathcal{D}(\mathbb{R}^N, \mathbb{R}^N)$. Then, $\forall\, t \geq 0$,*

$$\frac{1}{|\log \varepsilon|} \int_{\mathbb{R}^N \times \{t\}} \left( e_\varepsilon(u_\varepsilon)\delta_{ij} - \frac{\partial u_\varepsilon}{\partial x_i} \cdot \frac{\partial u_\varepsilon}{\partial x_j} \right) \frac{\partial X_i}{\partial x_j}\, dx = -\int_{\mathbb{R}^N \times \{t\}} \vec{X} \cdot \frac{-\partial_t u_\varepsilon \cdot \nabla u_\varepsilon}{|\log \varepsilon|}\, dx. \tag{2.2}$$

The proofs of the above identities are standard (see [6] and references therein). The l.h.s. of (2.2) involves the stress-energy matrix $A_\varepsilon$ given, in case $N = 2$, by

$$A_\varepsilon \equiv A_\varepsilon(u_\varepsilon) = T(u_\varepsilon) + V_\varepsilon(u_\varepsilon)\,\mathrm{Id}\,, \tag{2.3}$$

where the matrix $T(u)$ is defined by

$$T(u) = \frac{1}{2} \begin{pmatrix} |u_{x_2}|^2 - |u_{x_1}|^2 & -2u_{x_1} \cdot u_{x_2} \\ -2u_{x_1} \cdot u_{x_2} & |u_{x_1}|^2 - |u_{x_2}|^2 \end{pmatrix}, \tag{2.4}$$

and the function $V_\varepsilon$ denotes the potential

$$V_\varepsilon(u_\varepsilon) = \frac{(1 - |u_\varepsilon|^2)^2}{4\varepsilon^2}. \tag{2.5}$$

In dimension two, the product $T_{ij}\, \frac{\partial X_i}{\partial x_j}$ has a particularly simple expression using complex notation. Set[13]

$$X = X_1 + iX_2 \qquad \text{and} \qquad \omega = |u_{x_1}|^2 - |u_{x_2}|^2 - 2iu_{x_1} \cdot u_{x_2}.$$

Then, we have

$$\int_{\mathbb{R}^2} T_{ij}(u) \frac{\partial X_i}{\partial x_j} = \mathrm{Re} \left( -\int_{\mathbb{R}^2} \omega \frac{\partial X}{\partial \bar{z}} \right). \tag{2.6}$$

Combining Lemma 2.1 and Lemma 2.2 with the choice $\vec{X} = \nabla \chi$, we get rid of the time derivative on the r.h.s. of (2.1). More precisely

**Lemma 2.3.** *We have, for $t \geq 0$,*

$$\frac{d}{dt} \int_{\mathbb{R}^N} \chi(x)\, d\mu_\varepsilon^t = -\int_{\mathbb{R}^N \times \{t\}} \chi(x) \frac{|\partial_t u_\varepsilon|^2}{|\log \varepsilon|}\, dx + \frac{1}{|\log \varepsilon|} \mathcal{F}_S(t, \chi, u_\varepsilon), \tag{2.7}$$

*where*

$$\mathcal{F}_S(t, \chi, u_\varepsilon) = \int_{\mathbb{R}^N \times \{t\}} \left( D^2\chi \nabla u_\varepsilon \cdot \nabla u_\varepsilon - \Delta\chi e_\varepsilon(u_\varepsilon) \right) dx.$$

Another simple yet important consequence of Lemma 2.1 is

**Lemma 2.4.** *Let $\chi$ be as above. Then*

$$\frac{d}{dt} \int_{\mathbb{R}^N \times \{t\}} \chi^2(x) e_\varepsilon(u_\varepsilon) dx \leq 2\|\nabla \chi\|_{L^\infty}^2 M_0 |\log \varepsilon|\,. \tag{2.8}$$

[13] The quantity $\omega$ is usually termed the Hopf differential of $u$.

## 2.2 Phase-Vortex interaction

In this paragraph we consider a real-valued function $\phi_\varepsilon$ defined on $\mathbb{R}^2 \times \mathbb{R}^+$ satisfying the heat equation, the function $w_\varepsilon$ defined by $w_\varepsilon = u_\varepsilon \exp(-i\phi_\varepsilon)$, and the measure $\nu_\varepsilon^t$ defined by

$$\nu_\varepsilon^t = \frac{e_\varepsilon(w_\varepsilon(\cdot,t))}{|\log \varepsilon|} dx. \tag{2.9}$$

This kind of decomposition will be motivated in particular by Proposition 4.1. The purpose is to describe the evolution of $\nu_\varepsilon^t$ in the spirit of Lemma 2.3. In view of the r.h.s. of formula (2.7), we are led to consider the bilinear form

$$\mathcal{B}_\chi(A,B) = \int_{\mathbb{R}^N} D^2\chi A \cdot B - \frac{\Delta\chi}{2}(A \cdot B). \tag{2.10}$$

This quantity has remarkable algebraic properties, as the following formula shows.

**Lemma 2.5.** *Let $\chi \in \mathcal{D}(\mathbb{R}^N, \mathbb{R})$, and consider two 1-forms $A$ and $B$ belonging to $H^1_{loc}(\mathbb{R}^N)$. The following identity holds:*

$$\mathcal{B}_\chi(A,B) = \frac{1}{2}\int_{\mathbb{R}^N} dA \cdot (d\chi \wedge B) + dB \cdot (d\chi \wedge A) - \frac{1}{2}\int_{\mathbb{R}^N} d^*A(d\chi \cdot B) + d^*B(d\chi \cdot A). \tag{2.11}$$

*Proof.* First we write in coordinates

$$D^2\chi A \cdot B = \sum_{i,j} \partial^2_{ij}\chi A_i B_j = \frac{1}{2}\sum_{i,j}(\partial^2_{ij}\chi + \partial^2_{ji}\chi)A_i B_j.$$

Integrating by parts on $\mathbb{R}^N$ we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} D^2\chi A \cdot B = &-\frac{1}{2}\sum_{i,j}\int_{\mathbb{R}^N} \partial_i A_i \partial_j \chi B_j + A_i \partial_j \chi \partial_i B_j \\ &-\frac{1}{2}\sum_{i,j}\int_{\mathbb{R}^N} \partial_j A_i \partial_i \chi B_j + A_i \partial_i \chi \partial_j B_j.\end{aligned} \tag{2.12}$$

Similarly, we can write

$$-\frac{1}{2}\sum_{i,j}\int_{\mathbb{R}^N} \partial^2_{ii}\chi A_j B_j = \frac{1}{2}\sum_{i,j}\int_{\mathbb{R}^N} \partial_i A_j \partial_i \chi B_j + A_i \partial_j \chi \partial_j B_i.$$

The result follows adding the previous equalities. $\square$

Specifying (2.11) with $A = d\phi_\varepsilon$, $B = w_\varepsilon \times dw_\varepsilon$, we obtain

**Corollary 2.1.** *Set*

$$\mathcal{F}_I(t,\chi,\nabla\phi_\varepsilon,w_\varepsilon) = \mathcal{B}_\chi(d\phi_\varepsilon(\cdot,t)\,,\, w_\varepsilon \times dw_\varepsilon(\cdot,t)). \tag{2.13}$$

*If $N = 2$, we have the identity*

$$\mathcal{F}_I(t,\chi,\nabla\phi_\varepsilon,w_\varepsilon) = \int_{\mathbb{R}^2\times\{t\}} (\nabla\phi_\varepsilon \times \nabla\chi)Jw_\varepsilon + \mathcal{R}_I(t,\chi,\nabla\phi_\varepsilon,w_\varepsilon), \tag{2.14}$$

*where*

$$\mathcal{R}_I(t,\chi,\nabla\phi_\varepsilon,w_\varepsilon) = -\int_{\mathbb{R}^2\times\{t\}} \nabla\phi_\varepsilon \cdot \nabla\chi \operatorname{div}(w_\varepsilon \times \nabla w_\varepsilon) + \Delta\phi_\varepsilon \nabla\chi(w_\varepsilon \times \nabla w_\varepsilon). \tag{2.15}$$

After this digression we go back to the description of the evolution of $\nu_\varepsilon^t$.

**Lemma 2.6.** *We have*

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^2}\chi(x)d\nu_\varepsilon^t = &-\int_{\mathbb{R}^2}\chi(x)\frac{|\partial_t w_\varepsilon|^2}{|\log\varepsilon|}+\frac{1}{|\log\varepsilon|}\mathcal{F}_S(t,\chi,w_\varepsilon)\\ &+\frac{1}{|\log\varepsilon|}\mathcal{F}_I(t,\chi,\nabla\phi_\varepsilon,w_\varepsilon)+\frac{1}{|\log\varepsilon|}\mathcal{R}(t,\chi,\nabla\phi_\varepsilon,w_\varepsilon),\end{aligned}\tag{2.16}$$

*where*

$$\begin{aligned}\mathcal{R}(t,\chi,\nabla\phi_\varepsilon,w_\varepsilon) = &-\frac{d}{dt}\int_{\mathbb{R}^2\times\{t\}}\chi\left(\nabla\phi_\varepsilon\cdot w_\varepsilon\times\nabla w_\varepsilon+(|u_\varepsilon|^2-1)\frac{|\nabla\phi_\varepsilon|^2}{2}\right)\\ &+\int_{\mathbb{R}^2}2\chi\frac{\partial\phi_\varepsilon}{\partial t}\cdot w_\varepsilon\times\frac{\partial w_\varepsilon}{\partial t}+\int_{\mathbb{R}^2}(1-|u_\varepsilon|^2)\left(\chi\left|\frac{\partial\phi_\varepsilon}{\partial t}\right|^2-\Delta\chi\frac{|\nabla\phi_\varepsilon|^2}{2}\right).\end{aligned}\tag{2.17}$$

**Remark 2.1.** In formula (2.16) we have singled out two terms, whose significations in the case $N=2$ are the following:
- the term $\mathcal{F}_S$ will be interpreted as the force arising from the interaction between vortices (viewed, in our setting, as a self-interaction)
- the term $\mathcal{F}_I$ represents the interaction between the phase and the vortices.
The terms $\mathcal{R}$ and $\mathcal{R}_I$ will be shown to be of lower order (asymptotically), so that the main contribution in $\mathcal{F}_I$ is

$$\mathcal{F}_J(t,\chi,\nabla\phi_\varepsilon,w_\varepsilon)=\int_{\mathbb{R}^2\times\{t\}}(\nabla\phi_\varepsilon\times\nabla\chi)Jw_\varepsilon,\tag{2.18}$$

where $Jw_\varepsilon$ stands for the spatial Jacobian of $w_\varepsilon$, namely $Jw_\varepsilon$ is the scalar $\det(\nabla w_\varepsilon)\equiv\partial_1 w_\varepsilon\times\partial_2 w_\varepsilon$.

*Proof.* Since $u_\varepsilon = w_\varepsilon\exp(i\phi_\varepsilon)$, we have $\nabla u_\varepsilon=(\nabla w_\varepsilon+iw_\varepsilon\nabla\phi_\varepsilon)\exp(i\phi_\varepsilon)$, and $|w_\varepsilon|=|u_\varepsilon|$, so that

$$e_\varepsilon(u_\varepsilon)=e_\varepsilon(w_\varepsilon)+\frac{|\nabla\phi_\varepsilon|^2}{2}+\nabla\phi_\varepsilon\cdot w_\varepsilon\times\nabla w_\varepsilon+(|u_\varepsilon|^2-1)\frac{|\nabla\phi_\varepsilon|^2}{2},\tag{2.19}$$

and similarly

$$\left|\frac{\partial u_\varepsilon}{\partial t}\right|^2=\left|\frac{\partial w_\varepsilon}{\partial t}\right|^2+\left|\frac{\partial\phi_\varepsilon}{\partial t}\right|^2+2\frac{\partial\phi_\varepsilon}{\partial t}\cdot w_\varepsilon\times\frac{\partial w_\varepsilon}{\partial t}+(|u_\varepsilon|^2-1)\left|\frac{\partial\phi_\varepsilon}{\partial t}\right|^2.\tag{2.20}$$

Inserting these relations in identity (2.7), we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^2\times\{t\}}\chi(x)e_\varepsilon(w_\varepsilon) = A_0 + A_1 + A_2 + A_3 + A_4 + A_5,$$

$$A_0 = -\int_{\mathbb{R}^2}\chi\left|\frac{\partial w_\varepsilon}{\partial t}\right|^2 + \mathcal{F}_S(t,\chi,w_\varepsilon)$$

$$A_1 = -\frac{d}{dt}\int_{\mathbb{R}^2\times\{t\}}\chi\frac{|\nabla\phi_\varepsilon|^2}{2} - \int_{\mathbb{R}^N\times\{t\}}\left(\chi\left|\frac{\partial\phi_\varepsilon}{\partial t}\right|^2 + D^2\chi\nabla\phi_\varepsilon\cdot\nabla\phi_\varepsilon - \Delta\chi\frac{|\nabla\phi_\varepsilon|^2}{2}\right)$$

$$A_2 = \frac{d}{dt}\int_{\mathbb{R}^2\times\{t\}}\chi\left(\nabla\phi_\varepsilon\cdot w_\varepsilon\times\nabla w_\varepsilon + (|u_\varepsilon|^2-1)\frac{|\nabla\phi_\varepsilon|^2}{2}\right)$$

$$A_3 = \int_{\mathbb{R}^2\times\{t\}}2\chi\frac{\partial\phi_\varepsilon}{\partial t}\cdot w_\varepsilon\times\frac{\partial w_\varepsilon}{\partial t}$$

$$A_4 = 2\int_{\mathbb{R}^2\times\{t\}}\left(D^2\chi\nabla\phi_\varepsilon\cdot w_\varepsilon\times\nabla w_\varepsilon - \frac{\Delta\chi}{2}(\nabla\phi_\varepsilon\cdot w_\varepsilon\times\nabla w_\varepsilon)\right) = \mathcal{F}_I(t,\chi,\phi_\varepsilon,w_\varepsilon)$$

$$A_5 = \int_{\mathbb{R}^2\times\{t\}}(1-|u_\varepsilon|^2)\left(\chi\left|\frac{\partial\phi_\varepsilon}{\partial t}\right|^2 + D^2\chi\nabla\phi_\varepsilon\cdot\nabla\phi_\varepsilon - \Delta\chi\frac{|\nabla\phi_\varepsilon|^2}{2}\right).$$

Since $\phi_\varepsilon$ verifies the heat equation, $A_1$ vanishes and the conclusion follows. □

**Remark 2.2.** In Lemma 2.6, we have emphasized the evolution of the measures $\nu_\varepsilon^t$. Likewise, for $\mu_\varepsilon^t$ we have

$$\frac{d}{dt}\int_{\mathbb{R}^2}\chi(x)\,d\mu_\varepsilon^t = -\int_{\mathbb{R}^2\times\{t\}}\chi(x)\frac{|\partial_t w_\varepsilon|^2}{|\log\varepsilon|} + \mathcal{F}_S(t,\chi,w_\varepsilon) + \mathcal{F}_S(t,\chi,\phi_\varepsilon) + \mathcal{F}_I(t,\chi,\nabla\phi_\varepsilon,w_\varepsilon) + \mathcal{L}_0(t,|u_\varepsilon|,\chi,\phi_\varepsilon) \quad (2.21)$$

where

$$\mathcal{L}_0(t,|u_\varepsilon|,\chi,\phi_\varepsilon) = A_5 = \int_{\mathbb{R}^2\times\{t\}}(1-|u_\varepsilon|^2)\left(\chi\left|\frac{\partial\phi_\varepsilon}{\partial t}\right|^2 + D^2\chi\nabla\phi_\varepsilon\cdot\nabla\phi_\varepsilon - \Delta\chi\frac{|\nabla\phi_\varepsilon|^2}{2}\right).$$

## 2.3 Clearing-Out

We recall one version of the Clearing-Out theorem proved in [6]. It is used later to relate concentration sets and vorticity sets.

**Proposition 2.1.** *Let $u_\varepsilon$ be a solution of $(PGL)_\varepsilon$ verifying assumption $(H_0)$. Let $x_T \in \mathbb{R}^N$, $T > 0$ and $R \geq \sqrt{2\varepsilon}$. There exists a constant $\eta_0 > 0$ and a continuous function $\lambda$ defined on $\mathbb{R}_*^+$ such that, if*

$$\check{\eta}(x_T,T,R) \equiv \frac{1}{R^{N-2}|\log\varepsilon|}\int_{B(x_T,\lambda(T)R)} e_\varepsilon(u_\varepsilon(\cdot,T)) \leq \frac{\eta_0}{2}$$

*then*

$$|u_\varepsilon(x,t)| \geq \frac{1}{2} \qquad \textit{for } t\in[T+T_0,T+T_1] \quad \textit{and } x\in B(x_T,\frac{R}{2}).$$

*Here $T_0$ and $T_1$ are defined by*

$$T_0 = \max(2\varepsilon, \tau R^2), \qquad T_1 = R^2,$$

*where $\tau = 0$ if $N = 2$ and $\tau = \left(\frac{2\check{\eta}}{\eta_0}\right)^{\frac{2}{N-2}}$ otherwise.*

### 2.4 Pointwise estimates

First, we briefly recall some basic pointwise upper bounds.

**Proposition 2.2.** *Let $u_\varepsilon$ be a solution of $(PGL)_\varepsilon$ with $\mathcal{E}_\varepsilon(u_\varepsilon^0) < +\infty$. Then there exists a constant $K > 0$ depending only on $N$ such that, for $t \geq \varepsilon^2$ and $x \in \mathbb{R}^N$, we have*[14]

$$|u_\varepsilon(x,t)| \leq 3, \qquad |\nabla u_\varepsilon(x,t)| \leq \frac{C}{\varepsilon}, \qquad |\frac{\partial u_\varepsilon}{\partial t}(x,t)| \leq \frac{C}{\varepsilon^2}.$$

The proof relies essentially on some form of the maximum principle.

Our subsequent discussion requires also a careful analysis on the set where $|u_\varepsilon|$ is far from zero. For this purpose, we consider, for $T > 0$, $\Delta T > 0$, $R > 0$ given, the cylinder

$$\Lambda = B(x_0, R) \times [T, T + \Delta T] \subset \mathbb{R}^N \times \mathbb{R}^+,$$

and we assume that for some constant $0 < \sigma < \frac{1}{2}$,

$$|u_\varepsilon| \geq 1 - \sigma \qquad \text{on } \Lambda. \tag{2.22}$$

In particular, we may write $u_\varepsilon = \rho_\varepsilon \exp(i\varphi_\varepsilon)$ on $\Lambda$, where $\rho_\varepsilon = |u_\varepsilon|$ and where $\varphi_\varepsilon$ is a smooth real-valued map on $\Lambda$. For $0 < \alpha \leq 1$, set

$$\Lambda_\alpha = B(x_0, \alpha R) \times [T + (1 - \alpha^2)\Delta T, T + \Delta T].$$

The following higher-order regularity for $u_\varepsilon$ holds.

**Theorem 2.1.** *Assume (2.22) holds. There exists constants $0 < \sigma_0 \leq \frac{1}{2}$ and $0 < \alpha, \beta < 1$ depending only on $N$, such that if $\sigma < \sigma_0$, then*

$$\|\nabla \varphi_\varepsilon\|_{L^\infty(\Lambda_{\frac{3}{4}})} \leq C(\Lambda)\sqrt{M_0 |\log \varepsilon|} \tag{2.23}$$

$$\|1 - \rho_\varepsilon\|_{L^\infty(\Lambda_{\frac{1}{2}})} \leq C(\Lambda)\varepsilon^2 (1 + \|\nabla \varphi_\varepsilon\|^2_{L^\infty(\Lambda_{\frac{3}{4}})}) \tag{2.24}$$

$$\|\partial_t \rho_\varepsilon\|_{\mathcal{C}^{0,\alpha}(\Lambda_{\frac{1}{2}})} + \|\nabla \rho_\varepsilon\|_{\mathcal{C}^{0,\alpha}(\Lambda_{\frac{1}{2}})} \leq C(\Lambda) M_0 \varepsilon^\beta. \tag{2.25}$$

*There exists a real-valued function $\Phi_\varepsilon$ defined on $\Lambda_{\frac{1}{2}}$, and satisfying the heat equation, such that*

$$\|\partial_t \varphi_\varepsilon - \partial_t \Phi_\varepsilon\|_{\mathcal{C}^{0,\alpha}(\Lambda_{\frac{1}{2}})} + \|\nabla \varphi_\varepsilon - \nabla \Phi_\varepsilon\|_{\mathcal{C}^{0,\alpha}(\Lambda_{\frac{1}{2}})} \leq C(\Lambda) M_0 \varepsilon^\beta. \tag{2.26}$$

The proof is a little lengthy and requires some care. We postpone it to Appendix C.

[14] Note in particular that $C$ is independent of the initial data.

## 3 Asymptotics in the original time scale

As a preliminary step for the long-time analysis, we show that vortices do not move in the original time scale. Here we rely on our previous analysis in [6], which holds in any dimension $N \geq 2$. It asserts first as a consequence of the semi-decreasing property that up to a subsequence $\varepsilon_n \to 0$,

$$\mu^t_{\varepsilon_n} \to \mu^t_* \qquad \text{for all } t \geq 0,$$

for some Radon measures $\mu^t_*$. Moreover there exist a closed set $\Sigma_\mu = \cup_{t>0} \cup_{i=1}^{l} b_i(t)$ in $\mathbb{R}^2 \times \mathbb{R}^+_*$ such that $|u_\varepsilon| \to 1$ locally uniformly on $\mathbb{R}^2 \times \mathbb{R}^+_* \setminus \Sigma_\mu$ and such that for a.e. $t \geq 0$,

$$\mu^t_* = \frac{|\nabla \Phi_*|^2}{2}(.,t)dx + \nu^t_*, \qquad \text{where} \qquad \nu^t_* = \sum_{i=1}^{l} \sigma_i(t)\delta_{b_i(t)},$$

and

$$\text{either } \sigma_i(t) \geq \eta_0 \qquad \text{or} \qquad \sigma_i(t) = 0. \tag{3.1}$$

The function $\Phi_*$ satisfies the heat equation on $\mathbb{R}^2 \times \mathbb{R}^+_*$, and $l \leq CM_0$.

**Theorem 3.1.** *The points $b_i(t)$ do not move, i.e.*

$$b_i(t) = b_i \qquad \forall\, t > 0, \tag{3.2}$$

*and the functions $\sigma_i(t)$ are non-increasing.*

This last statement is consistent with Theorem B of [6] where it is shown that $\nu^t_*$ moves according to mean curvature: indeed, points have essentially zero mean curvature. Nevertheless, some arguments in [6] are not valid for $N = 2$ so that we present next the appropriate modifications.

Off the singular set $\Sigma_\mu$, the main contribution to the time derivative $\partial_t u_\varepsilon$ stems from the phase $\Phi_\varepsilon$. In this direction, the following proposition, motivated by Lemma 2.1, was stated without proof in [6]: we provide the details here for $N \geq 2$.[15]

**Proposition 3.1.** *Let $N \geq 2$, and $u_\varepsilon$ be a solution to $(PGL)_\varepsilon$. Then, as $\varepsilon \to 0$,*

$$\begin{aligned} \frac{|\partial_t u_\varepsilon|^2}{|\log \varepsilon|} &\to |\partial_t \Phi_*|^2 && \textit{in } \mathcal{C}^0_{\mathrm{loc}}(\mathbb{R}^N \times (0,+\infty) \setminus \Sigma_\mu) \\ \frac{\partial_t u_\varepsilon \cdot \nabla u_\varepsilon}{|\log \varepsilon|} &\to \partial_t \Phi_* \cdot \nabla \Phi_* && \textit{in } \mathcal{C}^0_{\mathrm{loc}}(\mathbb{R}^N \times (0,+\infty) \setminus \Sigma_\mu). \end{aligned} \tag{3.3}$$

*Proof.* Since $\mathbb{R}^N \times (0,+\infty) \setminus \Sigma_\mu$ is an open set, it suffices to establish the uniform convergences in (3.3) on a cylindrical domain $\Lambda_{\frac{1}{2}}$ such that $\Lambda \subset \mathbb{R}^N \times (0,+\infty) \setminus \Sigma_\mu$. Since $|u_\varepsilon| \to 1$ uniformly on $\Lambda$, we may apply Theorem 2.1 to $u_\varepsilon$ on $\Lambda$. Notice that, writing $u_\varepsilon = \rho_\varepsilon \exp(i\varphi_\varepsilon)$ on $\Lambda$, we have

$$\begin{aligned} \partial_t u_\varepsilon &= \partial_t \rho_\varepsilon \exp(i\varphi_\varepsilon) + i\rho_\varepsilon \exp(i\varphi_\varepsilon)\partial_t \varphi_\varepsilon = \partial_t \rho_\varepsilon \exp(i\varphi_\varepsilon) + i\rho_\varepsilon \exp(i\varphi_\varepsilon)\partial_t \Phi_\varepsilon \\ &\quad + i\rho_\varepsilon \exp(i\varphi_\varepsilon)(\partial_t \varphi_\varepsilon - \partial_t \Phi_\varepsilon) + i\exp(i\varphi_\varepsilon)(1-\rho_\varepsilon)\partial_t \varphi_\varepsilon = i\exp(i\varphi_\varepsilon)\partial_t \Phi_\varepsilon + O(\varepsilon^\beta), \end{aligned} \tag{3.4}$$

[15] The main part is actually Theorem 2.1, which is proved in Appendix C.

and analogously for the spatial gradient $\nabla u_\varepsilon$, we derive

$$\nabla u_\varepsilon = i \exp(i\varphi_\varepsilon)\nabla \Phi_\varepsilon + O(\varepsilon^\beta). \tag{3.5}$$

Combining (3.4) with (3.5), and invoking (2.24) of Theorem 2.1, the conclusion follows. □

We need next to establish some asymptotics for the measures

$$\frac{|\partial_t u_\varepsilon|^2}{|\log \varepsilon|} dxdt \qquad \text{and} \qquad \frac{\partial_t u_\varepsilon \cdot \nabla u_\varepsilon}{|\log \varepsilon|} dxdt.$$

For the first one we will use the inequality

$$\liminf_{\varepsilon \to 0} \int_{\mathbb{R}^2 \times \mathbb{R}^+} \frac{|\partial_t u_\varepsilon|^2}{|\log \varepsilon|} \chi(x) dxdt \geq \int_{\mathbb{R}^2 \times \mathbb{R}^+} |\partial_t \Phi_*|^2 \chi(x) dxdt\,, \tag{3.6}$$

which is a straightforward consequence of (3.3). The analysis of the second one requires a little more care. We have

**Lemma 3.1.** *Extracting possibly a further subsequence,*

$$\sigma_\varepsilon \equiv \frac{\partial_t u_\varepsilon \cdot \nabla u_\varepsilon}{|\log \varepsilon|} dxdt \rightharpoonup \sigma_* \equiv \partial_t \Phi_* \cdot \nabla \Phi_* dxdt + h\nu_*\,, \tag{3.7}$$

*weakly as measures on $\mathbb{R}^2 \times \mathbb{R}^+$, where $\nu_* = \nu_*^t dt = \mu_* \llcorner \Sigma_\mu$, and where $h \in L^2(\nu_*)$.*

*Proof.* Since $\sigma_\varepsilon$ is bounded on $\mathbb{R}^2 \times [0,T]$ for any $T > 0$, we may extract a further subsequence such that $\sigma_\varepsilon \rightharpoonup \sigma_*$ as measures in $\mathbb{R}^2 \times \mathbb{R}^+$. We claim that $\sigma_*$ is absolutely continuous with respect to $\mu_*$. In order to prove this, we follow [2] and work at the level $\varepsilon$: we compute the Radon-Nikodym derivative of $\sigma_\varepsilon$ with respect to $\mu_\varepsilon$, obtaining

$$\left| \frac{d\sigma_\varepsilon}{d\mu_\varepsilon} \right| \leq \frac{|\partial_t u_\varepsilon| \cdot |\nabla u_\varepsilon|}{e_\varepsilon(u_\varepsilon)}\,, \tag{3.8}$$

and therefore

$$\left|\left| \frac{d\sigma_\varepsilon}{d\mu_\varepsilon} \right|\right|^2_{L^2(\mu_\varepsilon)} \leq 2 \int_{\mathbb{R}^2 \times \mathbb{R}^+} \frac{|\partial_t u_\varepsilon|^2}{|\log \varepsilon|} dxdt \leq 2M_0\,. \tag{3.9}$$

Invoking a result of Reshetnyak [24] (see also [11]), the claim is proved.

It follows from Proposition 3.1 that on $\mathbb{R}^2 \times [0, +\infty) \setminus \Sigma_\mu$, $\sigma_* = \partial_t \Phi_* \cdot \nabla \Phi_* dxdt$, and the conclusion follows. □

In the same spirit, we have

**Lemma 3.2.** *Extracting possibly a further subsequence, we have*

$$\frac{A_\varepsilon}{|\log \varepsilon|} dxdt \rightharpoonup A_* = T(\Phi_*) dxdt + B\nu_*\,, \tag{3.10}$$

*weakly as measures on $\mathbb{R}^2 \times \mathbb{R}^+$, where $T$ is defined in (2.4), and $B \in L^\infty(\nu_*)$.*

The proof is identical to the proof of Lemma 3.1. Here the Radon-Nikodym derivative $\frac{dA_\varepsilon}{d\mu_\varepsilon}$ are even equibounded in $L^\infty$. The next result expresses the fact that points have "zero mean curvature".

**Proposition 3.2.** *The vector $h$ and the matrix $B$ given above are identically equal to zero.*

*Proof.* Let $X$ be a smooth, compactly supported vector field (independent of time). Passing to the limit in (2.2) we obtain, for any $0 < T_1 < T_2$,

$$\int_{\mathbb{R}^2\times[T_1,T_2]} A_{*\,ij}\cdot\frac{\partial X_i}{\partial x_j} = \int_{\mathbb{R}^2\times[T_1,T_2]} X\cdot\sigma_*\,. \tag{3.11}$$

Since $\Phi_*$ verifies the heat equation on $\mathbb{R}^2\times\mathbb{R}^+$, we have

$$\int_{\mathbb{R}^2\times[T_1,T_2]} T_{ij}(\Phi_*)\frac{\partial X_i}{\partial x_j}dxdt = \int_{\mathbb{R}^2\times[T_1,T_2]} X\cdot\partial_t\Phi_*\cdot\nabla\Phi_* dxdt\,, \tag{3.12}$$

so that

$$\int_{\mathbb{R}^2\times[T_1,T_2]} B_{ij}\cdot\frac{\partial X_i}{\partial x_j}d\nu_* = \int_{\mathbb{R}^2\times[T_1,T_2]} h\cdot X d\nu_*\,. \tag{3.13}$$

It follows that for a.e. $t > 0$,

$$\int_{\mathbb{R}^2\times\{t\}} B_{ij}\cdot\frac{\partial X_i}{\partial x_j}d\nu_*^t = \int_{\mathbb{R}^2\times\{t\}} h\cdot X d\nu_*^t\,. \tag{3.14}$$

Since the support of $\nu_*^t$ is a finite union of points, the preceding inequality, valid for any smooth vector field $X$, shows that $B = 0$ and $h = 0$. □

**Proof of Theorem 3.1.** We claim that for any function $\chi \geq 0$ compactly supported on $\mathbb{R}^2$, we have, for a.e. $t > 0$,

$$\frac{d}{dt}\int_{\mathbb{R}^2\times\{t\}} \chi d\nu_*^t \leq 0\,. \tag{3.15}$$

Indeed, passing to the limit in (2.1) and using (3.6), Lemma 3.1, Lemma 3.2 and Proposition 3.2, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^2\times\{t\}}\frac{|\nabla\Phi_*|^2}{2}\chi(x)dx + \frac{d}{dt}\int_{\mathbb{R}^2\times\{t\}}\chi d\nu_*^t \leq -\int_{R^2\times\{t\}} |\partial_t\Phi_*|^2\chi - \partial_t\Phi_*\cdot\nabla\Phi_*\cdot\nabla\chi\, dx\,. \tag{3.16}$$

On the other hand, since $\Phi_*$ solves the heat equation, we have

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^2\times\{t\}}\frac{|\nabla\Phi_*|^2}{2}\chi &= \int_{R^2\times\{t\}}\nabla(\partial_t\Phi_*)\cdot\nabla\Phi_*\chi = -\int_{R^2\times\{t\}}(\partial_t\Phi_*\cdot\Delta\Phi_*\chi - \partial_t\Phi_*\cdot\nabla\Phi_*\cdot\nabla\chi)\\ &= -\int_{R^2\times\{t\}}\Big(|\partial_t\Phi_*|^2\chi - \partial_t\Phi_*\cdot\nabla\Phi_*\cdot\nabla\chi\Big)\,,\end{aligned}$$

so that (3.15) follows. We deduce that

$$\nu_*^{t_1} \leq \nu_*^{t_0} \qquad \text{for any } 0 < t_0 \leq t_1.$$

The conclusion of Theorem 3.1 follows then easily in view of (3.15) and the uniform bounds $l \leq CM_0$. □

**Additional comments.** In contrast to the accelerated time scale, there is no compactness property for the functions $u_\varepsilon$ themselves in the original time scale. This is due to possible oscillations in the phase, which are reflected in the measure $|\nabla \Phi_*|^2$. However the degrees of the vortices are well defined in the original time scale, as follows from the fact that $|u_\varepsilon| \to 1$ outside of $\Sigma_\mu$.

We also would like to draw the attention to the fact that in the original time scale, the case $d_i = 0$ is not excluded as the following example shows. Take a "prepared" datum with two vortices of degree $+1$ and $-1$. In view of [17], it is known that these two vortices will not collide before a fixed time of order $C_0 |\log \varepsilon|$, whereas they disappear after this time as follows by Theorem 3. Contracting the initial datum by the factor $\sqrt{|\log \varepsilon|}$ and using the scaling of the equation it follows that the solution obtained disappears at time $C_0$ in the original time scale. Moreover,

$$\begin{cases} \sigma_i(t) = 2\pi & \text{if } t < C_0 \\ \sigma_i(t) = 0 & \text{if } t > C_0 \,. \end{cases} \tag{3.17}$$

This type of argument may be extended to derive arbitrary jumps of integer multiples of $2\pi$ at any prescribed times.

**Further properties at the $\varepsilon$-level.**

In the accelerated time scales considered in the next sections, we have to turn back to the $\varepsilon$ level (i.e. we cannot rely on the study of limiting measures introduced so far). In that analysis the concentration set $\Sigma_\mu^t$ is replaced by the sets $\Omega_\delta^\varepsilon(t)$, defined, for $\delta > 0$, by

$$\Omega_\delta^\varepsilon(t) = \left\{ x \in \mathbb{R}^2 \,, \int_{B(x,\delta)} e_\varepsilon(u(\cdot,t))dx \geq \frac{\eta_0}{2} |\log \varepsilon| \right\}, \tag{3.18}$$

where $\eta_0$ is the constant appearing in Proposition 2.1. A straightforward covering argument shows that there exist $l$ points $x_i^\varepsilon(t)$ such that

$$\Omega_\delta^\varepsilon(t) \subset \cup_{i=1}^l B(x_i^\varepsilon(t), 2\delta),$$

where the number $l$ of points is uniformly bounded by $l \leq C\frac{M_0}{\eta_0}$. The next lemma describes the evolution of the concentration sets $\Omega_\delta^\varepsilon(t)$ in the usual time scale.

**Lemma 3.3.** *Let $t_0 \geq \frac{1}{2}$ and $0 < \delta \leq C_0 \sqrt{\frac{\eta_0}{M_0}}$, where $C_0$ is some universal constant. There exists $\varepsilon_0 = \varepsilon_0(\delta)$ depending only on $\delta$ such that, if $\varepsilon \leq \varepsilon_0$ then*

$$dist\,(\Omega_\delta^\varepsilon(t), \Omega_\delta^\varepsilon(t_0)) \leq 2\delta \tag{3.19}$$

*for every $t_0 \leq t \leq t_0 + 2$.*

*Proof.* We argue by contradiction. Assume that (3.19) does not hold. Then, translating possibly the origin, we may assume that, for a sequence $\varepsilon_n \to 0$, there exists a time $t_n$, $t_0 \leq t_n \leq t_0 + 2$, such that $0 \in \Omega_\delta^\varepsilon(t)$ but

$$B(0, 2\delta) \cap \Omega_\delta^{\varepsilon_n}(t_0) = \emptyset \,, \qquad \text{for any } n \in \mathbb{N}. \tag{3.20}$$

We apply next Theorem 3.1 to the sequence $(u_{\varepsilon_n})_{n \in N}$. Extracting possibly a subsequence (still denoted $\varepsilon_n$), we may assume that $\mu_{\varepsilon_n}$ converges and that the conclusions of the invoked theorem hold. It follows from (3.20) and the lower density bound given by (3.1)[16] that

$$\Sigma_\mu^{t_0} \cap B(0, 2\delta) = \emptyset \,. \tag{3.21}$$

[16]The bound actually holds only for a.e. time, the reader will adapt the argument slightly changing $t_0$ if necessary.

By Theorem 3.1, $\Sigma_\mu^t \subset \Sigma_\mu^{t_0}$ for each $t \geq t_0$, so that $\Sigma_\mu^t \cap B(0,2\delta) = \emptyset$. Therefore

$$\mu_*^t = \frac{|\nabla \Phi_*|^2}{2} dx \qquad \text{on } B(0,2\delta) \qquad \text{for } t \geq t_0$$

and in particular

$$\begin{aligned} \int_{B(0,2\delta)} d\mu_*^t = \int_{B(0,2\delta)} \frac{|\nabla \Phi_*|^2}{2} &\leq C M_0 \delta^2 \\ \leq C M_0 \left( C_0^2 \frac{\eta_0}{M_0} \right) &\leq \frac{\eta_0}{4} \qquad \text{for } t \geq t_0 \end{aligned} \tag{3.22}$$

for an appropriate choice of the constant $C_0$. Passing possibly to a further subsequence, we may further assume that $t_n \to t_\infty$ as $n \to +\infty$, where $t_0 \leq t_\infty \leq t_0 + 2$. Let $0 \leq \chi \leq 1$ be a smooth function with compact support in $B(0,2\delta)$ such that $\chi \equiv 1$ on $B(0,\delta)$. Since $0 \in \Omega_\delta^{\varepsilon_n}(t_n)$, we have

$$\int_{\mathbb{R}^N} \chi d\mu_{\varepsilon_n}^{t_n} \geq \frac{\eta_0}{2}.$$

We next distinguish two cases.

**Case 1:** $t_\infty = t_0$. It follows from Lemma 2.4 that

$$\int_{\mathbb{R}^2} \chi d\mu_{\varepsilon_n}^{t_0} \geq \frac{\eta_0}{2} - C(\chi)(t_n - t_0) M_0 \,,$$

contradicting (3.22) for $n$ sufficiently large and $t = t_0$.

**Case 2:** $t_\infty \neq t_0$. Invoking Lemma 2.4 once more, we write

$$\int_{\mathbb{R}^2} \chi d\mu_{\varepsilon_n}^{t_\infty - \alpha} \geq \frac{\eta_0}{2} - C(\chi)(t_n - t_\infty + \alpha) M_0 \,.$$

This contradicts (3.22) for $\alpha$ sufficiently small (independently of $\varepsilon$) and $n$ sufficiently large. □

Our next results emphasize the connection between the concentration sets $\Omega_\delta^\varepsilon(t)$ and the vorticity set

$$\mathcal{V}_\varepsilon(t) = \{x \in \mathbb{R}^2 \,,\ |u_\varepsilon(x)| \leq \frac{1}{2}\} \,. \tag{3.23}$$

As an immediate consequence of the Clearing-Out, we have

**Lemma 3.4.** *Let $t_0 \geq \frac{1}{2}$ and $\delta > 0$ be given. For every $t \geq t_0$ we have*

$$\mathcal{V}_\varepsilon(t) \subset \Omega_\delta^\varepsilon(t_*), \qquad \textit{for any } t - C\delta^2 \leq t_* \leq t - 2\varepsilon \,. \tag{3.24}$$

Combining Lemma 3.3 and Lemma 3.4 we deduce

**Lemma 3.5.** *We have*

$$\mathcal{V}_\varepsilon(t) \subset \{x \in \mathbb{R}^2 \,,\ \mathit{dist}\,(x, \Omega_\delta^\varepsilon(t_0)) \leq 2\delta\} \subset \cup_{i=1}^{l} B(x_i^\varepsilon, 4\delta) \tag{3.25}$$

*for every $t_0 + 2\varepsilon \leq t \leq t_0 + 2$, where $x_i^\varepsilon = x_i^\varepsilon(t_0)$.*

Finally, the last result in this section is concerned with Jacobians. As a consequence of Theorem 3.1 and the previous analysis, we have

**Proposition 3.3.** *Let $N = 2$ and $u_\varepsilon$ be a solution of $(PGL)_\varepsilon$ satisfying the energy bound $(H_0)$, and let $R > 1$, $0 < \alpha < 1$. There exists a constant $C(\varepsilon, M_0, R)$ depending only on $\varepsilon$, $M_0$ and $R$, $l$ points $x_i^\varepsilon$ in $\mathbb{R}^2$ and $l$ integers $d_i \in \mathbb{Z}$ such that*

$$x_i^\varepsilon \in \mathcal{V}_\varepsilon(1) \qquad \forall i \in \{1, \cdots, l\}, \tag{3.26}$$

$$\|J_{x,t}u_\varepsilon - \sum_{i=1}^{l} d_i \delta_{x_i^\varepsilon} dx_1 \wedge dx_2\|_{[\mathcal{C}_c^{0,\alpha}(B(0,R)\times[1,R+1])]^*} \leq C_\alpha(\varepsilon, M_0, R). \tag{3.27}$$

*Moreover,*

$$\sum_{i=1}^{l} |d_i| \leq CM_0 \tag{3.28}$$

*and, for fixed $M_0$ and $R$,*

$$C_\alpha(\varepsilon, M_0, R) \to 0 \qquad \text{as } \varepsilon \to 0. \tag{3.29}$$

**Comments.** 1) Proposition 3.3 will be used in the proof of Theorem 1. In particular it will be used not only for $u_\varepsilon$ but also for translates (in space and time).

2) We would like to draw the attention to the fact that (3.28) implies that the space-time components of the 2-form $J_{x,t}u_\varepsilon$, namely $\partial_t u_\varepsilon \times \partial_i u_\varepsilon$, $i = 1, 2$, are vanishing with $\varepsilon$ in the norm considered.

*Proof.* The argument is by contradiction. Assume the result were false: then, for some $\delta > 0$ there would exist a sequence $\varepsilon_n \to 0$ and a sequence $u_{\varepsilon_n}$ of solutions to $(\mathrm{PGL})_\varepsilon$ satisfying $(H_0)$, and such that

$$\|J_{x,t}u_{\varepsilon_n} - \sum_{i=1}^{l} d_i \delta_{x_i^{\varepsilon_n}} dx_1 \wedge dx_2\|_{[\mathcal{C}_c^{0,\alpha}(B(0,R)\times[1,R+1])]^*} \geq \delta \tag{3.30}$$

for any points $x_i^\varepsilon \in \mathcal{V}_\varepsilon(1)$ and integers $d_i$. We invoke next the compactness results for Jacobians of [19, 1] to assert that, passing possibly to a further subsequence

$$J_{x,t}u_{\varepsilon_n} \rightharpoonup T \qquad \text{in } [\mathcal{C}_c^{0,\alpha}(B(0,R) \times [1, R+1])]^*,$$

where $\frac{1}{\pi}T$ is an integer multiplicity one-dimensional current. On the other hand, by Theorem 3.1 and the fact that the geometrical support of $T$ is contained in $\Sigma_\mu$, we infer that

$$T = \pi \sum_{i=1}^{l} d_i \delta_{b_i} dx_1 \wedge dx_2,$$

for some points $b_i$ and some multiplicities $d_i \in \mathbb{Z}$. This contradicts (3.30), since $|u_\varepsilon|$ converges uniformly to 1 outside $\Sigma_\mu$. $\square$

# 4 Long-time analysis

In this section we provide a number of estimates for $u_\varepsilon$ whose main feature is that they remain valid also for long time (in the original time scale). We assume throughout that $|\log \varepsilon| \geq 1$.

## 4.1 Identifying the linear mode

We will prove the following long-time variant of Theorem 3 in [6], valid in any dimension $N \geq 2$.

**Proposition 4.1.** *Let $N \geq 2$ and $u_\varepsilon$ be a solution to $(PGL)_\varepsilon$ satisfying $(H_0)$. There exists a real-valued function $\phi_\varepsilon$ and a complex-valued function $w_\varepsilon$, defined on $\mathbb{R}^N \times \mathbb{R}^+$ such that*
*(i) $u_\varepsilon = w_\varepsilon \exp(i\phi_\varepsilon)$*
*(ii) $\phi_\varepsilon$ verifies the heat equation on $\mathbb{R}^N \times \mathbb{R}^+_*$*
*(iii) for every $q > \frac{2N}{2N-3}$, $k \in \mathbb{N}^*$ and $t \geq 1$ we have*

$$\|\nabla^k \phi_\varepsilon\|_{L^\infty(\mathbb{R}^N \times \{t\})} \leq \frac{C(M_0)}{t^{N/4+(k-1)/2}} \sqrt{|\log \varepsilon|} + \frac{C(M_0, q)}{t^{N/2q+(k-1)/2}} \tag{4.1}$$

*(iv) $\|\nabla w_\varepsilon\|_{L^p(K \times [t,t+1])} \leq C(p, K, M_0)$, $\forall\, t \geq 1$, for every $1 \leq p < \frac{N+1}{N}$ and every compact subset $K \subset \mathbb{R}^N$.*

We would like to stress the main differences (and actually improvements) with Theorem 3 in [6]. The first point is that $w_\varepsilon$ and $\phi_\varepsilon$ are defined globally on $\mathbb{R}^N \times \mathbb{R}^+$. The second is that estimate (iv) is uniform in time: in view of propagation phenomena, this will require estimates on the whole of $\mathbb{R}^N$.

As in [6], the proof is based on appropriate Hodge-de Rham decompositions of $u_\varepsilon \times \nabla u_\varepsilon$. To this aim, we will denote $\delta$ and $\delta^*$ respectively the exterior differentiation operator for differential forms on $\mathbb{R}^N \times \mathbb{R}$ and its formal adjoint, while we will use the standard notations $d$ and $d^*$ when restricting to time slices $\mathbb{R}^N \times \{t\}$.

We extend first $u_\varepsilon$ to the whole of $\mathbb{R}^{N+1} = \mathbb{R}^N \times \mathbb{R}$ by standard reflection and consider its Jacobian $J_{x,t} u_\varepsilon$ defined by

$$J_{x,t} u_\varepsilon = \frac{1}{2} \delta(u_\varepsilon \times \delta u_\varepsilon) \qquad \text{on } \mathbb{R}^{N+1}\,. \tag{4.2}$$

We consider next the elliptic problem

$$-\Delta_{x,t} \psi = J_{x,t} u_\varepsilon \qquad \text{on } \mathbb{R}^{N+1}. \tag{4.3}$$

We first have

**Lemma 4.1.** *There exists a solution $\psi$ of (4.3) such that*

$$\nabla_{x,t} \psi = \nabla_{x,t} G * J_{x,t} u_\varepsilon \tag{4.4}$$

*and*

$$\sup_{t \in \mathbb{R}} \|\nabla_{x,t} \psi\|_{(L^p + L^q)(\mathbb{R}^N \times [t,t+1])} \leq C(p, q) M_0\,, \tag{4.5}$$

*for any $p > \frac{N}{N-1}$ and $1 \leq q < \frac{N+1}{N}$. For the space-time components*[17] *$\psi^{0j}$ of $\psi$, $j = 1, ..., N$, we have moreover*

$$\sup_{t \in \mathbb{R}} \|\nabla_{x,t} \psi^{0j}\|_{L^p(\mathbb{R}^N \times [t,t+1])} \leq C(p) M_0\,, \tag{4.6}$$

*for any $\frac{2N}{2N-1} < p < \frac{N+1}{N}$.*

[17] Here we write $\psi = \sum_{0 \leq i < j \leq N} \psi^{ij} dx_i \wedge dx_j$, with the convention that $x_0 = t$.

*Proof.* We invoke first Appendix B, which clearly applies since $u_\varepsilon$ satisfies conditions (B.1), (B.2) and (B.3). In particular, in view of Proposition B.2 we may write

$$2J_{x,t}u_\varepsilon = \omega_\varepsilon + \mathrm{div}_{x,t}\lambda_\varepsilon \,,$$

where $\omega_\varepsilon,\ \lambda_\varepsilon$ satisfy

$$\|\omega_\varepsilon\|_{L^1(\mathbb{R}^N\times[t,t+1])} \le CM_0 \qquad \forall\, t\in\mathbb{R}, \tag{4.7}$$

$$\|\lambda_\varepsilon\|_{L^p(\mathbb{R}^N\times[t,t+1])} \le C_pM_0\varepsilon^{\alpha_p} \qquad \forall\, t\in\mathbb{R}, \tag{4.8}$$

for every $1\le p<2$ and for some $\alpha_p>0$, and also

$$\left(\int_{\mathbb{R}} \|\omega_\varepsilon\|^q_{L^1(\mathbb{R}^N\times[t,t+1])}\right)^{\frac{1}{q}} \le C_qM_0 \tag{4.9}$$

for every $q>2$. We write $\psi=\psi_1+\psi_2$, where $\psi_1,\ \psi_2$ are the solutions of

$$-\Delta_{x,t}\psi_1=\omega_\varepsilon, \qquad -\Delta_{x,t}\psi_2=\mathrm{div}_{x,t}\lambda_\varepsilon \qquad \text{on } \mathbb{R}^{N+1} \tag{4.10}$$

respectively given by Lemma A.1 and Lemma A.2. In view of Lemma A.1, we may decompose $|\nabla\psi_1^{ij}|=g_1^{ij}+g_2^{ij}$, where

$$\sup_{t\in\mathbb{R}}\|g_1\|_{L^{p_1}(\mathbb{R}^N\times\{t\})} \le K(p_1)M_0 \qquad \text{for any } p_1>\frac{N}{N-1}, \tag{4.11}$$

with an improvement for the space-time components $g_1^{0j}$

$$\sup_{t\in\mathbb{R}}\|g_1^{0j}\|_{L^{p_1}(\mathbb{R}^N\times\{t\}} \le K(p_1)M_0 \qquad \text{for any } p_1>\frac{2N}{2N-1}, \tag{4.12}$$

and

$$\sup_{t\in\mathbb{R}}\|g_2^{0j}\|_{L^{p_2}(\mathbb{R}^N\times[t,t+1])} \le K(p_2)M_0 \qquad \text{for any } 1\le p_2<\frac{N+1}{N}. \tag{4.13}$$

Similarly, in view of Lemma A.2 we have

$$\sup_{t\in\mathbb{R}}\|\nabla\psi_2^{ij}\|_{L^{p_3}(\mathbb{R}^N\times[t,t+1])} \le K(p_3)M_0 \qquad \text{for every } 1<p_3<2. \tag{4.14}$$

The estimates of Lemma 4.1 follow noticing that $\frac{2N}{2N-1}<\frac{N+1}{N}$. □

**Lemma 4.2.** *We have*

$$\delta\psi=0 \qquad \textit{on } \mathbb{R}^{N+1}. \tag{4.15}$$

*Proof.* In view of the construction of $\psi$,

$$\delta\psi = 2G*\delta J_{x,t}u_\varepsilon.$$

Since $2\delta J_{x,t}u_\varepsilon=\delta(\delta(u_\varepsilon\times\delta u_\varepsilon))=0$, the conclusion (4.15) follows. □

In view of Lemma 4.2, since $-\Delta_{x,t} = \delta\delta^* + \delta^*\delta$, we deduce $\delta\delta^*\psi = 2J_{x,t}u_\varepsilon$. By subtraction, we obtain

$$\delta(u_\varepsilon \times \delta u_\varepsilon - \delta^*\psi) = 0 \qquad \text{on } \mathbb{R}^{N+1}.$$

We invoke the Poincaré lemma to assert that there exists some function $\Phi$ defined on $\mathbb{R}^{N+1}$ such that

$$u_\varepsilon \times \delta u_\varepsilon = \delta\Phi + \delta^*\psi \qquad \text{on } \mathbb{R}^{N+1}. \tag{4.16}$$

**Equation for the phase $\Phi$.** Taking the exterior product of $(\mathrm{PGL})_\varepsilon$ with $u_\varepsilon$, we are led to

$$u_\varepsilon \times \frac{\partial u_\varepsilon}{\partial t} - \operatorname{div}(u_\varepsilon \times \nabla u_\varepsilon) = 0 \qquad \text{on } \mathbb{R}^N \times \mathbb{R}^+. \tag{4.17}$$

On the other hand, in view of the decomposition (4.16),

$$\begin{cases} u_\varepsilon \times du_\varepsilon & = d\Phi + \delta^*\psi - P_t(\delta^*\psi)dt, \\ u_\varepsilon \times \dfrac{\partial u_\varepsilon}{\partial t} & = \dfrac{\partial \Phi}{\partial t} + P_t(\delta^*\psi). \end{cases} \tag{4.18}$$

Here, for a 1-form $\alpha$ on $\mathbb{R}^{N+1}$, we denote by $P_t(\alpha)$ its time component $\alpha_0$. Inserting into (4.17) we derive the equation

$$\frac{\partial \Phi}{\partial t} - \Delta\Phi = d^*(\delta^*\psi - P_t(\delta^*\psi)dt) - P_t(\delta^*\psi) \quad \text{ on } \mathbb{R}^N \times \mathbb{R}^+, \tag{4.19}$$

which is a heat equation with source terms bounded in appropriate norms, thanks to Lemma 4.1. The source terms can be decomposed into two contributions:

i) $A = d^*h \equiv d^*(\delta^*\psi - P_t(\delta^*\psi)dt)$, which is a derivative with respect to spatial coordinates. In view of estimate (4.5), we have

$$\sup_{t\in\mathbb{R}^+} \|h\|_{(L^p+L^q)(\mathbb{R}^N\times[t,t+1])} \leq \sup_{t\in\mathbb{R}^+} \|\nabla_{x,t}\psi\|_{(L^p+L^q)(\mathbb{R}^N\times[t,t+1])} \leq C(p,q)M_0\,, \tag{4.20}$$

for any $p > \frac{N}{N-1}$ and $1 \leq q < \frac{N+1}{N}$.

ii) $B = P_t(\delta^*\psi)$. In coordinates $B$ writes as

$$B = \sum_{i=1}^N (-1)^{i-1} \frac{\partial \psi^{0i}}{\partial x_i}\,. \tag{4.21}$$

It involves only spatial derivatives of space-time components $\psi^{0i}$ of $\psi$. This observation turns out to be important specially in dimension $N = 2$. In view of estimate (4.6) we have

$$\sup_{t\in\mathbb{R}} \|B\|_{L^p(\mathbb{R}^N\times[t,t+1])} \leq C(p)M_0\,, \tag{4.22}$$

for any $\frac{2N}{2N-1} < p < \frac{N+1}{N}$. Taking into account the previous discussion, we are now in position to complete the proof of Proposition 4.1.

**Proof of Proposition 4.1 completed.** We consider the initial-value parabolic problem

$$\begin{cases} \partial_t\Phi_0 - \Delta\Phi_0 = A + B & \text{in } \mathbb{R}^N \times (0, +\infty), \\ \Phi_0(x, 0) = 0 & \text{for any } x \in \mathbb{R}^N. \end{cases} \tag{4.23}$$

By Lemma A.3 and A.4 of the Appendix, as well as estimates (4.20) and (4.22), we deduce that $|\nabla\Phi_0| = g_1 + g_2$, with

$$\sup_{t\in\mathbb{R}^+} \|g_1\|_{(L^r+L^q)(\mathbb{R}^N\times[t,t+1])} \leq C(r,q)M_0, \tag{4.24}$$

for any numbers $q$ and $r$ such that $q \geq r$, $1 \leq r < \frac{N+1}{N}$ and $q > \frac{2N}{2N-3}$, and

$$\sup_{t\in\mathbb{R}^+} \|g_2\|_{L^p([t,t+1];L^{p^*}(\mathbb{R}^N))} \leq C(p)M_0 \qquad \text{for any } \frac{2N}{2N-1} < p < \frac{N+1}{N}. \tag{4.25}$$

In particular, for every compact subset $K \subset \mathbb{R}^N$, we have

$$\|\nabla\Phi_0\|_{L^p(K\times[t,t+1])} \leq C(p,K)M_0 \text{ for every } 1 \leq p < \frac{N+1}{N}. \tag{4.26}$$

We set

$$\phi_\varepsilon = \Phi - \Phi_0,$$

so that $\phi_\varepsilon$ verifies the heat equation on $\mathbb{R}^N \times \mathbb{R}^+$, and (ii) in Proposition 4.1 follows.

**Proof of (iii).** On every time slice $\mathbb{R}^N \times \{t\}$, we have, in view of (4.16) and the definition of $\phi_\varepsilon$,

$$d\phi_\varepsilon = u_\varepsilon \times du_\varepsilon - (\delta^*\psi)_\top - d\Phi_0\,. \tag{4.27}$$

Here and throughout, we denote by $\alpha_\top$ the restriction of a given form $\alpha$ defined on $\mathbb{R}^{N+1}$ to a time slice $\mathbb{R}^N \times \{t\}$.

In view of (4.20), we have

$$\|(\delta^*\psi)_\top\|_{(L^p+L^q)(\mathbb{R}^N\times[0,1])} \leq C(p,q)M_0 \tag{4.28}$$

for any $p > \frac{N}{N-1}$ and $1 \leq q < \frac{N+1}{N}$, and in view of (4.24) and (4.25)

$$\|\nabla\Phi_0\|_{L^1(\mathbb{R}^N\times[0,1])+L^p([0,1],L^{p^*}(\mathbb{R}^N))} \leq C(p)M_0 \tag{4.29}$$

for any $\frac{2N}{2N-1} < p < \frac{N+1}{N}$. In particular, for any $q > \frac{2N}{2N-3}$, we may choose some $t_0 \in [1/4, 1/2]$ such that

$$\|(\delta^*\psi)_\top + d\Phi_0\|_{(L^1+L^q)(\mathbb{R}^N\times\{t_0\})} \leq C(q)M_0. \tag{4.30}$$

On the other hand, on every slice $\mathbb{R}^N \times \{t\}$ for $t \geq 1/4$ we have $|u_\varepsilon| \leq 3$ (see Proposition 2.2), therefore using the energy inequality we obtain the estimate

$$\|u_\varepsilon \times du_\varepsilon\|_{L^2(\mathbb{R}^N\times\{t\})} \leq C\sqrt{M_0|\log\varepsilon|}. \tag{4.31}$$

In view of (4.27), (4.30) and (4.31) we may write, for any $q > \frac{2N}{2N-3}$,

$$\nabla\phi_\varepsilon(\cdot,t_0) = f_1 + f_2 + f_3 \qquad \text{on } \mathbb{R}^N, \tag{4.32}$$

where $f_1$ and $f_2$ satisfy

$$\|f_1\|_{L^1(\mathbb{R}^N)} \leq C(M_0)\,, \qquad \|f_2\|_{L^q(\mathbb{R}^N)} \leq C(M_0,q), \tag{4.33}$$

and $f_3$ satisfies

$$\|f_3\|_{L^2(\mathbb{R}^N)} \leq C\sqrt{M_0|\log\varepsilon|}. \tag{4.34}$$

Since $\phi_\varepsilon$ solves the heat equation, so does $\nabla^k\phi_\varepsilon$, and in particular for $k=1$ inequality (iii) follows using Lemma A.5 and inequality A.22 of Appendix A. For $k\geq 2$, we invoke likewise (A.23).

**Estimates for $w_\varepsilon$.** In view of (i), we set

$$w_\varepsilon = u_\varepsilon \exp(-i\phi_\varepsilon),$$

so that $|w_\varepsilon| = |u_\varepsilon|$. A simple computation shows that

$$\begin{aligned} w_\varepsilon \times \delta w_\varepsilon &= u_\varepsilon \times \delta u_\varepsilon - \delta\phi_\varepsilon + (1-|u_\varepsilon|^2)\delta\phi_\varepsilon \\ &= \delta^*\psi + \delta(\phi_\varepsilon + \Phi_0) - \delta\phi_\varepsilon + \zeta \\ &= \delta^*\psi + \delta\Phi_0 + \zeta, \end{aligned} \tag{4.35}$$

where we have defined $\zeta = (1-|u_\varepsilon|^2)\delta\phi_\varepsilon$. Clearly, $\zeta$ is a perturbation term. Indeed for $1 \leq p < \frac{N+1}{N}$ and $t \geq 1$,

$$\|\zeta\|_{L^p(K\times[t,t+1])} \leq C(K)\varepsilon\|\tfrac{1-|u_\varepsilon|^2}{\varepsilon}\|_{L^2(K\times[t,t+1])}\|\delta\phi_\varepsilon\|_{L^\infty(K\times[t,t+1])} \leq C(K,M_0)\varepsilon|\log\varepsilon|.$$

It follows from decomposition (4.35) and the various estimates for $\psi$, $\Phi_0$ and $\zeta$, that for every $1 \leq p < \frac{N+1}{N}$ and $t \geq 1$

$$\|w_\varepsilon \times \delta w_\varepsilon\|_{L^p(K\times[t,t+1])} \leq C(p,K,M_0)\,. \tag{4.36}$$

The proof of assertion (iv) is then completed as in the proof of [6], Theorem 3, deriving the corresponding bounds for $\nabla_{x,t}|u_\varepsilon|$ and $V_\varepsilon(u_\varepsilon)$.

**Remark 4.1.** a) It is tempting to believe that

$$\int_{\mathbb{R}^N\times\{t\}} |\nabla\phi_\varepsilon|^2 \leq C(M_0+1)|\log\varepsilon|, \qquad \text{for } t \geq 1\,, \tag{4.37}$$

but we have no proof of that fact.
b) Since $\phi_\varepsilon$ satisfies the heat equation, it follows from (4.1) that, for $q > \frac{2N}{2N-3}$ and $t \geq 1$,

$$\|\partial_t^m\nabla^k\phi_\varepsilon\|_{L^\infty(\mathbb{R}^N\times\{t\})} \leq \frac{C(M_0)}{t^{N/4+(k+2m-1)/2}}\sqrt{|\log\varepsilon|} + \frac{C(M_0,q)}{t^{N/2q+(k+2m-1)/2}}.$$

## 4.2 Improved properties for $w_\varepsilon$ and $u_\varepsilon$

In order to derive additional properties for $w_\varepsilon$, the first step is to derive an appropriate equation. We have

**Lemma 4.3.** *The function $w_\varepsilon$ verifies the equation*

$$w_\varepsilon \times \frac{\partial w_\varepsilon}{\partial t} - \operatorname{div}(w_\varepsilon \times \nabla w_\varepsilon) = r_\varepsilon \qquad \text{on } \mathbb{R}^N\times\mathbb{R}^+, \tag{4.38}$$

*where the function $r_\varepsilon$ is defined on $\mathbb{R}^N\times\mathbb{R}^+$ by*

$$r_\varepsilon = \nabla(1-|u_\varepsilon|^2)\cdot\nabla\phi_\varepsilon\,. \tag{4.39}$$

*Proof.* Since $w_\varepsilon = u_\varepsilon \exp(-i\phi_\varepsilon)$, we have $w_\varepsilon \times \nabla_{x,t} w_\varepsilon = u_\varepsilon \times \nabla_{x,t} u_\varepsilon + |u_\varepsilon|^2 \nabla_{x,t}\phi_\varepsilon$. Inserting this in identity (4.17) yields

$$w_\varepsilon \times \frac{\partial w_\varepsilon}{\partial t} - \operatorname{div}(w_\varepsilon \times \nabla w_\varepsilon) = |u_\varepsilon|^2 \frac{\partial \phi_\varepsilon}{\partial t} - \operatorname{div}(|u_\varepsilon|^2 \nabla \phi_\varepsilon)\,.$$

Since $\phi_\varepsilon$ verifies the heat equation, the conclusion follows. □

Let $\sqrt{2\varepsilon} \le R \le 1$ and consider for $(x_0, t_0) \in \mathbb{R}^N \times [1, +\infty)$ the parabolic cylinder

$$\Lambda = B(x_0, R) \times [t_0, t_0 + R^2]\,. \tag{4.40}$$

We have

**Proposition 4.2.** *Let $x_0$, $t_0$ and $R$ as above, and assume that $|u_\varepsilon| \ge \frac{1}{2}$ on $\Lambda$. Then we have*

$$|\nabla w_\varepsilon| \le \frac{C(M_0)}{R}\,, \qquad \left|\frac{\partial w_\varepsilon}{\partial t}\right| \le \frac{C(M_0)}{R^2} \qquad \text{on } \Lambda_{1/2}\,, \tag{4.41}$$

*where $\Lambda_{1/2} = B(x_0, \frac{R}{2}) \times [t_0 + \frac{3}{4}R^2, t_0 + R^2]$, and $C(M_0)$ depends only upon $M_0$.*

*Proof.* We assume $R = 1$, the general statement can be handled similarly by scaling. In view of Proposition 4.1, $u_\varepsilon = w_\varepsilon \exp(i\phi_\varepsilon)$. On the other hand, since $|u_\varepsilon| = |w_\varepsilon| \ge 1/2$ on $\Lambda_{\frac{1}{2}}$ there exists some real-valued function $\psi_\varepsilon$ such that[18] $w_\varepsilon = \rho_\varepsilon \exp(i\psi_\varepsilon)$. Equation (4.38) is transformed into the uniformly parabolic equation for $\psi_\varepsilon$

$$\rho_\varepsilon^2 \frac{\partial \psi_\varepsilon}{\partial t} - \operatorname{div}(\rho_\varepsilon^2 \nabla \psi_\varepsilon) = \nabla(1 - \rho_\varepsilon^2) \cdot \nabla \phi_\varepsilon = r_\varepsilon.$$

By Theorem 2.1, $\rho_\varepsilon^2 \in \mathcal{C}^{1,\alpha}(\Lambda_{\frac{1}{2}})$ and $r_\varepsilon \in \mathcal{C}^{0,\alpha}(\Lambda_{\frac{1}{2}})$. Invoking Schauder theory for parabolic equations with Hölder coefficients, we deduce

$$\|\frac{\partial \psi_\varepsilon}{\partial t}\|_{\mathcal{C}^{0,\alpha}(\Lambda_{\frac{1}{2}})} + \|\nabla \psi_\varepsilon\|_{\mathcal{C}^{0,\alpha}(\Lambda_{\frac{1}{2}})} \le C\|r_\varepsilon\|_{\mathcal{C}^{0,\alpha}(\Lambda_{\frac{1}{2}})} + C\|\nabla \psi_\varepsilon\|_{L^1(\Lambda_{\frac{3}{4}})} \le C.$$

□

Combining Proposition 4.2 and assertion iii) of Proposition 4.1 with $q = 5$ we immediately derive

**Corollary 4.1.** *Let $x_0$, $t_0$ and $R$ be as above and assume that $|u_\varepsilon| \ge \frac{1}{2}$ on $\Lambda$. We have*

$$e_\varepsilon(u_\varepsilon(x,t)) \le C(M_0)\left(\frac{1}{R^2} + \frac{|\log \varepsilon|}{t} + \frac{1}{t^{1/5}}\right), \tag{4.42}$$

*for every $(x,t) \in \Lambda_{1/2}$.*

[18] Hence, $u_\varepsilon = \rho_\varepsilon \exp(i\varphi_\varepsilon)$, where $\varphi_\varepsilon = \phi_\varepsilon + \psi_\varepsilon$.

### 4.3 On the evolution of $\nu_\varepsilon^t$.

Recall that $\nu_\varepsilon^t$ is defined in Section 2.2 by (2.9) and that its evolution in time is given by equation (2.16) in Lemma 2.6. We first give an estimate for the remainder term $\mathcal{R}(t,\chi,\nabla\phi_\varepsilon,w_\varepsilon)$.

**Lemma 4.4.** *Let $K$ be any compact subset of $\mathbb{R}^N$. Let $\chi\in\mathcal{D}(\mathbb{R}^N)$ be such that $\operatorname{supp}\chi\subset K$, and let $1\le t_1\le t_1+1\le t_2$. We have,*

$$\left|\int_{t_1}^{t_2}\mathcal{R}(t,\chi,\nabla\phi_\varepsilon,w_\varepsilon)\,dt\right|\le C(M_0,K)\|\chi\|_{\mathcal{C}^2}\Big[\sqrt{|\log\varepsilon|}\,\log(\frac{t_2}{t_1})+(t_2^{3/5}-t_1^{3/5})$$
$$+\sum_{i=1}^{2}\big(1+\|\nabla w_\varepsilon\|_{L^1(K\times\{t_i\})}\big)\big(\sqrt{|\log\varepsilon|}\,t_i^{-1/2}+t_i^{-1/5}\big)\Big].\quad(4.43)$$

*Proof.* In view of Lemma 2.6, it suffices to estimate the terms $B_1$, $B_2$, $B_3$, where

$$\begin{aligned}
B_1&=\sum_{i=1}^{2}(-1)^i\int_{\mathbb{R}^N\times\{t_i\}}\chi\left(\nabla\phi_\varepsilon\cdot w_\varepsilon\times\nabla w_\varepsilon+(|u_\varepsilon|^2-1)\frac{|\nabla\phi_\varepsilon|^2}{2}\right)\\
B_2&=\int_{t_1}^{t_2}\int_{\mathbb{R}^N}2\chi\frac{\partial\phi_\varepsilon}{\partial t}\cdot w_\varepsilon\times\frac{\partial w_\varepsilon}{\partial t}\\
B_3&=\int_{t_1}^{t_2}\int_{\mathbb{R}^N}(1-|u_\varepsilon|^2)\left(\chi\left|\frac{\partial\phi_\varepsilon}{\partial t}\right|^2+D^2\chi\nabla\phi_\varepsilon\cdot\nabla\phi_\varepsilon-\Delta\chi\frac{|\nabla\phi_\varepsilon|^2}{2}\right).
\end{aligned}$$

We handle each of those terms separately.

**Step 1: estimate for $B_1$.** We have, for $i=1,2$

$$\begin{aligned}
\int_{\mathbb{R}^N\times\{t_i\}}\chi\left||u_\varepsilon|^2-1\right|\frac{|\nabla\phi_\varepsilon|^2}{2}&\le\|\chi\|_{L^\infty(K)}\|\nabla\phi_\varepsilon\|^2_{L^\infty(K\times\{t_i\})}|K|^{1/2}\left(\int_{\mathbb{R}^N\times\{t_i\}}(|u_\varepsilon|^2-1)^2\right)^{1/2}\\
&\le C\varepsilon|\log\varepsilon|^{1/2}\|\nabla\phi_\varepsilon\|^2_{L^\infty(K\times\{t_i\})}|K|^{1/2}
\end{aligned}$$

and

$$\left|\int_{\mathbb{R}^N\times\{t_i\}}\chi\nabla\phi_\varepsilon\cdot w_\varepsilon\times\nabla w_\varepsilon\right|\le\|\chi\|_{L^\infty(K)}\|\nabla w_\varepsilon\|_{L^1(K\times\{t_i\})}\|\nabla\phi_\varepsilon\|_{L^\infty(K\times\{t_i\})}.$$

Hence, by iii) of Proposition 4.1 with $q=5$

$$|B_1|\le C\|\chi\|_{L^\infty(K)}\sum_{i=1}^{2}\left(1+\|\nabla w_\varepsilon\|_{L^1(K\times\{t_i\})}\right)\left(\sqrt{|\log\varepsilon|}\,t_i^{-1/2}+t_i^{-1/5}\right).$$

**Step 2: estimate for $B_2$.** We invoke Lemma 4.3 to assert that $w_\varepsilon\times\dfrac{\partial w_\varepsilon}{\partial t}=\operatorname{div}(w_\varepsilon\times\nabla w_\varepsilon)+r_\varepsilon$, so that

$$\begin{aligned}
B_2&=\int_{t_1}^{t_2}\int_{\mathbb{R}^N}2\chi\frac{\partial\phi_\varepsilon}{\partial t}\left(\operatorname{div}(w_\varepsilon\times\nabla w_\varepsilon)+\nabla(1-|u_\varepsilon|^2)\nabla\phi_\varepsilon\right)\\
&=\int_{t_1}^{t_2}\int_{\mathbb{R}^N}-\nabla\left(2\chi\frac{\partial\phi_\varepsilon}{\partial t}\right)w_\varepsilon\times\nabla w_\varepsilon+(1-|u_\varepsilon|^2)\operatorname{div}\left(2\chi\frac{\partial\phi_\varepsilon}{\partial t}\nabla\phi_\varepsilon\right).
\end{aligned}$$

Since $\partial_t \phi_\varepsilon = \Delta\phi_\varepsilon$, we have

$$\left|\nabla(\chi\frac{\partial\phi_\varepsilon}{\partial t})\right| \leq \|\chi\|_{\mathcal{C}^1}\left(|D^3\phi_\varepsilon| + |D^2\phi_\varepsilon|\right) \leq C(M_0)\|\chi\|_{\mathcal{C}^1}\left(\sqrt{|\log\varepsilon|}t^{-1} + t^{7/10}\right),$$

where we have used (4.1) with $q = 5$. Similarly,

$$\left|\mathrm{div}\left(2\chi\frac{\partial\phi_\varepsilon}{\partial t}\nabla\phi_\varepsilon\right)\right| \leq C\|\chi\|_{\mathcal{C}^1}\|\nabla\phi_\varepsilon\|_{\mathcal{C}^2}\,\|D^2\phi_\varepsilon\|_{\mathcal{C}^1} \leq C(M_0)\|\chi\|_{\mathcal{C}^1}\frac{|\log\varepsilon|}{t^{9/10}}.$$

It follows using iv) of Proposition 4.1 and the bounds on the potential term that the first one dominates and therefore

$$|B_2| \leq C(M_0,K)\|\chi\|_{\mathcal{C}^1}\left(\sqrt{|\log\varepsilon|}\log(\frac{t_2}{t_1}) + t_2^{3/10} - t_1^{3/10}\right).$$

**Step 3: estimate for $B_3$.** Using iii) and iv) of Proposition 4.1 with $q = 5$, we obtain

$$\begin{aligned}|B_3| &\leq C\|\chi\|_{\mathcal{C}^2}|K|^{1/2}\varepsilon|\log\varepsilon|^{1/2}\int_{t_1}^{t_2}\|\nabla_{x,t}\phi_\varepsilon\|^2_{L^\infty(K\times\{t\})}dt\\ &\leq C(M_0,K)\|\chi\|_{\mathcal{C}^2}\varepsilon|\log\varepsilon|^{3/2}\left(t_2^{3/5} - t_1^{3/5}\right).\end{aligned}$$

Combining the previous estimates the conclusion follows. □

Concerning the interaction term $\mathcal{F}_I$, we provide first a crude estimate, which will be needed in the Cylinders Lemma. At a later stage of our analysis (see Section 9), we will perform a refined decomposition of this term in dimension $N = 2$.

**Lemma 4.5.** *Let $K$ be any compact subset of $\mathbb{R}^N$. Let $\chi \in \mathcal{D}(\mathbb{R}^N)$ be such that $\mathrm{supp}\,\chi \subset K$, and let $1 \leq t_1 \leq t_1 + 1 \leq t_2$. We have,*

$$|\int_{t_1}^{t_2}\mathcal{F}_I(t,\chi,\nabla\phi_\varepsilon,w_\varepsilon)\,dt| \leq C(M_0,K)\|\chi\|_{\mathcal{C}^2}\Big((\sqrt{t_2} - \sqrt{t_1})\sqrt{|\log\varepsilon|} + (t_2 - t_1)\Big). \tag{4.44}$$

*Proof.* It follows from the definition of $\mathcal{F}_I$ that

$$|\int_{t_1}^{t_2}\mathcal{F}_I(t,\chi,\nabla\phi_\varepsilon,w_\varepsilon)\,dt| \leq C\|\chi\|_{\mathcal{C}^2}\int_{t_1}^{t_2}\left(\|\nabla w_\varepsilon\|_{L^1(K\times\{t\})}\|\nabla\phi_\varepsilon\|_{L^\infty(K\times\{t\})}\right)dt.$$

Using iii) and iv) of Proposition 4.1 with any admissible choice of $q$ the conclusion follows. □

Combining Lemmas 4.4 and 4.5 we derive

**Lemma 4.6.** *Let $K$ be any compact subset of $\mathbb{R}^N$. Let $\chi \in \mathcal{D}(\mathbb{R}^N)$ be such that $\mathrm{supp}\,\chi \subset K$, and let $1 \leq t_1 \leq t_1 + 1 \leq t_2$. We have,*

$$\int_{\mathbb{R}^N}\chi(x)d\nu_\varepsilon^{t_2} - \int_{\mathbb{R}^N}\chi(x)d\nu_\varepsilon^{t_1} \leq \frac{1}{|\log\varepsilon|}|\int_{t_1}^{t_2}\mathcal{F}_S(t,\chi,w_\varepsilon)\,dt| + \frac{C}{|\log\varepsilon|}\|\chi\|_{\mathcal{C}^2}\mathcal{R}_\varepsilon(t_1,t_2,w_\varepsilon), \tag{4.45}$$

*where $C = C(M_0,K)$, $\mathcal{F}_S$ is defined as in Lemma 2.3, that is*

$$\mathcal{F}_S(t,\chi,w_\varepsilon) = \int_{\mathbb{R}^N\times\{t\}}\left(D^2\chi\nabla w_\varepsilon\cdot\nabla w_\varepsilon - \Delta\chi e_\varepsilon(w_\varepsilon)\right)dx,$$

*and where*

$$\mathcal{R}_\varepsilon(t_1,t_2,w_\varepsilon) = \left(\sum_{i=1}^2\|\nabla w_\varepsilon\|_{L^1(K\times\{t_i\})} + (\sqrt{t_2} - \sqrt{t_1})\right)\sqrt{|\log\varepsilon|} + (t_2 - t_1). \tag{4.46}$$

## 4.4 Upper bounds for the velocity of concentration sets

In this section, we turn back to dimension $N = 2$, to the concentration sets $\Omega^\varepsilon_\delta(t)$ introduced in Section 3, and study their motion for long times. We recall that, by a standard covering argument,

$$\Omega^\varepsilon_\delta(t) \subset \cup_{i=1}^l B(x_i^\varepsilon, 2\delta)\,, \tag{4.47}$$

where the number $l$ of points $x_i^\varepsilon$ is uniformly bounded by $l \leq C\frac{M_0}{\eta_0}$.

Our main result, Proposition 4.3 below, is inspired by Lemma 5.1 of [17]. However, since the initial datum is not assumed to be well-prepared, we rely on different type of arguments, in particular the topological and regularity arguments in [17] are replaced here by the application of the Clearing-Out Lemma for vorticity and the decomposition in Section 4.2.

**Proposition 4.3. (Cylinders Lemma)** *Let $t_0 \geq 1$ and $r > 0$ be given. There exists positive constants $\sigma_0$, $\gamma_0$ and $r_0$ depending only on $M_0$ such that, if*

$$\Omega^\varepsilon(t_0) \equiv \Omega^\varepsilon_{r/4}(t_0) \subset \cup_{i=1}^l B(x_i^\varepsilon, r)\,, \tag{4.48}$$

*for some $|\log\varepsilon|^{-1/6} \leq r \leq r_0$ and some points $(x_i^\varepsilon)_{1\leq i\leq l}$ verifying*

$$|x_i^\varepsilon - x_j^\varepsilon| \geq \sigma_0 r\,, \qquad \forall\, i \neq j, \tag{4.49}$$

*then*

$$\Omega^\varepsilon(t) \subset \cup_{i=1}^l B(x_i^\varepsilon, \frac{\sigma_0 r}{8})\,, \tag{4.50}$$

*for every $t_0 \leq t \leq t_0 + \gamma_0 r^2|\log\varepsilon|$.*

*Proof.* The strategy is based on formula (4.45) of Lemma 4.3, and a suitable choice of function $\chi$. For that purpose, we first construct a smooth positive function $\Lambda$ defined on $\mathbb{R}^2$, satisfying

$$\begin{cases} \Lambda(x) = 8|x|^2 & \text{if } |x| \leq 1/4 \\ \Lambda(x) = 0 & \text{if } |x| \geq 1/2 \\ 0 \leq \Lambda(x) \leq 1 & \quad\text{on } \mathbb{R}^2 \end{cases} \tag{4.51}$$

Next, we consider the points $x_i^\varepsilon$ given by (4.48) and (4.49), we set

$$\chi(x) = \sum_{i=1}^l \Lambda\left(\frac{x - x_i^\varepsilon}{\sigma_0 r}\right),$$

for some constant $\sigma_0 > 0$ to be determined later, and we introduce the integral

$$A(t) = \int_{\mathbb{R}^2} \chi(x) d\nu_\varepsilon^t\,.$$

Let $t_e$ be the exit-time, i.e.

$$t_e = \sup\{t \geq t_0,\ \Omega^\varepsilon(s) \subset \cup_{i=1}^l B(x_i^\varepsilon, \frac{\sigma_0 r}{4}) \quad \text{for } t_0 \leq s < t\}\,.$$

Notice that in view of Lemma 3.3, $t_e \geq t_0 + 4$. Our purpose is to apply formula (4.45) and to prove that we are led to a contradiction if $t_e$ were too small, thanks to our special choice of function $\chi$. However, in view of the specific form of (4.45), we will first choose suitable times

$t_1 \in [t_0, t_0+1]$ and $t_2 \in [t_e-1, t_e]$ for which good estimates are available on $\|\nabla w_\varepsilon\|_{L^1(K\times\{t_i\})}$, $i = 1, 2$, where $K =$ supp $\chi$.

**Step 1.** There exists $t_1 \in [t_0 + \frac{1}{2}, t_0 + 1]$ and $t_2 \in [t_e - 1, t_e - \frac{1}{2}]$[19] such that

$$\|\nabla w_\varepsilon\|_{L^1(K\times\{t_i\})} \leq C(M_0)(\sigma_0^2 r^2 + 1)\,, \qquad \text{for } i = 1, 2\,,$$

and such that there exists some $x_e$ verifying, for some $i \in \{1, ..., l\}$,

$$x_e \in \Omega^\varepsilon(t_2) \qquad \text{and} \qquad (\frac{\sigma_0}{8} - 1)r \leq |x_e - x_i^\varepsilon| \leq (\frac{\sigma_0}{8} + 1)r. \tag{4.52}$$

*Proof.* In view of the definition of $t_e$, there exists some $\tilde{x}_e \in \Omega(t_e)$, and $i \in \{1, ..., l\}$ such that

$$(\frac{\sigma_0}{8} - \frac{1}{2})r \leq |\tilde{x}_e - x_i^\varepsilon| \leq (\frac{\sigma_0}{8} + \frac{1}{2})r. \tag{4.53}$$

It follows by Lemma 3.3 that for every $t \in [t_e - 1, t_e]$, there exists some $x(t)$ such that

$$(\frac{\sigma_0}{8} - 1)r \leq |\tilde{x}_e - x_i^\varepsilon| \leq (\frac{\sigma_0}{8} + 1)r. \tag{4.54}$$

The conclusion follows by averaging, since

$$\|\nabla w_\varepsilon\|_{L^1(K\times[t_0,t_0+1])} \leq C(M_0)\sigma_0^2 r^2 \quad \text{and} \quad \|\nabla w_\varepsilon\|_{L^1(K\times[t_e-1,t_e])} \leq C(M_0)\sigma_0^2 r^2.$$

**Step 2: upper bounds on $A(t_1)$.** We claim that

$$A(t_1) \leq \frac{C}{\sigma_0^2} M_0 + o(1), \qquad \text{where } o(1) \to 0 \text{ as } \varepsilon \to 0,$$

and where $C$ does not depend on $\sigma_0$.
*Proof.* By Lemma 3.5 we have

$$|u_\varepsilon| \geq \frac{1}{2} \qquad \text{on } \Big(\mathbb{R}^2 \setminus \cup_{i=1}^l B(x_i^\varepsilon, 2r)\Big) \times [t_0 + \frac{1}{4}, t_0 + 1].$$

Applying Proposition 4.2 we infer that

$$e_\varepsilon(w_\varepsilon) \leq \frac{C}{r^2} \qquad \text{on } \Big(\mathbb{R}^2 \setminus \cup_{i=1}^l B(x_i^\varepsilon, 3r)\Big) \times [t_0 + \frac{1}{2}, t_0 + 1].$$

In particular,

$$\int_{\mathbb{R}^2\setminus\cup_{i=1}^l B(x_i^\varepsilon,3r)} \chi d\nu_\varepsilon^{t_1} \leq \frac{C\sigma_0^2}{|\log\varepsilon|}. \tag{4.55}$$

On the other hand $|\chi| \leq \dfrac{C}{\sigma_0^2}$ on $\cup_{i=1}^l B(x_i^\varepsilon, 3r)$, and therefore we derive

$$\int_{\cup_{i=1}^l B(x_i^\varepsilon,3r)} \chi d\nu_\varepsilon^{t_1} \leq \frac{C\sigma_0^{-2}}{|\log\varepsilon|} M_0. \tag{4.56}$$

The conclusion follows combining (4.55) with (4.56).

[19]Notice in particular that $t_2 - t_1 \geq 1$.

**Step 3: lower bounds on $A(t_2)$.** We claim that $A(t_2) \geq \frac{\eta_0}{18} - C(M_0)(\sigma_0 r + \sigma_0^2 r^2)$.
*Proof.* Let $x_e$ be given by Step 1. We have, by definition of $\Omega(t_2)$,

$$\int_{B(x_e,\frac{r}{4})\times\{t_2\}} e_\varepsilon(u_\varepsilon)dx \geq \frac{\eta_0}{2}|\log\varepsilon|.$$

On the other hand, by (4.52), $\chi(x) \geq \frac{1}{9}$ on $B(x_e, \frac{r}{4})$, so that

$$\int_{B(x_e,\frac{r}{4})\times\{t_2\}} \chi e_\varepsilon(u_\varepsilon)dx \geq \frac{\eta_0}{18}|\log\varepsilon|.$$

It remains to compare $e_\varepsilon(u_\varepsilon)$ and $e_\varepsilon(w_\varepsilon)$. In view of (2.19) and estimate (ii) of Proposition 4.1, we have

$$\int_{B(x_e,\frac{r}{4})\times\{t_2\}} |\chi e_\varepsilon(u_\varepsilon) - \chi e_\varepsilon(w_\varepsilon)| \leq C(M_0)(\sigma_0 r + \sigma_0^2 r^2)|\log\varepsilon|,$$

and the conclusion follows.

**Step 4:** We claim that

$$|\int_{t_1}^{t_2} \mathcal{F}_S(t,\chi,w_\varepsilon)\,dt| \leq C(M_0)(1+\sigma_0^2 r^2)|t_2 - t_1|.$$

*Proof.* Since $\chi$ has compact support in $\mathcal{U} = \cup_{i=1}^{l} B(x_i^\varepsilon, \sigma_0 r/2)$, we may divide the integration into two disjoint contributions: the contribution on $\mathcal{U}_1 = \cup_{i=1}^{l} B(x_i^\varepsilon, \sigma_0 r/4)$, and that on $\mathcal{U}_2 = \mathcal{U} \setminus \mathcal{U}_1$, which is a union of annuli. On $\mathcal{U}_1$, $\chi(x) = \sum_{i=1}^{l} 8|x - x_i^\varepsilon|^2$, and this specific form implies a remarkable sign condition in the integration, namely

$$D^2\chi\nabla w_\varepsilon \cdot \nabla w_\varepsilon - \Delta\chi e_\varepsilon(w_\varepsilon) = -\Delta\chi V_\varepsilon(u_\varepsilon) \leq 0 \quad \text{on } \mathcal{U}_1. \tag{4.57}$$

The previous fact (and more generally, related identities for the squared distance function to a manifold) was remarked by De Giorgi [13], Rubinstein and Sternberg [25] and used extensively since then (see for example [29]).

Turning to $\mathcal{U}_2$ we have, in view of the definition of $t_e$ and our choice $t_2 \leq t_e$,

$$\Omega^\varepsilon_{\frac{r}{4}}(t) \subset \cup_{i=1}^{l} B(x_i^\varepsilon, \frac{\sigma_0 r}{8}) \qquad \forall t_1 - \frac{1}{2} \leq t \leq t_2.$$

Invoking Lemma 3.5 we deduce $|u_\varepsilon| \geq \frac{1}{2}$ on $\left(\mathcal{U} \setminus \cup_{i=1}^{l} B(x_i^\varepsilon, \frac{\sigma_0 r}{6})\right) \times [t_1 - \frac{1}{4}, t_2]$, and therefore, by Proposition 4.2 we obtain

$$e_\varepsilon(w_\varepsilon) \leq C(M_0)(1 + \frac{1}{\sigma_0^2 r^2}) \qquad \text{on } \mathcal{U}_2 \times [t_1, t_2], \tag{4.58}$$

so that combining (4.57) and (4.58) we derive

$$|\int_{t_1}^{t_2} \mathcal{F}_S(t,\chi,w_\varepsilon)\,dt| \leq C(M_0)(\sigma_0^2 r^2 + 1)|t_2 - t_1|,$$

which is the desired inequality.

**Step 5: bounds for $\mathcal{R}_\varepsilon$.** By Step 1 we have

$$\mathcal{R}_\varepsilon(t_1, t_2, w_\varepsilon) \le C(1 + \sigma_0^2 r^2 + (\sqrt{t_2} - \sqrt{t_1}))\sqrt{|\log\varepsilon|} + (t_2 - t_1).$$

**Step 6: proof of Proposition 4.3 completed.** Combining Step 2 and Step 3 we have

$$A(t_2) - A(t_1) \ge \frac{\eta_0}{18} - \frac{C}{\sigma_0^2} M_0 - C(M_0)(\sigma_0 r + \sigma_0^2 r^2) + o(1).$$

We choose first $\sigma_1$ such that

$$\frac{C}{\sigma_1^2} M_0 = \frac{\eta_0}{36}, \text{ i.e. } \sigma_1 = 6\sqrt{\frac{CM_0}{\eta_0}}, \tag{4.59}$$

and then finally set $\sigma_0 = \max\{100, \sigma_1\}$. For this choice of $\sigma_0$, we choose first $r_0$ in such a way that $\sigma_0 r_0 \le 1$ and

$$C(M_0)(\sigma_0 r_0 + \sigma_0^2 r_0^2) \le \frac{\eta_0}{72},$$

so that, if $r \le r_0$,

$$A(t_2) - A(t_1) \ge \frac{\eta_0}{72} + o(1). \tag{4.60}$$

On the other hand, by formula (4.45),

$$\begin{aligned} A(t_2) - A(t_1) &\le \frac{1}{|\log\varepsilon|} \left| \int_{t_1}^{t_2} \mathcal{F}_S(t, \chi, w_\varepsilon)\, dt \right| + \frac{C(M_0)}{|\log\varepsilon|} \frac{\mathcal{R}_\varepsilon(t_1, t_2, w_\varepsilon)}{\sigma_0^2 r^2} \\ &\le C(M_0) \left( \frac{|t_2 - t_1|}{|\log\varepsilon|} + \frac{\sqrt{t_2} - \sqrt{t_1} + 1}{\sqrt{|\log\varepsilon|}} \right). \end{aligned} \tag{4.61}$$

Combining (4.60) with (4.61) we deduce

$$\eta_0 \le \frac{C(M_0)}{\sigma_0^2 r^2} \left( \frac{|t_2 - t_1|}{|\log\varepsilon|} + \frac{\sqrt{t_2} - \sqrt{t_1} + 1}{\sqrt{|\log\varepsilon|}} \right).$$

Therefore, we obtain $t_2 \ge t_0 + C(M_0, \eta_0) r^2 |\log\varepsilon|$, and the proof is complete. □

It may occur that, as $\varepsilon$ tends to zero, some part of the set $\Omega^\varepsilon$ escapes to infinity. This however does not affect the asymptotics, since we have the following variant of Proposition 4.3 for which we omit the details.

**Proposition 4.4.** *Let $0 \le r \le r_0$, and $R > 10\sigma_0 r$ be given, assume the points $(x_i^\varepsilon)_{1\le i\le l}$ verify (4.49), and that*

$$\Omega^\varepsilon(t_0) \cap B(0, R) \subset \cup_{i=1}^{l} B(x_i^\varepsilon, r).$$

*Then*

$$\Omega^\varepsilon(t) \cap B(0, \frac{9R}{10}) \subset \cup_{i=1}^{l} B(x_i^\varepsilon, \frac{\sigma_0 r}{8})$$

*for $t_0 \le t \le \gamma_0 r^2 |\log\varepsilon|$.*

## 4.5 Consequences of the Cylinders Lemma

We assume in this subsection that (4.48) and (4.49) hold for $t_0 \geq 1$ and $0 < r < r_0$, $\sigma \geq \sigma_0$. In order to describe the energy evolution along concentration sets, we use the following

**Lemma 4.7.** *Let $\chi$ a smooth nonnegative function, and assume that* $\operatorname{supp} \nabla\chi \subset \mathbb{R}^2 \setminus \cup_{i=1}^{l} B(x_i^\varepsilon, \sigma r)$*. Then we have*

$$\frac{d}{dt}\int_{\mathbb{R}^2} \chi(x) d\mu_\varepsilon^t \leq C(M_0)|\operatorname{supp}\nabla\chi| \cdot \|D^2\chi\|_{L^\infty} \left(\frac{(\sigma r)^{-2}}{|\log\varepsilon|} + \frac{1}{t}\right), \tag{4.62}$$

*for every $t_0 + \frac{1}{2} \leq t \leq t_0 + \gamma_0 r^2 |\log\varepsilon|$.*

*Proof.* By Lemma 2.3, $\displaystyle\frac{d}{dt}\int_{\mathbb{R}^N} \chi(x)\, d\mu_\varepsilon^t \leq \frac{1}{|\log\varepsilon|}\mathcal{F}_S(t, \chi, u_\varepsilon)$. Since

$$\mathcal{F}_S(t,\chi,u_\varepsilon) = \int_{\mathbb{R}^2\times\{t\}} D^2\chi \nabla u_\varepsilon \cdot \nabla u_\varepsilon - \Delta\chi e_\varepsilon(u_\varepsilon) \leq C\|D^2\chi\|_{L^\infty} \int_{\operatorname{supp}\nabla\chi} e_\varepsilon(u_\varepsilon),$$

the conclusion follows by Proposition 4.3, Lemma 3.5 and Corollary 4.1. □

Concerning $w_\varepsilon$, we deduce from the Cylinders Lemma the pointwise estimate

**Lemma 4.8.** *We have, for some constant $K$ depending on $M_0$,*

$$|e_\varepsilon(w_\varepsilon)(x,t)| \leq K(\sigma r)^{-2} \quad \textit{and} \quad |e_\varepsilon(u_\varepsilon)(x,t)| \leq K\left((\sigma r)^{-2} + \frac{|\log\varepsilon|}{t}\right) \tag{4.63}$$

*for every $x \in \mathbb{R}^2 \setminus \cup_{i=1}^{l} B(x_i^\varepsilon, \sigma r)$ and every $t_0 + 1 \leq t \leq t_0 + \gamma_0 r^2|\log\varepsilon|$.*

*Proof.* A direct consequence of Proposition 4.3, Lemma 3.5 and Proposition 4.2. □

# 5 Limiting measures in the log time scale

Our purpose is to study the asymptotics for the measures $\mathfrak{v}_\varepsilon^s$. This will lead us to the proofs of Theorem 2, 4 and Theorem 5. From now on, we will work directly with the rescaled time $s = \frac{t}{|\log\varepsilon|}$. A first step in the argument is to consider limits for **fixed** $s$. In a second step, we prove some continuity property in time so that an abstract compactness argument leads finally to the existence of a limiting measure for **all** $s$. Both $\Sigma_{\mathfrak{v}}$ and $\mathfrak{v}_*^t$ are constructed at the same time.

## 5.1 Concentration points for fixed $s$

**Lemma 5.1.** *Let $s > 0$ be given. There exists a sequence $\varepsilon_n \to 0$ and $l_s$ points $(\mathfrak{a}_i(s))_{1\leq i\leq l_s}$ (depending only on s) with $l_s \leq C\frac{M_0}{\eta_0}$, such that for every $r > 0$ and $R > 2\sup_{1\leq i\leq l_s}|\mathfrak{a}_i|$ there exists $n_0 \in \mathbb{N}$ (depending only on $s, r$ and $R$) such that*

$$\Omega_{r/16}^{\varepsilon_n}(|\log\varepsilon_n|s) \cap B(0,R) \subset \cup_{i=1}^{l_s} B(\mathfrak{a}_i(s), r) \qquad \forall\, n \geq n_0. \tag{5.1}$$

*Moreover, for any $i = 1, ..., l_s$, and for any $n \geq n_0$,*

$$B(\mathfrak{a}_i(s), r) \cap \Omega_{r/16}^{\varepsilon_n}(|\log\varepsilon_n|s) \neq \emptyset. \tag{5.2}$$

*Proof.* Let $0 < \delta \leq 1$. We consider a covering of $\Omega_\delta^\varepsilon(|\log\varepsilon|s)$ as in (4.48), i.e. such that

$$\Omega_\delta^\varepsilon(|\log\varepsilon|s) \subset \cup_{i=1}^{l^{\varepsilon,\delta}} B(x_i^{\varepsilon,\delta}, 2\delta) \tag{5.3}$$

with $l^{\varepsilon,\delta} \leq \frac{CM_0}{\eta_0}$ and

$$x_i^{\varepsilon,\delta} \in \Omega_\delta^\varepsilon(|\log\varepsilon|s). \tag{5.4}$$

By a compactness argument, there exists a set $\{\mathfrak{a}_i^\delta\}_{1\leq i\leq l_\delta}\}$, with

$$l_\delta \leq \frac{CM_0}{\eta_0},$$

such that for a subsequence $\varepsilon_n \equiv \varepsilon_n^\delta \to 0$, $l^{\varepsilon_n,\delta} = l^\delta$ is independent of $n$, and such that, relabeling if necessary, we have

$$x_i^{\varepsilon_n,\delta} \to \mathfrak{a}_i^\delta \qquad \text{for } i = 1, ..., l_\delta,$$

$$|x_i^{\varepsilon_n,\delta}| \to +\infty \qquad \text{for } i = l_\delta + 1, ..., l^\delta.$$

We choose $\delta_m = 2^{-m}$ for $m \in \mathbb{N}$, and set $\mathfrak{a}_i^m = \mathfrak{a}_i^{\delta_m}$, $l^{\delta_m} = l^m$ and $l_{\delta_m} = l_m$. Since $\Omega_{\delta_{m+1}}^\varepsilon(|\log\varepsilon|s) \subset \Omega_{\delta_m}^\varepsilon(|\log\varepsilon|s)$, we notice that

$$\cup_{i=1}^{l_{m+1}}\{\mathfrak{a}_i^{m+1}\} \subset \cup_{i=1}^{l_m} B(\mathfrak{a}_i^m, 2\delta_m).$$

We deduce that, **without** need to pass to a subsequence, $\cup_{i=1}^{l_m}\{\mathfrak{a}_i^m\}$ converges to $\cup_{i=1}^{l_s}\{\mathfrak{a}_i(s)\}$ as $m \to +\infty$, and

$$\operatorname{dist}\left(\cup_{i=1}^{l_1}\{\mathfrak{a}_i^m\}, \cup_{i=1}^{l_1}\{\mathfrak{a}_i(s)\}\right) \leq 2^{-m+2}.$$

The subsequence $\varepsilon_n$ in the statement of the Lemma is easily constructed by a diagonal argument.

Let $0 < r < 1$ and $R > 2\sup_{1\leq i\leq l_s}|\mathfrak{a}_i|$ be given and let $m \in \mathbb{N}$ be such that $2^{-m-1} < \frac{r}{16} \leq 2^{-m}$. There exists $n_0 \in \mathbb{N}$ such that, for $n \geq n_0$,

$$|x_i^{\varepsilon_n,\delta_m} - \mathfrak{a}_i^m| \leq 2^{-m} \quad \text{for } i = 1, \dots, l_m, \qquad \text{and} \qquad |x_i^{\varepsilon_n,\delta_m}| > 2R \quad \text{for } i = l_m + 1, \dots, l^m.$$

Therefore, for $n \geq n_0$,

$$\begin{aligned}
\Omega_{r/16}^{\varepsilon_n}(|\log\varepsilon_n|s) \cap B(0,R) &\subset \Omega_{\delta_m}^{\varepsilon_n}(|\log\varepsilon_n|s) \cap B(0,R) \\
&\subset \left(\cup_{i=1}^{l^m} B(x_i^{\varepsilon_n,\delta_m}, 2\delta_m)\right) \cap B(0,R) = \cup_{i=1}^{l_m} B(x_i^{\varepsilon_n,\delta_m}, 2\delta_m) \\
&\subset \cup_{i=1}^{l_m} B(\mathfrak{a}_i^m, 3\delta_m) \subset \cup_{i=1}^{l_s} B(\mathfrak{a}_i(s), 3\delta_m + 2^{-m+2}) \\
&\subset \cup_{i=1}^{l_s} B(\mathfrak{a}_i(s), r).
\end{aligned} \tag{5.5}$$

The proof is complete. □

**Remark 5.1.** At a later stage, we will consider limiting measures $\mathfrak{v}_*^s$ for $\mathfrak{v}_{\varepsilon_n}^s$. A direct consequence of (5.2) is that

$$\mathfrak{v}_*^s(\{\mathfrak{a}_\mathfrak{i}\}) \geq \frac{\eta_0}{2}. \tag{5.6}$$

From Lemma 5.1 and a further diagonal argument we obtain

**Corollary 5.1.** *Let $\mathcal{Z} \subset \mathbb{R}^+$ be a countable set. There exists a sequence $\varepsilon_n \to 0$ (depending only on $\mathcal{Z}$) and, for each $s \in \mathcal{Z}$, $l_s$ points $\mathfrak{a}_1(s), ..., \mathfrak{a}_{l_s}(s)$ (with $l_s \leq C\frac{M_0}{\eta_0}$), such that for every $r > 0$ and $R(s) \geq \sup_{1\leq i\leq l_s} |\mathfrak{a}_i(s)|$, there exists $n_0 \in \mathbb{N}$ (depending only only on $s$, $r$ and $R(s)$) for which*

$$\Omega^{\varepsilon_n}_{r/16}(|\log \varepsilon_n|s) \cap B(0, R(s)) \subset \cup_{i=1}^{l_s} B(\mathfrak{a}_i(s), r) \qquad \forall\, n \geq n_0. \tag{5.7}$$

**Remark 5.2.** In the sequel, we take as sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ the one related to $\mathcal{Z} = \gamma_0\mathbb{Q}^+$, where $\gamma_0$ is the constant in Proposition 4.3.

An important part of our analysis will be devoted to prove that there exists a subsequence $\varepsilon_{\sigma(n)}$ for which (5.1) holds for **any** $s > 0$. The key ingredient is the Cylinders Lemma. In order to implement this technique we first need the following elementary covering argument.

**Lemma 5.2.** *Consider $l$ distinct points $a_1, ..., a_l$ in $\mathbb{R}^2$. Let $r_0 > 0$ and $\sigma \geq 2$ be given. Then, there exists $r > 0$ such that*

$$r_0 \leq r \leq (2\sigma)^l r_0 \tag{5.8}$$

*and a subset $\{a_j\}_{j\in J}$ of $\{a_i\}_{1\leq i\leq l}$ such that*

$$\cup_{i=1}^{l} B(a_i, r_0) \subset \cup_{j\in J} B(a_j, r)$$

*and*

$$|a_j - a_k| \geq \sigma r \qquad \forall\, j \neq k \text{ in } J. \tag{5.9}$$

*Proof.* The proof is by iteration, in at most $l$ steps. First, consider the collection $\{a_i\}_{1\leq i\leq l}$. If (5.9) is verified with $r = r_0$ there is nothing else to do. Otherwise, take two points, say $a_1, a_2$ such that $|a_1 - a_2| \leq \sigma r_0$, consider the collection $a_2, a_3, ..., a_l$, and set $r = 2\sigma r_0$. If (5.9) is verified, we stop. Otherwise we go on in the same way. If the process does not stop in $l-1$ steps, at the $l^{\text{th}}$ step we are left with one single ball of radius $r = (2\sigma)^l r_0$, and (5.9) is void. □

## 5.2 Continuity in time

**Proposition 5.1.** *Let $s_0 > 0$ and $0 < r_0 \leq 1$ and $R \geq 2\sup_{1\leq i\leq l(s_0)} |\mathfrak{a}_i(s_0)|$ be given. There exists $n_0 = n_0(s_0, r_0, R)$ such that for $n \geq n_0$*

$$\Omega^{\varepsilon_n}_{r_0/16}(|\log \varepsilon_n|s) \cap B(0, R) \subset \cup_{i=1}^{l(s_0)} B(\mathfrak{a}_i(s_0), \sigma_1 r_0) \qquad \forall\, s \in [s_0, s_0 + \gamma_0 r_0^2], \tag{5.10}$$

*where $\gamma_0$ is the constant in Proposition 4.3 and $\sigma_1$ is some constant depending only on $M_0$. Here the points $\{\mathfrak{a}_i(s_0)\}_{1\leq i\leq l(s_0)}$, the sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ and $n_0$ are given in Lemma 5.1. Moreover,*

$$e_{\varepsilon_n}(\mathfrak{u}_{\varepsilon_n}) \leq C(r_0^{-2} + \delta_0^{-1}) \text{ on } \Big[B(0, R) \setminus \cup_{i=1}^{l} B(\mathfrak{a}_i(s_0), \sigma_1 r_0)\Big] \times [s_0 + \frac{1}{|\log \varepsilon|}, s_0 + \gamma_0 r_0^2]. \tag{5.11}$$

*Proof.* We apply Lemma 5.1 with $s = s_0$ and $r = r_0$. Combining with Lemma 5.2, for the choice $\sigma = \sigma_0$, where $\sigma_0$ is the constant in Proposition 4.3, we are led to

$$\Omega^{\varepsilon_n}_{r_0/16}(|\log \varepsilon_n|s) \cap B(0, R) \subset \cup_{j\in J} B(a_i, r)$$

for some $r_0 \leq r \leq (2\sigma_0)^l r_0$, and $|\mathfrak{a}_j - \mathfrak{a}_k| \geq \sigma_0 r$ for $j \neq k \in J$. Conclusion (5.10) then follows from the Cylinders Lemma (Proposition 4.3). For (5.11) we invoke once more Lemma 4.8. □

### 5.3 Construction of $\Sigma_{\mathfrak{v}}$ and proof of Theorem 2

Given a length $r_0 > 0$ we consider the set

$$\Sigma^{\varepsilon}_{r_0} = \cup_{s>0} \Omega^{\varepsilon}_{r_0/16}(|\log \varepsilon| s)$$

and cover it by "chains" of cylinders of radius of order $r_0$ and height of order $r_0^2$. We then define the set $\Sigma_{\mathfrak{v}}$ as the intersection, as $r_0 \to 0$ and $\varepsilon \to 0$ of these chains. To implement this idea, we discretize time by slices of thickness $\gamma_0 r_0^2$. More precisely, we fix $r_0 \in \mathbb{Q}^+$ and consider the time slices $s_j = j\gamma_0 r_0^2$, for $j \in \mathbb{N},\ j \geq 1$. For $S > 0$ and $R > 0$ we set

$$\Sigma^{\varepsilon_n}_{r_0}(S, R) = \Sigma^{\varepsilon_n}_{r_0} \cap B(0, R) \times [0, S],$$

where $(\varepsilon_n)_{n\in\mathbb{N}}$ is the sequence considered in Remark 5.2. In view of Proposition 5.1, for given $S > 0$,

$$R(S) = 2 \sup_{s \in \mathcal{Z} \cap (0,S)} |\mathfrak{a}_i(s)| < +\infty.$$

As an immediate consequence of Proposition 5.1 we have

**Lemma 5.3.** *Consider the sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ given in Remark 5.2. Assume $r_0 \in \mathbb{Q}^+$ and $S > 0$, $R \geq R(S)$ be given. Then there exists $n_0 = n_0(S, R, r_0)$ depending only on $S$, $R$ and $r_0$ such that, for any $n \geq n_0$,*

$$\Sigma^{\varepsilon_n}_{r_0}(S, R) \subset \bigcup_{\substack{j\geq 1 \\ s_j \leq S}} \left( \bigcup_{i=1}^{l(s_j)} \overline{B}(\mathfrak{a}_i(s_j), \sigma_1 r_0) \times [s_j, s_{j+1}] \right).$$

We next specify the choice for $r_0$, taking namely $r_0 = \dfrac{1}{2^m}$, $m \in \mathbb{N}$, and set

$$\Sigma_{\mathfrak{v}} = \bigcap_{m\in\mathbb{N}} \bigcup_{\substack{j\geq 1 \\ 1\leq i\leq l}} \overline{B}(\mathfrak{a}_i(\frac{j\gamma_0}{2^{2m}}), \frac{\sigma_1}{2^m}) \times [\frac{j\gamma_0}{2^{2m}}, \frac{(j+1)\gamma_0}{2^{2m}}]. \tag{5.12}$$

By definition, $\Sigma_{\mathfrak{v}}$ is an intersection of closed sets, hence it is closed. Moreover, by definition of parabolic[20] Hausdorff measure $\mathcal{H}^k_P$, we have

$$\mathcal{H}^2_P(\Sigma_{\mathfrak{v}} \cap \mathbb{R}^2 \times [0, S]) \leq C(M_0) S\,. \tag{5.13}$$

This yields the first assertion of Theorem 2. Next, we first observe that for some $l$ depending only on $M_0$ [21]

$$\sharp \Sigma^s_{\mathfrak{v}} \leq l \qquad \text{for any } s > 0, \tag{5.14}$$

and the second assertion follows directly from the construction (5.12) of $\Sigma_\nu$, taking $\alpha = \frac{\gamma_0}{\sigma_1^2}$. At this stage we have established Theorem 2.[22]

[20] The parabolic ball $B_P(z, r) \subset \mathbb{R}^N \times \mathbb{R}$ of radius $r$ centered at $z = (x, t) \in \mathbb{R}^N \times \mathbb{R}$ is given by $B_P(z, r) = B(x, r) \times [t - r^2, t + r^2]$.

[21] Indeed, in view of the definition of $\Sigma_{\mathfrak{v}}$, $\Sigma^s_{\mathfrak{v}}$ is included the union of at most $l$ intervals of arbitrarily small size.

[22] Actually we have constructed a set $\Sigma_{\mathfrak{v}}$ satisfying the properties stated in Theorem 2. The important point of course is that the set $\Sigma_{\mathfrak{v}}$ satisfies **also** the properties stated in Theorem 1, in particular the convergence stated in (3). This will be established at a later stage of the analysis.

Notice that the above construction of $\Sigma_\nu$ also yields some properties stated in Theorem 1. Indeed, let $K \subset \mathbb{R}^2 \times \mathbb{R}^+ \setminus \Sigma_{\mathfrak{v}}$ be a compact set. By definition of $\Sigma_{\mathfrak{v}}$, there exists some $m \in \mathbb{N}$ such that

$$K \cap \bigcup_{\substack{j \geq 1 \\ 1 \leq i \leq l}} \overline{B}(\mathfrak{a}_i(\frac{j\gamma_0}{2^{2m}}), \frac{\sigma_1}{2^m}) \times [\frac{j\gamma_0}{2^{2m}}, \frac{(j+1)\gamma_0}{2^{2m}}] = \emptyset.$$

Therefore, by Proposition 5.1 we deduce that

$$e_{\varepsilon_n}(\mathfrak{u}_{\varepsilon_n}) \leq 2^{2m} C \qquad \text{on } K, \tag{5.15}$$

and hence

$$||\mathfrak{u}_{\varepsilon_n}| - 1| \leq 2^m C\varepsilon \qquad \text{on } K, \tag{5.16}$$

so that $|\mathfrak{u}_{\varepsilon_n}| \to 1$ uniformly on $K$ and moreover the energy is uniformly bounded on $K$.

**Remark 5.3.** Since, for any $s > 0$, $\Sigma_{\mathfrak{v}}^s$ is finite, we may write

$$\Sigma_{\mathfrak{v}}^s = \cup_{i=1}^{l(s)} \{a_i(s)\}. \tag{5.17}$$

It follows from the very construction of $\Sigma_{\mathfrak{v}}$ that for each $s \in \mathcal{Z}$ we have the inclusion

$$\{\mathfrak{a}_i(s)\}_{1 \leq i \leq l_s} \subset \{a_i(s)\}_{1 \leq i \leq l(s)}. \tag{5.18}$$

The two sets may not coincide for every $s$, in particular when collisions occur. However, in view of the above construction, we have

$$\Sigma_\mu = \cup_{s \in \mathcal{Z}} \cup_{i=1}^{l_s} \overline{\{\mathfrak{a}_i(s)\}},$$

and more precisely, for $s > 0$,

$$\cup_{i=1}^{l(s)} \{a_i(s)\} = \lim_{s' \to s,\ s' \in \mathcal{Z},\ s' < s} \cup_{i=1}^{l_{s'}} \{\mathfrak{a}_i(s')\}.$$

In particular, for any neighborhood $\mathcal{O}_i$ of $a_i(s)$,

$$\liminf_{s' \to s,\ s' < s} \nu_*^{s'}(\mathcal{O}_\rangle) \geq \frac{\eta_0}{2}. \tag{5.19}$$

Notice also that a consequence of Theorem 5 iii) will be that equality in (5.18) holds for all but finitely many times $s$.

## 5.4 The abstract compactness argument

The following is an easy variant of Helly's selection principle.

**Lemma 5.4.** *Let $I$ be an at most countable set, and let $(f_n^i)_{n \in \mathbb{N}, i \in I}$ be a collection of real-valued functions defined on some interval $(a, b)$. Assume that for each $i \in I$ the family $(f_n^i)_{n \in \mathbb{N}}$ is equibounded and satisfies the following semi-decreasing property*[23]

$$\begin{aligned} &\forall\, \delta > 0 \text{ there exists } \tau > 0 \text{ and } n_i \in \mathbb{N} \text{ such that, if } s_1, s_2 \in (a, b) \\ &\text{and } s_2 - \tau \leq s_1 \leq s_2, \text{ then } f_n^i(s_2) \leq f_n^i(s_1) + \delta,\ \forall\, n \geq n_i. \end{aligned} \tag{5.20}$$

[23]Such a condition appears in the literature under various forms, the one we adopt here does not require differentiability.

*Then there exists a subsequence $\sigma(n)$ and a family $(f^i)_{i\in I}$ of real-valued functions on $(a,b)$ such that*

$$f^i_{\sigma(n)}(s) \to f^i(s) \qquad \forall\, s\in(a,b), \quad \forall\, i\in I.$$

We apply the previous lemma to the following situation. Let $(\chi_i)_{i\in I}$ be a countable family of compactly supported nonnegative smooth functions on $\mathbb{R}^N$, and assume that $\mathrm{span}\,(\chi_i)_{i\in I}$ is dense in $\mathcal{C}^0_c(\mathbb{R}^N)$. For $s\in(a,b)$ and $n\in\mathbb{N}$, let $\{\mathfrak{v}^s_n\}$ be a family of measures on $\mathbb{R}^N$ and set

$$f^i_n(s) = \int_{\mathbb{R}^N} \chi_i d\mathfrak{v}^s_n, \qquad \text{for } s\in(a,b),\ n\in\mathbb{N},\ i\in I. \tag{5.21}$$

Assume that, for some constant $C>0$

$$\|\mathfrak{v}^s_n\| \leq C \qquad \forall\, s\in(a,b),\ \forall\, n\in\mathbb{N}. \tag{5.22}$$

**Lemma 5.5.** *Assume that the family $(f^i_n)_{n\in\mathbb{N}}$ defined by (5.21) satisfies (5.20). Then there exists a subsequence $(\sigma(n))_{n\in\mathbb{N}}$ and a family of measures $\{\mathfrak{v}^s_*\}_{s\in(a,b)}$ such that*

$$\mathfrak{v}^s_{\sigma(n)} \rightharpoonup \mathfrak{v}^s_* \qquad \textit{weakly as measures, as } n\to+\infty, \textit{ for all } s\in(a,b).$$

*Proof.* In view of Lemma 5.4, there exists a subsequence $(\sigma(n))_{n\in\mathbb{N}}$ such that

$$\mathfrak{v}^s_{\sigma(n)}(\chi_i) \quad \text{converges, as } n\to+\infty, \quad \text{for } \textbf{every } s\in(a,b). \tag{5.23}$$

Next let $s_0\in(a,b)$ be arbitrary but fixed. Since (5.22) holds and in view of (5.23), since $\mathrm{span}\,\{\chi_i\}$ is dense, for $s=s_0$ the family $\{\mathfrak{v}^{s_0}_{\sigma(n)}(\chi)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, hence it converges. This determines the measure $\mathfrak{v}^{s_0}_*$ and establishes the convergence for $s=s_0$. Since $s_0$ was arbitrary, the conclusion follows. □

## 5.5 Pseudo-decreasing property

Recall that at this stage the convergence of $\mathfrak{v}^s_{\varepsilon_n}$ to a limiting measure $\mathfrak{v}^s_*$ has already been established for $s\in\mathcal{Z}=\gamma_0\mathbb{Q}^+$ (see Corollary 5.1 and Remark 5.2). In this section we show that, extracting possibly a subsequence, convergence holds for **all** $s\in\mathbb{R}^+$. The main ingredient is a pseudo-decreasing property.

For $s\in\mathcal{Z}$ consider the class

$$Y(s) = \{\chi\in\mathcal{C}^1_c(\mathbb{R}^2,\mathbb{R}^+)\,, \quad \mathrm{supp}\,\nabla\chi\subset\mathbb{R}^2\setminus\bigcup_{i=1}^{l(s)}\{a_i(s)\}\ \}.$$

We have

**Lemma 5.6.** *Let $s_0\in\mathcal{Z}$ and $\chi\in Y(s_0)$ be given. Set*

$$r = \sigma_1^{-1}\cdot\mathrm{dist}\,(\mathrm{supp}\,\nabla\chi, \cup_{i=1}^{l(s_0)}\{a_i(s_0)\}).$$

*Then*

$$\frac{d}{ds}\int_{\mathbb{R}^2}\chi d\mathfrak{v}^s_{\varepsilon_n} \leq C \qquad \textit{on } \ [s_0+\frac{1}{|\log\varepsilon_n|}, s_0+\gamma_0 r^2], \tag{5.24}$$

*where the constant $C$ depends only on $\chi$.*

*Proof.* It is an immediate consequence of inequality (2.7) of Lemma 2.3 combined with inequality (5.11) of Proposition 5.1, and the fact that $\nabla\chi$ vanishes on $\cup_{i=1}^{l(s_0)} B(a_i(s_0), \sigma_1 r)$. $\square$

**Remark 5.4.** Notice that (5.24) is only valid on an interval depending on $\chi$.

We introduce next the class, for $s \in \mathcal{Z}$,

$$Y_r(s) = \{\chi \in Y(s)\,, \quad \text{dist}\,(\text{supp}\,\nabla\chi, \{a_i(s)\}_{1 \le i \le l(s)}) \ge \sigma_1 r\}.$$

The main step in the proof of Theorem 4 is the following

**Proposition 5.2.** *There exists a* **fixed** *subsequence of $(\varepsilon_n)_{n\in\mathbb{N}}$ (still denoted $(\varepsilon_n)_{n\in\mathbb{N}}$) such that for* **any** *$s_0 \in \mathcal{Z}$,* **any** *$r > 0$,* **any** *$\chi \in Y_r(s_0)$ and* **every** *$s \in [s_0, s_0 + \gamma_0 r^2]$,*

$$\mathfrak{v}^s_{\varepsilon_n}(\chi) \textit{ converges as } n \to +\infty.$$

*Proof.* Let $s_0 \in \mathcal{Z} = \gamma_0\mathbb{Q}^+$ and $r \in \mathbb{Q}^+$ be given. Thanks to Lemma 5.6 we can apply Lemma 5.4 with $[a,b] \subset (s_0, s_0+\gamma_0 r^2]$ and $f^i_n(s) = \mathfrak{v}^s_{\varepsilon_n}(\chi_i)$, where $\{\chi_i\}_{i\in I}$ is a countable dense subset of $Y_r(s_0)$. It follows that, for a subsequence $(\tilde\sigma(n))_{n\in\mathbb{N}} \equiv (\sigma_{s_0,r}(n))_{n\in\mathbb{N}}$ **depending** on $s_0$ and $r$,

$$f^i_{\tilde\sigma(n)}(s) \quad \text{converges on } [a,b]\,.$$

Using a diagonal argument for $s_0 \in \mathcal{Z}$ and $r \in \mathbb{Q}^+$ we get rid of the dependence on $s_0$ and $r$, and the conclusion follows by density of the family $\{\chi_i\}_{i\in I}$ in $Y(s_0)$. $\square$

### 5.6 Proof of Theorem 4 completed.

We inverse the role of $s$ and $s_0$, i.e. let $s > 0$ be given and fixed (whereas $s_0$ will vary). Define

$$Z(s) = \{\chi \in \mathcal{C}^0_c(\mathbb{R}^2, \mathbb{R}^+)\,, \quad \text{supp}\,\nabla\chi \subset \mathbb{R}^2 \setminus \Sigma^s_{\mathfrak{v}}\}\,.$$

Recall that for any $s > 0$, $\Sigma^s_{\mathfrak{v}}$ is a finite set. Let $\chi \in Z(s)$ and set $r = \text{dist}\,(\text{supp}\,\nabla\chi, \Sigma^s_{\mathfrak{v}})$. Next, we are going to choose $s_0 \in \mathcal{Z}$ such that $s_0 < s$ and $0 < s - s_0 < \dfrac{\alpha r^2}{16}$, so that in particular, since $\alpha = \frac{\gamma_0}{\sigma_1^2}$, $s \in (s_0, s_0 + \gamma_0 r^2)$. **We claim that** $\chi \in Y_{r/2}(s_0)$. Indeed, by construction $\Sigma^s_{\mathfrak{v}} \subset \mathbb{R}^2 \times \{s\} \cap \bigcup_{i=1}^{l(s_0)} \mathcal{P}_\alpha(a_i(s_0), s_0)$, that is

$$\Sigma^s_{\mathfrak{v}} \subset \bigcup_{i=1}^{l(s_0)} B(a_i(s_0), \sqrt{\frac{s-s_0}{\alpha}}) \subset \bigcup_{i=1}^{l} B(a_i(s_0), \frac{r}{4}),$$

and **the claim follows**. We apply next Proposition 5.2 to $s_0$ and $\frac{r}{2}$ to deduce that $\mathfrak{v}^s_{\varepsilon_n}(\chi)$ converges as $n \to +\infty$. Since $\chi$ was arbitrary in $Z(s)$ and since $Z(s)$ is dense in $\mathcal{C}^0_c(\mathbb{R}^2, \mathbb{R}^+)$, it follows that $\mathfrak{v}^s_{\varepsilon_n}(\chi)$ converges for every $\chi \in \mathcal{C}^0_c(\mathbb{R}^2, \mathbb{R}^+)$, and the proof of Theorem 4 is completed. $\square$

### 5.7 Proof of Theorem 5 i).

In view of (5.11), we have, for every $s > 0$ and $K \subset \mathbb{R}^2 \times \mathbb{R}^+ \setminus \Sigma_{\mathfrak{v}}$,

$$\mathfrak{v}_*^s(K \cap \mathbb{R}^2 \times \{s\}) = 0 \tag{5.25}$$

It follows from the fact that $\Sigma_{\mathfrak{v}}$ is closed and (5.25), that for any compact set $\Omega \subset \mathbb{R}^2 \setminus \Sigma_{\mathfrak{v}}^s = \mathbb{R}^2 \setminus \cup_{i=1}^{l(s)} \{a_i(s)\}$,

$$\mathfrak{v}_*^s(\Omega) = 0,$$

so that

$$\nu_*^s = \sum_{i=1}^{l(s)} \theta_i(s) \delta_{a_i(s)}$$

for some positive numbers $\theta_i(s)$, so that the first statement in Theorem 5 i) is established. Concerning the second statement (i.e. (8)), it requires to define the degrees $d_i$, and this will be done in Section 6. Once the degrees are defined, inequality (8) follows immediately from standard lower energy bounds (see e.g. [19]). $\square$

# 6 Convergence results for $u_\varepsilon$ in the log time scale

In order to prove Theorem 1, we use the decomposition given by Proposition 4.1, i.e. we write $u_\varepsilon = w_\varepsilon \exp i\phi_\varepsilon$ so that

$$u_\varepsilon \times \nabla u_\varepsilon = w_\varepsilon \times \nabla w_\varepsilon + \rho_\varepsilon^2 \nabla \phi_\varepsilon. \tag{6.1}$$

We handle each of the terms on the r.h.s. of (6.1) separately.

Recall that from Proposition 4.1 $\phi_\varepsilon$ solves the heat equation. Moreover, applying iii) and Remark 4.1 b) with $q = 5$, we have, for $s > 0$ and $s|\log \varepsilon| > 1$,

$$|\nabla \phi_\varepsilon(\cdot, |\log \varepsilon| s)| \le C(M_0) \left( \frac{1}{\sqrt{s}} + \frac{1}{(s|\log \varepsilon|)^{1/5}} \right), \tag{6.2}$$

$$|D^2 \phi_\varepsilon(\cdot, |\log \varepsilon| s)| \le \frac{C(M_0)}{\sqrt{s|\log \varepsilon|}} \left( \frac{1}{\sqrt{s}} + \frac{1}{(s|\log \varepsilon|)^{1/5}} \right), \tag{6.3}$$

$$|\partial_s \nabla \phi_\varepsilon(\cdot, |\log \varepsilon| s)| \le \frac{C(M_0)}{s|\log \varepsilon|} \left( \frac{1}{\sqrt{s}} + \frac{1}{(s|\log \varepsilon|)^{1/5}} \right). \tag{6.4}$$

We deduce

**Proposition 6.1.** *Extracting possibly a subsequence, there exists a function $c : \mathbb{R}_*^+ \to \mathbb{R}^2$ such that*

$$\nabla \phi_\varepsilon(\cdot, s|\log \varepsilon|) \to c(s)$$

*on every compact subset $K$ of $\mathbb{R}^2 \times (0, +\infty)$. Moreover,*

$$|c(s)| \le \frac{C(M_0)}{\sqrt{s}} \qquad \forall\, s > 0. \tag{6.5}$$

The proof is a straightforward consequence of (6.2),(6.3),(6.4) and Ascoli-Arzelà Theorem. Next we turn to $w_\varepsilon \times \nabla w_\varepsilon$, and recall the decomposition given in (4.35)

$$w_\varepsilon \times \delta w_\varepsilon = \delta^* \psi + \delta \Phi_0 + \zeta, \tag{6.6}$$

where $\zeta = (1 - |u_\varepsilon|^2)\delta\phi_\varepsilon$, the 2-form $\psi$ is defined on $\mathbb{R}^2 \times \mathbb{R}^+$ by the elliptic problem

$$-\Delta_{x,t}\psi = J_{x,t}u_\varepsilon \qquad \text{on } \mathbb{R}^N \times \mathbb{R}, \tag{6.7}$$

and where the function $\Phi_0$ is defined by the parabolic problem

$$\begin{cases} \partial_t \Phi_0 - \Delta \Phi_0 = A + B & \text{on } \mathbb{R}^2 \times \mathbb{R} \\ \Phi_\varepsilon(x,0) = 0 & \text{for } x \in \mathbb{R}^2\,, \end{cases} \tag{6.8}$$

where $A = d^*(\delta^*\psi - P_t(\delta^*\psi)dt)$ and $B = -P_t(\delta^*\psi)$. We would like to emphasize the fact that $\zeta$ is a perturbation term, whereas the definition of $\Phi_0$ involves only $\psi$, and thus $J_{x,t}u_\varepsilon$. Therefore, the system of equations for $\psi$ and $\Phi_0$ has $J_{x,t}u_\varepsilon$ as source term. On the other hand, we know by Proposition 3.3 that $J_{x,t}$ is essentially time-independent (in the original time scale). We will show that $\Phi_0$ tends to zero as $\varepsilon$ goes to zero in suitable norms, whereas $\psi$ is essentially the solution of a static 2-dimensional problem. At this stage, we still work in the original time variable $t$ and let $t_\varepsilon > 0$ be given. We start our analysis with $\psi$.

## 6.1 Relaxation of $\nabla\psi$ to static fields

Let $R > 10$ be given. We apply Proposition 3.3 to the translated function $u_\varepsilon(\cdot, t_\varepsilon - \frac{R}{2})$, assuming $t_\varepsilon \geq 2R$. This yields $l$ points $x_i^\varepsilon$ in $\mathcal{V}_\varepsilon(t_\varepsilon)$ and integers $d_i \in \mathbb{Z}$ such that

$$\|J_{x,t}u_\varepsilon - \sum_{i=1}^{l} d_i \delta_{x_i^\varepsilon} dx_1 \wedge dx_2\|_{[\mathcal{C}^{0,\alpha}(B(0,R)\times[t_\varepsilon - \frac{R}{4}, t_\varepsilon + \frac{R}{4}])]^*} \leq C_\alpha(\varepsilon, M_0, R).^{24} \tag{6.9}$$

We compare $\psi$ with the solution $\tilde\psi$ of the problem

$$-\Delta_{x,t}\tilde\psi = \pi \sum_{i=1}^{l} d_i \delta_{x_i^\varepsilon} dx_1 \wedge dx_2 \qquad \text{on } \mathbb{R}^2 \times \mathbb{R},$$

explicitly given by

$$\tilde\psi(x,t) = -\sum_{i=1}^{l} d_i \log|x - x_i^\varepsilon|\, dx_1 \wedge dx_2,$$

which is independent of time $t$. Notice however that the definition of $\tilde\psi$ depends on the choice of $t_\varepsilon$. We have

**Lemma 6.1.** *Let $1 \leq p < \frac{3}{2}$. There exists constants $C_1(\varepsilon, p, R)$ depending only on $\varepsilon$, $p$ and $R$, and $C_2(M_0)$, depending only on $M_0$, such that for every compact set $\Omega \subset B(0, \frac{R}{4}) \times [t_\varepsilon - \frac{R}{8}, t_\varepsilon + \frac{R}{8}]$, we have*

$$\|\nabla_{x,t}(\psi - \tilde\psi)\|_{L^p(\Omega)} \leq C_1(\varepsilon, p, R) + \frac{C_2(M_0)}{R^{9/20}}|\Omega|^{1/p}.$$

*Moreover, for fixed $p$ and $R$, $C_1(\varepsilon, p, R) \to 0$ as $\varepsilon \to 0$.*

[24] Recall that $C_\alpha(\varepsilon, M_0, R) \to 0$ as $\varepsilon \to 0$.

To illustrate the way Lemma 6.1 induces relaxation, take for example $\Omega_\varepsilon = B(0,1)\times(t_\varepsilon - 1, t_\varepsilon+1)$ and let $\varepsilon\to 0$, $t_\varepsilon\to+\infty$ and $R=R_\varepsilon = t_\varepsilon/4$. Then

*Proof.* In view of estimate (4.5) we have

$$\|\nabla_{x,t}(\psi-\tilde\psi)\|_{L^{4/3}+L^{5/2}(\mathbb{R}^2\times[t_\varepsilon-\frac{R}{2},t_\varepsilon+\frac{R}{2}])}\le CR^{3/4},$$

and therefore, by averaging, there exists some $R_0\in[\frac{7}{8}R,R]$ such that

$$\|\nabla_{x,t}(\psi-\tilde\psi)\|_{L^{4/3}+L^{5/2}(\partial B(0,R_0)\times[t_\varepsilon-\frac{R_0}{2},t_\varepsilon+\frac{R_0}{2}])}\le CR^{3/4-2/5}=CR^{7/20}.$$

Next, we decompose $\psi-\tilde\psi$ as

$$\psi-\tilde\psi=\xi_1+\xi_2,$$

where $\xi_1$ is harmonic on $P_{R_0}\equiv B(0,R_0)\times[t_\varepsilon-\frac{R_0}{2},t_\varepsilon+\frac{R_0}{2}]$ and where $\xi_2$ solves

$$\begin{cases} -\Delta_{x,t}\xi_2 &= J_{x,t}u_\varepsilon-\pi\sum_{i=1}^{l}\delta_{x_i^\varepsilon}dx_1\wedge dx_2 \quad \text{in } P_{R_0}\\ \xi_2 &=0 \quad \text{on } \partial P_{R_0}\end{cases}\tag{6.10}$$

By standard elliptic estimates and a straightforward scaling argument, we have

$$\|\nabla_{x,t}\xi_1\|_{L^\infty(B(0,\frac{R}{4})\times[t_\varepsilon-\frac{R}{8},t_\varepsilon+\frac{R}{8}])}\le CR^{-9/20}.$$

On the other hand, if $p<\frac{3}{2}$ there exists $0<\alpha<1$ such that $[\mathcal{C}^{0,\alpha}]^*\hookrightarrow W^{-1,p}$, so that, in view of (6.9) and standard elliptic estimates once more,

$$\|\nabla_{x,t}\xi_2\|_{L^p(B(0,\frac{R}{4})\times[t_\varepsilon-\frac{R}{8},t_\varepsilon+\frac{R}{8}])}\le C(\varepsilon,p,R),$$

where $C(\varepsilon,p,R)\to 0$ as $\varepsilon\to0$ by Proposition 3.3, and the proof is complete. $\square$

**Corollary 6.1.** *Let $1\le p<3/2$ and let $K\subset\mathbb{R}^2$ be a fixed compact set. For a given $\delta>0$ there exists $\varepsilon(\delta,K)>0$ and $T(\delta,K)>0$ such that, if $\varepsilon<\varepsilon(\delta,K)$ and $t_\varepsilon>T(\delta,K)$, then*

$$\|\nabla_{x,t}(\psi-\tilde\psi)\|_{L^p(K\times[t_\varepsilon-1,t_\varepsilon+1])}\le\delta.\tag{6.11}$$

## 6.2 Vanishing of $\nabla\Phi_0$

**Lemma 6.2.** *Let $K\subset\mathbb{R}^2$ be a fixed compact set. For a given $\delta>0$ there exists $\varepsilon(\delta,K)>0$ and $T(\delta,K)>0$ such that, if $\varepsilon<\varepsilon(\delta,K)$ and $t_\varepsilon>T(\delta,K)$, then*

$$\|\nabla\Phi_0\|_{L^{4/3}(K\times[t_\varepsilon-1,t_\varepsilon+1])}\le\delta.\tag{6.12}$$

*Proof.* We begin with the observation that, since $\tilde\psi$ is independent of time, we have, for the r.h.s. of (6.8),

$$A=d^*\tilde h=d^*(\delta^*(\psi-\tilde\psi)-P_t(\delta^*(\psi-\tilde\psi)dt))\qquad\text{and}\qquad B=P_t(\delta^*(\psi-\tilde\psi)),$$

so that we may take advantage of the smallness of $\psi-\tilde\psi$ derived in the previous paragraph.

We also recall the estimates obtained so far for $\varPhi_0$, namely $|\nabla \varPhi_0| = g_1 + g_2$, with[25]

$$\sup_{t\in\mathbb{R}^+} \|g_1\|_{(L^{4/3}+L^5)(\mathbb{R}^2\times[t,t+1])} \leq CM_0, \qquad \sup_{t\in\mathbb{R}^+} \|g_2\|_{L^{10/7}([t,t+1],L^5(\mathbb{R}^2))} \leq CM_0.$$

Let $R > 100$ and $1 \leq L \leq \sqrt{R}/4$ to be determined later, and let $t_\varepsilon > 2R$. By averaging, there exists some $t_0 \in [t_\varepsilon - (L^2+1), t_\varepsilon - L^2]$ such that

$$\|\nabla \varPhi_0\|_{(L^{4/3}+L^5)(\mathbb{R}^2\times\{t_0\})} \leq CM_0.$$

On $\mathbb{R}^2 \times [t_0, +\infty)$ we decompose $\varPhi_0$ as

$$\varPhi_0 = \varPhi_0^1 + \varPhi_0^2 + \varPhi_0^3, \tag{6.13}$$

where $\varPhi_0^1$ satisfies

$$\begin{cases} \partial_t \varPhi_0^1 - \Delta \varPhi_0^1 = 0 & \text{on } \mathbb{R}^2 \times [t_0, +\infty) \\ \varPhi_0^1(x, t_0) = \varPhi_0 & \text{for } x \in \mathbb{R}^2\,, \end{cases}$$

and $\varPhi_0^2$ satisfies

$$\begin{cases} \partial_t \varPhi_0^2 - \Delta \varPhi_0^2 = \tilde{A} + \tilde{B} & \text{on } \mathbb{R}^2 \times \mathbb{R} \\ \varPhi_0^2(x, t_0) = 0 & \text{for } x \in \mathbb{R}^2\,, \end{cases}$$

where $\tilde{A} = d^*(\chi\tilde{h})$, $\tilde{B} = \chi B$, with $0 \leq \chi \leq 1$ is a cut-off function on $\mathbb{R}^2$ such that $\chi \equiv 1$ on $B(0, L)$, $\chi \equiv 0$ on $\mathbb{R}^2 \setminus B(0, 2L)$, and $|\nabla\chi| \leq 2$. In view of estimate (A.22) we obtain, for every $t \geq t_\varepsilon - 1$,

$$\|\nabla \varPhi_0^1\|_{L^\infty(\mathbb{R}^2\times\{t\})} \leq \frac{C}{L^{2/5}}. \tag{6.14}$$

For $\varPhi_0^2$ we estimate $\tilde{A}$ using Lemma 6.1 with $p = 4/3$ and $\Omega = B(0, 2L) \times [t_0, t_\varepsilon]$. This yields

$$\|\nabla_{x,t}(\psi - \tilde{\psi})\|_{L^{4/3}(\Omega)} \leq C_1(\varepsilon, R) + C_2(M_0)\frac{L^3}{R^{9/20}}.$$

It follows from standard parabolic theory that there exists a constant $C_3(L)$ such that

$$\begin{aligned} \|\nabla \varPhi_0^2\|_{(L^{4/3}+L^4)(\mathbb{R}^2\times[t_\varepsilon-1,t_\varepsilon+1])} &\leq C_3(L)\|\nabla_{x,t}(\psi - \tilde{\psi})\|_{L^{4/3}(\Omega)} \\ &\leq C_3(L)[C_1(\varepsilon, R) + C_2(M_0)\frac{L^3}{R^{9/20}}]. \end{aligned} \tag{6.15}$$

where we have set $C_1(\varepsilon, R) \equiv C_1(\varepsilon, 4/3, R)$ (for $C_1$ given in Lemma 6.1). Finally, we turn to $\varPhi_0^3$. Arguing as in the proof of (4.24) and (4.25), we have $|\nabla \varPhi_0^3| = g_{1,3} + g_{2,3}$, with

$$\sup_{t\in\mathbb{R}^+} \|g_{1,3}\|_{(L^{4/3}+L^5)(\mathbb{R}^2\times[t,t+1])} \leq CM_0, \qquad \text{and} \qquad \sup_{t\in\mathbb{R}^+} \|g_{2,3}\|_{L^{10/7}([t,t+1],L^5(\mathbb{R}^2))} \leq CM_0.$$

On the other hand, $\varPhi_0^3$ satisfies the homogeneous heat equation on $B(0, L) \times [t_0, t_\varepsilon]$. It follows from standard heat equations, after scaling[26] and a few computations, that

$$\|\nabla \varPhi_0^3\|_{L^\infty(B(0,\frac{L}{2})\times[t_\varepsilon-\frac{L^2}{4},t_\varepsilon+1])} \leq \frac{C(M_0)}{L^{2/5}}. \tag{6.16}$$

[25] We specify estimates (4.24) with $r = 4/3$, $q = 5$ and (4.25) with $p^* = 5$, i.e. $p = 10/7$.

[26] Introduce the function $\tilde{\varPhi}_0^3(x, t) = \varPhi_0^3(x \cdot L, (t - t_0) \cdot L^2)$, which verifies the heat equation on $B(0, 1) \times [0, 1]$.

We collect the estimates for $\Phi_0^i$ $(i = 1, 2, 3)$ given in (6.13), (6.14), (6.16) to assert that for any compact set $K \subset B(0, L)$ we have

$$\|\nabla\Phi_0\|_{L^{4/3}(K\times[t_\varepsilon-1,t_\varepsilon+1])} \le C(K)[\frac{1}{L^{1/5}} + C_3(L)\cdot(C_1(\varepsilon, R) + C_2(M_0)\frac{L^3}{R^{9/20}})]. \tag{6.17}$$

In order to establish the vanishing of $\nabla\Phi_0$ we specify the values of $L$ and $R$. We first choose $L$ sufficiently large such that $K \subset B(0, L)$ and

$$\frac{C(K)}{L^{2/5}} \le \frac{\delta}{3}.$$

Next, determine $R$ so that

$$\frac{C(K)C_2(M_0)C_3(L)L^3}{R^{9/20}} \le \frac{\delta}{3}.$$

Finally, we invoke the fact (see Lemma 6.1) that $C_1(\varepsilon, R)$ tends to zero as $\varepsilon$ tends to zero, to derive (6.12). $\square$

### 6.3 Convergence of $u_\varepsilon \times \nabla u_\varepsilon$ to static fields

We express the results of sections 6.1 and 6.2 and in the log time scale. This straightforwardly yields

**Proposition 6.2.** *Let $\delta > 0$ and $s_0 > 0$ be given and let $K \subset \mathbb{R}^2$ be any compact subset. For every $s \ge s_0$ there exists $l$ points $x_i^\varepsilon(s)$ in $\mathcal{V}_\varepsilon(s|\log\varepsilon|)$, $l$ integers $d_i^\varepsilon(s)$ depending only on $s$ and a constant $\varepsilon(\delta, s_0, |K|) > 0$ depending only on $\delta$, $s_0$ and $|K|$ such that, if $\varepsilon < \varepsilon(\delta, s_0, |K|)$,*

$$\|u_\varepsilon \times \nabla u_\varepsilon - \nabla^\perp(-\sum_{i=1}^{l} d_i^\varepsilon(s)\log|x - x_i^\varepsilon(s)|) - c_\varepsilon(s)\|_{L^{4/3}(K\times[s|\log\varepsilon|-1,s|\log\varepsilon|+1])} \le \delta, \tag{6.18}$$

*where $c_\varepsilon : \mathbb{R}^+ \to \mathbb{R}^2$ is a function verifying*

$$|c_\varepsilon(s)| \le \frac{C(M_0)}{\sqrt{s}}. \tag{6.19}$$

Whereas estimate (6.18) provides an estimate in a weak norm but holds for arbitrary sets $K$, even those containing the concentration sets, better estimates can be deduced from (6.18) provided $K$ is far from the concentration sets. In this direction, we have, as a direct consequence of Lemma 3.5 and Theorem 2.1

**Proposition 6.3.** *Let $r > 0$, $\delta > 0$ and $s_0$ be given. Let $s \ge s_0$, $\varepsilon_1 > 0$ and let $K \subset \mathbb{R}^2$ be a compact set such that, if $\varepsilon \le \varepsilon_1$,*

$$dist(K, \Omega^\varepsilon(s|\log\varepsilon| - 2)) \ge 4r.$$

*There exists a constant $\varepsilon_0 \le \varepsilon_1$ depending only on $r$, $\delta$, $K$ and $s_0$ such that, for $\varepsilon < \varepsilon_0$,*

$$\|u_\varepsilon \times \nabla u_\varepsilon - \nabla^\perp(-\sum_{i=1}^{l} d_i^\varepsilon(s)\log|x - x_i^\varepsilon(s)|) - c_\varepsilon(s)\|_{\mathcal{C}^1(K\times\{s|\log\varepsilon|\})} \le \delta. \tag{6.20}$$

*Moreover, there exists $\tau_\varepsilon \in [0, 2\pi]$ such that*

$$\left\| w_\varepsilon - \exp(i\tau_\varepsilon)\prod_{i=1}^{l}\left(\frac{x - x_i^\varepsilon(s)}{|x - x_i^\varepsilon(s)|}\right)^{d_i(s)}\right\|_{\mathcal{C}^1(K\times\{s|\log\varepsilon|\})} \le \delta. \tag{6.21}$$

**Remark 6.1.** In estimates (6.20) and (6.21) one may replace the slice $t = s|\log\varepsilon|$ by $[s|\log\varepsilon| - 1, s|\log\varepsilon| + 1]$.

### 6.4 Proof of Theorem 1 completed

Formula (6.20) offers already a strong rigidity of possible behavior for $u_\varepsilon \times \nabla u_\varepsilon$. Indeed, it reduces the problem, for fixed time $s$, to finite dimensional objects, namely the points $x_i^\varepsilon(s)$ and the degrees $d_i^\varepsilon(s)$.[27] As for the construction in Section 5, the main point is to find a fixed subsequence for which convergence holds at all positive times. We developed in full details an argument for $\mu_\varepsilon$ in Section 5. Our argument here for $x_i^\varepsilon$ and $d_i^\varepsilon$ is somewhat parallel. Therefore we omit the details and point out the main adaptations.

First, whereas a semi-decreasing property was used in Section 5, here we invoke instead the fact that the topological degrees $d_i^\varepsilon(s)$ are constant on each of the pieces of the chains of cylinders. Second, concerning the points $x_i^\varepsilon$, by construction they are confined in the vorticity set, and hence in the concentration set $\Sigma_r^{\varepsilon_n}$ of Lemma 5.3, whose limit is precisely $\Sigma_\mu$. Once the fixed subsequence is determined, the conclusion is an immediate consequence of (6.20).

## 7 Computation of the interaction terms

In this section, we take advantage of the compactness and rigidity results of the previous section to derive explicit expansions of the various interaction terms, as functions of the points $a_i(s)$ and their degrees $d_i(s)$. To that aim, we restrict our attention here to test functions $\chi \in \mathcal{D}(\mathbb{R}^2)$ verifying the following assumption, for some $r > 0$,

$$\frac{\partial\chi}{\partial\bar{z}^2} = 0 \quad \text{on } \cup_{i=1}^{l(s)} B(a_i(s), r/8). \tag{$\mathrm{H}_r(s)$}$$

### 7.1 Refined estimates for the self-interaction term $\mathcal{F}_S$

In the log-time scale, we write the self-interaction term as

$$\mathfrak{F}_S(s,\chi,w_\varepsilon) = \mathcal{F}_S(s|\log\varepsilon|,\chi,w_\varepsilon) = \mathcal{A}_S(s,\chi,w_\varepsilon) - \int_{\mathbb{R}^2\times\{s|\log\varepsilon|\}} \Delta\chi V_\varepsilon(w_\varepsilon)$$

where we have set, for a complex valued function $w$

$$\mathcal{A}_S(s,\chi,w) = \int_{\mathbb{R}^2\times\{s|\log\varepsilon|\}} D^2\chi\nabla w\,\nabla w - \Delta\chi\frac{|\nabla w|^2}{2}.$$

We have

**Proposition 7.1.** *Let $s_0 > 0$, $r > 0$ and $\delta > 0$ be given. For some $s > s_0$ assume that $\chi \in \mathcal{D}(\mathbb{R}^2)$ verifies $H_r(s)$. There exists $\varepsilon_0 > 0$ depending only on $\delta$, $s_0$ and $\chi$ such that, for $0 < \varepsilon < \varepsilon_0$,*

$$\left|\mathcal{A}_S(s',\chi,w_\varepsilon) - \frac{\pi}{2}\sum_{i=1}^{l(s)} d_i^2(s)\Delta\chi(a_i(s)) - 4\mathrm{Re}\sum_{k<l}\tfrac{d_k(s)d_l(s)}{(a_l(s)-a_k(s))}\left(\tfrac{\partial\chi}{\partial\bar{z}}(a_k(s)) - \tfrac{\partial\chi}{\partial\bar{z}}(a_l(s))\right)\right| \le \delta, \tag{7.1}$$

[27] From subsection 6.1 we already know that the function $c_\varepsilon$ converges on $\mathbb{R}^+$ to a function $c$, extracting possibly a subsequence.

*for every* $s' \in (s - \frac{1}{|\log\varepsilon|}, s + \frac{1}{|\log\varepsilon|})$.

The proof of Proposition 7.1 is based on the asymptotics of $w_\varepsilon \times \nabla w_\varepsilon$ and of some properties of merely algebraic nature of $\mathcal{A}_S$. First we have

**Lemma 7.1.** *With the same assumptions as in Proposition 7.1, there exists* $\varepsilon_0$ *depending only on* $s_0$, $\delta$ *and* $\chi$, *such that for* $0 < \varepsilon < \varepsilon_0$,

$$\left| \int_{s|\log\varepsilon|-1}^{s|\log\varepsilon|+1} \mathcal{A}_S(\frac{t}{|\log\varepsilon|}, \chi, w_\varepsilon)\, dt - 4\,\mathrm{Re}\left( \int_{\mathrm{supp}\chi \setminus \cup_{i=1}^{l(s)} B(a_i(s), r/8)) \times \{s|\log\varepsilon|\}} \omega(w_*) \frac{\partial^2 \chi}{\partial \bar{z}^2} \right) \right| \leq \delta,$$

*where*

$$w_*(z) = \prod_{i=1}^{l(s)} \left( \frac{z - a_i(s)}{|z - a_i(s)|} \right)^{d_i(s)} \qquad \text{on } \mathbb{R}^2.$$

*Proof.* In view of (2.6), we have for $X = 2\frac{\partial \chi}{\partial \bar{z}}$

$$\mathcal{A}_S(s, \chi, w_\varepsilon) = 2\,\mathrm{Re}\left( \int_{\mathbb{R}^2 \times \{s|\log\varepsilon|\}} \omega(w_\varepsilon) \frac{\partial^2 \chi}{\partial \bar{z}^2} \right).$$

Note that on $B(a_i(s), r/8)$, $\frac{\partial^2 \chi}{\partial \bar{z}^2} = 0$ by assumption $H_r(s)$. On the other hand, from (6.21) we infer that on $\left(\mathrm{supp}\chi \setminus \cup_{i=1}^{l(s)} B(a_i(s), r/8)\right) \times [s|\log\varepsilon| - 1, s|\log\varepsilon| + 1]$

$$\omega(w_\varepsilon) \to \omega(w_*) \qquad \text{uniformly.}^{28}$$

The conclusion follows. □

**Proof of Proposition 7.1 completed.** In view of Lemma 7.1 it suffices to establish the formula

$$\begin{aligned} \int_{\left(\mathrm{supp}\chi \setminus \cup_{i=1}^{l(s)} B(a_i(s), r/8)\right) \times \{s|\log\varepsilon|\}} \omega(w_\varepsilon) \frac{\partial^2 \chi}{\partial \bar{z}^2} &= \pi \sum_{i=1}^{l(s)} d_i^2(s) \Delta\chi(a_i(s)) \\ &- 2 \sum_{k<l} \frac{d_k(s) d_l(s)}{(a_l(s) - a_k(s))} \left( \frac{\partial \chi}{\partial \bar{z}}(a_k(s)) - \frac{\partial \chi}{\partial \bar{z}}(a_l(s)) \right). \end{aligned} \tag{7.2}$$

Notice that $\omega(w_*)$ is not locally integrable, but however it defines a distribution in view of the formula[29]

$$\omega(w_*) = -\Big( \sum_{i=1}^{l(s)} \frac{d_i(s)}{z - a_i(s)} \Big)^2. \tag{7.3}$$

On the other hand, by assumption $H_r(s)$, $\frac{\partial^2 \chi}{\partial \bar{z}^2} = 0$ on $\cup_{i=1}^{l(s)} B(a_i(s), r/8)$ and therefore we obtain[30]

$$\int_{\mathbb{R}^2 \setminus B(a, \frac{r}{8}) \times \{s|\log\varepsilon|\}} \omega(w_*) \frac{\partial^2 \chi}{\partial \bar{z}^2} = \Big\langle \omega(w_*), \frac{\partial^2 \chi}{\partial \bar{z}^2} \Big\rangle_{\mathcal{D}', \mathcal{D}}. \tag{7.4}$$

[28] The domain here is not fixed, but identified modulo time translation to a fixed domain $K \times [-1, 1]$. Convergence here and in the sequel is meant in this last domain.

[29] see e.g. [4], chapter VIII.

[30] this is a standard exercise in distribution theory.

The expansion of (7.3) yields for (7.4)

$$\Big\langle \omega(w_*), \frac{\partial^2\chi}{\partial\bar z^2}\Big\rangle = -\sum_{k=1}^{l(s)}\Big\langle \frac{d_k^2(s)}{(z-a_k(s)^2)}, \frac{\partial^2\chi}{\partial\bar z^2}\Big\rangle - 2\sum_{k<l}\frac{d_kd_l}{a_l-a_k}\Big\langle \frac{1}{z-a_k}-\frac{1}{z-a_l}, \frac{\partial^2\chi}{\partial\bar z^2}\Big\rangle.$$

For the first terms, we integrate by parts

$$\begin{aligned}-\Big\langle \frac{1}{(z-a_k(s))^2}, \frac{\partial^2\chi}{\partial\bar z^2}\Big\rangle &= -\Big\langle \frac{1}{z-a_k}, \frac{\partial^3\chi}{\partial z\partial\bar z^2}\Big\rangle = \Big\langle \frac{\partial}{\partial\bar z}\Big(\frac{1}{z-a_k}\Big), \frac{\partial^2\chi}{\partial z\partial\bar z}\Big\rangle \\ &= \pi\Big\langle \delta_{a_k}, \frac{\partial^2\chi}{\partial z\partial\bar z}\Big\rangle = \pi\frac{\partial^2\chi}{\partial z\partial\bar z}(a_k) = \frac{\pi}{4}\Delta\chi(a_k).\end{aligned}$$

For the second terms, we obtain similarly

$$-\Big\langle \frac{1}{z-a_k}, \frac{\partial^2\chi}{\partial\bar z^2}\Big\rangle = \Big\langle \frac{\partial}{\partial\bar z}\left(\frac{1}{z-a_k}\right), \frac{\partial\chi}{\partial\bar z}\Big\rangle = \pi\frac{\partial\chi}{\partial\bar z}(a_k).$$

The conclusion (7.2) follows by summation. $\square$

## 7.2 Refined estimates for $\mathcal{F}_I$

Recall that $\mathcal{F}_I = \mathcal{F}_J + \mathcal{R}_I$, where $\mathcal{F}_J$ is given by (2.18) and $\mathcal{R}_I$ by (2.15). Concerning $\mathcal{F}_I$ we have, setting $\mathfrak{F}_J(s,\chi,\nabla\phi_\varepsilon,w_\varepsilon) = \mathcal{F}_J(s|\log\varepsilon|,\chi,\nabla\phi_\varepsilon,w_\varepsilon)$.

**Proposition 7.2.** *Let $s_0>0$, $r>0$ and $\delta>0$. For some $s>s_0$ assume that $\chi\in\mathcal{D}(\mathbb{R}^2)$ verifies $H_r(s)$. There exists $\varepsilon_0>0$ depending only on $\delta$, $s_0$ and $\chi$ such that, for $0<\varepsilon<\varepsilon_0$,*

$$\left|\mathfrak{F}_J(s,\chi,\nabla\phi_\varepsilon,w_\varepsilon) - \pi\sum_{i=1}^{l(s)} d_i(s)c(s)\times\nabla\chi(a_i(s))\right| \le \delta. \tag{7.5}$$

*Proof.* Recall that

$$\mathfrak{F}_J(s,\chi,\nabla\phi_\varepsilon,w_\varepsilon) = \int_{\mathbb{R}^2\times\{s|\log\varepsilon|\}} (\nabla\phi_\varepsilon\times\nabla\chi)Jw_\varepsilon.$$

The conclusion then follows from the convergence of $Jw_\varepsilon$ to $\pi\sum_{i=1}^{l(s)} d_i\delta_{a_i}$ in $(\mathcal{C}^1(\mathrm{supp}\chi))^*$ and the convergence of $\nabla\phi_\varepsilon$ to $c$ in $\mathcal{C}^1(\mathrm{supp}\chi)$. $\square$

We next show that $\mathcal{R}_I$ is of lower order.

**Proposition 7.3.** *Let $s_0>0$, $r>0$, $\delta>0$ and let $\chi\in\mathcal{D}(\mathbb{R}^2)$. There exists $\varepsilon_0>0$ depending only on $\delta$, $s_0$ and $\chi$ such that, for $0<\varepsilon<\varepsilon_0$,*

$$\left|\int_{s|\log\varepsilon|}^{s|\log\varepsilon|+1} \mathcal{R}_I(t,\chi,\nabla\phi_\varepsilon,w_\varepsilon)\,dt\right| \le \frac{\delta}{r}. \tag{7.6}$$

*Proof.* Recall that

$$\mathfrak{R}_I(s,\chi,\nabla\phi_\varepsilon,w_\varepsilon) = \int_{\mathbb{R}^2\times\{s|\log\varepsilon|\}} -\Delta\phi_\varepsilon\nabla\chi\cdot(w_\varepsilon\times\nabla w_\varepsilon) + \nabla\phi_\varepsilon\cdot\nabla\chi\mathrm{div}(w_\varepsilon\times\nabla w_\varepsilon). \tag{7.7}$$

For the first integrated term on the r.h.s. of (7.7) we invoke Proposition 4.1 iii) (for $k = N = 2$) and iv) to assert that

$$\left| \int_{\mathbb{R}^2 \times \{s|\log \varepsilon|\}} \Delta \phi_\varepsilon \nabla \chi \cdot (w_\varepsilon \times \nabla w_\varepsilon) \right| \leq \frac{C}{s\sqrt{|\log \varepsilon|} r}. \tag{7.8}$$

For the second term, we invoke the convergence (6.18) (see also remark 6.1)

$$w_\varepsilon \times \nabla w_\varepsilon \to \nabla^\perp \left( \sum_{i=1}^{l(s)} d_i(s) \log |x - a_i(s)| \right) \qquad \text{in } L^{4/3}_{\text{loc}}(\mathbb{R}^2 \times [s|\log \varepsilon|, s|\log \varepsilon| + 1])$$

so that

$$\operatorname{div}(w_\varepsilon \times \nabla w_\varepsilon) \to 0 \qquad \text{in } W^{-1,4/3}_{\text{loc}}(\mathbb{R}^2 \times [s|\log \varepsilon|, s|\log \varepsilon| + 1])$$

and the conclusion follows by Proposition 4.1 iii) once more. □

**Remark 7.1.** If we assume moreover that the test function $\chi$ satisfies

$$\operatorname{dist}(\operatorname{supp} \nabla \chi, \cup_{i=1}^{l}(s)\{a_i(s)\}) \geq r > 0,$$

then the integrated estimate (7.6) may be replaced by

$$|\mathcal{R}_I(s|\log \varepsilon|, \chi, \nabla \phi_\varepsilon, w_\varepsilon)| \leq \frac{\delta}{r}. \tag{7.9}$$

In the next two sections, we deduce consequences of the previous estimates with suitable choices of test functions $\chi$.

# 8 Proof of Theorem 5 ii)

Let $s_0 > 0$. Throughout this section, we choose the non-negative test function $\chi$ such that

$$\operatorname{dist}(\operatorname{supp} \nabla \chi, \cup_{i=1}^{l(s_0)} \{a_i(s_0)\}) \equiv 8r > 0. \tag{8.1}$$

In particular $\chi$ is constant on a neighborhood of the points $a_i(s_0)$, and assumption $H_r(s_0)$ is therefore satisfied.

The main point in the proof of Theorem 5 ii) is

**Proposition 8.1.** *Let $s_0 > 0$ be given, and assume that $\chi \in \mathcal{D}(\mathbb{R}^2)$ satisfies* (8.1). *There exists $\mu_0$ depending only on $s_0$, $M_0$, $\chi$ such that for $\delta > 0$ there exists $\varepsilon_0$ such that for $0 < \varepsilon < \varepsilon_0$,*

$$\frac{d}{ds} \int_{\mathbb{R}^2} \chi \, d\mathfrak{v}_\varepsilon^s \leq \delta$$

*for every $s \in (s_0, s_0 + \mu_0 r^2)$.*

*Proof.* Invoking formula (2.21)

$$\frac{d}{ds} \int_{\mathbb{R}^2} \chi \, d\mathfrak{v}_\varepsilon^s \leq \mathfrak{F}_S(s, \chi, w_\varepsilon) + \mathfrak{F}_S(s, \chi, \phi_\varepsilon) + \mathfrak{F}_I(s, \chi, \nabla \phi_\varepsilon, w_\varepsilon) + \mathfrak{L}_0(s, |u_\varepsilon|, \chi, \phi_\varepsilon).$$

For the first term on the right hand side, we invoke Proposition 7.1. For the third term, we invoke likewise Proposition 7.2 and Remark 7.1. $\mathfrak{L}_0 = A_5$ is clearly a perturbation term and can be shown to be arbitrarily small as in Step 3 of Lemma 4.4.

Finally, we turn to

$$\mathfrak{F}_S(s,\chi,\phi_\varepsilon) = \int_{\mathbb{R}^2\times\{s|\log\varepsilon|\}} D^2\chi\nabla\phi_\varepsilon\nabla\phi_\varepsilon - \Delta\chi\frac{|\nabla\phi_\varepsilon|^2}{2}.$$

Since $\nabla\phi_\varepsilon(s|\log\varepsilon|)$ converges in $\mathcal{C}^1$ to the function $c(s)$ which is constant in $x$, it follows that $\mathfrak{F}_S(s,\chi,\phi_\varepsilon)$ converges to

$$\left(\int_{\mathbb{R}^2} D^2\chi\right) c(s)c(s) - \left(\int_{\mathbb{R}^2}\Delta\chi\right)\frac{|c(s)|^2}{2} = 0.$$

□

It follows from Proposition 8.1, passing to the limit $\varepsilon_n \to 0$, that

$$\frac{d}{ds}\int_{\mathbb{R}^2}\chi\, d\mathfrak{v}_*^s \leq 0$$

(in the sense of distribution), and the proof of Theorem 5 ii) is completed.

# 9 Degree zero and collisions

The main focus of this section is to provide the proof of Theorem 3, Theorem 5 and Proposition 1. The starting point is once more the evolution equation for the energy : however here it will be used to derive estimates for the potential $V_\varepsilon(u_\varepsilon)$. In particular, its integral will be shown to be small on a vortex patch of total degree zero. Therefore, we use again the remarkable properties of the function $|x|^2 = z\bar{z}$ and specify throughout the choice of test function $\chi$ as follows. Let $a \in \mathbb{R}^2$, and $r > 0$ be given. We set

$$\chi_{a,r}(x) = \Lambda(\frac{x-a}{r}) \tag{9.1}$$

where $\Lambda$ is defined by (4.51) and modeled on $|x|^2$. Let $s > 0$. We say that $H_{a,r}(s)$ is satisfied if and only if, for every $i \in \{1,\cdots,l(s)\}$

$$(\mathrm{H}_{a,r}(s)) \qquad \text{either } \mathrm{dist}(a_i(s),a) \leq \frac{r}{8} \qquad \text{or} \qquad \mathrm{dist}(a_i(s),a) \geq r.$$

If $a$ and $r$ satisfy $H_{a,r}(s)$, we set

$$I = \{i\in\{1,\cdots,l(s)\},\ \text{s.t. } a_i(s)\in B(a,r/8)\} \qquad J = \{1,\cdots,l(s)\}\setminus I.$$

We also define

$$d(a,s,r) = \sum_{i\in I} d_i(s).$$

Notice that if $H_{a,r}(s)$ is met, then $\chi_{a,r}$ satisfies $H_r(s)$. In particular, Proposition 7.1 and 7.2 may be specified as follows :

**Lemma 9.1.** *Let $s_0 > 0$, $r > 0$ and $\delta > 0$ be given. There exists $\varepsilon_0$ depending only on $\delta$, $s_0$ and $r$ such that if $\varepsilon \le \varepsilon_0$, $s > s_0$ and $H_{a,r}(s)$ is satisfied, then*

$$\left|\mathcal{A}_S(s,\chi_{a,r},w_\varepsilon) - \frac{16\pi}{r^2}\left(d^2(a,s,r) + 2\sum_{i\in I,\ j\in J} d_i(s)d_j(s)\mathrm{Re}\frac{a_i(s)-a}{a_i(s)-a_j(s)}\right)\right| \le \delta.$$

*and*

$$\left|\mathfrak{F}_J(s,\chi_{a,r},\nabla\phi_\varepsilon,w_\varepsilon) - \frac{16\pi}{r^2}\sum_{i\in I} d_i(s)(a_i(s)-a)\times c(s)\right| \le \delta.$$

*Proof.* Notice that if $k \in I$, $\Delta\chi(a_k(s)) = \frac{32}{r^2}$, whereas if $k \in J$, $\frac{\partial^2\chi_{a,r}}{\partial z\partial\bar z}(a_k(s)) = 0$. Notice also that if $k \in I$, $\frac{\partial\chi_{a,r}}{\partial\bar z}(a_k(s)) = \frac{8}{r^2}(a_k(s)-a)$, whereas if $k \in J$, $\frac{\partial\chi_{a,r}}{\partial\bar z}(a_k(s)) = 0$. It suffices then to substitute these expressions in (7.1) and (7.5). ☐

### 9.1 Estimates for the potential $V_\varepsilon(u_\varepsilon)$

Combining the evolution of localized energies with the refined estimates of the previous subsection, we are led to

**Proposition 9.1.** *Let $s_0 > 0$, $a \in \mathbb{R}^2$, $r > 0$ be such that $H_{a,r}(s_0)$ holds. Let $\delta > 0$ and $\kappa \le 1/16$ be given, and assume the stronger confinement assumption*

$$|a - a_i(s_0)| \le \kappa r, \qquad \text{for every } i \in I. \tag{9.2}$$

*Then, there exists a constant $\varepsilon_0 > 0$ depending only on $\delta$, $r$, $\kappa$ and $s_0$, and constants $C_1, C_2 > 0$ depending only on $M_0$ such that for every $2\kappa \le \mu \le \frac{1}{8}$ and $0 < \varepsilon < \varepsilon_0$ we have*

$$\left|\frac{r^2}{32\Delta t}\left[\int_{\Lambda(\mu)}\chi_{a,r}(x)|\partial_t w_\varepsilon|^2 + \int_{\partial^+\Lambda(\mu)}\chi_{a,r}e_\varepsilon(w_\varepsilon)\right] + \frac{1}{\Delta t}\int_{\Lambda(\mu)} V_\varepsilon(u_\varepsilon) - \frac{\pi}{2}d^2(a,s,r)\right|$$
$$\le C_1\left((\frac{\kappa}{\mu})^2 + \mu + \delta r^2\right), \tag{9.3}$$

*where $\Lambda(\mu) = B(a,r/8)\times[s_0|\log\varepsilon|, s_0|\log\varepsilon| + \Delta t]$, $\partial^+\Lambda(\mu) = B(a,r/8)\times\{s_0|\log\varepsilon| + \Delta t\}$, and $\Delta t = C_2\mu^2 r^2|\log\varepsilon|$.*

*Proof.* The starting point is Lemma 2.6 specified with the choice $\chi = \chi_{a,r}$ given in (9.1). This yields, setting $t_1 = s_0|\log\varepsilon|$, $t_2 = (s_0 + C\mu^2 r^2)|\log\varepsilon|$, and after integration,

$$\int\limits_{\mathbb{R}^2\times\{t_2\}}\chi e_\varepsilon(w_\varepsilon) + \int\limits_{\mathbb{R}^2\times[t_1,t_2]}\chi|\partial_t w_\varepsilon|^2 + \Delta\chi\, V_\varepsilon(u_\varepsilon) = \int\limits_{\mathbb{R}^2\times\{t_1\}}\chi e_\varepsilon(w_\varepsilon)$$
$$+ \int_{t_1}^{t_2}\mathcal{A}_S(\frac{t}{|\log\varepsilon|},\chi,w_\varepsilon) + (\mathcal{F}_J + \mathcal{R}_I + \mathcal{R})(t,\chi,\nabla\phi_\varepsilon,w_\varepsilon)\,dt. \tag{9.4}$$

We next bound each of the terms on the r.h.s. of (9.4). For the first term, which involves only the initial time $t_1 = s_0|\log\varepsilon|$, we invoke hypothesis $H_{a,r}(s_0)$, together with the stronger confinement assumption (9.2), to obtain, if $\varepsilon$ is sufficiently small,

$$\left|\int_{\mathbb{R}^2\times\{t_1\}}\chi e_\varepsilon(w_\varepsilon)\right| \le C\kappa^2|\log\varepsilon|. \tag{9.5}$$

For $\mathcal{R}_I$ we invoke Proposition 7.3 (which does not rely on assumption $H_{a,r}(s_0)$) to assert that, if $\varepsilon$ is sufficiently small, then

$$\left|\int_{t_1}^{t_2} \mathcal{R}_I(t,\chi,\nabla\phi_\varepsilon,w_\varepsilon)\,dt\right| \leq C\delta\mu^2 r|\log\varepsilon|. \tag{9.6}$$

For $\mathcal{R}$ we use (4.43) together with the observation that, since $|\nabla\phi_\varepsilon|$ is bounded,

$$\|\nabla w_\varepsilon\|_{L^1(B(a,r)\times\{t_1\})} \leq \sqrt{\pi}r\|\nabla w_\varepsilon\|_{L^2(B(a,r)\times\{t_1\})} \leq Cr\sqrt{|\log\varepsilon|}.$$

This yields

$$\left|\int_{t_1}^{t_2} \mathcal{R}(t,\chi,\nabla\phi_\varepsilon,w_\varepsilon)\,dt\right| \leq \frac{C}{r^2}|\log\varepsilon|^{3/5}. \tag{9.7}$$

For the last terms $\mathcal{A}_S$ and $\mathcal{F}_J$ we rely on the Cylinders Lemma, which has the following consequence: if the constant $C_2 > 0$ is chosen sufficiently small (depending only on $M_0$), then assumption $H_{a,\mu r}(s)$ is satisfied for every $s \in [s_0, s_0 + C_2\mu^2 r^2]$ (for $2\sigma_0 \leq \mu \leq 1/8$). Therefore we may apply Propositions 7.1 and 7.2 with $r$ replaced by $\mu r$. We have, for $s \in [s_0, s_0 + C_2\mu^2 r^2]$ and every $i \in I$, $j \in J$,

$$\left|\frac{a_i - a}{a_i - a_j}\right| \leq C\mu\,,$$

so that

$$\left|\sum_{i\in I} d_i(s)(a_i(s) - a)\times c(s)\right| \leq C\mu r. \tag{9.8}$$

Hence, it follows from (7.1) and (7.5) that

$$\left|\int_{t_1}^{t_2} \mathcal{A}_S(\frac{t}{|\log\varepsilon|},\chi,w_\varepsilon)\,dt\right| \leq C(\frac{\mu}{r^2}+\delta)C_2\mu^2 r^2|\log\varepsilon|$$

and

$$\left|\int_{t_1}^{t_2} \mathcal{F}_J(t,\chi,\nabla\phi_\varepsilon,w_\varepsilon)\,dt\right| \leq C(\frac{\mu}{r}+\delta)C_2\mu^2 r^2|\log\varepsilon|. \tag{9.9}$$

Combining (9.4) to (9.9), we are led to

$$\int_{\Lambda(\mu)} \Delta\chi\cdot V_\varepsilon(u_\varepsilon) \leq C_1(\kappa^2 + \mu^2(\delta r^2 + \mu))|\log\varepsilon|\,. \tag{9.10}$$

Notice that on $B(a,r/8)$, $\Delta\chi = 32/r^2$. On the other hand, by the Cylinders Lemma and Theorem 2.1, (2.23), we know that

$$|V_\varepsilon(u_\varepsilon)(x,t)| \leq \frac{C}{r^2}\varepsilon^2|\log\varepsilon|^2$$

for $(x,t)$ in $B(a,r)\setminus B(a,r/8)\times[t_1,t_2]$. The conclusion (9.3) follows. $\square$

## 9.2 Clearing-Out via potential estimates

The philosophy of the Clearing-Out theorem presented in Section 2 was that smallness of integral energy bounds imply pointwise bounds. In this section we derive results in the same spirit, but based only on potential estimates. Our proofs rely heavily on the fact that $N = 2$. We begin with the following lemma, where time derivatives are treated as perturbation terms for the corresponding elliptic equations on time slices.

**Lemma 9.2.** *Let $u_\varepsilon$ be a solution of $(PGL)_\varepsilon$ on $\mathbb{R}^2 \times \mathbb{R}^+$ and let $t \geq 1$. Then, we have, for every $r > \sqrt{2\varepsilon}$,*

$$\int_{B(0,r)\times\{t\}} e_\varepsilon(u_\varepsilon) \leq C\Big[1 + |\log\varepsilon|(\int_{B(0,4r)\times\{t\}} V_\varepsilon(u_\varepsilon))^2 + \int_{B(0,4r)\times\{t\}} (r^2|\partial_t u_\varepsilon|^2 + r^{-2}(\int_{B(0,4r)\times\{t\}} |\nabla u_\varepsilon|^2)\Big], \tag{9.11}$$

*where $C$ depends only on $M_0$.*

*Proof.* We follow some arguments developed in Section 3.6 of [6]. We assume $r = 1$, the general case follows then by scaling. Let $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ be such that $0 \leq \chi \leq 1$, $\chi \equiv 1$ on $B(0,2)$ and $\chi \equiv 0$ on $\mathbb{R}^2 \setminus B(0,4)$. We assume moreover $\|\nabla\chi\|_\infty \leq 1$. We consider the 2-form $\psi_t$ defined on $\mathbb{R}^2 \times \{t\}$ by

$$\psi_t = -\frac{1}{2\pi}\log|x| * [d(u_\varepsilon \times du_\varepsilon)\chi],$$

so that in particular

$$-\Delta\psi_t = dd^*\psi_t = d(u_\varepsilon \times du_\varepsilon)\chi \qquad \text{on } \mathbb{R}^2 \times \{t\} \tag{9.12}$$

Since $\chi \equiv 1$ on $B(0,2)$ it follows that $d(u_\varepsilon \times du_\varepsilon - d^*\psi_t) = 0$ on $B(0,2) \times \{t\}$. Invoking Poincaré Lemma, there exists some real-valued function $\phi_t$ defined on $B(0,2) \times \{t\}$ such that

$$u_\varepsilon \times du_\varepsilon = d\phi_t + d^*\psi_t \qquad \text{on } B(0,2) \times \{t\}. \tag{9.13}$$

Applying the $d^*$ operator to (9.13) we obtain $d^*(u_\varepsilon \times du_\varepsilon) = -\Delta\phi_t$, so that by (4.17) we are led to the equation

$$-\Delta\phi_t = u_\varepsilon \times \frac{\partial u_\varepsilon}{\partial t}, \qquad \text{on } B(0,2) \times \{t\}. \tag{9.14}$$

**Step 1. Estimate for $\psi_t$.** We will prove that

$$\int_{B(0,2r)\times\{t\}} |\nabla\psi_t|^2 \leq C\Big[1 + |\log\varepsilon|(\int_{B(0,4r)\times\{t\}} V_\varepsilon(u_\varepsilon))^2\Big]. \tag{9.15}$$

To this aim, we first define a re-projection of $u_\varepsilon$ in the following way. Let $\tau$ be the real-valued function defined on $\mathbb{R}^2 \times (0, +\infty)$ by $\tau(x,t) = p(|u_\varepsilon(x,t)|)$, where $p : [0,3] \to [1/3, 2]$ is a function verifying the properties

$$p(s) = \tfrac{1}{s} \quad \text{if } s \geq \tfrac{1}{2}, \qquad p(s) = 1 \quad \text{if } 0 \leq s \leq \tfrac{1}{4}, \qquad |p'(s)| \leq 4 \quad \forall\, s. \tag{9.16}$$

By construction, $|1 - \tau^2(x)| \leq K(1 - |u_\varepsilon(x)|^2)$. Set $\tilde{u}_\varepsilon = \tau\, u_\varepsilon$, so that

$$\tilde{u}_\varepsilon = u_\varepsilon \quad \text{if } |u_\varepsilon| \leq \tfrac{1}{4}, \qquad |\tilde{u}_\varepsilon| = 1 \quad \text{if } |u_\varepsilon| \geq \tfrac{1}{2}. \tag{9.17}$$

Notice that, since $|\tilde{u}_\varepsilon| = 1$ if $|u_\varepsilon| \geq \frac{1}{2}$, we have

$$d(\tilde{u}_\varepsilon \times d\tilde{u}_\varepsilon) = 0 \qquad \text{if } |u_\varepsilon| \geq \tfrac{1}{2}. \tag{9.18}$$

On the other hand, since $|\nabla u_\varepsilon| \leq \frac{C}{\varepsilon}$, it follows[31]

$$|d(\tilde{u}_\varepsilon \times d\tilde{u}_\varepsilon)| \leq C V_\varepsilon(u_\varepsilon). \tag{9.19}$$

We decompose $\psi_t = \psi_{1,t} + \psi_{2,t}$ on $\mathbb{R}^2 \times \{t\}$, where

$$\begin{cases} -\Delta\psi_{1,t} = d(\tilde{u}_\varepsilon \times d\tilde{u}_\varepsilon)\,\chi & \text{on } \mathbb{R}^2 \times \{t\} \\ -\Delta\psi_{2,t} = d((1-\tau^2)u_\varepsilon \times du_\varepsilon)\,\chi & \text{on } \mathbb{R}^2 \times \{t\}. \end{cases} \tag{9.20}$$

By our previous estimates we have the pointwise inequality $((1-\tau^2)u_\varepsilon \times du_\varepsilon)^2 \leq CV_\varepsilon(u_\varepsilon)$, and hence

$$||(1-\tau^2)u_\varepsilon \times du_\varepsilon||_{L^2(B(0,4))\times\{t\}} \leq C\|V_\varepsilon(u_\varepsilon)\|_{L^1(B(0,4))\times\{t\}}.$$

It follows therefore by standard elliptic theory that

$$\int_{B(0,4)\times\{t\}} |\nabla\psi_{2,t}|^2 \leq C \int_{B(0,4)\times\{t\}} V_\varepsilon(u_\varepsilon). \tag{9.21}$$

In view of (9.19), we have

$$\|\Delta\psi_{1,t}\|_{L^1(\mathbb{R}^2\times\{t\})} \leq K \int_{B(0,4)\times\{t\}} V_\varepsilon(u_\varepsilon),$$

and therefore we obtain the $L^2$ estimate

$$\int_{B(0,4)\times\{t\}} |\nabla\psi_{1,t}| \leq C \int_{B(0,4)\times\{t\}} V_\varepsilon(u_\varepsilon). \tag{9.22}$$

To obtain an $L^2$ estimate for $\nabla\psi_1$, recall that by the Brezis-Gallouët inequality [9], for any $u \in H^2(\mathbb{R}^2)$,

$$\|u\|_{L^\infty(\mathbb{R}^2)} \leq K\|u\|_{H^1(\mathbb{R}^2)} \left[1 + \log^{\frac{1}{2}}(1 + \|u\|_{H^2(\mathbb{R}^2)})\right].$$

We apply the previous inequality to $\psi_{1,t}\chi$. Since $\|\psi_{1,t}\chi\|_{H^2(\mathbb{R}^2)} \leq \frac{K}{\varepsilon^2}$, we obtain

$$\|\psi_{1,t}\chi\|_{L^\infty(\mathbb{R}^2)} \leq K\|\psi_{1,t}\|_{H^1(\mathbb{R}^2)}\,|\log\varepsilon|^{\frac{1}{2}}. \tag{9.23}$$

On the other hand, we have

$$\Delta(\psi_{1,t}\chi) = (\Delta\psi_{1,t})\chi + 2\nabla\psi_{1,t}\nabla\chi + \psi_{1,t}\Delta\chi$$

so that by (9.22)

$$\|\Delta(\psi_{1,t}\chi)\|_{L^1(\mathbb{R}^2)} \leq K \int_{B(0,4)\times\{t\}} V_\varepsilon(u_\varepsilon). \tag{9.24}$$

[31] it suffices, in view of (9.18), to establish (9.19) for $|u_\varepsilon| \leq \frac{1}{2}$. In that case $V_\varepsilon(u_\varepsilon) \geq \frac{9}{64\varepsilon^2}$.

Using standard estimates, we finally write

$$\begin{aligned}\|\psi_{1,t}\chi\|^2_{H^1(\mathbb{R}^2)} &\leq K\|\Delta(\psi_{1,t}\chi)\|_{L^1(\mathbb{R}^2)}\,\|\psi_{1,t}\chi\|_{L^\infty(\mathbb{R}^2)}\\ &\leq K\|\Delta(\psi_{1,t}\chi)\|_{L^1(\mathbb{R}^2)}\,\|\psi_{1,t}\chi\|_{H^1(\mathbb{R}^2)}|\log\varepsilon|^{\frac{1}{2}},\end{aligned} \tag{9.25}$$

which combined with (9.24) yields

$$\int_{B(0,2)\times\{t\}} |\nabla\psi_{1,t}|^2 \leq C|\log\varepsilon|\left[\int_{B(0,4)\times\{t\}} V_\varepsilon(u_\varepsilon)\right]^2. \tag{9.26}$$

The claim (9.15) is proved.

**Step 2. Estimates for $\phi_t$.** We claim that

$$\int_{B(0,2)\times\{t\}}|\nabla\phi_t|^2 \leq C\Big[\|\partial_t u_\varepsilon\|^2_{L^2(B(0,4)\times\{t\})} + \|\nabla u_\varepsilon\|^2_{L^1(B(0,4)\times\{t\})} + \|V_\varepsilon(u_\varepsilon)\|^2_{L^1(B(0,4)\times\{t\})}\Big]. \tag{9.27}$$

Indeed, by Caccioppoli estimates we obtain from (9.14)

$$\int_{B(0,2)\times\{t\}}|\nabla\phi_t|^2 \leq C\left[\|\partial_t u_\varepsilon\|^2_{L^2(B(0,4)\times\{t\})} + \|\phi_t - \bar\phi_t\|^2_{L^2(B(0,4)\times\{t\})}\right],$$

where $\bar\phi_t$ denotes the mean value of $\phi_t$ on $B(0,4)\times\{t\}$. By Sobolev embedding,

$$\|\phi_t - \bar\phi_t\|^2_{L^2(B(0,4)\times\{t\})} \leq C\|\nabla\phi_t\|^2_{L^1(B(0,4)\times\{t\})},$$

so that

$$\int_{B(0,2)\times\{t\}}|\nabla\phi_t|^2 \leq C\left[\|\partial_t u_\varepsilon\|^2_{L^2(B(0,4)\times\{t\})} + \|\nabla\phi_t\|^2_{L^1(B(0,4)\times\{t\})}\right].$$

On the other hand, on $B(0,4)\times\{t\}$, by (9.13), $|\nabla\phi_t| \leq C(|\nabla u_\varepsilon| + |\nabla\psi_t|)$, and hence using (9.26) we obtain (9.27).

**Step 3. Estimates for $\nabla|u_\varepsilon|$.** We claim that

$$\int_{B(0,2)\times\{t\}} |\nabla|u_\varepsilon||^2 \leq C\left[\int_{B(0,4)\times\{t\}} V_\varepsilon(u_\varepsilon)\right]^{1/2}\left[(\int_{B(0,4)\times\{t\}} e_\varepsilon(u_\varepsilon))^{1/2} + \varepsilon|\log\varepsilon|\right]. \tag{9.28}$$

Set $\sigma_\varepsilon = 1 - |u_\varepsilon|^2$, so that

$$\partial_t\sigma_\varepsilon - \Delta\sigma_\varepsilon = 2|\nabla u_\varepsilon|^2 - \frac{2}{\varepsilon^2}\sigma_\varepsilon(1-\sigma_\varepsilon). \tag{9.29}$$

We multiply (9.29) by $\sigma_\varepsilon\chi^2$ and integrate by parts. This yields

$$\begin{aligned}\int_{B(0,2)\times\{t\}} |\nabla(\sigma_\varepsilon\chi)|^2 \leq 2\int_{B(0,4)\times\{t\}} |\nabla u_\varepsilon|^2\sigma_\varepsilon + C\int_{B(0,4)\times\{t\}} |\partial_t u_\varepsilon|\sigma_\varepsilon\\ + \int_{B(0,4)\times\{t\}} \sigma_\varepsilon^2\chi^2.\end{aligned} \tag{9.30}$$

Hence, in view of the estimate $|\nabla u_\varepsilon| \le \frac{C}{\varepsilon}$ we are led to

$$\int_{B(0,2)\times\{t\}} |\nabla(\sigma_\varepsilon)|^2 \le C \left[\int_{B(0,4)\times\{t\}} V_\varepsilon(u_\varepsilon)\right]^{1/2} \left[(\int_{B(0,4)\times\{t\}} |\nabla u_\varepsilon|^2)^{1/2} + \varepsilon|\log\varepsilon|\right] \tag{9.31}$$
$$+ \varepsilon\|\partial_t u_\varepsilon\|_{L^2(B(0,4)\times\{t\})}.$$

Since $|\nabla\sigma_\varepsilon| = |u_\varepsilon|\cdot|\nabla|u_\varepsilon|\,|$, we have

$$\int_{B(0,2)\times\{t\}} |\nabla|u_\varepsilon|\,|^2 \le 2\int_{B(0,2)\times\{t\}} |\nabla(\sigma_\varepsilon)|^2 + (1-|u_\varepsilon|^2)|\nabla u_\varepsilon|^2,$$

and using once more the bound $|\nabla u_\varepsilon| \le \frac{C}{\varepsilon}$ we derive (9.28).

Combining (9.26), (9.27), (9.28), the identity

$$4|u_\varepsilon|^2|\nabla u_\varepsilon|^2 = 4|u_\varepsilon \times \nabla u_\varepsilon|^2 + |\nabla|u_\varepsilon||^2$$

and the estimate

$$4|(1-|u_\varepsilon|^2)||\nabla u_\varepsilon|^2 \le C\frac{(1-|u_\varepsilon|^2)}{\varepsilon}|\nabla u_\varepsilon| \le 2|\nabla u_\varepsilon|^2 + CV_\varepsilon(u_\varepsilon)$$

conclusion (9.11) follows. □

As a consequence of Lemma 9.2 and the Cylinders Lemma, we have

**Proposition 9.2.** *Let $u_\varepsilon$ be a solution of $(PGL)_\varepsilon$, $s_0 > 0$, $R > 0$ and $\Delta s > 0$ be given. There exists a universal constant $\eta_v > 0$, and constants $\beta_0$, $\varepsilon_0$ and $C(M_0)$ depending only on $M_0$ such that, if*

$$|\log\varepsilon|^{-1/6} \le \frac{\sqrt{\Delta s}}{\beta_0} \le R \le |\log\varepsilon|^{1/6}, \tag{9.32}$$

*and*

$$\frac{1}{\Delta s}\int_\Lambda V_\varepsilon(u_\varepsilon) \le \eta_v|\log\varepsilon|, \tag{9.33}$$

*where $\Lambda = B(0,R)\times[s_0|\log\varepsilon|, (s_0+\Delta s)|\log\varepsilon|]$, then, for $\varepsilon \le \varepsilon_0$,*

$$e_\varepsilon(u_\varepsilon) \le \frac{C(M_0)}{\Delta s} \qquad \text{on} \qquad B(0,\tfrac{R}{2})\times[(s_0+\tfrac{3\Delta s}{4})|\log\varepsilon|, (s_0+\Delta s)|\log\varepsilon|].$$

*Proof.* By averaging, there exists some $s_1 \in [s_0, s_0+\Delta s]$ such that

$$\int_{B(0,R)\times\{s_1|\log\varepsilon|\}} V_\varepsilon(u_\varepsilon) \le C\eta_v, \quad \int_{B(0,R)\times\{s_1|\log\varepsilon|\}} |\partial_t u_\varepsilon|^2 \le C, \quad \int_{B(0,R)\times\{s_1|\log\varepsilon|\}} |\nabla u_\varepsilon| \le C(R^2+1),$$

where $C$ depends possibly on $M_0$. Invoking (9.11) and scaling, we deduce[32]

$$\int_{B(0,\frac{3R}{4})\times\{s_1|\log\varepsilon|\}} e_\varepsilon(u_\varepsilon) \le C\left(\eta_v^2|\log\varepsilon| + \frac{R^{-2}}{\Delta s} + C\frac{R^3}{\Delta s} + \frac{1}{\Delta s}\right).$$

[32] Actually, in (9.11) $4r$ can be replaced by $\alpha r$, for any arbitrary $\alpha > 1$, at a price of a larger constant $C_\alpha$.

Choosing $\eta_v$ and $\varepsilon_0$ sufficiently small, we obtain, for $\varepsilon \leq \varepsilon_0$,

$$\int_{B(0,\frac{3R}{4})\times\{s_1|\log\varepsilon|\}} e_\varepsilon(u_\varepsilon) \leq \frac{\eta_0}{4}|\log\varepsilon|. \tag{9.34}$$

In particular, for $r = \frac{R}{100}\sigma_0^{-1}$,

$$\Omega^\varepsilon_{r/4}(s_1|\log\varepsilon|) \cap B(0, \tfrac{5R}{8}) = \emptyset. \tag{9.35}$$

It follows from Proposition 4.4 that $\Omega^\varepsilon_{r/4}(s|\log\varepsilon|) \cap B(0, \frac{45R}{80}) = \emptyset$ for every $s \in [s_1, s_1 + C\sigma_0^{-2}\gamma_0 R^2]$. In particular, if $\beta_0$ is chosen sufficiently small, $C\sigma_0^{-2}\gamma_0 R^2 \geq \Delta s$. The proof is then completed as the one of Lemma 4.8. $\square$

## 9.3 Proof of Theorem 3

The proof of Theorem 3 is completed combining Proposition 9.1 and Proposition 9.2. We choose the parameters $\mu, \kappa, \delta$ so that the r.h.s. of inequality (9.3) is less than $\eta_v$. First let $r = 8R$, and choose $\delta$ so that $C_1\delta r^2 \leq \frac{\eta_v}{3}$. Set $\mu_0(\kappa) = \sqrt{\frac{3C_1}{\eta_v}}\kappa$ and $\mu_1 = \frac{\eta_v}{3C_1}$. For $\kappa \leq \kappa_1 = (\frac{\eta_v}{3C_1})^{3/2}$, we have $\mu_0(\kappa) \leq \mu_1$, and by construction

$$C_1((\tfrac{\kappa}{\mu})^2 + \mu + \delta r^2) \leq \eta_v \qquad \text{for } \mu_0(\kappa) \leq \mu \leq \mu_1. \tag{9.36}$$

In particular, it follows from Proposition 9.1 that if $\varepsilon$ is sufficiently small, then for every $\mu_0(\kappa) \leq \mu \leq \mu_1$, we have, since $d(a, s, r) = 0$,

$$\frac{1}{\Delta s(\mu)} \int_{\Lambda(\mu)} V_\varepsilon(u_\varepsilon) \leq \eta_v|\log\varepsilon|,$$

where $\Lambda(\mu) = B(a, R) \times [s_0|\log\varepsilon|, (s_0 + \Delta s)|\log\varepsilon|]$, and $\Delta s(\mu) = 64C_2\mu^2R^2$. On the other hand, if $\mu \leq \mu_2 = \frac{\beta_0}{8\sqrt{C_2}}$, we have $\sqrt{\Delta s} \leq \beta_0 R$. Set $\kappa_0 = \sqrt{\frac{\eta_v}{3C_1}} \cdot \min\{\mu_1, \mu_2\}$. For $\kappa \leq \kappa_0$, $\mu_0(\kappa) \leq \mu_3 = \min\{\mu_1, \mu_2\}$, so that for $\mu_0(\kappa) \leq \mu \leq \mu_3$ we may apply Proposition 9.2 which yields

$$|e_\varepsilon(u_\varepsilon)| \leq C(\mu) \qquad \text{on } B(a, \tfrac{R}{2}) \times [s_0 + \tfrac{3\Delta s(\mu)}{4}|\log\varepsilon|, s_0 + \Delta s(\mu)|\log\varepsilon|].$$

This completes the proof, setting $K_1 = \frac{144C_1C_2}{\eta_v}$ and $K_2 = \min\{\beta_0, \frac{64C_2\eta_v^2}{9C_1^2}\}$. $\square$

## 9.4 Proof of Theorem 5 iii) and Proposition 1

Let $s_0 > 0$ and $i \in \{1, \cdots, l(s_0)\}$ be such that $d_i(s_0) = 0$, and let $R > 0$ be such that $B(a_i(s_0), R) \cap \Sigma^s_{\mathfrak{v}} = \{a_i(s_0)\}$.

**Step 1.** We have

$$\limsup_{s\to s_0,\ s>s_0} \nu^s_*(B(a_i(s_0), \frac{R}{2}) = 0.$$

Indeed, assumption (6) of Theorem 3 is verified for every $0 < \kappa < 1$. In particular it follows from Theorem 3 that

$$\Sigma^s_{\mathfrak{v}} \cap B(a_i(s_0), \frac{R}{2}) = \emptyset$$

for every $s \in (s_0, s_0 + K_2R^2]$.

**Step 2.** We have

$$\liminf_{s\to s_0,\ s<s_0} \nu_*^s(B(a_i(s_0),\frac{R}{4})) \geq \frac{\eta_0}{2}.$$

This was already proved in (5.19).

**Step 3.** It follows from Step 1, Step 2 and Theorem 5 ii) that equation (9) is satisfied.

**Step 4.** It follows from Theorem 5 iii) that $d_i(s) = 0$ (for some $i \in \{1, \cdots, l(s)\}$) may happen for at most $\frac{2M_0}{\eta_0}$ times $s$. This yields the conclusion of Proposition 1.

# Appendix A : Linear elliptic and parabolic estimates

## A.1 Elliptic problems in $\mathbb{R}^{N+1}$

The first part of this Appendix is devoted to the study of elliptic problems on $\mathbb{R}^{N+1} = \mathbb{R}^N_x \times \mathbb{R}_t$ of the form

$$-\Delta\rho = \omega \qquad \text{on } \mathbb{R}^N_x \times \mathbb{R}_t, \tag{A.1}$$

where $\Delta \equiv \Delta_{x,t}$ denotes the Laplacian on $\mathbb{R}^{N+1}$. Whereas classical theory deals with sources $\omega$ for which some global bounds on $\mathbb{R}^{N+1}$ are assumed, here we focus on the case where we only have at our disposal bounds for each time slice $\mathbb{R}^N \times [t, t+1]$. Our first result in this direction is

**Lemma A.1.** *Assume that $\omega$ is a measure on $\mathbb{R}^{N+1}$, set*

$$\mu(t) = \|\omega\|(\mathbb{R}^N \times [t, t+1]), \qquad \text{for } t \in \mathbb{R},$$

*and assume that $\mu(t)$ belongs to $L^\infty \cap L^p(\mathbb{R})$ for some $1 \leq p \leq +\infty$. Then there exists a solution $\rho$ of (A.1) such that $|\nabla_{x,t}\rho| = g_1 + g_2$, where*

$$\sup_{t\in\mathbb{R}} \|g_1\|_{L^{p_1}(\mathbb{R}^N\times\{t\})} \leq K(p_1, p)\|\mu\|_{L^p(\mathbb{R})} \qquad \text{for any } p_1 > \frac{pN}{pN-(p-1)}, \tag{A.2}$$

$$\sup_{t\in\mathbb{R}} \|g_1\|_{L^{p_2}(\mathbb{R}^N\times[t,t+1])} \leq K(p_2)\|\mu\|_{L^\infty(\mathbb{R})} \qquad \text{for any } 1 \leq p_2 < \frac{N+1}{N}. \tag{A.3}$$

*Proof.* Let $G$ be the fundamental solution for the Laplacian on $\mathbb{R}^{N+1}$, so that in particular

$$|\nabla_{x,t}G(x,t)| \leq \sigma(x,t),$$

where the function $\sigma$ is explicitly defined by

$$\sigma(x,t) = \frac{1}{(x^2+t^2)^{N/2}}.$$

We next show that $G * \omega$ is a well-defined function. We write

$$\sigma = \sigma^{in} + \sigma^{out},$$

where $\sigma^{in} = \mathbf{1}_{B^N\times[-1,1]} \cdot \sigma$, where $B^N$ denotes the unit ball in $\mathbb{R}^N$, and $\sigma^{out} = \sigma - \sigma^{in}$. In particular $\sigma^{in}$ has compact support and $\sigma^{out}$ is bounded. Let $f^{in} = \sigma^{in} * \omega$ and $f^{out} = \sigma^{out} * \omega$. We bound each of the functions $f^{in}$ and $f^{out}$ in appropriate norms.

**Step 1.** We have

$$\sup_{t\in\mathbb{R}} \|f^{out}\|_{L^p(\mathbb{R}^N\times\{t\})} \leq K_p \sup_{t\in\mathbb{R}} \|\omega\|(\mathbb{R}^N\times[t,t+1]) \qquad \text{for each } p > \frac{N}{N-1}. \tag{A.4}$$

*Proof.* We may assume without loss of generality that $\omega$ is smooth. Since the norms involved in inequality (A.4) are invariant under time translations, we merely have to bound $\|f^{out}(\cdot,0)\|_{L^p(\mathbb{R}^N)}$. The starting point is an estimate for the kernel $\sigma^{out}$. We obviously have that

$$\sigma^{out}(x,t) \leq \frac{1}{|x|^{2\alpha}} \max_{|x|\geq 1} \frac{|x|^{2\alpha}}{(|x|^2+t^2)^{N/2}} \qquad \text{for } |x|\geq 1.$$

A simple computation shows that, if $\alpha < \frac{N}{2}$,

$$\max_{|x|\geq 1} \frac{|x|^{2\alpha}}{(|x|^2+t^2)^{N/2}} \leq C(1+|t|)^{2\alpha-N},$$

so that

$$\sigma^{out}(x,t) \leq C\frac{(1+|t|)^{2\alpha-N}}{(1+|x|^2)^{\alpha}} \qquad \forall\,(x,t)\in\mathbb{R}^N\times\mathbb{R}.$$

In particular, we obtain

$$|f^{out}(y,0)| \leq G_\alpha * H_\alpha(y), \tag{A.5}$$

where

$$G_\alpha(x) = \frac{C}{(1+|x|^2)^{\alpha}}, \qquad H_\alpha(x) = \int_{\mathbb{R}} (1+|t|)^{2\alpha-N}\omega(x,t)dt. \tag{A.6}$$

Notice that

$$G_\alpha \in L^{p_1}(\mathbb{R}^N) \qquad \text{for every } p_1 > \frac{N}{2\alpha}. \tag{A.7}$$

On the other hand, we may bound $\|H_\alpha\|_{L^1(\mathbb{R}^N)}$ by Fubini theorem:

$$\begin{aligned}
\int_{\mathbb{R}^N} |H_\alpha(x)dx| &= \int_{\mathbb{R}} (1+|t|)^{2\alpha-N}\|\omega(\cdot,t)\|_{L^1(\mathbb{R}^N\times\{t\})}dt \\
&\leq C\int_{\mathbb{R}} (1+|t|)^{2\alpha-N}\|\omega\|_{L^1(\mathbb{R}^N\times[t,t+1])}dt \\
&\leq C\int_{\mathbb{R}} (1+|t|)^{2\alpha-N}\mu(t)dt \\
&\leq C\left(\int_{\mathbb{R}} (1+|t|)^{(2\alpha-N)p'}\right)^{1/p'} \|\mu\|_{L^p(\mathbb{R})}.
\end{aligned}$$

If $(2\alpha-N)p' < -1$, that is $2\alpha < N+1-\frac{1}{p}$, then the explicit integral on the r.h.s. of the last inequality converges. Going back to (A.7), choosing $p_1 > \frac{N}{N+1-\frac{1}{p}}$ and invoking Young's inequality, (A.4) follows.

**Step 2.** We have, for every $1\leq q < \frac{N+1}{N}$,

$$\sup_{t\in\mathbb{R}} \|f^{in}\|_{L^q(\mathbb{R}^N\times[t,t+1])} \leq C_q \sup_{t\in\mathbb{R}} \|\omega\|(\mathbb{R}^N\times[t,t+1]). \tag{A.8}$$

*Proof.* By construction, $\sigma^{in}$ has compact support included in the strip $\mathbb{R}^N \times [-1,1]$. Therefore the restriction of $f^{in}$ to the strip $\mathbb{R}^N \times [t_0, t_0+1]$, for $t_0 \in \mathbb{R}$ is identical to the restriction on the same strip of the convolution $\sigma^{in} * \chi \cdot \omega$, where $\chi(x,t)$ verifies $\chi(x,t) = 1$ if $|t - t_0| \leq 2$, $\chi(x,t) = 0$ otherwise. We notice that $\chi \cdot \omega \in L^1(\mathbb{R}^{N+1})$, more precisely we have

$$\|\chi \cdot \omega\|_{L^1(\mathbb{R}^{N+1})} \leq C \sup_{t \in \mathbb{R}} \|\omega\|_{L^1(\mathbb{R}^N \times [t,t+1])}.$$

On the other hand,

$$\sigma^{in} \in L^q(\mathbb{R}^{N+1}) \qquad \text{for any } 1 \leq q < \frac{N+1}{N},$$

and the conclusion (A.8) follows once more by Young's inequality.

**Proof of Lemma A.1 completed.** It follows from Step 1 and Step 2 that $\nabla G * \omega$ is well-defined and may be written as $\nabla G * \omega = g_1 + g_2$, where $g_1$ and $g_2$ verify (A.2) and (A.3) respectively. The existence of $\rho$ follows by integration. $\square$

We next turn to the problem

$$-\Delta_{x,t} \zeta = \operatorname{div}_{x,t} h \qquad \text{on } \mathbb{R}^N \times \mathbb{R}, \tag{A.9}$$

where $h = (h_1, ..., h_N, h_{N+1})$ and $\Delta_{x,t}$ and $\operatorname{div}_{x,t}$ represent respectively the Laplacian and divergence operators on $\mathbb{R}^{N+1}$. We have

**Lemma A.2.** *Let $1 < p < +\infty$ and assume*

$$\sup_{t \in \mathbb{R}} \|h\|_{L^1 \cap L^p(\mathbb{R}^N \times [t,t+1])} < +\infty\,.$$

*Then there exists a solution $\zeta$ of (A.9) such that*

$$\sup_{t \in \mathbb{R}} \|\nabla_{x,t} \zeta\|_{L^p(\mathbb{R}^N \times [t,t+1])} \leq K_p \sup_{t \in \mathbb{R}} \|h\|_{L^1 \cap L^p(\mathbb{R}^N \times [t,t+1])}.$$

*Proof.* As in the proof of Lemma A.1, we consider $\nabla G * \operatorname{div} h$ and show that this is well-defined. We will first assume that $h$ is smooth and compactly supported, so that the convolution $\nabla G * \operatorname{div} h$ makes sense. Moreover, in this case we may integrate by parts, so that we have to consider the terms

$$f_{ij} = \frac{\partial^2 G}{\partial x_i \partial x_j} * h\,, \qquad \text{for } i,j = 1, ..., N+1.$$

We write once more, for $i,j = 1, ..., N+1$,

$$f_{ij} = f_{ij}^{in} + f_{ij}^{out},$$

where $f_{ij}^{in} = \sigma_{ij}^{in} * h$, $f_{ij}^{out} = \sigma_{ij}^{out} * h$, and $\sigma_{ij}^{in} = \chi \cdot \frac{\partial^2 G}{\partial x_i \partial x_j}$, $\sigma_{ij}^{out} = (1 - \chi) \cdot \frac{\partial^2 G}{\partial x_i \partial x_j}$. Here $\chi$ denotes some radial smooth function compactly supported in $B_2 = \{x \in \mathbb{R}^{N+1},\ |x| \leq 2\}$ and identically equal to 1 in the unit ball $B_1$ of $\mathbb{R}^{N+1}$.

By construction, $\sigma_{ij}^{in}$ has compact support included in the strip $\mathbb{R}^N \times [-1,1]$. Therefore the restriction of $f_{ij}^{in}$ to any strip $\mathbb{R}^N \times [t_0, t_0+1]$, $t_0 \in \mathbb{R}$, coincides with the convolution

$$\sigma_{ij}^{in} * \rho \cdot h,$$

where $\rho(x,t)$ verifies $\rho(x,t)=1$ if $|t-t_0|\leq 2$, $\rho(x,t)=0$ otherwise. We have

$$\|\rho\cdot h\|_{L^p(\mathbb{R}^{N+1})}\leq C\sup_{t\in\mathbb{R}}\|h\|_{L^p(\mathbb{R}^N\times[t,t+1])}.$$

On the other hand, convolution by $\sigma_{ij}^{in}$ is a bounded operator on $L^p(\mathbb{R}^{N+1})$ for any $1<p<+\infty$ in view of Calderòn-Zygmund theory. Hence

$$\sup_{t\in\mathbb{R}}\|f_{ij}^{in}\|_{L^p(\mathbb{R}^N\times[t,t+1])}\leq C_p\sup_{t\in\mathbb{R}}\|h\|_{L^p(\mathbb{R}^N\times[t,t+1])}.$$

The terms $f_{ij}^{out}$ are handled as in Lemma A.1. □

**Remark A.1.** One may wonder if the $L^1$ bound on $h$ in Lemma A.2 is necessary, and if $\nabla_{x,t}\zeta$ is bounded in $L^p(\mathbb{R}^N\times[t,t+1])$ under the only assumption that $h$ is bounded in $L^p$. In the case $p=2$, we will show that this is not the case. More precisely, we will exhibit some function $h$ verifying

$$\sup_{t\in\mathbb{R}}\|h\|_{L^2(\mathbb{R}^N\times[t,t+1])}<+\infty\,, \tag{A.10}$$

and such that

$$\sup_{t\in\mathbb{R}}\|f_{ij}\|_{L^2(\mathbb{R}^N\times[t,t+1])}=+\infty\,. \tag{A.11}$$

To this aim we work in Fourier variables and consider the Fourier transform $\hat{G}(\xi,\tau)=\frac{1}{|\xi|^2+\tau^2}$ with respect to space and time variables, and its Fourier transform with respect to the time variable only

$$\hat{G}_\tau(\xi)=\frac{1}{2\pi}\int_{\mathbb{R}}\frac{\exp(it\tau)}{\xi^2+\tau^2}d\tau=\frac{1}{2\pi^2|\xi|}\exp(-\sqrt{|\xi|t}).$$

Hence,

$$f_{ij}(\xi,0)=-\int_{\mathbb{R}}\xi_i\xi_j\hat{G}_\tau(\xi)\hat{h}_\tau(\xi)dt=\frac{1}{2\pi^2}\int_{\mathbb{R}}\frac{\xi_i\xi_j}{|\xi|}\exp(-\sqrt{|\xi|t})\hat{h}_\tau(\xi)dt. \tag{A.12}$$

For fixed $t$, the multiplier $\frac{\xi_i\xi_j}{|\xi|}\exp(-\sqrt{|\xi|t})$ achieves its maximum for $|\xi|\simeq\frac{c}{t}$, and the maximum value is proportional to $\frac{1}{t}$. It is clear from the proof of Lemma A.2 that difficulties stem from the lack of integrability at infinity in time, and therefore, in view of the previous relation $\xi_{max}\simeq\frac{c}{t}$, for small frequencies at large time. In view of this remark, we construct a function $h(\cdot,t)$ as follows:

$$\hat{h}_\tau(\xi)=t^{N/2}\mathbf{1}_{\{|\xi|\leq\frac{1}{\tau}\}}\qquad\text{for } t\geq 1,\qquad \hat{h}_\tau(\xi)=0\qquad\text{otherwise.}$$

Clearly

$$\int_{\mathbb{R}^N}|h|^2(x,t)dx=(2\pi)^N\int_{\mathbb{R}^N}|\hat{h}_\tau(\xi)|^2d\xi=(2\pi)^N|B_1|\qquad\text{for } t\geq 1,$$

and $\|h\|_{L^2(\mathbb{R}^N\times\{t\})}=0$ otherwise, so that (A.10) is satisfied. On the other hand, we claim that if $|\xi|\leq 1$

$$|\hat{f}(\xi,0)|\geq\frac{1}{|\xi|^{N/2}}. \tag{A.13}$$

Indeed, $|\xi|\exp(-\sqrt{|\xi|t}) \geq \frac{c}{t}$ for $\frac{1}{2|\xi|} \leq t \leq \frac{1}{|\xi|}$, and $\hat{h}_\tau(\xi) = t^{N/2}$ for $t \leq \frac{1}{|\xi|}$, and therefore

$$\int_{\mathbb{R}} |\xi|\exp(-\sqrt{|\xi|t})\hat{h}_\tau(\xi)dt \geq c\int_{\frac{1}{2|\xi|}}^{\frac{1}{|\xi|}} t^{\frac{N-2}{2}}dt \geq \frac{c}{|\xi|^{N/2}}.$$

This establish the claim (A.13), and hence $\hat{f}(0) \notin L^2(\mathbb{R}^N)$, $f \notin L^2(\mathbb{R}^N)$ and similarly one establishes (A.11).

**Remark A.2.** The same type of arguments shows that the high frequency part of $f$ remains bounded in $L^2(\mathbb{R}^N)$. For this purpose we consider the functions $g_{ij}$ defined in Fourier coordinates by

$$\hat{g}_{ij}(\xi, 0) = \int_{\{|t|\geq 1\}} \frac{\xi_i\xi_j}{|\xi|}\mathbf{1}_{\{|\xi|>0\}}\exp(-\sqrt{|\xi|t})dt.$$

The functions $g_{ij}$ represent the high-frequency terms in $f$ arising from the contribution of $h$ for $|t| \geq 1$.[33] Since for $|\xi| \geq 1$ and $|t| \geq 1$, $|\xi|\exp(-\sqrt{|\xi|t}) \leq \exp(-\frac{\sqrt{t}}{2})$, we have

$$\|\hat{g}_{ij}(\cdot, 0)\|_{L^2(\mathbb{R}^N)} \leq C\int_{\{|t|\geq 1\}} \exp(-\frac{\sqrt{t}}{2})\|\hat{h}\|_{L^2(\mathbb{R}^N)}dt,$$

so that

$$\|g(\cdot, 0)\|_{L^2(\mathbb{R}^N)} \leq C\sup_{t>0}\|h\|_{L^2(\mathbb{R}^N\times[t,t+1])}.$$

## A.2 Parabolic problems

We consider the initial value parabolic problem

$$\begin{cases} \dfrac{\partial\varphi}{\partial t} - \Delta\varphi = \omega & \text{on } \mathbb{R}^N \times (0, +\infty), \\ \quad\varphi(x, 0) = 0 & \text{for } x \in \mathbb{R}^N. \end{cases} \tag{A.14}$$

**Lemma A.3.** *Let $1 \leq p < N$ and assume that*

$$\sup\nolimits_{t\in\mathbb{R}^+}\|\omega\|_{L^p(\mathbb{R}^N\times[t,t+1])} < +\infty.$$

*Then, there exists a unique solution $\varphi$ to* (A.14) *such that $|\nabla_x\varphi| \leq g_1 + g_2$ where the functions $g_1$ and $g_2$ satisfy*

$$\sup\nolimits_{t\in\mathbb{R}^+}\|g_1\|_{L^r(\mathbb{R}^N\times\{t\})} \leq K(r)\sup\nolimits_{t\in\mathbb{R}^+}\|\omega\|_{L^p(\mathbb{R}^N\times[t,t+1])} \tag{A.15}$$

*where $r$ is any number satisfying $r > p^*$ and*

$$\sup\nolimits_{t\in\mathbb{R}^+}\|g_2\|_{L^p([t,t+1],L^{p^*})} \leq K(p)\sup\nolimits_{t\in\mathbb{R}^+}\|\omega\|_{L^p(\mathbb{R}^N\times[t,t+1])}, \tag{A.16}$$

*where $p^*$ is the Sobolev exponent in dimension $N$, i.e. $p^* = Np/(N-p)$.*

[33] The contribution for $|t| \leq 1$ is handled by standard estimates.

*Proof.* Let $G$ be the fundamental solution of the heat operator on $\mathbb{R}^N \times \mathbb{R}^+$, given by

$$G(x,t) = \frac{1}{(4\pi t)^{N/2}} \exp\left(-\frac{|x|^2}{4t}\right), \qquad \text{for } x \in \mathbb{R}^N,\ t > 0$$

so that for some explicit constant $C > 0$ we have

$$|\nabla_x G(x,t)\,| \leq C A_t(x) \equiv C \frac{|x|}{t^{\frac{N+2}{2}}} \exp\left(-\frac{|x|^2}{4t}\right).$$

Consider the function $\varphi$ defined by

$$\varphi(x,t) = G * \omega = \int_0^t e^{(t-s)\Delta}\omega_s\, ds.$$

We split this integral in two terms $\varphi_1$ and $\varphi_2$ by restricting the integration on the intervals $[0, t-1]$ and $[t-1, t]$ respectively. The term $\varphi_1$ is the contribution from the source $\omega_s$ in the "remote" past, and the term $\varphi_2$ is the contribution from the "near" past. We handle each of these terms in a different way.

**Step 1: Estimates for $\nabla\varphi_1$.** We have

$$|\nabla\varphi_1(x,t)| \leq \int_0^{t-1} A_{t-s} * |\omega_s|\, ds \equiv \int_0^{t-1} f_s(x)\, ds.$$

By Young's inequality,

$$\|f_s\|_{L^r(\mathbb{R}^N)} \leq \|A_{t-s}\|_{L^q(\mathbb{R}^N)} \|\omega_s\|_{L^p(\mathbb{R}^N)}$$

for any numbers $1 \leq p, q, r \leq +\infty$ satisfying the relation

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}. \tag{A.17}$$

An elementary computation shows that

$$\|A_t\|_{L^q(\mathbb{R}^N)} = C_q t^{-\gamma}, \qquad \text{where} \quad \gamma = \frac{(N+1)q - N}{2q}.$$

In particular, $\gamma > 1$ and

$$\int_1^{+\infty} \|A_s\|_{L^q(\mathbb{R}^N)}\, ds < +\infty \qquad \text{if and only if} \;\; q > \frac{N}{N-1}. \tag{A.18}$$

Therefore, if $p < N$, for any number $r$ satisfying the relation $r > (1/p - 1/N)^{-1}$ we may find some $q_r$ satisfying (A.17) and (A.18). In particular,

$$\begin{aligned}
\| \int_0^{t-1} f_s\, ds \|_{L^p(\mathbb{R}^N)} &\leq \int_0^{t-1} \|f_s\|_{L^p(\mathbb{R}^N)}\, ds \\
&\leq C_q \int_0^{t-1} (t-s)^{-\gamma} \|\omega_s\|_{L^p(\mathbb{R}^N)} \\
&\leq C \sup_{t \in \mathbb{R}^+} \|\omega\|_{L^p(\mathbb{R}^N \times [t, t+1])}.
\end{aligned}$$

**Step 2: Estimates for $\nabla\varphi_2$.** The function $\varphi_2$ satisfies the heat equation

$$\begin{cases} \dfrac{\partial\varphi_2}{\partial t} - \Delta\varphi_2 = \omega\,\mathbf{1}_{\mathbb{R}^N\times[t-1,t]} \\ \varphi_2(x,t-1) = 0. \end{cases}$$

By the classical $L^p - L^q$ theory for the heat operator, we thus obtain

$$\|\nabla\varphi_2\|_{L^p(\mathbb{R}^N\times[t-1,t])} \le C\|\omega\|_{L^p(\mathbb{R}^N\times[t-1,t])}$$

and relation (A.15) follows by the Sobolev embedding. □

We turn now to the problem

$$\begin{cases} \dfrac{\partial\varphi}{\partial t} - \Delta\varphi = \mathrm{div}_x\, h & \text{on } \mathbb{R}^N \times (0,+\infty), \\ \qquad \varphi(x,0) = 0 & \text{for } x \in \mathbb{R}^N, \end{cases} \tag{A.19}$$

where $h = (h_1, \cdots, h_N)$ and $\mathrm{div}_x$ represents the divergence operator on $\mathbb{R}^N$. We have

**Lemma A.4.** *Let $1 \le p < +\infty$ and assume that*

$$\sup_{t\in\mathbb{R}^+}\|h\|_{L^p(\mathbb{R}^N\times[t,t+1])} < +\infty.$$

*Then, there exists a unique solution $\varphi$ to* (A.19) *such that $|\nabla_x\varphi\,| \le g_1+g_2$ where the functions $g_1$ and $g_2$ satisfy*

$$\sup_{t\in\mathbb{R}^+}\|g_1\|_{L^r(\mathbb{R}^N\times\{t\})} \le K(r)\sup_{t\in\mathbb{R}^+}\|h\|_{L^p(\mathbb{R}^N\times[t,t+1])} \tag{A.20}$$

*for every $r > p$ and*

$$\sup_{t\in\mathbb{R}^+}\|g_2\|_{L^p(\mathbb{R}^N\times[t,t+1])} \le K(p)\sup_{t\in\mathbb{R}^+}\|h\|_{L^p(\mathbb{R}^N\times[t,t+1])}. \tag{A.21}$$

*Proof.* As in Lemma A.3, we decompose $\varphi = \varphi_1 + \varphi_2$, where

$$\varphi_1(t,.) = \int_0^{t-1} e^{(t-s)\Delta}\,\mathrm{div}\, h(.,s)\,ds, \qquad \varphi_2(t,.) = \int_{t-1}^{t} e^{(t-s)\Delta}\,\mathrm{div}\, h(.,s)\,ds,$$

The function $G$ still denoting the fundamental solution of the heat equation, we have

$$|D_x^2 G(x,t)\,| \le CB_t(x) \equiv C\left(\frac{|x|^2}{t^{\frac{N+4}{2}}} + \frac{1}{t^{\frac{N+2}{2}}}\right)\exp\left(-\frac{|x|^2}{4t}\right).$$

**Step 1: Estimates for $\nabla\varphi_1$.** We have

$$|\nabla\varphi_1(x,t)| \le \int_0^{t-1} B_{t-s} * |h_s|\,ds \equiv \int_0^{t-1} f_s(x)\,ds.$$

By Young's inequality,

$$\|f_s\|_{L^r(\mathbb{R}^N)} \le \|B_{t-s}\|_{L^q(\mathbb{R}^N)}\|h_s\|_{L^p(\mathbb{R}^N)}$$

for any numbers $1 \leq p, q, r \leq +\infty$ satisfying $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. We compute

$$\|B_t\|_{L^q(\mathbb{R}^N)} = C_q t^{-\gamma}, \qquad \text{where} \quad \gamma = \frac{(N+2)q - N}{2q}.$$

In particular, for every $q > 1$ we have $\gamma > 1$ so that $\int_1^{+\infty} \|B_s\|_{L^q(\mathbb{R}^N)}\, ds < +\infty$. Inequality (A.20) follows, setting $g_1 = |\nabla\varphi_1|$.

**Step 2: Estimates for $\boldsymbol{\nabla\varphi_2}$.** Estimate (A.21) for $g_2 = |\nabla\varphi_2\,|$ is derived as in Step 2 of Lemma A.3, using standard $L^p - L^q$ estimates for the heat operator. □

We end this section recalling some classical results concerning the initial value problem for the heat operator.

**Lemma A.5.** *We have, for every $t > 0$,*

$$\left\|e^{t\Delta}\right\|_{\mathcal{L}(L^2(\mathbb{R}^N), L^\infty(\mathbb{R}^N))} = \frac{C_N}{t^{N/4}}$$

*where the constant $C_N$ depends only on $N$.*

*Proof.* For $t = 1$, the estimate is a direct consequence of the Cauchy-Schwartz inequality and the fact that $G(.,1)$ is bounded in $L^2$. The estimate for arbitrary $t$ follows by scaling. □

**Remark A.3.** i) The supremum defining the norm in Lemma A.5 is achieved only by the Gaussian $\exp(-|x|^2/4t)$, its multiples and its translates.

ii) More generally, we also have, for $1 \leq p < +\infty$, the estimate

$$\left\|e^{t\Delta}\right\|_{\mathcal{L}(L^p(\mathbb{R}^N), L^\infty(\mathbb{R}^N))} = \frac{C(N,p)}{t^{N/2p}}. \tag{A.22}$$

and

$$\left\|\nabla^k e^{t\Delta}\right\|_{\mathcal{L}(L^p(\mathbb{R}^N), L^\infty(\mathbb{R}^N))} = \frac{C(N,p,k)}{t^{N/2p+k/2}}. \tag{A.23}$$

## A.3 Local parabolic estimates

In this section we provide some pointwise and smoothing estimates for the heat operator on bounded domains. Let

$$\Lambda = B(0,1) \times [0,1], \qquad \Lambda_{\frac{1}{2}} = B(0,\frac{1}{2}) \times [\frac{3}{4}, 1].$$

We first have

**Lemma A.6.** *Let $u$ and $a$ be respectively a smooth and a continuous real-valued function on $\Lambda$ such that $\bar{a} = \inf_\Lambda a \geq 2$ and let $b > 0$, $d > 0$. Assume that*

$$|u| \leq d \qquad \text{on } \partial_P\Lambda \equiv B(0,1) \times \{0\} \cup \partial B(0,1) \times [0,1]$$

*and*

$$|\partial_t u - \Delta u + au| \leq b \qquad \text{on } \Lambda.$$

*Then, there exists a constant $c > 0$ depending only on $N$ such that*

$$|u| \leq C\left(\frac{b+d}{\bar{a}}\right) \qquad \text{on } \Lambda_{\frac{1}{2}}.$$

*Proof.* By linearity, it suffices to consider the case $d = 1$. Let $\chi$ be a smooth cut-off function defined on $\mathbb{R}^N$ such that $0 \leq \chi \leq 1$ and $\chi \equiv 1$ on $B(0, \frac{1}{2})$, $\chi \equiv 0$ on $\mathbb{R}^N \setminus B(0, \frac{3}{4}$. Consider the function $\tau$ defined on $[0, 1]$ by $\tau(t) = 1 - \exp(-\bar{a}t)$, so that $0 \leq \tau(t) \leq 1$ and set $\sigma_0(x, t) = 1 - \tau(t)\chi(x)$. We have $\sigma_0 \geq 0$ on $\Lambda$, and

$$\partial_t \sigma_0 + a\sigma_0 \geq 0, \qquad |\Delta\sigma_0| \leq |\tau(t)| \cdot |\Delta\chi(x)| \leq C_0 \quad \text{on } \Lambda,$$

so that $\partial_t\sigma_0 - \Delta\sigma_0 + a\sigma_0 \geq -C_0$ on $\Lambda$. Finally set $\sigma = \sigma_0 + (\frac{C_0+b}{\bar{a}})$. By construction,

$$\partial_t\sigma - \Delta\sigma + a\sigma \geq b \geq \partial_t u - \Delta u + au \qquad \text{on } \Lambda.$$

On the other hand,

$$\sigma = 1 + \frac{C_0 + b}{a} \geq 1 \geq u \qquad \text{on } \partial_P\Lambda,$$

so that, by the maximum principle, $u \leq \sigma$ on $\Lambda$. Since $\chi \equiv 1$ on $B(0, 1/2)$, we have on $\Lambda_{\frac{1}{2}}$

$$u \leq \sigma \leq \exp(-\frac{3}{4}\bar{a}) + \frac{C_0 + b}{\bar{a}} \leq C\left(\frac{b+1}{\bar{a}}\right).$$

Applying the same argument to $-u$ we complete the proof. $\square$

**Lemma A.7.** *Let $u$ be a smooth real-valued function on $\Lambda$ and assume*

$$|\partial_t u - \Delta u| \leq b \qquad \text{on } \Lambda, \tag{A.24}$$

$$|u| \leq d \qquad \text{on } \Lambda. \tag{A.25}$$

*Then, there exists $0 < \alpha < 1$, $0 < \beta < 1$ and $c > 0$ depending only on $N$ such that*

$$\|\nabla u\|_{\mathcal{C}_P^{0,\alpha}(\Lambda_{\frac{1}{2}})} \leq C(b^\beta c^{1-\beta} + d).$$

*Here the norm $\mathcal{C}_P^{0,\alpha}$ denotes the parabolic Hölder norm defined by*

$$\|g\|_{\mathcal{C}_P^{0,\alpha}(\Lambda)} = \sup\{\frac{|g(x,t) - g(x',t')|}{(|x - x'| + |t - t'|^{1/2})^\alpha}, \quad (x,t)\,,\ (x',t') \in \Lambda\}.$$

*Proof.* Since (A.24) and (A.25) are $L^\infty$ bounds, we deduce from standard linear theory that, for every $1 < q_1, q_2 < +\infty$,

$$\|u\|_{W^{1,q_1}(I, L^{q_2}(B_{1/2}))} \leq C(b + d), \qquad \|u\|_{L^{q_1}(I, W^{1,q_2}(B_{1/2}))} \leq C(b + d),$$

where $I = [3/4, 1]$. Interpolating these inequalities we obtain $\|u\|_{W^{1/3,q_1}(I, W^{4/3,q_2}(B_{1/2}))} \leq C(b + d)$. Choosing $q_1$ and $q_2$ sufficiently large (in particular $q_1 > 3$, $q_2 > 3N$), we obtain that for every $0 < \gamma < 1$, $\|u\|_{\mathcal{C}^{0,1/4}(I, \mathcal{C}^{1,\gamma}(B_{1/2}))} \leq C_\gamma(b + d)$. On the other hand, (A.25) can be rephrased as $\|u\|_{L^\infty(I, L^\infty(B_{1/2}))} \leq d$, and therefore, by interpolation again, $\|u\|_{\mathcal{C}^{0,1/5}(I, \mathcal{C}^{1,\alpha}(B_{1/2}))} \leq C(b^\beta d^{1-\beta} + d)$, for some $\alpha < 1/5$ and $0 < \beta = \beta(\alpha) < 1$. $\square$

## Appendix B : Estimates for Jacobians

The fact that Jacobians have remarkable compensation properties, in particular in the context of the Ginzburg-Landau functional, has played an expanding role in recent years, after the pioneering work of Jerrard and Soner [19]. In this appendix we provide some variants, using the results of [7], adapted to the parabolic situation considered in this paper. Throughout this appendix, we assume that $w_\varepsilon$ is defined on $\mathbb{R}^N \times \mathbb{R}$ and satisfies the following bounds

$$\int_{\mathbb{R}^N \times [t,t+1]} e_\varepsilon(w_\varepsilon) \le CM_0|\log\varepsilon|, \qquad \forall\, t > 0 \tag{B.1}$$

$$\int_{\mathbb{R}^N \times \mathbb{R}} |\partial_t w_\varepsilon|^2 \le CM_0|\log\varepsilon|, \tag{B.2}$$

$$|w_\varepsilon| \le 3. \tag{B.3}$$

The following is a direct consequence of Theorem 2 of [7].

**Proposition B.1.** *Assume $w_\varepsilon$ verifies (B.1), (B.2) and (B.3). Then we may write*[34]

$$J_{x,t} w_\varepsilon = \omega_\varepsilon + \delta h_\varepsilon\,,$$

*where $\omega_\varepsilon$ and $h_\varepsilon$ verify*

$$\|\omega_\varepsilon\|_{L^1(\mathbb{R}^N \times [t,t+1])} \le CM_0\,, \qquad \forall t > 0, \tag{B.4}$$

$$\|h_\varepsilon\|_{L^p(\mathbb{R}^N \times [t,t+1])} \le C_p M_0 \varepsilon^{\alpha_p} \tag{B.5}$$

*for every $1 < p < 2$, where $\alpha_p > 0$ is some number depending only on $p$.*

*Proof.* We apply Theorem 2 of [7] to $w_\varepsilon$ restricted to the slices $\Lambda_n = \mathbb{R}^N \times [n - \frac{1}{4}, n + \frac{5}{4}]$, for $n \in \mathbb{N}^*$.[35] This provides a function $v_\varepsilon^n : \Lambda_n \to \mathbb{C}$ such that

$$\begin{aligned} &|v_\varepsilon^n| \le 1, && \int_{\Lambda_n} e_\varepsilon(v_\varepsilon^n) \le C \int_{\Lambda_n} e_\varepsilon(w_\varepsilon) \le CM_0|\log\varepsilon|, \\ &\|J_{x,t} v_\varepsilon^n\|_{L^1(\Lambda_n)} \le CM_0, && \|v_\varepsilon^n - w_\varepsilon\|_{L^2(\Lambda_n)} \le CM_0\varepsilon^\alpha, \end{aligned} \tag{B.6}$$

where $0 < \alpha < 1$ is some positive number. We set

$$\omega_\varepsilon^n = J_{x,t} v_\varepsilon^n\,, \qquad h_\varepsilon^n = \frac{1}{2}(v_\varepsilon^n - w_\varepsilon) \times (\delta v_\varepsilon^n + \delta w_\varepsilon)\,, \qquad \text{on } \Lambda_n$$

so that $J w_\varepsilon = \omega_\varepsilon^n + \delta h_\varepsilon^n$ on $\Lambda_n$. Clearly $\|\omega_\varepsilon^n\|_{L^1(\Lambda_n)} \le CM_0$. Moreover, by Cauchy-Schwarz inequality,

$$\|h_\varepsilon^n\|_{L^1(\Lambda_n)} \le CM_0\varepsilon^\alpha|\log\varepsilon|^{1/2}.$$

On the other hand, since $|v_\varepsilon^n| \le 1$, $|w_\varepsilon| \le 3$, we deduce $\|h_\varepsilon^n\|_{L^2(\Lambda_n)} \le CM_0|\log\varepsilon|^{1/2}$, so that by interpolation

$$\|h_\varepsilon^n\|_{L^p(\Lambda_n)} \le C_p M_0 \varepsilon^{\alpha_p} \qquad \text{for every } 1 \le p < 2.$$

[34] Here we will denote $\delta$ and $\delta^*$ respectively the exterior differentiation operator for differential forms on $\mathbb{R}^N \times \mathbb{R}$ and its formal adjoint, while we will use the standard notations $d$ and $d^*$ when restricting to time slices $\mathbb{R}^N \times \{t\}$.

[35] Although the domain $\Omega$ in [7] was assumed to be bounded, a careful reading of the proof shows that the arguments carry over to the situation considered here.

To complete the proof, we merely have to reconnect the functions $h^n_\varepsilon$, defined on the sets $\Lambda_n$, in the overlapping regions. For this purpose we use a partition of unity on the time axis. We write

$$1 = \sum_{i\in\mathbb{Z}} g(t-i), \qquad t \in \mathbb{R},$$

where the function $g$ has compact support on $\Lambda_0$ and is lipschitz. Hence

$$J_{x,t}w_\varepsilon = \sum_{i\in\mathbb{Z}} g(t-i)(\omega^i_\varepsilon + \delta h^i_\varepsilon) = \sum_{i\in\mathbb{Z}} g(t-i)\omega^i_\varepsilon + g'(t-i)dt\wedge h^i_\varepsilon + \sum_{i\in\mathbb{Z}} \delta(g(t-i)h^i_\varepsilon).$$

We set $\omega_\varepsilon = \sum_{i\in\mathbb{Z}} g(t-i)\omega^i_\varepsilon + g'(t-i)dt\wedge h^i_\varepsilon$, and $h_\varepsilon = \sum_{i\in\mathbb{Z}} g(t-i)h^i_\varepsilon$, and one easily verifies the desired estimates, since the sums involve a finite number of non-zero terms. □

If we restrict the attention to space-time components of the Jacobians, i.e. the quantities

$$J^{0i}_{x,t}w_\varepsilon = \frac{\partial w_\varepsilon}{\partial t} \times \frac{\partial w_\varepsilon}{\partial x_i}, \qquad \text{for } i = 1, ..., N\,,$$

then better estimates can be obtained in view of assumption (B.2). This important observation was already stressed in [26] (see also [16] and [6], Section 6, for related ideas).

**Proposition B.2.** *Let $w_\varepsilon$ verify conditions (B.1), (B.2) and (B.3). Then we may write*

$$J_{x,t}w_\varepsilon = \omega_\varepsilon + \mathrm{div}_{x,t}\lambda_\varepsilon,$$

*where $\omega_\varepsilon$ is a real-valued two-form and $\lambda_\varepsilon$ is a two-form with coefficients in $\mathbb{R}^N$ satisfying*[36]

$$\|w_\varepsilon\|_{L^1(\mathbb{R}^N\times[t,t+1]} \le CM_0, \tag{B.7}$$

$$\|\lambda_\varepsilon\|_{L^q(\mathbb{R}^N\times[t,t+1])} \le C_q M_0 \varepsilon^{\alpha_q} \tag{B.8}$$

*for every $1 < q < 2$, where $\alpha_q > 0$ is some number depending only on $p$. Moreover, writing*

$$\omega_\varepsilon = \sum_{i=1}^N \omega^{0i}_\varepsilon dt\wedge dx_i + \sum_{1\le i<j\le N} \omega^{ij}_\varepsilon dx_i \wedge dx_j,$$

*the space-time components $\omega^{0i}_\varepsilon$ verify, for $p > 2$,*

$$\left(\int_{\mathbb{R}} \|\omega^{0i}_\varepsilon\|^p_{L^1(\mathbb{R}^N\times[t,t+1])} dt\right)^{\frac{1}{p}} \le CM_0. \tag{B.9}$$

*Proof.* We consider again the slices $\Lambda_n = \mathbb{R}^N \times [n - \frac{1}{4}, n + \frac{5}{4}]$ and set

$$A_n = \int_{\Lambda_n} \left|\frac{\partial w_\varepsilon}{\partial t}\right|^2 dxdt, \qquad B_n = \int_{\Lambda_n} e_\varepsilon(w_\varepsilon(x,t))dxdt.$$

Let be given $p > 2$. We distinguish two cases:

[36] Writing $\lambda_\varepsilon = \sum_{i,j=0}^N \lambda^{ij}_\varepsilon dx_i \wedge dx_j$, we define $\mathrm{div}_{x,t}\lambda_\varepsilon = \sum_{i,j=0}^N \mathrm{div}_{x,t}\lambda^{ij}_\varepsilon dx_i\wedge dx_j$.

**Case 2.** $A_n \geq (M_0|\log\varepsilon|)^{-\frac{p+2}{p-2}}$. In this case, we rescale (as in [16, 6, 26]) the function $w_\varepsilon$ with respect to the time variable, and set

$$\tilde{w}_\varepsilon^n(x,s) = w_\varepsilon(x, \sqrt{\frac{B_n}{A_n}}s + n - \frac{1}{4}), \qquad \text{for } (x,s) \in \tilde{\Lambda}_n \equiv \mathbb{R}^N \times [0, \frac{3}{2}\sqrt{\frac{A_n}{B_n}}].$$

Notice that the width of the strip $\tilde{\Lambda}_n$ is larger than $|\log\varepsilon|^{-\tau}$ for some $\tau > 0$. We compute

$$\int_{\tilde{\Lambda}_n} \left|\frac{\partial \tilde{w}_\varepsilon^n}{\partial s}\right|^2 dxds = \sqrt{\frac{B_n}{A_n}} \int_{\Lambda_n} \left|\frac{\partial w_\varepsilon}{\partial t}\right|^2 = \sqrt{A_n B_n}.$$

and

$$\int_{\tilde{\Lambda}_n} e_\varepsilon(\tilde{w}_\varepsilon^n(x,s))dxds = \sqrt{\frac{A_n}{B_n}} \int_{\Lambda_n} e_\varepsilon(w_\varepsilon(x,t))dxdt = \sqrt{A_n B_n}.$$

We argue as in Proposition B.1 and apply[37] Theorem 2 of [7] to $\tilde{w}_\varepsilon^n$ on $\tilde{\Lambda}_n$. This yields a complex-valued function $\tilde{v}_\varepsilon^n$ on $\tilde{\Lambda}_n$ such that $|\tilde{v}_\varepsilon^n| \leq 1$, and

$$\int_{\tilde{\Lambda}_n} \frac{1}{2}\left|\frac{\partial \tilde{v}_\varepsilon^n}{\partial s}\right|^2 + e_\varepsilon(\tilde{v}_\varepsilon^n(x,s)dxds \leq C\int_{\tilde{\Lambda}_n} \frac{1}{2}\left|\frac{\partial \tilde{w}_\varepsilon^n}{\partial s}\right|^2 + e_\varepsilon(\tilde{w}_\varepsilon^n(x,s)dxds \leq C\sqrt{A_n B_n},$$

$$\|J_{x,s}\tilde{v}_\varepsilon^n\|_{L^1(\tilde{\Lambda}_n)} \leq \frac{C}{|\log\varepsilon|}\int_{\tilde{\Lambda}_n} \frac{1}{2}\left|\frac{\partial \tilde{w}_\varepsilon^n}{\partial s}\right|^2 + e_\varepsilon(\tilde{w}_\varepsilon^n(x,s)dxds \leq C\frac{\sqrt{A_n B_n}}{|\log\varepsilon|},$$

$$\|\tilde{v}_\varepsilon^n - \tilde{w}_\varepsilon^n\|_{L^2(\tilde{\Lambda}_n)} \leq C(A_n B_n)^{1/4}\varepsilon^\alpha.$$

We inverse next the scaling and go back to the original strip $\Lambda_n$, where we define the functions $v_\varepsilon^n$ as follows

$$v_\varepsilon^n(x,t) = \tilde{v}_\varepsilon^n(x, \sqrt{\frac{A_n}{B_n}}t - n + \frac{1}{4}), \qquad (x,t) \in \Lambda_n.$$

The integral of space-time components of $J_{x,s}\tilde{v}_\varepsilon^n$ are invariant under this transformation, that is

$$\|J_{x,t}^{0i}v_\varepsilon^n\|_{L^1(\Lambda_n)} = \|J_{x,s}^{0i}\tilde{v}_\varepsilon^n\|_{L^1(\tilde{\Lambda}_n)} \leq C\frac{\sqrt{A_n B_n}}{|\log\varepsilon|}, \tag{B.10}$$

whereas, for $1 \leq i \leq j \leq N$,

$$||J_{x,t}^{ij}v_\varepsilon^n\|_{L^1(\Lambda_n)} = \sqrt{\frac{B_n}{A_n}}\|J_{x,s}^{ij}\tilde{v}_\varepsilon^n\|_{L^1(\tilde{\Lambda}_n)} \leq CM_0. \tag{B.11}$$

On the other hand, we have $\|v_\varepsilon^n - w_\varepsilon\|_{L^2(\Lambda_n)} \leq C\sqrt{B_n}\varepsilon^\alpha$. We set

$$\omega_\varepsilon^n = J_{x,t}v_\varepsilon^n, \qquad h_\varepsilon^n = (v_\varepsilon^n - w_\varepsilon) \times (\delta v_\varepsilon^n + \delta w_\varepsilon),$$

and

$$\lambda_\varepsilon^{n,ij} = (0, \cdots, (v_\varepsilon^n - w_\varepsilon) \times \partial_{x_j}(v_\varepsilon^n + w_\varepsilon), \cdots, -(v_\varepsilon^n - w_\varepsilon) \times \partial_{x_i}(v_\varepsilon^n + w_\varepsilon)).$$

[37] This is possible because the width of the strip is not too small.

In view of (B.10), we have

$$\|\omega_\varepsilon^{n,0i}\|_{L^1(\Lambda_n)} \le CM_0^{1/2}\sqrt{\frac{A_n}{|\log\varepsilon|}}. \tag{B.12}$$

**Case 1.** $A_n < (M_0|\log\varepsilon|)^{-\frac{p+2}{p-2}}$. In this case, the previous method may not apply, since the width of the scaled strip $\tilde{\Lambda}_n$ might be too small. Therefore we argue differently, and distinguish spatial and space-time components. For $i = 1, \cdots, N$, we set

$$\omega_\varepsilon^{n,0i} = J_{x,t}^{0i} w_\varepsilon\,, \qquad \lambda_\varepsilon^{n,0i} = 0 \qquad \text{on } \Lambda_n.$$

By Cauchy-Schwarz inequality, we have in particular

$$\|\omega_\varepsilon^{n,0i}\|_{L^1(\Lambda_n)} \le A_n^{1/2} B_n^{1/2} \le CM_0^{1/2}|\log\varepsilon|^{1/2} A_n^{1/2}. \tag{B.13}$$

For the spatial components $\omega_\varepsilon^{n,ij}$ we use the construction of Proposition B.1, and set as above

$$\omega_\varepsilon^{n,ij} = J_{x,t}^{ij} v_\varepsilon^n, \qquad \lambda_\varepsilon^{n,ij} = (0, \cdots, (v_\varepsilon^n - w_\varepsilon) \times \partial_{x_j}(v_\varepsilon^n + w_\varepsilon), \cdots, -(v_\varepsilon^n - w_\varepsilon) \times \partial_{x_i}(v_\varepsilon^n + w_\varepsilon)),$$

where $v_\varepsilon^n$ is defined by Theorem 2 of [7] restricted to $\Lambda_n$ and verifying (B.6).

We need now to recombine the different strips. To that aim, set $I_1 = \{n \in \mathbb{Z}, \ \ A_n \le (M_0|\log\varepsilon|)^{-\frac{p+2}{p-2}}\}$, i.e. the set of indices $n$ where Case 2 holds, and $I_2 = \mathbb{Z} \setminus I_1$. In view of (B.12) and (B.2), we have

$$\sum_{n\in I_2} \|\omega_\varepsilon^{n,0i}\|_{L^1(\Lambda_n)}^2 \le C\frac{M_0}{|\log\varepsilon|}\sum_{n\in I_2} A_n \le CM_0^2. \tag{B.14}$$

On the other hand, by (B.13), we have

$$\sum_{n\in I_1} \|\omega_\varepsilon^{n,0i}\|_{L^1(\Lambda_n)}^p \le CM_0^{p/2}|\log\varepsilon|^{p/2}\sum_{n\in I_1} A_n^{p/2}.$$

We write

$$\sum_{n\in I_1} A_n^{p/2} \le \sup_{n\in I_1} A_n^{\frac{p-2}{2}} \sum_{n\in\mathbb{Z}} A_n \le CM_0|\log\varepsilon| \cdot |\log\varepsilon|^{-\frac{p+2}{2}},$$

so that finally

$$\sum_{n\in I_1} \|\omega_\varepsilon^{n,0i}\|_{L^1(\Lambda_n)}^p \le CM_0^p. \tag{B.15}$$

Combining (B.14) and (B.15), we are led to

$$\sum_{n\in\mathbb{Z}} \|\omega_\varepsilon^{n,0i}\|_{L^1(\Lambda_n)}^p \le CM_0^p. \tag{B.16}$$

The proof of Proposition B.2 is then completed as in Proposition B.1, reconnecting the $\omega_\varepsilon^n$ and $\lambda_\varepsilon^n$ using a partition of unity. Estimates (B.7) and (B.8) are derived as in Proposition B.1, whereas estimate (B.9) is a direct consequence of (B.16). $\square$

## Appendix C : Higher order regularity for $(\mathbf{PGL})_\varepsilon$

The aim of this section is to provide the proof of Theorem 2.1. The starting point of the analysis is the following Harnack-Moser-Struwe type inequality

**Proposition C.1.** *Assume* (2.22) *holds. There exists a constant* $0 < \sigma_0 < \frac{1}{2}$ *such that, if* $\sigma \le \sigma_0$*, then*

$$e_\varepsilon(u_\varepsilon)(x,t) \le C(\Lambda) \int_\Lambda e_\varepsilon(u_\varepsilon)\,, \tag{C.1}$$

*for any* $(x,t) \in \Lambda_{\frac{3}{4}}$.

The proof of Proposition C.1 is given in [6], Theorem 2. The case $\int_\Lambda e_\varepsilon(u_\varepsilon)$ is small was treated before by Struwe [31], whereas in [6] we allow a $|\log\varepsilon|$ divergence.

**Proof of Theorem 2.1.** By scaling, it suffices to consider the case $\Lambda = B(0,1) \times [0,1]$.

**Step 1: proof of i).** It is an immediate consequence of (C.1). For the proof of ii) and iii) we heavily rely on the system of equations for $\theta_\varepsilon \equiv 1 - \rho_\varepsilon$ and $\varphi_\varepsilon$

$$\partial_t\theta_\varepsilon - \Delta\theta_\varepsilon + a\theta_\varepsilon = (1-\theta_\varepsilon)|\nabla\varphi_\varepsilon|^2, \tag{C.2}$$

$$\rho_\varepsilon^2\partial_t\varphi_\varepsilon - \operatorname{div}(\rho_\varepsilon^2\nabla\varphi_\varepsilon) = 0, \tag{C.3}$$

where

$$a \equiv \frac{1 + (1-\theta_\varepsilon)^2}{\varepsilon^2}.$$

In particular, $\bar a = \inf_\Lambda a \ge \varepsilon^{-2}$.

**Step 2: proof of (2.24).** We apply Lemma A.6 to equation (C.2) on $\Lambda_{\frac{3}{4}}$ with $u = \theta_\varepsilon$, $b = 2\|\nabla\varphi_\varepsilon\|_{L^\infty(\Lambda_{3/4})}$, $d = 1$. We therefore obtain

$$\|1-\rho_\varepsilon\|_{L^\infty(\Lambda_{5/8})} \le C(\Lambda)\varepsilon^2(\|\nabla\varphi_\varepsilon\|^2_{L^\infty(\Lambda_{3/4})}), \tag{C.4}$$

so that (2.24) follows.

**Step 3: estimates on $|\nabla\rho_\varepsilon|$.** It follows from (C.4) that $\|a\theta_\varepsilon\|_{L^\infty(\Lambda_{5/8})} \le C|\log\varepsilon|$, and therefore

$$|\partial_t\theta_\varepsilon - \Delta\theta_\varepsilon| \le C|\log\varepsilon|.$$

We apply Lemma A.7 on $\Lambda_{5/8}$ to $\theta_\varepsilon$ with $b = C|\log\varepsilon|$ and $d = C\varepsilon^2|\log\varepsilon|$. This yields

$$\|\nabla\rho_\varepsilon\|_{\mathcal{C}^{0,\alpha}(\Lambda_{9/16})} = \|\nabla\theta_\varepsilon\|_{\mathcal{C}^{0,\alpha}(\Lambda_{9/16})} \le C\varepsilon^{\beta_1} \tag{C.5}$$

for some $0 < \alpha, \beta_1 < 1$, which gives the desired estimate for the right-hand side of (2.25). We turn next to $\partial_t\theta_\varepsilon$. For that purpose, we will differentiate (C.2) according to time: however, this requires higher order estimates on $\varphi_\varepsilon$.

**Step 4: estimates on $D_x^2\varphi_\varepsilon$.** We turn to (C.3); expanding the r.h.s. and dividing by $\rho_\varepsilon$ we are led to

$$\partial_t\varphi_\varepsilon - \Delta\varphi_\varepsilon = 2\frac{\nabla\rho_\varepsilon}{\rho_\varepsilon}\cdot\nabla\varphi_\varepsilon. \tag{C.6}$$

Since $\nabla\rho_\varepsilon \in \mathcal{C}^{0,\alpha}(\Lambda_{9/16})$ and $\rho_\varepsilon \geq \frac{1}{2}$ we obtain, invoking standard Schauder theory (see e.g. [15]) that

$$\|D_x^2\varphi_\varepsilon\|_{\mathcal{C}^{0,\alpha}(\Lambda_{17/36})} \leq C|\log\varepsilon|^{1/2}. \tag{C.7}$$

**Step 5: estimates on $D_x^2\rho_\varepsilon$.** We differentiate (C.2) with respect to $x$. Setting $u = \nabla\theta_\varepsilon$ we obtain

$$\partial_t u - \Delta u + \frac{2}{\varepsilon^2}u = -\nabla\rho_\varepsilon|\nabla\varphi_\varepsilon|^2 + 2(1-\theta_\varepsilon)D_x^2\varphi_\varepsilon\nabla\varphi_\varepsilon + (4-3\theta_\varepsilon)\frac{\theta_\varepsilon}{\varepsilon^2}\nabla\theta_\varepsilon.$$

In view of estimates (C.4), (C.5) and (C.7), the r.h.s. of the previous equation is uniformly bounded on $\Lambda_{\frac{17}{36}}$ by $C|\log\varepsilon|$. On the other hand, we already know that $u$ is bounded by $C\varepsilon^\beta$. Invoking Lemma A.6 once more, we deduce

$$\|\nabla u\|_{L^\infty(\Lambda_{33/64})} = \|D_x^2\rho_\varepsilon\|_{L^\infty(\Lambda_{33/64})} \leq C\varepsilon^{\beta_2}, \tag{C.8}$$

for some $0 < \beta_2 < 1$.

**Step 6: estimates on $\partial_t\nabla\varphi_\varepsilon$.** Differentiating (C.6) with respect to $x$ and by Step 5 we obtain

$$\|\partial_t\nabla\varphi_\varepsilon\|_{L^\infty(\Lambda_{65/128})} \leq C|\log\varepsilon|^{1/2}. \tag{C.9}$$

**Step 7: proof of (2.25) completed.** In view of Step 6, we may now differentiate equation (C.2) with respect to $t$. Setting $u = \partial_t\theta_\varepsilon$ we obtain

$$\partial_t u - \Delta u + au = 2(1-\theta_\varepsilon)\nabla\varphi_\varepsilon\partial_t\nabla\varphi_\varepsilon, \tag{C.10}$$

where

$$a = \frac{2}{\varepsilon^2} - |\nabla\varphi_\varepsilon|^2 - \frac{4-3\theta_\varepsilon}{\varepsilon^2}\theta_\varepsilon.$$

We notice that $a \geq \frac{1}{\varepsilon^2}$ (if $\varepsilon$ is sufficiently small). In view of (C.9) we obtain

$$|\partial_t u - \Delta u + au| \leq C|\log\varepsilon| \qquad \text{on } \Lambda_{\frac{65}{128}}.$$

On the other hand,

$$|u| = |\partial_t\theta_\varepsilon| = |\Delta\theta_\varepsilon - \frac{1+(1-\theta_\varepsilon)^2}{\varepsilon^2}\theta_\varepsilon + (1-\theta_\varepsilon)|\nabla\varphi_\varepsilon|^2| \leq C|\log\varepsilon|.$$

Invoking Lemma A.6 we obtain $|u| \leq C\varepsilon^2|\log\varepsilon| \quad$ on $\Lambda_{\frac{129}{256}}$. Applying Lemma A.7 we are led to

$$\|u\|_{\mathcal{C}^{0,\alpha}(\Lambda_{1/2})} = \|\partial_t\nabla\rho_\varepsilon\|_{\mathcal{C}^{0,\alpha}(\Lambda_{1/2})} \leq C\varepsilon^{\beta_1}.$$

In particular, this completes the proof of (2.25), and hence ii).

**Proof of iii).** We introduce the solution $\Phi_\varepsilon$ of the boundary value problem on $\Lambda_{\frac{1}{2}}$

$$\begin{cases} \partial_t\Phi_\varepsilon - \Delta\Phi_\varepsilon = 0 & \text{on } \Lambda_{\frac{1}{2}}, \\ \Phi_\varepsilon(x,t) = \varphi_\varepsilon(x,t) & \text{on } \partial_P\Lambda_{\frac{1}{2}}. \end{cases} \tag{C.11}$$

$$\begin{cases} \partial_t\Phi_1 - \Delta\Phi_1 = 2\frac{\nabla\rho_\varepsilon}{\rho_\varepsilon}\cdot\nabla\varphi_\varepsilon & \text{on } \Lambda_{\frac{1}{2}}, \\ \Phi_1(x,0) = 0 & \text{on } \partial_P\Lambda_{\frac{1}{2}}. \end{cases} \tag{C.12}$$

The r.h.s. of (C.12) is estimated by $C\varepsilon_3^\beta$ for some $0 < \beta_3 < 1$ in $\mathcal{C}^{0,\alpha}(\Lambda_{\frac{1}{2}})$. Estimate iii) follows immediately.

# References


[1] G. Alberti, S. Baldo and G. Orlandi, *Variational convergence for functionals of Ginzburg-Landau type*, Indiana Math. Journal, submitted.

[2] L. Ambrosio and M. Soner, *A measure theoretic approach to higher codimension mean curvature flow*, Ann. Sc. Norm. Sup. Pisa, Cl. Sci. **25** (1997), 27-49.

[3] P. Baumann, C-N. Chen, D. Phillips, P. Sternberg, *Vortex annihilation in nonlinear heat flow for Ginzburg-Landau systems*, Eur. J. Appl. Math. **6** (1995), 115–126.

[4] F. Bethuel, H. Brezis and F. Hélein, *Ginzburg-Landau vortices*, Birkhäuser, Boston, 1994.

[5] F. Bethuel, G. Orlandi and D. Smets, *Vortex rings for the Gross-Pitaevskii equation*, Jour. Eur. Math. Soc. **6** (2004), 17-94.

[6] F. Bethuel, G. Orlandi and D. Smets, *Convergence of the parabolic Ginzburg-Landau equation to motion by mean curvature*, Annals of Math., to appear.

[7] F. Bethuel, G. Orlandi and D. Smets, *Approximations with vorticity bounds for the Ginzburg-Landau functional*, Comm. Contemp. Math. **6** (2004), 803-832.

[8] K. Brakke, *The motion of a surface by its mean curvature*, Princeton University Press, 1978.

[9] H. Brezis and T. Gallouet, *Nonlinear Schrödinger evolution equations,* Nonlinear Anal. **4** (1980), 677-681.

[10] L. Bronsard and R.V. Kohn, *Motion by mean curvature as the singular limit of Ginzburg-Landau dynamics,* J. Differential Equations **90** (1991), 211-237.

[11] G. Buttazzo, *Integral representation theory for some classes of local functions,* Optimization and nonlinear analysis (Haifa, 1990), 64-75, Pitman Res. Notes Math. Ser., 244, Longman Sci. Tech., Harlow, 1992.

[12] X. Chen, *Generation and propagation of interfaces for reaction-diffusion equations,* J. Differential Equations **96** (1992), 116-141.

[13] E. De Giorgi, *Some conjectures on flow by mean curvature*, Proc. of the Capri Workshop 1990, Benevento-Bruno-Sbordone (eds.), 1990.

[14] W. E, *Dynamics of vortices in Ginzburg-Landau theories with applications to superconductivity,* Phys. D **77** (1994), no. 4, 383-404.

[15] A. Friedman, *Partial differential equations of parabolic type,* Prentice-Hall, Englewood Cliffs, N.J. 1964.

[16] R.L. Jerrard, *Vortex dynamics for the Ginzburg-Landau wave equation,* Calc. Var. Partial Diff. Eq. **9** (1999), no. 1, 1-30.

[17] R.L. Jerrard and H.M. Soner, *Dynamics of Ginzburg-Landau vortices,* Arch. Rational Mech. Anal. **142** (1998), 99-125.

[18] R.L. Jerrard and H.M. Soner, *Scaling limits and regularity results for a class of Ginzburg-Landau systems,* Ann. Inst. H. Poincaré Anal. Non Linéaire **16** (1999), 423-466.

[19] R.L. Jerrard and H.M. Soner, *The Jacobian and the Ginzburg-Landau energy,* Calc. Var. PDE **14** (2002), 151-191.

[20] F.H. Lin, *Some dynamical properties of Ginzburg-Landau vortices*, Comm. Pure Appl. Math. **49** (1996), 323–359.

[21] F.H. Lin, *Complex Ginzburg-Landau equations and dynamics of vortices, filaments, and codimension-*2 *submanifolds,* Comm. Pure Appl. Math. **51** (1998), 385-441.

[22] J.C. Neu, *Vortices in complex scalar fields,* Phys. D **43** (1990), no.2-3, 385-406.

[23] L.M. Pismen and J. Rubinstein, *Motion of vortex lines in the Ginzburg-Landau model,* Phys. D **47** (1991), 353-360.

[24] Y. Reshetnyak, *Weak convergence of completely additive functions on a set,* Siberian Math. J. **9** (1968), 487-498.

[25] J. Rubinstein and P. Sternberg, *On the slow motion of vortices in the Ginzburg-Landau heat-flow,* SIAM J. Appl. Math. **26** (1995), 1452-1466.

[26] E. Sandier and S. Serfaty, *A product-estimate for Ginzburg-Landau and corollaries,* J. Funct. Anal., to appear.

[27] E. Sandier and S. Serfaty, *Gamma-convergence of gradient flows with applications to Ginzburg-Landau,* Comm. Pure App. Math., to appear.

[28] S. Serfaty, *Vortex Collision and Energy Dissipation Rates in the Ginzburg-Landau Heat Flow,* in preparation.

[29] H.M. Soner, *Ginzburg-Landau equation and motion by mean curvature. I. Convergence,* and *II. Development of the initial interface,* J. Geom. Anal. **7** (1997), no. 3, 437-475 and 477-491.

[30] D. Spirn, *Vortex dynamics of the full time-dependent Ginzburg-Landau equations,* Comm. Pure Appl. Math. **55** (2002), no. 5, 537-581.

[31] M. Struwe, *On the evolution of harmonic maps in higher dimensions,* J. Diff. Geom. **28** (1988), 485–502.

## Addresses

**Fabrice Bethuel**, Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, 4 place Jussieu BC 187, 75252 Paris, France & Institut Universitaire de France.
E-mail : `bethuel@ann.jussieu.fr`

**Giandomenico Orlandi**, Dipartimento di Informatica, Università di Verona, Strada le Grazie, 37134 Verona, Italy.
E-mail : `orlandi@sci.univr.it`

**Didier Smets**, Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, 4 place Jussieu BC 187, 75252 Paris, France.
E-mail : `smets@ann.jussieu.fr`