# Overdetermined problems with possibly degenerate ellipticity, a geometric approach

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**Abstract**

Given an open bounded connected subset $\Omega$ of $\mathbb{R}^n$, we consider the overdetermined boundary value problem obtained by adding both zero Dirichlet and constant Neumann boundary data to the elliptic equation $-\mathrm{div}(A(|\nabla u|)\nabla u) = 1$ in $\Omega$. We prove that, if this problem admits a solution in a suitable weak sense, then $\Omega$ is a ball. This is obtained under fairly general assumptions on $\Omega$ and $A$. In particular, $A$ may be degenerate and no growth condition is required. Our method of proof is quite simple. It relies on a maximum principle for a suitable $P$-function, combined with some geometric arguments involving the mean curvature of $\partial\Omega$.

**Keywords:** overdetermined boundary value problem, degenerate elliptic operators.

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## 1 Introduction

For a bounded, connected, open set $\Omega \subset \mathbb{R}^n$ and for a parameter $c > 0$, consider the elliptic boundary value problem

$$\begin{cases} -\mathrm{div}(A(|\nabla u|)\nabla u) = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ |\nabla u| = c & \text{on } \partial\Omega\ . \end{cases} \tag{1}$$

Imposing boundary conditions for both $u$ and $\nabla u$ on $\partial\Omega$ makes problem (1) *overdetermined*, so that in general it has no solution. On the other hand, it is not difficult to verify that, under reasonable assumptions on $A$, if $\Omega$ is a ball then problem (1) admits a unique solution, which is radially symmetric (see Proposition 2.2 below). A natural question which arises is to determine if this condition is also necessary, namely whether the following statement holds true:

$$\text{if (1) admits a solution, then } \Omega \text{ is a ball.} \tag{2}$$

In the linear case, when $A \equiv 1$ and the equation becomes $-\Delta u = 1$, (1) may be used to describe both the motion of a viscous incompressible fluid moving in straight parallel streamlines through a pipe with planar section $\Omega$ or the torsion of a solid straight bar of given cross section $\Omega$. For these models, (2) has the following meaning, which we quote from [34]: "the tangential stress on the pipe wall is the same at all points of the wall if and only if the pipe has a circular cross section" and "when a solid straight bar is subject to torsion, the magnitude of the resulting traction which occurs at the surface of the bar is independent of position if and only if the bar has a circular cross section".

When $A(t) = (1 + t^2)^{-1/2}$ the solution of (1) describes the shape of a capillary surface in absence of gravity, adhering to a given plane with constant contact angle. In this case, (2) means that the wetted area on the plane is necessarily spherical, see [34, 38].

For degenerate elliptic operators, further physical applications may be pointed out. For instance, when $A(t) = t^{p-2}$ for some $p > 1$, problem (1) models torsional creep with constant stress on the boundary [21]. When $A(t) = 1 + \alpha t^{p-2}$ (with $\alpha > 0$, $p > 1$), equation (1) has applications in Born-Infeld theory for electrostatic fields [14], and its solutions are static critical points of an action functional with Lorentz-invariant Lagrangian density proposed by Derrick [12] as a model for elementary particles. We also refer to [3, 4] for more general applications to quantum physics.

Indeed, the problem of proving (2) has raised a good deal of attention in the last decades. The first fundamental contribution is due to Serrin. In his celebrated paper [34], (2) is obtained in the uniformly elliptic case, when solutions of (1) are classical. Serrin's proof is based on what is now known as the "moving planes method". This method has subsequently been used in many further symmetry results for elliptic equations, see [17, 26, 32]. In its original version, the method applies under the requirement that $\partial\Omega \in C^2$. Later this assumption was weakened; we refer to [5] and [31] for the case of domains with Lipschitz boundary and with one possible corner or cusp.

In the special (and simplest) case where $A \equiv 1$, a different method to obtain (2) was discovered by Weinberger [37] whose proof is the first successful attempt to use an associated "$P$-function". By using some integral identities and the maximum principle, he shows that a certain function of $u$ is constant in all of $\Omega$ (see Remark 5.4 below). As a consequence the Hessian matrix of $u$ is a multiple of the identity, which gives (2). This approach requires very weak assumptions on the regularity of the boundary. Let us also mention that for smooth domains, an alternative proof still valid only for the case $A \equiv 1$ has been obtained by Choulli-Henrot [10] via shape derivatives.

All these methods, including the original one of moving planes, fail when $A$ is a general elliptic operator, possibly degenerate. In this case, solutions of (1) may lose regularity and must be intended in some weak sense. For instance, when $A(t) = t^{p-2}$ for some $p > 1$ (which corresponds to the $p$-Laplacian operator), solutions are generally of class $C^{1,\alpha}$ but not $C^2$. In fact, as far as we are aware, the existing results about (1) in the degenerate case cover just "$p$-Laplacian type" equations. More precisely, assuming that $A(t) \approx t^{p-2}$ as $t \to \infty$ for some $p > 1$ (see Remark 5.1 below), in [15] Garofalo-Lewis deal with solutions of (1) which belong to $W^{1,p}(\Omega)$ and satisfy the boundary conditions in a fairly weak sense. In their ingenious proof of (2), inspired by

Weinberger's approach, the asymptotic behaviour of $A$ is used to obtain gradient bounds and to apply elliptic regularity. Later, under the same assumptions on $A$ but only for $p \geq 2$, (2) was derived via continuous Steiner symmetrization by Brock-Henrot [6], assuming initially that $\Omega$ is convex and that solutions are in $C^1(\overline{\Omega})$. Finally let us mention the paper by Damascelli-Pacella [13], where (2) is proved when $A(t) = t^{p-2}$ and $p \in (1, 2)$. In this special case the authors are able to adapt the moving planes method because, at critical points of solutions, the operator is more likely to be singular rather than degenerate.

The scope of the present paper is to provide a new simple unifying proof of (2) for very general problems, possibly degenerate. Dealing with $C^1(\overline{\Omega})$ solutions, we make fairly weak assumptions (in particular, no growth restrictions) on the function $A$. The price that we must pay for these general assumptions are some geometric restrictions on the admissible domains $\Omega$, *i.e.* simple connectedness for planar domains (see Theorem 2.4) and star-shapedness in higher space dimensions $n \geq 3$ (see Theorem 2.3). If we make no geometric assumptions on $\Omega$, we may only prove a much weaker version of (2), namely that $\Omega$ coincides with its Cheeger set, see Theorem 2.5.

Our approach combines analytical and geometrical arguments. It is based on Alexandrov characterization of spheres [1, 2]. In order to apply his principle, we use a suitable $P$-function, which enables us to obtain a uniform upper bound for the mean curvature of $\partial\Omega$. Then we employ two crucial tools from geometry, a sharp estimate for the radius of the maximal inscribed disk in dimension $n = 2$ (see Lemma 3.4 below) and a so-called Minkowski identity in any space dimension (see the first identity in formula (21) below).

The outline of the paper is as follows. The main results are stated in Section 2 and proved in Section 4. Section 3 contains some crucial preliminary lemmata. In Section 5 we gather some concluding remarks.

## 2 Main results

Throughout the paper we assume that $\partial\Omega \in C^{2,\alpha}$. This ensures that solutions of (1) are $C^{2,\alpha}$ in a neighbourhood of $\partial\Omega$, see Lemma 3.1. In particular, the Neumann condition reads

$$-u_\nu = c \qquad \text{on } \partial\Omega \ ,$$

where $\nu$ denotes the exterior unit normal to $\partial\Omega$. We remark that less regularity on $\partial\Omega$ could be required thanks to the results in [36], but it is not our purpose to discuss here the optimal assumptions on the boundary. Our attention is mainly focused on the operator $A$. We ask it to satisfy the regularity requirement

$$A \in C^2(0, +\infty) \tag{3}$$

and the (possibly degenerate) ellipticity conditions

$$\lim_{t \to 0^+} tA(t) = 0 \ , \qquad \big(tA(t)\big)' > 0 \qquad \text{for } t > 0. \tag{4}$$

Note that the first condition in (4) is necessary for the existence of a $C^1(\overline{\Omega})$ solution of (1). To see this, let $r \to 0^+$ in (18) in the proof of Proposition 2.2 below. Note also that, in view of Theorem 2.5, the second condition in (4) could be assumed to hold only for $t \in (0, c)$.

Under assumptions (3)-(4) on $A$, we consider $C^1$ distributional solutions of (1). More precisely we give the following:

**Definition 2.1** We say that $u$ is a *solution* of (1) if $u \in C_0^1(\overline{\Omega})$, $u_\nu = -c$ on $\partial\Omega$ and

$$\int_\Omega A(|\nabla u|)\nabla u \nabla\varphi = \int_\Omega \varphi \qquad \text{for all } \varphi \in C_c^\infty(\Omega) \ .$$

To investigate the existence of a solution of (1), one starts in a natural way from the easiest case when $\Omega$ is, *by assumption*, a ball. In such case, the radius of the ball and the corresponding solution can be uniquely and explicitly determined according to the following simple proposition, that is proved for completeness in Section 4.

**Proposition 2.2** *Let $A$ satisfy* (3) *and* (4) *and assume that $\Omega = B_R$ is a ball of radius $R$ in $\mathbb{R}^n$. Then problem* (1) *admits a solution $u$ if and only if $R = ncA(c)$. In this case, $u$ is radially symmetric and decreasing in $B_R$ and, when $B_R$ is centered at the origin, $u$ can be written as*

$$u(x) = \int_{|x|}^R \mathcal{A}\left(\frac{s}{n}\right) ds \ ,$$

*where $\mathcal{A}$ is the inverse of the map $t \mapsto tA(t)$.*

The delicate matter is to prove the converse statement of Proposition 2.2, namely that (2) holds. We are able to prove this implication under the initial assumption that $\Omega$ is a star-shaped domain, *i.e.* that there exists a point $x_0 \in \Omega$ such that $(x - x_0) \cdot \nu \geq 0$ on $\partial\Omega$.

**Theorem 2.3** *Let $A$ satisfy* (3) *and* (4) *and assume that $\Omega \subset \mathbb{R}^n$ is star-shaped with $C^{2,\alpha}$ boundary. If problem* (1) *admits a solution, then $\Omega$ is a ball of radius $R = ncA(c)$.*

In dimension $n = 2$ the result remains valid if $\Omega$ is merely assumed to be simply connected.

**Theorem 2.4** *Let $A$ satisfy* (3) *and* (4) *and assume that $\Omega \subset \mathbb{R}^2$ is simply connected with $C^{2,\alpha}$ boundary. If problem* (1) *admits a solution, then $\Omega$ is a disk of radius $R = 2cA(c)$.*

In arbitrary dimension, if $\Omega$ is not assumed to be star-shaped, we can prove a result weaker than Theorem 2.3. According to [24] and [25] we say that $\Omega$ coincides with its *Cheeger set* if

$$\frac{|\partial\Omega|}{|\Omega|} = \min_D \frac{|\partial D|}{|D|} := h(\Omega)$$

where the minimum is taken over all open, nonempty, simply connected subdomains $D$ of $\overline{\Omega}$. $h(\Omega)$ is named after [9] and called the Cheeger constant of $\Omega$.

**Theorem 2.5** *Let $A$ satisfy* (3) *and* (4) *and assume that $\Omega \subset \mathbb{R}^n$ has a $C^{2,\alpha}$ boundary. If problem* (1) *admits a solution $u$, then $|\nabla u(x)| \le c$ for all $x \in \overline{\Omega}$, and $\Omega$ coincides with its Cheeger set.*

In [25] it was shown that convex plane domains satisfy $h(\Omega) = |\partial\Omega|/|\Omega|$ if and only if the curvature of their boundary is bounded from above by $|\partial\Omega|/|\Omega|$. For instance, ellipses and other domains coincide with their Cheeger sets. But not all domains coinciding with their Cheeger sets are star-shaped. Annuli or rounded $L$-shaped domains can serve as counterexamples. Thus, combining Theorems 2.3 and 2.5 is unfortunately not sufficient to conclude that the star-shapedness assumption in Theorem 2.3 may be removed.

# 3 Preliminary results

Throughout this section we assume without further mention that $\partial\Omega \in C^{2,\alpha}$ and that $A$ satisfies (3)-(4). We first show that, if a solution of (1) exists, it is unique and it gains regularity.

**Lemma 3.1** *There exists at most one solution $u$ of* (1) *in the sense of Definition 2.1. If it exists, it satisfies*

$$u \in C^{2,\alpha}(\overline{\Omega} \setminus \{x \,:\, \nabla u(x) \neq 0\}) \,. \tag{5}$$

*Proof.* To prove uniqueness, observe that any solution of (1) is a critical point of the integral functional

$$J(u) = \int_\Omega \Big[B(|\nabla u|) - u\Big] \,, \qquad u \in C^1_0(\overline{\Omega}) \,, \tag{6}$$

where $B(s) := \int_0^s tA(t)dt$. Due to (4), the map $s \mapsto B(s)$ is strictly convex, then so is $J$, and $u$ must coincide with its unique minimizer. Condition (5) follows from standard elliptic regularity theory. □

Now we put $\Phi(t) := 2\int_0^t (A(s) + sA'(s))s\,ds$ and we assume that $u$ solves (1) in the sense of Definition 2.1. Then, we consider the $P$-function defined by

$$P(x) := \Phi(|\nabla u(x)|) + \frac{2}{n}u(x) \qquad (x \in \overline{\Omega}) \,. \tag{7}$$

Clearly, $P$ is continuous in $\overline{\Omega}$ and, by Lemma 3.1, it is of class $C^1$ in a neighbourhood of $\partial\Omega$. The next lemma is an extension of a known result on $P$-functions to possibly degenerate equations. Let us stress that it *does not* exclude that $P$ might attain its maximum also in critical points of $u$ (which are in the interior of $\Omega$).

**Lemma 3.2** ($P$-FUNCTION)
*If $u$ solves* (1) *in the sense of Definition 2.1, then the $P$-function defined by* (7) *is either constant in $\overline{\Omega}$ or it satisfies $P_\nu > 0$ on $\partial\Omega$.*

*Proof.* Throughout the proof we assume that $P$ is not constant in $\Omega$. We first claim that $P$ attains its maximum on $\partial\Omega$ and that if $P$ also attains its maximum in a point $\overline{x} \in \Omega$ then necessarily $\nabla u(\overline{x}) = 0$. We divide the proof of this claim into two steps. In the former we assume that $A$ is uniformly elliptic, in the latter we proceed by approximation.
*Step 1.* In this step we follow essentially [28] and [35] and prove the statement when $A$ satisfies the uniform ellipticity conditions (which imply (3) and (4))

$$A \in C^2[0, +\infty) \ , \qquad \big(tA(t)\big)' > 0 \quad \text{for } t \geq 0. \tag{8}$$

We set

$$\mathcal{L}P := \Delta P + \frac{A'(|\nabla u|)}{|\nabla u| A(|\nabla u|)} \nabla^2 P \nabla u \cdot \nabla u \ .$$

By some long but straightforward computations, one may obtain the explicit expression of $\mathcal{L}P$ and write it down as

$$\mathcal{L}P + L \cdot \nabla P = g \tag{9}$$

where $L = L(u)$ is a suitable vector-valued function, and $g = g(u)$ contains all the "remainder terms", see [35, Section 7]. Via an application of Schwarz's inequality, one gets that the function $g$ is nonnegative on $\Omega$, so that $P$ turns out to satisfy the second order differential inequality

$$\mathcal{L}P + L \cdot \nabla P \geq 0 \qquad \text{in } \Omega \ . \tag{10}$$

Clearly, since (8) holds, the operator $\mathcal{L}$ has bounded coefficients, and also the vector field $L = L(u)$ remains bounded, see *e.g.* [35, Theorem 7.3]. Moreover, thanks to (8), the operator $\mathcal{L}$ is strongly elliptic, because there exists $\mu > 0$ such that

$$|\xi|^2 + \frac{A'(|\nabla u|)}{|\nabla u| A(|\nabla u|)} (\nabla u \cdot \xi)^2 \geq \mu |\xi|^2 \qquad \text{for all } \xi \in \mathbb{R}^n \ , \ x \in \Omega \ . \tag{11}$$

Indeed, for those $x \in \Omega$ such that $A'(|\nabla u(x)|) \geq 0$, (11) is satisfied with $\mu = 1$. Otherwise, we may apply Schwarz's inequality to obtain

$$|\xi|^2 + \frac{A'(|\nabla u|)}{|\nabla u| A(|\nabla u|)} (\nabla u \cdot \xi)^2 \geq \frac{A(|\nabla u|) + A'(|\nabla u|)|\nabla u|}{A(|\nabla u|)} |\xi|^2 \ .$$

Hence, (11) holds for all $x \in \Omega$ with

$$\mu = \inf \left\{ \frac{\big(tA(t)\big)'}{A(t)} \ : \ t \in \big[0, \max_{x \in \overline{\Omega}} |\nabla u(x)|\big] \right\} > 0 \ .$$

Thus, under assumption (8), (10) is an elliptic inequality of second order with bounded differentiable coefficients. Hence, the second order operator may be written in divergence form. Then, the classical maximum principle (see *e.g.* [18, Theorem 8.1]) proves that $P$ attains its maximum over $\overline{\Omega}$ on $\partial\Omega$. Notice that in this case, since $P$ is assumed nonconstant, the maximum of $P$ over $\overline{\Omega}$ is attained only on $\partial\Omega$.

*Step 2.* Let us turn to the case when $A$ is possibly degenerate. If (8) is weakened to (3) and (4), the coefficients of the inequality (10) may become singular on the set $\mathcal{C} := \{x \in \Omega \, : \, \nabla u(x) = 0\}$, and also (11) may fail. Thus, we proceed through a careful perturbation argument. We set

$$\mathcal{L}_\varepsilon P := \Delta P + \frac{A'(|\nabla u| + \varepsilon)}{(|\nabla u| + \varepsilon) A(|\nabla u| + \varepsilon)} \nabla^2 P \nabla u \cdot \nabla u \ ,$$

and we choose a sequence $\{L_\varepsilon\}$ in $C^0(\overline{\Omega}; \mathbb{R}^n)$ which converges to $L$ locally uniformly in $\overline{\Omega} \setminus \mathcal{C}$ as $\varepsilon \to 0$. For every $\varepsilon > 0$ we consider now the solution $P_\varepsilon$ to the boundary value problem

$$\begin{cases} \mathcal{L}_\varepsilon P_\varepsilon + L_\varepsilon \cdot \nabla P_\varepsilon = g & \text{in } \Omega \\ P_\varepsilon = P = \Phi(c) & \text{on } \partial\Omega \ , \end{cases}$$

with $g$ defined by (9). By construction, for every $\varepsilon$ the operator $\mathcal{L}_\varepsilon$ has bounded coefficients, and also the vector field $L_\varepsilon = L_\varepsilon(u)$ remains bounded. Moreover, arguing as in Step 1, we find an ellipticity constant given by

$$\mu_\varepsilon = \inf \left\{ \frac{\big(tA(t)\big)'}{A(t)} \ : \ t \in \big[\varepsilon, \varepsilon + \max_{x \in \overline{\Omega}} |\nabla u(x)|\big] \right\} > 0 \ .$$

Thus, by the maximum principle, $P_\varepsilon$ attains its maximum on $\partial\Omega$. In particular, for every neighbourhood $\mathcal{U}$ of $\mathcal{C}$, there holds

$$\max_{\overline{\Omega}} P_\varepsilon = \max_{\partial\Omega} P_\varepsilon = \max_{\overline{\Omega \setminus \mathcal{U}}} P_\varepsilon \ .$$

Now, since $\mathcal{C} \cap \partial\Omega = \emptyset$, and since $P_\varepsilon$ converges to $P$ uniformly on compact subsets of $\overline{\Omega} \setminus \mathcal{C}$, we deduce that

$$\max_{\partial\Omega} P = \lim_\varepsilon \max_{\partial\Omega} P_\varepsilon = \lim_\varepsilon \max_{\overline{\Omega \setminus \mathcal{U}}} P_\varepsilon = \max_{\overline{\Omega \setminus \mathcal{U}}} P = \max_{\overline{\Omega \setminus \mathcal{C}}} P \ ,$$

where the last equality follows from the arbitrariness of $\mathcal{U}$. We infer that, if $P(x^*) > \max_{\partial\Omega} P$ for some $x^* \in \Omega$, then $x^*$ belongs to the interior of $\mathcal{C}$. But such interior is empty, as otherwise integrating the first equation in (1) on a ball $B \subset \mathcal{C}$ would give a contradiction via the divergence theorem. Hence $P$ assumes its maximum on $\partial\Omega$. Moreover, the above approximation method shows that any maximum point for $P$ in $\Omega$ belongs necessarily to $\mathcal{C}$. This completes the proof of the claim.

In order to complete the proof of the lemma, note that since $u$ solves (1), we have $|\nabla u| \neq 0$ in a (closed) neighbourhood $D \subset \overline{\Omega}$ of $\partial\Omega$. By the just proved claim, $P$ attains its maximum in $D$ only on $\partial\Omega$. Moreover, the equation (9) is uniformly elliptic in $D$ and therefore $P$ satisfies the classical boundary point principle. This shows that $P_\nu > 0$ on $\partial\Omega$. □

As a consequence of Lemma 3.2, we obtain a uniform upper bound for the mean curvature of $\partial\Omega$ for those domains where (1) admits a solution.

**Lemma 3.3** (Upper bound for the mean curvature)
*If problem* (1) *admits a solution, then the mean curvature $H(x)$ of $\partial\Omega$ satisfies*

$$\text{either} \qquad H(x) < \frac{1}{n\ c\ A(c)} \qquad \text{for all } x \in \partial\Omega \qquad \text{or} \qquad H(x) \equiv \frac{1}{n\ c\ A(c)} \ .$$

*Proof.* Since $c \neq 0$, the first equation in (1) is nondegenerate in a neighbourhood of $\partial\Omega$, and by (5) it may be rewritten pointwise on $\partial\Omega$ as

$$\big[A(c) + cA'(c)\big]\, u_{\nu\nu} - (n-1)cA(c)H(x) = -1 \ . \tag{12}$$

Consider now the $P$-function associated with $u$ defined in (7). According to Lemma 3.2, two cases may occur. Let us first consider the case where

$$P_\nu = \big[A(c) + cA'(c)\big]\, 2u_\nu u_{\nu\nu} + \frac{2}{n} u_\nu > 0 \qquad \text{on } \partial\Omega. \tag{13}$$

Since $u_\nu < 0$ on $\partial\Omega$, we can divide by $2u_\nu$ and obtain

$$\big[A(c) + cA'(c)\big]\, u_{\nu\nu} + \frac{1}{n} < 0 \qquad \text{on } \partial\Omega. \tag{14}$$

By combining (12) and (14) we readily obtain $H(x) < [n\, c\, A(c)]^{-1}$ for all $x \in \partial\Omega$.

The second case of Lemma 3.2 turns (13) into an equality. Then, arguing as above, also (14) becomes an equality so that $H(x) = [n\, c\, A(c)]^{-1}$ for all $x \in \partial\Omega$. This proves the lemma. □

Lemma 3.3 states that in any case

$$H(x) \leq \frac{1}{n\ c\ A(c)} \qquad \text{for all } x \in \partial\Omega \ . \tag{15}$$

For planar domains, inequality (15) has the following intuitive geometrical consequence.

**Lemma 3.4** (Maximal inscribed ball)
*Assume that $n = 2$ and that $\Omega$ is a simply connected domain. If* (15) *holds, then $\Omega$ contains a ball of radius $R = 2cA(c)$.*

*Proof.* See [8, Section 30.4.1] and also the previous papers [29, 19] for a complete proof. □

**Remark 3.5** In dimension $n \geq 3$ the analogue of Lemma 3.4 is false. For a (sharp) lower bound on the radius of the maximal inscribed ball in arbitrary dimension, see [8, Section 30.4.2].

Lemma 3.4 will be exploited for the proof of Theorem 2.4, by inscribing a ball of radius $R = 2cA(c)$ inside $\Omega$ and using comparison principles which we prove below. Here and in the sequel, we use the notation

$$Qu := -\text{div}(A(|\nabla u|)\nabla u) \ .$$

**Definition 3.6** Let $u_1, u_2 \in C^1(\overline{\Omega})$. We say that $Qu_1 = Qu_2$ in $\Omega$ if

$$\int_\Omega A(|\nabla u_1|)\nabla u_1 \nabla\varphi = \int_\Omega A(|\nabla u_2|)\nabla u_2 \nabla\varphi \qquad \text{for all } \varphi \in C_c^\infty(\Omega).$$

We say that $Qu_1 \leq Qu_2$ in $\Omega$ if

$$\int_\Omega \Big[A(|\nabla u_1|)\nabla u_1 - A(|\nabla u_2|)\nabla u_2\Big]\nabla\varphi \leq 0 \qquad \text{for all } \varphi \in C_c^\infty(\Omega),\ \varphi \geq 0\ . \tag{16}$$

With this definition, we may state our first comparison result:

**Lemma 3.7** (Weak comparison principle)
*Assume that $u_1, u_2 \in C^1(\overline{\Omega})$ satisfy*

$$\begin{cases} Qu_1 \leq Qu_2 & \text{in } \Omega \\ u_1 \leq u_2 & \text{on } \partial\Omega. \end{cases}$$

*Then $u_1 \leq u_2$ in $\Omega$.*

*Proof.* Let $v := (u_1 - u_2)^+$. By assumption, $v \in W_0^{1,\infty}(\Omega)$ so that by a density argument it can be used as a test function in (16). By subtracting, we infer that

$$\int_{\{u_1>u_2\}} [A(|\nabla u_1|)\nabla u_1 - A(|\nabla u_2|)\nabla u_2] \cdot (\nabla u_1 - \nabla u_2) \leq 0\ . \tag{17}$$

Thanks to Schwarz's inequality and assumption (4), there holds

$$\begin{aligned} & [A(|\nabla u_1|)\nabla u_1 - (A(|\nabla u_2|)\nabla u_2] \cdot (\nabla u_1 - \nabla u_2) \\ \geq\ & A(|\nabla u_1|)|\nabla u_1|^2 + A(|\nabla u_2|)|\nabla u_2|^2 - [A(|\nabla u_1|) + A(|\nabla u_2|)]|\nabla u_1||\nabla u_2| \\ =\ & [A(|\nabla u_1|)|\nabla u_1| - A(|\nabla u_2|)|\nabla u_2|](|\nabla u_1| - |\nabla u_2|) \geq 0, \end{aligned}$$

the latter inequality being strict for $|\nabla u_1| \neq |\nabla u_2|$. This combined with (17) gives a contradiction unless $v \equiv 0$. □

Let us now prove a boundary point principle. We are grateful to J. Serrin for making us aware that, in the same spirit of his paper [33], the following statement holds under the mere assumption $A \in C^1(0, +\infty)$ instead of $A \in C^2(0, +\infty)$. Therefore, also Theorem 2.4 remains valid under this weaker regularity assumption on $A$.

**Lemma 3.8** (Boundary point principle)
*Let $B$ be a ball with center $0$ and let $u_1 \in C^1(\overline{B}) \cap C^2(B \setminus \{0\})$, with $|\nabla u_1| \neq 0$ on $\partial B$. Assume that there exists a function $u_2 \in C^1(\overline{B})$ and a point $x^* \in \partial B$ such that*

$$\begin{cases} Qu_1 & \leq Qu_2 & \text{in } B \\ u_1 & < u_2 & \text{in } B \\ u_1(x^*) & = u_2(x^*)\ . \end{cases}$$

*Then $\nabla u_1(x^*) \neq \nabla u_2(x^*)$.*

*Proof.* Assume without loss of generality that $B$ is the unit ball $B_1$ and $|\nabla u_1(x)| \neq 0$ for $|x| \in [1/2, 1]$. For $\alpha > 0$, set

$$v(x) := \frac{e^{-\alpha|x|^2} - e^{-\alpha}}{\alpha^2} \ .$$

Clearly, for sufficiently large $\alpha$ we have $u_1 + v \leq u_2$ on $\partial(B_1 \setminus B_{1/2})$. We claim that (still for sufficiently large $\alpha$) there holds $Q(u_1 + v) \leq Qu_2$ in $B_1 \setminus B_{1/2}$. Were this claim proved, the statement would follow at once. Indeed, since

$$\begin{cases} Q(u_1+v) & \leq Qu_2 \qquad \text{in } B_1 \setminus B_{1/2} \\ \quad u_1 + v & \leq u_2 \qquad \text{on } \partial(B_1 \setminus B_{1/2}) \ , \end{cases}$$

Lemma 3.7 ensures that $u_1 + v \leq u_2$ in $B_1 \setminus B_{1/2}$ and the claim follows.

Since $u_1 \in C^2(B_1 \setminus B_{1/2})$, we may argue pointwise. With some tedious but straightforward computations we obtain $Q(u_1 + v) = -\text{I} - \text{II}$, where, in the asymptotic expansion as $\alpha \to +\infty$,

$$\text{I} := A\big(|\nabla(u_1+v)|\big)\Delta(u_1+v) = A\big(|\nabla u_1|\big)\Delta u_1 + 4|x|^2 A(|\nabla u_1|)e^{-\alpha|x|^2} + o(e^{-\alpha})$$

and

$$\begin{aligned} \text{II} \ &:= \frac{A'\big(|\nabla(u_1+v)|\big)}{2|\nabla(u_1+v)|}\nabla\big(|\nabla(u_1+v)|^2\big)\cdot\nabla(u_1+v) \\ &= \frac{A'\big(|\nabla u_1|\big)}{2|\nabla u_1|}\nabla\big(|\nabla u_1|^2\big)\cdot\nabla u_1 + 4(x\cdot\nabla u_1)^2\frac{A'\big(|\nabla u_1|\big)}{|\nabla u_1|}e^{-\alpha|x|^2} + o(e^{-\alpha}) \ . \end{aligned}$$

We point out that, in order to perform the above asymptotic expansion, one needs the fact that $\nabla u_1 \neq 0$ in $B_1 \setminus B_{1/2}$. Hence

$$Q(u_1+v) - Q(u_1) = -4e^{-\alpha|x|^2}\Big[|x|^2 A(|\nabla u_1|) + (x\cdot\nabla u_1)^2\frac{A'\big(|\nabla u_1|\big)}{|\nabla u_1|}\Big] + o(e^{-\alpha}) \ .$$

We claim that the term inside square brackets in the above expansion is positive. This is trivially true when $A'(|\nabla u_1|) \geq 0$ (recall $\nabla u_1 \neq 0$). Otherwise, by Schwarz's inequality and assumption (4), we obtain as well

$$|x|^2 A(|\nabla u_1|) + (x\cdot\nabla u_1)^2\frac{A'\big(|\nabla u_1|\big)}{|\nabla u_1|} \geq |x|^2\Big[A(|\nabla u_1|) + |\nabla u_1|A'\big(|\nabla u_1|\big)\Big] > 0 \ .$$

This shows that for $\alpha$ sufficiently large we have $Q(u_1 + v) \leq Qu_1$ pointwise in $B_1 \setminus B_{1/2}$ and, *a fortiori*, the same inequality holds in the weak sense of (16). Hence, in the same weak sense, $Q(u_1 + v) \leq Qu_2$ in $B_1 \setminus B_{1/2}$, which concludes the proof. □

## 4 Proofs of the main results

**Proof of Proposition 2.2.** If (1) admits a solution $u$, then by Lemma 3.1 it is unique and coincides with the minimizer of the functional $J$ in (6). Standard symmetrization arguments (see *e.g.* [7] or [20]) then show that such a minimizer is radially symmetric and radially decreasing. Now, any radial $C^1$-solution $u = u(r)$ must satisfy the ordinary differential equation

$$\big(r^{n-1}A(|u_r|)u_r\big)_r = -r^{n-1} \qquad \text{on } [0, R],$$

whose first integral is easily computed as

$$A(|u_r(r)|)\ u_r(r) = -\frac{r}{n} \qquad \text{on } [0, R].$$

This tells us that $u_r < 0$, and that the above equation may be rewritten as

$$A(|u_r(r)|)\ |u_r(r)| = \frac{r}{n} \qquad \text{on } [0, R]. \tag{18}$$

Hence, if $\mathcal{A}$ is the inverse function of $t \mapsto tA(t)$, we have $u_r(r) = -\mathcal{A}\big(\frac{r}{n}\big)$ and, subsequently,

$$u(r) = \int_r^R \mathcal{A}\left(\frac{s}{n}\right)\, ds\ .$$

Finally, writing (18) for $r = R$ we obtain $R = ncA(c)$. □

**Proof of Theorem 2.3.** The claim of Theorem 2.3 follows from Alexandrov's characterization of spheres [1, 2] once we show that

$$H(x) \equiv \frac{1}{n\ c\ A(c)} \qquad \text{on } \partial\Omega\ . \tag{19}$$

Assume for contradiction that (19) is false. In view of Lemma 3.3, this means that

$$H(x) < \frac{1}{n\ c\ A(c)} \quad \text{on } \partial\Omega\ . \tag{20}$$

Up to a translation, we may assume that $\Omega$ is star-shaped with respect to the origin. Inspired by [16], we now point out that

$$\int_{\partial\Omega} H(x)\, x \cdot \nu = |\partial\Omega|\ , \qquad \int_{\partial\Omega} x \cdot \nu = n|\Omega|\ , \tag{21}$$

where the first identity is a so-called Minkowski formula (see for instance Section 2A in [27]), and the second one is immediate from the divergence theorem. In particular, (21) and starshapedness with respect to the origin tell us that $x \cdot \nu \geq 0$ on $\partial\Omega$ with $x \cdot \nu > 0$ on a subset of positive $(n-1)$ measure. Therefore, multiplying inequality (20) by $x \cdot \nu$ and integrating over $\partial\Omega$ yields

$$\int_{\partial\Omega} H(x)\, x \cdot \nu < \int_{\partial\Omega} \frac{x \cdot \nu}{n\ c\ A(c)}\ . \tag{22}$$

By (21) and (22) we get

$$cA(c)|\partial\Omega| < |\Omega| \ . \tag{23}$$

On the other hand, integrating the differential equation in (1) and using again the divergence theorem gives

$$|\Omega| = -\int_\Omega \mathrm{div}(A(|\nabla u|)\nabla u) = cA(c)|\partial\Omega| \ . \tag{24}$$

This contradicts (23) and completes the proof. □

**Proof of Theorem 2.4.** By Lemmata 3.3 and 3.4, $\Omega$ contains a disk $B$ of radius $R = 2cA(c)$ and center, say, 0. Without loss of generality we may assume that $B$ touches $\partial\Omega$ tangentially in a point $x^*$, so that they have the same outward normal $\nu^*$. Otherwise we shift $B$. By Proposition 2.2 the boundary value problem (1) on $B$ admits a unique solution $v \in C_0^1(\overline{B}) \cap C^2(\overline{B} \setminus \{0\})$. In particular, this solution $v$ satisfies

$$\nabla v(x^*) = -c\nu^* = \nabla u(x^*) \ , \tag{25}$$

and $Qv = Qu$ in $B$. Moreover, $v \le u$ on $\partial B$, because by Lemma 3.7 we know that $u \ge 0$ in $\Omega$. Hence, again by Lemma 3.7 applied now to $B$, we deduce that $v \le u$ in $B$. After setting $E := \{x \in B \ : \ v(x) = u(x)\}$, three cases may occur: $E = \emptyset$, $\emptyset \ne E \ne B$, and $E = B$. Let us exclude the first two cases.

In the first case, we have

$$\begin{cases} Qv = Qu & \text{in } B \\ v < u & \text{in } B \\ v(x^*) = u(x^*) \ , \end{cases}$$

and then by Lemma 3.8 we infer that $\nabla u(x^*) \ne \nabla v(x^*)$. This contradicts (25).

In the second case, we can find a disk $B_0 \subset B$ (not containing the origin) and a point $x_0 \in B \cap \partial B_0$ such that

$$\begin{cases} Qv = Qu & \text{in } B_0 \\ v < u & \text{in } B_0 \\ v(x_0) = u(x_0) \ , \end{cases}$$

but then Lemma 3.8, now applied on $B_0$, gives $\nabla v(x_0) \ne \nabla u(x_0)$. This contradicts the fact that $x_0$ is a minimum point for $u - v$ in $B$.

Hence, the third case $E = B$ necessarily holds, and so $v \equiv u$ in $B$. In particular, the conditions $u = v = 0$ and $u_\nu = v_\nu < 0$ hold on $\partial B$. If $B \subsetneq \Omega$, this would imply that $u$ is negative somewhere in $\Omega$, while we know from Lemma 3.7 that $u \ge 0$ in $\Omega$. Therefore, $\Omega = B$ and the proof is complete. □

**Proof of Theorem 2.5.** By Lemma 3.7 we know that $u(x) \ge 0$ for all $x \in \overline{\Omega}$. This, together with Lemma 3.2, shows that

$$\Phi(|\nabla u(x)|) \le \Phi(|\nabla u(x)|) + \frac{2}{n}u(x) \le \Phi(c) \qquad \text{for all } x \in \overline{\Omega} \ .$$

Since $t \mapsto \Phi(t)$ is strictly increasing in view of (4), we deduce that the first statement in Theorem 2.5 holds, namely $|\nabla u(x)| \leq c$ for all $x \in \overline{\Omega}$. Hence, for any subdomain $D \subseteq \Omega$ an integration of the differential equation (1) over $D$ and an integration by parts yields

$$|D| = -\int_{\partial D} A(|\nabla u|)u_\nu \leq \int_{\partial D} A(|\nabla u|)|\nabla u| \leq cA(c)|\partial D|.$$

This, combined with (24), shows that

$$\frac{|\partial\Omega|}{|\Omega|} = \frac{1}{cA(c)} \leq \frac{|\partial D|}{|D|} \qquad \text{for all } D \subseteq \Omega$$

and proves the second statement in Theorem 2.5. □

# 5 Concluding remarks

**Remark 5.1** The assumptions made on the operator $A$ in [15] (and in [6]) were that $A(t) = f'(t)/t$, where $f$ is a positive convex function of class $C^2(0, +\infty)$ satisfying

$$c_1(t^r - 1) \leq tf'(t) \leq c_2(t^r + 1) \ , \qquad c_1 \leq tf''(t)/f'(t) \leq c_2$$

for all $t > 0$, some $r \in (1, +\infty)$ and some positive constants $c_1$ and $c_2$. It is immediate to check that these hypotheses imply the validity of (4), while the converse is clearly false. In terms of $f$, we require no growth conditions besides the ellipticity inequality $f''(t) > 0$ on $(0, +\infty)$. For instance, given real exponents $p > 1$ and $q \geq 0$, consider an operator $A$ of the kind

$$A(t) := \frac{t^{p-2}}{(1+t^2)^{q/2}} \ .$$

As special cases, $A$ becomes the $p$-Laplacian when $q = 0$, and the mean curvature operator when $p = 2$ and $q = 1$. It is easy to check that (3)-(4) are satisfied as soon as $p > 1$ and $p - 1 - q \geq 0$, while the case $p - 1 - q = 0$ (including the mean curvature operator) is not covered by the setting of Garofalo-Lewis and Brock-Henrot. On the other hand, if $p \neq 2$ the operator is degenerate and it is not covered by the setting of Serrin.

**Remark 5.2** In some sense, our proof of Theorem 2.3 is reminiscent of Pohožaev's identity [30]. In its proof, Pohožaev multiplies the PDE with $x \cdot \nabla u$ and integrates over $\Omega$. In our proof we multiply with essentially the same thing, namely $x \cdot \nu$, but in contrast we multiply the curvature bound and integrate over $\partial\Omega$ (where $\nabla u = -c\nu$).

**Remark 5.3** Solutions of (1) are minimizers for the functional $J$ defined in (6). Under very weak assumptions on $A$ and $\Omega$, it has recently been shown by Crasta [11] that if the minimizer is a web function (in other words, if it only depends on the distance to the boundary), then $\Omega$ is a ball. Of course, requiring the minimizer of $J$ to be a web function is much more stringent than just requiring the additional boundary condition $u_\nu = -c$.

**Remark 5.4** In the linear case where $A \equiv 1$, Weinberger [37] proved that, if (1) admits a solution, then the $P$-function $P(x)$ given by Lemma 3.2 satisfies $P(x) \equiv \Phi(c)$ on all of $\Omega$. To that aim, since $P(x)$ assumes its maximum on $\partial\Omega$ and it is constantly equal to $\Phi(c)$ there, he managed to bring the integral inequality

$$\int_\Omega P(x) < \Phi(c)|\Omega| \tag{26}$$

to a contradiction. We tried to follow the same approach, but it does not work for general $A$ satisfying just (3)-(4). Since it seems instructive to see where the proof breaks down, for the benefit of the reader let us present the line of argument in the case of the $p$-Laplacian, for which

$$P(x) = \frac{2(p-1)}{p}|\nabla u(x)|^p + \frac{2}{n}u(x).$$

Testing (1) with $u$, it is easy to see that (26) can be rewritten as

$$\left(n + \frac{p}{p-1}\right)\int_\Omega u\ dx < n\ c^p\ |\Omega|. \tag{27}$$

Now one would like to relate $\int_\Omega u$ to $|\Omega|$. To this end, set $r := |x|$. Since $\Delta(r^2) = 2n$, via integration by parts we obtain

$$-2n\int_\Omega u(x) = \int_\Omega \nabla\left(r^2\right)\nabla u = 2\int_\Omega r\frac{\partial u}{\partial r}\ . \tag{28}$$

On the other hand, since $\Delta(r\frac{\partial u}{\partial r}) = -2$, by Green's formula and using (1) we have

$$\int_\Omega \left[2u - r\frac{\partial u}{\partial r}\right] \ = \int_\Omega \left[-u\Delta(r\frac{\partial u}{\partial r}) + r\frac{\partial u}{\partial r}\mathrm{div}(|\nabla u|^{p-2}\nabla u)\right]$$

$$= d + \int_{\partial\Omega}\left[-u\frac{\partial}{\partial\nu}\left(r\frac{\partial u}{\partial r}\right) + r\frac{\partial u}{\partial r}|\nabla u|^{p-2}\frac{\partial u}{\partial\nu}\right] = d + c^p\int_{\partial\Omega} r\frac{\partial r}{\partial\nu} = d + n\ c^p|\Omega|\ , \tag{29}$$

where

$$d := \int_\Omega \nabla u\nabla\left(r\frac{\partial u}{\partial r}\right)\left[-1 + |\nabla u(x)|^{p-2}\right]\ dx\ .$$

Now, only for $p = 2$ the extra term $d$ vanishes and then (28) and (29) contradict (27).

Let us also mention that a symmetry proof showing that the $P$-function is constant on all of $\Omega$ cannot extend to general semilinear equations of the type $\Delta u = f(u)$ either. This was explained in [22].

**Remark 5.5** Once it is known that the $P$-function satisfies $P(x) \equiv \Phi(c)$ in $\Omega$, the function $u$ satisfies the system of two autonomous equations

$$\mathrm{div}\big(A(|\nabla u|)\big) = 1 \qquad \text{and} \qquad |\nabla u|\Phi^{-1}\Big(\Phi(c) - \frac{2}{n}u\Big) =: g(u)\ .$$

The first equation is of second order and the second one of first order and both equations hold in $\Omega$, an extremely overdetermined situation. Therefore the level surfaces $\{x \in \Omega : u(x) = c\}$ must be *isoparametric*, *i.e.* their nonzero principal curvatures are all equal. They can be spheres, cylinders or planes, but for positive solutions to homogeneous Dirichlet problems they can only be spheres. This observation was pointed out in [23] in the context of Weinberger's proof of Serrin's result and for two other symmetry problems.

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