Vortex rings for the Gross-Pitaevskii equation

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Abstract

We provide a mathematical proof of the existence of traveling vortex rings solutions to the Gross-Pitaevskii (GP) equation in dimension $N \geq 3$. We also extend the asymptotic analysis of the free field Ginzburg-Landau equation to a larger class of equations, including the Ginzburg-Landau equation for superconductivity as well as the traveling wave equation for GP. In particular we rigorously derive a curvature equation for the concentration set (i.e. line vortices if $N = 3$).

1 Introduction

In this paper, we consider the Gross-Pitaevskii equation

$$i\frac{\partial \psi}{\partial t} + \Delta\psi + (1 - |\psi|^2)\psi = 0, \tag{1}$$

where $\psi : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C}$ and $N \geq 3$. In dimension 3, this equation, or close variants, are often used as models in various areas of physics : nonlinear optics, superfluidity, Bose-Einstein condensation (see e.g. [21, 35, 38] for surveys). At least formally, it possesses a Hamiltonian structure, whose energy is given by

$$E(\psi) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla\psi(.,t)|^2 + \frac{1}{4}\int_{\mathbb{R}^N} \left(1 - |\psi(.,t)|^2\right)^2. \tag{2}$$

Another important quantity conserved by the flow (1) is the momentum $\vec{P} \in \mathbb{R}^N$, given, again formally, by

$$\vec{P}(\psi) := \mathrm{Im}\int_{\mathbb{R}^N} \psi \cdot \overline{\nabla\psi} = \int_{\mathbb{R}^N} (i\psi, \nabla\psi), \tag{3}$$

where $(,)$ stands for the scalar product in $\mathbb{R}^2$. The first component of the vector $\vec{P}$ will be denoted by P, i.e. $P = \vec{P} \cdot \vec{e}_1$.

Traveling wave solutions to (1) are known to play an important role in the full dynamics of (1). More precisely, these are solutions of (1) of the form (up to rotation)

$$\psi(x,t) = U(x_1 - Ct, x_2, \cdots, x_N), \tag{4}$$

where $C > 0$ is the wave's speed and $U : \mathbb{R}^N \to \mathbb{C}$. One easily verifies that ψ is a solution of (1) iff the "profile" U is a solution to the equation

$$iC\frac{\partial U}{\partial x_1} = \Delta U + U - |U|^2 U. \tag{5}$$

The focus of this paper is on **finite** energy solutions to (5). Our purpose is twofold. First, we embed equation (5) in a larger class of equations (which contain in particular the equations of superconductivity) and study qualitative properties of solutions in an asymptotic regime which is described below. Since these results are of independent interest (and will be used in forthcoming works), we devote a large appendix to this analysis. It will then enter in a crucial way in our second scope, namely the existence problem for (5). The existence of solutions in the case $N = 2$ was considered in [14], our main existence result here concerns its extension to higher dimensions. For that purpose, consider in cylindrical coordinates (x_1, r, θ), where $r := \sqrt{x_2^2 + \cdots + x_N^2}$, the sphere $S := \{(0, 1, \theta)\}$, and on the upper half plane $H_+ := \{(x_1, r),\ r > 0\}$, the operator

$$L\Psi = r^{N-2}\partial_r \left(r^{2-N}\partial_r \Psi \right) + \partial^2_{x_1}\Psi.$$

The linear problem

$$\begin{cases} -L\Psi = 2\pi\delta_q & q = (0,1) \\ \Psi(x_1, 0) = 0 \end{cases}$$

has a unique solution Ψ_* bounded at infinity. Up to a phase change, there also exists (see e.g. [9]) a unique function $\omega_* \in \mathcal{C}^\infty(H_+ \setminus \{q\})$, such that $|\omega_*| = 1$ and

$$\left(\omega_* \times \frac{\partial \omega_*}{\partial x_1}, \omega_* \times \frac{\partial \omega_*}{\partial r}\right) = \left(-\frac{\partial \Psi_*}{\partial r}, \frac{\partial \Psi_*}{\partial x_1}\right)$$

(here $a \times b := a_1 b_2 - a_2 b_1$ is the exterior product of two vectors $a, b \in \mathbb{R}^2 \simeq \mathbb{C}$). Finally, we consider the function U_* defined by

$$U_*(x_1, r, \theta) := \omega_*(x_1, r).$$

The function U_* is cylindrically symmetric, smooth on $\mathbb{R}^N \setminus S$, with values into the circle S^1. In particular, in dimension 3, U_* is singular on a circle (often refered to as a "concentrated vortex ring"). Our main result states that, after scalings, there are solutions of (5) close to U_*.

Theorem 1. *There exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ there exists a solution U_ε to (5) with $C = C(\varepsilon)$ verifying*

$$\frac{C(\varepsilon)}{\varepsilon|\log\varepsilon|} \to N - 2 \qquad \text{as } \varepsilon \to 0, \tag{6}$$

and, for $E(\varepsilon) := E(U_\varepsilon)$, $P(\varepsilon) := P(U_\varepsilon)$, we have

$$\frac{P(\varepsilon)}{2\pi\varepsilon^{1-N}} = |B^{N-1}|, \qquad \frac{E(\varepsilon)}{\pi\varepsilon^{2-N}|\log\varepsilon|} \to |S^{N-2}|, \tag{7}$$

and

$$|U_\varepsilon(x)| \to 1 \quad as \quad |x| \to +\infty. \tag{8}$$

Moreover, for every $k \in \mathbb{N}$,

$$U_\varepsilon(\frac{x}{\varepsilon}) \to U_* \quad in \quad \mathcal{C}^k_{\text{loc}}(\mathbb{R}^N \setminus S). \tag{9}$$

Remark 1. Notice that both the energy $E(\varepsilon)$ and the momentum $P(\varepsilon)$ diverge as $\varepsilon \to 0$, and instead that $C(\varepsilon) \to 0$ as $\varepsilon \to 0$.

A few comments are in order. First, observe that (1) corresponds to a defocusing nonlinear Schrödinger equation (NLS); it has been widely studied with respect to the Cauchy problem in case the initial data are in $L^2(\mathbb{R}^N)$ (see e.g. [40]). In this (different) situation, due to dispersion, any solution vanishes as time tends to infinity. This phenomenon of course excludes traveling wave solutions except for the trivial one. Instead, in our situation, the L^2-norm is not bounded (this is incompatible with the fact that E is bounded) for the solution U_ε; we have seen that $|U_\varepsilon(x)| \to 1$ as $|x| \to +\infty$, and dispersion effects are balanced by the nonlinearity. Our results provide some rigorous mathematical proofs to the study in [30].

Second, the Cauchy problem for (1) with an initial data in $H^1(\mathbb{R}^N)+\{1\}$ having its vorticity concentrating on round spheres has been considered by Jerrard [28]. Although our results are of a different nature, some of the arguments there are closely related to ours.

Third, some properties of (1) can be usefully analyzed through the Madelung transform

$$\psi(x,t) = \sqrt{\rho}\exp(i\varphi),$$

which is meaningful if $|\psi|$ is not zero. In the ρ and $v := \nabla\varphi$ variables, equation (1) can be written as

$$\begin{cases} \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho v) = 0, \\ \rho(\frac{\partial v}{\partial t} + v \cdot \nabla v) + \nabla\left(\frac{\rho^2}{4}\right) = -\rho\nabla\left(\frac{|\nabla\rho|^2}{8\rho^2} - \frac{\Delta\rho}{4\rho}\right). \end{cases} \tag{10}$$

Neglecting the term of the right-hand side of (10) (which is often termed the "quantum pressure" in the physics literature), this system reduces to the Euler equations for compressible ideal fluids with pressure given by $\frac{\rho^2}{4}$. The full system (10) enters in the larger class of quantum fluids equations (see e.g. [36]).

The existence of traveling wave solutions for the incompressible Euler equations was already considered by Helmholtz in his celebrated paper of 1858 [27]; more precisely, the solutions he proposed have vorticity concentrated on a ring of small cross-section (like "smoke rings"). Later, Lord Kelvin computed the relations between the cross-section, the radius of the ring, and its propagation speed. The first rigorous proofs of existence of such steady vortex-rings (steady in a traveling frame) were given by Fraenkel and Berger [22] in the seventies, and later by Ambrosetti and Struwe [4].

Concerning the compressible Euler equation, we are only aware of numerical results in this direction [33].

We will turn later to the properties of the solutions in Theorem 1. In view of the last statement of the theorem, it is clear that they behave like vortex rings. The remainder of this introduction is a detailed description of the strategy of the analysis.

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1.1 The variational approach

Since, as already mentioned, (1) is Hamiltonian, it follows that (5) is variational. At least two different variational approaches are under hand. First, as considered in [14], one could introduce the Lagrangian

$$F_C(U) := E(U) - C\,P(U),$$

whose critical points are solutions to (5). This approach has the advantage that the wave speed C is prescribed a priori. It was shown in [14], for $N = 2$, that for some $C_0 > 0$, F_C has the mountain-pass geometry for $C < C_0$, providing existence in a full interval of speed $]0, C_0[$. In this approach however, the question of stability seems more difficult to address.

The second possible approach, the one we will use here, is by minimizing the energy E keeping the momentum P fixed. It is convenient to perform the following rescaling for $0 < \varepsilon < 1/2$,

$$u_\varepsilon(x) := U_\varepsilon(\frac{x}{\varepsilon}), \qquad c(\varepsilon) := \frac{C(\varepsilon)}{\varepsilon|\log\varepsilon|},$$

so that if U_ε is a solution of (5), then u_ε solves the equation

$$ic(\varepsilon)|\log\varepsilon|\frac{\partial u_\varepsilon}{\partial x_1} = \Delta u_\varepsilon + \frac{1}{\varepsilon^2}u_\varepsilon(1 - |u_\varepsilon|^2), \tag{11}$$

and

$$E_\varepsilon(u_\varepsilon) := \varepsilon^{2-N}E(U_\varepsilon) = \int_{\mathbb{R}^N} \frac{1}{2}|\nabla u_\varepsilon|^2 + \frac{1}{4\varepsilon^2}(1 - |u_\varepsilon|^2)^2 \equiv \int_{\mathbb{R}^N} e_\varepsilon(u_\varepsilon).$$

The energy E_ε is often called the Ginzburg-Landau energy, and has been extensively studied, in particular in the asymptotic limit $\varepsilon \to 0$ (see e.g. [9]). Likewise, the momentum rescales as

$$\vec{p}(u_\varepsilon) := \varepsilon^{1-N}P(U_\varepsilon) = \int_{\mathbb{R}^N}(iu_\varepsilon, \nabla u_\varepsilon).$$

One major difficulty comes from the fact that in the natural energy space

$$X := \left\{ u \in H^1_{\text{loc}}(\mathbb{R}^N),\ E_\varepsilon(u) < +\infty \right\},$$

the momentum $\vec{p}$ is not well defined. Indeed, consider for example the function $w := \exp(i\varphi)$, where φ is smooth and $\varphi(x) = |x|^\alpha$ for some $(1-N)/2 < \alpha < (2-N)/2$ and $|x| > 1$. Notice that $|w| = 1$ and $|\nabla w| = |\nabla \varphi| \in L^2(\mathbb{R}^N)$ so that $w \in X$. On the other hand, $(iw, \nabla w) = \nabla \varphi \notin L^1(\mathbb{R}^N)$ and similarly $(i(w-1), \nabla w) \notin L^1(\mathbb{R}^N)$; hence $\vec{p}(w)$ is not well defined in the Lebesgue sense. To overcome this difficulty, we will introduce a series of approximate problems $(\mathcal{P}_n^\varepsilon)$ on expanding tori. A price has to be paid, however :

- one has to find uniform bounds for both the Lagrange multipliers and the solutions associated to $(\mathcal{P}_n^\varepsilon)$,
- some information (energy, momentum,...) could be lost in the limit (see the discussion on stability later).

1.2 The approximating problems

Setting. For $n \in \mathbb{N}^*$, consider the flat torus

$$\Pi_n \simeq \Omega_n \equiv [-n, n]^N,$$

with opposite faces identified, and the space

$$X_n := H^1(\Pi_n, \mathbb{C}) \simeq H^1_{\text{per}}(\Omega_n, \mathbb{C})$$

of $2n$-periodic H^1 functions. Since Π_n is compact, we can define the (first component of the) momentum as

$$p(u) := \int_{\Pi_n} (iu, \partial_1 u),$$

and this clearly defines a quadratic functional on X_n.

Let

$$\Gamma_n := \{ u \in X_n,\ p(u) = 2\pi |B^{N-1}| \},$$

and consider the minimization problem :

$$(\mathcal{P}_n^\varepsilon) \qquad\qquad I_{n,\varepsilon} := \inf_{u \in \Gamma_n} E_\varepsilon(u).$$

The constraint is easily seen to be non void. It is also straightforward to prove existence of a minimizer for $(\mathcal{P}_n^\varepsilon)$.

Proposition 1. *There exists a minimizer $u_{n,\varepsilon} \in X_n$ for $(\mathcal{P}_n^\varepsilon)$ and some constant $c_{n,\varepsilon} \in \mathbb{R}$ such that $u_{n,\varepsilon}$ verifies (11), i.e.*

$$ic_{n,\varepsilon} |\log \varepsilon| \frac{\partial u_{n,\varepsilon}}{\partial x_1} = \Delta u_{n,\varepsilon} + \frac{1}{\varepsilon^2} u_{n,\varepsilon} (1 - |u_{n,\varepsilon}|^2) \qquad on\ \Pi_n.$$

In the sequel, for sake of simplicity, we will skip the subscripts n or ε when this is not misleading.

Remark 2. There is presumably some freedom in the choice of the approximate problem. A natural candidate might have been

$$Y_n := \{u \in H^1(\Omega_n, \mathbb{C}),\ u \equiv 1 \text{ on } \partial\Omega_n\}.$$

One advantage of Y_n is that

$$p(u) = m(u) := \int_{\Omega_n} \langle Ju, \xi_1 \rangle, \qquad \text{for all } \ u \in Y_n,$$

which follows easily by integration by parts. Here, Ju (the Jacobian of u) denotes the 2-form on Ω_n

$$Ju := \frac{1}{2} d(u \times du) = \sum_{i<j} (\partial_i u \times \partial_j u) dx_i \wedge dx_j, \tag{12}$$

and the 2-form ξ_1 is defined on Ω_n by

$$\xi_1(x) := \tfrac{2}{N-1} \sum_{i=2}^{N} x_i dx_1 \wedge dx_i. \tag{13}$$

Finally $\langle \cdot, \cdot \rangle$ stands for the scalar product of 2-forms. As we will see later, m has a convenient geometric interpretation which we will use throughout. On the torus Π_n however, m is **not** well defined (due to ξ_1), and we will have to circumvent this difficulty by choosing suitable unfoldings.
Whereas part of the analysis is somewhat simpler in Y_n, the main disadvantage is that the translation invariance of our original problem is broken in Y_n.

Upper bound for $I_{n,\varepsilon}$ and $c_{n,\varepsilon}$. The upper bound on $I_{n,\varepsilon}$ is obtained using appropriate comparison functions for $(\mathcal{P}_n^\varepsilon)$. As already mentioned, in the limit $\varepsilon \to 0$, the solution u_ε (and also $u_{n,\varepsilon}$) will ultimately look like thin vortex rings. In the sequel, for $R > 0$ $(2R < n)$ we propose a simple construction of such a vortex ring $w_{\varepsilon,R}$ of radius R, which will turn out to be an almost optimal candidate.
We carry out the construction in cylindrical coordinates (x_1, r, θ) where $r := \sqrt{x_2^2 + \cdots + x_N^2}$. The function $w_{\varepsilon,R}$ will be independent of θ (i.e. cylindrically symmetric); we therefore just need to describe it in the (x_1, r) half-plane H_+. For that purpose, consider in the complex plane the point $z_R := iR$ and the function ω_R defined on B_{2R} by

$$\omega_R(z) = \frac{z - z_R}{|z - z_r|} \frac{z + z_R}{|z + z_r|} \exp(i\varphi),$$

where φ is a real harmonic function such that $\omega_R \equiv 1$ on ∂B_{2R} (see [9]). Then we set

$$w_{\varepsilon,R}(x_1, r, \theta) := \begin{cases} \omega_R(x_1 + ir) & \text{if } x_1 + ir \in B_{2R} \setminus B_\varepsilon(z_R), \\ \varepsilon^{-1} |x_1 + i(r - R)| \omega_R(x_1 + ir) & \text{if } x_1 + ir \in B_\varepsilon(z_R). \end{cases}$$

By standard computations,

$$\frac{1}{2}\int_{\mathbb{R}^N} |\nabla w_{\varepsilon,R}|^2 = \frac{1}{2}|S^{N-2}|\int_{H_+} |\nabla w_{\varepsilon,R}|^2 r^{N-2} dx_1\, dr$$
$$= \pi R^{N-2}|S^{N-2}||\log\varepsilon| + O(1),$$

and similarly

$$\frac{1}{4\varepsilon^2}\int_{\mathbb{R}^N} (1-|w_{\varepsilon,R}|^2)^2 = O(1),$$

so that

$$E_\varepsilon(w_{\varepsilon,R}) = \pi|S^{N-2}|R^{N-2}|\log\varepsilon| + O(1). \tag{14}$$

For the momentum $p(w_{\varepsilon,R}) = m(w_{\varepsilon,R})$, we have

$$\begin{aligned} p(w_{\varepsilon,R}) &= |S^{N-2}|\int_{H_+}\left(\frac{\partial w_{\varepsilon,R}}{\partial x_1}\times\frac{\partial w_{\varepsilon,R}}{\partial r}\right)\frac{2r}{N-1}\, r^{N-2} dx_1\, dr \\ &= 2\pi\tfrac{|S^{N-2}|}{N-1}R^{N-1} + o(1) = 2\pi|B^{N-1}|R^{N-1} + o(1), \end{aligned} \tag{15}$$

since

$$Jw_{\varepsilon,R}(x_1,r,\theta) = (\partial_{x_1} w_{\varepsilon,R}\times\partial_r w_{\varepsilon,R})\, dx_1\wedge dr$$

and since

$$\partial_{x_1} w_{\varepsilon,R}\times\partial_r w_{\varepsilon,R} \rightharpoonup \pi\delta_{(0,R)}$$

in the sense of measures on H_+. The detailed computations to obtain estimates (14) and (15) are standard and can be found in many places (see e.g. [9]), we do not repeat them here. With the help of these estimates, it is then fairly easy to obtain a (sharp) upper bound for $I_{n,\varepsilon}$.

Lemma 1. *There exists some constant K_0, which is independent of n and ε, such that*

$$|I_{n,\varepsilon}| \le K_0|\log\varepsilon|. \tag{16}$$

Moreover,

$$\limsup_{\varepsilon\to 0}\left(\sup_{n\in\mathbb{N}^*}\frac{I_{n,\varepsilon}}{|\log\varepsilon|}\right) \le \pi|S^{N-2}|. \tag{17}$$

We turn next to $c_{n,\varepsilon}$. As a consequence of the Pohozaev identity for (11) and some careful analysis of the boundary terms relating $p(u)$ and $m(u)$ in X_n we obtain the following.

Lemma 2. *There exists $\varepsilon_1 > 0$ such that for $0<\varepsilon<\varepsilon_1$ and $n\ge n(\varepsilon)$, where $n(\varepsilon)\in\mathbb{N}$ depends only on ε, we have*

$$|c_{n,\varepsilon}| \le K_1. \tag{18}$$

Here K_1 is some constant which is independent of n and ε.

Remark 3. It follows from our proof of Lemma 2 that an upper bound for $n(\varepsilon)$ is $K_2|\log\varepsilon|\varepsilon^{3-N}$ where K_2 is some sufficiently large constant. With a little more work, one should be able to prove that a large (but independent of ε) constant is a valid upper bound. Since our final goal is to let $n\to+\infty$ at fixed ε, the first upper bound is sufficient.

1.3 Some properties of the Euler-Lagrange equation

An important part of our results relies on the analysis of the Euler-Lagrange equation (11). Since we believe that it is of interest in related topics, as, for instance, superconductivity (see Remark 4 iv) we will be more general than what is strictly needed for the proof of Theorem 1. Therefore, we will consider solutions w_ε to the class of equations

$$i|\log\varepsilon|\vec{c}(x)\cdot\nabla w = \Delta w + \frac{1}{\varepsilon^2}w(1-|w|^2) - |\log\varepsilon|^2 d(x)w \qquad \text{on } \Omega, \tag{19}$$

where $\Omega \subseteq \mathbb{R}^N$ is a piecewise $\mathcal{C}^1$ simply connected domain, $\vec{c}:\overline{\Omega}\to\mathbb{R}^N$ is a bounded Lipschitz vector field and $d:\overline{\Omega}\to\mathbb{R}$ is Lipschitz non-negative and bounded [in our original problem $\vec{c}$ is constant and $d=0$]. If we allow $\vec{c}$ and d to depend on ε (in view in particular of the application to superconductivity), then we require that there exists some constant $\Lambda_0>0$ not depending on ε such that

$$|\vec{c}|^2_{L^\infty(\Omega)} + |\nabla\vec{c}|^2_{L^\infty(\Omega)} + |d|^2_{L^\infty(\Omega)} + |\nabla d|^2_{L^\infty(\Omega)} \le \Lambda_0^2.$$

Notice that (19) can be rewritten as

$$i|\log\varepsilon|\vec{c}(x)\cdot\nabla w = \Delta w + \frac{1}{\varepsilon^2}w(a_\varepsilon(x)-|w|^2), \tag{20}$$

where

$$a_\varepsilon(x) := 1 - d(x)\varepsilon^2|\log\varepsilon|^2.$$

When $\operatorname{div}\vec{c}\equiv 0$ it is also equivalent to

$$(\nabla - i|\log\varepsilon|\frac{\vec{c}}{2})^2 w + \frac{1}{\varepsilon^2}w(b_\varepsilon(x)-|w|^2) = 0, \tag{21}$$

where

$$b_\varepsilon(x) := a_\varepsilon(x) + \varepsilon^2|\log\varepsilon|^2\frac{c^2(x)}{4}.$$

Equation (19) is variational when $\vec{c}$ is divergence free; we will make this assumption throughout. It is likely however that a large part of the analysis can be done in the general case. Notice also that no boundary condition is prescribed here so that the focus in this section will be on local properties.

The outline of our analysis of (19) follows closely the corresponding theory for the Ginzburg-Landau equation developed in [9, 41, 13, 37, 31, 32, 10, 29, 5, 16] and the references therein. In particular, the emphasis is placed there on the set

$$S_\varepsilon := \{x\in\Omega,\ |w_\varepsilon(x)|\le\frac{1}{2}\},$$

where vorticity and energy will eventually concentrate in the limit $\varepsilon\to 0$. Notice that for the proof of Theorem 1, the structure of S_ε for ε fixed but expanding Ω will also play a key role. We first start with the following standard pointwise estimates.

Lemma 3. *Let K be any compact subset of Ω. Then, for any solution w_ε of (19) we have :*

$$\begin{cases} |w_\varepsilon|_{L^\infty(K)} \le 1 + c_\infty^2\,\varepsilon^2 |\log\varepsilon|^2 + C\frac{\varepsilon^2}{dist(K,\partial\Omega)^2}, \\ |\nabla w_\varepsilon|_{L^\infty(K)} \le \frac{C_K}{\varepsilon}, \end{cases}$$

where $c_\infty := |\vec{c}|_{L^\infty}$, C is a constant depending only on N, and C_K is a constant depending only on N, c_∞ and K.

In order to describe the properties of S_ε, monotonicity formulas play an important role (as in the works quoted above). More generally, they have been extensively used in the context of regularity for various problems in PDE's and geometry (see e.g. [34, 23]).
For $x_0 \in \Omega$ and $r > 0$ such that $B_r(x_0) \subset \Omega$, consider the scaled energy

$$\tilde{E}_\varepsilon(w_\varepsilon, x_0, r) := \frac{1}{r^{N-2}} E_\varepsilon(w_\varepsilon, x_0, r) \equiv \frac{1}{r^{N-2}} \int_{B_r(x_0)} \frac{|\nabla w_\varepsilon|^2}{2} + \frac{(a_\varepsilon(x) - |w_\varepsilon|^2)^2}{4\varepsilon^2}. \tag{22}$$

When this does not lead to a confusion, we will also note it by $\tilde{E}_\varepsilon(x_0, r)$ or even $\tilde{E}_\varepsilon(r)$.

Lemma 4. *There exists $C > 0$ depending only on N such that for*

$$\Lambda := C\,(c_\infty + 1)\,|\log\varepsilon|, \qquad Q := C\Lambda_0 |\log\varepsilon|^2 \varepsilon,$$

and for any w_ε satisfying (19) on $B_R(x_0) \subset \Omega$, we have

$$\frac{d}{dr}\left(\exp(\Lambda r)(\tilde{E}_\varepsilon(x_0, r) + \frac{Q^2}{\Lambda})\right) \ge \frac{1}{r^{N-2}} \int_{\partial B_r} \left|\frac{\partial w_\varepsilon}{\partial n}\right|^2 + \frac{1}{r^{N-1}} \int_{B_r} \frac{(a_\varepsilon(x) - |w_\varepsilon|^2)^2}{4\varepsilon^2} \ge 0$$

for $0 < r < R$. In particular, $\exp(\Lambda r)(\tilde{E}_\varepsilon(x_0, r) + Q^2/\Lambda)$ is increasing.

The above inequality is obtained using a crude estimate for the Jacobian Jw_ε. This restricts somehow its usefulness to balls of size $O(1/|\log\varepsilon|)$. In order to handle balls of radius $O(1)$, refined estimates on Jacobian integrals are needed (see [29, 1]).

Proposition 2. *There exists $C > 0$ and $\beta > 0$ depending only on N such that for any w_ε satisfying (19) on $B_R(x_0) \subset \Omega$, we have*

$$\tilde{E}_\varepsilon(x_0, \theta r) \le C\left(\tilde{E}_\varepsilon(x_0, r) + (1 + \Lambda_0)^{N-1} \varepsilon^\beta\right) \tag{23}$$

for $0 < \theta < 1/2$ and $0 < r < \min(R, \frac{2}{\Lambda_0 + 1})$.

Using the two previous results, and following the arguments of [10] (see also [37, 31, 32]), we derive the following result, which plays an important role in the analysis.

Theorem 2. *Let w_ε be a solution of (19) on Ω and $\sigma > 0$ be given. There exists constants $\eta > 0$ and $\varepsilon_0 > 0$, depending only on N, σ and Λ_0, such that if $x_0 \in \Omega$, $\varepsilon \leq \varepsilon_0$, $\sqrt{\varepsilon} \leq r \leq 1/(1+\Lambda_0)$, $B_{2r}(x_0) \subset \Omega$, and*

$$\tilde{E}_\varepsilon(x_0, r) \leq \eta |\log \varepsilon| \,,$$

then

$$|w_\varepsilon(x_0)| \geq 1 - \sigma.$$

Asymptotic analysis of concentrating measures

We assume from now on that w_ε verifies the bound

$$\boldsymbol{E_\varepsilon(w_\varepsilon) = \int_\Omega e_\varepsilon(w_\varepsilon) \leq M_0 |\log \varepsilon|}, \tag{24}$$

where M_0 is some fixed constant. In this regime, one of the main consequences of Theorem 2 is that as ε tends to zero, the set S_ε concentrates on a rectifiable limiting set S_*, of locally finite $N-2$ Hausdorff measure. It is convenient to introduce the following measures :

$$\begin{cases} \mu_\varepsilon := \frac{e_\varepsilon(w_\varepsilon)}{|\log \varepsilon|}\, dx, \\ \eta_\varepsilon := \varepsilon^{-2} 1_{S_\varepsilon}\, dx, \\ J_\varepsilon := J w_\varepsilon. \end{cases}$$

In view of assumption (24), μ_ε is bounded. Therefore, up to a subsequence we may assume that

$$\mu_\varepsilon \rightharpoonup \mu_* \qquad \text{as measures.}$$

Using Theorem 2 again, combined with a Besicovitch covering argument, it follows that $\varepsilon^{-2} 1_{S_\varepsilon}$ is locally bounded in $L^1(\Omega)$. Extracting possibly a further subsequence, we may thus assume that

$$\eta_\varepsilon \rightharpoonup \eta_* \qquad \text{as measures.}$$

Concerning J_ε (a measure with values in 2-forms), it is tempting to believe that it is also bounded in $L^1_{loc}(\Omega)$. We have no proof of that fact, however we may invoke Jerrard Soner's [29] compactness result (valid for arbitrary functions verifying (24), see also [1]) to assert that J_ε is bounded in $[\mathcal{C}^{0,\alpha}(K)]^*$ for any compact $K \subset \Omega$ and any $0 < \alpha < 1$. Going possibly to a third subsequence, we thus have

$$J_\varepsilon \rightharpoonup J_* \qquad \text{in } [\mathcal{C}^{0,\alpha}(K, \Lambda_2 \mathbb{R}^N)]^*, \qquad \text{for every compact } K \subset \Omega.$$

It is proved moreover in [29, 1] that

$$\|J_*\| \leq \mu_*, \tag{25}$$

and that the current $[\![J_*]\!]$ associated to J_* is an integer multiplicity $(N-2)$-current. In particular, its geometrical support

$$\Sigma_J := \{x \in \Omega \text{ s.t. } \Theta_{N-2}(\|J_*\|, x) > 0\}$$

is an $(N-2)$-rectifiable set. Here, for a Radon measure $\nu \in \mathcal{M}(\Omega)$ and $m > 0$, the m dimensional density of ν at $x \in \Omega$ is defined by

$$\Theta_m(\nu, x) := \liminf_{r \to 0} \frac{\nu(B(x,r))}{r^m}.$$

Likewise we set

$$\Sigma_\mu = \{x \in \Omega \text{ s.t. } \Theta_{N-2}(\mu_*, x) > 0\}$$

and similarly we define Σ_η.

In the next theorem we will clarify the structure of the measure μ_* and we will specify its relation to J_*. We emphasize that **no boundary condition** has been prescribed on $\partial\Omega$.

Theorem 3. *The following properties hold.*
i) The set Σ_μ is closed in Ω and $(N-2)$-rectifiable. There exists $\eta_0 > 0$ such that for each $x_0 \in \Sigma_\mu$,

$$\Theta_*(x_0) := \Theta_{N-2}(\mu_*, x_0) = \liminf_{r \to 0} \frac{\mu_*(B_r(x_0))}{r^{N-2}} \geq \eta_0. \tag{26}$$

Moreover, for every compact set $F \subset \Omega \setminus \Sigma_\mu$,

$$|w_\varepsilon(x)| \to 1 \qquad \textit{uniformly on } F \textit{ as } \varepsilon \to 0. \tag{27}$$

ii) The measure μ_* can be decomposed as

$$\mu_* = |\nabla h(x)|^2 \cdot \mathcal{H}^N + \Theta_*(x) \cdot \mathcal{H}^{N-2} \llcorner \Sigma_\mu, \tag{28}$$

where h is some harmonic function.
iii) Let $K \subset \Omega$ be any compact set. There exists some constant C_K, depending only on K, such that

$$(C_K)^{-1}\, J_* \leq \eta_* \leq C_K\, M_0\, \mu_*.$$

iv) The varifold $V := V(\Sigma_\mu, \Theta_*)$ satisfies the equation

$$\vec{H}(x) = * \left(\vec{c}(x) \wedge * \frac{dJ_*}{d\mu_*} \right) \qquad \textit{for } \mu_*\textit{-a.e. } x \textit{ in } \Sigma_\mu, \tag{29}$$

where $\vec{H}(x)$ denotes the generalized mean curvature of V at x, $*$ refers to the Hodge duality, and $\frac{dJ_*}{d\mu_*}$ is the Radon-Nikodym derivative of J_* with respect to μ_*.

A short comment is needed concerning the interpretation of (29). The generalized mean curvature H of the varifold V is defined by (see [39])

$$\int_\Omega \operatorname{div}_{\Sigma_\mu} \vec{X} = -\int_\Omega \vec{H} \cdot \vec{X} \qquad \text{for all } \vec{X} \in \mathcal{C}_c^\infty(\Omega, \mathbb{R}^N),$$

where $\operatorname{div}_{\Sigma_\mu}$ denotes the divergence restricted to the tangent space. Moreover, we identify vector fields and 1-forms.

Remark 4. i) In the case $\vec{c} \equiv 0$ and $d \equiv 0$, (19) is the standard Ginzburg-Landau equation and then (29) means that V is a stationary varifold (see [5, 10]).
ii) Equation (29) is very reminiscent of the prescribed mean curvature equation in codimension 1. However here, in codimension 2, an important difference is that the right hand side of (29) **does** depend on V through its tangent space. To give a flavour of the structure of (29), let us first consider the case $N = 3$, and $\frac{d\|J_*\|}{d\mu_*} = 1$. Then V is a smooth curve and (29) writes

$$\vec{\kappa} = \vec{c} \times \vec{\tau}, \tag{30}$$

where $\vec{\tau}$ is the unit tangent vector to V and $\vec{\kappa}$ its curvature vector. In the case $\vec{c} \equiv \vec{c}_0$ is a constant vector field, the solutions are

- Straight lines parallel to $\vec{c}_0$,
- Circles of radius $1/c_0$ in a plane orthogonal to $\vec{c}_0$,
- Helicoidals of axis parallel to $\vec{c}_0$.

On the other hand, any constant mean curvature hypersurface in dimension $N - 1$ yields a solution of (29) for some constant vector field $\vec{c}_0$. In dimension 3, this yields, as already mentioned, the round circle as unique compact solution. In higher dimension however, there is a rich class of constant mean curvature hypersurfaces besides spheres (e.g. Wente's tori in dimension $N = 4$). It is tempting to believe that any compact solution of (29) with a constant vector field $\vec{c}_0$ is contained in an affine hyperplane (and is thus a constant mean curvature hypersurface).
iii) In the case $\frac{d\|J_*\|}{d\mu_*} = 1$, V has integer multiplicity. In the optimal case where J_* has constant multiplicity, it follows from (29) and Allard's theorem (see [2, 39]) that V is a $\mathcal{C}^{1,\alpha}$ manifold.
iv) Equation (19) with $\vec{c}(x) = A(x)$ and $d(x) = |A(x)|^2/4$ is the first equation in the Ginzburg-Landau system of superconductivity, namely

$$(\nabla - iA|\log\varepsilon|/2)^2 u = \frac{1}{\varepsilon^2} u(1 - |u|^2).$$

In particular, for solutions verifying the energy bound (24) in the Coulomb gauge, vortices will be curved according to the equation

$$\vec{\kappa} = A \times \vec{\tau},$$

provided $\frac{d\|J_*\|}{d\mu_*} = 1$.

Theorem 3 states some compactness properties for the measures. However, without assumptions on the boundary data, we cannot expect compactness for the functions w_ε, as noticed in [17]. The presence in the decomposition (28) of one part which is absolutely continuous with respect to the Lebesgue measure is precisely due to possible wild oscillations of w_ε on the boundary.

Asymptotics for w_ε

If we impose boundary conditions on $\partial\Omega$, then we may obtain compactness properties for the sequence w_ε. In this subsection, we will focus only on the case which is of interest for Theorem 1, namely

$$\Omega := \Pi_n \simeq \Omega_n,$$

with the convention that $\Pi_\infty := \mathbb{R}^N$; we refer to Appendix A for more general statements. We make the assumption that

$$\boldsymbol{n \geq (M_0 + 1)|\log \varepsilon|}. \tag{31}$$

The main point here is that we would like to obtain estimates which are uniform with respect to the domain size (i.e. independent of n). In this situation we obtain :

Theorem 4. *Let w_ε be a solution of (11) such that (24) and (31) are satisfied.*
i) Let $1 \leq p < \frac{N}{N-1}$. Then there exists some constant C depending only on p, Λ_0 and M_0, but independent of ε and n, such that for any $x_0 \in \Pi_n$ we have

$$\int_{B(x_0,1)} |\nabla w_\varepsilon|^p \leq C.$$

ii) There exist $R > 0$, $C > 0$ and $l \in \mathbb{N}$ depending only on Λ_0 and M_0, and q points $x_{1,\varepsilon}, \cdots, x_{q,\varepsilon}$ $(q \leq l)$ in Π_n such that $S_\varepsilon \subset \cup_{i=1}^q B(x_{i,\varepsilon}, R)$, $B(x_{i,\varepsilon}, 8R) \cap B(x_{j,\varepsilon}, 8R) = \emptyset$ if $i \neq j$, and

$$\int_{\Pi_n \setminus \cup B(x_{i,\varepsilon},R)} e_\varepsilon(w_\varepsilon) \leq C.$$

1.4 The isoperimetric problem

After this rather lenghty discussion on the Euler-Lagrange equation, we go back to our original problem and consider from now on only minimizers $u_{n,\varepsilon}$ of $(\mathcal{P}_n^\varepsilon)$. Since our ultimate goal is to provide the existence of a solution u_ε of (11) as well as some qualitative properties (see Theorem 1), we will eventually let n go to $+\infty$ keeping ε **fixed** (in particular, we assume throughout that (31) is verified). In order to describe properly the behavior of u_ε (including the stability properties, which will be discussed later), it is extremely important, in this approach, to get more information than a simple H^1_{loc} convergence.

The first crucial observation in this section, is the relation of the energy $E_\varepsilon(u_{n,\varepsilon})$ and the flux $p(u_{n,\varepsilon})$ with geometrical properties of $Ju_{n,\varepsilon}$ (as already observed in [14] and [28]). This relation is best understood taking the limit as ε tends to 0 when n is fixed (note however that this is incompatible with $n \geq n(\varepsilon)$ of Lemma 2 !). It follows from the analysis of [29, 1] that, up to a subsequence,

$$Ju_{n,\varepsilon} \to \pi T_n = \pi \partial R_n \qquad \text{in } [\mathcal{C}^{0,1}(\Pi_n)]^*$$

where $T_n = \partial R_n$ is an $(N-2)$ dimensional integral boundary, i.e. T_n is a rectifiable current with integer multiplicities (of course the choice of the rectifiable current R_n is not unique). Moreover,

$$p(u_{n,\varepsilon}) \to \pi \partial R_n(*\xi_1) = \pi R_n(*d^*\xi_1) = 2\pi R_n(*dx_1) \equiv \mathcal{F}(T_n)\,,$$

where $\mathcal{F}(T_n)$ represents 2π times the flux of the vector $\vec{e}_1$ through R_n. Notice that in particular $\mathcal{F}(T_n) \le 2\pi \mathbf{M}(R_n)$. On the other hand, it is also proved in [29, 1] that

$$\liminf_{\varepsilon \to 0} \frac{E_\varepsilon(u_{n,\varepsilon})}{\pi|\log\varepsilon|} \ge \mathbf{M}(T_n)\,.$$

This establishes immediately the inequality

$$\frac{\mathbf{M}(\partial R_n)^{\frac{N-1}{N-2}}}{\mathbf{M}(R_n)} \le \liminf_{\varepsilon\to 0} \frac{2\pi E_\varepsilon(u_{n,\varepsilon})^{\frac{N-1}{N-2}}}{(\pi|\log\varepsilon|)^{\frac{N-1}{N-2}} p(u_{n,\varepsilon})}\,. \tag{32}$$

Using Lemma 1 we deduce that

$$\frac{\mathbf{M}(\partial R_n)^{\frac{N-1}{N-2}}}{\mathbf{M}(R_n)} \le \lambda_N \equiv \frac{|S^{N-2}|^{\frac{N-1}{N-2}}}{|B^{N-1}|}\,. \tag{33}$$

Since the right hand side of (33) is the best constant in the isoperimetric inequality it follows that $T_n = \partial R_n$ is a round $(N-2)$-sphere (contained in a $(N-1)$-hyperplane orthogonal to $\vec{e}_1$).

In our situation, we will obtain an inequality similar to (33), but uniformly for n large. To be more precise, assume from now on that $n \ge n(\varepsilon)$ where $n(\varepsilon)$ was defined in Lemma 2. Then we have

Lemma 5. *For every $n \ge n(\varepsilon)$ there exists an $(N-2)$ dimensional integral boundary $T_{n,\varepsilon} = \partial R_{n,\varepsilon}$ supported in at most ℓ balls of radius R (ℓ, R being independent of n and ε), such that*

i) $\|Ju_{n,\varepsilon} - \pi T_{n,\varepsilon}\|_{[\mathcal{C}^{0,1}(\Pi_n)]^*} \le r(\varepsilon)\,,$

ii) $|p(u_{n,\varepsilon}) - \mathcal{F}(T_{n,\varepsilon})| \le r(\varepsilon),$

iii) $\mathbf{M}(T_{n,\varepsilon}) \le \dfrac{E_\varepsilon(u_{n,\varepsilon})}{\pi|\log\varepsilon|} + r(\varepsilon)\,,$

where $r(\varepsilon) \to 0$ as $\varepsilon \to 0$, independently of n.

As mentioned, the choice of a current $R_{n,\varepsilon}$ such that $T_{n,\varepsilon} = \partial R_{n,\varepsilon}$ is not unique. We may therefore additionally require that

$$\mathbf{M}(R_{n,\varepsilon}) = \inf\{\mathbf{M}(R)\,,\ \partial R = T_{n,\varepsilon}\}\,. \tag{34}$$

For such a choice (which is always possible by [20], 4.1.12), the following isoperimetric inequality is valid (see [3]):

$$\frac{\mathbf{M}(T_{n,\varepsilon})^{\frac{N-1}{N-2}}}{\mathbf{M}(R_{n,\varepsilon})} \ge \lambda_N\,. \tag{35}$$

Proposition 3. *We have*

$$\frac{\mathbf{M}(\partial R_{n,\varepsilon})^{\frac{N-1}{N-2}}}{\mathbf{M}(R_{n,\varepsilon})} = \lambda_N + r(\varepsilon)\,, \tag{36}$$

where $r(\varepsilon) \to 0$ as $\varepsilon \to 0$, independently of n. In particular, for all sequences $\varepsilon_j \to 0$ and $n_j \geq n(\varepsilon_j)$ there exist subsequences (still denoted ε_j and n_j) and translations τ_j in Π_{n_j} such that

$$\tau_j T_{n_j,\varepsilon_j} \to S^{N-2} \qquad \text{in } [\mathcal{C}_c^{0,1}(\mathbb{R}^N)]^* \qquad \text{as } j \to +\infty, \tag{37}$$

where S^{N-2} is the unit round $(N-2)$-sphere contained in the hyperspace orthogonal to $\vec{e}_1$.

Remark 5. Actually, as $j \to +\infty$ we have $\tau_j R_{n_j,\varepsilon_j} \to B^{N-1}$ and also $\tau_j T_{n_j,\varepsilon_j} \to S^{N-2}$ in the flat norm sense (see [20], 4.1.12), with $\mathbf{M}(R_{n_j,\varepsilon_j}) \to |B^{N-1}|$ and $\mathbf{M}(T_{n_j,\varepsilon_j}) \to |S^{N-2}|$.

Note that (37) states a rather weak convergence. In particular, it does not exclude very small structures even far from the limit S^{N-2}. The next lemma, which improves statement (ii) of Theorem 4, excludes such structures.

Lemma 6. *There exist $R > 0$, $C > 0$ independent of ε and n, and $x_{n,\varepsilon} \in \Pi_n$ such that*

i) $S_\varepsilon(u_{n,\varepsilon}) \subset B(x_{n,\varepsilon}, R)\,,$

ii) $\displaystyle\int_{\Pi_n \setminus B(x_{n,\varepsilon},R)} e_\varepsilon(u_{n,\varepsilon}) \leq C\,,$

for every $\varepsilon \leq \varepsilon_0$ and $n \geq n(\varepsilon)$, ε_0 being independent of n.

As already mentioned, our problem is invariant under translation. We now remove this invariance. To that aim, in view of Lemma 6 and Proposition 3, we assume that the identification $\Pi_n \simeq [-n, n]^N$ is such that

$$\boldsymbol{x_{n,\varepsilon} = 0} \qquad \textbf{and} \qquad \boldsymbol{Ju_{n_j,\varepsilon_j} \to \pi S^{N-2}} \quad \textbf{in } [\boldsymbol{\mathcal{C}_c^{0,1}}(\mathbb{R}^{\boldsymbol{N}})]^*$$

for all sequences $\varepsilon_j \to 0$ and $n_j \geq n(\varepsilon_j)$, where S^{N-2} is the unit $(N-2)$-sphere contained in the subspace orthogonal to $\vec{e}_1$.

1.5 Limits of growing tori

It remains at this stage, for fixed ε (but choosen sufficiently small), to let $n \to +\infty$. Since $E_\varepsilon(u_{n,\varepsilon})$ is bounded uniformly in n by Lemma 1 (but not in ε!), up to a possible subsequence we may assume

$$u_{n,\varepsilon} \rightharpoonup u_\varepsilon \quad \text{in } H^1_{\text{loc}}(\mathbb{R}^N) \quad \text{as } n \to +\infty,$$

so that

$$E_\varepsilon(u_\varepsilon) \le \liminf_{n\to+\infty} E_\varepsilon(u_{n,\varepsilon}).$$

Moreover, by standard elliptic estimates (ε is fixed),

$$u_{n,\varepsilon} \to u_\varepsilon \quad \text{strongly in } H^1_{\rm loc}(\mathbb{R}^N) \quad \text{as } n \to +\infty.$$

Note also that since $(u_{n,\varepsilon})_{n\in\mathbb{N}}$ is bounded in L^∞, so is u_ε, and we may pass to the limit in the equation. Hence, u_ε verifies (11) with

$$c(\varepsilon) = \lim_{n\to+\infty} c_{n,\varepsilon}.$$

Since we also have $Ju_{n,\varepsilon} \to \pi S^{N-2}$ as $\varepsilon \to 0$, for fixed but small ε the jacobian $Ju_{n,\varepsilon}$ is far from zero (for all $n \ge n(\varepsilon)$) and therefore u_ε is not a trivial solution. Hence, existence of $U_\varepsilon(x) := u_\varepsilon(\varepsilon x)$ in Theorem 1 is established. Properties (6), (7), (8) and (9) follow then from the analysis of Subsection 1.3 (see Section 4 for the details).

The definition of $P(U_\varepsilon)$ needs some clarification. For this purpose, we consider the class of functions

$$W = \left\{ u \in L^\infty(\mathbb{R}^N),\ E_\varepsilon(u) < +\infty \text{ and } \exists R > 0 \ \text{ s.t. } \inf_{|x|\ge R} |u(x)| \ge 1/2 \right\}.$$

If $u \in W$, we may write, for $|x| > R$,

$$u = \rho \exp i\varphi$$

where φ is a real function on $\mathbb{R}^N \setminus B_R(0)$ defined modulo a multiple of 2π. We define

$$p(u) := \int_{\mathbb{R}^N} (iu, \partial_1 u)\chi + \int_{\mathbb{R}^N} (1-\chi)(\rho^2 - 1)\partial_1\varphi + \int_{\mathbb{R}^N} \varphi\partial_1(1-\chi) \tag{38}$$

where χ is an arbitrary smooth function with compact support such that $\chi \equiv 1$ on $B_R(0)$ and $0 \le \chi \le 1$. On checks immediately that the definition makes sense in W and is independent of the choice of χ and φ. [To motivate this choice, notice that formally

$$\begin{aligned}
\int_{\mathbb{R}^N} (iu, \partial_1 u) &= \int_{\mathbb{R}^N} (iu, \partial_1 u)\chi + \int_{\mathbb{R}^N} (iu, \partial_1 u)(1-\chi) \\
&= \int_{\mathbb{R}^N} (iu, \partial_1 u)\chi + \int_{\mathbb{R}^N} (1-\chi)\rho^2\partial_1\varphi \\
&= \int_{\mathbb{R}^N} (iu, \partial_1 u)\chi + \int_{\mathbb{R}^N} (1-\chi)(\rho^2-1)\partial_1\varphi + \int_{\mathbb{R}^N} \varphi\partial_1(1-\chi)
\end{aligned}$$

so that we recover the usual formula when $\nabla u \in L^1(\mathbb{R}^N)$]. Clearly, in view of our analysis, $u_\varepsilon \in W$ so that $P(U_\varepsilon) := \varepsilon^{N-1} p(u_\varepsilon)$ is well defined.

Remark 6. Consider the affine space

$$Y = H^1(\mathbb{R}^N) + \{1\} = \{u, \ s.t. \ u = 1 + v, \ v \in L^2(\mathbb{R}^N), \ \nabla v \in L^2(\mathbb{R}^N) \},$$

equipped with the H^1-distance. For functions in Y, one may set

$$\tilde{p}_1(u) = \int_{\mathbb{R}}^N (u - 1, \frac{\partial u}{\partial x_1})$$

as a definition of the momentum. It is straightforward to see that $\tilde{p}_1$ is continuous on Y (for the H^1 norm). On the other hand, $\mathcal{C}_c^\infty(\mathbb{R}^N) + \{1\}$ is dense in Y, and included in W. One verifies, in view of the definition of p, that

$$\tilde{p}_1(u) = p_1(u), \ \forall\, u \in \mathcal{C}_c^\infty(\mathbb{R}^N) + \{1\} \subset W.$$

1.6 Discussion on stability

The discussion about stability of special solutions for dynamical systems is a fundamental issue, in particular if one argues about some physical relevance. This is a vast topic, and the very notion of stability appears in different places with different meanings. We want to stress first that we are not yet able to state any trully satisfactory result concerning the stability of U_ε. We next explain the main difficulty in this direction, and the partial results we have obtained.

When dealing with PDE's, a first step commonly needed for stability is to solve the Cauchy problem, at least in a neighborhood of the special solution. In particular, one has to define a suitable functional space, and this usually requires some knowledge of the decay properties of the solution. In our case, it can be proved (see [14]) that the Cauchy problem is well defined on $Y = H^1(\mathbb{R}^N) \cap L^4(\mathbb{R}^N) + \{1\}$ and that both energy E_ε and momentum $\tilde{p}$ are conserved during the flow. However, it is not known that the solution u_ε belongs to Y (see however results by Gravejeat [25] for the asymptotic behavior of finite energy travelling waves), and the possibility to solve the Cauchy problem in other spaces has not been investigated yet.

Assume that in some way one is able to overcome this difficulty. Then in our context the notion of (nonlinear) orbital stability seems to be the most adequate (see e.g. [6, 15, 18, 26]). Indeed, recall that our solution is obtained as a limit of constrained minimizers for which both the constraint and the minimized quantity are conserved by the flow. **We will show that u_ε is itself a constrained minimizer.** For this purpose, set

$$\Gamma_\infty := \left\{ u \in W \ \text{ s.t. } \ p(u) = 2\pi |B^{N-1}| \right\}.$$

Theorem 5. *We have*

$$p(u_\varepsilon) = 2\pi |B^{N-1}| \tag{39}$$

so that $u_\varepsilon \in W$ and

$$E_\varepsilon(u_\varepsilon) := \inf_{u \in \Gamma_\infty} E_\varepsilon(u). \tag{40}$$

The proof of Theorem 5 relies essentially on the following proposition which provides a decay of the energy at infinity.

Proposition 4. *There exists constants* $\lambda > 0$ *and* $C > 0$*, independent of* $n \geq n(\varepsilon)$*, such that*

$$\int_{\Omega_n \setminus B(R)} e_\varepsilon(u_{n,\varepsilon}) \leq CR^{-\lambda}. \tag{41}$$

In particular,

$$\lim_{n \to +\infty} E_\varepsilon(u_{n,\varepsilon}) = E_\varepsilon(u_\varepsilon) \tag{42}$$

and

$$\lim_{n \to +\infty} p(u_{n,\varepsilon}) = p(u_\varepsilon) = 2\pi |B^{N-1}|. \tag{43}$$

Recall that the definition of $p(u_\varepsilon)$ was given in (38).

Remark 7. i) The result of Proposition 4 is an **exact** result for **fixed** ε, and has to be compared with the weaker asymptotic result

$$E_\varepsilon(u_{n,\varepsilon}) = E_\varepsilon(u_\varepsilon) + O(1) \qquad \text{as } \varepsilon \to 0$$

which is an easy consequence of Theorems 1 and 4.
ii) In fact, it follows from the proof of Proposition 4 that (41) holds for any $\lambda < \sqrt{N-1}$, provided ε is sufficiently small. One might expect however, that the gradient of u_ε decays as the gradient of U_*, and $\lambda = N$ should be the optimal constant in (41).
iii) The statements in Proposition 4 essentially mean that there is no loss of compactness at infinity (it excludes for example a sliding bump "escaping" towards infinity, or vanishing but widespread oscillations).

Comments. i) The existence of a unique solution for the Cauchy problem in $H^1(\Pi_n)$ is standard. Moreover, it is easily proved that the set of minimizers for $(\mathcal{P}_{n,\varepsilon})$ (which contains $u_{n,\varepsilon}$) is orbitally stable. In particular, the uniqueness of $u_{n,\varepsilon}$ (up to translation and multiplication by a complex number of modulus one) would imply its orbital stability.
ii) One may wonder wether there is no direct proof (i.e. avoiding the approximate problems) of Theorem 5, and thus also of Theorem 1. This seems to be a difficult task, mainly since W is not open.
iii) A rigorous proof of the orbital stability of U_ε would require, in addition to solving the Cauchy problem, to obtain compactness properties for minimizing sequences for (40). We will not tackle this problem here.

Added in proof. After the completion of this work, P. Gravejeat was able to prove that any finite energy solution (in particular u_ε) belongs to Y. It follows therefore from Remark 6 that

$$E_\varepsilon(u_\varepsilon) = \inf\{E_\varepsilon(u),\ u \in Y,\ \tilde{p}(u) = 2\pi|B^{N-1}|\},$$

which is certainly an important step towards orbital stability, since, as mentioned, the Cauchy problem is well-defined on Y.

1.7 Cylindrically symmetric solutions

Since equation (11) is invariant under rotations preserving the x_1 axis, it is tempting to believe that up to a translation U_ε inherits this symmetry; i.e. that $U_\varepsilon(x_1, x')$ depends only on x_1 and $|x'|$, where $x' = (x_2, \cdots, x_N)$. We have no proof of this fact. However, the following variant of Theorem 1 can be easily established with minor changes in the proof.

Theorem 6. *There exists $\varepsilon_1 > 0$ such that for every $0 < \varepsilon < \varepsilon_1$ there is a solution $\mathcal{U}_\varepsilon$ to equation (5) with $C = C(\varepsilon)$ verifying (6), (7), (8), (9) and such that $\mathcal{U}_\varepsilon$ is cylindrically symmetric.*

The slight change is to introduce the space Z_n of axially symmetric functions on $[-n, n]^N$ with periodic boundary conditions :

$$Z_n := \Big\{ u \in H^1([-n,n]^N), \; u = u(x_1, |x'|) \text{ and } \forall k \in \{1, \cdots, N\} \\ u(x_1, \cdots, x_{k-1}, -n, x_{k+1}, \cdots, x_N) = u(x_1, \cdots, x_{k-1}, n, x_{k+1}, \cdots, x_N) \Big\}$$

and to consider the minimization problem

$$\inf \Big\{ E_\varepsilon(u), \; u \in Z_n, \; p(u) = 2\pi |B^{N-1}| \Big\}.$$

All the arguments in the proof of Theorem 1 can be carried out similarly working with Z_n instead of X_n, yielding the proof of Theorem 6.

We emphasize however an important difference, concerning stability. Stability properties of $\mathcal{U}_\varepsilon$ can be obtained (in the same way) for axially symmetric perturbations only. This is a rather restricted class, and it seems difficult to obtain stability results for general perturbations.

Remark 8. i) As already mentioned, we nevertheless suspect that, up to translation and multiplication by a complex of modulus one, $U_\varepsilon = \mathcal{U}_\varepsilon$.
ii) An alternate proof of Theorem 6 would be to work directly in the upper half-plane (x_1, r), where $r = |x'|$, at the cost of introducing a degenerate elliptic operator. Since this approach is basically two dimensional, the results of the Appendix could possibly be replaced by easier two dimensional analysis.

2 The approximating problems

The main purpose of this section is to present the proofs of Proposition 1 and of Lemmas 1 and 2. In particular, we stress the fact that Lemma 2 provides an important upper bound for the Lagrange multiplier $c_{n,\varepsilon}$. This is the first required step in order to implement the PDE analysis of the Appendix.

Before we start with the proofs, we wish first to clarify the identification $\Pi_n \simeq [-n, n]^N \equiv \Omega_n$, as well as the notion of unfolding.

Unfolding the torus. We start with the usual definition $\Pi_n = \mathbb{R}^N/(2n\mathbb{Z})^N$ obtained by the identification $x \sim x'$ iff $x - x' \in (2n\mathbb{Z})^N$. For a fixed $\alpha = (\alpha_1, \cdots, \alpha_N) \in \mathbb{R}^N$, the cube $C_\alpha := \prod_{i=1}^N [-n + \alpha_i, n + \alpha_i[$ contains a unique element of each equivalence class (C_α is often termed a fundamental domain); it may therefore be identified with Π_n. Given $\alpha \in \mathbb{R}^N$, the unfolding τ_α of Π_n associated to α is by definition the one to one mapping

$$\begin{aligned}\tau_\alpha :\quad & \Pi_n \longrightarrow \Omega_n \equiv [-n, n[^N \\ & p = [(x_1 + \alpha_1, \cdots, x_N + \alpha_N)] \longmapsto (x_1, \cdots, x_N).\end{aligned}$$

This corresponds to a translation of the origin in $\mathbb{R}^N$, and thus on the torus. For a given function f defined on Π_n, each unfolding τ_α induces a 2n-periodic function f_α defined on Ω_n.

In some computations (in particular dealing with integration by parts for functions which are not necessarily all periodic), we will need to estimate boundary integrals. The following Lemma provides a choice of a "good" unfolding of the torus, by averaging.

Lemma 2.1. *Let $f \in L^1(\Pi_n)$ be given. There exists an unfolding of the torus Π_n such that*

$$|\int_{\partial\Omega_n} f_\alpha(x)\,dx\,| \le \frac{2^{N-1}}{n} \int_{\Omega_n} |f_\alpha(x)|\,dx. \tag{44}$$

Moreover, for any $0 < \sigma < 1$ there exists a subset D_α, of Ω_n of measure larger than $\sigma|\Omega_n|$, such that for any $\alpha \in D_\alpha$ we have

$$|\int_{\partial\Omega_n} f_\alpha(x)\,dx\,| \le \frac{C(\sigma)}{n} \int_{\Omega_n} |f_\alpha(x)|\,dx. \tag{45}$$

Proof. Integrate the left hand side of (44) for $\alpha \in [-n, n[^N$ and use the mean value theorem to get (44). For (45), argue similarly. $\square$

[Notice that the trace of f_α is well defined for almost every unfolding] In the sequel, we will no longer distinguish f and f_α : hopefully this will not lead to a confusion.

Proof of Proposition 1. Let $(u^k_{n,\varepsilon})_{k\in\mathbb{N}}$ be a minimizing sequence for $(\mathcal{P}^\varepsilon_n)$. Since $E_\varepsilon(u^k_{n,\varepsilon})$ is uniformly bounded with respect to k, $(u^k_{n,\varepsilon})_{k\in\mathbb{N}}$ is bounded in $H^1(\Pi_n)$ so that up to a subsequence we may assume

$$u^k_{n,\varepsilon} \rightharpoonup u_{n,\varepsilon} \quad \text{in } H^1(\Pi_n) \quad \text{as} \quad k \to +\infty,$$

for some $u_{n,\varepsilon}$ in $H^1(\Pi_n)$. By weak lower semi-continuity and Rellich compactness theorem, we infer that

$$E_\varepsilon(u_{n,\varepsilon}) \le \liminf_{k\to+\infty} E_\varepsilon(u^k_{n,\varepsilon}) = I_{n,\varepsilon}.$$

On the other hand, Rellich compactness theorem also yields

$$p(u_{n,\varepsilon}) = \lim_{k\to+\infty} \int_{\Pi_n} (iu^k_{n,\varepsilon}, \partial_1 u^k_{n,\varepsilon}) = 2\pi|B^{N-1}|.$$

Hence $u_{n,\varepsilon}$ is a minimizer for $(\mathcal{P}^\varepsilon_n)$. The Lagrange multiplier rule implies that for some $\lambda_{n,\varepsilon} \in \mathbb{R}$,

$$dE_\varepsilon(u_{n,\varepsilon}) = \lambda_{n,\varepsilon} \cdot dp(u_{n,\varepsilon}).$$

Define $c_{n,\varepsilon} := 2\lambda_{n,\varepsilon}/|\log\varepsilon|$. The previous equality is precisely the weak formulation for the equation

$$ic_{n,\varepsilon}|\log\varepsilon|\frac{\partial u_{n,\varepsilon}}{\partial x_1} = \Delta u_{n,\varepsilon} + \frac{1}{\varepsilon^2}u_{n,\varepsilon}(1-|u_{n,\varepsilon}|^2) \qquad \text{on } \Pi_n.$$

This ends the proof. □

Proof of Lemma 1. We will use the test functions $w_{\varepsilon,R}$ constructed in Subsection 1.2. Notice that $E(w_{\varepsilon,R})$ and $p(w_{\varepsilon,R})$ depend continuously on R. It then follows from (15) that

$$\exists\, R(\varepsilon) > 0 \qquad \text{such that} \;\; w_{\varepsilon,R(\varepsilon)} \in \Gamma_n$$

for each large enough n and that

$$R(\varepsilon) \to 1 \quad \text{as} \;\; \varepsilon \to 0.$$

The conclusion of Lemma 1 then follows from (14). □

We turn now to the proof of Lemma 2. As often in elliptic PDE's, Pohozaev's identity (also termed virial identity in the physics literature) leads to useful estimates. In our case, after unfolding it reads (see Lemma A.2)

$$\begin{aligned}\tfrac{N-2}{2}\int_{\Omega_n}|\nabla u_{n,\varepsilon}|^2 + \frac{N}{4\varepsilon^2}\int_{\Omega_n}(1-|u_{n,\varepsilon}|^2)^2 - c_{n,\varepsilon}\tfrac{N-1}{2}|\log\varepsilon|\int_{\Omega_n}\langle Ju_{n,\varepsilon},\xi_1\rangle \\ = \int_{\partial\Omega_n}[n\frac{|\nabla u_{n,\varepsilon}|^2}{2} + \frac{n}{4\varepsilon^2}(1-|u_{n,\varepsilon}|^2)^2 - \frac{\partial u_{n,\varepsilon}}{\partial\nu}\cdot(\sum_{i=1}^N x_i\frac{\partial u_{n,\varepsilon}}{\partial x_i})], \quad (46)\end{aligned}$$

where ξ_1 is the two form defined in (13). Notice that ξ_1 is **not** periodic and therefore (46) depends on the choice of unfolding. In order to bound $c_{n,\varepsilon}$, we thus need to provide a lower bound for the quantity

$$\left|\int_{\Omega_n}\langle Ju_{n,\varepsilon},\xi_1\rangle\right|. \qquad (47)$$

As we have already noticed in the Introduction (see Remark 2), (47) is related to the momentum $p(u_{n,\varepsilon})$ (actually they would even be equal if $u_{n,\varepsilon}$ was constant on $\partial\Omega_n$). In the situation which is of interest for us, we have the following.

Lemma 2.2. *Let $M_0 > 0$. There exists a constant $K_2 > 0$ (depending only on M_0) such that for any $n \in \mathbb{N}$ and $u \in H^1(\Pi_n)$ verifying*

$$\begin{cases} n \geq K_2 |\log \varepsilon| \varepsilon^{3-N}, \\ E_\varepsilon(u) \leq M_0 |\log \varepsilon|, \end{cases}$$

there exists an unfolding of Π_n such that

$$\left| \int_{\Pi_n} (iu, \partial_1 u) - \int_{\Omega_n} \langle Ju, \xi_1 \rangle \right| \leq r(\varepsilon),$$

where $r(\varepsilon) \to 0$ as $\varepsilon \to 0$, independently of n, and

$$n \int_{\partial \Omega_n} e_\varepsilon(u) \leq C \int_{\Pi_n} e_\varepsilon(u).$$

Proof. We first claim that there exists $v \in X_n$ such that

$$|\nabla v|_\infty \leq \frac{C}{\varepsilon}, \qquad |v|_\infty \leq 1, \qquad E_\varepsilon(v) \leq 2M_0 |\log \varepsilon|$$

and

$$\left| \int_{\Pi_n} (iu, \partial_1 u) - \int_{\Pi_n} (iv, \partial_1 v) \right| \leq r(\varepsilon),$$

where $r(\varepsilon) \to 0$ as $\varepsilon \to 0$, independently of n. Indeed, consider first the function v_1 defined by

$$v_1(x) := \begin{cases} u(x) & \text{if } |u(x)| \leq 1 \\ \frac{u(x)}{|u(x)|} & \text{if not.} \end{cases}$$

Clearly, $E_\varepsilon(v_1) \leq E_\varepsilon(u)$ and

$$\begin{aligned} \left| \int_{\Pi_n} (iu, \partial_1 u) - \int_{\Pi_n} (iv_1, \partial_1 v_1) \right| &\leq \int_{\Pi_n} |u - v_1| \cdot |\nabla u| + \left| \int_{\Pi_n} (iv_1, \partial_1 u - \partial_1 v_1) \right| \\ &= \int_{\Pi_n} |u - v_1| \cdot |\nabla u| + \left| \int_{\Pi_n} (i\partial_1 v_1, u - v_1) \right| \\ &\leq C \left(\int_{\Pi_n} |u - v_1|^2 \right)^{1/2} E_\varepsilon(u)^{1/2} \qquad (48) \\ &\leq C \left(\int_{|u|>1} (1 - |u|^2)^2 \right)^{1/2} E_\varepsilon(u)^{1/2} \\ &\leq C\varepsilon |\log \varepsilon| := r_1(\varepsilon). \end{aligned}$$

Next, consider a function v_2 defined as a solution of the minimization problem

$$\min_{w \in H^1(\Pi_n)} E_\varepsilon(w) + \int_{\Pi_n} \frac{|w - v_1|^2}{2\varepsilon}.$$

Clearly, we also have $E_\varepsilon(v_2) \le E_\varepsilon(v_1)$ and

$$\begin{aligned}\left|\int_{\Pi_n}(iv_1,\partial_1 v_1) - \int_{\Pi_n}(iv_2,\partial_1 v_2)\right| &\le C\left(\int_{\Pi_n}|v_1-v_2|^2\right)^{1/2} E_\varepsilon(v_1)^{1/2} \\ &\le C\sqrt{\varepsilon}\, E_\varepsilon(v_1)^{1/2}\, E_\varepsilon(u)^{1/2} \\ &\le C\sqrt{\varepsilon}|\log\varepsilon| := r_2(\varepsilon).\end{aligned} \tag{49}$$

On the other hand, v_2 satisfies the equation

$$\Delta v_2 + \frac{1}{\varepsilon^2}v_2(1-|v_2|^2) = \frac{v_2-v_1}{\varepsilon},$$

so that $\tilde{v}_2(x) := v_2(\varepsilon x)$ satisfies

$$\Delta\tilde{v}_2 + \tilde{v}_2(1-|\tilde{v}_2|^2) = \varepsilon\,(\tilde{v}_2-\tilde{v}_1)$$

($\tilde{v}_1$ is defined similarly). Since $|\tilde{v}_1|_\infty, |\tilde{v}_2|_\infty \le 1$, it follows from standard elliptic estimates that

$$|\nabla\tilde{v}_2|_\infty \le C \qquad \text{and so} \qquad |\nabla v_2|_\infty \le \frac{C}{\varepsilon}.$$

Combining (48) and (49) we obtain that $v := v_2$ satisfies the conditions of the claim with $r(\varepsilon) := r_1(\varepsilon) + r_2(\varepsilon)$.

We will now choose a suitable unfolding. Notice first that for any unfolding of Π_n,

$$\int_{\Pi_n}(iu,\partial_1 u) - \int_{\Omega_n}\langle Ju,\xi_1\rangle = n\int_{\partial\Omega_n}(iu,\partial_1 u). \tag{50}$$

By Lemma 2.1, there exists an unfolding such that

$$\int_{\partial\Omega_n}\frac{|u-v_2|^2}{\sqrt{\varepsilon}} + \sqrt{\varepsilon}(e_\varepsilon(u)+e_\varepsilon(v_2)) \le \frac{2^{N-1}}{n}\int_{\Pi_n}\frac{|u-v_2|^2}{\sqrt{\varepsilon}} + \sqrt{\varepsilon}(e_\varepsilon(u)+e_\varepsilon(v_2)). \tag{51}$$

Hence, arguing as in (48),

$$\begin{aligned} n\,\left|\int_{\partial\Omega_n}(iu,\partial_1 u) - (iv_2,\partial_1 v_2)\right| &\le C\cdot n\int_{\partial\Omega_n}|u-v_2|\cdot(|\nabla u|+|\nabla v_2|) \\ &\le C\cdot n\int_{\partial\Omega_n}\left[\frac{|u-v_2|^2}{\sqrt{\varepsilon}} + \sqrt{\varepsilon}(e_\varepsilon(u)+e_\varepsilon(v_2))\right] \\ &\le C\int_{\Pi_n}\left[\frac{|u-v_2|^2}{\sqrt{\varepsilon}} + \sqrt{\varepsilon}(e_\varepsilon(u)+e_\varepsilon(v_2))\right] \\ &\le C\sqrt{\varepsilon}|\log\varepsilon| = r_2(\varepsilon).\end{aligned} \tag{52}$$

If $n \ge C_0\varepsilon^{3-N}|\log\varepsilon|$, then it follows from (51) that

$$\int_{\partial\Omega_n} e_\varepsilon(v_2) \le \frac{C}{K_2}\varepsilon^{3-N}. \tag{53}$$

It is an easy matter to verify that the last inequality, combined with the estimate $|\nabla v_2|_\infty \leq \frac{C}{\varepsilon}$ implies that for K_2 sufficiently large,

$$|v_2(x)| \geq \frac{1}{2} \qquad \text{for all } x \in \partial\Omega_n.$$

We may thus write $v_2 = \rho \exp(i\varphi)$ on $\partial\Omega_n$ and from (53) it follows that φ is 2n-periodic (see step 4 of Theorem 4 in Appendix C for a detailed proof of this last statement). Hence,

$$\begin{aligned}\left|\int_{\partial\Omega_n}(iv_2, \partial v_2)\right| &= \left|\int_{\partial\Omega_n}\rho^2\partial_1\varphi\right| = \left|\int_{\partial\Omega_n}(\rho^2-1)\partial_1\varphi\right| \\ &\leq C\varepsilon\left(\int_{\partial\Omega_n}\frac{(\rho^2-1)^2}{\varepsilon^2}\right)^{1/2}\cdot\left(\int_{\partial\Omega_n}|\nabla v_2|^2\right)^{1/2} \\ &\leq C\varepsilon\frac{E_\varepsilon(v_2)}{n} = \frac{r_1(\varepsilon)}{n}.\end{aligned} \tag{54}$$

Combining (50),(52) and (54) we finally obtain

$$\left|\int_{\Pi_n}(iu, \partial_1 u) - \int_{\Omega_n}\langle Ju, \xi_1\rangle\right| \leq r_2(\varepsilon) + r_1(\varepsilon),$$

which finishes the proof. □

We are now in position to obtain the expected upper bound for the Lagrange multiplier $c_{n,\varepsilon}$.

Proof of Lemma 2. We deduce form (46), that for each unfolding we have

$$c_{n,\varepsilon}|\log\varepsilon|\left|\int_{\Omega_n}\langle Ju_{n,\varepsilon}, \xi_1\rangle\right| \leq C\left[n\int_{\partial\Omega_n}e_\varepsilon(u_{n,\varepsilon}) + \int_{\Pi_n}e_\varepsilon(u_{n,\varepsilon})\right]. \tag{55}$$

By Lemma 2.2, there exist an unfolding such that

$$n\int_{\partial\Omega_n}e_\varepsilon(u) \leq C\int_{\Pi_n}e_\varepsilon(u) \tag{56}$$

and

$$\left|\int_{\Pi_n}(iu, \partial_1 u) - \int_{\Omega_n}\langle Ju, \xi_1\rangle\right| \leq \pi|B^{N-1}|$$

provided ε is chosen sufficiently small and $n \geq n(\varepsilon)$. Therefore, since $u_{n,\varepsilon}$ verifies the constraint $\int_{\Pi_n}(iu_{n,\varepsilon}, \partial_1 u_{n,\varepsilon}) = 2\pi|B^{N-1}|$, we obtain

$$\left|\int_{\Omega_n}\langle Ju, \xi_1\rangle\right| \geq \pi|B^{N-1}|. \tag{57}$$

Combining (55), (56) and (57) we deduce

$$|c_{n,\varepsilon}| \leq C\,\frac{E_\varepsilon(u_{n,\varepsilon})}{|\log\varepsilon|} \leq K_1,$$

where we have used Lemma 1 for the last inequality.

3 Relation with the isoperimetric problem

In this section, we specify the geometrical interpretation of both the momentum and the energy,in the asymptotic limit $\varepsilon \to 0$. Roughly speaking, for $N = 3$, $E_\varepsilon(u_{n,\varepsilon})$ is proportional to the length of the concentration set, whereas $p(u_{n,\varepsilon})$ is proportionnal to the flux (along $\vec{e}_1$) through the concentration set. As we emphasized in the Introduction, the concepts of Geometric Measure Theory are appropriate to express these properties.

We start with the proof of Lemma 5. Recall that in view of Theorem 4, there exists $\ell \in \mathbb{N}$, $R > 0$ and q points $x_{1,\varepsilon}, \cdots, x_{q,\varepsilon}$ with $q \leq l$ such that

$$|u_{n,\varepsilon}(x)| \geq \frac{1}{2} \qquad \text{on } \Pi_n \setminus \cup_{i=1}^q B(x_{i,\varepsilon}, R). \tag{58}$$

Without loss of generality, we may assume that the balls $B(x_{i,\varepsilon}, 8R)$ are disjoint. For a map $u \in H^1(\Pi_n, \mathbb{C})$, let $\tilde{u}$ be defined by

$$\tilde{u}(x) := \begin{cases} u(x) & \text{if } x \in \cup_{i=1}^q B(x_{i,\varepsilon}, R) \\ \lambda(x)u(x) + (1-\lambda(x))\bar{u}(x) & \text{if } x \in \cup_{i=1}^q B(x_{i,\varepsilon}, 2R) \setminus B(x_{i,\varepsilon}, R) \\ \bar{u}(x) & \text{otherwise,} \end{cases}$$

where

$$\bar{u}(x) := \begin{cases} \frac{u(x)}{|u(x)|} & \text{if } |u(x)| \geq \frac{1}{2} \\ 2u(x) & \text{otherwise,} \end{cases}$$

and $\lambda(x) := \frac{2R - |x - x_{i,\varepsilon}|}{R}$ if $x \in B(x_{i,\varepsilon}, 2R) \setminus B(x_{i,\varepsilon}, R)$. In view of (58),

$$J\tilde{u}_{n,\varepsilon} = 0 \qquad \text{on } \Pi_n \setminus \cup_{i=1}^q B(x_{i,\varepsilon}, 2R) \tag{59}$$

and

$$\int_{B(x_{i,\varepsilon},2R)} J\tilde{u}_{n,\varepsilon} = 0, \tag{60}$$

this last inequality follows by integration by parts, using the fact that $|\tilde{u}_{n,\varepsilon}| = 1$ on $\partial B(x_{i,\varepsilon}, 2R)$. These localization properties of $J\tilde{u}_{n,\varepsilon}$ will be useful in the sequel. On the other hand, $J\tilde{u}_{n,\varepsilon}$ and $Ju_{n,\varepsilon}$ are close in view of the following lemma.

Lemma 3.1. *Let $u \in H^1(\Pi_n, \mathbb{C})$ such that $E_\varepsilon(u) \leq M_0|\log\varepsilon|$. Then, there exists an unfolding of Π_n such that for every $\varphi \in \mathcal{C}^\infty(\Omega_n, \Lambda^2\mathbb{R}^N)$, we have*

$$\left|\int_{\Omega_n} \langle Ju - J\tilde{u}, \varphi\rangle\right| \leq (\frac{1}{n}||\varphi||_\infty + ||d^*\varphi||_\infty)C\varepsilon|\log\varepsilon|\,, \tag{61}$$

and in particular

$$||Ju - J\tilde{u}||_{[\mathcal{C}^{0,1}(\Pi_n)]^*} \leq C\varepsilon|\log\varepsilon|\,, \tag{62}$$

where C depends only on N and M_0 but is independent of n.

Proof. According to Lemma 2.1 there exists an unfolding of Π_n such that

$$\int_{\partial\Omega_n} |u \times du - \tilde{u} \times d\tilde{u}\,| \leq \frac{2^{N-1}}{n} \int_{\Omega_n} |u \times du - \tilde{u} \times d\tilde{u}\,| \,. \tag{63}$$

Let $\varphi \in \mathcal{C}^\infty(\Omega_n, \Lambda^2\mathbb{R}^N)$. Integrating by parts on Ω_n, we obtain

$$\int_{\Omega_n} \langle Ju - J\tilde{u}, \varphi\rangle = \frac{1}{2}\int_{\partial\Omega_n} (u \times du - \tilde{u} \times d\tilde{u})_\top \wedge (*\varphi)_\top - \frac{1}{2}\int_{\Omega_n} \langle u \times du - \tilde{u} \times d\tilde{u}, d^*\varphi\rangle \,.$$

Hence, we deduce from (63) that

$$\left|\int_{\Omega_n} \langle Ju - J\tilde{u}, \varphi\rangle\right| \leq C(\frac{2^{N-1}}{n}||\varphi||_\infty + ||d^*\varphi||_\infty)||u \times du - \tilde{u} \times d\tilde{u}||_{L^1(\Omega_n)} \,. \tag{64}$$

The proof is completed using the estimate for $\|u \times du - \tilde{u} \times d\tilde{u}\|_{L^1(\Omega_n)}$ given in the next lemma. ☐

Lemma 3.2. *There exists an absolute constant $C > 0$ such that*

$$\|u \times du - \tilde{u} \times d\tilde{u}\|_{L^1(\Omega_n)} \leq C\varepsilon E_\varepsilon(u). \tag{65}$$

Proof. Let $A = \{|u| \geq 1/2\}$, $B = \Omega \setminus A$. A simple computation gives

$$\begin{aligned} \|u \times du - \tilde{u} \times d\tilde{u}\|_{L^1(A)} &\leq C\int_A |(1 - \frac{1}{|u|^2})u \times du| \\ &\leq C\varepsilon \left(\int_{\Omega_n} \frac{(1-|u|^2)^2}{\varepsilon^2}\right)^{1/2} \left(\int_{\Omega_n} |\nabla u|^2\right)^{1/2} \\ &\leq C\varepsilon E_\varepsilon(u) \,. \end{aligned} \tag{66}$$

On the other hand, we have

$$\begin{aligned} \|u \times du - \tilde{u} \times d\tilde{u}\|_{L^1(B)} &\leq C\int_B |u \times du| \\ &\leq C|B|^{1/2}\left(\int_{\Omega_n} |\nabla u|^2\right)^{1/2} \\ &\leq C\varepsilon \left(\int_{\Omega_n} \frac{(1-|u|^2)^2}{\varepsilon^2}\right)^{1/2} E_\varepsilon(u)^{1/2} \\ &\leq C\varepsilon E_\varepsilon(u) \,. \end{aligned} \tag{67}$$

Combining (66) with (67) yields (65). ☐

In view of (59), $J\tilde{u}_{n,\varepsilon}$ is localized in balls $B(x_{i,\varepsilon}, 2R)$. Concerning the existence of integral boundaries close to $J\tilde{u}_\varepsilon$, we will make use of recent works on the geometry of the Jacobians [29, 1]. In particular the Γ-convergence results contained in the above quoted works lead to the following.

Lemma 3.3. *Let* $M_0 > 0$, $R > 0$ *and* $X := \{u \in H^1(B_{4R}, \mathbb{C}),\ |u| \geq 1/2 \text{ on } B_{4R} \backslash B_R\}$. *Then, for every* $\delta > 0$, *there exists* $\varepsilon_0 > 0$ *(depending only on* δ, R *and* M_0*), such that for any* $\varepsilon < \varepsilon_0$, *and for any* $u \in X$ *such that* $E_\varepsilon(u) \leq M_0|\log \varepsilon|$, *there exists an* $(N-2)$*-dimensional integral boundary* $T_u = \partial R_u$ *supported in* B_R *verifying*

i) $||Ju - \pi T_u||_{[\mathcal{C}_c^{0,1}(B_{4R})]^*} \leq \delta$

ii) $\mathbf{M}(T_u) \leq \dfrac{E_\varepsilon(u)}{\pi|\log \varepsilon|} + \delta.$

Proof. We argue by contradiction. Assume that there exists some $\delta > 0$, a sequence $\varepsilon_j \to 0$, and maps $u_j \in X$ satisfying the bound

$$E_{\varepsilon_j}(u_j) \leq M_0|\log \varepsilon_j| \tag{68}$$

and such that for every integral boundary T supported in B_R and verifying $i)$, statement $ii)$ is contradicted, i.e.

$$\mathbf{M}(T) > \frac{E_\varepsilon(u)}{\pi|\log \varepsilon|} + \delta. \tag{69}$$

According to the Γ-convergence results in [29, 1] (see e.g. Theorem 3.1 and Remark 3.2 in [1]), there exists an integral boundary T^* supported in B_{4R} such that

$$\|Ju_j - \pi T^*\|_{[\mathcal{C}_c^{0,1}(B_{4R})]^*} \to 0 \quad \text{as } j \to +\infty \tag{70}$$

and

$$\mathbf{M}(T) \leq \liminf_{\varepsilon_j \to 0} \frac{E_{\varepsilon_j}(u_j)}{\pi|\log \varepsilon_j|}. \tag{71}$$

We deduce from (70) that $i)$ is satisfied for $T = T^*$ and j sufficiently large, so that (71) contradicts (69) [indeed the fact that T^* is supported in B_R, and therefore can be used as a test current in $i)$, follows from its construction in [1]]. ☐

Proof of Lemma 5 completed. We apply Lemma 3.3 to $\tilde{u}_{n,\varepsilon}$ restricted to the balls $B(x_{i,\varepsilon}, 4R)$, for $i = 1, \cdots, q$. This yields integral boundaries T_i (depending of course on ε and n) supported in $B(x_{i,\varepsilon}, R)$ such that

$$\|J\tilde{u}_{n,\varepsilon} - \pi T_i\|_{[\mathcal{C}_c^{0,1}(B(x_{i,\varepsilon},4R))]^*} \leq r(\varepsilon) \tag{72}$$

and

$$\mathbf{M}(T_i) \leq \frac{E_\varepsilon(\tilde{u}_{n,\varepsilon}; B(x_{i,\varepsilon}, 4R))}{\pi|\log \varepsilon|} + r(\varepsilon) \leq \frac{E_\varepsilon(u_{n,\varepsilon}; B(x_{i,\varepsilon}, 4R))}{\pi|\log \varepsilon|} + r(\varepsilon). \tag{73}$$

[here and in the following, $r(\varepsilon)$ denotes a generic function such that $r(\varepsilon) \to 0$ as $\varepsilon \to 0$, independently of n, but whose exact value may differ from place to place]
Set $T = \sum_{i=1}^q T_i$. By (73),

$$\mathbf{M}(T) \leq \frac{E_\varepsilon(u_{n,\varepsilon})}{\pi|\log \varepsilon|} + r(\varepsilon), \tag{74}$$

so that $iii)$ is established. Concerning $i)$, since $J\tilde{u}_{n,\varepsilon}$ is supported in the balls of radius $2R$, we deduce form (72) that

$$\|J\tilde{u}_{n,\varepsilon} - \pi T\|_{[C^{0,1}(\Pi_n))]^*} \leq \sum_{i=1}^{q} \|J\tilde{u}_{n,\varepsilon} - \pi T_i\|_{[C^{0,1}(B(x_{i,\varepsilon},2R))]^*}$$
$$\leq C \sum_{i=1}^{q} \|J\tilde{u}_{n,\varepsilon} - \pi T_i\|_{[C_c^{0,1}(B(x_{i,\varepsilon},4R))]^*} \leq r(\varepsilon).$$

Since $\|Ju_{n,\varepsilon} - J\tilde{u}_{n,\varepsilon}\|_{[C^{0,1}(\Pi_n)]^*} \leq r(\varepsilon)$ in view of Lemma 3.1, we derive $i)$ from the previous inequality.

Finally, we turn to $ii)$. For any unfolding we have

$$|\int_{\Omega_n} \langle Ju_{n,\varepsilon} - \pi T, \xi_1\rangle| \leq |\int_{\Omega_n} \langle J\tilde{u}_{n,\varepsilon} - \pi T, \xi_1\rangle| + |\int_{\Omega_n} \langle J\tilde{u}_{n,\varepsilon} - Ju_{n,\varepsilon}, \xi_1\rangle|\,. \tag{75}$$

Notice that

$$\begin{aligned} |\int_{\Omega_n} \langle J\tilde{u}_{n,\varepsilon} - \pi T, \xi_1\rangle| &\leq \sum_{i=1}^{q} |\int_{B(x_{i,\varepsilon},2R)} \langle J\tilde{u}_{n,\varepsilon} - \pi T_i, \xi_1\rangle| \\ &= \sum_{i=1}^{q} |\int_{B(x_{i,\varepsilon},2R)} \langle J\tilde{u}_{n,\varepsilon} - \pi T_i, \xi_1 - \xi_1^i\rangle|, \end{aligned} \tag{76}$$

where ξ_1^i denotes the constant form

$$\xi_1^i := \frac{2}{N-1} \sum_{j=1}^{N} (x_{i,\varepsilon})_j dx_i \wedge dx_j,$$

and $(x_{i,\varepsilon})_j$ denotes the j-component of the point $x_{i,\varepsilon}$. For the last inequality, we have used (60). By construction,

$$\|\xi_1 - \xi_1^i\|_{L^\infty(B(x_{i,\varepsilon},2R)} \leq \frac{4R}{N-1}$$

(whereas $\|\xi_1\|_{L^\infty(B(x_{i,\varepsilon},2R))}$ diverges as $n \to +\infty$). Hence, we obtain the estimate

$$|\int_{B(x_{i,\varepsilon},2R)} \langle J\tilde{u}_{n,\varepsilon} - \pi T_i, \xi_1 - \xi_1^i\rangle| \leq C\|J\tilde{u}_{n,\varepsilon} - \pi T_i\|_{[C_c^{0,1}(4R)]^*} \leq r(\varepsilon)\,. \tag{77}$$

We now choose the particular unfolding given by Lemma 3.1, and similarly we obtain

$$\left|\int_{\Omega_n} \langle Ju_{n,\varepsilon} - J\tilde{u}_{n,\varepsilon}, \xi_1\rangle\right| \leq r(\varepsilon), \tag{78}$$

so that $ii)$ follows from (75), (76), (77) and (78). □

Proof of Proposition 3. First, observe that

$$|\int_{\Omega_n} \langle \pi T_{n,\varepsilon}, \xi_1\rangle| = |\int_{\Omega_n} \langle \pi R_{n,\varepsilon}, 2dx_1\rangle| \leq 2\pi \mathbf{M}(R_{n,\varepsilon})\,. \tag{79}$$

In view of *ii*) and *iii*) of Lemma 5 and (79), we deduce

$$\frac{\mathbf{M}(T_{n,\varepsilon})^{\frac{N-1}{N-2}}}{\mathbf{M}(R_{n,\varepsilon})} \leq \frac{2\pi E_\varepsilon(u_{n,\varepsilon})^{\frac{N-1}{N-2}}}{(\pi|\log\varepsilon|)^{\frac{N-1}{N-2}} p(u_{n,\varepsilon})} + r(\varepsilon)\,, \tag{80}$$

where $r(\varepsilon) \to 0$ as $\varepsilon \to 0$, independently of n. The last inequality together with (17) proves (36).

Moreover, from Lemma 1 and (35), we deduce

$$\Big|\mathbf{M}(T_{n,\varepsilon}) - |S^{N-2}|\Big| \leq r(\varepsilon)\,, \quad \Big|\mathbf{M}(R_{n,\varepsilon}) - |B^{N-1}|\Big| \leq r(\varepsilon)\,, \tag{81}$$

and

$$\Big|\int_{\Pi_n} \langle R_{n,\varepsilon}, dx_1\rangle - |B^{N-1}|\Big| \leq r(\varepsilon)\,, \tag{82}$$

where $r(\varepsilon) \to 0$ as $\varepsilon \to 0$, independently of n. From (81) we infer in particular (see [20], 4.2.17) that for each sequences $\varepsilon_j \to 0$ and $n_j \geq n(\varepsilon_j)$ there exists subsequences (still denoted ε_j and n_j) and translations τ_j in Π_{n_j} such that

$$T_{n_j,\varepsilon_j} \to T_\infty \qquad \text{and} \qquad R_{n_j,\varepsilon_j} \to R_\infty \qquad \text{in } [\mathcal{C}_c^{0,1}(\mathbb{R}^N)]^*$$

as $j \to +\infty$, where $T_\infty = \partial R_\infty$ satisfy

$$\frac{\mathbf{M}(T_\infty)^{\frac{N-1}{N-2}}}{\mathbf{M}(R_\infty)} = \lambda_N\,. \tag{83}$$

From (83) and (81) we conclude that $T_\infty = S^{N-2}$, $R_\infty = B^{N-1}$. Combining (79) with (82) we also obtain

$$|\int_{\mathbb{R}^N} \langle R_\infty, dx_1\rangle| = \mathbf{M}(R_\infty)\,, \tag{84}$$

i.e. R_∞ is contained in a hyperplane orthogonal to $\vec{e}_1$. The proof is complete. □

Proof of Lemma 6. We claim first that $T_{n,\varepsilon}$ is contained in a single ball $B(x_{i,\varepsilon}, R)$. The other statements are then direct consequences of Theorem 4.
We argue by contradiction. Assume there exists sequences $\varepsilon_j \to 0$ and $n_j \geq n(\varepsilon_j)$ for which the claim is false. In particular, for every $R > 0$ and every sequence $x_j \in \Pi_{n_j}$,

$$\Big(\Pi_{n_j} \setminus B(x_j, R)\Big) \cap S_{n_j,\varepsilon_j} \neq \emptyset \tag{85}$$

for j sufficiently large. By Proposition 3, up to some subsequence we have

$$\tau_j T_{n_j,\varepsilon_j} \to S^{N-2}, \tag{86}$$

where τ_j is a translation in Π_{n_j}. Let $x_{n_j,\varepsilon_j} := \tau_j^{-1}(0)$ and $r > 1$ be such that

$$|u_{n_j,\varepsilon_j}| \geq 1/2 \qquad \text{on } B(x_{n_j,\varepsilon_j}, 4r) \setminus B(x_{n_j,\varepsilon_j}, r)$$

(the fact that such an r always exists follows easily by Theorem 2). From (85) with $R = 8r$ we infer that

$$\Big(\Pi_{n_j} \setminus B(x_{n_j,\varepsilon_j}, 8r)\Big) \cap S_{n_j,\varepsilon_j} \neq \emptyset$$

for j sufficiently large. We deduce from Theorem 2 the inequality

$$\int_{B(x_{n_j,\varepsilon_j},4r)} \frac{e_{\varepsilon_j}(u_{n_j,\varepsilon_j})}{\pi|\log\varepsilon_j|} \le \int_{\Pi_{n_j}} \frac{e_{\varepsilon_j}(u_{n_j,\varepsilon_j})}{\pi|\log\varepsilon_j|} - \frac{\eta}{\pi}, \tag{87}$$

where $\eta > 0$ is the constant given by Theorem 2 for $\sigma = \frac{1}{2}$. Taking the limit $j \to +\infty$ we obtain, using respectively (86), Lemma 3.3 *ii*) with $\delta := \frac{\eta}{2\pi}$, and (17),

$$\begin{aligned}
|S^{N-2}| &\le \liminf_{j\to+\infty} \int_{B(x_{n_j,\varepsilon_j},4r)} \frac{e_{\varepsilon_j}(u_{n_j,\varepsilon_j})}{\pi|\log\varepsilon_j|} + \frac{\eta}{2\pi} \\
&\le \liminf_{j\to+\infty} \frac{E_{\varepsilon_j}(u_{n_j,\varepsilon_j})}{\pi|\log\varepsilon_j|} - \frac{\eta}{\pi} + \frac{\eta}{2\pi} \\
&\le |S^{N-2}| - \frac{\eta}{2\pi}.
\end{aligned}$$

This is a contradiction. □

4 Proof of Theorem 1 completed

Recall that in Section 1.5 of the introduction we have already constructed, for $0 < \varepsilon < \varepsilon_0$ small but **fixed**, a subsequence of $u_{n,\varepsilon}$ (still denoted here $u_{n,\varepsilon}$) such that

$$\begin{cases} u_{n,\varepsilon} \to u_\varepsilon & \text{strongly in } H^1_{\rm loc}(\mathbb{R}^N), \\ c_{n,\varepsilon} \to c_\varepsilon & \text{in } \mathbb{R}, \end{cases}$$

as $n \to +\infty$. Moreover, as $\varepsilon \to 0$, we have

$$Ju_\varepsilon \to \pi S^{N-2} \tag{88}$$

and u_ε is a solution on $\mathbb{R}^N$ of

$$ic_\varepsilon|\log\varepsilon|\frac{\partial u_\varepsilon}{\partial x_1} = \Delta u_\varepsilon + \frac{1}{\varepsilon^2}u_\varepsilon(1-|u_\varepsilon|^2). \tag{89}$$

In view of (88), u_ε is non trivial (non constant) at least for small ε.

Theorem 1 is stated with $U_\varepsilon(x) := u_\varepsilon(\varepsilon x)$. We will prove the equivalent statements for u_ε; it is then straighforward to come back to U_ε. We decompose the remaining of the proof in several steps.

Step 1. We have

$$\limsup_{n\to+\infty} |E_\varepsilon(u_\varepsilon) - E_\varepsilon(u_{n,\varepsilon})| \le C,$$

where C is independent of ε.

Proof. This is a direct consequence of Theorem 4 and of the strong $H^1_{\rm loc}$ convergence at ε fixed. □

Step 2. We have

$$\frac{E_\varepsilon(u_\varepsilon)}{\pi|\log\varepsilon|} = |S^{N-2}| + r(\varepsilon),$$

where $r(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Proof. This is a direct consequence of Step 1, Lemma 1, assertion *iii)* of Lemma 5, and (36). □

Step 3. Similarly, we have

$$p(u_\varepsilon) = p(u_{n,\varepsilon}) + r(\varepsilon) = 2\pi|B^{N-1}| + r(\varepsilon)$$

where $r(\varepsilon) \to 0$ as $\varepsilon \to 0$, independently of n.

Proof. Recall that by Lemma 6

$$|u_{n,\varepsilon}| \geq \frac{1}{2} \qquad \text{on } \Pi_n \setminus B(0,R),$$

so that we may write

$$u_{n,\varepsilon} = \rho_{n,\varepsilon}\exp(i\varphi_{n,\varepsilon}) \qquad \text{on } \Pi_n \setminus B(0,R).$$

The definition of $p(u_\varepsilon)$ is then given by (see (38))

$$p(u_\varepsilon) = \int_{\mathbb{R}^N} (iu_\varepsilon, \partial_1 u_\varepsilon)\chi + \int_{\mathbb{R}^N} (1-\chi)(\rho_\varepsilon^2 - 1)\partial_1\varphi_\varepsilon + \int_{\mathbb{R}^N} \varphi_\varepsilon \partial_1(1-\chi), \tag{90}$$

where χ is an arbitrary smooth function with compact support such that $\chi \equiv 1$ on $B_R(0)$ and $0 \leq \chi \leq 1$. On the other hand, we have, for n sufficiently large,

$$p(u_{n,\varepsilon}) = \int_{\mathbb{R}^N} (iu_{n,\varepsilon}, \partial_1 u_{n,\varepsilon})\chi + \int_{\Omega_n} (1-\chi)(\rho_{n,\varepsilon}^2 - 1)\partial_1\varphi_{n,\varepsilon} + \int_{\mathbb{R}^N} \varphi_{n,\varepsilon}\partial_1(1-\chi). \tag{91}$$

By strong $H^1_{\rm loc}$ convergence, the first and third terms in (91) converge to the corresponding terms in (90). For the second one, we have

$$\begin{aligned}\left|\int_{\Omega_n} (1-\chi)(\rho_{n,\varepsilon}^2 - 1)\partial_1\varphi_{n,\varepsilon}\right| &\leq \left(\int_{\Omega_n} (\rho_{n,\varepsilon}^2 - 1)^2\right)^{1/2} \left(\int_{\Omega_n\setminus B(0,R)} |\nabla\varphi_{n,\varepsilon}|^2\right)^{1/2} \\ &\leq C\varepsilon E_\varepsilon(u_{n,\varepsilon}).\end{aligned}$$

A similar estimate holds for the second term in (90), so that the proof is complete. □

Step 4. We have

$$c(\varepsilon) \to N-2 \qquad \text{as } \varepsilon \to 0.$$

Proof. The proof relies (as in Lemma 2) on Pohozaev identity; however we are now in position to use Theorem 4 and Lemma 6, which provide a better decay of the energy at infinity. Set $B := B(0,R)$. By Lemma 2.1, there exists an unfolding of the torus such that $\partial\Omega_n \cap B = \emptyset$ and

$$\left| n \int_{\partial\Omega_n} e_\varepsilon(u_{n,\varepsilon}) \right| \le \int_{\Omega_n} e_\varepsilon(u_{n,\varepsilon}) \cdot 1_{\Omega_n \setminus B} \le C, \tag{92}$$

the last inequality being a consequence of Theorem 4. On the other hand, by Corollary A.1 of the Appendix, we know that

$$\int_{\Pi_n} \frac{(1-|u_{n,\varepsilon}|^2)^2}{\varepsilon^2} = o(|\log\varepsilon|) \quad \text{as } \varepsilon \to 0. \tag{93}$$

Finally, using Lemma 2.2, we may choose our unfolding such that it verifies the additional condition

$$\left| \int_{\Pi_n} (iu_{n,\varepsilon}, \partial_1 u_{n,\varepsilon}) - \langle Ju_{n,\varepsilon}, \xi_1 \rangle \right| \le r(\varepsilon).$$

Hence, by Step 2,

$$\int_{\Pi_n} \langle Ju_{n,\varepsilon}, \xi_1 \rangle = p(u_{n,\varepsilon}) + r(\varepsilon) = 2\pi |B^{N-1}| + r(\varepsilon). \tag{94}$$

Going back to (46), we have by (92), for fixed ε,

$$\left| \tfrac{N-2}{2} \int_{\Omega_n} |\nabla u_{n,\varepsilon}|^2 + \tfrac{N}{4\varepsilon^2} \int_{\Omega_n} (1-|u_{n,\varepsilon}|^2)^2 - c_{n,\varepsilon} \tfrac{N-1}{2} |\log\varepsilon| \int_{\Omega_n} \langle Ju_{n,\varepsilon}, \xi_1 \rangle \right| \le C.$$

Dividing by $|\log\varepsilon|$ and using(93) and (94) we are led to

$$\frac{N-1}{2} p(u_{n,\varepsilon}) c_{n,\varepsilon} = (N-2) \frac{E_\varepsilon(u_{n,\varepsilon})}{|\log\varepsilon|} + r(\varepsilon). \tag{95}$$

The conclusion follows from Step 2 and Step 3. □

From now on, we will not consider $u_{n,\varepsilon}$ anymore in this Section, and derive asymptotic properties of u_ε as ε goes to zero.

Step 5. Up to a subsequence, there exists some map $U_* \in W^{1,p}_{\text{loc}}(\mathbb{R}^N, S^1)$, ($1 \le p < \frac{N}{N-1}$) such that

$$u_\varepsilon \rightharpoonup U_* \quad \text{weakly in } W^{1,p}_{\text{loc}}(\mathbb{R}^N) \ \text{ as } \varepsilon \to 0,$$

where U_* is defined (up to a constant phase) in the statement of Theorem 1.

Proof. By Theorem 4 *i*), u_ε is bounded in $W^{1,p}_{\text{loc}}(\mathbb{R}^N)$. Therefore, up to a subsequence, there exists some map $u_* \in W^{1,p}_{\text{loc}}(\mathbb{R}^N)$ such that $u_\varepsilon \rightharpoonup u_*$ weakly in $W^{1,p}_{\text{loc}}(\mathbb{R}^N)$ and almost everywhere. Moreover,

$$\int_{\mathbb{R}^N \setminus B_R(0)} |\nabla u_*|^2 \le C \tag{96}$$

since the same inequality holds for u_ε. We show next that $u_* = U_*$. Since u_ε satisfies the equation (89), taking the exterior product of (89) with u_ε and iu_ε respectively we are led to

$$\begin{cases} d^*(u_\varepsilon \times du_\varepsilon) = ic_\varepsilon|\log\varepsilon|\frac{\partial}{\partial x_1}(|u_\varepsilon|^2 - 1) \\ d(u_\varepsilon \times du_\varepsilon) = 2Ju_\varepsilon. \end{cases}$$

Passing to the limit $\varepsilon \to 0$ [notice that $c_\varepsilon|\log\varepsilon|(|u_\varepsilon|^2 - 1) \to 0$ in $L^2(\mathbb{R}^N)$ so that the right hand side of the first equation here above converges to zero in $H^{-1}(\mathbb{R}^N)$], we obtain

$$\begin{cases} d^*(u_* \times du_*) = 0 \\ d(u_* \times du_*) = 2\pi S^{N-2}. \end{cases}$$

This elliptic system, together with (55) determines u_* uniquely (up to a constant phase). Indeed, from the first equation and classical Hodge-de Rham theory (see e.g. the Appendix of [10]) there exists a two form ψ such that

$$u_* \times du_* = d^*\psi, \qquad d\psi = 0, \qquad \text{and} \quad \nabla\psi \in L^2(\mathbb{R}^N \setminus B_R(0)).$$

Inserting this in the second equation satisfied by u_* we obtain

$$\Delta\psi = 2\pi S^{N-2}$$

so that $\psi = \psi_*$ (ψ_* is defined before Theorem 1), and the conclusion follows. □

Step 6. Let $K \subset \mathbb{R}^N \setminus S^{N-2}$ be compact and simply connected. For ε sufficiently small, we have

$$|u_\varepsilon(x)| \geq \frac{1}{2} \qquad \text{on } K.$$

Proof. We apply Theorem 3 with the sequence $(u_\varepsilon)_{\varepsilon>0}$. Indeed, we have

$$\Sigma_\mu = S^{N-2}$$

[This can be established arguing as in the proof of Lemma 4]. The claim follows then directly for (27). □

We may now write

$$u_\varepsilon(x) := \rho_\varepsilon(x)\exp(i\varphi_\varepsilon(x)) \qquad \text{on } K.$$

For convenience we skip the subscripts ε in the sequel. It remains to prove the stronger convergence in the compact K. In contrast with the case $c = 0$, where $\frac{1-|u|^2}{\varepsilon^2}$ remains bounded as ε goes to zero (see [8]), this is not the case here (it diverges like $|\log\varepsilon|$). We rely instead on a cancellation effect.

Step 7. We have

$$i) \qquad \|\nabla\varphi\|_{\mathcal{C}^k(K)} \leq C_k \quad \forall k \geq 0 \tag{97}$$

$$ii) \qquad \|\frac{2(1-\rho)}{\varepsilon^2} + c_\varepsilon|\log\varepsilon|\frac{\partial\varphi}{\partial x_1}\|_{\mathcal{C}^k(K)} \leq C_k \quad \forall k \geq 0. \tag{98}$$

In particular

$$\|\nabla \rho\|_{\mathcal{C}^k(K)} \le C_k \varepsilon^2 |\log \varepsilon| \quad \forall k \ge 0 \tag{99}$$

and the convergence claim in Theorem 1 follows from (97) and (99).

Proof. The first important point is to obtain uniform $\mathcal{C}^{0,\alpha}$ bounds, namely

$$\|u\|_{\mathcal{C}^{0,\alpha}(K)} \le C, \tag{100}$$

for some $\alpha > 0$. This is achieved as in [10] Theorem IV.1, obtaining first a monotonicity property

$$\tilde{E}_\varepsilon(\delta r, x_0) \le \frac{1}{2}\tilde{E}_\varepsilon(r, x_0) \qquad \text{for all } 0 < r < r_0,$$

for every $x_0 \in K$ and for some $\delta > 0$, and then using the Morrey embedding theorem. We skip the details [see however Step 1 of the proof of Theorem 3 for a very similar proof to obtain (B-9)]. The analysis of the further regularity properties of u is long and technical in the case of the general equation (19). For equation (89), we make use of the following trick which gives rather directly some first rough (in the sense non uniform) estimates for all the derivatives (see also [19]). The remaining analysis is then substantially simplified.

Let $v := \exp(-ic_\varepsilon |\log \varepsilon| x_1) u$. Then v satisfies the equation

$$\Delta v + \frac{1}{\varepsilon^2} v(h_\varepsilon - |v|^2) = 0,$$

where $h_\varepsilon := 1 + c_\varepsilon^2 \varepsilon^2 |\log \varepsilon|^2$. Set $w(x) := (h_\varepsilon)^{-1/2} v(x)$ and $\tilde{w}(x) := w(\frac{x}{|\log \varepsilon|})$. We have

$$\Delta \tilde{w} + \frac{1}{\tilde{\varepsilon}} \tilde{w}(1 - |\tilde{w}|^2) = 0,$$

where $\tilde{\varepsilon}^2 := \varepsilon^2 |\log \varepsilon|^2 h_\varepsilon^{-1}$. By (100) and the construction of $\tilde{w}$,

$$\|\tilde{w}\|_{\mathcal{C}^{0,\alpha}_{\text{loc}}} \le C.$$

Using the regularity theory for the Ginzburg-Landau equation, we thus infer (see [10] Theorem IV.1) that

$$\|\tilde{w}\|_{\mathcal{C}^k_{\text{loc}}} \le C_k \qquad \text{and} \qquad \|\frac{1 - |\tilde{w}|^2}{\tilde{\varepsilon}^2}\|_{\mathcal{C}^k_{\text{loc}}} \le C_k$$

for all $k \ge 0$. Coming back to u, this yields

$$\|u\|_{\mathcal{C}^k_{\text{loc}}} \le C_k |\log \varepsilon|^k \qquad \text{and} \qquad \|\frac{1 - |u|^2}{\varepsilon^2}\|_{\mathcal{C}^k_{\text{loc}}} \le C_k |\log \varepsilon|^{2+k}. \tag{101}$$

Starting with these rough estimates we are now going to prove (97) and (98) using a bootstrap argument. Define

$$B_\varepsilon := 2(1 - \rho) + c_\varepsilon |\log \varepsilon| \varepsilon^2 \frac{\partial \varphi}{\partial x_1} \qquad \text{and} \qquad A_\varepsilon := \varepsilon^{-2} B_\varepsilon.$$

The set of equations needed for the bootstrap are

$$\mathrm{div}(\rho^2\nabla\varphi) = c_\varepsilon|\log\varepsilon|\tfrac{\partial}{\partial x_1}(\rho^2-1) \tag{102}$$

$$-\Delta\rho = A_\varepsilon + c_\varepsilon|\log\varepsilon|(\rho-1)\tfrac{\partial\varphi}{\partial x_1} - \tfrac{2(1-\rho)^2}{\varepsilon^2}(\rho+2) \tag{103}$$

$$-\Delta B_\varepsilon + \tfrac{\rho(1+\rho)}{\varepsilon^2}B_\varepsilon = 2\rho|\nabla\varphi|^2 + \rho(\rho-1)|\log\varepsilon|c_\varepsilon\tfrac{\partial\varphi}{\partial x_1} - (1-\rho^2)\varepsilon^2\Delta\tfrac{\partial\varphi}{\partial x_1}. \tag{104}$$

Since ρ is bounded in $\mathcal{C}^{0,\alpha}$ by (100), we infer from Schauder regularity theory, (101) and (102) that

$$\|\nabla\varphi\|_{\mathcal{C}^{0,\alpha}_{\mathrm{loc}}} \leq C. \tag{105}$$

Using (101) and (105), we deduce that the right hand side of (104) is bounded in L^∞_{loc}. Hence, using (105) and standard arguments,

$$\|B_\varepsilon\|_{\mathcal{C}^0_{\mathrm{loc}}} \leq C\varepsilon^2 \quad \text{and thus} \quad \|A_\varepsilon\|_{\mathcal{C}^0_{\mathrm{loc}}} \leq C. \tag{106}$$

Using (101) and (106), we deduce from (103) and then from (102) that

$$\|\nabla\rho\|_{\mathcal{C}^{0,\alpha}_{\mathrm{loc}}} \leq C, \quad \text{and} \quad \|\nabla\varphi\|_{\mathcal{C}^{1,\alpha}_{\mathrm{loc}}} \leq C. \tag{107}$$

We are now in position to differentiate (104) once. This leads us, using (101),(107), to the estimate

$$\|B_\varepsilon\|_{\mathcal{C}^1_{\mathrm{loc}}} \leq C\varepsilon^2, \qquad \text{i.e.} \qquad \|A_\varepsilon\|_{\mathcal{C}^1_{\mathrm{loc}}} \leq C. \tag{108}$$

We have thus proved that *i)* and *ii)* hold for $k = 1$. The estimates for the next derivatives are obtained following exactly the same steps. This finishes the proof of Theorem 1. □

5 Proof of Theorem 5

The main ingredient in the proof of Proposition 4 is the following inequality

Lemma 5.1. *There exists a constant $C > 0$ such that*

$$\int_{\Omega_n\setminus B(R)} e_\varepsilon(u_{n,\varepsilon}) \leq C\,R\int_{\partial B(R)} e_\varepsilon(u_{n,\varepsilon})$$

for $R \geq 2$, $n \geq n(\varepsilon)$, and ε sufficiently small.

Proof. We multiply the equation

$$\mathrm{div}(\rho^2\nabla\varphi) = -\frac{c_\varepsilon}{2}|\log\varepsilon|\frac{\partial}{\partial x_1}(\rho^2-1)$$

by $\varphi - \bar\varphi$, where

$$\bar\varphi := \frac{1}{|\partial B(R)|}\int_{\partial B(R)}\varphi$$

denotes the mean value of the phase on $\partial B(0,R)$. Integrating by parts on $\Omega_n \setminus B(R)$, we obtain

$$\begin{aligned}\int_{\Omega_n\setminus B(R)} \rho^2|\nabla\varphi|^2 &= \int_{\partial B(R)} \rho^2 \frac{\partial\varphi}{\partial\nu}(\varphi-\bar\varphi) + \frac{c_\varepsilon}{2}|\log\varepsilon| \int_{\Omega_n\setminus B(R)} \frac{\partial(\rho^2-1)}{\partial x_1}(\varphi-\bar\varphi)\\ &= \int_{\partial B(R)} \rho^2 \frac{\partial\varphi}{\partial\nu}(\varphi-\bar\varphi) - \frac{c_\varepsilon}{2}|\log\varepsilon| \int_{\partial B(R)} (\rho^2-1)(\varphi-\bar\varphi)n_1\\ &\qquad + \frac{c_\varepsilon}{2}|\log\varepsilon| \int_{\Omega_n\setminus B(R)} (1-\rho^2)\frac{\partial\varphi}{\partial x_1}.\end{aligned}$$

We estimate each of the three terms on the right hand side separately. For the first term, we invoke the Poincaré-Wirtinger inequality to assert

$$\begin{aligned}\left|\int_{\partial B(R)} \rho^2 \frac{\partial\varphi}{\partial\nu}(\varphi-\bar\varphi)\right| &\le C\left(\int_{\partial B(R)} |\nabla\varphi|^2\right)^{1/2} \left(\int_{\partial B(R)} (\varphi-\bar\varphi)^2\right)^{1/2}\\ &\le C\,R\left(\int_{\partial B(R)} |\nabla\varphi|^2\right) \le C\,R\int_{\partial B(R)} e_\varepsilon(u_{n,\varepsilon}).\end{aligned}\tag{109}$$

Similarly, we obtain

$$\begin{aligned}&\frac{c_\varepsilon}{2}|\log\varepsilon|\left|\int_{\partial B(R)} (\rho^2-1)(\varphi-\bar\varphi)n_1\right| \le\\ &\qquad C\varepsilon|\log\varepsilon|R\left(\int_{\partial B(R)}\frac{(1-\rho^2)^2}{\varepsilon^2}\int_{\partial B(R)}|\nabla\varphi|^2\right)^{1/2} \le C\varepsilon|\log\varepsilon|R\int_{\partial B(R)} e_\varepsilon(u_{n,\varepsilon}),\end{aligned}\tag{110}$$

and

$$\frac{c_\varepsilon}{2}|\log\varepsilon|\left|\int_{\Omega_n\setminus B(R)}(1-\rho^2)\frac{\partial\varphi}{\partial x_1}\right| \le C\varepsilon|\log\varepsilon|\int_{\Omega_n\setminus B(R)} e_\varepsilon(u_{n,\varepsilon}).\tag{111}$$

Combining (109),(110),(111) we are led to

$$\int_{\Omega_n\setminus B(R)} |\nabla\varphi|^2 \le C\,R\int_{\partial B(R)} e_\varepsilon(u_{n,\varepsilon}) + C\varepsilon|\log\varepsilon|\int_{\Omega_n\setminus B(R)} e_\varepsilon(u_{n,\varepsilon}).\tag{112}$$

We now turn to the equation for ρ,

$$-\Delta\rho + \rho|\nabla\varphi|^2 + c_\varepsilon|\log\varepsilon|\rho\frac{\partial\varphi}{\partial x_1} = \rho\frac{(1-\rho^2)}{\varepsilon^2}.$$

Multiplying by ρ^2-1 and integrating by parts on $\Omega_n\setminus B(R)$ gives

$$\begin{aligned}\int_{\Omega_n\setminus B(R)} 2\rho|\nabla\rho|^2 + \rho\frac{(1-\rho^2)^2}{\varepsilon^2} &= \int_{\partial B(R)} \frac{\partial\varphi}{\partial\nu}(1-\rho^2)\\ &\quad + c_\varepsilon|\log\varepsilon|\int_{\Omega_n\setminus B(R)} \rho(1-\rho^2)\frac{\partial\varphi}{\partial x_1} + \int_{\Omega_n\setminus B(R)} \rho(1-\rho^2)|\nabla\varphi|^2.\end{aligned}\tag{113}$$

We have,

$$\left| \int_{\partial B(R)} \frac{\partial \varphi}{\partial \nu}(1-\rho^2) \right| \leq C\varepsilon \int_{\partial B(R)} e_\varepsilon(u_{n,\varepsilon}), \tag{114}$$

and

$$c_\varepsilon |\log \varepsilon| \left| \int_{\Omega_n \setminus B(R)} \rho(1-\rho^2)\frac{\partial \varphi}{\partial x_1} \right| \leq C\varepsilon |\log \varepsilon| \int_{\Omega_n \setminus B(R)} e_\varepsilon(u_{n,\varepsilon}). \tag{115}$$

For the third term, we invoke the fact (see Theorem 1) that $|\nabla \varphi| \leq C$ in $\Omega_n \setminus B(R)$ so that

$$\left| \int_{\Omega_n \setminus B(R)} \rho(1-\rho^2)|\nabla \varphi|^2 \right| \leq C \int_{\Omega_n \setminus B(R)} \rho(1-\rho^2)|\nabla \varphi| \leq C\varepsilon \int_{\Omega_n \setminus B(R)} e_\varepsilon(u_{n,\varepsilon}). \tag{116}$$

Combining (113),(114),(115),(116), we are led to

$$\int_{\Omega_n \setminus B(R)} |\nabla \rho|^2 + \frac{(1-\rho^2)^2}{\varepsilon^2} \leq C\varepsilon \int_{\partial B(R)} e_\varepsilon(u_{n,\varepsilon}) + C\varepsilon |\log \varepsilon| \int_{\Omega_n \setminus B(R)} e_\varepsilon(u_{n,\varepsilon}). \tag{117}$$

Finally, from (112) and (117) we derive the conclusion. □

Proof of Proposition 4. Set, for $R > 2$,

$$f_n(R) := \int_{\Omega_n \setminus B(R)} e_\varepsilon(u_{n,\varepsilon}).$$

We infer from Lemma 5.1 that the f_n verify the following differential inequality

$$f_n(s) \leq -C\, s f_n'(s), \qquad \text{for all } s > 2.$$

Integrating between 2 and R yields

$$f_n(R) \leq f_n(2) \left(\frac{2}{R}\right)^\lambda,$$

where $\lambda := \frac{1}{C}$. This proves (41). The other statements (42) and (43) follow directly from this decay. □

Remark 5.1. In the previous computations, we have not tried to optimize the constants. Using the best constant in Poincaré-Wirtinger inequality for (109), we may prove that (41) is valid with any $\lambda < \sqrt{N-1}$ provided ε is sufficiently small (depending on λ).

Proof of Theorem 5. Equality (39) has already been established in Proposition 4. For (40), we argue by contradiction and assume it is false. Then, there exists $v \in W$ such that

$$E_\varepsilon(v) < E_\varepsilon(u_\varepsilon) = \lim_{n \to +\infty} E_\varepsilon(u_{n,\varepsilon}) \tag{118}$$

and

$$p(v) = 2\pi |B^{N-1}|. \tag{119}$$

If v were constant outside some large ball $B(R)$, then its restriction to Ω_n, for $n \geq R$, would be well defined on Π_n and therefore, in view of (119), a test function for $(\mathcal{P}_n^\varepsilon)$. This contradicts (118) for n sufficiently large.

In the general situation, we will construct from v a function $\tilde{v}$, constant outside some large ball $B(R)$, and verifying

$$p(\tilde{v}) = 2\pi |B^{N-1}| \tag{120}$$

and

$$E_\varepsilon(\tilde{v}) < E_\varepsilon(u_\varepsilon), \tag{121}$$

so that a contradiction holds similarly.

Construction of $\tilde{v}$. Since $v \in W$, we may write

$$v = \eta \exp(i\psi) \qquad \text{on } \mathbb{R}^N \setminus B(R),$$

provided R is sufficiently large. We begin by the construction of a function $\check{v}_R$, constant outside $B(3R)$, but which will not yet satisfy (120). For that purpose, consider the functions $\check{\eta}_R$ and $\check{\psi}_R$ defined on $\mathbb{R}^N \setminus B(R)$ by

$$\begin{aligned} \check{\eta}_R(x) &:= \sigma(x)\eta(x) + (1 - \sigma(x)), & \text{with } \sigma(x) := \tfrac{2R - |x|}{R}, \\ \check{\psi}_R(x) &:= \tau(x)\psi(x) + (1 - \tau(x)) \left(\tfrac{1}{|\partial B(R)|} \textstyle\int_{\partial B(R)} \psi \right), & \text{with } \tau(x) := \tfrac{3R - |x|}{R}. \end{aligned}$$

Set,

$$\check{v}_R(x) := \begin{cases} v(x) & \text{if } |x| \leq R, \\ \check{\eta}_R(x) \exp(i\psi(x)) & \text{if } R \leq |x| \leq 2R, \\ \exp(i\check{\psi}_R) & \text{if } 2R \leq |x| \leq 3R, \\ \exp\left(i \frac{1}{|\partial B(R)|} \int_{\partial B(R)} \psi \right) & \text{otherwise.} \end{cases}$$

A few computations show that, for some constant $C > 0$ independent of R,

$$|E_\varepsilon(\check{v}_R) - E_\varepsilon(v)| + |p(\check{v}_R) - p(v)| \leq C \left[\int_{\partial B(R)} e_\varepsilon(v) + \int_{\Omega_n \setminus B(R)} e_\varepsilon(v) \right].$$

We may take next a sequence $(R_m)_{m \in \mathbb{N}}$, such that $R_m \to +\infty$ and

$$\int_{\partial B(R_m)} e_\varepsilon(v) \to 0 \qquad \text{as } m \to +\infty,$$

so that

$$p(\check{v}_{R_m}) = 2\pi |B^{N-1}| + o(1) \qquad \text{as } m \to +\infty \tag{122}$$

and

$$E_\varepsilon(\check{v}_{R_m}) = E_\varepsilon(v) + o(1) \qquad \text{as } m \to +\infty. \tag{123}$$

We finally complete the construction of $\tilde{v}$ setting

$$\tilde{v}_{R_m}(x) := \check{v}_{R_m}(\alpha_m x),$$

where $\alpha_m > 0$ is uniquely defined by the relation $p(\tilde{v}_{R_m}) = 2\pi |B^{N-1}|$. It follows from (122) that $\alpha_m = 1 + o(1)$ as $m \to +\infty$. Hence, if we choose $\tilde{v} := \tilde{v}_{R_m}$, we verify that $\tilde{v}$ satisfies the required conditions for m sufficiently large. $\square$

Appendices

The purpose of these appendices is to develop the asymptotic analysis of the equation

$$i|\log\varepsilon|\vec{c}(x)\cdot\nabla w = \Delta w + \frac{1}{\varepsilon^2}w(1-|w|^2) - |\log\varepsilon|^2 d(x)w \qquad \text{on } \Omega, \tag{124}$$

where $\Omega \subseteq \mathbb{R}^N$ is a piecewise $\mathcal{C}^1$ simply connected domain, $\vec{c}:\overline{\Omega}\to\mathbb{R}^N$ is a bounded Lipschitz vector field and $d:\overline{\Omega}\to\mathbb{R}$ is Lipschitz non negative and bounded. The main results of this analysis have been stated in Lemmas 3 and 4, Proposition 2 and Theorems 2, 3 and 4. We will provide proofs here. Notice that (124) can be rewritten as

$$i|\log\varepsilon|\vec{c}(x)\cdot\nabla w = \Delta w + \frac{1}{\varepsilon^2}w(a_\varepsilon(x)-|w|^2) \tag{125}$$

where

$$a_\varepsilon(x) := 1 - d(x)\varepsilon^2|\log\varepsilon|^2.$$

When $\operatorname{div}\vec{c}=0$ it is also equivalent to

$$(\nabla - i|\log\varepsilon|\frac{\vec{c}}{2})^2 w + \frac{1}{\varepsilon^2}w(b_\varepsilon(x)-|w|^2) = 0, \tag{126}$$

where

$$b_\varepsilon(x) := a_\varepsilon(x) + \varepsilon^2|\log\varepsilon|^2\frac{c^2(x)}{4}.$$

In the sequel, we assume throughout that

$$\operatorname{div}\vec{c} = 0.$$

Appendix A: the PDE analysis

In this first Appendix we establish some basic estimates, in particular we give th proof of Lemma 3.

Proof of Lemma 3. Let w satisfy (124) and $\rho(x) := |w(x)|$. Then we have

$$\begin{aligned}\Delta\rho^2 &= (2w,\Delta w) + 2|\nabla w|^2\\ &= -\frac{2}{\varepsilon^2}\rho^2(a_\varepsilon(x)-\rho^2) + |\log\varepsilon|\,(2iw,\vec{c}\cdot\nabla w) + 2|\nabla w|^2\\ &\geq -\frac{2}{\varepsilon^2}\rho^2(a_\varepsilon(x)-\rho^2) + (\sqrt{2}|\nabla w| - \frac{1}{\sqrt{2}}c_\infty|\log\varepsilon||w|)^2 - \frac{1}{2}c_\infty^2|\log\varepsilon|^2\rho^2\\ &\geq -\frac{2}{\varepsilon^2}\rho^2(b_\varepsilon^\infty - \rho^2)\end{aligned} \tag{A-1}$$

where $b_\varepsilon^\infty := |b_\varepsilon|_{L^\infty(K)}$. Hence the function $W(x) := \rho^2(x) - b_\varepsilon^\infty$ satisfies the inequality

$$\Delta W \geq \frac{2}{\varepsilon^2}W(W+b_\varepsilon^\infty) \qquad \text{on } \Omega.$$

If $x_0 \in K$ and $R := \mathrm{dist}(x_0, \partial\Omega)$, the rescaled function

$$Y(x) := W(R(x - x_0))$$

is thus a subsolution to the equation

$$\Delta y = \frac{2}{\tilde{\varepsilon}^2} y(y + b_\varepsilon^\infty) \qquad \text{on } B(0,1), \tag{A-2}$$

where $\tilde{\varepsilon} := \frac{\varepsilon}{R}$. On the other hand, it is easy to check that there exists a constant $C > 0$ depending only on N such that the function

$$Z(x) := \begin{cases} C\tilde{\varepsilon}^2(|x| - 1)^{-2} & \text{if } |x| \in [\frac{1}{3}, 1] \\ \frac{9}{8}C\tilde{\varepsilon}^2 + \frac{81}{8}C\tilde{\varepsilon}^2|x|^2 & \text{if } |x| \in [0, \frac{1}{3}] \end{cases}$$

is a supersolution to (A-2) [notice that $Y(x) \to +\infty$ as $|x| \to 1$]. It then follows from the maximum principle that $Y(x) \leq Z(x)$ for all x in $B(0,1)$, and in particular

$$W(x_0) = Y(0) \leq C\tilde{\varepsilon}^2 \leq C \frac{\varepsilon^2}{\mathrm{dist}(K, \partial\Omega)^2}.$$

Hence, we obtain the desired estimate

$$|w|_{L^\infty(K)} \leq |W|_{L^\infty(K)} + b_\varepsilon^\infty \leq 1 + c_\infty^2 \varepsilon^2 |\log \varepsilon|^2 + C \frac{\varepsilon^2}{\mathrm{dist}(K, \partial\Omega)^2}.$$

Concerning the estimate on the gradient, let $r := \mathrm{dist}(K, \partial\Omega)$ and

$$\tilde{K} := \{x \in \Omega \text{ s.t. } \mathrm{dist}(x, K) \leq r/2\}.$$

By the first step, $|w|_{L^\infty(\tilde{K})} \leq C_K$ where C_K does not depend on w or ε. Let U be the solution of

$$\begin{cases} (\nabla - i|\log \varepsilon|\frac{\vec{\varepsilon}}{2})^2 U = 0 & \text{on } \tilde{K} \\ U = w & \text{on } \partial\tilde{K}. \end{cases} \tag{A-3}$$

Since $w - U \in H^2(\tilde{K}) \cap H_0^1(\tilde{K})$, we deduce from the Gagliardo-Nirenberg inequality that

$$\begin{aligned} |\nabla(w - U)|_{L^\infty(\tilde{K})} &\leq |\Delta(w - U)|_{L^\infty(\tilde{K})}^{\frac{1}{2}} \cdot |(w - U)|_{L^\infty(\tilde{K})}^{\frac{1}{2}} \\ &\leq C_K(\frac{1}{\varepsilon} + |\log \varepsilon|^{\frac{1}{2}} c_\infty^{\frac{1}{2}} |\nabla(w - U)|_{L^\infty(\tilde{K})}^{\frac{1}{2}}) \\ &\leq C_K(\frac{1}{\varepsilon} + \frac{1}{2C_K}|\nabla(w - U)|_{L^\infty(\tilde{K})} + 2C_K c_\infty |\log \varepsilon|), \end{aligned} \tag{A-4}$$

so that

$$|\nabla(w - U)|_{L^\infty(\tilde{K})} \leq \frac{C_K}{\varepsilon}.$$

Hence, since U satisfies (A-3)

$$
\begin{aligned}
|\nabla w|_{L^\infty(K)} &\leq |\nabla U|_{L^\infty(K)} + \frac{C_K}{\varepsilon} \leq \frac{C_K}{\varepsilon}|U|_{L^\infty(\tilde{K})} + \frac{C_K}{\varepsilon} \\
&\leq \frac{C_K}{\varepsilon},
\end{aligned}
\tag{A-5}
$$

where C_K depends only on K, N and c_∞. The lemma is proved. □

Let us now define the 2-forms on $\mathbb{R}^N$,

$$
\xi_j(x) := \frac{2}{N-1}\sum_{i\neq j} x_i dx_j \wedge dx_i \qquad \text{for } j = 1, \dots, N
$$

which satisfy the equations $d^*\xi_j = 2dx_j$.

Lemma A.2 (Pohozaev identity). *Let w be a solution of equation (124) on Ω, then*

$$
\begin{aligned}
\tfrac{N-2}{2}\int_\Omega |\nabla w|^2 + \tfrac{N}{4\varepsilon^2}\int_\Omega (a_\varepsilon(x) - |w|^2)^2 - \tfrac{N-1}{2}|\log\varepsilon| \int_\Omega \langle Jw, \sum_i c_i(x)\xi_i(x)\rangle \\
= \int_{\partial\Omega} [x\cdot\nu\frac{|\nabla w|^2}{2} + \frac{x\cdot\nu}{4\varepsilon^2}(a_\varepsilon(x) - |w|^2)^2 - \frac{\partial w}{\partial\nu}\cdot(\sum x_i \frac{\partial w}{\partial x_i})] \\
+ \frac{1}{2}|\log\varepsilon|^2 \int_\Omega (a_\varepsilon(x) - |w|^2)\, x\cdot\nabla d(x).
\end{aligned}
\tag{A-6}
$$

In particular, for $B_r(x_0) \subset \Omega$ we have

$$
\begin{aligned}
\frac{N-2}{2}\int_{B_r(x_0)} |\nabla w|^2 + \frac{N}{4\varepsilon^2}\int_{B_r(x_0)} (a_\varepsilon(x) - |w|^2)^2 \\
= \frac{N-1}{2}|\log\varepsilon| \int_{B_r(x_0)} \langle Jw, \sum_i c_i(x)\xi_i(x - x_0)\rangle \\
+ \frac{1}{2}|\log\varepsilon|^2 \int_{B_r(x_0)} (a_\varepsilon(x) - |w|^2)\,(x - x_0)\cdot\nabla d(x) \\
+ \int_{\partial B_r(x_0)} [r\frac{|\nabla_\top w|^2}{2} - \frac{r}{2}\left|\frac{\partial w}{\partial n}\right|^2 + \frac{r}{4\varepsilon^2}(a_\varepsilon(x) - |w|^2)^2].
\end{aligned}
\tag{A-7}
$$

For $x_0 \in \Omega$ and $r > 0$ such that $B_r(x_0) \subset \Omega$, consider the scaled energy

$$
\tilde{E}_\varepsilon(w, x_0, r) := \frac{1}{r^{N-2}} E_\varepsilon(w, x_0, r) \equiv \frac{1}{r^{N-2}}\int_{B_r(x_0)} \frac{1}{2}|\nabla w|^2 + \frac{(a_\varepsilon(x) - |w|^2)^2}{4\varepsilon^2}.
$$

When this will not lead to a confusion, we will also note it $\tilde{E}_\varepsilon(x_0, r)$ or even $\tilde{E}_\varepsilon(r)$.

Lemma A.3. *Let w satisfy (124) on $B_R(x_0) \subset \Omega$, then for $0 < r < R$*

$$\begin{aligned}\frac{d}{dr}(\tilde{E}_\varepsilon(x_0,r)) = &\frac{1}{r^{N-2}}\int_{\partial B_r(x_0)} \left|\frac{\partial w}{\partial n}\right|^2 + \frac{1}{r^{N-1}}\int_{B_r(x_0)} \frac{(a_\varepsilon(x)-|w|^2)^2}{2\varepsilon^2} \\ &- \frac{N-1}{2r^{N-1}}|\log\varepsilon| \int_{B_r(x_0)} \langle Jw, \sum_i c_i(x)\xi_i(x-x_0)\rangle \\ &- \frac{1}{2r^{N-1}}|\log\varepsilon|^2 \int_{B_r(x_0)} ((x-x_0)\cdot\nabla d(x))(a_\varepsilon(x)-|w|^2).\end{aligned} \tag{A-8}$$

Proof. Without loss of generality, we can assume that $x_0 = 0$. First one has,

$$\begin{aligned}\frac{d}{dr}(E_\varepsilon(r)) &= \int_{\partial B_r} \frac{|\nabla w|^2}{2} + \frac{1}{4\varepsilon^2}\int_{\partial B_r}(a_\varepsilon(x)-|w|^2)^2 \\ &= \int_{\partial B_r} \frac{|\nabla_\top w|^2}{2} + \frac{1}{2}\left|\frac{\partial w}{\partial n}\right|^2 + \frac{1}{4\varepsilon^2}(a_\varepsilon(x)-|w|^2)^2\ .\end{aligned}$$

Hence,

$$\begin{aligned}\frac{d}{dr}(\tilde{E}_\varepsilon(r)) &= -\frac{N-2}{r^{N-1}}E_\varepsilon(r) + \frac{1}{r^{N-2}}\int_{\partial B_r}\frac{|\nabla_\top w|^2}{2} + \frac{1}{2}\left|\frac{\partial w}{\partial n}\right|^2 + \frac{(a_\varepsilon(x)-|w|^2)^2}{4\varepsilon^2} \\ &= -\left(\frac{N-2}{r^{N-1}}\int_{B_r}\frac{|\nabla w|^2}{2} + \frac{N-2}{4\varepsilon^2 r^{N-1}}\int_{B_r}(a_\varepsilon(x)-|w|^2)^2\right) \\ &+ \frac{1}{r^{N-2}}\int_{\partial B_r}\frac{|\nabla_\top w|^2}{2} + \frac{1}{2}\left|\frac{\partial w}{\partial n}\right|^2 + \frac{1}{4\varepsilon^2}(a_\varepsilon(x)-|w|^2)^2 \\ &= -\left(\frac{1}{r^{N-1}}[\int_{B_r}\frac{N-2}{2}|\nabla w|^2 + \frac{N}{4\varepsilon^2}\int_{B_r}(a_\varepsilon(x)-|w|^2)^2]\right) \\ &+ \frac{1}{2\varepsilon^2 r^{N-1}}\int_{B_r}(a_\varepsilon(x)-|w|^2)^2 \\ &+ \frac{1}{r^{N-2}}\int_{\partial B_r}\frac{|\nabla_\top w|^2}{2} + \frac{1}{2}\left|\frac{\partial w}{\partial n}\right|^2 + \frac{1}{4\varepsilon^2}(a_\varepsilon(x)-|w|^2)^2 .\end{aligned}$$

Using Lemma A.2, we obtain

$$
\begin{aligned}
\frac{d}{dr}(\tilde{E}_\varepsilon(r)) = &- \left[\frac{1}{r^{N-2}} \int_{\partial B_r} \frac{|\nabla_\top w|^2}{2} - \frac{1}{2}\left|\frac{\partial w}{\partial n}\right|^2 + \frac{1}{4\varepsilon^2}(a_\varepsilon(x) - |w|^2)^2\right] \\
&- \frac{N-1}{2r^{N-1}}|\log\varepsilon| \int_{B_r} \langle Jw, \sum_i c_i(x)\xi_i(x)\rangle \\
&- \frac{1}{2r^{N-1}}|\log\varepsilon|^2 \int_{B_r} (x\cdot\nabla d(x))(a_\varepsilon(x) - |w|^2) \\
&+ \frac{1}{r^{N-2}}\left[\int_{\partial B_r} \frac{|\nabla_\top w|^2}{2} + \frac{1}{2}\left|\frac{\partial w}{\partial n}\right|^2 + \frac{1}{4\varepsilon^2}(a_\varepsilon(x) - |w|^2)^2\right] \\
= &\frac{1}{r^{N-2}} \int_{\partial B_r} \left|\frac{\partial w}{\partial n}\right|^2 + \frac{1}{r^{N-1}} \int_{B_r} \frac{(a_\varepsilon(x) - |w|^2)^2}{2\varepsilon^2} \\
&- \frac{N-1}{2r^{N-1}}|\log\varepsilon| \int_{B_r} \langle Jw, \sum_i c_i(x)\xi_i(x)\rangle \\
&- \frac{1}{2r^{N-1}}|\log\varepsilon|^2 \int_{B_r} (x\cdot\nabla d(x))(a_\varepsilon(x) - |w|^2),
\end{aligned}
$$

which yields the result. □

Proof of Lemma 4 (Monotonicity at small scales). Again we can assume that $x_0 = 0$. In view of the previous Lemma, we need to estimate the last two terms in (A-8). For the first one, notice that

$$
\|Jw(x)\| \leq C|\nabla w(x)|^2 \quad \text{and} \quad \|\xi_j(x)\| \leq C\, r \quad \text{for all } x \in B_r,
$$

where $\|\cdot\|$ refers e.g. to the Euclidean norm on two-forms. Hence,

$$
\begin{aligned}
&\frac{N-1}{2r^{N-1}}|\log\varepsilon| \left|\int_{B_r} \langle Jw, \sum_i c_i(x)\xi_i(x)\rangle\right| \\
\leq &C\, c_\infty |\log\varepsilon| \frac{1}{r^{N-2}} \int_{B_r} |\nabla w|^2 \leq C\, c_\infty |\log\varepsilon| \tilde{E}_\varepsilon(r),
\end{aligned} \tag{A-9}
$$

where C depends only on N. For the second term we have,

$$
\begin{aligned}
&\frac{1}{2r^{N-1}}|\log\varepsilon|^2 \left|\int_{B_r} (x\cdot\nabla d(x))(a_\varepsilon(x) - |w|^2)\right| \\
&\leq \frac{C}{r^{N-2}}\varepsilon|\log\varepsilon|^2 \left(\int_{B_r} |\nabla d|^2\right)^{1/2} \cdot \left(\int_{B_r} \frac{(a_\varepsilon(x) - |w|^2)^2}{4\varepsilon^2}\right)^{1/2} \\
&\leq C\Lambda_0 r^{\frac{2-N}{2}} \varepsilon|\log\varepsilon|^2 \left(\int_{B_r} \frac{(a_\varepsilon(x) - |w|^2)^2}{4\varepsilon^2}\right)^{1/2} \\
&\leq C\Lambda_0 \varepsilon |\log\varepsilon|^2 \tilde{E}_\varepsilon^{1/2}(r) \\
&\leq \tilde{E}_\varepsilon(r) + C^2\Lambda_0^2\varepsilon^2|\log\varepsilon|^4.
\end{aligned} \tag{A-10}
$$

Set $\Lambda := C\,(c_\infty+1)|\log\varepsilon|$, then using Lemma A.3, (A-9) and (A-10),

$$
\begin{aligned}
&\frac{d}{dr}\Big(\exp(\Lambda r)\tilde{E}_\varepsilon(r)\Big) = \Lambda\exp(\Lambda r)\tilde{E}_\varepsilon(r) + \exp(\Lambda r)\frac{d}{dr}(\tilde{E}_\varepsilon(r)) \\
&\geq \Lambda\exp(\Lambda r)\tilde{E}_\varepsilon(r) - \exp(\Lambda r)(C\,c_\infty|\log\varepsilon|\tilde{E}_\varepsilon(r) + \tilde{E}_\varepsilon(r) + C^2\Lambda_0^2\varepsilon^2|\log\varepsilon|^4) \\
&\geq -\exp(\Lambda r)C^2\Lambda_0^2\varepsilon^2|\log\varepsilon|^4) = -\frac{d}{dr}\left(\frac{Q^2}{\Lambda}\exp(\Lambda r)\right).
\end{aligned}
\tag{A-11}
$$

This finishes the proof. □

As already mentioned, the pointwise estimate on the Jacobian used in the previous proof is far from being optimal. In order to obtain a monotonicity formula valid on larger balls, we will use the following estimate due to Jerrard and Soner [29] (see [29] for a more quantitative version).

Lemma A.4 (Jerrard & Soner). *Let $w \in H^1_{loc}(\Omega,\mathbb{C})$, $\varphi \in \mathcal{C}^{0,1}_c(\Omega,\Lambda^2\mathbb{R}^N)$ and set $K := supp\,\varphi$. Then there exists constants $C>0$ (depending only on N) and $0<\alpha<1$ such that*

$$
\left|\int_\Omega \langle Jw,\varphi\rangle\right| \leq \frac{C}{|\log\varepsilon|}\|\varphi\|_{L^\infty}\int_K e_\varepsilon(w) + C\varepsilon^\alpha\|d\varphi\|_{L^\infty}(1+\int_K e_\varepsilon(w))(1+|K|^2). \tag{A-12}
$$

The big advantage of (A-12) with respect to estimate (A-9) is the factor $1/|\log\varepsilon|$ which appears in front of the energy. However, since (A-12) contains a second term involving a derivative of φ, we need to adapt temporarily the definition of $\tilde{E}_\varepsilon$.

We define a cut-off function f on $\mathbb{R}_+\times\mathbb{R}_+$ by

$$
f(a,b) = \begin{cases} 1 & \text{if } b\leq a \\ 2-b/a & \text{if } a\leq b\leq 2a \\ 0 & \text{if } b\geq 2a. \end{cases}
$$

For $x_0\in\Omega$ and $r>0$ such that $B_{2r}(x_0)\subset\Omega$, we will consider the quantity

$$
\bar{E}_\varepsilon(x_0,r) := \frac{1}{r^{N-2}}\int_{B_{2r}(x_0)} e_\varepsilon(w)f(r,|x-x_0|)\,dx. \tag{A-13}
$$

Lemma A.5. *Let w satisfy (124) on $B_R(x_0)\subset\Omega$, then for $0<r<R/2$*

$$
\begin{aligned}
&\frac{d}{dr}(\bar{E}_\varepsilon(x_0,r)) \\
&= \frac{1}{r^{N-2}}\int_1^2 t\int_{\partial B_{tr}(x_0)}\left|\frac{\partial w}{\partial n}\right|^2 + \frac{1}{r^{N-1}}\int_{B_{2r}(x_0)}\frac{(a_\varepsilon(x)-|w|^2)^2}{2\varepsilon^2}f(r,|x-x_0|) \\
&- \frac{N-1}{2r^{N-1}}|\log\varepsilon|\int_{B_{2r}(x_0)}\langle Jw,\sum_i c_i(x)\xi_i(x-x_0)f(r,|x-x_0|)\rangle \\
&- \frac{1}{2r^{N-1}}|\log\varepsilon|^2\int_{B_{2r}(x_0)}((x-x_0)\cdot\nabla d(x))(a_\varepsilon(x)-|w|^2)f(r,|x-x_0|).
\end{aligned}
\tag{A-14}
$$

Proof. For $x_0 = 0$ we have,

$$\begin{aligned}
\frac{d}{dr}(\bar{E}_\varepsilon(r)) &= -\frac{N-2}{r^{N-1}}\int_{B_{2r}} e_\varepsilon(w) f(r,|x|)\,dx + \frac{1}{r^{N-2}}\int_{B_{2r}} e_\varepsilon(w)\partial_r f(r,|x|)\,dx \\
&= -\frac{N-2}{r^{N-1}}\int_1^2\int_{B_{tr}} e_\varepsilon(w)\,dx\,dt + \frac{1}{r^{N-2}}\int_r^{2r}\int_{\partial B_t} e_\varepsilon(w)\frac{t}{r^2}\,dx\,dt \\
&= -\frac{N-2}{r^{N-1}}\int_1^2\int_{B_{tr}} e_\varepsilon(w)\,dx\,dt + \frac{1}{r^{N-2}}\int_1^2 t\int_{\partial B_{tr}} e_\varepsilon(w)\,dx\,dt \\
&= \int_1^2 t^{N-1}\frac{d}{d(tr)}\tilde{E}_\varepsilon(tr)\,dt.
\end{aligned} \tag{A-15}$$

It suffices then to use Lemma A.3 and to integrate in t. The case $x_0 \neq 0$ is reduced to the first one by a change of variable. ☐

Lemma A.6 (Monotonicity at large scales). *There exists a constant $C > 0$ such that for any w satisfying (124) and $x_0 \in \Omega$, $r > 0$ such that $B_{2r}(x_0) \subset \Omega$,*

$$\bar{E}_\varepsilon(\theta\, r, x_0) \le C \exp(C\,\Lambda_0\, r)\left(\bar{E}_\varepsilon(r, x_0) + \frac{\varepsilon^\alpha|\log\varepsilon|}{(\theta r)^{N-1}} + \Lambda_0\varepsilon^2|\log\varepsilon|^4\right)$$

for every $0 < \theta \le 1$.

Proof. The proof bears some resemblance with the one of Lemma 4. Once more we restrict to the case $x_0 = 0$; we first need to estimate the last two terms in (A-14). The second one is treated as before,

$$\begin{aligned}
&\frac{1}{2r^{N-1}}|\log\varepsilon|^2\int_{B_{2r}}(x\cdot\nabla d(x))(a_\varepsilon(x) - |w|^2)f(r,|x|) \\
&\le \frac{C}{r^{N-2}}\varepsilon|\log\varepsilon|^2\left(\int_{B_{2r}}|\nabla d|^2\right)^{1/2}\cdot\left(\int_{B_{2r}}\frac{(a_\varepsilon(x) - |w|^2)^2}{4\varepsilon^2}\right)^{1/2} \\
&\le C\Lambda_0 r^{\frac{2-N}{2}}\varepsilon|\log\varepsilon|^2\left(\int_{B_{2r}}\frac{(a_\varepsilon(x) - |w|^2)^2}{4\varepsilon^2}\right)^{1/2} \\
&\le C\Lambda_0\varepsilon|\log\varepsilon|^2\bar{E}_\varepsilon^{1/2}(2r) \\
&\le \bar{E}_\varepsilon(2r) + C^2\Lambda_0^2\varepsilon^2|\log\varepsilon|^4.
\end{aligned} \tag{A-16}$$

Concerning the first term, notice that the 2-form

$$\varphi(x) := \sum_i c_i(x)\xi_i(x) f(r,|x|)$$

satisfies the bounds

$$\|\varphi\|_{L^\infty(B_{2r})} \le C c_\infty r \qquad \text{and} \qquad \|d\varphi\|_{L^\infty(B_{2r})} \le C\Lambda_0.$$

Hence, using Lemma A.4, we obtain

$$\begin{aligned}
&\frac{N-1}{2r^{N-1}}|\log\varepsilon| \int_{B_{2r}} \langle Jw, \sum_i c_i(x)\xi_i(x) f(r,|x|)\rangle \\
&\leq C\, c_\infty \bar{E}_\varepsilon(2r) + C\,\Lambda_0 \varepsilon^\alpha |\log\varepsilon| (\frac{1}{r^{N-1}} + \bar{E}_\varepsilon(2r)) \\
&\leq C\,\Lambda_0 \bar{E}_\varepsilon(2r) + \frac{C\,\Lambda_0}{r^{N-1}} \varepsilon^\alpha |\log\varepsilon|.
\end{aligned} \tag{A-17}$$

From (A-14), (A-16) and (A-17) we thus infer that

$$\frac{d}{dr}\bar{E}_\varepsilon(r) \geq -C\,\Lambda_0 \bar{E}_\varepsilon(2r) - C\left(\frac{\Lambda_0}{r^{N-1}}\varepsilon^\alpha|\log\varepsilon| + \Lambda_0^2\varepsilon^2|\log\varepsilon|^4\right). \tag{A-18}$$

The conclusion then follows form a discrete version of Gronwall's lemma given hereafter. □

Lemma A.7 (Discrete Gronwall inequality). *Let $h : (0,1] \to \mathbb{R}_+$ be continuously differentiable and such that*

$$\begin{cases} h(s) \leq \theta^{N-2} h(\theta s) & \text{for all } \theta \in [1,2] \\ h'(s) \geq -Ch(2s) - D & \text{for all } s \leq 1/2, \end{cases}$$

where C and D are positive constants. Then,

$$h(s) \leq 2^{N-2} \exp(Ct)\,(h(t) + D/C) \qquad \text{for all } 0 < s < t < 1. \tag{A-19}$$

Proof. Let $g(s) := h(s) + D/C$. We have

$$\begin{cases} g(s) = h(s) + \frac{D}{C} \leq \theta^{N-2} h(\theta s) + \theta^{N-2}\frac{D}{C} = \theta^{N-2} g(\theta s) & \text{for all } \theta \in [1,2] \\ g'(s) = h'(s) \geq -Ch(2s) - D = -Cg(2s) & \text{for all } s \leq 1/2, \end{cases} \tag{A-20}$$

so that we just need to consider the case $D = 0$. Let $0 < s < t < 1$ be given. If $s \in [t/2, t]$, then by (A-20),

$$g(s) \leq 2^{N-2} g(t).$$

By induction, assume that for some $k \in \mathbb{N}_*$ it holds

$$g(s) \leq 2^{N-2} g(t) \prod_{i=2}^{k} (1 + \frac{Ct}{2^i}) \qquad \forall\, s \in [\frac{t}{2^k}, \frac{t}{2^{k-1}}].$$

Then, if $s \in \left[\frac{t}{2^{k+1}}, \frac{t}{2^k}\right]$,

$$\begin{aligned}
g(s) &\leq g(\frac{t}{2^k}) + C\int_s^{\frac{t}{2^k}} g(2r)\,dr \\
&\leq 2^{N-2} g(t) \prod_{i=2}^{k}(1 + \frac{Ct}{2^i}) + \frac{Ct}{2^{k+1}} 2^{N-2} \prod_{i=2}^{k}(1 + \frac{Ct}{2^i}) \\
&= 2^{N-2} g(t) \prod_{i=2}^{k+1}(1 + \frac{Ct}{2^i}).
\end{aligned} \tag{A-21}$$

The conclusion then follows using the fact that

$$\prod_{i=1}^{m}(1+\frac{Ct}{2^i}) \leq \exp(Ct) \quad \text{for all } m \in \mathbb{N}_*.$$

Coming back to h, we obtain

$$h(s) \leq g(s) \leq 2^{N-2}\exp(Ct)g(t) = 2^{N-2}\exp(Ct)(h(t)+D/C),$$

and the proof is complete. □

Notice that whereas Lemma 4 was appropriate for balls of radius of the order of $1/|\log\varepsilon|$, Lemma A.6 is only appropriate for balls of radius larger than $O(\varepsilon^{\alpha/(N-1)})$. This is caused by the oscillation term or order $\varepsilon^\alpha/r^{N-1}$. Fortunately, these two conditions complement perfectly to obtain Proposition 2.

Proof of Proposition 2. We first consider the case

$$\theta r < \rho := (|\log\varepsilon|(c_\infty+1))^{-1} < r/2,$$

the other ones being easier to treat. Using Lemma 4, we deduce that

$$\tilde{E}_\varepsilon(\theta r) \leq C(\tilde{E}_\varepsilon(\rho) + \Lambda_0|\log\varepsilon|^3\varepsilon^2). \tag{A-22}$$

Next, using Lemma A.6 and the definition of ρ,

$$\begin{aligned} \tilde{E}_\varepsilon(\rho) \leq \bar{E}_\varepsilon(\rho) &\leq C(\bar{E}_\varepsilon(r/2) + \frac{\varepsilon^\alpha|\log\varepsilon|}{\rho^{N-1}} + \Lambda_0|\log\varepsilon|^4\varepsilon^2) \\ &\leq C(2^{N-2}\tilde{E}_\varepsilon(r) + \varepsilon^\alpha|\log\varepsilon|^N(c_\infty+1)^{N-1} + \Lambda_0|\log\varepsilon|^4\varepsilon^2). \end{aligned} \tag{A-23}$$

It suffices then to take $\beta = \alpha/2$ and combining (A-22) and (A-23) we get the desired estimate (23). In the case $\theta r \geq \rho$ (resp. $r \leq \rho$), it suffices to use Lemma A.6 (resp. Lemma 4) to obtain directly (23). This finishes the proof. □

Proof of Theorem 2.

Through a scaling, we first show that we can assume without loss of generality that $x_0 = 0,\ r = 1$ and $\Lambda_0 \leq 1$. Indeed, let

$$u(x) := w_\varepsilon(r(x-x_0)),$$

then u satisfies the equation

$$\Delta u + \frac{1}{\tilde{\varepsilon}^2}u(1-|u|^2) = i\tilde{c}\cdot\nabla u|\log\tilde{\varepsilon}| + \tilde{d}|\log\tilde{\varepsilon}|^2 u \tag{A-24}$$

on $B(0,2)$, where $\tilde{\varepsilon} := \varepsilon/r,\ \ \tilde{c}(x) := \vec{c}(r(x-x_0))r|\log\varepsilon|/|\log\tilde{\varepsilon}|$, and $\tilde{d}(x) := d(r(x-x_0))r^2|\log\varepsilon|^2/|\log\tilde{\varepsilon}|^2$. Since $r \geq \sqrt{\varepsilon}$, we have $|\log\varepsilon| \leq 2|\log\tilde{\varepsilon}|$ so that $\Lambda_0(\tilde{c},\tilde{d}) \leq 1$. We conclude noticing that $\tilde{E}_\varepsilon(w_\varepsilon, x_0, r) = \tilde{E}_{\tilde{\varepsilon}}(u, 0, 1)$.

From now on, we thus assume that $x_0 = 0$, $r = 1$ and $\Lambda_0 \le 1$. For the ease of presentation, we follow closely the lines of [10]. Let $0 < \delta < 1/32$ a constant to be determined later (and depending only on N), in the sequel we will denote by C generic constants not depending on the choice of δ.

Part A: Choosing a "good" radius.

Lemma A.8. *Assume that $0 < \varepsilon < \delta^{2(N-1)/\alpha}$. Then there exists some constant $C > 0$ and a radius $r_0 \in (\varepsilon^{\alpha/(2N-2)}, 1)$ such that*

- $\displaystyle \frac{1}{r_0^{N-2}} \int_{B_{r_0}} \frac{(a_\varepsilon - |w|^2)^2}{2\varepsilon^2} \le C(\eta|\log\delta| + \varepsilon^\beta),$
- $\tilde{E}_\varepsilon(r_0) - 2^{N-2}\tilde{E}_\varepsilon(\delta r_0) \le C(\eta|\log\delta| + \varepsilon^\beta).$

Proof. We will essentially make use of (A-14) together with a covering argument. First notice that

$$r_0 > \varepsilon^{\alpha/(2N-2)} \text{ implies } \frac{\varepsilon^\alpha}{r^{N-1}} \le \varepsilon^\beta \text{ for } r \ge r_0.$$

Hence, from (A-14) and following the lines of Lemma A.6 we obtain

$$\left| \frac{d}{dr}\bar{E}_\varepsilon(r) - A(r) \right| \le C\bar{E}_\varepsilon(2r) + C\varepsilon^\beta, \tag{A-25}$$

where

$$A(r) := \frac{1}{r^{N-2}} \int_1^2 t \int_{\partial B_{tr}} \left|\frac{\partial w}{\partial n}\right|^2 + \frac{1}{r^{N-1}} \int_{B_{2r}} \frac{(a_\varepsilon(x) - |w|^2)^2}{2\varepsilon^2} f(r, |x - x_0|).$$

From (A-25) and the monotonicity formula of Proposition 2 we thus infer that

$$\int_{\varepsilon^{\alpha/(2N-2)}}^{1/4} A(r) + C\bar{E}_\varepsilon(2r) + C\varepsilon^\beta \, dr \le C(\eta|\log\varepsilon| + \varepsilon^\beta). \tag{A-26}$$

Let k be the greatest integer such that $\varepsilon^{\alpha/(2N-2)}(\frac{\delta}{4})^{-k} \le \frac{1}{4}$, and define the intervals

$$I_j := \left(\varepsilon^{\alpha/(2N-2)}(\frac{\delta}{4})^{-j+1}, \varepsilon^{\alpha/(2N-2)}(\frac{\delta}{4})^{-j}\right) \qquad 1 \le j \le k.$$

Clearly, these intervals are disjoint and $\cup_{j=1}^k I_j \subset (\varepsilon^{\alpha/(2N-2)}, \frac{1}{4})$. Since

$$k \ge C^{-1}\frac{|\log\varepsilon|}{|\log\delta|}$$

we deduce form (A-26) that there exists some $j_0 \in \{1, \cdots, k\}$ such that

$$\int_{I_{j_0}} A(r) + C\bar{E}_\varepsilon(2r) + C\varepsilon^\beta \, dr \le C(\eta|\log\delta| + \varepsilon^\beta). \tag{A-27}$$

In particular, by the mean-value formula there exists some

$$r_0 \in \left(\frac{1}{2} \varepsilon^{\alpha/(2N-2)} (\frac{\delta}{4})^{-j}, \varepsilon^{\alpha/(2N-2)} (\frac{\delta}{4})^{-j} \right)$$

such that

$$\frac{1}{r_0^{N-2}} \int_{B_{r_0}} \frac{(a_\varepsilon(x) - |w|^2)^2}{2\varepsilon^2} \leq C(\eta |\log \delta| + \varepsilon^\beta),$$

which establishes the first claim. Notice that $\frac{\delta}{2} r_0 \in I_0$, hence

$$\begin{aligned} \tilde{E}_\varepsilon(r_0) - 2^{N-2} \tilde{E}_\varepsilon(\delta r_0) &\leq \bar{E}_\varepsilon(r_0) - \bar{E}_\varepsilon(\frac{\delta}{2} r_0) \\ &\leq \int_{I_{j_0}} A(r) + C\bar{E}_\varepsilon(2r) + C\varepsilon^\beta \, dr \qquad \text{(A-28)} \\ &\leq C(\eta |\log \delta| + \varepsilon^\beta). \end{aligned}$$

The lemma is proved. □

Part B: δ-Energy decay.

In this second part, we present an estimate valid for any solution u of (124) with $\Lambda_0 \leq 1$. We will apply it later in Part C to an appropriate dilation of w. Let $0 < \gamma < 1/8$ be constant to be determined later.

Lemma A.9. *There exist constants $\varepsilon_N > 0$ (depending only on γ and N) and $C > 0$ such that for any $0 < \varepsilon < \varepsilon_N$ and any solution u of (124) on $B(0,2)$ for some $\vec{c}$ and d satisfying $\Lambda_0(\vec{c}, d) \leq 1$ we have :*

$$E_\varepsilon(\delta) \leq C \Bigg((\gamma^2 + \delta^N + \gamma^{-4} \int_{B_1} \frac{(a_\varepsilon - |u|^2)^2}{\varepsilon^2}) E_\varepsilon(1) + \gamma^{-4} (\int_{B_1} \frac{(a_\varepsilon - |u|^2)^2}{\varepsilon^2} + \varepsilon^\beta) \Bigg).$$

Proof. The starting point is the identity

$$4|u|^2 |\nabla u|^2 = 4|u \times \nabla u|^2 + |\nabla |u|^2|^2, \qquad \text{(A-29)}$$

which holds for any map from $\mathbb{R}^N$ to $\mathbb{R}^k$; in the special case where $k = 2$, $|u(x_0)| \neq 0$, we may write near x_0

$$u(x) = \rho \exp(i\varphi),$$

and then

$$u \times \nabla u = \rho^2 \nabla \varphi,$$

i.e. $u \times \nabla u$ plays the role of the gradient of the phase. The advantage of the form (A-29) is that $u \times \nabla u$ is always globally well defined, while the phase need not to be well-defined when u vanishes somewhere.

Since u is a solution on $B(0,2)$, we infer from Lemma 3 that there exists ε_N depending only on N and γ and $C>0$ such that if $0<\varepsilon<\varepsilon_N$ then

$$\|u\|_\infty \leq 1+\gamma/2, \qquad \|\nabla u\|_\infty \leq \frac{C}{\varepsilon} \qquad \text{in } B(0,1). \tag{A-30}$$

By the mean-value inequality, we may find some $r_1 \in [\frac{1}{16}, \frac{1}{8}]$ such that

$$\begin{aligned} \int_{\partial B_{r_1}} |\nabla u|^2 &\leq 32 \int_{B_1} |\nabla u|^2, \\ \int_{\partial B_{r_1}} (a_\varepsilon - |u|^2)^2 &\leq 32 \int_{B_1} (a_\varepsilon - |u|^2)^2. \end{aligned} \tag{A-31}$$

We divide the estimate in several steps.

Step 1: Hodge-de Rham decomposition of $u \times \nabla u$.

Observe that since u is a solution of (124),

$$\begin{aligned} d^*(u \times du) = u \times \Delta u &= (u, \vec{c}\cdot \nabla u)|\log\varepsilon| \\ &= d^*\left((|u|^2-1)\textstyle\sum c_i(x)dx_i|\log\varepsilon|\right). \end{aligned} \tag{A-32}$$

Let ξ be the solution of the auxiliary Neumann problem

$$\begin{cases} \Delta\xi = 0 & \text{in } B_{r_1} \\ \frac{\partial\xi}{\partial n} = u \times \frac{\partial u}{\partial n} - (|u|^2-1)\vec{c}\cdot\vec{n}|\log\varepsilon| & \text{on } \partial B_{r_1}. \end{cases}$$

Notice that ξ exists since $\mathrm{div}(u\times\nabla u - (|u|^2-1)\vec{c}|\log\varepsilon|) = 0$ implies by integration $\int_{\partial B_{r_1}}(u\times\nabla u - (|u|^2-1)\vec{c}|\log\varepsilon|)\cdot n = 0$. Moreover, we have

$$\int_{B_{r_1}} |\nabla\xi|^2 \leq C\int_{B_1}|\nabla u|^2 + C\varepsilon^2|\log\varepsilon|^2\int_{B_1}\frac{(1-|u|^2)}{\varepsilon^2} \leq C(E_\varepsilon(1)+\varepsilon^\beta).$$

Since ξ is harmonic on B_{r_1}, we have by standard elliptic estimates, for $0<\delta\leq r_1$,

$$\int_{B_\delta}|\nabla\xi|^2 \leq C\delta^N\int_{B_{r_1}}|\nabla\xi|^2 \leq C\delta^N(E_\varepsilon(1)+\varepsilon^\beta). \tag{A-33}$$

By construction we verify that

$$d^*[(u\times du) - (|u|^2-1)\textstyle\sum c_i(x)dx_i|\log\varepsilon| - d\xi)1_{B_{r_1}}] = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N),$$

where 1_A denotes the characteristic function of the set A. By classical Hodge theory (see [10] Proposition A.7) there exists some 2-form φ on $\mathbb{R}^N$ such that $\varphi\in H^1_{loc}(\mathbb{R}^N)$ and

$$d^*\varphi = (u\times du - (|u|^2-1)\textstyle\sum c_i(x)dx_i|\log\varepsilon| - d\xi)\,1_{B_{r_1}} \text{ in } \mathcal{D}'(\mathbb{R}^N), \tag{A-34}$$

$$d\varphi = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N), \tag{A-35}$$

$$\|\nabla\varphi\|_{L^2(\mathbb{R}^N)} \leq C\left(E_\varepsilon(r_1) + \|\nabla\xi\|_{L^2(B_{r_1})}\right), \tag{A-36}$$

$$|\varphi(x)|\cdot|x|^{N-1} \text{ tends to zero at infinity.} \tag{A-37}$$

We therefore have

$$u \times du = d^*\varphi + d\xi + (|u|^2 - 1)\textstyle\sum c_i(x)dx_i|\log\varepsilon| \quad \text{in } B_{r_1}\,. \tag{A-38}$$

In order to bound the L^2-norm of $u \times du$ on B_δ, we turn next to estimates for $d^*\varphi$.

Step 2: Improved estimates for $\nabla\varphi$ on B_δ.

Let $f : \mathbb{R}^+ \to (1, \frac{1}{1-\gamma})$ be any smooth function such that

$$\begin{cases} f(t) = \frac{1}{t} & \text{if } t \geq 1-\gamma \\ f(t) = 1 & \text{if } t \leq 1-2\gamma \\ |f'(t)| \leq 4 & \text{for any } t \in \mathbb{R}^+\,. \end{cases}$$

Define on $\mathbb{R}^N$ the function τ by

$$\tau(x) = \begin{cases} f^2(|u(x)|) & \text{in } B_{r_1} \\ 1 & \text{outside,} \end{cases}$$

so that, taking (A-30) into account,

$$0 \leq \tau - 1 \leq 4\gamma \quad \text{in } \mathbb{R}^N. \tag{A-39}$$

Notice that

$$f^2(|u|)u \times du = f(|u|)u \times d(f(|u|)u),$$

hence

$$d(\tau u \times du) = d(f^2(|u|)u \times du) = d(f(|u|)u \times d(f(|u|)u)) \quad \text{in } B_{r_1},$$

i.e.

$$d(\tau u \times du) = \sum_{i<j} 2(f(|u|)u)_{x_i} \times (f(|u|)u)_{x_j} dx_i \wedge dx_j.$$

Now we turn to φ. We have

$$\begin{aligned} -\Delta\varphi = dd^*\varphi &= d(1_{B_{r_1}}\tau u \times du) - d(1_{B_{r_1}}d\xi) - d\Big(1_{B_{r_1}}(|u|^2 - 1) \\ &\qquad \textstyle\sum c_i dx_i|\log\varepsilon|\Big) + d(1_{B_{r_1}}(1-\tau)u \times du) \qquad \text{in } \mathcal{D}'(\mathbb{R}^N) \\ &= \omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5, \end{aligned}$$

where

$$\begin{aligned} \omega_1 &= 1_{B_{r_1}} d(\tau u \times du) = 1_{B_{r_1}} \textstyle\sum_{i<j} 2(f(|u|)u)_{x_i} \times (f(|u|)u)_{x_j} dx_i \wedge dx_j\,, \\ \omega_2 &= \sigma_{\partial B_{r_1}} f(|u|)u \times du \wedge dr\,, \qquad (r = |x|)\,, \\ \omega_3 &= -d(1_{B_{r_1}}d\xi) = \sigma_{\partial B_{r_1}} dr \wedge d\xi, \\ \omega_4 &= -d(1_{B_{r_1}}(|u|^2 - 1)\textstyle\sum c_i dx_i|\log\varepsilon|), \\ \omega_5 &= d(1_{B_{r_1}}(1-\tau)u \times du). \end{aligned}$$

Here $\sigma_{\partial B_{r_1}}$ stands for the surface measure on ∂B_{r_1}. Set $\varphi_i := G * \omega_i$, where $G(x) := c_N|x|^{2-N}$ is the fundamental solution of $-\Delta$ in $\mathbb{R}^N$. Since φ tends to zero at infinity by (A-37) and each φ_i tends to zero at infinity (because each ω_i has compact support), we conclude that

$$\varphi = \sum_{i=1}^{5} \varphi_i \ .$$

We now proceed to estimate separately each φ_i.

Estimate for φ_5. We have

$$\int_{\mathbb{R}^N} |\nabla\varphi_5|^2 \leq C\gamma^2 \int_{B_1} |\nabla u|^2. \tag{A-40}$$

Indeed, we have

$$-\Delta\varphi_5 = \omega_5 = d(1_{B_{r_1}}(1-\tau)u \times du).$$

Multiplying by φ_5 and integrating we obtain

$$\int_{\mathbb{R}^N} |\nabla\varphi_5|^2 \leq \|1-\tau\|_{L^\infty(B_1)}\|u\|_{L^\infty(B_1)}\|\nabla u\|_{L^2(B_1)}\|\nabla\varphi_5\|_{L^2},$$

and thus

$$\int_{\mathbb{R}^N} |\nabla\varphi_5|^2 \leq C\gamma\|\nabla u\|_{L^2(B_1)}\|\nabla\varphi_5\|_{L^2(\mathbb{R}^N)},$$

by (A-30) and (A-39), which yields the result.

Estimate for φ_4. We have

$$\int_{\mathbb{R}^N} |\nabla\varphi_4|^2 \leq C\int_{B_1} \frac{(1-|u|^2)^2}{\varepsilon^2} \leq C(\int_{B_1} \frac{(a_\varepsilon-|u|^2)^2}{\varepsilon^2} + \varepsilon^\beta). \tag{A-41}$$

Indeed, we have

$$-\Delta\varphi_4 = \omega_4 = -d(1_{B_{r_1}}(|u|^2-1)\sum c_i dx_i|\log\varepsilon|).$$

Multiplying by φ_4 and integrating we obtain

$$\int_{\mathbb{R}^N} |\nabla\varphi_4|^2 \leq \varepsilon|\log\varepsilon|\cdot\|\vec{c}\|_{L^\infty(B_1)}(\int_{B_1} \frac{(1-|u|^2)^2}{\varepsilon^2})^{1/2}\|\nabla\varphi_4\|_{L^2},$$

which yields the result since $\Lambda_0 \leq 1$.

Estimate for φ_3. We have

$$\int_{B_\delta} |\nabla\varphi_3|^2 \leq C\delta^N(E_\varepsilon(1)+\varepsilon^\beta). \tag{A-42}$$

Indeed, we have

$$-\Delta\varphi_3 = \omega_3 = -d(1_{B_{r_1}}d\xi).$$

Multiplying by φ_3 and integrating we obtain

$$\int_{\mathbb{R}^N} |\nabla \varphi_3|^2 \leq \|\nabla \xi\|_{L^2(B_{r_1})} \|\nabla \varphi_3\|_{L^2}.$$

Since φ_3 is harmonic on B_{r_1} ($r_1 \geq 1/16$), we also have

$$\|\nabla \varphi_3\|_{L^\infty(B_{\frac{1}{32}})} \leq C \|\nabla \varphi_3\|_{L^2(B_{r_1})},$$

so that ($\delta \leq 1/32$)

$$\int_{B_\delta} |\nabla \varphi_3|^2 \leq C\delta^N \|\nabla \xi\|^2_{L^2(B_{r_1})} \leq C\delta^N (E_\varepsilon(1) + \varepsilon^\beta).$$

Estimate for φ_2. We have

$$\int_{B_\delta} |\nabla \varphi_2|^2 \leq C\delta^N \int_{B_1} |\nabla u|^2. \qquad \text{(A-43)}$$

Indeed, we have

$$-\Delta \varphi_2 = \omega_2 = \sigma_{\partial B_{r_1}} f(|u|) u \times du \wedge dr.$$

By standard elliptic estimates for harmonic functions with measure data it holds

$$\|\nabla \varphi_2\|_{L^\infty(B(1/32))} \leq C\|\omega_2\| \leq C(\int_{\partial B_{r_1}} |\nabla u|^2)^{1/2},$$

so that using (A-31) we finally obtain

$$\int_{B_\delta} |\nabla \varphi_2|^2 \leq C\delta^N \int_{B_1} |\nabla u|^2.$$

Estimate for φ_1. We start with the crucial observation that

$$|\omega_1| \leq C\gamma^{-2} \frac{(a_\varepsilon - |u|^2)^2}{\varepsilon^2} \quad \text{in } B_1. \qquad \text{(A-44)}$$

Indeed, we have to distinguish the two regions

$$V_\gamma = \{x \in B_1;\, |u(x)| \geq 1 - \gamma\}, \qquad W_\gamma = \{x \in B_1;\, |u(x)| \leq 1 - \gamma\}.$$

Recall that

$$\omega_1 = 1_{B_{r_1}} d(\tau u \times du) = 1_{B_{r_1}} \sum_{i<j} 2(f(|u|)u)_{x_i} \times (f(|u|)u)_{x_j} dx_i \wedge dx_j.$$

On V_γ we have $f(|u(x)|) = \frac{1}{|u(x)|}$ and therefore

$$(f(|u|)u)_{x_i} \times (f(|u|)u)_{x_j} = 0, \quad \text{for } i \neq j.$$

On W_γ we have, by (A-30)

$$|(f(|u|)u)_{x_i}| \leq \frac{C}{\varepsilon},$$

so that

$$|\omega_1| \leq \frac{C}{\varepsilon^2} \leq \frac{C}{\varepsilon^2}\gamma^{-2}\gamma^2 \leq \frac{C}{\varepsilon^2}\gamma^{-2}(1-|u|)^2 \leq C\gamma^{-2}\frac{(1-|u|^2)^2}{\varepsilon^2}.$$

Decreasing ε_N if necessary, we have

$$(1-|u|^2)^2 \leq 2(a_\varepsilon - |u|^2)^2 \qquad \text{on } W_\gamma,$$

which yields (A-44).

The final crucial estimate is

$$\|\varphi_1\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C}{\gamma^2}(E_\varepsilon(0,1)+\varepsilon^\beta)\,. \tag{A-45}$$

Indeed,

$$\varphi_1(x) = \int_{\mathbb{R}^N} \frac{c_N}{|x-y|^{N-2}}\omega_1(y)dy = \int_{B_{r_1}} \frac{c_N}{|x-y|^{N-2}}\omega_1(y)dy,$$

so that

$$|\varphi_1(x)| \leq \frac{C}{\gamma^2}\int_{B_{r_1}} \frac{(a_\varepsilon - |u(y)|^2)^2}{\varepsilon^2|x-y|^{N-2}}dy.$$

Assume $|x| \leq r_1 \leq 1/8$. Since $B_{r_1} \subset B_{\frac{1}{4}}(x)$ we have

$$|\varphi_1(x)| \leq \frac{C}{\gamma^2}\int_{B_{\frac{1}{4}}(x)} \frac{(a_\varepsilon - |u(y)|^2)^2}{\varepsilon^2|x-y|^{N-2}}dy.$$

Next, we observe that

$$\begin{aligned}\int_{B_{\frac{1}{4}}(x)} \frac{(a_\varepsilon - |u(y)|^2)^2}{\varepsilon^2|x-y|^{N-2}}dy &= \int_0^{\frac{1}{4}} \frac{1}{r^{N-2}}\int_{\partial B_r} \frac{(a_\varepsilon - |u|^2)^2}{\varepsilon^2}dr \\ &= (N-2)\int_0^{\frac{1}{4}} \frac{1}{r^{N-1}}\int_{B_r} \frac{(a_\varepsilon - |u|^2)^2}{\varepsilon^2}\,dr + \left[\frac{1}{r^{N-2}}\int_{B_r} \frac{(a_\varepsilon - |u|^2)^2}{\varepsilon^2}\,dr\right]_0^{\frac{1}{4}}.\end{aligned} \tag{A-46}$$

Using the monotonicity formulae (A-8) when $r \in (0, 1/|\log\varepsilon|)$ and (A-14) when $r \in (1/|\log\varepsilon|, 1/4)$, together with the estimates in Lemmas 4 and A.6, we thus infer that

$$\int_{B_{\frac{1}{4}}(x)} \frac{(a_\varepsilon - |u(y)|^2)^2}{\varepsilon^2|x-y|^{N-2}}dy \leq C(\tilde{E}_\varepsilon(x,\frac{1}{2})+\varepsilon^\beta) \leq C(E_\varepsilon(0,1)+\varepsilon^\beta), \tag{A-47}$$

since $B(x,\frac{1}{2}) \subset B(0,1)$. Hence for every $x \in B_{r_1}$

$$|\varphi_1(x)| \leq C\gamma^{-2}(E_\varepsilon(0,1)+\varepsilon^\beta).$$

Recall that $\Delta\varphi_1 = 0$ outside B_{r_1}, so that by the maximum principle

$$\|\varphi_1\|_{L^\infty(\mathbb{R}^N)} = \|\varphi_1\|_{L^\infty(B_{r_1})} \leq C\gamma^{-2}(E_\varepsilon(0,1) + \varepsilon^\beta),$$

which is (A-45).

Going back to the equation

$$-\Delta\varphi_1 = \omega_1 \quad \text{ in } \mathbb{R}^N \,,$$

we conclude

$$\int_{\mathbb{R}^N} |\nabla\varphi_1|^2 \leq \|\varphi_1\|_{L^\infty(\mathbb{R}^N)} \int_{B_{r_1}} |\omega_1|,$$

so that

$$\int_{\mathbb{R}^N} |\nabla\varphi_1|^2 \leq C\gamma^{-4} \int_{B_1} \frac{(a_\varepsilon - |u|^2)^2}{\varepsilon^2} (E_\varepsilon(0,1) + \varepsilon^\beta). \tag{A-48}$$

We gather now the different estimates for $\varphi_1, \cdots, \varphi_5$ to obtain

$$\int_{\mathbb{R}^N} |\nabla\varphi|^2 \leq C\left((\gamma^2 + \delta^N + \gamma^{-4} \int_{B_1} \frac{(a_\varepsilon - |u|^2)^2}{\varepsilon^2}) E_\varepsilon(1) \right. \\ \left. + \gamma^{-4} (\int_{B_1} \frac{(a_\varepsilon - |u|^2)^2}{\varepsilon^2} + \varepsilon^\beta) \right). \tag{A-49}$$

Step 3: Improved estimates for $\nabla(|u|^2)$ on B_δ.

The equation for $|u|^2$ reads

$$\Delta(|u|^2) + 2\frac{(a_\varepsilon - |u|^2)|u|^2}{\varepsilon^2} = 2|\nabla u|^2 + 2|\log\varepsilon|(i\vec{c}\cdot\nabla u, u).$$

Multiplying by $a_\varepsilon - |u|^2$ and integrating on B_{r_1} we obtain

$$\int_{B_{r_1}} |\nabla|u|^2|^2 + 2\frac{(a_\varepsilon - |u|^2)^2|u|^2}{\varepsilon^2} = 2\int_{B_{r_1}} (a_\varepsilon - |u|^2)|\nabla u|^2 \\ + \int_{\partial B_{r_1}} (a_\varepsilon - |u|^2)\frac{\partial|u|^2}{\partial n} + \int_{B_{r_1}} \nabla|u|^2 \cdot \nabla a_\varepsilon \\ + \int_{B_{r_1}} 2|\log\varepsilon|(i\vec{c}\cdot\nabla u, u)(a_\varepsilon - |u|^2). \tag{A-50}$$

From (A-31) we deduce

$$\left| \int_{\partial B_{r_1}} (a_\varepsilon - |u|^2)\frac{\partial|u|^2}{\partial n} \right| \leq C\varepsilon \left(\int_{B_1} \frac{(a_\varepsilon - |u|^2)^2}{\varepsilon^2} \right)^{1/2} \left(\int_{B_1} |\nabla u|^2 \right)^{1/2}. \tag{A-51}$$

We also have

$$\left|\int_{B_{r_1}}(a_\varepsilon-|u|^2)|\nabla u|^2\right| \le C\int_{V_{\gamma^2}}\gamma^2|\nabla u|^2 + C\gamma^{-2}\int_{W_{\gamma^2}}\frac{(a_\varepsilon-|u|^2)^2}{\varepsilon^2} \\ \le C\gamma^2\int_{B_1}|\nabla u|^2 + C\gamma^{-2}\int_{B_1}\frac{(a_\varepsilon-|u|^2)^2}{\varepsilon^2}. \tag{A-52}$$

On the other hand,

$$\left|\int_{B_{r_1}} 2|\log\varepsilon|(i\vec{c}\cdot\nabla u, u)(a_\varepsilon-|u|^2)\right| \\ \le C\varepsilon|\log\varepsilon|\left(\int_{B_1}\frac{(a_\varepsilon-|u|^2)^2}{\varepsilon^2}\right)^{1/2}\left(\int_{B_1}|\nabla u|^2\right)^{1/2}, \tag{A-53}$$

and

$$\left|\int_{B_{r_1}}\nabla|u|^2\cdot\nabla a_\varepsilon\right| \le \frac{1}{2}\int_{B_{r_1}}|\nabla|u|^2|^2 + 2\varepsilon^4|\log\varepsilon|^4\int_{B_{r_1}}|\nabla d|^2. \tag{A-54}$$

Inserting (A-51),(A-52),(A-53) and (A-54) in (A-50) we finally obtain the estimate

$$\int_{B_{r_1}}|\nabla|u|^2|^2 \le C\left(\gamma^2\int_{B_1}|\nabla u|^2 + \gamma^{-2}\int_{B_1}\frac{(a_\varepsilon-|u|^2)^2}{\varepsilon^2} + \varepsilon^\beta\right). \tag{A-55}$$

Step 4: Proof of Lemma A.9 completed.

Recall that

$$4|u|^2|\nabla u|^2 = 4|u\times\nabla u|^2 + |\nabla|u|^2|^2,$$

and thus

$$(3+a_\varepsilon)|\nabla u|^2 = 4|u\times\nabla u|^2 + |\nabla|u|^2|^2 + 4(a_\varepsilon-|u|^2)|\nabla u|^2 \\ \le 8(|\nabla\varphi|^2 + |\nabla\xi|^2 + (1-|u|^2)^2|\textstyle\sum c_i(x)dx_i|^2|\log\varepsilon|^2) \\ + |\nabla|u|^2|^2 + 4(a_\varepsilon-|u|^2)|\nabla u|^2,$$

by (A-38). Combining (A-49), (A-33), (A-55), (A-52) and the easy estimate

$$\int_{B_\delta}(1-|u|^2)^2|\textstyle\sum c_i(x)dx_i|^2|\log\varepsilon|^2 \le C(\varepsilon^2|\log\varepsilon|^2\displaystyle\int_{B_1}\frac{(a_\varepsilon-|u|^2)^2}{\varepsilon^2} + \varepsilon^\beta),$$

we finally obtain

$$E_\varepsilon(\delta) \le C\Bigg((\gamma^2+\delta^N+\gamma^{-4}\int_{B_1}\frac{(a_\varepsilon-|u|^2)^2}{\varepsilon^2})E_\varepsilon(1) \\ +\gamma^{-4}(\int_{B_1}\frac{(a_\varepsilon-|u|^2)^2}{\varepsilon^2}+\varepsilon^\beta)\Bigg),$$

which is the desired estimate. This ends the proof. □

Part C: Proof of Theorem 2 completed.

Remember that we are concerned with a solution w of (124) with $\Lambda_0 \le 1$ on B_1 satisfying the estimate

$$E_\varepsilon(w,0,1) \le \eta|\log\varepsilon|. \tag{A-56}$$

Recall also that in Part A we have exhibited some $r_0 \in (\varepsilon^{\alpha/(2N-2)}, 1)$, such that

$$\frac{1}{r_0^{N-2}} \int_{B_{r_0}} \frac{(a_\varepsilon - |w|^2)^2}{2\varepsilon^2} \le C(\eta|\log\delta| + \varepsilon^\beta), \tag{A-57}$$

$$\tilde{E}_\varepsilon(r_0) - 2^{N-2}\tilde{E}_\varepsilon(\delta r_0) \le C(\eta|\log\delta| + \varepsilon^\beta), \tag{A-58}$$

where δ is fixed but to be determined later. The function $u(x) := w(r_0 x)$ defined on B_1 satisfies the equation

$$\Delta u + \frac{1}{\tilde{\varepsilon}^2} u(1-|u|^2) = i\tilde{c}\cdot\nabla u|\log\tilde{\varepsilon}| + \tilde{d}|\log\tilde{\varepsilon}|^2 u,$$

where $\tilde{\varepsilon} := \varepsilon/r_0$ and $\Lambda_0(\tilde{c},\tilde{d}) \le 1$. Since $r_0 \ge \varepsilon^{\alpha/(2N-2)}$, we have $\tilde{\varepsilon} \le \varepsilon^{1/2}$. By scaling we also have the identities

$$E_{\tilde{\varepsilon}}(u,0,1) = \tilde{E}_\varepsilon(w,0,r_0),$$
$$E_{\tilde{\varepsilon}}(u,0,\delta) = \frac{1}{r_0^{N-2}} E_\varepsilon(w,0,\delta r_0) = \delta^{2-N}\tilde{E}_\varepsilon(w,0,\delta r_0),$$

and

$$\int_{B_1} \frac{(a_{\tilde{\varepsilon}} - |u|^2)^2}{\tilde{\varepsilon}^2} = \frac{1}{r_0^{N-2}} \int_{B_{r_0}} \frac{(a_\varepsilon - |w|^2)^2}{\varepsilon^2}.$$

We now apply Lemma A.9 to u, and using the previous identities we find

$$\frac{1}{r_0^{N-2}} E_\varepsilon(\delta r_0) \le C((\gamma^2 + \delta^N + \gamma^{-4}\frac{1}{r_0^{N-2}} \int_{B_{r_0}} \frac{(a_\varepsilon - |w|^2)^2}{\varepsilon^2})\tilde{E}_\varepsilon(r_0)$$
$$+ \gamma^{-4}(\frac{1}{r_0^{N-2}} \int_{B_{r_0}} \frac{(a_\varepsilon - |w|^2)^2}{\varepsilon^2} + \varepsilon^\beta)).$$

Using (A-57) and (A-58) we obtain

$$\begin{aligned}\tilde{E}_\varepsilon(r_0) &\le 2^{N-2}\tilde{E}_\varepsilon(\delta r_0) + C(\eta|\log\delta| + \varepsilon^\beta) \\ &\le C\delta^{N-2}\left(\gamma^2 + \gamma^{-4}(\eta|\log\delta| + \varepsilon^\beta)\right)\tilde{E}_\varepsilon(r_0) + C\delta^2\tilde{E}_\varepsilon(r_0) \\ &\quad + C\gamma^{-4}(\eta|\log\delta| + \varepsilon^\beta).\end{aligned}$$

We now fix the values of δ and γ. First, choose δ small enough so that

$$C\delta^2 \le 1/4.$$

Next, choose γ small enough so that

$$C\delta^{N-2}\gamma^2 \le 1/4.$$

There exist also ε_N and η_N such that if $\varepsilon < \varepsilon_N$ and $\eta \le \eta_N$ we have

$$C\delta^{N-2}\gamma^{-4}(\eta|\log\delta| + \varepsilon^\beta) \le 1/4.$$

Hence,

$$\tilde{E}_\varepsilon(r_0) \le C\gamma^{-4}(\eta|\log\delta| + \varepsilon^\beta) \quad \text{for } \varepsilon < \varepsilon_N,\ \eta < \eta_N. \tag{A-59}$$

Using the monotonicity formula of Proposition 2, we thus obtain

$$\begin{aligned}\frac{1}{\varepsilon^N}\int_{B_\varepsilon}(1-|u|^2)^2 &\le C(\frac{1}{\varepsilon^N}\int_{B_\varepsilon}(a_\varepsilon - |u|^2)^2 + \Lambda_0^2\varepsilon^\beta) \le C(\tilde{E}_\varepsilon(\varepsilon) + \Lambda_0^2\varepsilon^\beta)\\ &\le C(\tilde{E}_\varepsilon(r_0) + \Lambda_0^2\varepsilon^\beta) \le C\gamma^{-4}(\eta|\log\delta| + \Lambda_0^2\varepsilon^\beta).\end{aligned}$$

The conclusion then follows form the next lemma taken from [10]. □

Lemma A.10. *Let w be a solution of (124) on B_1. Then*

$$1 - |w(0)| \le C\left(\frac{1}{\varepsilon^N}\int_{B_\varepsilon}(1-|w|^2)^2\right)^{1/(N+2)}.$$

Proof. Set $k = |w(0)|$ and assume that $k \le 1$ (otherwise there is nothing to be proved). By (A-30) we have

$$|w(x) - w(0)| \le \frac{C}{\varepsilon}|x| \le 1 - \frac{k}{2},$$

provided $|x| \le \frac{\varepsilon(1-k)}{2C} \equiv \lambda$. Therefore $|w(x)| \le \frac{1+k}{2}$ on B_λ. We distinguish two cases.
Case 1: $\lambda < \varepsilon$. Then

$$\int_{B_\lambda}(1-|w|^2)^2 \le \int_{B_\varepsilon}(1-|w|^2)^2.$$

On the other hand

$$\int_{B_\lambda}(1-|w|^2)^2 \ge \int_{B_\lambda}(1-|w|)^2 \ge (\frac{1-k}{2})^2|B_\lambda| = C\varepsilon^N(1-k)^{N+2},$$

by definition of λ. Consequently

$$(1-k)^{N+2} \le \frac{C}{\varepsilon^N}\int_{B_\varepsilon}(1-|w|^2)^2,$$

and the conclusion follows.
Case 2: $\lambda \ge \varepsilon$. Then

$$|w(x)| \le \frac{1+k}{2} \quad \text{in } B_\varepsilon,$$

and

$$\int_{B_\varepsilon}(1-|w|^2)^2 \ge \left(\frac{1-k}{2}\right)^2|B_\varepsilon|.$$

Therefore

$$(1-k)^{N+2} \le (1-k)^2 \le \frac{C}{\varepsilon^N}\int_{B_\varepsilon}(1-|w|^2)^2,$$

and the lemma is proved. □

Corollary A.1. *Let $0 < \sigma < 1$ and the corresponding $\eta > 0$ and $\varepsilon_0 > 0$ given by Theorem 2. Let $x_0 \in \Omega$ and $r > 0$ such that $B(x_0, 2r) \subset \Omega$ and $4\sqrt{\varepsilon} < r < 4/(1+\Lambda_0)$. Then for all $\varepsilon < \varepsilon_0$, if w is a solution of (124) in Ω and*

$$\tilde{E}_\varepsilon(x_0, r) \leq 4^{2-N}\eta|\log\varepsilon| \tag{A-60}$$

then

$$|1 - |w(x)|\,| \leq \sigma \qquad \text{for all } x \in B(x_0, 3r/4). \tag{A-61}$$

Proof. If $x \in B(x_0, 3r/4)$, then $B(x, r/4) \subset B(x_0, r)$ so that

$$\tilde{E}_\varepsilon(x, r/4) = 4^{N-2}\frac{1}{r^{N-2}}E_\varepsilon(x, r/4) \leq 4^{N-2}\tilde{E}_\varepsilon(x_0, r) \leq \eta|\log\varepsilon|,$$

and the conclusion follows by Theorem 2. □

Concerning the asymptotic of the potential part in the energy, namely

$$\int_\Omega \frac{(a_\varepsilon(x) - |w|^2)^2}{\varepsilon^2},$$

it is tempting to believe that it remains bounded as $\varepsilon \to 0$ (at least away from the boundary). We have no proof of that fact, however the following Proposition holds.

Proposition A.1. *Let $K \subset \Omega$ be a compact subset and w a solution of (A.1) satisfying (24). Then,*

$$\int_K \frac{(a_\varepsilon(x) - |w|^2)^2}{\varepsilon^2} \leq C\, r(\varepsilon)|\log\varepsilon|, \tag{A-62}$$

where $r(\varepsilon) \to 0$ as $\varepsilon \to 0$ and C depends only on M_0.

Proof. Let $\rho := |w|$. If w verifies (124) then ρ verifies

$$-\Delta\rho^2 + 2|\nabla w|^2 = \frac{2}{\varepsilon^2}\rho^2(a_\varepsilon - \rho^2) - (w, i\vec{c}\cdot\nabla w)|\log\varepsilon|. \tag{A-63}$$

Let $0 < \sigma < \frac{1}{2}$. Define $A := \{x \in K,\ \rho(x) > 1 - \sigma\}$, and $\bar{\rho} := \max(\rho, 1 - \sigma)$, so that $\rho = \bar{\rho}$ on A. Let also $\zeta \in \mathcal{D}(\Omega)$ such that $0 \leq \zeta \leq 1$ on Ω, $\zeta \equiv 1$ on K, and $|\nabla\zeta| \leq C$, where C depends only on K. Multiplying equation (A-63) by $\zeta(\bar{\rho}^2 - 1)$ (which is compactly supported in Ω), and integrating over Ω we obtain

$$\begin{aligned}\int_\Omega \nabla\rho^2\nabla\bar{\rho}^2\zeta + \int_\Omega \frac{2\rho(1-\rho^2)(1-\bar{\rho}^2)}{\varepsilon^2}\zeta &= \int_\Omega (1-\bar{\rho}^2)|\nabla w|^2 \\ &+ \int_\Omega \nabla\rho^2\nabla\zeta(1-\bar{\rho}^2) + \int_\Omega 2\rho|\log\varepsilon|^2 d(x)(1-\bar{\rho}^2)\zeta \\ &+ \int_\Omega (w, i\vec{c}\cdot\nabla w)(\bar{\rho}^2 - 1)\zeta|\log\varepsilon|.\end{aligned}$$

It follows that

$$\begin{aligned}
&\int_\Omega \frac{2\rho(1-\rho^2)(1-\bar\rho^2)}{\varepsilon^2}\zeta\rho^2\nabla\bar\rho^2 \\
&\leq 2\sigma\int_\Omega |\nabla w|^2 + C\sigma\int_\Omega |\nabla\rho||a_\varepsilon-\rho^2| + C\Lambda_0 M_0\varepsilon|\log\varepsilon|^2 \\
&\leq 2\sigma\int_\Omega |\nabla w|^2 + C\sigma\left[\int_\Omega |\nabla\rho|^2 + \int_\Omega \frac{(a_\varepsilon-\rho^2)^2}{4\varepsilon^2}\right] + C\Lambda_0 M_0\varepsilon|\log\varepsilon|^2,
\end{aligned}$$

hence, since $\rho \geq 1/2$ and $\zeta = 1$ on A, we obtain

$$\int_A \frac{(a_\varepsilon-\rho^2)^2}{\varepsilon^2} \leq C\sigma E_\varepsilon(w) + C\Lambda_0 M_0\varepsilon|\log\varepsilon|^2. \tag{A-64}$$

Define also $B := K \setminus A$. We claim that

$$\int_B \frac{(a_\varepsilon-\rho^2)^2}{\varepsilon^2} \leq C, \tag{A-65}$$

where C depends only on σ, M_0 and K. The proof of (A-65) follows from Theorem 2, the monotonicity formula in Lemma 4 and Besicovitch covering theorem, following the same outlines as the proof of Proposition 1 in [10]. Indeed, only the afore mentioned ingredients are used and hence the proof there applies also to our equation]. In particular, we infer from (A-65) that there exists $\varepsilon_\sigma > 0$ such that

$$\int_B \frac{(a_\varepsilon-\rho^2)^2}{\varepsilon^2} \leq \sigma|\log\varepsilon|, \tag{A-66}$$

for all $0 < \varepsilon < \varepsilon_\sigma$, where ε_σ depends only on σ, M_0 and K. Combining (A-64) and (A-66) we finally obtain

$$\int_K \frac{(a_\varepsilon(x)-|w|^2)^2}{\varepsilon^2} \leq C\sigma|\log\varepsilon| \qquad \text{for } 0 < \varepsilon < \varepsilon_\sigma.$$

Clearly we can assume that the mapping $t : \sigma \mapsto \varepsilon_\sigma$ is strictly increasing. The function $r := t^{-1}$ fulfills the statement of the proposition, so that the proof is complete. □

Appendix B : Properties of the concentration set Σ_μ

Recall that

$$\Sigma_\mu = \{x \in \Omega, \Theta_*(\mu_*, x) > 0\,.$$

The purpose of this section is to describe and prove the properties of Σ_μ stated in Theorem 3. We first have

Lemma B.11. *There exists $\eta_0 > 0$ such that if $x_0 \in \Sigma_\mu$, then*

$$\Theta_*(\mu_*, x_0) \geq \eta_0\,.$$

Proof. Let $\sigma > 0$ to be determined later, and let $\eta > 0$ and $\varepsilon_0 > 0$ be the corresponding constants provided by Theorem 2. Set

$$\eta_0 = 4^{2-N}\eta\,.$$

Assume by contradiction that

$$\Theta_*(x_0) < 4^{2-N}\eta. \tag{B-1}$$

Then for each $r_0 > 0$ there exists $0 < r < r_0$ such that $B(x_0, 2r) \subset \Omega$ and $\varepsilon_1 \leq \min(\varepsilon_0, r^2/16)$ such that

$$\tilde{E}_\varepsilon(x_0, r) < 4^{2-N}\eta|\log\varepsilon| \qquad \forall\varepsilon \leq \varepsilon_1. \tag{B-2}$$

From Corollary A.1 we thus infer that

$$|1 - |w(x)|\,| \leq \sigma \qquad \forall x \in B(x_0, 3r/4).$$

We write

$$w(x) = \rho(x)\exp(i\varphi(x)) \qquad \text{in } B(x_0, 3r/4).$$

The phase φ satisfies the equation

$$-\Delta\varphi = -\mathrm{div}((1-\rho^2)\nabla\varphi) + \frac{1}{2}|\log\varepsilon|\vec{c}\cdot\nabla(\rho^2 - 1) \qquad \text{in } B(x_0, 3r/4). \tag{B-3}$$

Let $\tilde{\varphi}$ be the harmonic function defined on $B(x_0, 3r/4)$ such that $\tilde{\varphi} = \varphi$ on the boundary of $B(x_0, 3r/4)$. In particular, we have

$$\int_{B(x_0,3r/4)} |\nabla\tilde{\varphi}|^2 \leq \int_{B(x_0,3r/4)} |\nabla\varphi|^2$$

and for all $\delta > 0$,

$$\int_{B(x_0,\delta 3r/4)} |\nabla\tilde{\varphi}|^2 \leq C\delta^N \int_{B(x_0,3r/4)} |\nabla\tilde{\varphi}|^2 \leq C\delta^N \int_{B(x_0,3r/4)} |\nabla\varphi|^2. \tag{B-4}$$

Multiplying equation (B-3) by $\varphi - \tilde{\varphi}$ and integrating over $B(x_0, 3r/4)$ we obtain, similarly as in the proof of Theorem 2,

$$\int_{B(x_0,\delta 3r/4)} |\nabla(\varphi - \tilde{\varphi})|^2 \leq C(\sigma + \Lambda_0\varepsilon|\log\varepsilon|)E_\varepsilon(x_0, 3r/4). \tag{B-5}$$

Combining (B-4) and (B-5) we finally obtain

$$\int_{B(x_0,\delta 3r/4)} |\nabla\varphi|^2 \leq C(\delta^N + \sigma + \Lambda_0\varepsilon|\log\varepsilon|)E_\varepsilon(x_0, 3r/4). \tag{B-6}$$

Concerning the modulus, let $\xi_r \in \mathcal{D}(B(x_0, 3r/4), [0,1])$ such that $\xi \equiv 1$ on $B(x_0, 3r/8)$ and $|\nabla\xi_r| \leq C/r$, multiplying the equation

$$-\Delta\rho + \rho|\nabla\varphi|^2 = \frac{1}{\varepsilon^2}\rho(a_\varepsilon - \rho^2) + |\log\varepsilon|\rho\vec{c}\cdot\nabla\varphi$$

by $\xi_r(1-\rho)$ and integrating over $B(x_0, 3r/4)$ we obtain

$$\begin{aligned}\int_{B(x_0,3r/8)} |\nabla\rho|^2 + \frac{(a_\varepsilon - \rho^2)^2}{\varepsilon^2} \leq C\sigma \int_{B(x_0,3r/4)} |\nabla w|^2 \\ + C\frac{\varepsilon}{r}E_\varepsilon(x_0, 3r/4) + C\Lambda_0\varepsilon|\log\varepsilon|^2.\end{aligned} \tag{B-7}$$

Hence, since $r \geq 4\sqrt{\varepsilon}$, from (B-6) and (B-7) we have

$$\tilde{E}_\varepsilon(x_0, \delta 3r/4) \leq C(\delta^2 + \delta^{2-N}(\sigma + \Lambda_0\varepsilon|\log\varepsilon| + \varepsilon^{1/2})\tilde{E}_\varepsilon(x_0, r) + C\frac{\Lambda_0\varepsilon|\log\varepsilon|^2}{(\delta r)^{N-2}}. \tag{B-8}$$

Now choose δ such that $C\delta^2 < 1/4$ and then σ such that $C\delta^{2-N}\sigma < 1/4$. Letting ε tend to zero in the previous inequality keeping r fixed yields

$$\frac{\mu_*(B(x_0, \delta 3r/4)}{(\delta 3r/4)^{N-2}} \leq \frac{1}{2}\frac{\mu_*(B(x_0, r))}{r^{N-2}}. \tag{B-9}$$

Since $r < r_0$, and r_0 was arbitrary small, we infer taking a sequence $r_0 \to 0$ that

$$\Theta_*(x_0) \leq \frac{1}{2}\Theta_*(x_0), \qquad \text{i.e.} \qquad \Theta_*(x_0) = 0.$$

This contradicts the definition of Σ_μ and the proof is complete. □

Lemma B.12. *Σ_μ is closed in Ω.*

Proof. It follows directly from the upper-semicontinuity of Θ_*, the lower density. □

Lemma B.13. (Uniform convergence away from Σ_μ) *Let $K \subset \Omega \setminus \Sigma_\mu$ be any compact subset. For any $\sigma > 0$, there exists $\tilde{\varepsilon} > 0$ depending only on K and σ such that, if $0 < \varepsilon < \tilde{e}ps$, then*

$$|1 - |w|| \leq \sigma \qquad\qquad \text{on } K.$$

Proof. Let $\sigma > 0$ and the corresponding $\eta > 0$ and $\varepsilon_0 > 0$ given by Theorem 2. For each $x \in K$, we deduce from Lemma B.11 that there exists $r(x) > 0$ and $\varepsilon(x) > 0$ such that

$$\tilde{E}_\varepsilon(x, r(x)) \leq 4^{2-N}\eta|\log\varepsilon| \qquad \forall \varepsilon \leq \varepsilon(x).$$

Let $x_1, \cdots, x_k$ be such that

$$K \subset \cup_{i=1}^k B(x_i, r(x_i)/2)$$

and let $\tilde{\varepsilon} := \min(\varepsilon_0, \varepsilon(x_1), \cdots, \varepsilon(x_k))$. From Corollary A.1, it follows that for $\varepsilon \leq \tilde{\varepsilon}$,

$$|1 - |w|| \leq \sigma \ \text{ on } B(x_i, r(x_i)/2) \qquad \forall i = 1 \cdots k.$$

This proves the Lemma. □

Lemma B.14. (Structure of μ_*) *We have*

$$\mu_* = g(x)\mathcal{H}^N + h(x)\mathcal{H}^{N-2}\llcorner\Sigma_\mu\,,$$

where g and h are locally bounded on Ω and h verifies

$$\eta_0 \leq \Theta_*(x) < h(x) \leq \Theta^*(x) \equiv \limsup_{r\to 0}\frac{\mu_*(B(x,r))}{r^{N-2}} \leq c(x)M_0\,.$$

Proof. Since Σ_μ is closed in Ω and hence measurable, we have

$$\mu_* = \mu_*\llcorner\Sigma_\mu + \mu_*\llcorner(\Omega \setminus \Sigma_\mu).$$

As in [10] Theorem VIII.1, we infer from Corollary A.1 that $\mathcal{H}^{N-2}(\Sigma_\mu) \leq CM_0$. It also follows from the monotonicity formula of Proposition 2 that for all $x \in \Omega$,

$$\Theta^*(x) := \limsup_{r\to 0}\frac{\mu_*(B(x,r)}{r^{N-2}} \leq CM_0.$$

Using the Radon-Nikodym theorem, we thus obtain

$$\mu_*\llcorner\Sigma_\mu = h(x)\cdot\mathcal{H}^{N-2}\llcorner\Sigma_\mu, \tag{B-10}$$

for some $\Theta_* \leq h \leq \Theta^*$. We will prove that in fact $\Theta_* = \Theta^*$.

Now, let $x_0 \in \Omega \setminus \Sigma_\mu$ and $r > 0$ such that $\overline{B(x_0, 2r)} \subset \Omega \setminus \Sigma_\mu$. By Lemma B.13, we know that

$$\sigma := |1 - |w||_{L^\infty(\overline{B(x_0,2r)})} = o(1) \ \text{ as } \varepsilon \to 0.$$

The same computation as in Lemma B.11 (see (B-8)) shows that for each $0 < \delta < 1/2$,

$$E_\varepsilon(x_0, \delta 3r/4) \leq C(\delta^N + \sigma + \Lambda_0\varepsilon|\log\varepsilon| + \varepsilon^{1/2})E_\varepsilon(x_0, r) + C\Lambda_0\varepsilon|\log\varepsilon|^2, \tag{B-11}$$

but now we know that $\sigma = \sigma(\varepsilon) = o(1)$. Hence, dividing both sides by $|\log \varepsilon|$ and sending ε to zero we obtain

$$\mu_*(B(x_0, \delta 3r/4)) \leq C\delta^N \mu_*(B(x_0, r)).$$

This implies that $\mu_* \llcorner (\Omega \setminus \Sigma_\mu)$ is absolutely continuous with respect to the Lebesgue measure, and using the Radon-Nikodym theorem once more we finally deduce that

$$\mu_* = g(x) \cdot \mathcal{H}^N + h(x) \cdot \mathcal{H}^{N-2} \llcorner \Sigma_\mu \tag{B-12}$$

for some locally bounded function g. $\square$

Lemma B.15. *We have*

$$g(x) = |\nabla h_*(x)|^2 \qquad \text{a.e. in } \Omega\,,$$

where h_* is some harmonic function.

Proof. The argument is similar to the one carried out in [12] for Theorem A, statement iv). Since the proof is rather lenghty we briefly sketch the main steps.

First, one has to prove that, if $|w_\varepsilon| \geq 1 - \sigma_0$ on some ball $B(x_0, R)$ (where σ_0 is some suitable constant), then

$$|\nabla w_\varepsilon|^2 \simeq |\nabla \phi_\varepsilon|^2 \qquad \text{on } B(x_0, \frac{3R}{4})\,,$$

where ϕ_ε is harmonic and verifies

$$|\nabla \phi_\varepsilon|^2 \leq C\sqrt{M_0 |\log \varepsilon|}\,.$$

Then

$$\frac{\phi_\varepsilon}{\sqrt{|\log \varepsilon|}} \rightharpoonup h_*\,,$$

which is thus harmonic on $B(x_0, 3R/4)$.

A second important step is to prove that h_* is globally well-defined and harmonic on Ω. Here the argument is readily the same as in [12]. $\square$

Proof of the curvature equation and the rectifiability of Σ_μ.

Let $\vec{X} \in \mathcal{D}(\Omega, \mathbb{R}^N)$ be a smooth vector field and

$$e_\varepsilon(w) := \frac{1}{2}|\nabla w|^2 + \frac{1}{4\varepsilon^2}(a_\varepsilon - |w|^2)^2.$$

We have

$$\begin{aligned} \int_\Omega e_\varepsilon(w) \mathrm{div} \vec{X} &= -\int_\Omega \nabla e_\varepsilon(w) \cdot \vec{X} \\ &= -\int_\Omega \frac{1}{2}\nabla(|\nabla w|^2) + \frac{1}{2\varepsilon^2}(a_\varepsilon - |w|^2)(-2w\nabla w + \nabla a_\varepsilon) \cdot \vec{X}, \end{aligned} \tag{B-13}$$

and

$$\int_\Omega \sum_{i,j} \frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j}\frac{\partial X^i}{\partial x_j} = -\int_\Omega \sum_{i,j} (\frac{\partial^2 w}{\partial x_i \partial x_j}\frac{\partial w}{\partial x_j} + \frac{\partial w}{\partial x_i}\frac{\partial^2 w}{\partial x_j^2})X^i$$
$$= -\int_\Omega \nabla w \cdot \vec{X}\Delta w - \int_\Omega \sum_{i,j} \frac{\partial}{\partial x_i}\left|\frac{\partial w}{\partial x_j}\right|^2 X^i \qquad \text{(B-14)}$$
$$= -\int_\Omega \nabla w \cdot \vec{X}\Delta w - \int_\Omega \frac{1}{2}\nabla(|\nabla w|^2)\cdot \vec{X}.$$

Since w is a solution of (124), we deduce from (B-13) and (B-14) that

$$\frac{1}{|\log\varepsilon|}\int_\Omega \left(e_\varepsilon(w)\delta_{ij} - \frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j}\right)\frac{\partial X^i}{\partial x_j}$$
$$= \frac{1}{|\log\varepsilon|}\int_\Omega (\nabla w\cdot\vec{X})\Big(\Delta w + \frac{1}{\varepsilon^2}w(a_\varepsilon - |w|^2)\Big) + \frac{1}{2}(a_\varepsilon - |w|^2)|\log\varepsilon|^2\nabla d\cdot\vec{X} \qquad \text{(B-15)}$$
$$= \int_\Omega (\nabla w\cdot\vec{X}, i\vec{c}\cdot\nabla w) + \int_\Omega \frac{1}{2}(a_\varepsilon - |w|^2)|\log\varepsilon|\nabla d\cdot\vec{X}$$
$$= -\int_\Omega \langle *\,(\vec{c}\wedge *Jw)\,,\vec{X}\rangle + \int_\Omega \frac{1}{2}(a_\varepsilon - |w|^2)|\log\varepsilon|\nabla d\cdot\vec{X}.$$

Set

$$\alpha_\varepsilon^{ij} := \frac{1}{|\log\varepsilon|}\left(e_\varepsilon(w)\delta_{ij} - \frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j}\right).$$

Notice that α_ε^{ij} is a symmetric matrix with trace larger than $(N-2)\mu_\varepsilon$, and a little linear algebra shows that its eigenvalues are less or equal to μ_ε. Moreover,

$$|\alpha_\varepsilon^{ij}| \le N\mu_\varepsilon. \qquad \text{(B-16)}$$

Going if necessary to a subsequence, we may thus assume that

$$\alpha_\varepsilon^{ij} \to \alpha_*^{ij} \qquad \text{in the sense of measures.}$$

In view of (B-16) we have $|\alpha_*^{ij}| \le N\mu_*$, therefore we may write

$$\alpha_*^{ij}(x) = A^{ij}(x)\mu_* \qquad \text{for } \mu_* \text{ a.e. } x\in\Omega,$$

where the matrix $A^{ij}(x)$ is symmetric, with trace **equal** to $N-2$ and eigenvalues less or equal to one [The fact that the trace is equal to $N-2$ and not just less than $N-2$ follows from Proposition A.1]. From (B-16) we also have

$$A^{ij} \ge -N\delta^{ij} \qquad \text{for } \mu_* \text{ a.e. } x\in\Omega. \qquad \text{(B-17)}$$

Notice that

$$\left|\int_\Omega \frac{1}{2}(a_\varepsilon - |w|^2)|\log\varepsilon|\nabla d\cdot\vec{X}\right| \le C\Lambda_0\varepsilon|\log\varepsilon| \to 0 \quad \text{ as } \varepsilon\to 0,$$

so that passing to the limit in (B-15) we obtain

$$\begin{aligned}\int_\Omega A^{ij}(x)\frac{\partial X^i}{\partial x_j}d\mu_*(x) &= -\int_\Omega \langle *\left(\vec{c}(x)\wedge *dJ_*(x)\right), \vec{X}\rangle \\ &= -\int_\Omega \langle *\left(\vec{c}\wedge *\frac{dJ_*}{d\mu_*}\right), \vec{X}\rangle\, d\mu_*(x).\end{aligned} \tag{B-18}$$

We decompose the r.h.s. of (B-18) as

$$\begin{aligned}\int_\Omega A^{ij}(x)\frac{\partial X^i}{\partial x_j}d\mu_*(x) &= \int_\Omega A^{ij}(x)\frac{\partial X^i}{\partial x_j}d\mu_*(x)\llcorner\Sigma_\mu \\ &+ \int_\Omega \left(\frac{|\nabla h_*|^2}{2}\delta_{ij} - \frac{\partial h_*}{\partial x_i}\frac{\partial h_*}{\partial x_j}\right)\frac{\partial X^i}{\partial x_j}dx\,.\end{aligned} \tag{B-19}$$

Since h_* is harmonic, the last term in (B-19) vanishes. Hence, the support of J_* being included in Σ_μ, using (B-12) we obtain that

$$\int_\Omega A^{ij}(x)\frac{\partial X^i}{\partial x_j}d\mu_*(x)\llcorner\Sigma_\mu = -\int_\Omega \langle *\left(\vec{c}\wedge *\frac{dJ_*}{d\mu_*}\right), \vec{X}\rangle\, d\mu_*(x)\llcorner\Sigma_\mu. \tag{B-20}$$

Since $\vec{X}$ was arbitrary, the previous equality means in particular that the generalized $(N-2)$-varifold (see [5])

$$\tilde{V} := \delta_{A^{ij}(x)}\mu_*\llcorner\Sigma_\mu(x)$$

has a first variation. From Step 1 and [5] Theorem 3.8 c) we thus infer that $\tilde{V}$ is indeed a real rectifiable $(N-2)$-varifold. In particular, the geometrical support Σ_μ of $\mu_*\llcorner\Sigma_\mu$ is rectifiable. From the rectifiability of Σ_μ, we deduce that

$$\Theta_*(x) = \Theta^*(x) \qquad \mu_*\text{-a.e. } x \text{ in } \Sigma_\mu,$$

so that

$$\mu_* = g(x)\cdot\mathcal{H}^N + \Theta_*(x)\cdot\mathcal{H}^{N-2}\llcorner\Sigma_\mu,$$

and

$$V(\Sigma_\mu,\Theta_*) = \tilde{V}.$$

Equation (B-20) then precisely states that $V(\Sigma_\mu,\Theta_*)$ satisfies the mean curvature equation

$$\vec{H}(x) = *\left(\vec{c}(x)\wedge *\frac{dJ_*}{d\mu_*}\right) \qquad \text{for } \mu_*\text{-a.e. } x \text{ in } \Sigma_\mu.$$

The proof of Theorem 3 is now complete. □

Appendix C : Compactness

If some additional conditions are imposed on the boundary data, we may then obtain compactness properties for w_ε. In this part, we will assume

$$\int_{\partial\Omega} e_\varepsilon(w) \leq M_0 \qquad \text{and} \qquad \|w\|_{H^{1/2}(\partial\Omega)} \leq M_0. \tag{C-1}$$

[There are however many variants of condition (C-1), see [7, 11]].

Proposition C.2. *Let $1 \leq p < \frac{N}{N-1}$. There exists a constant $C > 0$ depending on p, M_0, Λ_0 and Ω but independent of ε such that if w is a solution of (124) satisfying (24) and (C-1) then*

$$\int_\Omega |\nabla w|^p \leq C.$$

Proof. We follow the lines of [7, 11]. Let $\rho := |w|$. From the identity

$$\rho^2|\nabla w|^2 = \rho^2|\nabla \rho|^2 + |w \times \nabla w|^2,$$

and the inequality $|\nabla w| \geq |\nabla \rho|$, we deduce that

$$\begin{aligned} |\nabla w|^2 &= |\nabla \rho|^2 + |w \times \nabla w|^2 + (1 - |w|^2)(|\nabla w|^2 - |\nabla \rho|^2) \\ &\leq |\nabla \rho|^2 + |w \times \nabla w|^2 + |a_\varepsilon - |w|^2||\nabla w|^2 + \Lambda_0 \varepsilon^2 |\log \varepsilon||\nabla w|^2 \\ &\leq |\nabla \rho|^2 + |w \times \nabla w|^2 + (\sqrt{2}\varepsilon + \Lambda_0 \varepsilon^2 |\log \varepsilon|) e_\varepsilon(w). \end{aligned} \tag{C-2}$$

Hence, since (24) is satisfied,

$$\int_\Omega |\nabla w|^p \leq C \left[\int_\Omega |\nabla \rho|^p + \int_\Omega |w \times \nabla w|^p + 1 \right], \tag{C-3}$$

where C depends only on p, Λ_0, M_0 and Ω.

Step 1: Estimates for the modulus. Notice that ρ satisfies the equation

$$-\Delta \rho^2 + 2|\nabla w|^2 = \frac{2}{\varepsilon^2}\rho^2(a_\varepsilon - \rho^2) - (w, i\vec{c} \cdot \nabla w)|\log \varepsilon|. \tag{C-4}$$

Let us introduce the set

$$A = \{x \in \Omega,\ \rho(x) > 1 - \varepsilon^{1/2}\}$$

and the function

$$\bar{\rho} = \max\{\rho, 1 - \varepsilon^{1/2}\},$$

so that $\bar{\rho} = \rho$ on A and $0 \leq 1 - \bar{\rho} \leq \varepsilon^{1/2}$ in Ω.

Next let ζ_ε be a function in $\mathcal{D}(\Omega)$ such that $0 \leq \zeta_\varepsilon \leq 1$ on Ω, $\zeta_\varepsilon \equiv 1$ on $\Omega_\varepsilon \equiv \{x \in \Omega,\ \text{dist}\,(x, \partial\Omega) \geq \varepsilon^{1/2}\}$, and $|\nabla \zeta_\varepsilon| \leq C\varepsilon^{-1/2}$, where C depends only on Ω.

By multiplying equation (C-4) by $\zeta_\varepsilon(\bar{\rho}^2-1)$ (which is compactly supported in Ω), and integrating over Ω we obtain

$$\int_\Omega \nabla\rho^2\nabla\bar{\rho}^2\zeta_\varepsilon + \int_\Omega \frac{2\rho(1-\rho^2)(1-\bar{\rho}^2)}{\varepsilon^2}\zeta_\varepsilon = \int_\Omega (1-\bar{\rho}^2)|\nabla w|^2$$
$$+\int_\Omega \nabla\rho^2\nabla\zeta_\varepsilon(1-\bar{\rho}^2) + \int_\Omega 2\rho|\log\varepsilon|^2 d(x)(1-\bar{\rho}^2)\zeta_\varepsilon$$
$$+\int_\Omega (w, i\vec{c}\cdot\nabla w)(\bar{\rho}^2-1)\zeta_\varepsilon|\log\varepsilon|.$$

It follows that on the set $A_\varepsilon = \Omega_\varepsilon \cap A$ we have

$$\int_{A_\varepsilon} |\nabla\rho^2|^2 = \int_{A_\varepsilon} \nabla\rho^2\nabla\bar{\rho}^2$$
$$\leq 2\varepsilon^{1/2}\int_\Omega |\nabla w|^2 + \frac{C}{\varepsilon^{1/2}}\int_\Omega |\nabla\rho||a_\varepsilon-\rho^2| + C\Lambda_0 M_0\varepsilon|\log\varepsilon|^2$$
$$\leq 2\varepsilon^{1/2}\int_\Omega |\nabla w|^2 + C\varepsilon^{1/2}\left[\int_\Omega |\nabla\rho|^2 + \int_\Omega \frac{(a_\varepsilon-\rho^2)^2}{4\varepsilon^2}\right] + C\Lambda_0 M_0\varepsilon|\log\varepsilon|^2,$$

hence, since $\rho \geq 1-\varepsilon^{1/2}$ on A_ε, we have, for $\varepsilon \leq 1/4$,

$$\int_{A_\varepsilon} |\nabla\rho|^2 \leq 4\int_{A_\varepsilon} |\nabla\rho^2|^2 \leq C\varepsilon^{1/2}E_\varepsilon(w) + C\Lambda_0 M_0\varepsilon|\log\varepsilon|^2 \leq C. \tag{C-5}$$

Set $W_\varepsilon = \Omega\setminus\Omega_\varepsilon$, $B = \Omega\setminus A$, so that

$$\Omega = B\cup A_\varepsilon\cup W_\varepsilon.$$

From (24) we deduce $\int_B(1-\rho^2)^2 \leq 4M_0\varepsilon^2|\log\varepsilon|$ and hence, since $(1-\rho)\geq\varepsilon^{1/2}$ on B, it follows $|B|\leq 4M_0\varepsilon|\log\varepsilon|$. Thus

$$\int_B |\nabla\rho|^p \leq \left(\int_\Omega |\nabla\rho|^2\right)^{p/2}|B|^{1-p/2} \leq C|\log\varepsilon|^{p/2}(\varepsilon|\log\varepsilon|)^{1-p/2},$$

i.e.

$$\int_B |\nabla\rho|^p \leq C\varepsilon^{1-p/2}|\log\varepsilon|. \tag{C-6}$$

Finally, we turn to W_ε. Clearly, by construction $|W_\varepsilon|\leq C\varepsilon^{1/2}$. Hence

$$\int_{W_\varepsilon} |\nabla\rho|^p \leq \left(\int_\Omega |\nabla\rho|^2\right)^{p/2}|W|^{1-p/2} \leq C\varepsilon^{1/2-p/4}|\log\varepsilon|^{p/2}. \tag{C-7}$$

Combining (C-5) with (C-6) and (C-7) we get the estimate for the modulus

$$\int_\Omega |\nabla\rho|^p \leq C, \tag{C-8}$$

where C does not depend on ε.

Step 2: Estimates for the pre Jacobian. Consider the Hodge - de Rham decomposition of $w \times \nabla w$:

$$w \times \nabla w = d\varphi + d^*\psi \tag{C-9}$$

where the function φ satisfies $\varphi = 0$ on $\partial\Omega$ and the 2-form ψ satisfies $d\psi = 0$ on Ω and $\psi_\top = 0$ on $\partial\Omega$. Applying respectively the operators d^* and d to (C-9) we obtain the equation for φ (resp. ψ) :

$$\begin{cases} \Delta\varphi = \vec{c}\cdot\nabla(|w|^2-1)|\log\varepsilon| & \text{in } \Omega \\ \quad \varphi = 0 & \text{on } \partial\Omega \end{cases} \tag{C-10}$$

and

$$\begin{cases} \Delta\psi = 2Jw & \text{in } \Omega \\ \quad \psi_\top = 0, \quad (d^*\psi)_\top = (w \times dw)_\top & \text{on } \partial\Omega. \end{cases} \tag{C-11}$$

From (24), (C-1), (C-11) and Proposition III.1 in [11] we infer that

$$\int_\Omega |\nabla\psi|^p \leq C. \tag{C-12}$$

Indeed, the estimate (C-12) is valid even without assuming that w is a solution of (124) (see [11]). Notice however that the constant C may depend on Ω; in the proof of Theorem 4 we will see how to use the extra information that w satisfies (124) to obtain estimates independent of the domain.

Concerning φ, multiplying equation (C-10) by φ and integrating over Ω we get

$$\begin{aligned} \int_\Omega |\nabla\varphi|^2 &= |\log\varepsilon| \int_\Omega \operatorname{div}((|w|^2-1)\vec{c})\varphi \\ &= |\log\varepsilon| \int_\Omega (|w|^2-1)\, \vec{c}\cdot\nabla\varphi \\ &\leq C\Lambda_0\varepsilon|\log\varepsilon|(\int_\Omega \frac{(1-|w|^2)^2}{4\varepsilon^2})^{1/2}(\int_\Omega |\nabla\varphi|^2)^{1/2} \\ &\leq C\Lambda_0(M_0+1)\varepsilon|\log\varepsilon|^{3/2}(\int_\Omega |\nabla\varphi|^2)^{1/2}, \end{aligned} \tag{C-13}$$

so that

$$\int_\Omega |\nabla\varphi|^p \leq (\int_\Omega |\nabla\varphi|^2)^{p/2}|\Omega|^{1-p/2} \leq C, \tag{C-14}$$

where C does not depend on ε.
Combining (C-8), (C-12) and (C-14) we get the desired conclusion from (C-3). □

Proof of Theorem 4.

Recall that w_ε is a solution of (11) on Π_n such that (24) and (31) are satisfied. For simplicity, we omit the subscripts ε in the sequel, i.e. we set $w \equiv w_\varepsilon$.

Step 1: Extracting the "bad" balls. From Theorem 2, we infer that there exists $\eta > 0$ and $R_0 > 0$ such that for each $x \in S_\varepsilon$,

$$E_\varepsilon(x, R_0) \geq \eta |\log \varepsilon|. \tag{C-15}$$

It follows from Vitali's covering theorem that there exist an at most countable family of points $(y_{i,\varepsilon})_{i\in I}$ in S_ε such that

$$S_\varepsilon \subset \cup_{i\in I} B(y_{i,\varepsilon}, 5R_0)$$

and

$$B(y_{i,\varepsilon}, R_0) \cap B(y_{j,\varepsilon}, R_0) = \emptyset \quad \text{if } i \neq j.$$

We deduce from (24),(C-15) and the previous equality that

$$\sharp I \leq l := \frac{M_0}{\eta}.$$

We claim that there exists a constant $10 \leq \kappa \leq C(\sharp I)$ (where $C(\sharp I)$ depends only on $\sharp I$) and q points $x_{1,\varepsilon}, \cdots, x_{q,\varepsilon} \in \Pi_n$ $(q \leq l)$ such that, setting $R := \kappa R_0$,

$$S_\varepsilon \subset \cup_{i=1}^q B(x_{i,\varepsilon}, R) \quad \text{and} \quad \text{dist}(x_{i,\varepsilon}, x_{j,\varepsilon}) \geq 10R \ \text{ if } i \neq j.$$

Indeed, set $R_1 := 10R_0$. If $\text{dist}(y_{i,\varepsilon}, y_{j,\varepsilon}) \geq 10R_1$ there is nothing to prove. If not, consider the equivalence relation

$$y_{i,\varepsilon} \sim y_{j,\varepsilon} \qquad \text{if} \quad \text{dist}(y_{i,\varepsilon}, y_{j,\varepsilon}) \leq 10R_1,$$

and denote $\mathcal{C}_j,\ j \in J$ the different equivalence classes. We define $B(z_j, R_{2,j})$ for each $j \in J$ as the smallest ball such that

$$\overline{\cup_{y_{i,\varepsilon}\in\mathcal{C}_j} B(y_{i,\varepsilon}, R_1)} \subset \overline{B(z_j, R_{2,j})},$$

and we set $R_2 := \max_j R_{2,j}$. If $\text{dist}(z_{j,\varepsilon}, z_{k,\varepsilon}) \geq 10R_2$ for each $j \neq k$ we are done, otherwise we repeat inductively the previous growing argument. Since at each step, the number of equivalence classes decreases at least by one, the process finishes after at most $\sharp I$ steps.

Step 2: Choosing a good unfolding of the torus. Since (24) and (31) are satisfied, we infer from Lemma 2.1 that there exists a good unfolding of the torus Π_n such that

$$\int_{\partial\Omega_n} e_\varepsilon(w) \leq \frac{2^{N-1}M_0|\log\varepsilon|}{n} \leq C, \tag{C-16}$$

where C does not depend on n or ε. In particular, $\|w\|_{H^1(\partial\Omega_n)}$ is uniformly bounded.

Step 3: Uniform $W^{1,p}_{loc}$ estimates. Let $x_0 \in \Omega_n$ and $1 \leq p < \frac{N}{N-1}$ be given. As in the proof of Proposition C.2 (C-3) we obtain

$$\int_{B(x_0,1)} |\nabla w|^p \leq C\left[\int_{B(x_0,1)} |\nabla\rho|^p + \int_{B(x_0,1)} |w \times \nabla w|^p + 1\right], \tag{C-17}$$

where $\rho := |w|$ and C depends only on p, Λ_0 and M_0.
The estimate for the modulus is also obtained as in Proposition C.2 replacing Ω by $B(x_0, 1)$; we have

$$\int_{B(x_0,1)} |\nabla\rho|^p \leq C, \tag{C-18}$$

where C does not depend on n or ε.

Consider the Hodge - de Rham decomposition of $w \times \nabla w$ in Ω_n :

$$w \times \nabla w = d\varphi + d^*\psi \tag{C-19}$$

where the function φ satisfies $\varphi = 0$ on $\partial\Omega_n$ and the 2-form ψ satisfies $d\psi = 0$ on Ω_n and $\psi_\top = 0$ on $\partial\Omega_n$. Applying respectively the operators d^* and d to (C-19) we obtain the equation for φ (resp. ψ) :

$$\begin{cases} -\Delta\varphi = c(\varepsilon)\frac{\partial}{\partial x_1}(|w|^2 - 1)|\log\varepsilon| & \text{in } \Omega_n \\ \quad\varphi = 0 & \text{on } \partial\Omega_n \end{cases} \tag{C-20}$$

and

$$\begin{cases} -\Delta\psi = 2Jw & \text{in } \Omega_n \\ \quad\psi_\top = 0, \quad (d^*\psi)_\top = (w \times dw)_\top & \text{on } \partial\Omega_n. \end{cases} \tag{C-21}$$

Still the estimate for φ follows as in Proposition C.2, we obtain

$$\int_{B(x_0,1)} |\nabla\varphi|^p \leq C, \tag{C-22}$$

where C does not depend on n or ε (and $C \to 0$ as $\varepsilon \to 0$).

The estimate for ψ is more delicate since the embedding constants used in the proof of Proposition C.2 heavily depends on n. We will overcome this difficulty by taking advantage of the confinement of Jw described in Step 1. Let $\tilde{w}$ be defined by

$$\tilde{w}(x) := \begin{cases} 2w(x) & \text{if } |w(x)| \leq 1/2, \\ \frac{w(x)}{|w(x)|} & \text{if } |w(x)| \geq 1/2. \end{cases}$$

Notice that $E_\varepsilon(\tilde{w}) \leq 4M_0|\log\varepsilon|$ and that $J\tilde{w}$ is supported in S_ε. We also define, for $1 \leq i \leq q$, the two forms

$$\omega_i := 2J\tilde{w} \llcorner B(x_{i,\varepsilon}, R).$$

Let $\psi_{0,i}$ be the solution of the problem

$$\begin{cases} -\Delta\psi_{0,i} = \omega_i & \text{in } \Omega_n \\ \quad\psi_{0,i} = 0 & \text{on } \partial\Omega_n \end{cases} \tag{C-23}$$

(note the different kind of boundary conditions here). Let ψ_1 be the solution of

$$\begin{cases} -\Delta\psi_1 = 2(Jw - J\tilde{w}) & \text{in } \Omega_n \\ \quad(\psi_1)_\top = 0, \qquad (d^*\psi_2)_\top = 0 & \text{on } \partial\Omega_n, \end{cases} \tag{C-24}$$

and ψ_2 the solution of

$$\begin{cases} -\Delta\psi_2 = 0 & \text{in } \Omega_n \\ (\psi_2)_\top = 0, \qquad (d^*\psi_2)_\top = (w \times dw)_\top - \sum_{i=1}^q (d^*\psi_{0,i})_\top & \text{on } \partial\Omega_n. \end{cases} \tag{C-25}$$

Clearly,

$$\psi = \sum_{i=1}^q \psi_{0,i} + \psi_1 + \psi_2.$$

We also set

$$U_1^i := B(x_0, 1) \cap (\Omega_n \setminus B(x_{i,\varepsilon}, 2R)) \qquad \text{and} \qquad U_2^i := \overline{B(x_{i,\varepsilon}, 2R)}.$$

Estimate for $\psi_{0,i}$. From the Green formula

$$\psi_{0,i}(x) = \int_{\mathrm{supp}(\omega_i)} \langle \omega_i(x), G_{\Omega_n}(x,y)\rangle \, dy \tag{C-26}$$

we deduce that

$$\|\psi_{0,i}\|_{\mathcal{C}^k(U_1^i)} \leq C(k)\|\omega_i\|_{[\mathcal{C}^{0,\alpha}(U_1^i)]^*} \leq C(k). \tag{C-27}$$

Indeed, for any x in U_1^i and $y \in \mathrm{supp}(\omega_i)$ one has

$$\min(\mathrm{dist}(x, \partial\Omega_n), \mathrm{dist}(y, \partial\Omega_n), \mathrm{dist}(x,y)) \geq R,$$

so that (C-27) follows from standard estimates on the Green functions (which is even explicit in the case of the cube Ω_n).
For U_2^i, consider the solution $\tilde{\psi}_{0,i}$ of

$$\begin{cases} -\Delta\tilde{\psi}_{0,i} = \omega_i & \text{in } B(x_{i,\varepsilon}, 3R) \\ \psi_{0,i} = 0 & \text{on } \partial B(x_{i,\varepsilon}, 3R). \end{cases} \tag{C-28}$$

Following the lines of Proposition C.2 we obtain

$$\int_{B(x_{i,\varepsilon})} |\nabla\tilde{\psi}_{0,i}|^p \leq C(R)\|\omega_i\|_{[\mathcal{C}^{0,\alpha}]^*} \leq C(R, M_0). \tag{C-29}$$

On the other hand, for $x \in U_2^i$ we have

$$\tilde{\psi}_{0,i}(x) - \psi_{0,i}(x) = \int_{\mathrm{supp}(\omega_i)} \langle \omega_i(y), \left[R_{B(x_{i,\varepsilon},3R)}(x,y) - R_{\Omega_n}(x,y)\right]\rangle \, dy$$

where R_{Ω_n} stands for the regular part of the green function G_{Ω_n} and similarly for $B(x_{i,\varepsilon}, 3R)$. Note that for all $x \in U_2^i$ and for all $y \in \mathrm{supp}(\omega_i)$,

$$\min(\mathrm{d}(x, \partial\Omega_n), \mathrm{d}(y, \partial\Omega_n), \mathrm{d}(x, \partial B(x_{i,\varepsilon}, 3R)), \mathrm{d}(y, \partial B(x_{i,\varepsilon}, 3R))) \geq R,$$

so that again using standard estimates

$$\|\tilde{\psi}_{0,i} - \psi_{0,i}\|_{\mathcal{C}^k(U_2^i)} \leq C(k)\|\omega_i\|_{[\mathcal{C}^{0,\alpha}]^*} \leq C(R, M_0). \tag{C-30}$$

Combining (C-27), (C-29) and (C-30) we obtain

$$\int_{B(x_0,1)} |\nabla \psi_{0,i}|^p \leq C$$

where C does not depend on n or ε.
Estimate for ψ_1. From standard elliptic estimates we have

$$\|\psi_1\|_{W_0^{1,p}(\Omega_n)} \leq C\|Jw - J\tilde{w}\|_{[W_0^{1,p'}(\Omega_n)]^*}, \tag{C-31}$$

where C does not depend on n (indeed the previous inequality is invariant under scaling of the domain and of the corresponding equation). On the other hand,

$$\|Jw - J\tilde{w}\|_{[W_0^{1,p}(\Omega_n)]^*} = \sup_{h \in W_0^{1,q}(\Omega_n, \Lambda^2\mathbb{R}^N), \|h\|=1} \int_{\Omega_n} \langle (Jw - J\tilde{w}), h \rangle,$$

and

$$\begin{aligned} \int_{\Omega_n} \langle (Jw - J\tilde{w}), h \rangle &= \int_{\Omega_n} \langle w \times dw - \tilde{w} \times d\tilde{w}, d^*h \rangle \\ &\leq C(\int_{\Omega_n} |\nabla h|^q)^{\frac{1}{q}} \cdot (\int_{\Omega_n} |w \times dw - \tilde{w} \times d\tilde{w}|^p)^{\frac{1}{p}}. \end{aligned} \tag{C-32}$$

On S_ε,

$$|w \times dw - \tilde{w} \times d\tilde{w}| \leq C|w| \cdot |\nabla w|$$

so that since $|S_\varepsilon| \leq C\varepsilon^2 |\log \varepsilon|$,

$$\begin{aligned} (\int_{S_\varepsilon} |w \times dw - \tilde{w} \times d\tilde{w}|^p)^{\frac{1}{p}} &\leq C(\int_{S_\varepsilon} |\nabla w|^2)^{\frac{1}{2}} \cdot (\int_{S_\varepsilon} |w|^s)^{\frac{1}{s}} \\ &\leq C|\log \varepsilon|^{1/2} (\varepsilon^2 |\log \varepsilon|)^{1/s}, \end{aligned}$$

where $s := 2p/(2-p) \geq 2$. Outside S_ε, we have

$$|w \times dw - \tilde{w} \times d\tilde{w}| = \left| \frac{|w|^2 - 1}{|w|^2} w \times dw \right| \leq 4|(|w|^2 - 1)| \cdot |\nabla w|$$

so that

$$\begin{aligned} (\int_{\Omega_n \setminus S_\varepsilon} |w \times dw - \tilde{w} \times d\tilde{w}|^p)^{\frac{1}{p}} &\leq C(\int_{\Omega_n \setminus S_\varepsilon} |\nabla w|^2)^{\frac{1}{2}} \cdot (\int_{\Omega_n \setminus S_\varepsilon} (|w|^2 - 1)^s)^{\frac{1}{s}} \\ &\leq C\varepsilon^{2/s} |\log \varepsilon|^{1/2} (\int_{\Omega_n \setminus S_\varepsilon} \frac{(|w|^2 - 1)^2}{\varepsilon^2})^{\frac{1}{s}} \\ &\leq C\varepsilon^{2/s} |\log \varepsilon|^{1/2} |\log \varepsilon|^{1/s} \\ &\leq C. \end{aligned} \tag{C-33}$$

Combining these two estimates with (C-31) we thus obtain

$$\int_{B(x_0,1)} |\nabla \psi_1|^p \leq C,$$

where C does not depend on n or ε.
Estimate for ψ_2. We deduce from Step 2 that

$$\|(w \times dw)_\top\|_{L^2(\partial\Omega_n)} \leq C.$$

On the other hand, since $\mathrm{dist}(\partial\Omega_n, \mathrm{supp}(\omega_i)) \geq R$ we have

$$\|\nabla\psi_{0,i}\|_{L^\infty(\partial\Omega_n)} \leq C$$

(this again follows from standard estimates on the Green function for the cube Ω_n). Since ψ_2 is harmonic on Ω_n, we thus obtain

$$\|\psi_2\|_{C^k(B(x_0,1))} \leq C$$

where C depends on k but not on n or ε.

Combining the estimates for $\psi_{0,i}$, ψ_1 and ψ_2 with (C-17) and (C-18) we conclude that

$$\int_{B(x_0,1)} |\nabla w|^p \leq C. \tag{C-34}$$

This establishes claim i) of the Theorem.
Next, we prove estimate ii) of the Theorem, i.e. provide uniform energy bounds away from the bad balls. Here, we will work directly on Π_n (as a manifold). Therefore, the Hodge - de Rham decomposition will involve also harmonic forms. The next step will be useful to control these forms.

Step 4: Degree estimate. Since $|w| \geq \frac{1}{2}$ out of S_ε, we may write $w(x) = \rho(x)\exp(i\varphi(x))$ out of S_ε, where $\varphi(x) \in S^1$. Moreover, since

$$\tilde{\Omega}_n := \Omega_n \setminus \cup_{i=1}^q B(x_{i,\varepsilon}, R)$$

is simply connected, the phase $\varphi(x)$ can be lifted as a function form $\tilde{\Omega}_n$ to $\mathbb{R}$. If the coordinates $(y_2, \cdots, y_N)$ are such that

$$[-n,n] \times (y_2, \cdots, y_N) \cap \cup_{i=1}^q B(x_{i,\varepsilon}, R) = \emptyset$$

then the degree of the map $s \mapsto \varphi(s, y_2, \cdots, y_n)$, i.e.

$$d := \deg(s \mapsto \varphi(s, y_2, \cdots, y_n)).$$

is well defined. Clearly it follows from the invariance of the degree under homotopy that d does not depend on the particular choice of an admissible $(y_2, \cdots, y_N)$.
We claim that $d = 0$ (an elementary way to rephrase this is that the lifted phase φ takes the same values on opposite faces of Ω_n).
Indeed, from Step 1 we infer that the set of admissible $(y_2, \cdots, y_N) \in [-n,n]^{N-1}$ has

measure larger than n^{N-1} for n sufficiently large (and thus ε sufficiently small). On the other hand, if $d \neq 0$ we obtain for each admissible $(y_2, \cdots, y_N)$,

$$\int_{-n}^{n} |\nabla w(s, y_2, \cdots, y_N)|^2 \, ds \geq \frac{1}{4} \int_{-n}^{n} |\frac{\partial \varphi}{\partial x_1}(s, y_2, \cdots, y_N)|^2 \, ds$$
$$\geq 2n \frac{1}{4} (\frac{2\pi}{2n})^2 = \frac{\pi^2}{2n},$$

so that using Fubini's theorem,

$$\int_{\Omega_n} |\nabla w|^2 \geq n^{N-1} \frac{\pi^2}{2n} = \frac{\pi^2}{2} n^{N-2} \geq 2(M_0 + 1)|\log \varepsilon|.$$

This contradicts hypothesis (24) and proves the claim. Obviously the corresponding degree computed with respect to the other coordinates is also zero.

Step 5: Local uniform energy estimates. Let $x \in \Omega_n$ and $r > 0$ such that $B(x,r) \subset \Omega_n \setminus S_\varepsilon$. As in the previous step, we write $w(x) = \rho(x) \exp(i\varphi(x))$ in $B(x,r)$, and we have

$$\operatorname{div}(\rho^2 \nabla \varphi) = c|\log \varepsilon| \frac{\partial}{\partial x_1} (\rho^2 - 1).$$

Let $\tilde{\varphi}$ be the solution of

$$\begin{cases} \operatorname{div}(\rho^2 \nabla \tilde{\varphi}) = c|\log \varepsilon| \frac{\partial}{\partial x_1} (\rho^2 - 1), & \text{in } B(x,r) \\ \tilde{\varphi} = 0 & \text{on } \partial B(x,r). \end{cases} \tag{C-35}$$

Multiplying (C-35) by $\tilde{\varphi}$ and integrating by parts leads to

$$\int_{B(x,r)} |\nabla \tilde{\varphi}|^2 \leq C \varepsilon^2 |\log \varepsilon| \leq C. \tag{C-36}$$

On the other hand, $\bar{\varphi} := \varphi - \tilde{\varphi}$ satisfies

$$\operatorname{div}(\rho^2 \nabla \bar{\varphi}) = 0 \qquad \text{on } B(x,r). \tag{C-37}$$

Since φ is defined up to a constant multiple of 2π, we may assume without loss of generality that

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} \bar{\varphi} \in [0, 2\pi). \tag{C-38}$$

Combining (C-37) with (C-38) and the $W^{1,p}_{loc}$ estimates in Step 4 we obtain, using standard elliptic regularity theory

$$\int_{B(x,r/2)} |\nabla \bar{\varphi}|^2 \leq C, \tag{C-39}$$

so that finally using (C-36)

$$\int_{B(x,r/2)} |\nabla \varphi|^2 \leq C. \tag{C-40}$$

Next, let $\xi \in \mathcal{D}(B(x,r/2)),\ 0 \le \xi \le 1$, such that $\xi \equiv 1$ on $B(x,r/4)$. Multiplying the equation

$$\Delta\rho - \rho|\nabla\varphi|^2 + \frac{1}{\varepsilon^2}\rho(1-\rho^2) = -c|\log\varepsilon|\rho\frac{\partial\varphi}{\partial x_1}, \tag{C-41}$$

by $(1-\rho^2)\xi^2$ and integrating by parts we obtain

$$\int_{B(x,r/2)} 2\rho|\nabla\rho|^2\xi^2 + \rho\frac{(1-\rho^2)^2}{\varepsilon^2} = \int_{B(x,r/2)} 2\xi(1-\rho^2)\nabla\rho\cdot\nabla\xi + \int_{B(x,r/2)} \rho(1-\rho^2)\xi^2|\nabla\varphi|^2 - c|\log\varepsilon|\rho\frac{\partial\varphi}{\partial x_1}(1-\rho^2)\xi^2. \tag{C-42}$$

On the other hand, we have

$$\int_{B(x,r/2)} 2\xi(1-\rho^2)\nabla\rho\cdot\nabla\xi \le \frac{1}{10}\int_{B(x,r/2)}|\nabla\rho|^2\xi^2 + 10\int_{B(x,r/2)}(1-\rho^2)^2|\nabla\xi|^2,$$

and from (C-40),

$$\int_{B(x,r/2)} \rho(1-\rho^2)\xi^2|\nabla\varphi|^2 \le C$$

and

$$\int_{B(x,r/2)} c|\log\varepsilon|\rho\frac{\partial\varphi}{\partial x_1}(1-\rho^2)\xi^2 \le C\Big(\int_{B(x,r/2)}|\nabla\varphi|^2\Big)^{1/2}\Big(\int_{B(x,r/2)}\frac{(1-\rho^2)^2}{\varepsilon^2}\Big)\varepsilon|\log\varepsilon| \le C.$$

Hence, from (C-42) and since $\rho \ge 1/2$ on $B(x,r)$,

$$\int_{B(x,r/4)} |\nabla\rho|^2 + \frac{(1-\rho^2)^2}{4\varepsilon^2} \le C, \tag{C-43}$$

which, combined with (C-40) leads to

$$\int_{B(x,r/4)} e_\varepsilon(w) \le C. \tag{C-44}$$

Step 6: Proof of estimate ii).

In order to conclude the proof of Theorem 4 it remains to show that

$$\int_{\Omega_n\setminus\cup_{i=1}^q B(x_{i,\varepsilon},R)} e_\varepsilon(w) \le C.$$

As in Proposition C.2, we have

$$\int_{\Pi_n\setminus\cup_{i=1}^q B(x_{i,\varepsilon},R)} |\nabla w|^2 \le C\Big(1 + \int_{\Pi_n\setminus\cup_{i=1}^q B(x_{i,\varepsilon},R)} |\nabla\rho|^2 + |w\times dw|^2\Big). \tag{C-45}$$

Here we consider the Hodge - de Rham decomposition of $w \times \nabla w$ in Π_n (as a manifold) :

$$w \times \nabla w = d\varphi + d^*\psi + \sum_{i=1}^{N} \alpha_i dx_i \tag{C-46}$$

where the 2-form ψ satisfies $d\psi = 0$ on Π_n, each α_i is a real number and the dx_i represent the canonical harmonic 1-forms on Π_n. Applying respectively the operators d^* and d to (C-46) we obtain the equation for φ (resp. ψ) :

$$-\Delta\varphi = c(\varepsilon)\frac{\partial}{\partial x_1}(\rho^2 - 1)|\log\varepsilon| \quad \text{ in } \Pi_n \tag{C-47}$$

and

$$-\Delta\psi = 2Jw \quad \text{ in } \Pi_n. \tag{C-48}$$

Still the estimate for φ follows as in Proposition C.2 (C-13), we obtain

$$\int_{\Pi_n} |\nabla\varphi|^2 \leq C, \tag{C-49}$$

where C does not depend on n or ε (and $C \to 0$ as $\varepsilon \to 0$).

The estimate for ψ has to be slightly adapted with respect to Step 3.
Let $\tilde{w}$ be defined by

$$\tilde{w}(x) := \begin{cases} w(x) & \text{if } x \in \cup_{i=1}^q B(x_{i,\varepsilon}, \frac{R}{2}), \\ (\frac{4s}{R} - 2)w(x) + (3 - \frac{4s}{R})\frac{w(x)}{|w(x)|} & \text{if s:=dist}(x, \cup_i x_{i,\varepsilon}) \in (\frac{R}{2}, \frac{3R}{4}), \\ \frac{w(x)}{|w(x)|} & \text{otherwise.} \end{cases}$$

Notice that $E_\varepsilon(\tilde{w}) \leq CM_0|\log\varepsilon|$ and that $J\tilde{w}$ is supported in the set $\cup_{i=1}^q B(x_{i,\varepsilon}, 3R/4)$. We also have

$$w \times dw = \tilde{w} \times d\tilde{w} \quad \text{ on } \cup_{i=1}^q B(x_{i,\varepsilon}, R/2). \tag{C-50}$$

We also define, for $1 \leq i \leq q$, the two forms

$$\omega_i := 2J\tilde{w} \llcorner B(x_{i,\varepsilon}, 3R/4),$$

and denote by $\psi_{0,i}$ the Newtonian potential of ω_i on Π_n (i.e. $\psi_{0,i} := G_n * \omega_i$ where G_n is the Green function on Π_n.) Similarly, ψ_1 denotes the Newtonian potential of $2(Jw - J\tilde{w})$ on Π_n. Clearly,

$$\psi(x) = \sum_{i=1}^{q} \psi_{0,i}(x) + \psi_1(x).$$

We claim that

$$|\nabla\psi_{0,i}(x)| \leq C(\text{dist}(x, x_{i,\varepsilon}))^{1-N} \qquad \forall\, x \in \Pi_n \setminus B(x_{i,\varepsilon}, R), \tag{C-51}$$

where C does not depend on n or ε. Indeed, this is a direct consequence of the formula

$$\nabla\psi_{0,i}(x) = \int_{\Pi_n} \frac{\partial G}{\partial y}(x,y)\omega_i(y)\,dy,$$

of the $[\mathcal{C}^{0,\alpha}]^*$ uniform bound on ω_i, and of classical estimates on G_n.
Hence, since $N \geq 3$, we obtain

$$\int_{\Pi_n\setminus\cup_{i=1}^q B(x_{i,\varepsilon},R)} |\nabla\psi_{0,i}|^2 \leq \int_{\mathbb{R}^N\setminus B(0,R)} C|x|^{2-2N}\,dx \leq C. \tag{C-52}$$

We turn next to **the estimate for ψ_1.** We have

$$\|\nabla\psi_1\|_{L^2(\Pi_n)} \leq C \sup_{h\in\mathcal{C}^\infty(\Pi_n,\Lambda^2\mathbb{R}^N)} \left\{\int_{\Pi_n} \langle Jw - J\tilde{w}, h\rangle,\ \int_{\Pi_n} |\nabla h|^2 = 1\right\}.$$

On the other hand, taking (C-50) into account,

$$\int_{\Pi_n} \langle Jw - J\tilde{w}, h\rangle = \int_{\Pi_n} \langle w\times dw - \tilde{w}\times d\tilde{w}, d^*h\rangle = \int_{\tilde{\Pi}_n} \langle w\times dw - \tilde{w}\times d\tilde{w}, d^*h\rangle$$

where $\tilde{\Pi}_n := \Pi_n \setminus \cup_{i=1}^q B(x_{i,\varepsilon}, R/2)$. Notice that $|w| \geq 1/2$ in $\tilde{\Pi}_n$, hence

$$\int_{\tilde{\Pi}_n} \langle w\times dw - \tilde{w}\times d\tilde{w}, d^*h\rangle \leq C\|\rho^2-1\|_{L^\infty(\tilde{\Pi}_n)}(\int_{\tilde{\Pi}_n}|\nabla w|^2)^{1/2}(\int_{\tilde{\Pi}_n}|\nabla h|^2)^{1/2}.$$

From Theorem 3 we know that $|w|$ uniformly converges to 1 on $\tilde{\Pi}_n$ (and actually uniformly with respect to n as can be seen examining Step 2 of the proof of Theorem 3). Hence, we obtain

$$\|\nabla\psi_1\|_{L^2(\Pi_n)} \leq C(1 + r(\varepsilon)\int_{\tilde{\Pi}_n}|\nabla w|^2) \tag{C-53}$$

where $r(\varepsilon) \to 0$ as $\varepsilon \to 0$, uniformly in n. Finally, we turn to the components of the harmonic forms.
We claim that

$$|\alpha_i| \leq \frac{C}{|\Pi_n|^2}.$$

Indeed, since

$$\alpha_i = \frac{1}{|\Pi_n|^2}\int_{\Pi_n} \langle w\times dw, dx_i\rangle, \tag{C-54}$$

it suffices to prove that

$$\left|\int_{\Pi_n} \langle w\times dw, dx_i\rangle\right| \leq C.$$

Let $R' \in [R/2, R]$ and $\tilde{\Pi}'_n := \Pi_n \setminus \cup_{i=1}^q B(x_{i,\varepsilon}, R')$. The phase φ of w is well defined in $\tilde{\Pi}'_n$; we extend it as a continuous function φ' on Π_n by considering its harmonic extension inside each ball $B(x_{i,\varepsilon}, R')$. We have

$$\left|\int_{\Pi_n} \langle w \times dw, dx_i\rangle\right| \leq \left|\int_{\cup_{i=1}^q B(x_{i,\varepsilon},R')} \langle w \times dw, dx_i\rangle\right| + \left|\int_{\tilde{\Pi}'_n} \rho^2 \frac{\partial \varphi}{\partial x_1}\right|$$
$$\leq C\int_{\cup_{i=1}^q B(x_{i,\varepsilon},R')} |\nabla w| + C\left|\int_{\tilde{\Pi}'_n} (\rho^2-1)|\nabla\varphi|\right|$$
$$+ C\left|\int_{\Pi_n} \frac{\partial \varphi'}{\partial x_1}\right| + \int_{\cup_{i=1}^q B(x_{i,\varepsilon},R')} |\nabla \varphi'|.$$

From Step 3 we infer that

$$\int_{\cup_{i=1}^q B(x_{i,\varepsilon},R')} |\nabla w| \leq C.$$

An averaging argument shows that there exists $R' \in [R/2, R]$ such that

$$\int_{\cup_{i=1}^q B(x_{i,\varepsilon},R')} |\nabla \varphi'| \leq C(\int_{\cup_{i=1}^q B(x_{i,\varepsilon},R')} |\nabla\varphi'|^2)^{1/2}$$
$$\leq C(\int_{\cup_{i=1}^q B(x_{i,\varepsilon},R)\cap\tilde{\Pi}_n} |\nabla w|^2)^{1/2}$$

so that using Step 5,

$$\int_{\cup_{i=1}^q B(x_{i,\varepsilon},R')} |\nabla\varphi'| \leq C.$$

We also have,

$$\left|\int_{\tilde{\Pi}'_n} (\rho^2-1)|\nabla\varphi|\right| \leq C\varepsilon|\log\varepsilon| \leq C$$

and by virtue of Step 4,

$$\int_{\Pi_n} \frac{\partial\varphi'}{\partial x_1} = 0.$$

This proves the claim. Coming back to $w \times dw$, we obtain combining (C-47),(C-52), (C-53) and the previous claim,

$$\int_{\Pi_n\setminus\cup_{i=1}^q B(x_{i,\varepsilon},R)} |w\times dw|^2 \leq C(1 + r(\varepsilon)\int_{\tilde{\Pi}_n} |\nabla w|^2), \qquad \text{(C-55)}$$

where $r(\varepsilon) \to 0$ when $\varepsilon \to 0$, uniformly in n.
We still need **the estimate for the modulus.** Let $\xi \in \mathcal{D}(\tilde{\Pi}_n)$, $0 \leq \xi \leq 1$, such that $\xi \equiv 1$ on $\Pi_n \setminus \cup_{i=1}^q B(x_{i,\varepsilon}, R)$. Multiplying the equation

$$\Delta\rho - \rho|\nabla\varphi|^2 + \frac{1}{\varepsilon^2}\rho(1-\rho^2) = -c|\log\varepsilon|\rho\frac{\partial\varphi}{\partial x_1},$$

by $(1-\rho^2)\xi^2$ and integrating by parts we obtain

$$\int_{\tilde{\Pi}_n} 2\rho|\nabla\rho|^2\xi^2 + \rho\frac{(1-\rho^2)^2}{\varepsilon^2} = \int_{\tilde{\Pi}_n} 2\xi(1-\rho^2)\nabla\rho\cdot\nabla\xi$$
$$+ \int_{\tilde{\Pi}_n} \rho(1-\rho^2)\xi^2|\nabla\varphi|^2 - c|\log\varepsilon|\rho\frac{\partial\varphi}{\partial x_1}(1-\rho^2)\xi^2. \qquad \text{(C-56)}$$

Arguing as in Step 5, we deduce from (C-56) that

$$\int_{\Pi_n \setminus \cup_{i=1}^q B(x_{i,\varepsilon},R)} |\nabla \rho|^2 + \frac{(1-\rho^2)^2}{4\varepsilon^2} \leq C(1 + r(\varepsilon) \int_{\tilde{\Pi}_n} |\nabla w|^2), \tag{C-57}$$

where $r(\varepsilon) \to 0$ when $\varepsilon \to 0$, uniformly in n.
We can now complete the proof. Adding (C-55) to (C-57) we obtain, using (C-45) and Step 5,

$$\begin{aligned}\int_{\Pi_n \setminus \cup_{i=1}^q B(x_{i,\varepsilon},R)} e_\varepsilon(w) &\leq C(1 + r(\varepsilon) \int_{\tilde{\Pi}_n} |\nabla w|^2) \\ &\leq C(1 + r(\varepsilon) \int_{\Pi_n \setminus \cup_{i=1}^q B(x_{i,\varepsilon},R)} |\nabla w|^2).\end{aligned}$$

For $\varepsilon \leq \varepsilon_0$ sufficiently small, $Cr(\varepsilon) < \frac{1}{2}$, which yields the desired estimate

$$\int_{\Pi_n \setminus \cup_{i=1}^q B(x_{i,\varepsilon},R)} e_\varepsilon(w) \leq C.$$

For $\varepsilon \geq \varepsilon_0$ the previous inequality is clearly also verified, and the proof is complete. □

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