PARTIAL REGULARITY OF FINITE MORSE INDEX SOLUTIONS TO THE LANE-EMDEN EQUATION

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Abstract. We prove regularity and partial regularity results for finite Morse index solutions $u \in H^1(\Omega) \cap L^p(\Omega)$ to the Lane-Emden equation $-\Delta u = |u|^{p-1}u$ in Ω.

1. Introduction

Given an open set $\Omega \subset \mathbb{R}^N$, $N \geq 1$, and $p > 1$, consider the Lane-Emden equation

$$-\Delta u = |u|^{p-1}u \qquad \text{in } \Omega. \tag{1.1}$$

We are interested in the classical question of regularity of solutions to (1.1). Namely, given a class of weak solutions $\mathcal{C}$, we ask: what is the largest exponent $p > 1$, such that

$$u \in \mathcal{C} \Longrightarrow u \in C^2(\Omega) \ ? \tag{1.2}$$

Consider first $\mathcal{C} = L^p_{loc}(\Omega)$ and assume (1.1) is understood in the sense of distributions. Then, as follows from a well-known bootstrap argument, (1.2) holds true for all $p < p_0(N)$, where

$$p_0(N) = \begin{cases} +\infty, & \text{if } 1 \leq N \leq 2, \\ \dfrac{N}{N-2}, & \text{if } 3 \leq N. \end{cases}$$

The exponent $p_0(N)$ is sharp. Indeed, for all $p > p_0(N)$, $u(x) = c_{N,p}|x|^{-\frac{2}{p-1}}$ is a singular solution belonging to $\mathcal{C}$. A (radial) singular solution also exists if $p = p_0(N)$, see [2, 14].

Consider next the case $\mathcal{C} = H^1_{loc}(\Omega) \cap L^p_{loc}(\Omega)$. Then (1.2) holds true for all $p \leq p_S(N)$ where

$$p_S(N) = \begin{cases} +\infty, & \text{if } 1 \leq N \leq 2, \\ \dfrac{N+2}{N-2}, & \text{if } 3 \leq N. \end{cases}$$

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When $p < p_S(N)$, the proof uses the same bootstrap argument and the extra assumption that $u \in H^1_{loc}(\Omega)$ for the initial step. See [6] for the critical case $p = p_S(N)$. The Sobolev exponent $p_S(N)$ is again sharp in the considered class, using the same counter-example.

Restrict at last to the class $\mathcal{C}$ of energy solutions having finite Morse index, i.e., $u \in \mathcal{C}$ if $u \in H^1_{loc}(\Omega) \cap L^p_{loc}(\Omega)$ and the maximal dimension of a vector space $X \subset C^1_c(\Omega)$ such that

$$Q_u(\varphi) := \int_\Omega |\nabla\varphi|^2 \, dx - p\int_\Omega |u|^{p-1}\varphi^2 \, dx < 0, \qquad \text{for all } \varphi \in X \setminus \{0\}$$

is an integer k, called the Morse index of u. If $k = 0$, we say that u is stable. Note that this class of weak solutions is a natural choice, since any C^2 solution to (1.1) is bounded on any open set $\omega \subset\subset \Omega$ and so must have finite Morse index on ω. We obtain the following result.

Theorem 1.1. *Let $u \in H^1_{loc}(\Omega) \cap L^p_{loc}(\Omega)$ be a solution to* (1.1) *of finite Morse index. If $p < p_c(N)$, where*

$$p_c(N) = \begin{cases} +\infty, & \text{for } 1 \le N \le 10, \\ \dfrac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)}, & \text{for } 11 \le N, \end{cases}$$

then, $u \in C^2(\Omega)$.

Under the stronger assumptions that Ω is smoothly bounded, u is stable, and $u|_{\partial\Omega} = 1$, the smoothness of u was first proved in [7,13]. At no surprise, the proof of Theorem 1.1 is by bootstrap (using the additional information that u has finite Morse index in the initial step). The Joseph-Lundgren exponent $p_c(N)$ is again sharp in the considered class, using the same counter-example.

In the supercritical cases ($p \ge p_0(N)$, $p > p_S(N)$, $p \ge p_c(N)$), solutions can be singular at a point, as discussed earlier, but also on larger sets. E.g. if there exists an integer k such that $0 \le k \le N-1$ and $k < N - 2\frac{p}{p-1}$, then

$$u(x) = C_{N,p,k}\left(x_1^2 + \dots x_{N-k}^2\right)^{-\frac{1}{p-1}} \tag{1.3}$$

is a solution having k-dimensional singular set. See also an example in [18] where the Hausdorff dimension of the singular set is not an integer.
Nevertheless, if solutions are assumed to be positive and stationary, the singular set can be estimated as follows.

Theorem 1.2 ([16])**.** *Let $N \ge 3$ and $p \ge p_S(N)$. Let $u \in H^1_{loc}(\Omega) \cap L^{p+1}_{loc}(\Omega)$ be a positive weak solution to* (1.1)*. Assume in addition that u is stationary. Then, $u \in C^2(\Omega \setminus \Sigma)$, where Σ is a closed set of Hausdorff dimension bounded above by*

$$\mathcal{H}_{dim}(\Sigma) \le N - 2\frac{p+1}{p-1}.$$

The precise definition of stationary solution can be found in [16]. Smooth solutions (and limits thereof in the $H^1(\Omega) \cap L^{p+1}(\Omega)$ topology) are stationary.
When the solution has finite Morse index, we prove that the singular set is in fact much smaller.

Theorem 1.3. *Let $N \geq 11$ and $p > p_c(N)$. Let $u \in H^1_{loc}(\Omega) \cap L^p_{loc}(\Omega)$ be a positive solution to* (1.1) *of finite Morse index. Then, $u \in C^2(\Omega \setminus \Sigma)$, where Σ is a closed set of Hausdorff dimension bounded above by*

$$\mathcal{H}_{dim}(\Sigma) \leq N - 2\frac{p+\gamma}{p-1},$$

with $\gamma = 2p + 2\sqrt{p(p-1)} - 1$.

Remark 1.4. *The dimension of the singular set computed in Theorem 1.3 is optimal at least when it is an integer. Indeed, the solution given by* (1.3) *is stable in $\mathbb{R}^N$ if $p > p_c(N-k)$, while $p_c(N-k)$ solves*

$$N - 2\frac{p+\gamma}{p-1} = k.$$

Remark 1.5. *If $u \in H^1_{loc}(\Omega) \cap L^p_{loc}(\Omega)$ is a positive solution with finite Morse index, we prove in the next section that for any point $x \in \Omega$, there exists a ball $B = B(x,r)$ such that u is the limit of C^2 solutions in the $H^1(B) \cap L^{p+1}(B)$ topology. Hence, u is stationary in B.*

Theorem 1.3 remains valid for sign-changing solutions, provided they are stationary.

We discuss at last the question of universal *a priori* estimates.

Theorem 1.6. *Assume $1 < p < p_c(N)$ and $p \neq p_S(N)$. Assume $u \in H^1_{loc}(\Omega) \cap L^p_{loc}(\Omega)$ is a solution to* (1.1) *of finite Morse index m. Then, there exists a constant C depending on N, p, m only, such that for all $x \in \Omega$,*

$$|u(x)| + |\nabla u(x)|^{\frac{2}{p+1}} \leq C\, dist(x, \partial\Omega)^{-\frac{2}{p-1}}. \tag{1.4}$$

Remark 1.7. *Assume $p = p_S(N)$. Then,* (1.4) *remains true for stable solutions (see* [10]*). However, the estimate is false for solutions of finite Morse index, since for $\lambda > 0$,*

$$u_\lambda(x) = \left(\frac{\lambda\sqrt{N(N-2)}}{\lambda^2 + |x|^2}\right)^{\frac{N-2}{2}}$$

provides an unbounded family of solutions of constant Morse index.

The universal estimate (1.4) was first proved for positive solutions and $1 < p < p_S(N)$ (see [4, 8, 11, 19]), with a constant C independent of the Morse index m. Note however that for such p, there do exist sign-changing solutions of arbitrary large Morse index, for which the dependance of the constant C to m must be kept (see [1, 3, 17]). Estimate (1.4) was then proved in [10] for C^2 solutions which are stable, for the full range $1 < p < p_c(N)$.

We provide at last a universal estimate for C^2 solutions of the more general problem:

$$-\Delta u = f(u) \quad \text{in } \Omega \tag{1.5}$$

where $f \in C^1(\mathbb{R}, \mathbb{R})$ behaves like a power of u at infinity. More precisely,

Theorem 1.8. *Suppose*

$$\lim_{t \to \pm\infty} \frac{f'(t)}{p|t|^{p-1}} = a \tag{1.6}$$

for some $a > 0$ and $1 < p < p_c(N)$ and $p \neq p_S(N)$. Let $u \in H^1_{loc}(\Omega) \cap L^p_{loc}(\Omega)$ be a solution of (1.5) *of finite Morse index m. Then, there exists a constant C depending on N, f, m only, such that for all $x \in \Omega$,*

$$|u(x)| + |\nabla u(x)|^{\frac{2}{p+1}} \leq C(1 + dist(x, \partial\Omega)^{-\frac{2}{p-1}}).$$

Theorem 1.8 was proved in [20] for positive solutions, $1 < p < p_S(N)$, and with a constant C independent of the Morse index m. In the case $p = p_S(N)$, the theorem remains valid for stable solutions, but fails for solutions of finite Morse index (see the counter-example in Remark 1.7). Similar statements can be derived for the nonlinearity $f(u) = e^u$, as we shall demonstrate in a future publication.

2. Preliminary results

2.1. **Reduction to the case of stable solutions.**

Proposition 2.1. *Let $u \in H^1_{loc}(\Omega) \cap L^p_{loc}(\Omega)$ be a solution to* (1.1) *with finite Morse index. Then, for every $x_0 \in \Omega$, there exists $r_0 > 0$ such that u is stable in $B(x_0, r_0)$.*

Proof. When $N = 1$, any function $u \in H^1_{loc}(\Omega)$ is locally bounded by Morrey's inequality. In particular, the linearized operator $L = -\Delta - p|u|^{p-1}$ has positive principal eigenvalue in any sufficiently small ball, whence u is stable on such a ball. Assume now $N \geq 2$. We may always assume that $B(0, 1) \subset \Omega$ and it suffices to prove that u is stable near the origin. Assume first that u has Morse index 1. Either u is stable in $B(0, 1/n)$ for some $n \geq 2$ and we are done. Or, for all $n \geq 2$, there exists a direction $\varphi_n \in C^1_c(B(0, 1/n))$ such that $Q_u(\varphi_n) < 0$. Since u has index 1, this implies that u is stable in $B(0, 1) \setminus \overline{B(0, 1/n)}$. This being true for all $n \geq 2$, we deduce that u is stable in $B(0, 1) \setminus \{0\}$. In fact, since $N \geq 2$, points have zero Newtonian capacity and so u is stable in $B(0, 1)$. So, every solution of index 1 is stable in a neighborhood of 0. Take now a solution u of index $k \geq 2$. Working exactly as above, we deduce that u has index $k - 1$ in some ball $B(0, r_1)$. Working inductively on k, we deduce that u is stable in some ball $B(0, r_k)$.

□

2.2. **Approximation of singular stable solutions.**

Lemma 2.2. *Suppose $u \in H^1_{loc}(\Omega) \cap L^p_{loc}(\Omega)$ is a nonnegative stable weak solution to* (1.1)*. Then, there exists a sequence of nonnegative stable solutions $u_n \in C^2(\Omega)$ to* (1.1)*, such that $u_n \nearrow u$ a.e. and in $H^1_{loc}(\Omega)$.*

Proof. The proof is a refinement of a concave truncation technique found in [5].

Let us first observe that since $u \in H^1_{loc}(\Omega)$ and u solves (1.1), we have $u \in L^{p+1}_{loc}(\Omega)$. Take now $\omega \subset\subset \Omega$ with smooth boundary, so that $u \in H^1(\omega) \cap L^{p+1}(\omega)$. We are going to produce a sequence u_n converging to u in $H^1(\omega)$. By a standard diagonal argument, we then reach the desired conclusion.

In the sequel, we write $\omega = \Omega$ for notational convenience. Given $c > 0$, consider the function

$$\phi_c(t) = \left(c + t^{-(p-1)}\right)^{-\frac{1}{p-1}}, \quad \text{defined for } t > 0.$$

We set also $\phi_c(0) = 0$. Then, ϕ_c is increasing, concave, and smooth for $t > 0$. In addition, $\phi_c(t) \nearrow t$ as $c \searrow 0^+$, and $\phi_c(t) \leq t$, for all $t \geq 0$. Also, if $c > 0$, then ϕ_c,

ϕ_c' are uniformly bounded. We have

$$\phi_c'(t) = \frac{\phi_c(t)^p}{t^p} \quad \forall t > 0.$$

Let w_c denote the unique solution to

$$\begin{cases} -\Delta w_c = 0 & \text{in } \Omega \\ w_c = \phi_c(u) & \text{on } \partial\Omega. \end{cases}$$

Then, $w_c \geq 0$, $w_c \in L^\infty(\Omega) \cap H^1(\Omega)$. Moreover, w_c is non-increasing with respect to c. We claim that $w_c \to w$ in $H^1(\Omega)$ as $c \to 0$, where w is the solution to

$$\begin{cases} -\Delta w = 0 & \text{in } \Omega \\ w = u & \text{on } \partial\Omega. \end{cases}$$

To see this, consider the problem

$$\begin{cases} -\Delta v = (v + w_c)^p & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Since $w_c \in L^\infty(\Omega)$, (2.1) has a minimal nonnegative solution v_c, which can be constructed by the method of sub and super-solutions, as follows. Note that $\underline{v} = 0$ is a sub-solution, since $w_c \geq 0$. Moreover, by Kato's inequality, $\overline{v} = \phi_c(u) - w_c$ is a bounded super-solution:

$$-\Delta(\phi_c(u) - w_c) = -\Delta\phi_c(u) \geq -\phi_c'(u)\Delta u = \phi_c(u)^p = (\phi_c(u) - w_c + w_c)^p.$$

In particular, (2.1) has a minimal nonnegative solution v_c. This minimal solution is bounded and by elliptic regularity, v_c belongs to $C^{1,\alpha}(\overline{\Omega})$. Moreover, v_c is stable in the sense that

$$p\int_\Omega (v_c + w_c)^{p-1}\varphi^2 \, dx \leq \int_\Omega |\nabla\varphi|^2 \, dx, \quad \text{for all } \varphi \in C_c^1(\Omega).$$

Since v_c is minimal and w_c is non-increasing with respect to c, we deduce that v_c is also non-increasing with respect to c. It follows that $v(x) = \lim_{c\to 0} v_c(x)$ is well-defined for all $x \in \Omega$. Since $v_c \in C^1(\overline{\Omega})$, we have

$$\int_\Omega |\nabla v_c|^2 \, dx = \int_\Omega (v_c + w_c)^p v_c \, dx \leq \int_\Omega u^{p+1} \, dx.$$

In particular, v_c is bounded in $H_0^1(\Omega)$. It follows that $v_c \rightharpoonup v$ weakly in $H_0^1(\Omega)$. Multiplying (2.1) by $\varphi \in C_c^\infty(\Omega)$, integrating, and passing to the limit as $c \to 0$, we see that v is a weak solution to

$$\begin{cases} -\Delta v = (v + w)^p & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Let $\varphi_k \in C_c^{0,1}(\Omega)$ be a sequence such that $\varphi_k \to v$ in $H_0^1(\Omega)$. Since $v \geq 0$ we can assume $\varphi_k \geq 0$. We can also assume that $\varphi_k \to v$ a.e. in Ω. Multiplying (2.2) by φ_k and integrating, we obtain

$$\int_\Omega \nabla v \nabla \varphi_k \, dx = \int_\Omega (v + w)^p \varphi_k \, dx$$

By Fatou's lemma,

$$\int_\Omega (v + w)^p v \, dx \leq \liminf_{k\to\infty} \int_\Omega \nabla v \nabla\varphi_k \, dx = \int_\Omega |\nabla v|^2 \, dx.$$

By monotone convergence,

$$\lim_{c\to 0}\int_\Omega |\nabla v_c|^2\,dx = \lim_{c\to 0}\int_\Omega (v_c+w_c)^p v_c\,dx = \int_\Omega (v+w)^p v\,dx.$$

Hence,

$$\lim_{c\to 0}\int_\Omega |\nabla v_c|^2\,dx = \int_\Omega (v+w)^p v\,dx \le \int_\Omega |\nabla v|^2\,dx.$$

Since $v_c \rightharpoonup v$ weakly in $H_0^1(\Omega)$, the reverse inequality

$$\int_\Omega |\nabla v|^2\,dx \le \liminf_{c\to 0}\int_\Omega |\nabla v_c|^2\,dx$$

also holds, which proves that $v_c \to v$ in $H_0^1(\Omega)$.

We claim that $u = v + w$, from which Lemma 2.2 follows. By construction, $v = \lim v_c \le \lim(\phi_c(u) - w_c) = u - w$. We need thus only prove that $u \le v + w$. Note that $\tilde v = u - w$ solves (2.2). Let $z = \tilde v - v \ge 0$. Then, $z \in H_0^1(\Omega)$, and since u is stable,

$$p\int_\Omega (\tilde v + w)^{p-1}(\tilde v - v)^2\,dx \le \int_\Omega |\nabla(\tilde v - v)|^2\,dx. \tag{2.3}$$

Now, $\tilde v - v$ satisfies

$$\int_\Omega \nabla(\tilde v - v)\nabla\varphi\,dx = \int_\Omega ((\tilde v + w)^p - (v+w)^p)\varphi\,dx, \qquad \forall\varphi \in C_c^\infty(\Omega).$$

We would like to take $\varphi = \tilde v - v$. First, we claim that we can take $\varphi \in H_0^1(\Omega)\cap L^\infty(\Omega)$. These functions can be approximated in $H_0^1(\Omega)$ by functions in $C_c^\infty(\Omega)$ with a uniform bound. Then, take $\varphi = \min(\tilde v - v, t)$, $t > 0$, which belongs to $H_0^1(\Omega)\cap L^\infty(\Omega)$. We get

$$\int_{[\tilde v - v \le t]} |\nabla(\tilde v - v)|^2\,dx = \int_\Omega ((\tilde v + w)^p - (v+w)^p)\min(\tilde v - v, t)\,dx.$$

Now let $t \to \infty$. Then,

$$\int_\Omega |\nabla(\tilde v - v)|^2\,dx = \int_\Omega ((\tilde v + w)^p - (v+w)^p)(\tilde v - v)\,dx$$

Combined with (2.3) we find

$$\int_\Omega (\tilde v - v)\left[p(\tilde v + w)^{p-1}(\tilde v - v) - (\tilde v + w)^p + (v+w)^p\right]\,dx \le 0.$$

By convexity, $p(\tilde v + w)^{p-1}(\tilde v - v) - (\tilde v + w)^p + (v+w)^p \ge 0$ with strict inequality, unless $\tilde v \equiv v$. □

2.3. **Some well-known ingredients.** Proofs of all the results in this section can be found in [9]. We begin with a so-called ε-regularity result for weak solutions to (1.1) in Morrey spaces. Recall the following definition.

Definition 2.3. *Let Ω be a bounded open set of $\mathbb{R}^N$, $N \ge 1$. Given $p > 1$ and $\lambda \in [0, N]$, the Morrey space $L^{p,\lambda}(\Omega)$ is the set of functions u in $L^p(\Omega)$ such that the following norm is finite:*

$$\|u\|^p_{L^{p,\lambda}(\Omega)} = \sup_{x_0\in\Omega,\, r>0} r^{-\lambda}\int_{B(x_0,r)\cap\Omega} |u|^p\,dx < \infty.$$

Then,

Theorem 2.4 ([12,16]). *Let $N \geq 3$, $p > 1$, and $\lambda = N - 2\frac{p+1}{p-1}$. Let also $B(x_0, r_0)$ be a ball. There exists $\varepsilon = \varepsilon(N,p) > 0$ such that for any weak solution $u \in H^1(B(x_0,r_0)) \cap C(\overline{B(x_0,r_0)})$ to (1.1) satisfying*

$$\|u\|_{L^{p+1,\lambda}(B(x_0,r_0))} \leq \varepsilon, \tag{2.4}$$

there holds

$$\|u\|_{L^\infty(B(x_0,r_0/2))} \leq \left(\frac{4}{r_0}\right)^{\frac{2}{p-1}}.$$

Also recall the following classical result from geometric measure theory.

Theorem 2.5. *Let Ω denote an open set of $\mathbb{R}^N$, $N \geq 1$, u a function in $L^1_{loc}(\Omega)$ and $0 \leq s < N$. Set*

$$E_s = \left\{ x \in \Omega \ : \ \limsup_{r \to 0^+} r^{-s} \int_{B_r(x)} |u(y)| \, dy > 0 \right\}.$$

Then,

$$\mathcal{H}^s(E_s) = 0,$$

where $\mathcal{H}^s$ denotes the Hausdorff measure of dimension s.

The next ingredient in the proof of Theorem 1.3 is the following monotonicity formula.

Theorem 2.6 ([15]). *Let $u \in H^1(\Omega) \cap L^{p+1}(\Omega)$ denote a stationary weak solution to (1.1). For $x \in \Omega$, $r > 0$, such that $B(x,r) \subset \Omega$, consider the energy $\mathcal{E}_u(x,r)$ given by*

$$\mathcal{E}_u(x,r) = r^{-\mu} \int_{B(x,r)} \left(\frac{1}{2}|\nabla u|^2 - \frac{1}{p+1}|u|^{p+1} \right) dx + \frac{r^{-\mu-1}}{p-1} \int_{\partial B(x,r)} |u|^2 \, d\sigma, \tag{2.5}$$

where

$$\mu = N - 2\frac{p+1}{p-1}.$$

Then,

- *$\mathcal{E}_u(x,r)$ is nondecreasing in r.*
- *$\mathcal{E}_u(x,r)$ is continuous in $x \in \Omega$ and $r > 0$.*

Remark 2.7 ([15]). *The energy $\mathcal{E}_u(x,r)$ can be equivalently written as*

$$\mathcal{E}_u(x,r) = \tag{2.6}$$

$$\frac{p-1}{p+3} r^{-\mu} \int_{B(x,r)} \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{p+1}|u|^{p+1} \right) dy + \frac{1}{p+3} \frac{d}{dr} \left(r^{-\mu} \int_{\partial B(x,r)} |u|^2 \, d\sigma \right).$$

We shall use at last the following capacitary estimate.

Proposition 2.8 ([10]). *Let Ω be an open set of $\mathbb{R}^N$, $p > 1$. Let $u \in H^1_{loc}(\Omega) \cap L^p_{loc}(\Omega)$ denote a stable solution to (1.1). Then, for any $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1)$, any $\psi \in C^1_c(\Omega)$, $0 \leq \psi \leq 1$, and any integer $m \geq \max\left\{\frac{p+\gamma}{p-1}, 2\right\}$, there exists a constant $C_{p,m,\gamma} > 0$ such that*

$$\int_\Omega \left(\left| \nabla \left(|u|^{\frac{\gamma-1}{2}} u \right) \right|^2 + |u|^{p+\gamma} \right) \psi^{2m} \, dx \leq C_{p,m,\gamma} \int_\Omega |\nabla \psi|^{2\left(\frac{p+\gamma}{p-1}\right)} \, dx.$$

In the case where $u \in C^2(\Omega)$, the proof of this result is given in [10]. This proof can be adapted to the case $u \in H^1_{loc}(\Omega) \cap L^p_{loc}(\Omega)$ as follows: multiply (1.1) with $|T_k(u)|^{\gamma-1} u\varphi^2$, where $T_k(s) = \max(-k, \min(u,k))$ and $\varphi \in C^2_c(\Omega)$ and apply stability with test function $|T_k(u)|^{\frac{\gamma-1}{2}} u\varphi$.

3. Proofs of Theorems 1.1 and 1.3.

Proof of Theorem 1.1. Thanks to Proposition 2.8, $u \in L^{p+\gamma}_{loc}(\Omega)$ for all $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1)$. Using elliptic estimates and a standard bootstrap argument, we deduce that $u \in C^2(\Omega)$, provided $N \le 10$ or $N \ge 11$ and $p < p_c(N)$. □

Proof of Theorem 1.3 . By Proposition 2.1, we may assume that u is a nonnegative stable weak solution to (1.1). Given $\varepsilon > 0$, define

$$\Sigma_\varepsilon = \left\{ x \in \Omega : \forall r > 0 \quad \int_{B(x,r)} (u^{p+1} + |\nabla u|^2)\, dx \ge \varepsilon r^{N-2\frac{p+1}{p-1}} \right\}$$

Step 1. There exists a fixed value of $\varepsilon > 0$ such that for every $x \notin \Sigma_\varepsilon$, u is bounded (hence C^2) in a neighborhood of x.

To see this, let $x_0 \notin \Sigma_\varepsilon$: there exists $r_0 > 0$ such that

$$r_0^{-\mu} \int_{B(x_0,r_0)} (u^{p+1} + |\nabla u|^2)\, dx < \varepsilon,$$

where $\mu = N - 2\frac{p+1}{p-1}$. By (2.5), for $r < r_0$,

$$\begin{aligned}
\mathcal{E}_u(x_0, r) &\le r^{-\mu} \int_{B(x_0,r)} \frac{1}{2}|\nabla u|^2\, dy + \frac{r^{-\mu-1}}{p-1} \int_{\partial B(x_0,r)} u^2\, d\sigma \\
&\le r^{-\mu} \int_{B(x_0,r_0)} \frac{1}{2}|\nabla u|^2\, dy + \frac{r^{-\mu-1}}{p-1} \int_{\partial B(x_0,r)} u^2\, d\sigma \\
&\le \frac{\varepsilon}{2} \left(\frac{r}{r_0}\right)^{-\mu} + \frac{r^{-\mu-1}}{p-1} \int_{\partial B(x_0,r)} u^2\, d\sigma
\end{aligned}$$

Integrating between $r = r_0/2$ and r_0, and recalling that $\mathcal{E}_u(x, r)$ is nondecreasing in r, we deduce that

$$\begin{aligned}
\frac{r_0}{2}\mathcal{E}_u(x_0, r_0/2) &\le 2^{\mu-2}\varepsilon r_0 + \frac{1}{p-1} \int_{r_0/2}^{r_0} r^{-\mu-1} \left(\int_{\partial B(x_0,r)} u^2\, d\sigma \right) dr \\
&\le C\varepsilon r_0 + C r_0^{-\mu-1} \int_{B(x_0,r_0)} u^2\, dy \\
&\le C\varepsilon r_0 + C r_0^{-\mu-1} \left(\int_{B(x_0,r_0)} u^{p+1}\, dy \right)^{\frac{2}{p+1}} r_0^{N(1-\frac{2}{p+1})} \\
&< C\varepsilon r_0.
\end{aligned}$$

Hence,

$$\mathcal{E}_u(x_0, r_0/2) < C\varepsilon.$$

Since $\mathcal{E}_u$ is continuous in x, there exists $r_1 < r_0/2$ such that $\mathcal{E}_u(x, r_0/2) < 2C\varepsilon$, for $x \in B(x_0, r_1)$. Since $\mathcal{E}_u$ is nonincreasing in r, we deduce that for all $x \in B(x_0, r_1)$ and all $r < r_1$,

$$\mathcal{E}_u(x, r) < 2C\varepsilon. \tag{3.1}$$

Now take an approximating sequence u_n given by Lemma 2.2. Integrating (2.6) between 0 and $r_2 < r_1$, we find

$$\frac{p-1}{p+3}\int_0^{r_2} r^{-\mu}\left(\int_{B(x,r)}\left(\frac{1}{2}|\nabla u_n|^2 + \frac{1}{p+1}u_n^{p+1}\right)\,dy\right)\,dr + \\ + \frac{r_2^{-\mu}}{p+3}\int_{\partial B(x,r_2)} u_n^2\,d\sigma \le r_2 \mathcal{E}_{u_n}(x, r_2).$$

It follows that

$$\begin{aligned} Cr_2\mathcal{E}_u(x,r_2) &\ge \int_0^{r_2}\left(r^{-\mu}\int_{B(x,r)} u^{p+1}\,dy\right)\,dr \\ &\ge \int_{r_2/2}^{r_2}\left(r^{-\mu}\int_{B(x,r)} u^{p+1}\,dy\right)\,dr \end{aligned}$$

By the fundamental theorem of calculus, we deduce that there exists $r_3 \in (r_2/2, r_2)$ such that

$$C\mathcal{E}_u(x, r_2) \ge r_3^{-\mu}\int_{B(x,r_3)} u^{p+1}\,dy \ge r_2^{-\mu}\int_{B(x,r_2/2)} u^{p+1}\,dy$$

Apply now (3.1). Then,

$$r^{-\mu}\int_{B(x,r)} u^{p+1}\,dy \le C\varepsilon,$$

for all $x \in B(x_0, r_1)$ and all $r < r_1/2$. Taking ε sufficiently small, it follows from Theorem 2.4 that (u_n) is uniformly bounded near x_0 and so, u is C^2 in a neighborhood of x_0.

Step 2. For all $\gamma \ge 1$, there exists $\varepsilon' > 0$ such that

$$\Sigma_{\varepsilon'} \supseteq \tilde{\Sigma}_{\varepsilon'} := \left\{x \in \Omega : \forall r > 0 \quad \int_{B(x,r)} u^{p+\gamma}\,dx \ge \varepsilon' r^{N-2\frac{p+\gamma}{p-1}}\right\}.$$

Indeed, suppose $x \notin \tilde{\Sigma}_{\varepsilon'}$. Then,

$$\int_{B(x,r)} u^{p+\gamma}\,dx < \varepsilon' r^{N-2\frac{p+\gamma}{p-1}}$$

for some $r > 0$. By Hölder's inequality,

$$\begin{aligned} \int_{B(x,r)} u^{p+1}\,dx &\le C\left(\int_{B(x,r)} u^{p+\gamma}\,dx\right)^{\frac{p+1}{p+\gamma}} r^{N(1-\frac{p+1}{p+\gamma})} \\ &< C\left(\varepsilon' r^{N-2\frac{p+\gamma}{p-1}}\right)^{\frac{p+1}{p+\gamma}} r^{N(1-\frac{p+1}{p+\gamma})} = C(\varepsilon')^{\frac{p+1}{p+\gamma}} r^{N-2\frac{p+1}{p-1}}. \end{aligned} \tag{3.2}$$

Take a function $\varphi \in C^2_c(\Omega)$ and multiply the Lane-Emden equation (1.1) by $u\varphi^2$. Then,

$$\int_\Omega |\nabla u|^2 \varphi^2 \, dx + \int_\Omega u \nabla u \cdot \nabla \varphi^2 \, dx = \int_\Omega u^{p+1} \varphi^2 \, dx$$

i.e.

$$\int_\Omega |\nabla u|^2 \varphi^2 \, dx = \int_\Omega u^{p+1} \varphi^2 \, dx + \frac{1}{2} \int_\Omega u^2 \Delta \varphi^2 \, dx$$

Choose now φ such that $\varphi = 1$ in $B(x, r/2)$, $\varphi = 0$ outside $B(x,r)$, and $|\Delta\varphi^2| \leq C/r^2$. Then,

$$\int_{B(x,r/2)} |\nabla u|^2 \, dx \leq C \int_{B(x,r)} u^{p+1} \, dx + \frac{C}{r^2} \int_{B(x,r)} u^2 \, dx$$

We estimate

$$\begin{aligned} \frac{1}{r^2} \int_{B(x,r)} u^2 \, dx &\leq \frac{C}{r^2} \left(\int_{B(x,r)} u^{p+\gamma} \, dx \right)^{\frac{2}{p+\gamma}} r^{1-\frac{2}{p+\gamma}} \\ &< \frac{C}{r^2} \left(\varepsilon' r^{n-2\frac{p+\gamma}{p-1}} \right)^{\frac{2}{p+\gamma}} r^{1-\frac{2}{p+\gamma}} = C(\varepsilon')^{\frac{2}{p+\gamma}} r^{N-2\frac{p+1}{p-1}}. \end{aligned}$$

Using (3.2), we deduce that

$$\int_{B(x,r/2)} (u^{p+1} + |\nabla u|^2) \, dx < C(\varepsilon')^{\frac{2}{p+\gamma}} r^{N-2\frac{p+1}{p-1}}.$$

Choosing ε' such that $C(\varepsilon')^{\frac{2}{p+\gamma}} \leq \varepsilon$, we deduce that $x \notin \Sigma_\varepsilon$. And so, $\tilde{\Sigma}_{\varepsilon'} \subset \Sigma_\varepsilon$.

Step 3. By the capacitary estimate (Proposition 2.8), $u \in L^{p+\gamma}_{loc}(\Omega)$ if $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1)$. By Theorem 2.5 It follows that for $\varepsilon' > 0$ small,

$$\mathcal{H}^{N-2\frac{p+\gamma}{p+1}}(\tilde{\Sigma}_{\varepsilon'}) = 0.$$

This being true for all $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1)$, Theorem 1.3 follows.

□

4. Proof of the apriori estimates

Proof of Theorem 1.6. The proof of (1.4) is the same as the one given in [20], except for the use of Theorem 2 of [10] stating that there are no entire solutions of finite Morse index if p is in the range of Theorem 1.1. □

Proof of Theorem 1.8. By Theorem 1.1, any finite Morse index solution to (1.1) is C^2, provided $p < p_c(N)$. Working by contradiction, as in the proof of Theorem 2.3 of [20], we can find a sequence (u_k) of solutions of (1.5) (with Morse index at most m) and a sequence of points (x_k) such that by setting

$$\lambda_k = (|u_k(x_k)|^{\frac{p-1}{2}} + |\nabla u_k(x_k)|^{\frac{p-1}{p+1}})^{-1}$$

$$v_k(y) = \lambda_k^{\frac{2}{p-1}} u_k(x_k + \lambda_k y)$$

we have $\lambda_k \to 0$,

$$-\Delta v_k = f_k(v_k) \quad \text{in } B(0,k)$$

where

$$f_k(v) = \lambda_k^{\frac{2p}{p-1}} f(\lambda_k^{-\frac{2}{p-1}} v),$$

and

$$|v_k|^{\frac{p-1}{2}} + |\nabla v_k|^{\frac{p-1}{p+1}} \le 2 \quad \text{in } B(0,k)$$

$$|v_k(0)|^{\frac{p-1}{2}} + |\nabla v_k(0)|^{\frac{p-1}{p+1}} = 1.$$

Note that $f_k(v_k)$ and $\nabla[f_k(v_k)]$ are both uniformly bounded. Then, up to subsequence $v_k \to v$ in the $C^1_{loc}(\mathbb{R}^N)$ topology, $f_k(v_k) \to g$ in the $C^{0,\alpha}_{loc}(\mathbb{R}^N)$ topology (for some $\alpha \in (0,1)$) and $-\Delta v = g$ in the sense of distributions. By standard elliptic estimates v is then a classical $C^{2,\alpha}_{loc}(\mathbb{R}^N)$ solution of $-\Delta v = g$ in $\mathbb{R}^N$.

We claim that v satisfies

$$-\Delta v = a|v|^{p-1}v \quad \text{in } \mathbb{R}^N. \tag{4.1}$$

To this end it is enough to prove that $g = a|v|^{p-1}v \quad$ in $\mathbb{R}^N$. The assumption (1.6) implies

$$\lim_{t\to\pm\infty} \frac{f(t)}{|t|^{p-1}t} = a. \tag{4.2}$$

Therefore, on the open set $[v \ne 0]$, $f_k(v_k(x)) \to a|v(x)|^{p-1}v(x)$ pointwise, hence $g = |v|^{p-1}v$ on $[v \ne 0]$ and also on $\overline{[v \ne 0]}$ by continuity. If $y \notin \overline{[v \ne 0]}$ then v is zero in a neighborhood U_y of y and hence $0 = -\Delta v = g$ in U_y, giving in particular that $g(y) = 0$.

It remains to verify that v has Morse index at most m. We first prove that $\lim_{k\to+\infty} f_k'(v_k(y)) = ap|v(y)|^{p-1}$ pointwise in $\mathbb{R}^N$. This is clearly true for $y \in [v \ne 0]$, thanks to (1.6). On the other hand, for $y \in [v = 0]$, the desired conclusion holds true since $\limsup_{k\to+\infty} |f_k'(v_k(y))| = 0$. Indeed, let us suppose the contrary, then $\lim_{n\to+\infty} |f_{k_n}'(v_{k_n}(y))| > 0$ for a sequence $k_n \nearrow +\infty$. Since $f_{k_n}'(v_{k_n}(y)) = \lambda_{k_n}^2 f'(\lambda_{k_n}^{-\frac{2}{p-1}} v_{k_n}(y))$, the sequence $\lambda_{k_n}^{-\frac{2}{p-1}}|v_{k_n}(y)|$ must be unbounded. Hence, up to a subsequence, $\lambda_{k_n}^{-\frac{2}{p-1}}|v_{k_n}(y)| \to +\infty$ and then, by (1.6), $|f_{k_n}'(v_{k_n}(y))| \le C|v_{k_n}(y)|^{p-1} \to 0$. A contradiction.

To conclude we work again by contradiction. Assume there exist $m+1$ linearly independent functions $\varphi_j \in C_c^\infty(\mathbb{R}^N)$ such that $Q_v(\varphi_j) < 0$, $j = 1, \dots, m+1$, where

$$Q_v(\varphi) = \int_{\mathbb{R}^N} \left(|\nabla\varphi|^2 - ap|v|^{p-1}\varphi^2\right)\, dx.$$

Then, since $\lim_{k\to+\infty} f_k'(v_k(y)) = ap|v(y)|^{p-1}$ pointwise in $\mathbb{R}^N$, we have

$$\lim_{k\to\infty} \int_{B(0,k)} \left(|\nabla\varphi|^2 - f_k'(v_k)\varphi^2\right)\, dx = Q_v(\varphi),$$

for any $\varphi \in C_c^\infty(\mathbb{R}^N)$. Therefore, for k large enough, u_k has Morse index greater or equal than $m+1$, which is a contradiction.

We have constructed a nontrivial C^2 solution of (4.1) of finite Morse index, which is not possible by Theorem 2 of [10] if $p \ne p_S(N)$ (and by Theorem 1 of [10] when $p = p_S(N)$ and $m = 0$). $\square$

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