A stability result for nonlinear Neumann problems in Reifenberg flat domains in $\mathbb{R}^N$.

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Abstract

In this paper we prove that if Ω_k is a sequence of Reifenberg-flat domains in $\mathbb{R}^N$ that converges to Ω for the complementary Hausdorff metric and if in addition the sequence Ω_k has a "uniform size of holes", then the solutions u_k of a Neumann problem of the form

$$\begin{cases} -\mathrm{div}\, a(x, \nabla u_k) + b(x, u_k) = 0 & \text{in } \Omega_k \\ a(x, \nabla u_k) \cdot \nu = 0 & \text{on } \partial\Omega_k \end{cases} \tag{0.1}$$

converge to the solution u of the same Neumann problem in Ω. The result is obtained by proving the Mosco convergence of some Banach spaces. As an application, in the second part of the paper we prove a decay estimate on the gradient for solutions of nonlinear Neumann problems. The estimate is initially established when the boundary is flat and then a similar estimate for perturbed boundaries using the stability property is obtained.

AMS classification. 35J65, 47H05, 49J45.

Key words. Boundary value problems, nonlinear elliptic equations, Hausdorff metric, Reifenberg-flat sets, Mosco convergence.

Contents

Introduction

In this paper we study the stability of solutions for the following nonlinear Neumann problem

$$\begin{cases} -\mathrm{div}\, a(x,\nabla u) + b(x,u) = 0 & \text{in } \Omega \\ a(x,\nabla u)\cdot \nu = 0 & \text{on } \partial\Omega \end{cases} \tag{0.2}$$

where Ω is a bounded subset of $\mathbb{R}^N$, $a : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ and $b : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are two Carathéodory functions satisfying suitable monotonicity, coerciveness and growth conditions (see (1.2)-(1.3) below). More precisely, we are interested in the following question. Let Ω_k be a sequence of open sets in $\mathbb{R}^N$ that converges to Ω for the Hausdorff distance. Let u_k be the sequence of solutions for the problem (0.2) in Ω_k and let u be the solution associated to Ω. Is it true that u_k converges to u ? If the answer is positive we say that the problem (0.2) is stable along the sequence Ω_k (see Definition 3).

This question was studied by many authors in the last decade, principally in dimension 2, for smooth and non smooth domains. The case of non smooth domain, and more specifically domains with cracks, appears typically in applications from fracture mechanics. In addition, the problem of stability is linked to the notions of Mosco convergence or Gamma convergence, which are powerful tools to study the semicontinuity of functionals in shape optimization problems or for instance the Mumford-Shah functional in image processing.

Recently, G. Dal Maso, F. Ebobisse and M. Ponsiglione [6] proved that in dimension 2, if Ω_k tends to Ω for the complementary Hausdorff distance in $\mathbb{R}^2$, $|\Omega_k|$ tends to $|\Omega|$ and if the number of the connected components of the complements Ω_k^c are uniformly bounded, then the problem (0.2) is stable. This stability property extends the corresponding results of A. Chambolle and F. Doveri [5] and also D. Bucur and N. Varchon [4]. In particular, the authors showed that the stability is equivalent to the convergence of some Banach spaces in the sense of Mosco.

According to the authors knowledge, only very few results have been proven in higher dimensions. In [12], A. Giacomini proves a stability result in $\mathbb{R}^N$ for fractured locally Lipschitz domains where an approach involving the Mosco convergence is also used.

A famous example called the "Neumann Sieve" ([10], [19], [21]), shows that in many cases the stability is not true. The general idea, for instance in dimension 2, is that if one considers a sequence of 1-dimensional sets E_k in $B(0,1)$ with more and more holes of size $\frac{1}{k}$ but with always same global length, it could happen that E_k would converge to a segment while the problem (0.2) is not stable along the sequence $B(0,1)\backslash E_k$. One can notice that this example is very close to the one that makes the Hausdorff measure $K \mapsto H^1(K)$ not lower semicontinuous with respect to the Hausdorff distance.

In this paper, we partially extend the results of [6] and [12]. We prove that if Ω_k converges to Ω for the complementary Hausdorff distance in $\mathbb{R}^N$ and if $\{\Omega_k\}$ is a sequence of Reifenberg flat domains with "uniform size of holes", then problem (0.2) is stable along the sequence $\{\Omega_k\}$. The main point is to prove that the limit in the sense of Mosco, of the space of restrictions of functions in $W^{1,p}(\Omega_k)$ is the space of restrictions of functions in $W^{1,p}(\Omega)$.

The notion of "uniform size of holes" (Definition 8) is here to avoid the "Neumann Sieve" problem as described above. Indeed, we want to allow crack domains but according to our definition of uniform size of holes, the "tips" of the cracks can never become arbitrary closer to each other at the limit.

Our theorem partially extends the result of [6] in any dimension, and it also can be considered as an extension of [12] since the regularity of our sets is weaker than Lipschitz. Indeed, Reifenberg flat sets are ones that are locally well-approximated by hyperplanes, at every scale. This allows for instance some "Hölderian spirals" or fractal boundaries (see [9]).

Reifenberg flat sets are naturally used in the study of boundary regularity and the regularity of free boundaries coming from a minimization problem (see for instance [7], [8], [15], [18] and references therein). As we shall see in the sequel, Reifenberg flat domains are moreover very well-adapted in the construction of good extensions for functions in the Sobolev space, which is the key ingredient for proving the Mosco convergence. This sort of technics were also used in [14], and probably appeared for the first time in [13] while Peter Jones was seeking some geometrical conditions to define a class of extension domains.

In the second part of the paper, we prove a decay estimate for the solution to nonlinear elliptic operators of divergence form at the boundary. At first glance we prove the decay when the boundary is flat. This is done by establishing a differential inequality and using some estimates on the smallest eigenvalue in the half-sphere. Then we use the stability to prove that this estimate is still true when the boundary is a Reifenberg flat set close enough to a hyperplane.

This decay result gives a nice example of how the stability property can be used. It can be considered as a partial extension of [14] for nonlinear minimizers. In addition, such a decay estimate is useful for studying minimizers of fully nonlinear convex functionals. We anticipate that a similar argument is needed in our study for the regularity of minimizers of the Mumford-Shah functional with a nonlinear term of energy. This fact along with a slightly different approach will be the content of a forthcoming paper [16].

Notation :

The real numbers p and q satisfy $1 < p \leq 2 \leq q < +\infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

$\partial\Omega := \overline{\Omega}\backslash\Omega$, the topological boundary of Ω.

$W^{1,p}(\Omega) :=$ the Sobolev space.

$C^1_c(\Omega) :=$ space of $C^1(\Omega)$ functions with compact support.

$\nu :=$ the outward normal vector.

$|A| :=$ the Lebesgue measure of the Borel set A.

$d_H :=$ the Hausdorff distance (defined in (1.4)).

$A^c :=$ the complement of the set A.

$A \triangle B := A\backslash B \cup B\backslash A$.

$B(x,r) :=$ the ball with center at x and radius r.

$C :=$ a positive constant, that could vary from line to line, and that only could depend on dimension.

1 Stability for the nonlinear Neumann problem

Let Ω be an open bounded subset of $\mathbb{R}^N$. In this section we prove the stability for problem (0.2) where $a : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ and $b : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are two Carathéodory functions satisfying the following assumptions : there exist $0 < c_1 \leq c_2$, $\alpha \in L^q(\mathbb{R}^N)$, and $\beta \in L^1(\mathbb{R}^N)$ such that, for almost every $x \in \mathbb{R}^N$ and for every $\xi, \xi_1, \xi_2 \in \mathbb{R}^N$ with $\xi_1 \neq \xi_2$

$$\langle a(x,\xi_1) - a(x,\xi_2), (\xi_1 - \xi_2)\rangle > 0\ ; \tag{1.1}$$

$$|a(x,\xi)| \leq \alpha(x) + c_2|\xi|^{p-1}\ ; \tag{1.2}$$

$$a(x,\xi)\cdot\xi \geq -\beta(x) + c_1|\xi|^p. \tag{1.3}$$

We assume that b satisfies the same inequalities.

Throughout the paper we assume that (0.2) is satisfied in the usual weak sense of Sobolev spaces, that is $u \in W^{1,p}(\Omega)$ and

$$\int_\Omega \langle a(x,\nabla u), \nabla\varphi\rangle + \langle b(x,u), \varphi\rangle \,\mathrm{d}x = 0$$

for all $\varphi \in W^{1,p}(\Omega)$. It is well known that problem (0.2) has a unique solution in $W^{1,p}(\Omega)$ (see [17]).

If A and B are two nonempty closed subsets of $\mathbb{R}^N$, we define the Hausdorff distance between A and B by

$$d_H(A,B) := \max\{\sup_{x\in B} d(x,A), \sup_{x\in A} d(x,B)\} \tag{1.4}$$

where $d(x,A) = \mathrm{dist}(x,A)$.

Definition 1. *Let $\{\Omega_k\}_{k\in\mathbb{N}}$ and Ω be some nonempty open subsets of $\mathbb{R}^N$. We say that Ω_k converges to Ω for the complementary Hausdorff distance if $d_H(\Omega_k^c, \Omega^c)$ tends to* 0.

Remark 2. Notice that if Ω_k tends to Ω for the complementary Hausdorff distance, then $d_H(\partial\Omega_k, \partial\Omega)$ tends to zero, but observe that the converse is false in general (for instance take $\Omega_k := (\frac{1}{k}, +\infty)$ and $\Omega := (-\infty, 0)$).

Suppose that Ω_k converges to Ω for the complementary Hausdorff distance, and let u_k be a weak solution of the problem

$$\begin{cases} -\text{div } a(x, \nabla u_k) + b(x, u_k) = 0 & \text{in } \Omega_k \\ a(x, \nabla u_k) \cdot \nu = 0 & \text{on } \partial\Omega_k. \end{cases} \tag{1.5}$$

Definition 3. *We say that problem* (0.2) *is stable along the sequence* Ω_k*, if the following holds: Let* $u_k \in W^{1,p}(\Omega_k)$ *be a sequence of solutions of the problem* (1.5) *in* Ω_k*. Then* $(u_k 1_{\Omega_k}, \nabla u_k 1_{\Omega_k})$ *converges strongly to* $(v1_\Omega, \nabla v 1_\Omega)$ *in* $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N, \mathbb{R}^N)$ *and* v *is a solution of problem* (0.2)*.*

We seek conditions on Ω_k to make the problem (0.2) stable.

Definition 4. *A* (δ, r_0)*-Reifenberg-flat domain* $\Omega \subset \mathbb{R}^N$ *is an open bounded set such that for each* $x \in \partial\Omega$ *and for any* $r \leq r_0$ *there exist a hyperplane* $P(x,r)$ *containing* x *such that*

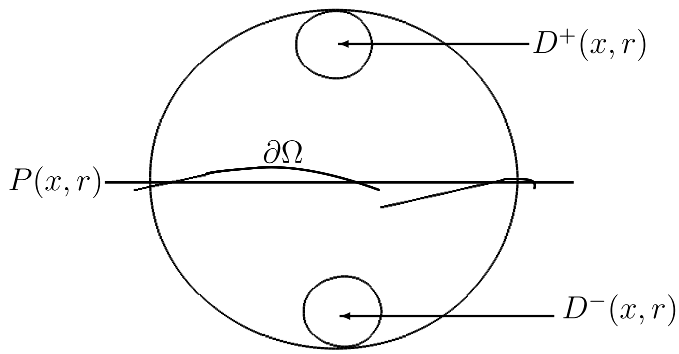
$$\frac{1}{r} \sup_{y \in \partial\Omega \cap B(x,r)} dist(y, P(x,r)) \leq \delta. \tag{1.6}$$

Our definition of Reifenberg-flat domains is not exactly the same that could be found in the literature, essentially from the fact that we didn't take a bilateral definition of the distance in (1.6) in order to allow cracks (when the domain lies in each side of its boundary). Using the terminology of [14], our Reifenberg-flat domains should be called "weak Reifenberg-flat domains". Observe that our definition allows the fact that $\partial\Omega$ could have a infinite number of connected components. Moreover in our definition, Ω is not supposed to be connected.

The topological disc theorem of Reifenberg [20] says that, under some additional separation conditions and if δ is small enough, then the boundary of a Reifenberg-flat domain is locally the bi-hölderian image of a $N-1$ dimensional unit disc. In addition, this is optimal since a Reifenberg-flat domain can admit some Hölder spiral. It is worth mentioning that a Reifenberg-flat domain could have a fractal "snowflake-like" boundary as it is shown in [9]. As a consequence, the regularity of our domains is weaker than the one considered in [12] which are essentially locally Lipschitz domains.

Now we consider a topological assumption in order to avoid the problem of "Neumann Sieves" (see [19]). For a Reifenberg-flat domain Ω and for any ball $B(x,r)$ centered at $\partial\Omega$

and with radius $r \leq r_0$, let us define the sets $D^+(x,r)$ and $D^-(x,r)$ by the following way. Let $P(x,r)$ be the hyperplane given by the definition of Reifenberg flatness of Ω. Denote by $z^{\pm}(x,r)$ two points of $B(x,r)$ that lie at distance $3r/4$ from $P(x,r)$ and whose orthogonal projections on $P(x,r)$ are equal to x. Then we set $D^{\pm}(x,r) := B(z^{\pm}, r/4)$ as in the following picture.



Definition 5. *Let Ω be a (δ, r_0)-Reifenberg flat domain. We say that Ω has a uniform size of holes with constant $C_0 \geq 1/2$, if for every ball $B(x,r)$ centered at $\partial\Omega$ with radius $r \leq \frac{1}{2}r_0$ and such that $D^+(x,r)$ and $D^-(x,r)$ lie in the same connected component of $B(x,r) \cap \Omega$, there exists a ball $B(y,s)$ centered on $P(x,2r) \cap B(x,2r)$ with radius $s > \frac{1}{C_0}r$ such that $B(y,s) \cap \partial\Omega = \emptyset$. By convention if Ω is such that $D^{\pm}(x,r)$ always lie in different connected components we say that Ω has a uniform size of holes with $C_0 = 1/3$.*

Remark 6. If Ω has a uniform size of holes with constant $C_0 \geq 1/2$ then Ω has a uniform size of holes with any constant $C > C_0$. Therefore, we will usually take C_0 as being the minimal constant satisfying the property of Definition 5 which can never be less than $1/2$ for obvious geometrical reasons. Moreover when we mean that a sequence Ω_k has a uniform size of holes with same constant C_0, we say that all the minimal constants associated to Ω_n are bounded by a same constant C_0 (see also the examples given below).

The rest of the section is devoted to the proof of the following result.

Theorem 7. *Let $C_0 \in [1/2, +\infty) \cup \{1/3\}$ be a constant and assume that $\{\Omega_k\}_{k\in\mathbb{N}}$, Ω are $(\delta, r_0)-$Reifenberg flat domains with $\delta < 10^{-3}C_0^{-1}$, having a uniform size of holes with same* *constant C_0, and such that Ω_k converges to Ω for the complementary Hausdorff distance. Then the Neumann problem* (0.2) *is stable along the sequence Ω_k.*

Before passing to the details of the proof of Theorem 7, we give two examples for domains described in the discussion above.

1.1 Examples

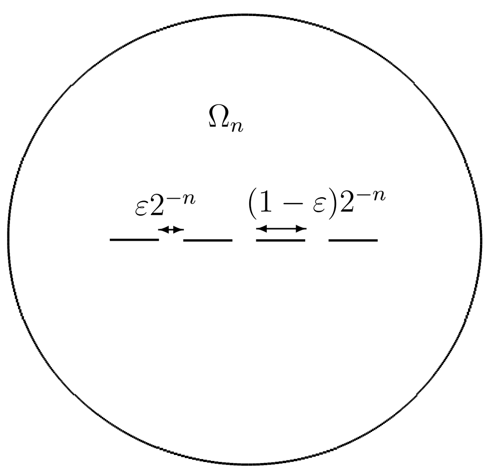
We would like to emphasize the fact that according to Definition 5, in Theorem 7 we do not consider "sieve domains" with holes in the boundary that are becoming smaller when Ω_k tends to Ω, but only "crack domains" which contain some holes but with a fixed size bigger than $C_0^{-1}r_0$ and controlled shape. To illustrate this, let us show two basic examples in dimension 2.

1.1.1 A counterexample

Let us consider the classical 2 dimensional Sieve Domain defined as follow. For a fixed ε set $r_n := \varepsilon 2^{-n}$,

$$\Gamma_n := [0,1]\backslash \bigcup_{k=1}^{2^n} [k2^{-n} - r_n, k2^{-n}]$$

and $\Omega_n := B((1/2,0),1)\backslash\Gamma_n$.



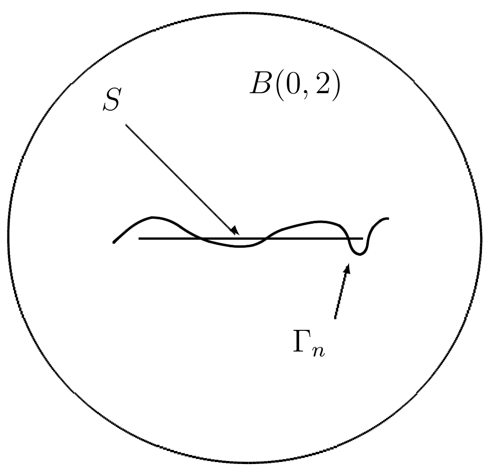
Since the boundary of Ω_n is "smooth", for every $\delta \in (0,1)$, one can easily obtain a radius $r_\delta \in (0, \frac{1}{10})$ such that all the Ω_n are (δ, r_δ)-Reifenberg flat domains. Now for n big enough, we claim that the minimal constant C_0^n of uniform size of holes in Ω_n is bounded from below by $C2^n$. Indeed, we denote $x_0 := (1/2, 0) \in \mathbb{R}^2$ and consider the ball $B(x_0, r_\delta/2)$. The corresponding domains $D^{\pm}(x_0, r_\delta/2)$ lie in the same connected components of Ω and $P(x, r_\delta/2)$ is the first axis. Now since $r_\delta < \frac{1}{10}$, any ball $B(y, s)$ centered on $P(x_0, r_\delta) \cap B(x_0, r_\delta)$ and such that $B(y, s) \cap \partial\Omega_n = \emptyset$ must have a radius $s < \varepsilon 2^{-(n+1)}$. This means that

$$C_0^n \geq r_\delta \frac{2^{n+1}}{\varepsilon}$$

thus the sequence Ω_n cannot have a uniform size of hole with same constant C_0.

1.1.2 An example with fractured domains

Further, we give another example where the domains Ω_n have now a uniform size of holes with same constant. Of course we could also take some "non-fractured" domains for which the constant is fixed to $\frac{1}{3}$ by convention, but let us consider the following more instructive crack situation. Let S be the segment $[-1, 1] \times \{0\} \subset \mathbb{R}^2$ and $\Omega := B(0, 2) \backslash S$. Now let $\Gamma_n := \{(t, f_n(t)); t \in [-a_n, a_n] \subset \mathbb{R}\}$ be a sequence of Lipschitz graphs with same Lipschitz constant L and such that $d_H(\Gamma_n, S) \leq 2^{-n-3}$.



It is not difficult to see that the sequence $\Omega_n := B(0, 2) \backslash \Gamma_n$ is a sequence of (δ, r_0)-Reifenberg-flat domains converging to $\Omega \backslash S$ for the complementary Hausdorff distance. Indeed, they are Reifenberg-flat with a good choice of δ and r_0 depending only on the Lipschitz constant L. Moreover we have that $d_H(\Omega_n^c, \Omega^c) = d_H(\Gamma_n, S) \leq 2^{-n-3}$. Finaly, Ω_n and Ω have all a uniform size of holes with constant $C_0 \leq 10$. This is easy to prove for Ω because the only way for a ball $B(x,r)$ to be centered on $\partial\Omega$ and having the property that $B(x,r)\cap\Omega$ is connected is to be centered on S with a radius $r > dist(x,E(S))$ where $E(S)$ are the two endpoints of S. In this case it is clear that $B(x,2r)$ contains a ball of radius $s > 10^{-1}r$ centered on the first axis that does not meet $\partial\Omega$. Now one can see this also for Ω_n by exactly the same argument, replacing the endpoints of S by the endpoints of Γ_n, using also that Γ_n is a graph.

Of course in our example, Lipschitz graphs was assumed for convenience and one could try to weaken the regularity assumption on Γ_n taking for instance a sequence of connected Reifenberg-flat sets but the proof become more technical in this case.

1.2 Whitney Extension

The main ingredient for proving the Mosco convergence will be the extension Lemma contained in this section. For every function $u \in L^1_{loc}(B(x,r))$ we denote by $m^{\pm}(u)$ the average of u on $D^{\pm}(x,r)$

$$m^{\pm}(u) := \frac{1}{|D^{\pm}(x,r)|}\int\limits_{D^{\pm}(x,r)} u(x)\,\mathrm{d}x. \tag{1.7}$$

We begin by the following fact.

Proposition 8. *Let $C_0 \geq 1/2$ and let Ω be a $(\delta, r_0)-$Reifenberg flat domain with $\delta < 10^{-3}C_0^{-1}$ and having uniform size of holes with constant C_0. Then for every ball $B(x,r)$ centered at $\partial\Omega$ with $r \leq \frac{1}{4}r_0$ such that $D^+(x,r)$ and $D^-(x,r)$ lie in the same connected component of $B(x,r)\cap\Omega$, we have that*

$$|m^+(u) - m^-(u)| \leq C\frac{1}{r^{N-1}}\int\limits_{B(x,3r)} |\nabla u|\,\mathrm{d}x \tag{1.8}$$

for any function $u \in W^{1,1}(B(x,3r)\cap\Omega)$, and where C is depending on dimension N and constant C_0.

Proof. The proof is an easy consequence of the classical Poincaré inequality. Indeed, the definition of Uniform size of holes implies that there exist a ball $B(x,s)$ centered on $P(x,2r)\cap B(x,2r)$ such that $B(x,s)\cap\partial\Omega = \emptyset$, and with $s > \frac{1}{C_0}r$. Now the proposition follows from the following fact : since $\partial\Omega$ is at distance less than $2\delta r < 2.10^{-3}C_0^{-1}r$ from $P(x,2r)$ in $B(x,2r)$, one can define a Lipschitz domain A contained in $B(x,3r)$, that contains $D^+(x,r)\cup D^-(x,r)\cup B(x,\frac{s}{2})$, and such that the Poincaré constant in A is less than $C(C_0,N)r$, where $C(C_0,N)$ is only depending on C_0 and N and is independent from all the possible positions of $B(x,s)$. □

Next we give the Extension lemma.

Lemma 9. *Let r_0 be a positive radius, $\tau \in (0,r_0)$ and $C_0 \in [1/2,+\infty)\cup\{1/3\}$. Let Ω_1 and Ω_2 be two (δ,r_0)-Reifenberg-flat domains with $\delta < 10^{-3}C_0^{-1}$, having a uniform size of holes with same constant C_0 and such that*

$$d_H(\Omega_1^c,\Omega_2^c) \leq 10^{-3}C_0^{-1}\tau.$$

Then by setting

$$W(\tau) := \{y \in \mathbb{R}^N; d(y,\partial\Omega_1) \leq \tau\},$$

for any $v \in W^{1,p}(\Omega_1)$ there exists a function $\tilde{v} \in W^{1,p}(\Omega_2)$ such that $v = \tilde{v}$ in $\Omega_1\backslash W(10\tau)$ and

$$\|\tilde{v}\|_{L^p(\Omega_2\cap W(10\tau))} \leq C\|v\|_{L^p(\Omega_1\cap W(10\tau))} \tag{1.9}$$

$$\|\nabla\tilde{v}\|_{L^p(\Omega_2\cap W(10\tau))} \leq C_1\|\nabla v\|_{L^p(\Omega_1\cap W(60\tau))} \tag{1.10}$$

with C depending only on the dimension N, while C_1 depends on constant C_0 and on dimension N.

Proof. Let $\tau \in (0,r_0)$ be fixed and let $B_i := B(x_i,\tau)$ be a family of balls of radius τ, centered at $x_i \in \partial\Omega_1$, and maximal for the property that $\frac{1}{10}B_i \cap \frac{1}{10}B_j = \emptyset$ for all $i,j \in I$ with $i \neq j$. Notice that by this way,

$$W(9\tau) \subset \bigcup_{i\in I} 10B_i \subset W(10\tau).$$

We want to construct a partition of unity associated to $\{B_i\}_{i\in I}$. For all i, define a function $\varphi_i \in C^1_c(10B_i)$, such that $\varphi = 1$ in $8B_i$, $|\nabla\varphi| \leq \tau^{-1}$ and let φ_0 be a function that is equal to 1 in $\Omega_1\backslash\bigcup_{i\in I}10B_i$, $\varphi_0 = 0$ in $\bigcup_{i\in I}8B_i$ and $\varphi_0 + \sum_{i\in I}\varphi_i \geq 1$ in $\Omega_1\cup\bigcup_{i\in I}10B_i$. Moreover, we can assume that there is a constant C such that for all $x \in 10B_i \backslash 8B_i$, $|\nabla \varphi_0(x)| \leq C\tau^{-1}$. Indeed, such a function φ_0 can be obtained by setting

$$\varphi_0(x) := \prod_{i \in I} l(d(x, x_i)/\tau)$$

where l is a Lipschitz function equal to 0 in $[0, 8]$, equal to 1 in $[10, +\infty)$ and $l'(x) \leq 10$. Finally, define

$$\theta_i := \frac{\varphi_i}{\varphi_0 + \sum_{i \in I} \varphi_i} \quad \text{for } i \in I \cup \{0\}$$

thus we now have a partition of unity in $\Omega_1 \cup \bigcup_{i \in I} B_i$. To define a function $\tilde{v} \in W^{1,p}(\Omega_2)$, it suffices to define an extension of v in $\Omega_2 \cap \bigcup_{i \in I} 10B_i$ since $\Omega_1 \triangle \Omega_2$ is contained in $\bigcup_{i \in I} 10B_i$.

Let $P_i := P(x_i, 10\tau)$ be the hyperplane associated to Ω_1 given by definition 4. Recall that since $d_H(\Omega_1^c, \Omega_2^c) \leq 10^{-3} C_0^{-1} \tau$, we have that $10B_i \cap (\partial \Omega_2) \subset \{y \in 10B_i; d(y, P_i) \leq 2.10^{-2} C_0^{-1} \tau\}$.

For each ball B_i, we denote by $D^{\pm}(x_i, 10\tau)$ the two balls defined just before Definition 8 associated to Ω_1. We need to consider three cases.

• Cracktip case : $D^+(x_i, 10\tau)$ and $D^-(x_i, 10\tau)$ lie both in the same connected component of $B(x, 10\tau) \cap \Omega_2$. Let I_c be the set of indices corresponding to the balls in this situation and for every $i \in I_c$ define $m_i := m^+(v)$ (as in (1.7)).

• Boundary case 1: $D^+(x_i, 10\tau)$ and $D^-(x_i, 10\tau)$ lie in different connected components of $B(x, 10\tau) \cap \Omega_2$. Let I_{b_1} be the set of indices corresponding to the balls in this situation. For every $i \in I_{b_1}$, let A_i^+ and A_i^- be the two connected components of $10B_i \backslash \partial \Omega_2$ that contain respectively $D^+(x_i, 10\tau)$ and $D^-(x_i, 10\tau)$ and define $m_i^{\pm} := m^{\pm}(v)$.

• Boundary case 2: One of $D^{\pm}(x_i, 10\tau)$ lies in Ω_2 while the other one lies in Ω_2^c. Let I_{b_2} be the set of indices corresponding to the balls in this situation. For every $i \in I_{b_2}$, let m_i be equal to the one of $m^{\pm}(v)$ corresponding to ball $D^{\pm}(x_i, 10\tau)$ that lies in Ω_2. Finally as for I_{b_1}, let A_i^+ and A_i^- be the two connected components of $10B_i \backslash \partial \Omega_2$ that contain respectively $D^+(x_i, 10\tau)$ and $D^-(x_i, 10\tau)$.

We are now ready to define $\tilde{v}$. It could be that Ω_2 has some tiny connected components hidden in some $10B_i \backslash (A_i^+ \cup A_i^-)$ for $i \in I_{b_1}$. In those components, let us define $\tilde{v}$ to be equal to 0. Then, anywhere else, i.e. for all $x \in A := (\Omega_2 \setminus \bigcup_{i\in I_{b_1}} 10B_i) \cup (\bigcup_{i\in I_{b_1}} A_i^+ \cup A_i^-)$, define

$$\tilde{v}(x) := \theta_0 v(x) + \sum_{i\in I_c\cup I_{b_2}} \theta_i(x) m_i + \sum_{i\in I_{b_1}} \theta_i(x)(m_i^+ 1_{A_i^+}(x) + m_i^- 1_{A_i^-}(x)). \tag{1.11}$$

The function $\tilde{v}$ is now well defined for every $x \in \Omega_2$. We claim that $\tilde{v} \in W^{1,p}(\Omega_2)$ and that (1.9) and (1.10) are satisfied. Let us first show (1.9). Using Hölder inequality and extending v by 0 outside of Ω_1, we have for all $i \in I_{b_2} \cup I_c$,

$$|m_i|^p \leq C\Big(\frac{1}{|B_i|}\int\limits_{B_i} |v| \,\mathrm{d}x\Big)^p \leq |B_i|^{-1}\int\limits_{B_i} |v|^p \tag{1.12}$$

and by the same way for $i \in I_{b_1}$ we also have

$$|m_i^\pm|^p \leq |B_i|^{-1}\int\limits_{B_i} |v|^p. \tag{1.13}$$

In addition, since B_i are in a bounded cover (with a universal constant C), the sums in (1.11) are locally finite. Thus, since $\theta_i(x) \leq 1_{10B_i}(x)$ and using (1.12) and (1.13),

$$\| \sum_{i\in I_c\cup I_{b_2}} \theta_i(x) m_i \|^p_{L^p(\Omega_2)} \leq C \sum_{i\in I_c\cup I_{b_2}} \int\limits_{10B_i} |m_i|^p \leq C \sum_{i\in I} \int\limits_{10B_i} |v|^p \,\mathrm{d}x \leq C\|v\|^p_{W(10\tau)}$$

and by the same way

$$\| \sum_{i\in I_{b_1}} \theta_i(x)(m_i^+ 1_{A_i^+}(x) + m_i^- 1_{A_i^-}(x)) \|^p_{L^p(\Omega_2)} \leq C \sum_{i\in I} \int\limits_{10B_i} |v|^p \,\mathrm{d}x \leq C\|v\|^p_{W(10\tau)}$$

thus taking the L^p norm in (1.11) we deduce that (1.9) holds. Let us now prove (1.10), which will also imply that $\tilde{v} \in W^{1,p}(\Omega_2)$. By definition, $\tilde{v} = v$ in $\Omega_2 \setminus \bigcup_{i\in I} 10B_i$, thus all we have to prove is that

$$\int\limits_{A\cap\bigcup_{i\in I} 10B_i} |\nabla \tilde{v}|^p \leq \int\limits_{\Omega_1\cap W(10\tau)} |\nabla v|^p. \tag{1.14}$$

Let $x \in A \cap \bigcup_{i\in I} 10B_i$ and let I_x be the finite set of indices $i \in I$ such that $x \in B_i$. All balls B_i for $i \in I_x$ are contained in $B(x_{i_0}, 20\tau)$ for any $i_0 \in I_x$. Let such an i_0 be fixed. Let us define two convex domains D_x^+ and D_x^- such that $D_x^\pm$ is the convex hull of all the $D^\pm(x_i, 10\tau)$ for $i \in I_x$. Since $(\partial\Omega_1 \cup \partial\Omega_2) \cap B(x_{i_0}, 20\tau)$ is contained in $\{y; d(y, P_0) \leq \frac{2\tau}{10}\}$

for some hyperplane P_0, we have that $D_x^\pm$ do not meet $(\partial\Omega_1 \cup \partial\Omega_2) \cap B(x_{i_0}, 20\tau)$. Suppose firstly that if there exists an $i \in I_x \cap (I_{b_1} \cup I_{b_2})$, then x lies in A_i^+. Define

$$m_x^+ := \frac{1}{|D_x^+|} \int\limits_{D_x^+} v(x)\, \mathrm{d}x.$$

Since $\nabla\theta_0 + \sum_{i\in I} \nabla\theta_i = 0$ we have

$$\begin{aligned}
\nabla v(x) &= \theta_0(x)\nabla v(x) + \nabla\theta_0(x)v(x) + \sum_{i\in I_c\cup I_{b_2}} \nabla\theta_i m_i + \sum_{i\in I_{b_1}} \nabla\theta_i m_i^+ 1_{A_i^+}(x) \\
&= \theta_0(x)\nabla v(x) + \nabla\theta_0(x)(v(x) - m_x^+) + \sum_{i\in I_c\cup I_{b_2}} \nabla\theta_i (m_i - m_x^+) + \sum_{i\in I_{b_1}} \nabla\theta_i (m_i^+ - m_x^+).
\end{aligned}$$

Now by the Poincaré inequality,

$$|m_i^+ - m_x^+| \leq \frac{C}{\tau^N} \int\limits_{D^+(x_i,10\tau)} |u - m_x^+| \leq \frac{C}{\tau^N} \int\limits_{D_x^+} |u - m_x^+| \leq \frac{C}{\tau^{N-1}} \int\limits_{D_x^+} |\nabla u|. \tag{1.15}$$

If $i \in I_c$, we have the same inequality with m_i instead of m_i^+. On the other hand, since D_x^+ is convex, and after a mollification of v if necessary,

$$\begin{aligned}
|v(x) - m_x^+| &\leq \frac{1}{|D_x^+|} \int\limits_{D_x^+} |v(x) - v(y)| dy \\
&\leq \frac{1}{|D_x^+|} \int\limits_{D_x^+} \int\limits_{[0,1]} |x - y||\nabla v(x + t(y - x))| dt dy \\
&\leq \frac{1}{|D_x^+|} \int\limits_{D_x^+} \int\limits_{D_x^+} |\nabla v(z)| \frac{dz}{|x - y|^{N-1}} dy \leq C \frac{1}{\tau^{N-1}} \int\limits_{D_x^+} |\nabla v(z)| dz.
\end{aligned}$$

Therefore we have proved,

$$|\nabla \tilde{v}(x)| \leq \theta_0(x)|\nabla v(x)| + C\frac{1}{\tau^N} \int\limits_{D_x^+} |\nabla v|\, \mathrm{d}x. \tag{1.16}$$

Now suppose that if there exists an $i \in I_x \cap (I_{b_1} \cup I_{b_2})$, then x lies in A_i^-. Then we can proceed as above, except for inequality (1.15) with $i \in I_c$ since by convention in this case we set $m_i := m^+(v)$. However, since the boundary of $\partial\Omega_1$ has a uniform size of holes, we can estimate the difference $|m^+(v) - m^-(v)|$ by (1.8). Indeed, we know that $D_{\Omega_1}^\pm(x_i, 10\tau)$ lie in the same connected component of $B(x_i, 10\tau) \cap \Omega_2$. This means that there is a ball $B(y_i, s)$

centered on $P(x_i,10\tau)\cap B(x_i,20\tau)$ with radius $s > C_0^{-1}20\tau$, such that $B(y_i,s)\cap\partial\Omega_1=\emptyset$. But since $d_H(\Omega_1^c,\Omega_2^c) < 10^{-3}C_0^{-1}\tau$, we deduce that $B(y_i,\frac{s}{2})\cap\partial\Omega_2=\emptyset$ thus $D^{\pm}_{\Omega_2}(x_i',20\tau)$ (the one associated to Ω_2 for a $x_i'\in\partial\Omega_2$ satisfying $dist(x_i,x_i')\leq 10^{-3}C_0^{-1}\tau$), lie also in the same connected components of $B(x_i',20\tau)\cap\Omega_2$. Then Proposition 8, together with an other application of the Poincaré inequality to estimate the difference between the average on $D^{\pm}_{\Omega_2}(x_i',20\tau)$ and the average on $D^{\pm}_{\Omega_1}(x_i,10\tau)$, gives

$$|m_i - m_x^-| \leq |m_i - m^-(v)| + |m^-(v) - m_x^-| \leq \frac{C}{\tau^{N-1}}\int_{B(x_i,60\tau)} |\nabla u|$$

where C is now depending on C_0.

In conclusion, extending ∇v by 0 out of Ω_1 we have obtained, for all $x\in\Omega_2\cap\bigcup_{i\in I}10B_i$

$$|\nabla\tilde{v}(x)| \leq \theta_0(x)|\nabla v(x)| + C\frac{1}{\tau^N}\int_{B(x_{i_0},60\tau)} |\nabla v|\,\mathrm{d}x. \tag{1.17}$$

Following, we will denote i_x instead of i_0. Then we have

$$\|\nabla\tilde{v}(x)\|_{L^p(\Omega_2\cap\bigcup_{i\in I}10B_i)} \leq \|\nabla v\|_{L^p(\Omega_1\cup\bigcup_{i\in I}10B_i)} + \frac{C}{\tau^N}\Big(\int_{\bigcup_{i\in I}10B_i}\Big(\int_{B(x_{i_x},60\tau)}|\nabla v|\Big)^p\Big)^{\frac{1}{p}}.$$

On the other hand,

$$\begin{aligned}
\frac{C}{\tau^{pN}}\int_{\bigcup_{i\in I}10B_i}\Big(\int_{B(x_{i_x},30\tau)}|\nabla v|\,\mathrm{d}y\Big)^p\mathrm{d}x &\leq \frac{C}{\tau^{pN}}\int_{\bigcup_{i\in I}10B_i}\tau^{N\frac{p}{q}}\int_{B(x_{i_x},60\tau)}|\nabla v|^p\,\mathrm{d}y\,\mathrm{d}x\\
&\leq \frac{C}{\tau^N}\int_{\bigcup_{i\in I}60B_i}|\nabla v|^p\int_{\{x;y\in B(x_{i_x},60\tau)\}}\mathrm{d}x\,\mathrm{d}y\\
&\leq C\int_{\bigcup_{i\in I}60B_i}|\nabla v|^p\,\mathrm{d}x\\
&\leq C\int_{W(60\tau)}|\nabla v|^p\,\mathrm{d}x
\end{aligned}$$

thus (1.10) follows and the proof is complete. □

1.3 Mosco-convergence

For every open set $\Omega \subset \mathbb{R}^N$ we define the closed linear subspace X_Ω of $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N, \mathbb{R}^N)$ by

$$X_\Omega := \{(u1_\Omega, \nabla u 1_\Omega); u \in W^{1,p}(\Omega)\}. \tag{1.18}$$

Definition 10 (Mosco-convergence)**.** *Let Ω_k and Ω be open subsets of $\mathbb{R}^N$ and let X_{Ω_k} and X_Ω be the corresponding subspaces of $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N, \mathbb{R}^N)$ defined by* (1.18)*. We say that X_{Ω_k} converges to X_Ω in the sense of Mosco if the following two properties hold:*

$(M1)$ *for every $u \in W^{1,p}(\Omega)$, there exists a sequence $u_k \in W^{1,p}(\Omega_k)$ such that $u_k 1_{\Omega_k}$ converges to $u1_\Omega$ strongly in $L^p(\mathbb{R}^N)$ and $\nabla u_k 1_{\Omega_k}$ converges to $\nabla u 1_\Omega$ strongly in $L^p(\mathbb{R}^N, \mathbb{R}^N)$;*

$(M2)$ *if h_k is a sequence of indices converging to ∞, u_k is a sequence such that $u_k \in W^{1,p}(\Omega_{h_k})$ for every k, and $u_k 1_{\Omega_{h_k}}$ converges weakly in $L^p(\mathbb{R}^N)$ to a function ϕ, while $\nabla u_k 1_{\Omega_{h_k}}$ converges weakly in $L^p(\mathbb{R}^N, \mathbb{R}^N)$ to a function ψ, then there exists $u \in W^{1,p}(\Omega)$ such that $\phi = u1_\Omega$ and $\psi = \nabla u 1_\Omega$ a.e. in $\mathbb{R}^N$.*

The Mosco convergence is a great tool to study stability for Neumann problems. In particular, we have the following result coming from [6] (Theorem 2.3.), which is stated in $\mathbb{R}^2$ in [6] but can be extended in $\mathbb{R}^N$ with the same proof.

Theorem 11. [6] *Let Ω_k and Ω be open subsets of $\mathbb{R}^N$. Then Ω is stable for the problems* (0.2) *along the sequence Ω_k, if and only if X_{Ω_k} converges to X_Ω in the sense of Mosco.*

According to Theorem 11, Theorem 7 will be a consequence of the following result.

Theorem 12. *Let $r_0 > 0$, $C_0 \in [1/2, +\infty) \cup \{1/3\}$ and let $\{\Omega_k\}_{k \in \mathbb{N}}$ and Ω be $(\delta, r_0)-$Reifenberg flat domains with $\delta < 10^{-3} C_0^{-1}$ and having a uniform size of holes with same constant C_0. Assume that Ω_k converges to Ω for the complementary Hausdorff distance. Then X_{Ω_k} converges to X_Ω in the sense of Mosco.*

Proof. Let $u \in W^{1,p}(\Omega)$ and assume without loss of generality that $\tau_k := d_H(\Omega_k^c, \Omega^c) < C_0^{-1} 10^{-4} r_0$. Set $W(t) := \{x; d(x, \partial\Omega) \leq t\}$ and let $\tilde{u}_k$ be the extension function given in

Lemma 9 for $\Omega_1 := \Omega$, $\Omega_2 := \Omega_k$, $v := u$, $\tau := C_0 10^3 \tau_k$. Since $\tilde{u}_k \in W^{1,p}(\Omega_k)$, all we have to prove is that $(\tilde{u}_k 1_{\Omega_k}, \nabla \tilde{u}_k 1_{\Omega_k})$ converges strongly to $(u 1_\Omega, \nabla u 1_\Omega)$ in $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N, \mathbb{R}^N)$. Since $u = \tilde{u}_k$ in $\Omega \backslash W(10\tau_k)$ we have that

$$\begin{aligned}\Big(\int_{\mathbb{R}^N} |\tilde{u} 1_{\Omega_k} - u 1_\Omega|^p \,\mathrm{d}x\Big)^{\frac{1}{p}} &= \Big(\int_{W(10\tau_k)} |\tilde{u} 1_{\Omega_k} - u 1_\Omega|^p \,\mathrm{d}x\Big)^{\frac{1}{p}} \\ &\leq \Big(\int_{W(10\tau_k)} |\tilde{u} 1_{\Omega_k}|^p \,\mathrm{d}x\Big)^{\frac{1}{p}} + \Big(\int_{W(10\tau_k)} |u 1_\Omega|^p \,\mathrm{d}x\Big)^{\frac{1}{p}} \\ &\leq C\|u\|_{L^p(W(10\tau_k))}\end{aligned}$$

which tends to zero when k tends to $+\infty$ because $\partial\Omega$ is closed and $|\partial\Omega| = 0$. For the gradients, a similar argument can be done, using (1.10). That is,

$$\begin{aligned}\Big(\int_{\mathbb{R}^N} |\nabla \tilde{u} 1_{\Omega_k} - \nabla u 1_\Omega|^p \,\mathrm{d}x\Big)^{\frac{1}{p}} &= \Big(\int_{W(10\tau_k)} |\nabla \tilde{u} 1_{\Omega_k} - \nabla u 1_\Omega|^p \,\mathrm{d}x\Big)^{\frac{1}{p}} \\ &\leq \Big(\int_{W(10\tau_k)} |\nabla \tilde{u} 1_{\Omega_k}|^p \,\mathrm{d}x\Big)^{\frac{1}{p}} + \Big(\int_{W(10\tau_k)} |\nabla u 1_\Omega|^p \,\mathrm{d}x\Big)^{\frac{1}{p}} \\ &\leq C\|\nabla u\|_{L^p(W(60\tau_k))}\end{aligned}$$

which tends to 0, thus $(M1)$ is proved.

Let us now prove $(M2)$. Let $\varphi \in C^\infty(\mathbb{R}^N)$ be compactly supported in Ω. Then we know by the weak convergence that

$$\begin{cases} \int_\Omega u_k 1_{\Omega_{h_k}} \varphi \,\mathrm{d}x \xrightarrow{k \to +\infty} \int_\Omega \phi \varphi \,\mathrm{d}x \\ \int_\Omega \langle \nabla u_k 1_{\Omega_{h_k}}, \varphi \rangle \,\mathrm{d}x \xrightarrow{k \to +\infty} \int_\Omega \psi \varphi \,\mathrm{d}x. \end{cases} \tag{1.19}$$

On the other hand, since Ω_{h_k} converges to Ω for the complementary Hausdorff distance, for k large enough $1_{\Omega_{h_k}}$ is equal to 1 everywhere on the support of ϕ. Thus (1.19) shows that u_k converges to ϕ in $\mathcal{D}'(\Omega)$ and ∇u_k converges to ψ in $\mathcal{D}'(\Omega)$. By uniqueness of the limit in $\mathcal{D}'(\Omega)$ we conclude that

$$\psi = \nabla \phi \qquad \text{in} \quad \mathcal{D}'(\Omega). \tag{1.20}$$

Moreover since $\psi \in L^p(\mathbb{R}^N, \mathbb{R}^N)$, we deduce that (1.20) in true in $L^p(\Omega, \mathbb{R}^N)$ thus $\psi = \nabla \phi$ a.e. in Ω and therefore $\phi|_\Omega \in W^{1,p}(\Omega)$. To conclude, all we have to show is that $\varphi = \psi = 0$ in Ω^c.

To see this, we use a similar argument as above by defining a function φ compactly supported in Ω^c. By the weak convergence, and because Ω_k converges to Ω for the complementary Hausdorff distance, we deduce that $\int_\Omega \phi\varphi \,\mathrm{d}x = 0$. This holds for any function φ compactly supported in Ω^c. Since $\phi \in L^p(\mathbb{R}^N)$ we conclude that $\phi = 0$ in Ω^c. In a similar way we obtain that $\psi = 0$ in Ω^c and since the Lebesgue measure of $\partial\Omega$ is zero, the proof is complete. □

2 Stability for problems with mixed boundary conditions

Following the proof of Theorem 6.3 in [6] and using the Mosco convergence (Theorem 12), one can show a similar stability result for the problem with mixed boundary conditions. The argument is based on taking some judicious extensions as in [6]. Let us recall here the definitions and statements since it will be used in the sequel. Let $A \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary ∂A, and let $\partial_D A$ be a relatively open subset of ∂A. For every compact set $K \subset \bar{A}$, for every $g \in W^{1,p}(A)$, and for every pair of function a and b satisfying the properties (1.2)-(1.3) we consider the solutions u of the mixed problem

$$\begin{cases} -\mathrm{div}\, a(x,\nabla u) + b(x,u) = 0 & \text{in } A\backslash K, \\ u = g & \text{on } \partial_D A\backslash K, \\ a(x,\nabla u)\cdot\nu = 0 & \text{on } \partial(A\backslash K)\backslash(\partial_D A\backslash K) \end{cases} \tag{2.1}$$

Definition 13. *We say that the problem* (2.1) *is stable along the sequence* (K_k, g_k), *if the following holds: Let* $u_k \in W^{1,p}(A\backslash K)$ *be a sequence of solutions of the problem* (2.1) *corresponding to* K_k *and* g_k. *Then* $(u_k 1_{A\backslash K_k}, \nabla u_k 1_{A\backslash K_k})$ *converges strongly to* $(v 1_{A\backslash K}, \nabla v 1_{A\backslash K})$ *in* $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N,\mathbb{R}^N)$ *and* v *is a solution of problem* (2.1).

For all $g \in W^{1,p}(A)$ we denote

$$W^{1,p}_g(A\backslash K, \partial_D A\backslash K) := \{u \in W^{1,p}(A\backslash K) : u = g \text{ on } \partial_D\backslash K\}.$$

As in (1.18) we define the closed linear subspace $X^g_K(A)$ of $L^p(A)\times L^p(A,\mathbb{R}^N)$ by

$$X^g_K(A) := \{(u1_{K^c}, \nabla u 1_{K^c}); u \in W^{1,p}_g(A\backslash K, \partial_D A\backslash K)\}. \tag{2.2}$$

Definition 14 (Mosco-convergence)**.** *We say that $X^{g_k}_{K_k}(A)$ converges to $X^g_K(A)$ in the sense of Mosco if the following two properties hold:*

($M1'$) *for every $u \in W^{1,p}_g(A\backslash K, \partial_D A\backslash K)$, there exists a sequence $u_k \in W^{1,p}_{g_k}(A\backslash K_k, \partial_D A\backslash K_k)$ such that $u_k 1_{K_k^c}$ converges to $u 1_{K^c}$ strongly in $L^p(A)$ and $\nabla u_k 1_{K_k^c}$ converges to $\nabla u 1_{K^c}$ strongly in $L^p(A, \mathbb{R}^N)$;*

($M2'$) *if h_k is a sequence of indices converging to ∞, u_k is a sequence such that $u_k \in W^{1,p}_{g_k}(A\backslash K_k)$ for every k, and $u_k 1_{K_k^c}$ converges weakly in $L^p(A)$ to a function ϕ, while $\nabla u_k 1_{K_k^c}$ converges weakly in $L^p(A, \mathbb{R}^N)$ to a function ψ, then there exists $u \in W^{1,p}_g(A\backslash K, \partial_D A\backslash K)$ such that $\phi = u 1_{K^c}$ and $\psi = \nabla u 1_{K^c}$ a.e. in $\mathbb{R}^N$.*

Remark 15. As it is pointed out in [6], by adopting the same proof as in Theorem 11, the Mosco convergence of $X^{g_k}_{K_k}(A)$ to $X^g_K(A)$ is equivalent to the stability of the mixed problem (2.1) along the sequence (K_k, g_k).

Finally we obtain the corresponding result in the case of the mixed boundary conditions.

Theorem 16. *Let A be a bounded open subset of $\mathbb{R}^N$ with Lipschitz boundary ∂A, let $\partial_D A$ be a relatively open subset of ∂A and let C_0 be a positive constant. Let g_h be a sequence in $W^{1,p}(A)$ converging strongly to a function $g \in W^{1,p}(A)$, and let K_h be a sequence of compact subsets of $\bar{A}$ converging to a set K in the Hausdorff metric. Assume in addition that for every h, $A\backslash K_h$ is a $(\delta, r_0)-$Reifenberg flat domain with $\delta < 10^{-3}C_0^{-1}$ and having a uniform size of holes with constant C_0. Then $X^{g_h}_{K_h}(A)$ converges to $X^g_K(A)$ in the sense of Mosco.*

Proof. The proof is essentially the same as the proof of Theorem 6.3 in [6], consisting of an idea due to Chambolle. The only difference is that the sets

$$\Omega_h := \Sigma \setminus (K_h \cup (\partial A \setminus \partial_D A)) \quad \text{and} \quad \Omega := \Sigma \setminus (K \cup (\partial A \setminus \partial_D A))$$

where Σ is an open ball in $\mathbb{R}^N$ such that $\bar{A} \subset \Sigma$, are now satisfying the assumptions of Theorem 12 and thus the Mosco convergence in that setting. We refer the reader to [6], Theorem 6.3 for the details in that approach. □

3 A decay estimate and its stability

In this section we prove a boundary estimate on the gradient of solutions for a non-linear Neumann problem similar to (0.2). In the first part we will prove the estimate when the boundary is flat, and then using the stability of problem (2.1) we will extend the result for Reifenberg flat domains, whose boundaries are close enough to a hyperplane for the Hausdorff distance.

Let Ω be an open subset of $\mathbb{R}^N$ and let I be a relatively open and Lipschitz subset of $\partial\Omega$. For every $g \in W^{1,2}(\Omega)$ we denote by $W^{1,2}_g(\Omega)$ the functions of $W^{1,2}(\Omega)$ that are equal to g on I (in terms of trace). We focus on minimizers $u \in W^{1,2}_g(\Omega)$ for the following functional

$$\mathcal{F}(u) := \int_\Omega F(|\nabla u|^2)dx \tag{3.1}$$

where F is a $C^{2,1}([0,\infty))$ function satisfying

$$\begin{cases} F(0) = 0, \quad c \le F'(t) \le C \\ 0 \le F''(t) \le \frac{C}{1+t} \end{cases} \tag{3.2}$$

for some positive constants c, C. We define $f(p) := F(|p|^2)$, which is a convex function due to (3.2). We also assume hereby that f is uniformly elliptic that is, there exists positive constants λ, Λ such that

$$\lambda|\xi|^2 \le \sum \frac{\partial^2 f(p)}{\partial p_i \partial p_j}\xi_i\xi_j \le \Lambda|\xi|^2$$

for all $\xi \in \mathbb{R}^N$. If we combine the ellipticity condition with (3.2) we get that f satisfies

$$\begin{cases} \lambda|p|^2 \le \nabla f(p)\cdot p \\ \lambda|p|^2 \le f(p) \le \Lambda|p|^2. \end{cases} \tag{3.3}$$

Lemma 17. *Let Ω be an open subset of $\mathbb{R}^N$ and $u \in W^{1,2}_g(\Omega)$ be a minimizer for the functional $\mathcal{F}$. Then u is a weak solution for the problem*

$$\begin{cases} -\operatorname{div}\,(F(|\nabla u|^2)\nabla u) = 0 & \text{in } \Omega \\ u = g & \text{on } I \\ F'(|\nabla u|^2)\nabla u\cdot\nu = 0 & \text{on } \partial\Omega\backslash I. \end{cases} \tag{3.4}$$

Proof. This is a standard variational argument. Let u be such a minimizer and let $\varphi \in C^\infty(\bar{\Omega})$, such that $\varphi = 0$ on I. Then, comparing u with $u + t\varphi$, we obtain that

$$\int_\Omega F(|\nabla u|^2)\,\mathrm{d}x \le \int_\Omega F(|\nabla u + t\nabla\varphi|^2)\,\mathrm{d}x = \int_\Omega F(|\nabla u|^2 + t\langle \nabla u, \nabla \varphi\rangle + t^2|\nabla\varphi|^2)\,\mathrm{d}x.$$

Then, using that

$$F(|\nabla u|^2 + t\langle \nabla u, \nabla \varphi\rangle + t^2|\nabla\varphi|^2) = F(|\nabla u|^2) + tF'(|\nabla u|^2)\langle \nabla u, \nabla\varphi\rangle + o(t^2),$$

dividing by t and letting t goes to 0 we obtain that

$$\int_\Omega F'(|\nabla u|^2)\langle \nabla u, \nabla \varphi\rangle \ge 0.$$

By a similar argument dividing by $-t$ we obtain the reverse inequality and the lemma follows. □

Remark 18. By our assumptions, $L := -\mathrm{div}(F(|\nabla u|^2 \nabla u))$ is a nonlinear operator of the divergence form as in Problem (0.2) and Problem (2.1) with $a(x,\xi) = F'(|\xi|^2)\cdot\xi$ and $b(x,\xi) = 0$. One can easily check that (1.2)-(1.3) hold for these choices of a and b.

Let us recall this result from [11] that can be also found in [1] p. 347.

Theorem 19. [11] *If f satisfies the assumptions (3.3) and $u \in W^{1,2}_{loc}(\Omega)$ is a local minimizer of the functional*

$$w \mapsto \int_\Omega F(|\nabla w|^2)dx,$$

then u is locally Lipschitz in Ω and

$$\sup_{x\in B(x_0,r/2)} |\nabla u|^2 \le C_1 \frac{1}{r^N} \int_{B(x_0,r)} |\nabla u|^2 dx, \tag{3.5}$$

for all balls $B(x_0,r) \subset \Omega$ where C_1 depends only on N, λ, and Λ.

Following, we want to prove the same sort of estimate but for a local minimizer in $\Omega \setminus K$ where K is a compact set such that $\Omega\setminus K$ is a Reifenberg flat domain and for balls centered on $\partial K \cap \partial\Omega$. We begin with the case when K is a half-space.

3.1 In the complement of a hyperplane

Let $B(0,1)$ be the unit ball in $\mathbb{R}^N$, and define

$$P_0 := \{(x', x_N) \in \mathbb{R}^N; x_N = 0\}.$$

For all $r \leq 1$ we also set

$$B_r^+ := B(0,r) \cap \{(x', x_N) \in \mathbb{R}^N; x_N \geq 0\}.$$

We split the boundary of B_r^+ in two parts. The flat part is denoted by $B' := P_0 \cap B(0,r)$ and the spherical part is $\partial B_r^+ := \partial B(0,r) \cap \{(x', x_N) \in \mathbb{R}^N; x_N > 0\}$. We denote by γ the trace operator from $W^{1,2}(B_r^+)$ to $W^{\frac{1}{2},2}(\partial B_r^+)$. In order to present our argument in a simple way we first concentrate on the case when $f(p) = |\nabla p|^2$. This is a well-known result and the proof is simpler just by using the mean value Theorem (extend by reflection and use Exercise 7.6 of [1]). On the other hand the following proof can be slightly changed in order to be used in the nonlinear case, in Lemma 21 below. The proof consists of considering a monotonicity formula for a specific quantity on the hyperplane (of dimension N) which has been recently presented by Athanasopoulos and Caffarelli, [2] in their study on the optimal regularity of solutions in lower dimensional (Signorini-type) obstacle problems (see also [3] for a different setting in the case of the heat equation).

Lemma 20. *Let $g \in W^{1,2}(B_1^+)$ and let $u \in W_g^{1,2}(B_1^+)$ be a minimizer of the functional $\mathcal{F}$ with $F(|\nabla u|^2) = |\nabla u|^2$. Then u is locally Lipschitz in B_1^+ and for all $r < 1$ and for all $0 < a < 1$ we have*

$$\int_{B_{ar}^+} |\nabla u|^2 dx \leq C_1 a^{N-1+\beta} \int_{B_r^+} |\nabla u|^2 dx, \tag{3.6}$$

for a positive exponent β, where $C_1 = C_1(N, \lambda, \Lambda, L)$ where L is the Lipschitz constant of u.

Proof. The fact that u is locally Lipschitz in B_1^+ comes directly from Theorem 19. Assume without loss of generality that $u(0) = 0$. We extend u in $B(0,r)$ by a reflection argument, therefore we have $u_{x_N} = 0$ on $\{x_N = 0\} \cap B(0,r)$. Define

$$\phi(r) = \frac{1}{r} \int_{B_r^+} \frac{|\nabla u|^2}{|x|^{N-2}} dx$$

and observe that $\Delta u^2 = 2u\Delta u + 2|\nabla u|^2 = 2|\nabla u|^2$. Next we follow the approach presented in [2] by computing

$$\begin{aligned}
\phi'(r) &= -\frac{1}{r^2}\int\limits_{B_r^+} \frac{|\nabla u|^2}{|x|^{N-2}}dx + \frac{1}{r^{N-1}}\int\limits_{\partial B_r^+} |\nabla u|^2 dS \\
&= -\frac{1}{2r^2}\int\limits_{B_r^+} \frac{\Delta u^2}{|x|^{N-2}}dx + \frac{1}{r^{N-1}}\int\limits_{\partial B_r^+} |\nabla u|^2 dS \\
&= -\frac{1}{2r^2}\Big\{\int\limits_{\partial B_r^+} \frac{2uu_r}{r^{N-2}}dx + \frac{N-2}{r^{N-1}}\int\limits_{\partial B_r^+} u^2 dS\Big\} + \frac{1}{r^{N-1}}\int\limits_{\partial B_r^+} |\nabla u|^2 dS.
\end{aligned}$$

Therefore,

$$\phi'(r) = -\frac{1}{r^N}\int\limits_{\partial B_r^+} uu_r dx - \frac{N-2}{2r^{N+1}}\int\limits_{\partial B_r^+} u^2 dS + \frac{1}{r^{N-1}}\int\limits_{\partial B_r^+} |\nabla u|^2 dS, \tag{3.7}$$

since $u_{x_N} = 0$ on $\{x_N = 0\} \cap B_r(0)$. From the Schwartz inequality we obtain,

$$-\frac{1}{r^N}\int\limits_{\partial B_r^+} uu_r dx \geq -\Big(\int\limits_{\partial B_r^+} \frac{u^2}{2r^{N+1}}dS\Big)^{1/2}\Big(\int\limits_{\partial B_r^+} \frac{2u_r^2}{r^{N-1}}dS\Big)^{1/2}. \tag{3.8}$$

It is also clear that

$$\frac{1}{r^{N-1}}\int\limits_{\partial B_r^+} |\nabla u|^2 dS \geq \frac{1}{r^{N-1}}\int\limits_{\partial B_r^+} u_r^2 dS + \frac{1}{r^{N-1}}\int\limits_{\partial B_r^+} |\nabla_\tau u|^2 dS, \tag{3.9}$$

where by τ we denote the tangential direction. Using (3.8), (3.9) in (3.7), we conclude

$$\phi'(r) \geq -\frac{2N-3}{4r^{N+1}}\int\limits_{\partial B_r^+} u^2 dS + \frac{1}{r^{N-1}}\int\limits_{\partial B_r^+} |\nabla_\tau u|^2 dS. \tag{3.10}$$

Now we consider function $w := -(u - Cr^\alpha)^-$, where $C,\ \alpha$ will be chosen later. Note that u is Lipschitz continuous which will allow us to choose α in the sequel.

By construction,

$$\int\limits_{\partial B_r^+} |\nabla_\tau w|^2 dS \leq \int\limits_{\partial B_r^+} |\nabla_\tau u|^2 dS.$$

From (3.10), we have

$$\phi'(r) \geq -\frac{2N-3}{4r^{N+1}}\int\limits_{\partial B_r^+} (u - w + w)^2 dS + \frac{1}{r^{N-1}}\int\limits_{\partial B_r^+} |\nabla_\tau w|^2 dS. \tag{3.11}$$

Now by solving the corresponding eigenvalue problem as in [2], we find that for $w \in H^{\frac{1}{2}}(\partial B_1^+)$ and $w = 0$ on $(\partial B_1')^-$,

$$\inf \frac{\int_{\partial B_1^+} |\nabla_\tau w|^2 dS}{\int_{\partial B_1^+} w^2 dS} = \frac{2N-3}{4}.$$

Therefore (3.11) is reduced to

$$\begin{aligned} \varphi'(r) &\geq -\frac{2N-3}{4r^{N+1}} \int\limits_{\partial B_r^+} \Big[(u-w)^2 + 2(u-w)w\Big] dS - \frac{2n-3}{4r^{N+1}} \int\limits_{\partial B_r^+} w^2 dS \\ &+ \frac{1}{r^{N-1}} \int\limits_{\partial B_r^+} |\nabla_\tau w|^2 dS \\ &\geq -\frac{2N-3}{4r^{N+1}} \int\limits_{\partial B_r^+} \Big[(u-w)^2 + 2(u-w)w\Big] dS. \end{aligned}$$

Finally, since $w = -(u - Cr^\alpha)^-$, we obtain

$$\varphi'(r) \geq -\frac{2N-3}{4} C \frac{r^{N-1}}{r^{N+1}} r^{2\alpha} = -Cr^{2\alpha-2},$$

or

$$\varphi(1) - \varphi(r) \geq -\frac{C}{2\alpha-1} + \frac{C}{2\alpha-1} r^{2\alpha-1}. \tag{3.12}$$

Since $|x| \leq r$ we have that

$$\frac{1}{r^{N-2}} \int\limits_{B_r^+} |\nabla u|^2 dx \leq \int\limits_{B_r^+} \frac{|\nabla u|^2}{|x|^{N-2}} dx$$

which for the appropriate selection of C, α in the definition of the truncated function w along with (3.12) gives

$$\int\limits_{B_r^+} |\nabla u|^2 dx \leq C_1 r^{N-1+\beta} \int\limits_{B_1^+} |\nabla u|^2 dx$$

where C_1 depends on N and the Lipschitz constant of u and β depends on α. □

Next we prove a similar estimate in the general case when $f(p) := F(|p|^2)$.

Lemma 21. *Let f be satisfying the assumptions (3.3) and let $u \in W_g^{1,2}(B_1^+)$ be a minimizer of the functional $\mathcal{F}$. Then u is locally Lipschitz in B_1^+ and for all $r < 1$ and for all $0 < a < 1$*

we have

$$\int_{B_{ar}^+} |\nabla u|^2 dx \le C_1 a^{N-1+\beta} \int_{B_r^+} |\nabla u|^2 dx, \tag{3.13}$$

for a positive exponent β, where C_1 depends only on N, the Lipschitz constant of u, λ and Λ.

Proof. We follow the lines of the proof of Lemma 20. We extend u in $B(0,r)$ as before and denote by $Lv := \nabla(A\nabla v)$ where $A(x) = F'(|\nabla u(x)|^2)$. It is enough to establish the corresponding estimate for v solving the problem (3.4).

We define

$$\varphi(r) = \frac{e^{\overline{C}r^\alpha}}{r} \int_{B_r^+} \frac{|\nabla v|^2}{|x|^{N-2}} dx$$

where $\overline{C}, \alpha < 1$ are positive constants to be chosen later. Let us denote by V the fundamental solution of $L^* v = 0$. Then

$$\begin{aligned} V &= C\rho^{2-N} + O(\rho^{2-N+\alpha}) \\ \nabla V &= (2-N)C\rho^{-N}x + O(\rho^{1-N+\alpha}) \end{aligned}$$

where $\rho = |x|$ and $\alpha \in (0,1)$. Next we compute (after a mollification, if necessary)

$$\begin{aligned} \varphi'(r) &= \frac{\alpha\overline{C}r^\alpha e^{\overline{C}r^\alpha} - e^{\overline{C}r^\alpha}}{r^2} \int_{B_r^+} \frac{|\nabla v|^2}{|x|^{N-2}} dx + \frac{e^{\overline{C}r^\alpha}}{r^{N-1}} \int_{\partial B_r^+} |\nabla v|^2 dS \\ &= \frac{\alpha\overline{C}r^\alpha e^{\overline{C}r^\alpha} - e^{\overline{C}r^\alpha}}{r^2} I_1 + \frac{e^{\overline{C}r^\alpha}}{r^{N-1}} I_2 \end{aligned}$$

that is,

$$e^{-\overline{C}r^\alpha}\varphi'(r) = \alpha\overline{C}r^{\alpha-2} I_1 - \frac{1}{r^2} I_1 + \frac{1}{r^{N-1}} I_2.$$

Now

$$\begin{aligned} \frac{1}{r^2} I_1 &\le \frac{1}{r^2} \int_{B_r^+} \left[\frac{1}{2} \frac{D_i(A_{ij}D_j v^2)}{|x|^{N-2}} + c_1 r^\alpha \frac{|\nabla v|^2}{|x|^{N-2}} \right] dx \\ &= \frac{1}{2r^2} \int_{B_r^+} \frac{D_i(A_{ij}D_j v^2)}{|x|^{N-2}} dx + c_1 r^{\alpha-2} I_1 \end{aligned}$$

therefore

$$e^{-\overline{C}r^{\alpha}}\varphi'(r) \geq (\alpha\overline{C}r^{\alpha-2} - c_1 r^{\alpha-2})I_1 - \frac{1}{2r^2}\int\limits_{B_r^+}\frac{D_i(A_{ij}D_jv^2)}{|x|^{N-2}}dx + \frac{1}{r^{N-1}}I_2.$$

Note that

$$\begin{aligned}
\frac{1}{2r^2}\int\limits_{B_r^+}\frac{D_i(A_{ij}D_jv^2)}{|x|^{N-2}}dx &\leq \frac{1+c_2r^{\alpha}}{2r^2}\int\limits_{B_r^+}D_i(A_{ij}D_jv^2)V\,dx \\
&= \frac{1+c_2r^{\alpha}}{2r^2}\Big[\int\limits_{\partial B_r^+}A_{ij}D_jv^2\nu_iV\,dS - \int\limits_{\partial B_r^+}v^2A_{ij}D_iV\nu_j\,dS\Big] \\
&\leq \frac{1+Cr^{\alpha}}{2r^2}\Big[\frac{2}{r^{N-2}}\int\limits_{\partial B_r^+}vv_r\,dS + \frac{N-2}{r^{N-1}}\int\limits_{\partial B_r^+}v^2\,dS \\
&+ C\frac{r^{\alpha}}{r^{N-2}}\int\limits_{\partial B_r^+}v|\nabla_\tau v|\,dS\Big].
\end{aligned}$$

The proof now follows that of Lemma 20 by introducing the appropriate truncated function. Note also that Lipschitz continuity holds due to Theorem 19. If the constant C is chosen large enough, it controls all the r^{α} error terms in the above expression. □

3.2 In Reifenberg flat Domains

As we have already mentioned, our purpose is to use the stability property in order to obtain a decay estimate in the case of complement Reifenberg flat domains. More precisely we will prove the following Theorem. Recall that $\bar{B}^- := \bar{B}(0,1)\cap\{x_N \leq 0\}$

Theorem 22. *For each choice of $r_0, a_0 > 0$, $C_0 \in [1/2,+\infty)\cup\{1/3\}$ and $g \in W^{1,2}(B(0,1))$, there exists a positive constant ε_0 such that the following holds. Let $K \subset \bar{B}(0,1)$ be a compact set such that $B(0,1)\backslash K$ is a (δ, r_0)-Reifenberg flat domain with $\delta < 10^{-3}C_0^{-1}$, having uniform size of holes with constant C_0, and such that*

$$d_H(K, \bar{B}^-) \leq \varepsilon_0.$$

Then for any solution $u \in W^{1,2}_g(B(0,1)\backslash K)$ of the problem (3.4) with $I := \partial B_1^+\backslash K$ and $\Omega := B(0,1)\backslash K$ we have that for all $a_0 \leq a \leq 1$,

$$\int\limits_{B_a\backslash K}|\nabla u|^2dx \leq C_1 a^{N-1+\beta}\int\limits_{B\backslash K}|\nabla u|^2dx, \tag{3.14}$$

for a positive exponent β, where C_1 depends only on N, the Lipschitz constant of u, λ and Λ.

Proof. Let r_0 and C_0 be fixed, β is an exponent strictly greater than the one in (3.13), and C_1 is the constant in (3.13). Suppose Theorem 22 is false. Then for every $\varepsilon_k > 0$ there exists a compact set K_k of $\bar{B}(0,1)$ such that $B(0,1)\backslash K_k$ is a Reifenberg flat Domain with maximum radius r_0, with uniform size of holes with constant C_0 and

$$d_H(K_k, \bar{B}^-) \leq \varepsilon_k.$$

In addition there exists a sequence of solutions $u_k \in W^{1,2}_g(A\backslash K_k)$ of the problem (3.4) with $I := \partial B_1^+\backslash K_k$ and $\Omega := B(0,1)\backslash K_k$ and a sequence of radii $a_k \in (a_0, 1)$ such that

$$\int_{B_a\backslash K_k} |\nabla u_k|^2 dx > C_1 a_k^{N-1+\beta} \int_{B\backslash K_k} |\nabla u_k|^2 dx. \tag{3.15}$$

By extracting a subsequence (not relabeled), we may assume that a_k converges to a certain $a' \leq 1$. Now we claim that

$$\liminf_{k\to+\infty} \int_{B\backslash K_k} |\nabla u_k|^2 dx \geq E\lambda^{-1} > 0$$

Indeed, let us consider the compact set $K_0 := \bar{B}(0,1)\cap\{x_N \leq 3\varepsilon_0\}$ and let $v \in W^{1,2}_g(B(0,1)\backslash K_0)$ be the solution of the problem (3.4). Since for every k, u_k is a competitor for v we have that

$$E := \int_{B(0,1)\backslash K_0} F(|\nabla v|)^2 \leq \int_{B(0,1)\backslash K_0} F(|\nabla u_k|^2) \leq \int_{B(0,1)\backslash K_k} F(|\nabla u_k|^2) \leq \Lambda \int_{B(0,1)\backslash K_k} |\nabla u_k|^2$$

which proves the claim.

Then, by the stability of Problem (2.1) we know that $\tilde{u}_k 1_{K^c}$ converges strongly in $L^2(A)$ to a function u, and $\nabla\tilde{u}_k 1_{K^c}$ converges strongly in $L^2(A)$ to $\nabla u 1_{B^+}$. Moreover, u is a solution of the problem (3.4) in B^+ with $I = \partial B^+$ and boundary values on I equal to g. Thus by passing to the limit in (3.15) we obtain a contradiction according to Lemma 21. □

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