# An extension theorem in $SBV$ and an Application to the homogenization of the Mumford-Shah functional in perforated domains

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**Abstract**

The aim of this paper is to prove the existence of extension operators for $SBV$ functions from periodically perforated domains. This result will be the fundamental tool to prove the compactness in a non coercive homogenization problem.

Le but de cet article est de prouver l'existence d'opérateurs de prolongation pour des fonctions en $SBV$ définies sur des domaines périodiquement perforés. Ce résultat sera l'outil fondamental pour prouver la compacité dans un problème non coercitif d'homogénéisation.

*Keywords:* extension theorem, homogenization, Γ-convergence, integral representation, brittle fracture, Mumford-Shah functional

## 1. Introduction

In this paper we show the existence of an extension operator for special functions of bounded variation with a careful energy estimate. Our main motivation comes from the study of effective properties of elastic porous media where fractures are allowed. More precisely, we are interested in the asymptotic behaviour of the minimisers of the energy associated to a displacement in a periodically perforated brittle body, as the size $\varepsilon$ of the microstructure vanishes.

### *1.1. Classical Results*

The analogous of this problem in the absence of fracture (i.e., in the Sobolev setting) has been extensively studied and it is one of the most classical examples in Homogenization Theory. We briefly recall the expression of the elastic energy in the Sobolev case.

Let $E \subset \mathbb{R}^n$ be a periodic, open and connected set with Lipschitz boundary. For a bounded open set $\Omega \subset \mathbb{R}^n$ let $\Omega(\varepsilon) := \Omega \cap (\varepsilon E)$, where $\varepsilon > 0$. The set $\Omega(\varepsilon)$ describes a perforated body with holes of size of order $\varepsilon$ (see Figure 1).

In the context of linearised elasticity, in the case of generalised antiplanar shear, the classical expression for the energy associated to a (scalar) displacement $u$ of the elastic body filling the region $\Omega(\varepsilon)$ is given by

$$\mathcal{F}^{\varepsilon}_{el}(u,\Omega) := \begin{cases} \displaystyle\int_{\Omega(\varepsilon)} |\nabla u|^2 dx & \text{if } u_{|_{\Omega(\varepsilon)}} \in H^1(\Omega(\varepsilon)), \\ +\infty & \text{otherwise in } L^2(\Omega). \end{cases} \tag{1.1}$$

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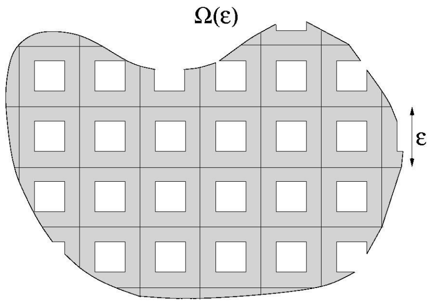


Figure 1: The perforated set $\Omega(\varepsilon)$, in the case $n = 2$.

The goal of Homogenization Theory is to provide a good description of the overall properties of the perforated domain for small $\varepsilon$ via a simpler functional, independent of $\varepsilon$, which is obtained from the family $(\mathcal{F}^{\varepsilon}_{el})$ through a limit procedure. This is often done by means of $\Gamma$-convergence, a variational convergence that enjoys the following stability property for the minima. If the family $(\mathcal{F}^{\varepsilon}_{el})$ is equicoercive, that is if every sequence $(u^{\varepsilon})$ with energy $\mathcal{F}^{\varepsilon}_{el}(u^{\varepsilon})$ uniformly bounded in $\varepsilon$ is compact, then minimisers of $\mathcal{F}^{\varepsilon}_{el}$ converge to a minimum point of the $\Gamma$-limit.

What makes the problem (1.1) complicated is the lack of coerciveness of the functionals $\mathcal{F}^{\varepsilon}_{el}$, due to the presence of the holes. Indeed, for a sequence $(u^{\varepsilon})$ with bounded energy $\mathcal{F}^{\varepsilon}_{el}(u^{\varepsilon})$, one cannot immediately obtain a uniform bound of the $L^2$-norm of the gradients in the whole of $\Omega$, as there is no control on the behaviour of the sequence in the set $\Omega \setminus \Omega(\varepsilon)$. Only in the special case where $(u^{\varepsilon})$ satisfies homogeneous boundary conditions on $\partial\Omega(\varepsilon)$, one can trivially extend each $u^{\varepsilon}$ to zero in $\Omega \setminus \Omega(\varepsilon)$, so that

$$\int_{\Omega} |\nabla u^{\varepsilon}|^2 dx = \mathcal{F}^{\varepsilon}_{el}(u^{\varepsilon}, \Omega),$$

giving the required bound for the gradients in $\Omega$, and therefore compactness for $(u^{\varepsilon})$. Otherwise, in the general case there is no obvious way to provide an extension from $\Omega(\varepsilon)$ to $\Omega$ preserving the control on the $L^2$-norm of the gradients. We notice that, instead of considering the problem in $\Omega(\varepsilon)$, one could focus on a single periodicity cell. Indeed, to solve the problem it is sufficient to construct an extension satisfying the required estimate for the gradients in a fixed periodicity cell, in a way that does not depend on $\varepsilon$.

More in general, given an open connected set $D$ with Lipschitz boundary, and an open and bounded set $A$, what is needed is the existence of an extension operator $L : H^1(D) \to H^1(A)$ such that, for every $u \in H^1(D)$, $Lu = u$ a.e. in $A \cap D$ and

$$\int_{A} |\nabla(Lu)|^2 dx \le c \int_{D} |\nabla u|^2 dx \tag{1.2}$$

for some constant $c$ depending only on the dimension $n$ and on the sets $D$ and $A$, and invariant under dilations. The well-known extension results in Sobolev spaces (see [2] for instance) are not the appropriate tool to solve this difficulty. Indeed, they usually provide only an estimate of the $H^1$-norm of the extended function in terms of the *whole* $H^1$-norm of the original function.

Estimate (1.2) was firstly proved in 1977 by Tartar (see [22] and [9]), with a clever use of the classical Poincaré-Wirtinger inequality. For the extension result in its most general form and an application to the homogenization of (1.1) we refer to [1] (see also [20] for the special case of $\varepsilon E$ disconnecting $\Omega$).

### *1.2. The SBV case*

The main feature of the present situation is that we model a porous media where fractures can occur, and therefore deformations are allowed to have discontinuities. The classical functional setting for problems of this kind is the space of Special functions of Bounded Variations (see

[5]), $SBV$ in short. We will assume, following Griffith's model for brittle fractures (see [19]), that the energy needed to create a crack is proportional to its length. Thus, the total energy associated to a displacement $u$ of a brittle elastic body filling an open bounded region $U \subset \mathbb{R}^n$ is the Mumford-Shah functional (see [21]), defined as

$$MS(u,U) := \int_U |\nabla u|^2\, dx + \mathcal{H}^{n-1}(S_u \cap U).$$

Here $\nabla u$ and $S_u$ denote the absolutely continuous part of the gradient and the jump set of $u$, respectively, while $\mathcal{H}^{n-1}$ stands for the $(n-1)$-dimensional Hausdorff measure.

In the $SBV$ setting, instead of the energy functionals in (1.1), it is therefore natural to consider

$$\mathcal{F}^\varepsilon(u,\Omega) := \begin{cases} MS(u,\Omega(\varepsilon)) & \text{if } u_{|\Omega(\varepsilon)} \in L^\infty(\Omega(\varepsilon)) \cap SBV^2(\Omega(\varepsilon)), \\ +\infty & \text{otherwise in } L^2(\Omega), \end{cases} \tag{1.3}$$

where $\Omega$ and $\Omega(\varepsilon)$ are defined as above (see Section 2 for the definition of the space $SBV^2$). The restriction of the functional to bounded functions is done for technical reasons.

Our goal is to study the asymptotic behaviour of the family $(\mathcal{F}^\varepsilon)$ as $\varepsilon \to 0$ via $\Gamma$-convergence (see [11]). To this aim, we need the analogue in the $SBV$ framework of (1.2) and of the general extension estimates obtained in [1]. This is provided by the following theorem, that is the main result of the paper.

**Theorem 1.1.** *Let $D$, $A$ be open subsets of $\mathbb{R}^n$. Assume that $A$ is bounded and that $D$ is connected and has Lipschitz boundary. Then there exists an extension operator $L : SBV^2(D) \cap L^\infty(D) \to SBV^2(A) \cap L^\infty(A)$ and a constant $c = c(n, D, A) > 0$ such that*

$$\begin{aligned} &(i)\ Lu = u \quad \text{a.e. in } A \cap D,\\ &(ii)\ ||Lu||_{L^\infty(A)} \le ||u||_{L^\infty(D)},\\ &(iii)\ MS(Lu, A) \le c\, MS(u, D), \end{aligned} \tag{1.4}$$

*for every $u \in SBV^2(D) \cap L^\infty(D)$. The constant $c$ is invariant under translations and dilations.*

We want to underline that in general one cannot replace condition $(iii)$ in the theorem above with an estimate involving only the (absolutely continuous part of the) gradients, like (1.2). Indeed, the classical Poincaré-Wirtinger inequality, which was the crucial argument to prove (1.2), does not hold true in the $SBV$ setting. This is because it is possible to construct non constant $SBV$ functions whose absolutely continuous gradient is zero almost everywhere. On the other hand, the available version in $SBV$ of the Poincaré-Wirtinger inequality (see [14]) does not lead directly to $(iii)$. Let us explain the main idea of the present work in the following simplified version of Theorem 1.1.

**Theorem 1.2.** *Let $D, A \subset \mathbb{R}^n$ be bounded open sets with Lipschitz boundary and assume that $D$ is connected, $D \subset A$ and $A \setminus D \subset\subset A$. Then there exists an extension operator $L : SBV^2(D) \cap L^\infty(D) \to SBV^2(A) \cap L^\infty(A)$ and a constant $c = c(n, D, A) > 0$ such that*

$$\begin{aligned} &(i)\ Lu = u \quad \text{a.e. in } D,\\ &(ii)\ ||Lu||_{L^\infty(A)} \le ||u||_{L^\infty(D)},\\ &(iii)\ MS(Lu, A) \le c\, MS(u, D), \end{aligned} \tag{1.5}$$

*for every $u \in SBV^2(D) \cap L^\infty(D)$. The constant $c$ is invariant under translations and dilations.*

We want to emphasize that without the assumption that the set $D$ is connected both Theorem 1.1 and its simplified version Theorem 1.2 do not hold. Indeed, for every $r > 0$ let $B_r(0)$ denote the open $n$-dimensional ball of $\mathbb{R}^n$ centered at the origin with radius $r$. If we choose $A = B_2(0)$ and $D = A \setminus \partial B_1(0)$, the function

$$u(x) := \begin{cases} 0 & \text{if } x \in B_1(0) \\ 1 & \text{if } x \in B_2(0) \setminus \overline{B}_1(0) \end{cases}$$

belongs to $SBV^2(D) \cap L^\infty(D)$ and satisfies $MS(u, D) = 0$. Nevertheless, it is clear that there exists no extension $Lu$ in $A$ satisfying the requirement $(iii)$ of the theorems.

To prove Theorem 1.2, we first consider a local minimiser of $MS$, that is a solution $\hat{v}$ of the following problem:

$$\min \big\{ MS(w, D \cup W) : w \in SBV^2(D \cup W),\, w = u \text{ a.e. in } D \big\},$$

where $W \subset\subset A$ is a sufficiently small neighbourhood of $\partial D \cap A$ (see Fig. 2).

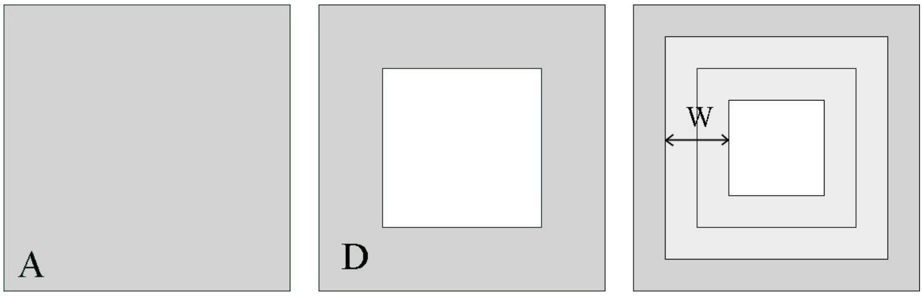


Figure 2: The set $A$; the set $D$; the neighbourhood $W$.

Then, we carry out a delicate analysis of the behaviour of the function $\hat{v}$ in the set $W$. More precisely, we define the extension $Lu$ in $A \setminus D$ modifying the function $\hat{v}$ in different ways, according to the measure of the set $S_{\hat{v}} \cap (W \setminus \overline{D})$.

If this measure is *large* enough, then we consider $Lu$ defined as $\hat{v}$ in $D \cup W$ and zero in the remaining part of $A$. In this way we have essentially increased the energy in the surface term only, of an amount that is comparable to the measure of $S_u \cap D$. This guarantees that properties $(i)$–$(iii)$ are satisfied in this case.

On the other hand, if $\mathcal{H}^{n-1}\big(S_{\hat{v}} \cap (W \setminus \overline{D})\big)$ is *small*, then we may use the *elimination property* proved in [14, 12] to detect a subset $\Delta$ of $W \setminus \overline{D}$ where the function $\hat{v}$ has no jump (see also Theorem 2.5). This allows us to apply the extension property proved in the Sobolev setting in each connected component of $\Delta$.

As already mentioned, Theorem 1.1 finds an application in the study of the asymptotic behaviour of the functionals $\mathcal{F}^\varepsilon$ defined in (1.3), as made precise by the following theorem.

**Theorem 1.3.** *Let $E$ be a periodic, connected, open subset of $\mathbb{R}^n$, with Lipschitz boundary, let $\varepsilon > 0$, and set $E^\varepsilon := \varepsilon E$. Given a bounded open set $\Omega \subset \mathbb{R}^n$, set $\Omega(\varepsilon) := \Omega \cap E^\varepsilon$. Then, there exists an extension operator $T^\varepsilon : SBV^2(\Omega(\varepsilon)) \cap L^\infty(\Omega(\varepsilon)) \to SBV^2(\Omega) \cap L^\infty(\Omega)$ and a constant $k_0 > 0$, depending on $E$ and $n$, but not on $\varepsilon$ and $\Omega$, such that*

- $T^\varepsilon u = u$ *a.e. in* $\Omega(\varepsilon)$,
- $||T^\varepsilon u||_{L^\infty(\Omega)} \leq ||u||_{L^\infty(\Omega(\varepsilon))}$,
- $MS(T^\varepsilon u, \Omega) \leq k_0 \big( MS(u, \Omega(\varepsilon)) + \mathcal{H}^{n-1}(\partial\Omega) \big)$

*for every $u \in SBV^2(\Omega(\varepsilon)) \cap L^\infty(\Omega(\varepsilon))$.*

This means that we can *fill* the holes of $\Omega(\varepsilon)$ by means of an extension of $u$, whose Mumford-Shah energy is kept bounded by $k_0\left(MS(u,\Omega(\varepsilon))+\mathcal{H}^{n-1}(\partial\Omega)\right)$, where the constant $k_0=k_0\,(n,E)$ depends on $n$, and $E$, but is *independent of* $\Omega$, $\varepsilon$ *and* $u$. This is the key estimate to prove compactness of minimising sequences for $(\mathcal{F}^\varepsilon)$, and to identify a class of functions where the $\Gamma$-limit is finite. Within this class, we give a more explicit expression for the $\Gamma$-limit, characterizing the volume and the surface densities by means of two separate homogenization formulas (see Theorem 7.2).

For completeness we mention that a previous work (see [15]) shows that a very different situation occurs when the homogeneous Neumann boundary conditions on $\partial\Omega(\varepsilon)$ are replaced by homogeneous Dirichlet boundary conditions. In particular, in this case an extension theorem is not needed, since every function $u\in SBV^2(\Omega(\varepsilon))\cap L^\infty(\Omega(\varepsilon))$ admits a natural extension by zero to the whole $\Omega$, as already observed for the Sobolev setting.

We finally remark that the same homogenization result has been independently obtained in the recent paper [16], where the lack of coerciveness has been solved in an alternative way, bypassing the construction of an extension operator. In the quoted paper the authors first truncate the function around each perforation, and then extend the truncated function inside the hole using standard cut-off techniques. Strictly speaking, the function obtained in this way is not an extension. Nevertheless, it coincides with the original function in a set that is sufficiently large for the purpose of proving compactness of minimising sequences. Indeed, the authors are able to obtain a good control of the total energy, providing suitable Poincaré-type inequalities in $SBV$.

The plan of the paper is the following. In Section 2 we recall the basic properties of special functions with bounded variation and the extension results available in the Sobolev setting. In order to simplify the exposition, in Sections 3 and 4 we focus on the case in which the set $A\setminus D$, where the extension has to be performed, is compactly contained in $A$. More precisely, in Section 3 we prove Theorem 1.2, while Section 4 is devoted to the corresponding simplified version of Theorem 1.3 (see Theorem 4.1). Then, we face the general case, proving Theorem 1.1 and Theorem 1.3 in Sections 5 and 6, respectively. In Section 7 we study the $\Gamma$-limit of the sequence of functionals (1.3). Finally, we postpone some technical lemmas in the Appendix.

## 2. Preliminaries

Let us give some definitions and results that will be widely used throughout the paper.

We denote with $Q$ the unit cube in $\mathbb{R}^n$, i.e. $Q=\left(-\frac{1}{2},\frac{1}{2}\right)^n$, while $(e_i)_{i=1,\dots,n}$ stands for the canonical basis of $\mathbb{R}^n$. We use the following compact notation for the opposite hyperfaces of the cube:

$$\partial Q_{\pm,i}:=\partial Q\cap\left\{x_i=\pm\frac{1}{2}\right\}\quad i=1,\dots,n.$$

We say that a set $E\subset\mathbb{R}^n$ is periodic if $E+e_i=E$ for every $i=1,\dots,n$.

Moreover, we say that an open set $E\subset\mathbb{R}^n$ has a Lipschitz boundary at a point $x\in\partial E$ (or equivalently, that $\partial E$ is locally Lipschitz at $x$) if there exist an orthogonal coordinate system $(y_1,\dots,y_n)$, a coordinate rectangle $R=(a_1,b_1)\times\ldots\times(a_n,b_n)$ containing $x$, and a Lipschitz function $\Psi:(a_1,b_1)\times\ldots\times(a_{n-1},b_{n-1})\to(a_n,b_n)$ such that $E\cap R=\{y\in R:\,y_n<\Psi(y_1,\dots,y_{n-1})\}$. If this property holds true for every $x\in\partial E$ with the same Lipschitz constant, we say that $E$ has Lipschitz boundary (or equivalently, that $\partial E$ is Lipschitz).

We will denote with $\mathbb{M}^n$ the set of all the $n\times n$ matrices with real entries. For the identity map we use the notation $Id$, i.e., $Id(x)=x$ for every $x\in\mathbb{R}^n$. For an open set $A$, $C_0^\infty(A)$ denotes the class of $C^\infty$ functions with compact support in $A$. Finally, int$A$ is the interior of a set $A\subset\mathbb{R}^n$.

We recall now some properties of rectifiable sets and of the space $SBV$ of special functions with bounded variation. We refer the reader to [5] for a complete treatment of these subjects.

A set $\Gamma\subset\mathbb{R}^n$ is rectifiable if there exist $N_0\subset\Gamma$ with $\mathcal{H}^{n-1}(N_0)=0$, and a sequence $(M_i)_{i\in\mathbb{N}}$ of $C^1$-submanifolds of $\mathbb{R}^n$ such that

$$\Gamma\setminus N_0\subset\bigcup_{i\in\mathbb{N}}M_i.$$

Let $x \in \Gamma \setminus N_0$, and let $i \in \mathbb{N}$ such that $x \in M_i$. We define the normal to $\Gamma$ at $x$ as the normal $\nu_{M_i}(x)$ to $M_i$ at $x$. It turns out that the normal is well defined (up to the sign) for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$.

Let $U \subset \mathbb{R}^n$ be an open bounded set with Lipschitz boundary. We define $SBV(U)$ as the set of functions $u \in L^1(U)$ such that the distributional derivative $Du$ is a Radon measure which, for every open set $A \subset U$, can be represented as

$$Du(A) = \int_A \nabla u \, dx + \int_{A \cap S_u} [u](x) \, \nu_u(x) \, d\mathcal{H}^{n-1}(x),$$

where $\nabla u$ is the approximate differential of $u$, $S_u$ is the set of jump of $u$ (which is a rectifiable set), $\nu_u(x)$ is the normal to $S_u$ at $x$, and $[u](x)$ is the jump of $u$ at $x$.
For every $p \in (1, +\infty)$ we set

$$SBV^p(U) = \big\{ u \in SBV(U) : \nabla u \in L^p(U; \mathbb{R}^n), \mathcal{H}^{n-1}(S_u) < +\infty \big\}.$$

If $u \in SBV(U)$ and $\Gamma \subset U$ is rectifiable and oriented by a normal vector field $\nu$, then we can define the traces $u^+$ and $u^-$ of $u \in SBV(U)$ on $\Gamma$, which are characterised by the relations

$$\lim_{r \to 0} \frac{1}{r^n} \int_{U \cap B_r^{\pm}(x)} |u(y) - u^{\pm}(x)| \, dy = 0 \quad \text{for } \mathcal{H}^{n-1} - \text{a.e. } x \in \Gamma,$$

where $B_r^{\pm}(x) := \{ y \in B_r(x) : (y - x) \cdot \nu \gtrless 0 \}$.

The following extension theorems are the Sobolev versions of Theorem 1.1 and Theorem 1.3, respectively (see [1, Lemma 2.6] and [1, Theorem 2.1]).

**Theorem 2.1.** *Let $D$, $A$ be open subsets of $\mathbb{R}^n$. Assume that $A$ is bounded and that $D$ is connected and has Lipschitz boundary at each point of $\partial D \cap \bar{A}$. Then, there exists a linear and continuous operator $\tau : H^1(D) \to H^1(A)$ such that, for every $u \in H^1(D)$*

$$\begin{gathered} \tau u = u \quad \text{a.e. in } A \cap D, \\ \int_A |\tau u|^2 dx \le k_1 \int_D |u|^2 dx, \\ \int_A |\nabla(\tau u)|^2 dx \le k_2 \int_D |\nabla u|^2 dx, \end{gathered} \tag{2.1}$$

*where $k_1 = k_1(n, D, A)$ and $k_2 = k_2(n, D, A)$ are positive constants depending only on $n$, $D$, and $A$.*

**Theorem 2.2.** *Let $E$ be a periodic, connected, open subset of $\mathbb{R}^n$, with Lipschitz boundary, let $\varepsilon > 0$, and set $E^\varepsilon := \varepsilon E$. Given a bounded open set $\Omega \subset \mathbb{R}^n$, set $\Omega(\varepsilon) := \Omega \cap E^\varepsilon$. Then, there exists a linear and continuous extension operator $\tau^\varepsilon : H^1(\Omega(\varepsilon)) \to H^1_{loc}(\Omega)$ and three constants $k_3, k_4, k_5 > 0$ depending on $E$ and $n$, but not on $\varepsilon$ and $\Omega$, such that*

$$\begin{gathered} \tau^\varepsilon u = u \ \text{a.e. in } \Omega(\varepsilon), \\ \int_{\Omega_{\varepsilon k_3}} |\tau^\varepsilon u|^2 dx \le k_4 \int_{\Omega(\varepsilon)} |u|^2 dx, \\ \int_{\Omega_{\varepsilon k_3}} |\nabla(\tau^\varepsilon u)|^2 dx \le k_5 \int_{\Omega(\varepsilon)} |\nabla u|^2 dx, \end{gathered}$$

*for every $u \in H^1(\Omega(\varepsilon))$. Here we used the notation $\Omega_{\varepsilon k_3} := \{ x \in \Omega : dist(x, \partial\Omega) > \varepsilon k_3 \}$.*

We give now the definition of a local minimiser for the Mumford-Shah functional. We recall that for an open set $U \subset \mathbb{R}^n$ and for $w \in SBV^2(U)$

$$MS(w, U) = \int_U |\nabla w|^2 dx + \mathcal{H}^{n-1}(S_w \cap U). \tag{2.2}$$

**Definition 2.3.** Let $\Omega \subset \mathbb{R}^n$ be open. We say that $w \in SBV^2(\Omega)$ is a *local minimiser* for the functional $MS(\cdot, \Omega)$ if $MS(w, A) \leq MS(v, A)$ for every open set $A \subset\subset \Omega$, whenever $v \in SBV^2(\Omega)$ and $\{v \neq w\} \subset\subset A \subset\subset \Omega$.

Next theorem provides an estimate of the measure of the jump set for a local minimiser of the Mumford-Shah functional (see [5, Theorem 7.21] and [14]).

**Theorem 2.4** (Density lower bound)**.** *There exists a strictly positive dimensional constant $\vartheta_0 = \vartheta_0(n)$ with the property that if $u \in SBV^2(\Omega)$ is a local minimiser for the functional $MS(\cdot, \Omega)$ defined in (2.2) for an open set $\Omega \subset \mathbb{R}^n$, $n \geq 2$, then*

$$\mathcal{H}^{n-1}(S_u \cap B_\varrho(x)) > \vartheta_0 \varrho^{n-1}$$

*for every ball $B_\varrho(x) \subset \Omega$ with centre $x \in S_u$ and radius $\varrho > 0$.*

An equivalent but more appealing formulation of the previous theorem is the following elimination property (see [12]).

**Theorem 2.5** (Elimination property)**.** *Let $\Omega \subset \mathbb{R}^n$ be open. There exists a strictly positive dimensional constant $\beta = \beta(n)$ independent of $\Omega$ such that, if $u \in SBV^2(\Omega)$ is a local minimiser for the functional $MS(\cdot, \Omega)$ defined in (2.2) and $B_\varrho(x_0) \subset \Omega$ is any ball with centre $x_0 \in \Omega$ with*

$$\mathcal{H}^{n-1}(S_u \cap B_\varrho(x_0)) < \beta \varrho^{n-1},$$

*then $S_u \cap B_{\varrho/2}(x_0) = \emptyset$.*

We state now a theorem which provides an approximation result for $SBV$ functions, with the property that the value of the Mumford-Shah functional along the approximating sequence converges to the value of the Mumford-Shah functional on the limit function. For the proof we refer to [10].

**Theorem 2.6.** *Let $\Omega \subset \mathbb{R}^n$ be open. Assume that $\partial\Omega$ is locally Lipschitz and let $u \in SBV^2(\Omega)$. Then there exists a sequence $(u_h) \subset SBV^2(\Omega)$ such that for every $h \in \mathbb{N}$*

$$\begin{aligned} &(i)\ S_{u_h} \text{ is essentially closed;}\\ &(ii)\ \bar{S}_{u_h} \text{ is a polyhedral set;}\\ &(iii)\ u_h \in W^{k,\infty}(\Omega \setminus \bar{S}_{u_h}) \text{ for every } k \in \mathbb{N};\end{aligned}$$

*and such that $(u_h)$ approximates $u$ in the following sense:*

$$\begin{aligned} &(iv)\ u_h \to u \text{ strongly in } L^2(\Omega),\\ &(v)\ \nabla u_h \to \nabla u \text{ strongly in } L^2(\Omega),\\ &(vi)\ \mathcal{H}^{n-1}(S_{u_h}) \to \mathcal{H}^{n-1}(S_u).\end{aligned}$$

## 3. Compactly contained hole: extension for a fixed domain

In this section we prove Theorem 1.2. This is a simplified version of Theorem 1.1, under the additional assumption that the set $A \setminus D$, where the extension has to be performed, is compactly contained in $A$ (see Fig. 2a) and 2b)). In this way, it will be possible to highlight the main ideas of the present work, without facing the further difficulties of the general case, that will be treated in Section 5.

In order to prove the extension result we need to define, for every open set, a *reflection* map with respect to bounded Lipschitz subsets of the boundary, as made clear in the following theorem.

**Theorem 3.1.** *Let $D \subset \mathbb{R}^n$ be an open set, and assume that $\Lambda \subset \partial D$ is a bounded, relatively open, nonempty Lipschitz set, with $\Lambda \subset\subset \{x \in \partial D : \partial D$ has Lipschitz boundary at $x\}$. Then, there exists a bounded open set $W \subset \mathbb{R}^n$ with Lipschitz boundary, such that $\Lambda = W \cap \partial D$, and a bilipschitz map $\phi : W \to W$ with $\phi^2 = Id$, $\phi_{|\Lambda} = Id$ and $\phi(W^\pm) = W^\mp$, where $W^+ := W \cap D$ and $W^- := W \cap (\mathbb{R}^n \setminus \overline{D})$.*

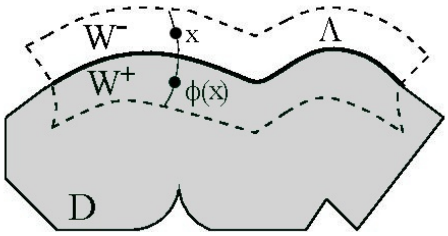


Figure 3: The sets $W^+$, $W^-$, $\Lambda$ and the bilipschitz map $\phi$.

A pictorial idea of the previously stated reflection result is illustrated in Figure 3.

*Proof.* Since $\overline{\Lambda}$ is Lipschitz and compact, we can find a finite open cover $U_1, \dots, U_m$ of $\overline{\Lambda}$ such that we can associate to every $U_j$ a vector $u_j^0 \in \mathbb{R}^n$ and a parameter $\eta_j \in (0,1]$ with the following property. If $x \in \overline{\Lambda} \cap U_j$ for some $j$, then for every $t \in (0,1]$ and for every $u_j \in \mathbb{R}^n$ such that $|u_j - u_j^0| < \eta_j$ it turns out that $x + t\,u_j \in D$ and $x - t\,u_j \in \mathbb{R}^n \setminus \overline{D}$.

Set $\eta := \min_j \eta_j$. Now, for every index $j$ we fix an open set $V_j \subset\subset U_j$ such that $V_1, \dots, V_m$ is still a covering of $\overline{\Lambda}$. Let $(\psi_j)_{j=1,\dots,m}$ be a partition of unity for $\Lambda$ subordinate to $(V_j)_{j=1,\dots,m}$, i.e.,

$$\psi_j \in C_0^\infty(\mathbb{R}^n), \quad \text{supp}\, \psi_j \subset V_j, \quad 0 \le \psi_j \le 1 \quad \text{in } \mathbb{R}^n, \quad \sum_{j=1}^m \psi_j = 1 \text{ on } \overline{\Lambda}.$$

Let us fix $\alpha_0 > 0$ so that for every collection of vectors $\{u_1, \dots, u_m\}$ satisfying $|u_i - u_i^0| < \eta$ for every $i$, we have

$$\alpha_0 \sum_{i=1}^m |u_i| < \text{dist}(V_j, \partial U_j) \quad \text{for } j = 1, \dots, m.$$

Let us define $B_\eta^m(u^0) := \{u = (u_1, \dots, u_m) \in (\mathbb{R}^n)^m : |u_i - u_i^0| < \eta \text{ for every } i\}$. For every $\alpha \in [-\alpha_0, \alpha_0]$ and for every $u \in B_\eta^m(u^0)$, we define the $C^\infty$ function $r_u^\alpha : \mathbb{R}^n \to \mathbb{R}^n$ as

$$r_u^\alpha(x) := x + \alpha \sum_{j=1}^m \psi_j(x)\, u_j.$$

It turns out that, by construction, $r_u^\alpha - Id$ has compact support and $r_u^\alpha - Id \to 0$ in $C_0^\infty(\mathbb{R}^n; \mathbb{R}^n)$ as $\alpha \to 0$. Let us set $\Psi_u(x) := \sum_{j=1}^m \psi_j(x)\, u_j$ and $\Psi^0(x) := \sum_{j=1}^m \psi_j(x)\, u_j^0$. Following the argument used in [13, Proposition 1.2], it is possible to show that, for every $x \in \overline{\Lambda}$, we have that for every $u \in B_\eta^m(u^0)$, $x + \alpha\, \Psi_u(x) \in D$ if $0 < \alpha \le \alpha_0$ and $x + \alpha\, \Psi_u(x) \in \mathbb{R}^n \setminus \overline{D}$ if $-\alpha_0 \le \alpha < 0$.

We claim that there exists $\eta_0 \in (0, \eta]$ such that for every $x \in \overline{\Lambda}$ we have the following property:

$$|v - \Psi^0(x)| < \eta_0 \;\Rightarrow\; \begin{cases} x + \alpha\, v \in D & \text{if} \quad 0 < \alpha \le \alpha_0, \\ x + \alpha\, v \in \mathbb{R}^n \setminus \overline{D} & \text{if} \; -\alpha_0 \le \alpha < 0. \end{cases} \tag{3.1}$$

We notice that in order to obtain (3.1) it is sufficient to prove that

$$\text{if } v \text{ satisfies } |v - \Psi^0(x)| < \eta_0, \text{ then } v = \Psi_u(x) \quad \text{for some } u \in B_\eta^m(u^0). \tag{3.2}$$

Let us show (3.2). Let us fix $x \in \overline{\Lambda}$; we define the linear map $\mathcal{I}^x : \left(\mathbb{R}^n\right)^m \to \mathbb{R}^n$ as

$$u = (u_1, \dots, u_m) \mapsto \mathcal{I}^x(u) := \Psi_u(x) = \sum_{j=1}^m \psi_j(x)\, u_j.$$

Since $x \in \overline{\Lambda}$, we have that $\sum_j \psi_j(x) = 1$. Hence, there exists $\bar{\imath} \in \{1, \dots, m\}$ such that $\psi_{\bar{\imath}}(x) \ge \frac{1}{m}$.

We claim that $\mathcal{I}^x(B^m_\eta(u^0))$ contains a neighbourhood of $\mathcal{I}^x(u^0)$. First of all, let us notice that

$$\mathcal{I}^x(B^m_\eta(u^0)) = \mathcal{I}^x\big(B_\eta(u^0_1) \times \cdots \times B_\eta(u^0_m)\big) \supseteq A, \tag{3.3}$$

where $A := \mathcal{I}^x\big(\{u^0_1\} \times \cdots \times \{u^0_{\bar{\imath}-1}\} \times B_\eta(u^0_{\bar{\imath}}) \times \{u^0_{\bar{\imath}+1}\} \times \cdots \times \{u^0_m\}\big)$. Easy computations show that

$$\big\{y - \mathcal{I}^x(u_0) : y \in A\big\} = B_{\eta\,\psi_{\bar{\imath}}(x)}(0).$$

Therefore we can rewrite $A$ as

$$A = \mathcal{I}^x(u^0) + B_{\eta\,\psi_{\bar{\imath}}(x)}(0) = B_{\eta\,\psi_{\bar{\imath}}(x)}(\mathcal{I}^x(u^0)) \supseteq B_{\frac{\eta}{m}}(\mathcal{I}^x(u^0)). \tag{3.4}$$

The same argument can be repeated for every $x \in \overline{\Lambda}$.

Let us now show that (3.2) holds true with $\eta_0 := \frac{\eta}{m}$. Let $x \in \overline{\Lambda}$ and $v \in \mathbb{R}^n$ such that $|v - \Psi^0(x)| < \eta_0$, i.e., $v \in B_{\eta_0}(\Psi^0(x)) = B_{\eta_0}(\mathcal{I}^x(u^0))$. From (3.3) and (3.4) we have that $v \in A \subset \mathcal{I}^x(B^m_\eta(u^0))$, hence there exists $u \in B^m_\eta(u^0)$ such that $v = \mathcal{I}^x(u) = \Psi_u(x)$. This proves (3.2).

For every $x_0 \in \overline{\Lambda}$ let us consider the following Cauchy problem:

$$\begin{cases} \dot{x}(t) = \Psi^0(x(t)), \\ x(0) = x_0. \end{cases} \tag{3.5}$$

We denote by $(x_0, t) \mapsto \Phi(x_0, t)$ the flow associated to (3.5). Using (3.1) and the compactness of $\overline{\Lambda}$, we have that there exists $t_0 > 0$, independent of $x_0 \in \overline{\Lambda}$, such that $\{\Phi(x_0, t) : t \in (0, t_0)\} \subset D$ and $\{\Phi(x_0, -t) : t \in (0, t_0)\} \subset \mathbb{R}^n \setminus \overline{D}$. Clearly, the restriction $\Phi_{|\overline{\Lambda}\times(-t_0,t_0)}$ is bijective. In particular we have that $\{\Phi(x_0, 0) : x_0 \in \overline{\Lambda}\} = \overline{\Lambda}$. Now we define $W, W^+, W^-$ as

$$W := \big\{\Phi(x_0, t) : (x_0, t) \in \Lambda \times (-t_0, t_0)\big\} \tag{3.6}$$

$$W^+ := W \cap D = \big\{\Phi(x_0, t) : (x_0, t) \in \Lambda \times (0, t_0)\big\}, \tag{3.7}$$

$$W^- := W \cap (\mathbb{R}^n \setminus \overline{D}) = \big\{\Phi(x_0, t) : (x_0, t) \in \Lambda \times (-t_0, 0)\big\}. \tag{3.8}$$

Using classical properties of the flow, and the fact that $\Lambda$ is Lipschitz, it is possible to show that the map $\Phi_{|\Lambda\times(-t_0,t_0)} : \Lambda \times (-t_0, t_0) \to W$ is bilipschitz.

We define $\phi : W \to W$ in the following way. Let $x \in W$. Then, by definition of $W$, there exists a pair $(x_0, t) \in \Lambda \times (-t_0, t_0)$ such that $x = \Phi(x_0, t)$. We set $\phi(x) := \Phi(x_0, -t)$. This map is bijective and bilipschitz, and satisfies the required properties. Hence the theorem is proved. □

In the periodic case, the previous theorem can be modified in the following way.

**Corollary 3.2.** *Let $E \subset \mathbb{R}^n$ be a periodic open connected set with Lipschitz boundary. Then, there exists a periodic neighbourhood $W$ of $\partial E$ with Lipschitz boundary, and a bilipschitz periodic map $\phi : W \to W$ such that $\phi_{|\partial E} = Id$ and $\phi(W^\pm) = W^\mp$, where $W^+ := W \cap E$ and $W^- := W \cap (\mathbb{R}^n \setminus \overline{E})$.*

*Proof.* We repeat the proof of Theorem 3.1, with $D = E$ and $\Lambda = \partial E \cap Q$. We observe that, due to the periodicity of $E$, the open covering $U_1, \ldots, U_m$ of $\overline{\Lambda}$ and the vectors $u^0_1, \ldots, u^0_m$ can be chosen to be periodic, in the following sense.

If $j \in \{1, \ldots, m\}$ and $i \in \{1, \ldots, n\}$ are such that $U_j \cap \partial Q_{+,i} \neq \emptyset$, then there exists $k \in \{1, \ldots, m\}$ such that $U_k = U_j + e_i$ and $u^0_j = u^0_k$. Similarly, If $j \in \{1, \ldots, m\}$ and $i \in \{1, \ldots, n\}$ are such that $U_j \cap \partial Q_{-,i} \neq \emptyset$, then there exists $k \in \{1, \ldots, m\}$ such that $U_k = U_j - e_i$ and $u^0_j = u^0_k$.

By the previous argument, proceeding as in the proof of Theorem 3.1, it is possible to construct a vector field $\Psi^0 : \mathbb{R}^n \to \mathbb{R}^n$ such that ${\Psi^0}_{|\partial Q_{+,i}} = {\Psi^0}_{|\partial Q_{-,i}}$ for every $i = 1, \ldots, n$. Therefore without loss of generality we can assume that $\Psi^0$ is Lipschitz and periodic. Indeed, we can otherwise replace it with the Lipschitz periodic extension of ${\Psi^0}_{|\overline{Q}}$.

As in the proof of Theorem 3.1, for every $x_0 \in \overline{\partial E \cap Q}$ we consider the Cauchy problem (3.5) and we denote with $\Phi(x_0, t)$ the associate flow. Then we define a positive real number $t_0$, the set

$$K := \{\Phi(x_0, t) : (x_0, t) \in \overline{\partial E \cap Q} \times (-t_0, t_0)\},$$

and a periodic bilipschitz map $\hat{\phi} : K \to K$ such that $\hat{\phi}_{|\overline{\partial E \cap Q}} = Id$. Moreover, $K$ and $\hat{\phi}$ can be extended by periodicity, so that the set

$$W := \text{int}\left( \bigcup_{h \in \mathbb{Z}^n} (K + h) \right)$$

and the function $\phi : W \to W$ defined as $\phi(x) := h + \hat{\phi}(x - h)$ for $x \in (K + h)$ and $h \in \mathbb{Z}^n$ have the required properties.

□

We can now prove Theorem 1.2.

*Proof of Theorem 1.2.* Let $u \in SBV^2(D) \cap L^\infty(D)$.

By Theorem 3.1 applied to $\Lambda = \partial D \cap A$, we can find an open set $W$ containing $\partial D \cap A$, and a bilipschitz map $\phi : W \to W$ such that $\phi_{|\partial D \cap A} = Id$ and $\phi(W^\pm) = W^\mp$, where $W^+ = W \cap D$ and $W^- = W \cap (\mathbb{R}^n \setminus \overline{D})$. Without loss of generality, we can assume $W \subset\subset A$. We define $v : D \cup W \to \mathbb{R}$ as

$$v(x) := \begin{cases} u(x) & \text{if } x \in D, \\ u(\phi(x)) & \text{if } x \in W^-. \end{cases}$$

It turns out that $v \in SBV^2(D \cup W)$ and that the following estimate holds true:

$$MS(v, D \cup W) \leq (1 + C_1) MS(u, D), \tag{3.9}$$

where, setting $\psi := \phi^{-1}$, the constant $C_1 = C_1(n, D, A)$ is given by

$$C_1 := \| \det\nabla\psi \, (\nabla\psi)^{-T} \|_{L^\infty(W;\mathbb{M}^n)}. \tag{3.10}$$

For the rigorous proof of (3.9) we refer to Theorem 8.1 in the Appendix.

Now, let us consider a solution $\hat{v}$ of the following problem:

$$\min\left\{ \int_{D \cup W} |\nabla w|^2 dx + \mathcal{H}^{n-1}(S_w) : w \in SBV^2(D \cup W), w = u \ \text{ a.e. in } \ D \right\}.$$

By definition of $\hat{v}$ and using (3.9), we have that $\hat{v} = u$ a.e. in $D$ and

$$MS(\hat{v}, D \cup W) \leq MS(v, D \cup W) \leq (1 + C_1) MS(u, D). \tag{3.11}$$

By a truncation argument, it follows that $||\hat{v}||_{L^\infty(D \cup W)} = ||u||_{L^\infty(D)}$.

Let us analyze more carefully the structure of $W$. By (3.6), (3.7) and (3.8), we have

$$\begin{aligned} W &= \{\Phi(x_0, t) : (x_0, t) \in (\partial D \cap A) \times (-t_0, t_0)\}, \\ W^+ &= W \cap D = \{\Phi(x_0, t) : (x_0, t) \in (\partial D \cap A) \times (0, t_0)\}, \\ W^- &= W \cap (\mathbb{R}^n \setminus \overline{D}) = \{\Phi(x_0, t) : (x_0, t) \in (\partial D \cap A) \times (-t_0, 0)\}, \end{aligned}$$

where the function $(x_0, t) \mapsto \Phi(x_0, t)$ is the flux associated to problem (3.5). Now we set

$$\Gamma := \{\Phi(x_0, -t_0/2) : x_0 \in \partial D \cap A\} \subset W^-.$$

For every $z \in \Gamma$ let $\varrho(z)$ be defined as $\varrho(z) := \sup \{\varrho > 0 : B_\varrho(z) \subset W^-\}$, and let $\gamma$ be the positive constant given by

$$\gamma := \frac{1}{2} \inf_{z \in \Gamma} \varrho(z).$$

The situation is shown in Fig. 4.

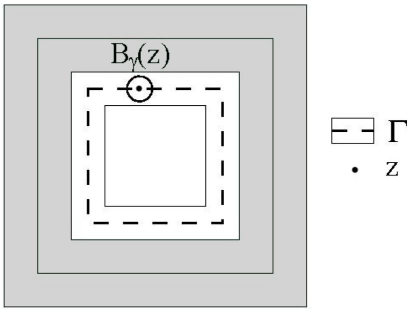


Figure 4: A point $z \in \Gamma$ and the ball $B_\gamma(z)$. Here $A, D$ and $W$ are those shown in Figure 2.

Let $\omega > 0$ be defined as $\omega := \beta\,\gamma^{n-1}$, where $\beta > 0$ is the constant given by the Elimination Theorem 2.5. In order to construct the required extension, we need to distinguish two cases, that will be treated in a different way.

**First case: large jump set**

We assume that $\mathcal{H}^{n-1}(S_{\hat{v}} \cap W^-) \geq \omega$. Let us define $Lu$ as

$$(Lu)(x) := \begin{cases} \hat{v}(x) & \text{if } x \in D \cup W, \\ 0 & \text{if } x \in A \setminus (D \cup W). \end{cases} \tag{3.12}$$

It turns out that $Lu \in SBV^2(A)$. Moreover, using (3.11) we have

$$\begin{aligned} MS(Lu, A) &\leq MS(\hat{v}, D \cup W) + \mathcal{H}^{n-1}(\partial W \setminus D) \\ &= MS(\hat{v}, D \cup W) + C_2\,\omega \\ &\leq MS(\hat{v}, D \cup W) + C_2\;\mathcal{H}^{n-1}(S_{\hat{v}} \cap W^-) \\ &\leq \max\{1, C_2\} MS(\hat{v}, D \cup W) \\ &\leq (1 + C_1) \max\{1, C_2\} MS(u, D), \end{aligned} \tag{3.13}$$

where $C_2 = C_2(n, D, A)$ is the positive constant given by

$$C_2 := \frac{\mathcal{H}^{n-1}(\partial W \setminus D)}{\omega}.$$

**Second case: small jump set**

We assume that $\mathcal{H}^{n-1}(S_{\hat{v}} \cap W^-) < \omega$. Let us fix $z \in \Gamma$ and let us consider the ball $B_\gamma(z) \subset W^-$. Clearly, $\mathcal{H}^{n-1}(S_{\hat{v}} \cap B_\gamma(z)) \leq \mathcal{H}^{n-1}(S_{\hat{v}} \cap W^-) < \omega$. By our definition of $\omega$, this implies that

$$\mathcal{H}^{n-1}(S_{\hat{v}} \cap B_\gamma(z)) < \beta\,\gamma^{n-1}.$$

Hence, by Theorem 2.5 we have that $S_{\hat{v}} \cap B_{\gamma/2}(z) = \emptyset$ (see Fig. 5)). The same argument can be repeated for every $z \in \Gamma$. Therefore, we deduce that the set $\Delta \subset W^-$ defined as

$$\Delta := \bigcup_{z \in \Gamma} B_{\gamma/2}(z)$$

does not intersect the jump set of $\hat{v}$ (see Fig. 6)).

Without loss of generality, we can assume $\Delta$ connected, otherwise the same argument can be repeated for every connected component of $\Delta$. We observe that $\Delta$ disconnects $A \setminus \overline{D}$. Indeed, we can write $(A \setminus \overline{D}) \setminus \overline{\Delta} = U' \cup U''$, where $U'$ and $U''$ are open, connected disjoint sets, with $\partial D \cap \partial U' \neq \emptyset$. Now, let us define $Lu$ as

$$(Lu)(x) := \begin{cases} \hat{v}(x) & \text{if } x \in A \setminus (\Delta \cup \overline{U''}), \\ (\tau\hat{v})(x) & \text{if } x \in (\Delta \cup \overline{U''}), \end{cases} \tag{3.14}$$

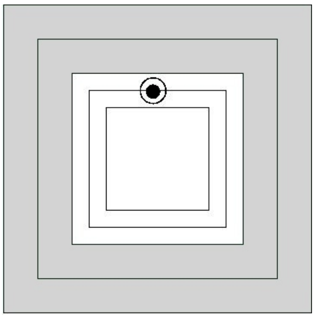

Figure 5: The ball $B_{\frac{\gamma}{2}}(z)$.

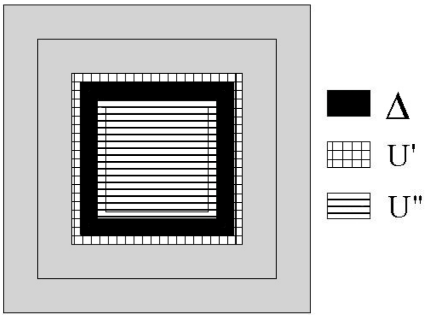


Figure 6: The sets $\Delta$, $U'$ and $U''$.

where $\tau$ denotes the extension operator from $H^1(\Delta)$ to $H^1(\Delta\cup\overline{U''})$ provided by Theorem 2.1. By relation (2.1), we have that

$$\int_{\Delta\cup\overline{U''}}|\nabla(\tau\hat{v})|^2dx \le k_2\int_{\Delta}|\nabla\hat{v}|^2dx, \tag{3.15}$$

where $k_2 = k_2(n,\Delta,\Delta\cup\overline{U''})$. Furthermore, up to truncation, we can always assume that the $L^\infty$ bound is preserved. Then, it turns out that $Lu\in SBV^2(A)$, $Lu = u$ a.e. in $D$ and $||Lu||_{L^\infty(A)} = ||u||_{L^\infty(D)}$. By (3.15), we have

$$\begin{aligned} MS(Lu,A) &= MS(\hat{v}, D\cup\overline{U'}\cup\Delta) + \int_{U''}|\nabla(\tau\hat{v})|^2dx \\ &\le MS(\hat{v}, D\cup W) + k_2\, MS(\hat{v},\Delta) \\ &\le \max\{1,k_2\}MS(\hat{v}, D\cup W) \\ &\le (1+C_1)\max\{1,k_2\}MS(u,D), \end{aligned} \tag{3.16}$$

where in the last inequality we used (3.11).

**Estimate in the general case**.

The function $Lu$ defined in (3.12) and (3.14) respectively clearly satisfies properties $(i)$ and $(ii)$ of Theorem 1.2. By (3.13) and (3.16), estimate (1.5) holds true setting

$$c(n,D,A) := (1+C_1)\max\{1,C_2,k_2\}.$$

The arguments used in the proof are clearly invariant under translations. Thus, it remains to prove that the constant $c$ is invariant under dilations. Let $w\in SBV^2(D)\cap L^\infty(D)$ and let $\lambda>0$. We define the function $w^\lambda:\lambda D\to\mathbb{R}$ as $w^\lambda(x) := \sqrt{\lambda}\,w(\frac{x}{\lambda})$ for every $x\in\lambda D$. Let $Lw\in SBV^2(A)\cap L^\infty(A)$ denote the extension provided by the theorem just proved. Then, we can define an extension operator $L^\lambda$ from $SBV^2(\lambda D)\cap L^\infty(\lambda D)$ to $SBV^2(\lambda A)\cap L^\infty(\lambda A)$ as

$$(L^\lambda w^\lambda)(x) := \sqrt{\lambda}\,(Lw)\left(\frac{x}{\lambda}\right) \qquad \text{for every } x\in\lambda A.$$

This concludes the proof, since

$$MS(L^\lambda w^\lambda,\lambda A) = \lambda^{n-1}MS(Lw,A) \le c\,\lambda^{n-1}MS(w,D) = c\,MS(w^\lambda,\lambda D).$$

□

## 4. Compactly contained hole: $\varepsilon$-periodic extension

In this section we prove a simplified version of Theorem 1.3. We will consider the case in which the set $E$ is obtained removing from the unit square a compactly contained hole and repeating this construction by periodicity. More precisely, let $F \subset\subset Q$ be an open Lipschitz set (see Figure 1 where, for simplicity, $F$ is a cube and $F$ and $Q$ are concentric). We assume that $E$ is given by

$$E := \mathbb{R}^n \setminus \bigcup_{h\in\mathbb{Z}^n} (\overline{F} + h). \tag{4.1}$$

We state now the main result of this section.

**Theorem 4.1.** *Fix $\varepsilon > 0$. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary, and let $E \subset \mathbb{R}^n$ the periodic set defined as in (4.1). Set $E^\varepsilon := \varepsilon E$ and $\Omega(\varepsilon) := \Omega \cap E^\varepsilon$. Then there exists an extension operator $T^\varepsilon : SBV^2(\Omega(\varepsilon)) \cap L^\infty(\Omega(\varepsilon)) \to SBV^2(\Omega) \cap L^\infty(\Omega)$ and a constant $k_0 > 0$ depending on $E$ and $n$, but not on $\varepsilon$ and $\Omega$, such that*

- $T^\varepsilon u = u$ *a.e. in* $\Omega(\varepsilon)$,
- $||T^\varepsilon u||_{L^\infty(\Omega)} \le ||u||_{L^\infty(\Omega(\varepsilon))}$,
- $MS(T^\varepsilon u, \Omega) \le k_0\big(MS(u, \Omega(\varepsilon)) + \mathcal{H}^{n-1}(\partial\Omega)\big)$

*for every $u \in SBV^2(\Omega(\varepsilon)) \cap L^\infty(\Omega(\varepsilon))$.*

*Proof.* Let $u \in SBV^2(\Omega(\varepsilon)) \cap L^\infty(\Omega(\varepsilon))$. Let $\mathbb{Z}_\varepsilon$ be the set of vectors $h \in \mathbb{Z}^n$ such that the $\varepsilon$-homothetic of the domain $h + \overline{Q}$ has a nonempty intersection with $\Omega$, and let us introduce an ordering of its elements. More precisely, we set

$$\mathbb{Z}_\varepsilon := \{h \in \mathbb{Z}^n : \varepsilon(h + \overline{Q}) \cap \Omega \neq \emptyset\} = \{h_1, h_2, \dots, h_{N(\varepsilon)}\}, \tag{4.2}$$

where with $N(\varepsilon) \in \mathbb{N}$ we denoted the cardinality of $\mathbb{Z}_\varepsilon$. For shortening the notation, we set

$$Q_k := h_k + Q, \quad Q_k^\varepsilon := \varepsilon\, Q_k \qquad k = 1, \dots, N(\varepsilon), \tag{4.3}$$

and

$$\Omega_Q(\varepsilon) := \mathrm{int}\left(\bigcup_{k=1}^{N(\varepsilon)} \overline{Q_k^\varepsilon}\right), \tag{4.4}$$

where "int" stands for the interior of the set in brackets. We define $\tilde{u} : E^\varepsilon \to \mathbb{R}$ as

$$\tilde{u} := \begin{cases} u & \text{in } \Omega(\varepsilon) \\ 0 & \text{in } E^\varepsilon \setminus \Omega(\varepsilon). \end{cases}$$

Clearly, the function $\tilde{u}$ satisfies $\tilde{u} = u$ a.e. in $\Omega(\varepsilon)$, $||\tilde{u}||_{L^\infty(E^\varepsilon)} \le ||u||_{L^\infty(\Omega(\varepsilon))}$, and

$$MS(\tilde{u}, E^\varepsilon) \le MS(u, \Omega(\varepsilon)) + \mathcal{H}^{n-1}(\partial\Omega). \tag{4.5}$$

Notice that we can write

$$MS(\tilde{u}, E^\varepsilon) = \sum_{k=1}^{N(\varepsilon)} MS(\tilde{u}, Q_k^\varepsilon \cap E^\varepsilon) + \mathcal{H}^{n-1}\left(S_{\tilde{u}} \cap E^\varepsilon \cap \left(\bigcup_{k=1}^{N(\varepsilon)} \partial Q_k^\varepsilon\right)\right). \tag{4.6}$$

Let us denote with $L_k^\varepsilon : SBV^2(Q_k^\varepsilon \cap E^\varepsilon) \cap L^\infty(Q_k^\varepsilon \cap E^\varepsilon) \to SBV^2(Q_k^\varepsilon) \cap L^\infty(Q_k^\varepsilon)$ the extension operator provided by Theorem 1.2, with $k = 1, \dots, N(\varepsilon)$, and we define $v^\varepsilon \in SBV^2(\Omega_Q(\varepsilon)) \cap L^\infty(\Omega_Q(\varepsilon))$ as

$$v^\varepsilon(x) := (L_k^\varepsilon \tilde{u})(x) \quad \text{if } x \in Q_k^\varepsilon,\ k \in \{1, \dots, N(\varepsilon)\}.$$

We have that for every $k = 1, \dots, N(\varepsilon)$

- $v^\varepsilon = \tilde{u}$ a.e. in $Q_k^\varepsilon \cap E^\varepsilon$,
- $\|v^\varepsilon\|_{L^\infty(Q_k^\varepsilon)} \le \|\tilde{u}\|_{L^\infty(Q_k^\varepsilon \cap E^\varepsilon)}$,
- $MS(v^\varepsilon, Q_k^\varepsilon) \le c\, MS(\tilde{u}, Q_k^\varepsilon \cap E^\varepsilon)$. (4.7)

Since the constant provided by the theorem is invariant under translations and homotheties, $c = c(n, E, Q)$ is independent of $k$ and $\varepsilon$. Then, using (4.6) and (4.7), we get

$$\begin{aligned} MS(v^\varepsilon, \Omega_Q(\varepsilon)) &= \sum_{k=1}^{N(\varepsilon)} MS(v^\varepsilon, Q_k^\varepsilon) + \mathcal{H}^{n-1}\left(S_{v^\varepsilon} \cap E^\varepsilon \cap \left(\bigcup_{k=1}^{N(\varepsilon)} \partial Q_k^\varepsilon\right)\right) \\ &\leq c \sum_{k=1}^{N(\varepsilon)} MS(\tilde{u}, Q_k^\varepsilon \cap E^\varepsilon) + \mathcal{H}^{n-1}\left(S_{\tilde{u}} \cap E^\varepsilon \cap \left(\bigcup_{k=1}^{N(\varepsilon)} \partial Q_k^\varepsilon\right)\right) \\ &\leq k_0\, MS(\tilde{u}, E^\varepsilon), \end{aligned}$$

where $k_0 := \max\{c, 1\}$. Combining the previous expression with (4.5) we have

$$MS(v^\varepsilon, \Omega_Q(\varepsilon)) \leq k_0\big(MS(u, \Omega(\varepsilon)) + \mathcal{H}^{n-1}(\partial\Omega)\big),$$

therefore the claim follows defining $T^\varepsilon u := v^\varepsilon{}_{|\Omega}$.

□

## 5. General connected sets: extension for a fixed domain

In this section we prove Theorem 1.1. Apart from some technical difficulties, the strategy of the proof remains the same as in Theorem 1.2. First, we need to state two lemmas that follow, up to some slight variations, from [1, Lemma 2.2] and from [1, Lemma 2.3], respectively.

**Lemma 5.1.** *Let $P$, $\omega$, $\omega'$ be open subsets of $\mathbb{R}^n$. Assume that $\omega$, $\omega'$ are bounded, with $\omega \subset\subset \omega'$ and that $P$ has Lipschitz boundary at each point of $\partial P \cap \overline{\omega}$. Then the number of connected components of $P \cap \omega'$ that intersect $P \cap \omega$ is finite.*

We notice that Lemma 5.1 implies in particular that if infinitely many connected components of $P$ accumulate on a point $z \in \partial P$, then the boundary of $P$ at $z$ is not Lipschitz.

We will apply the previous result in the proof Theorem 1.1 with $P = \mathbb{R}^n \setminus \overline{D}$ and $\omega = A$, to conclude that the number of holes to "fill" (i.e., the holes of $D$ that intersect $A$) is necessarily finite. The latter conclusion could be misleading, as it seems to suggest that the proof of Theorem 1.1 follows by simply applying Theorem 1.2 a finite number of times, one for each hole. This is true, however, only if every hole that has to be filled is "well contained" in D, that is, only if every hole belongs to a *bounded* connected component of $\mathbb{R}^n \setminus \overline{D}$. Indeed, in this special case, we can "surround" each hole with a stripe all contained in $D$, and then apply Theorem 1.1. Anyway, there may be holes that do not satisfy this property (see for instance Figure 7, where $U_4$ belongs to an unbounded connected component of $\mathbb{R}^n \setminus \overline{D}$).

**Lemma 5.2.** *Let $D$ be a connected open subset of $\mathbb{R}^n$, with Lipschitz boundary, and let $A \subset \mathbb{R}^n$ be open and bounded, with $A \cap D \neq \emptyset$. Then, there exists $\mathbf{k} \in \mathbb{N}$, $\mathbf{k} \geq 2$, such that $A \subset\subset \mathbf{k}Q$ and $A \cap D$ is contained in a single connected component of $\mathbf{k}Q \cap D$.*

*Proof of Theorem 1.1.* Since in the case $A \cap D = \emptyset$ the function $u$ can be trivially extended from $D$ to $A$ setting $Lu \equiv 0$ in $A$, we can assume from now on that $A \cap D \neq \emptyset$. Let $\mathbf{k}$ be given by Lemma 5.2; applying Lemma 5.1 with $P = \mathbb{R}^n \setminus \overline{D}$, $\omega = A$ and $\omega' = (\mathbf{k}+1)Q$, we have that the number of connected components of $(\mathbb{R}^n \setminus \overline{D}) \cap ((\mathbf{k}+1)Q)$ that intersect $A$ is finite, say $M \in \mathbb{N}$. Let us denote these connected components by $U_1, \ldots, U_M$. The situation is described in Figure 7.

Notice that, since $A \subset\subset \mathbf{k}Q$, then $\delta := \operatorname{dist}(A, \partial(\mathbf{k}Q)) > 0$. We want to extend the function $u$ to the sets $U_1 \cap A, \ldots, U_M \cap A$, in such a way that conditions $(i)$, $(ii)$ and $(iii)$ of Theorem 1.1 are satisfied.

Let $W, W^\pm$, $\Phi$, $\phi$ and $t_0$ be those defined in the proof of Lemma 3.1 with

$$\Lambda := \bigcup_{i=1}^{M} \Lambda_i, \quad \text{where } \ \Lambda_i := \partial U_i \cap \partial D, \ \text{ for } \ i = 1, \ldots, M.$$

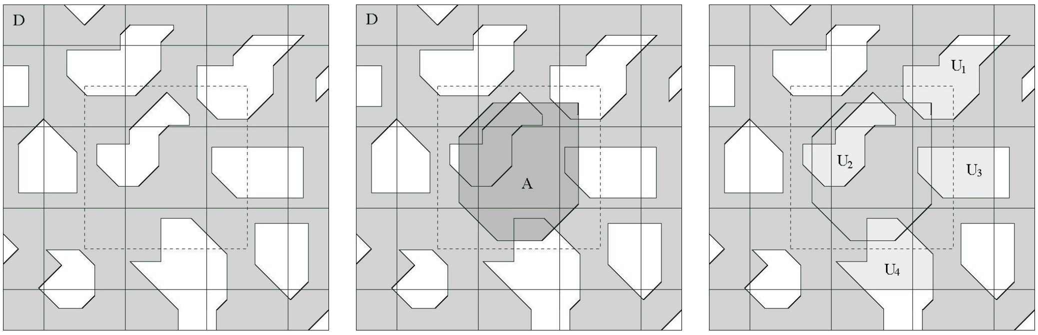


Figure 7: The set $D$; the set $A$ (here $\mathbf{k} = 2$); the sets $U_i$'s (notice that $M = 4$).

Let us define the sets

$$\Theta_1 := \bigcup_{i=1}^{M} \Theta_1^i, , \qquad \Theta_2 := \bigcup_{i=1}^{M} \Theta_2^i,$$

where, for $i = 1, \ldots, M$,

$$\Theta_1^i := \Lambda_i \cap \partial(\mathbf{k}Q), \qquad \Theta_2^i := \Lambda_i \cap \partial((\mathbf{k}+1)Q).$$

• *Possible restriction of the interval* $[-t_0, t_0]$.

In the case $\Theta_2 \neq \emptyset$ we restrict the interval $[-t_0, t_0]$ to some $[-\eta, \eta]$, with $\eta \in (0, t_0]$, to guarantee that the image under the map $\Phi$ of an $\eta$-neighbourhood of $\Theta_1$ is well separated by $\partial A$. More precisely, we proceed in the following way.

If $\Theta_2 = \emptyset$ we just set $\eta := t_0$. If instead $\Theta_2 \neq \emptyset$, we have also $\Theta_1 \neq \emptyset$. Then, for every $x_0 \in \Theta_1$, $x_0 = \Phi(x_0, 0)$ and $\mathrm{dist}(x_0, \partial A) \geq \delta$. Since $(x,t) \mapsto \Phi(x,t)$ is uniformly continuous in the compact set $\Theta_1 \times [-t_0, t_0]$, and $x \mapsto \mathrm{dist}(x, \partial A)$ is continuous (in fact Lipschitz), there exists $\eta \in (0, t_0]$ such that

$$\mathrm{dist}(\Phi(x_0, t), \partial A) > \delta/2 \text{ for every } (x_0, t) \in \Theta_1 \times [-\eta, \eta],$$

i.e.,

$$\mathrm{dist}\big(\Phi\big(\Theta_1 \times [-\eta, \eta]\big), \partial A\big) > \delta/2.$$

Notice that, since $\Theta_1 \cap \Theta_2 = \emptyset$, the sets $\Theta_1 \times [-\eta, \eta]$ and $\Theta_2 \times [-\eta, \eta]$ are mapped by the flow $\Phi$ into two "parallel" (in the sense of the flow $\Phi$) disjoint sets $\Phi(\Theta_1 \times [-\eta, \eta])$ and $\Phi(\Theta_2 \times [-\eta, \eta])$. Thus,

$$\mathrm{dist}\big(\,\Phi(\Theta_1 \times [-\eta, \eta]), \Phi(\Theta_2 \times [-\eta, \eta])\,\big) > 0. \tag{5.1}$$

• *Definition of an auxiliary minimum problem.*

We define the following subsets of $W$

$$W_* := \Phi(\Lambda \times (-\eta, \eta)), \qquad W_*^+ := \Phi(\Lambda \times (0, \eta)), \qquad W_*^- := \Phi(\Lambda \times (-\eta, 0)).$$

Without loss of generality, we assume that $W_* \subset (\mathbf{k}+2)Q$. This will be useful in order to prove Theorem 1.3. Notice that $W_* \subset W$, and that $\phi(W_*^-) = W_*^+$.

Now we define $v : D \cup W_* \to \mathbb{R}$ as

$$v(x) = \begin{cases} u(x) & \text{if } x \in D, \\ u(\phi(x)) & \text{if } x \in W_*^-. \end{cases}$$

It turns out that $v \in SBV^2(D \cup W_*)$ and

$$MS(v, D \cup W_*) \leq (1 + C_1) MS(u, D), \tag{5.2}$$

where, setting $\psi := \phi^{-1}$, $C_1 = C_1(n, D, A)$ is given by (see Theorem 8.1)

$$C_1 := \| \det\nabla\psi \, (\nabla\psi)^{-T} \|_{L^\infty(W_*;\mathbb{M}^n)}. \tag{5.3}$$

Let us consider a solution $\hat{v}$ of the following problem:

$$\min\left\{ \int_{D\cup W_*} |\nabla w|^2 dx + \mathcal{H}^{n-1}(S_w) : w \in SBV^2(D \cup W_*), w = u \text{ a.e. in } D \right\}.$$

Using the definition of $\hat{v}$ and the estimate (5.2), we have $\hat{v} = u$ a.e. in $D$ and

$$MS(\hat{v}, D \cup W_*) \le MS(v, D \cup W_*) \le (1 + C_1)MS(u, D). \tag{5.4}$$

By a truncation argument, we can choose $\hat{v}$ such that $||\hat{v}||_{L^\infty(D\cup W_*)} \le ||u||_{L^\infty(D)}$.

Now, for every $i = 1, \dots, M$, we denote with $\Gamma_i$ the connected component of $\Phi(\Lambda_i, -\eta/2) \cap \mathbf{k}Q$ intersecting $A$. We notice that $\Gamma_i \subset W_*^-$, for $i = 1, \dots, M$. Then we set

$$\Gamma := \bigcup_{i=1}^{M} \Gamma_i \subset W_*^-. \tag{5.5}$$

For every $z \in \Gamma$, let $\varrho(z)$ be defined as $\varrho(z) := \sup\{\varrho \in (0, \mu) : B_\varrho(z) \subset W_*^-\}$, where

$$\mu := \begin{cases} \dfrac{\delta}{2} & \text{if } \Theta_2 \neq \emptyset, \\ +\infty & \text{otherwise,} \end{cases}$$

and let $\gamma$ be given by $\gamma := \frac{1}{2} \inf\{\varrho(z) : z \in \Gamma\}$. Thanks to (5.1), we have $\gamma > 0$.

In the case $\Theta_2 \neq \emptyset$ we require $\gamma < \delta/2$ since, as will be clear in the sequel, we need to disconnect the sets $U_i \cap \mathbf{k}Q$ (for $i = 1, \dots, M$), by covering $\Gamma$ with balls of radius $\frac{\gamma}{2}$.

Let $\omega > 0$ be defined as $\omega := \beta\, \gamma^{n-1}$, where $\beta > 0$ is the constant given by the Elimination Theorem 2.5. In order to construct the required extension, we need to distinguish two cases, that will be treated in a different way.

**First case: large jump set**

We assume that $\mathcal{H}^{n-1}(S_{\hat{v}} \cap W_*^-) \ge \omega$. Let us define $Lu$ as

$$(Lu)(x) := \begin{cases} \hat{v}(x) & \text{if } x \in A \cap (D \cup W_*), \\ 0 & \text{if } x \in A \setminus (D \cup W_*). \end{cases} \tag{5.6}$$

It turns out that $Lu \in SBV^2(A)$ and, by construction, $||Lu||_{L^\infty(A)} \le ||\hat{v}||_{L^\infty(D\cup W_*)} = ||u||_{L^\infty(D)}$. Moreover, using relations (5.4) and (5.9),

$$\begin{aligned} MS(Lu, A) &\le MS(\hat{v}, A \cap (D \cup W_*)) + \mathcal{H}^{n-1}(\partial W_* \setminus D) \\ &\le MS(\hat{v}, D \cup W_*) + C_2\, \omega \\ &\le MS(\hat{v}, D \cup W_*) + C_2\, \mathcal{H}^{n-1}(S_{\hat{v}} \cap W_*^-) \\ &\le \max\{1, C_2\} MS(\hat{v}, D \cup W_*) \\ &\le (1 + C_1) \max\{1, C_2\} MS(u, D), \end{aligned} \tag{5.7}$$

where we set

$$C_2 := \frac{\mathcal{H}^{n-1}(\partial W_* \setminus D)}{\omega}.$$

**Second case: small jump set**

We assume that $\mathcal{H}^{n-1}(S_{\hat{v}} \cap W_*^-) < \omega$. Let us fix $i \in \{1, \dots, M\}$ and $z \in \Gamma_i$, and let us consider the ball $B_\gamma(z) \subset W_*^-$. Clearly, $\mathcal{H}^{n-1}(S_{\hat{v}} \cap B_\gamma(z)) \le \mathcal{H}^{n-1}(S_{\hat{v}} \cap W_*^-) < \omega$. By our definition of $\omega$, this implies that

$$\mathcal{H}^{n-1}(S_{\hat{v}} \cap B_\gamma(z)) < \beta\, \gamma^{n-1}.$$

Hence, by Theorem (2.5) we have that $S_{\hat{v}} \cap B_{\gamma/2}(z) = \emptyset$. The same argument can be repeated for every $z \in \Gamma_i$. Therefore we deduce that, for every $i = 1, \ldots, M$, the set $\Delta_i$ defined as

$$\Delta_i := \bigcup_{z \in \Gamma_i} B_{\gamma/2}(z) \subset W_*^-$$

does not intersect the jump set of $\hat{v}$. Moreover, by definition, $\Delta_i$ is Lipschitz, connected, and disconnects the set $U_i \cap \mathbf{k}Q$. Indeed, for every $i = 1, \ldots, M$, we can write $(U_i \cap \mathbf{k}Q) \setminus \overline{\Delta_i} = U_i' \cup U_i''$, where $U_i'$ and $U_i''$ are open, disjoint, and $\partial D \cap \partial U_i' \neq \emptyset$.

The situation is illustrated in Figure 8, where for simplicity we focused on the set $U_4$ of the previous Figure 7. Then, we define

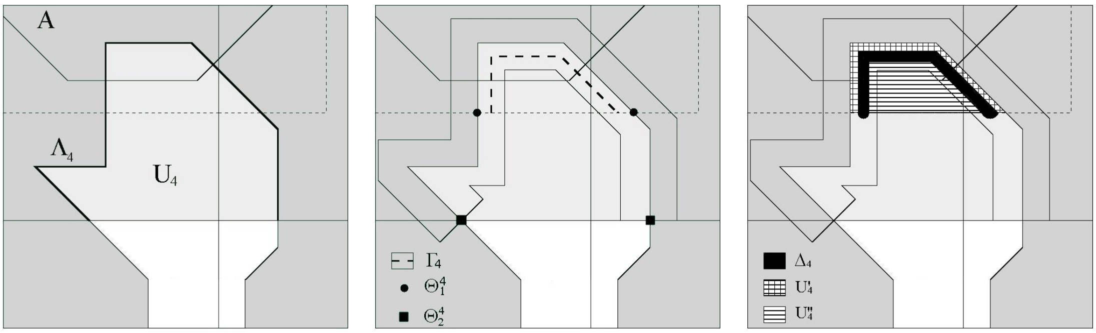


Figure 8: In this figure we present a step-by-step construction of the set $\Delta_4$.

$$\Delta := \bigcup_{i=1}^{M} \Delta_i.$$

Notice that, by construction, $\Delta_1, \ldots, \Delta_M$ are the connected components of the set $\Delta$. We underline that this fact is crucial in order to get the desired extension, since we are going to apply $M$ times Theorem 2.1, by extending the function $u$ from the sets $\Delta_i$.
Now, let us define the function $Lu$ as

$$(Lu)(x) := \begin{cases} \hat{v}(x) & \text{if } x \in A \setminus U_i'' \quad \text{for } i = 1, \ldots, M, \\ (\tau_i \hat{v})(x) & \text{if } x \in A \cap U_i'' \quad \text{for } i = 1, \ldots, M, \end{cases} \tag{5.8}$$

where, for every $i = 1, \ldots, M$, $\tau_i$ denotes the extension operator provided by Theorem 2.1 from $H^1(\Delta_i)$ to $H^1\left(\Delta_i \cup (\partial\Delta_i \cap \partial U_i'') \cup U_i''\right)$. By (2.1), we have that for every $i = 1, \ldots, M$

$$\int_{\Delta_i \cup (\partial\Delta_i \cap \partial U_i'') \cup U_i''} |\nabla(\tau_i \hat{v})|^2 dx \le K_2 \int_{\Delta_i} |\nabla \hat{v}|^2 dx, \tag{5.9}$$

where we set

$$K_2 := \max_{i=1,\ldots,M} \left\{ k_2(n, \Delta_i, \Delta_i \cup (\partial\Delta_i \cap \partial U_i'') \cup U_i'') \right\}. \tag{5.10}$$

Furthermore, up to truncation, we can always assume that the $L^\infty$ bound is preserved. Then, $Lu \in SBV^2(A)$, $Lu = u$ a.e. on $A \cap D$ and $||Lu||_{L^\infty(A)} \le ||u||_{L^\infty(D)}$.

To conclude the proof of the theorem in the case of a small jump set it remains to estimate the Mumford-Shah functional of the extended function $Lu$ on $A$ in terms of the function $u$ on $D$.

By (5.4) and (5.9),

$$
\begin{aligned}
MS(Lu, A) &= MS\left(\hat{v}, \bigcup_{i=1}^{M} (A \setminus \overline{U_i''})\right) + \sum_{i=1}^{M} \int_{A \cap U_i''} |\nabla(\tau_i \hat{v})|^2 dx \\
&\leq MS(\hat{v}, D \cup W_*) + K_2 \sum_{i=1}^{M} MS(\hat{v}, \Delta_i) \\
&\leq \max\{1, K_2\} MS(\hat{v}, D \cup W_*) \\
&\leq (1 + C_1) \max\{1, K_2\} MS(u, D). \qquad (5.11)
\end{aligned}
$$

**Estimate in the general case.**
The function $Lu$ defined in (5.6) and (5.8) clearly satisfies properties $(i)$ and $(ii)$ of Theorem 1.1. By (5.7) and (5.11), we obtain (1.4) setting

$$c(n, D, A) := (1 + C_1) \max\{1, C_2, K_2\}.$$

The invariance of the constant $c$ under translations and homotheties follows as in the proof of Theorem 1.2. $\square$

## 6. General connected domains: $\varepsilon$-periodic extension

We now prove Theorem 1.3, stated in the Introduction. For a pictorial idea of the set $E$, see Figure 9.

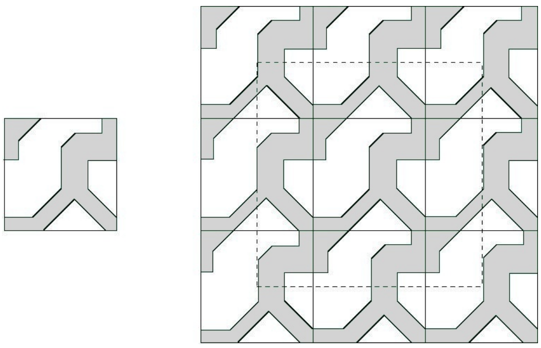

Figure 9: A periodic connected set with its periodicity cell. Notice that $E \cap Q$ is not connected and that, in this case, $\mathbf{k} = 2$.

*Proof.* Following closely the proof of Theorem 4.1, we define $\mathbb{Z}_\varepsilon$, $Q_k$, $Q_k^\varepsilon$ and $\Omega_Q(\varepsilon)$ as in (4.2), (4.3) and (4.4), respectively. From now on, we will consider the positive constants

$$M, \quad \mathbf{k}, \quad \omega, \quad C_1, \quad K_2, \quad C_2,$$

the sets

$$W_*, \quad W_*^+, \quad W_*^-, \quad \Gamma, \quad \Delta, \quad \Delta_1, \dots, \Delta_M, \quad U_1'', \dots, U_M'',$$

and the bilipschitz function $\phi : W_* \to W_*$ defined in the proof of Theorem 1.1, with $D = E$ and $A = Q$. We introduce also the sets

$$\mathcal{W}_* := \bigcup_{k=1}^{N(\varepsilon)} (h_k + W_*), \quad \mathcal{W}_*^{\pm} := \bigcup_{k=1}^{N(\varepsilon)} (h_k + W_*^{\pm}),$$

and the function $\phi^\varepsilon : \varepsilon\mathcal{W}_* \longrightarrow \varepsilon\mathcal{W}_*$ given by

$$\phi^\varepsilon(y) := \varepsilon\phi\left(\frac{y-\varepsilon h_k}{\varepsilon}\right) + \varepsilon h_k \qquad y \in \varepsilon(h_k + W_*), \qquad k = 1, \dots N(\varepsilon).$$

By Corollary 3.2, the sets $\mathcal{W}_*$, $\mathcal{W}_*^\pm$ are Lipschitz, and the function $\phi^\varepsilon$ is well defined and bilipschitz. Setting $\psi := \phi^{-1}$ and $\psi^\varepsilon := (\phi^\varepsilon)^{-1}$, we have

$$\psi^\varepsilon(z) = \varepsilon\psi\left(\frac{z-\varepsilon h_k}{\varepsilon}\right) + \varepsilon h_k$$

for every $z \in \varepsilon(h_k + W_*)$. Notice that

$$(\nabla_z\psi^\varepsilon)(z) = (\nabla\psi)\left(\frac{z-\varepsilon h_k}{\varepsilon}\right) \qquad z \in \varepsilon(h_k + W_*), \qquad k = 1, \dots N(\varepsilon), \tag{6.1}$$

where $\nabla_z$ denotes the gradient with respect to the variable $z$. Let $\tilde{u} : E^\varepsilon \to \mathbb{R}$ be defined as

$$\tilde{u} := \begin{cases} u & \text{in } \Omega(\varepsilon), \\ 0 & \text{in } E^\varepsilon \setminus \Omega(\varepsilon). \end{cases}$$

Clearly the function $\tilde{u}$ satisfies $\tilde{u} = u$ in $\Omega(\varepsilon)$, $||\tilde{u}||_{L^\infty(E^\varepsilon)} = ||u||_{L^\infty(\Omega(\varepsilon))}$, and

$$MS(\tilde{u}, E^\varepsilon) \leq MS(u, \Omega(\varepsilon)) + \mathcal{H}^{n-1}(\partial\Omega). \tag{6.2}$$

We define the extension $v : E^\varepsilon \cup \varepsilon\mathcal{W}_* \to \mathbb{R}$ as

$$v(x) := \begin{cases} \tilde{u}(x) & \text{if } x \in E^\varepsilon, \\ \tilde{u}\big(\phi^\varepsilon(x)\big) & \text{if } x \in \varepsilon\mathcal{W}_*^-. \end{cases}$$

It turns out that $v \in SBV^2(E^\varepsilon \cup \varepsilon\mathcal{W}_*)$; moreover, by Theorem 8.1 and using (6.2),

$$MS(v, E^\varepsilon \cup \varepsilon\mathcal{W}_*) \leq (1 + C_1)MS(\tilde{u}, E^\varepsilon) \leq (1 + C_1)\big(MS(u, \Omega(\varepsilon)) + \mathcal{H}^{n-1}(\partial\Omega)\big), \tag{6.3}$$

where, thanks to (6.1), the constant $C_1$ is independent of $\varepsilon$ and is given by

$$C_1 = \|\det\nabla\psi\,(\nabla\psi)^{-T}\|_{L^\infty(W_*;\mathbb{M}^n)} = \|\det(\nabla_z\psi^\varepsilon)\,(\nabla_z\psi^\varepsilon)^{-T}\|_{L^\infty(\varepsilon W_*;\mathbb{M}^n)}.$$

Consider now a solution $\hat{v}^\varepsilon$ to the minimum problem

$$\min\left\{\int_{E^\varepsilon\cup\varepsilon\mathcal{W}_*} |\nabla w|^2 dx + \mathcal{H}^{n-1}(S_w) : \; w \in SBV^2\left(E^\varepsilon \cup \varepsilon\mathcal{W}_*\right), \; w = \tilde{u} \text{ a.e. in } E^\varepsilon\right\}.$$

As in the proof of Theorem 1.1, we get that $||\hat{v}^\varepsilon||_{L^\infty(E^\varepsilon\cup\varepsilon\mathcal{W}_*)} = ||u||_{L^\infty(\Omega(\varepsilon))}$. Moreover, since $v$ is a competitor for the minimum problem defining $\hat{v}^\varepsilon$, using (6.3) we have

$$\begin{aligned} MS(v, E^\varepsilon \cup \varepsilon\mathcal{W}_*) &\leq MS(\hat{v}^\varepsilon, E^\varepsilon \cup \varepsilon\mathcal{W}_*) \leq (1 + C_1)MS(\tilde{u}, E^\varepsilon) \\ &\leq (1 + C_1)\big(MS(u, \Omega(\varepsilon)) + \mathcal{H}^{n-1}(\partial\Omega)\big). \end{aligned} \tag{6.4}$$

In order to construct the required extension, we divide the cubes into two groups, that will be treated in a different way. More precisely, we enumerate the vectors $h_1, \dots, h_{N(\varepsilon)}$ as follows:

$$\mathcal{H}^{n-1}\left(\varepsilon(h_k + W_*^-) \cap S_{\hat{v}^\varepsilon}\right) \geq \omega\,\varepsilon^{n-1} \qquad k = 1, \dots, N_1(\varepsilon), \tag{6.5}$$

where $N_1(\varepsilon) \in \{0, 1, \dots, N(\varepsilon)\}$, and

$$\mathcal{H}^{n-1}\left(\varepsilon(h_k + W_*^-) \cap S_{\hat{v}^\varepsilon}\right) < \omega\,\varepsilon^{n-1} \qquad k = N_1(\varepsilon) + 1, \dots, N(\varepsilon). \tag{6.6}$$

**First case: large jump set.**
We start proving a bound for the number $N_1(\varepsilon)$ of cubes with large jump set, showing that they cannot be "too many" as $\varepsilon$ approaches zero. Indeed, by (6.4) we have

$$\begin{aligned} MS(\tilde{u}, E^\varepsilon) &\geq \frac{1}{(1+C_1)} MS(\hat{v}^\varepsilon, E^\varepsilon \cup \varepsilon\mathcal{W}_*) \geq \frac{1}{(1+C_1)} MS(\hat{v}^\varepsilon, \varepsilon\mathcal{W}_*) \\ &\geq \frac{1}{(1+C_1)C_3} \sum_{k=1}^{N(\varepsilon)} MS\big(\hat{v}^\varepsilon, \varepsilon(h_k + W_*) \cap S_{\hat{v}^\varepsilon}\big) \\ &\geq \frac{1}{(1+C_1)C_3} \sum_{k=1}^{N_1(\varepsilon)} \mathcal{H}^{n-1}\big(\varepsilon(h_k + W_*^-) \cap S_{\hat{v}^\varepsilon}\big) \\ &\geq \frac{N_1(\varepsilon)\,\omega\,\varepsilon^{n-1}}{(1+C_1)C_3}, \end{aligned}$$

where, recalling that $W_* \subset (\mathbf{k}+2)Q$, we denoted with $C_3 = C_3(n, \mathbf{k})$ the smallest integer such that each point $x \in \mathbb{R}^n$ is contained in at most $C_3$ different cubes of the form $(h + (\mathbf{k}+2)Q)_{h \in \mathbb{Z}^n}$. From the previous estimate it follows that

$$N_1(\varepsilon) \leq \frac{(1+C_1)C_3}{\omega\,\varepsilon^{n-1}} MS(\tilde{u}, E^\varepsilon). \tag{6.7}$$

**Second case: small jump set.**
Once again, following the proof of Theorem 1.1, and defining

$$F := \bigcup_{k=N_1(\varepsilon)+1}^{N(\varepsilon)} (h_k + \Delta), \quad F^\varepsilon := \varepsilon F,$$

we have that $S_{\hat{v}^\varepsilon} \cap F^\varepsilon = \emptyset$. Notice that, arguing as it has been done to prove Corollary 3.2, one can show that $F$ is Lipschitz. We also set

$$G := \bigcup_{k=N_1(\varepsilon)+1}^{N(\varepsilon)} \bigcup_{j=1}^{M} (h_k + U_j''), \quad G^\varepsilon := \varepsilon G.$$

Next lemma, whose proof is postponed to the Appendix, gives the correct estimate for the cubes with "small jump set".

**Lemma 6.1.** *There exists an extension operator* $J^\varepsilon : H^1(F^\varepsilon) \to H^1(F^\varepsilon \cup G^\varepsilon)$ *and a constant* $C_4 = C_4(n, E)$*, independent of* $\varepsilon$ *and* $\Omega$*, such that, for every* $w \in H^1(F^\varepsilon)$*,*

- $J^\varepsilon w = w$ a.e. in $F^\varepsilon$,
- $\|J^\varepsilon w\|_{L^\infty(F^\varepsilon \cup G^\varepsilon)} = \|w\|_{L^\infty(F^\varepsilon)}$,
- *the following estimate holds true:*

$$\int_{F^\varepsilon \cup G^\varepsilon} |\nabla(J^\varepsilon w)|^2 dx \leq C_4 \int_{F^\varepsilon} |\nabla w|^2 dx.$$

**Estimate in the general case.**
Let us denote with $L_k^\varepsilon : SBV^2(Q_k^\varepsilon \cap E^\varepsilon) \cap L^\infty(Q_k^\varepsilon \cap E^\varepsilon) \to SBV^2(Q_k^\varepsilon) \cap L^\infty(Q_k^\varepsilon)$ the extension operator provided by Theorem 1.1, with $k = 1, \ldots, N_1(\varepsilon)$.

We define the function $v^\varepsilon : \Omega_Q(\varepsilon) \to \mathbb{R}$ as

$$v^\varepsilon(x) := \begin{cases} (L_k^\varepsilon \tilde{u})(x) & \text{if } x \in Q_k^\varepsilon,\ k = 1, \ldots, N_1(\varepsilon), \\ (J^\varepsilon \hat{v}^\varepsilon)(x) & \text{if } x \in F^\varepsilon \cup G^\varepsilon, \\ \hat{v}^\varepsilon(x) & \text{otherwise in } \Omega_Q(\varepsilon). \end{cases}$$

Notice that $v^\varepsilon = \hat{v}^\varepsilon = \tilde{u} = u$ a.e. in $\Omega(\varepsilon)$ and $\|v^\varepsilon\|_{L^\infty\Omega_Q(\varepsilon))} = \|u\|_{L^\infty(\Omega(\varepsilon))}$. Moreover,

$$MS(v^\varepsilon, Q_k^\varepsilon) \le c\, MS(\tilde{u}, Q_k^\varepsilon \cap E^\varepsilon) \quad k = 1, \ldots, N(\varepsilon). \tag{6.8}$$

Recalling that the constant provided by Theorem 1.1 is invariant under translations and dilations, $c = c(n, E, Q)$ is independent of $k$ and $\varepsilon$. We notice that the function $v^\varepsilon$ can possibly jump along the boundaries of the cubes $Q_k^\varepsilon$, for $k = 1, \ldots, N_1(\varepsilon)$, and this contribution is controlled by $N_1(\varepsilon)\varepsilon^{n-1}$. Therefore we have, by (6.2), (6.4), (6.7) and (6.8),

$$\begin{aligned}
MS(v^\varepsilon, \Omega_Q(\varepsilon)) &\le \sum_{k=1}^{N_1(\varepsilon)} MS(L_k^\varepsilon \tilde{u}, Q_k^\varepsilon) + N_1(\varepsilon)\varepsilon^{n-1} + MS(J^\varepsilon \hat{v}^\varepsilon, F^\varepsilon \cup G^\varepsilon) + MS(\hat{v}^\varepsilon, E^\varepsilon \cup \varepsilon\mathcal{W}_*) \\
&\le c \sum_{k=1}^{N_1(\varepsilon)} MS(\tilde{u}, Q_k^\varepsilon \cap E^\varepsilon) + \frac{(1+C_1)C_3}{\omega} MS(\tilde{u}, E^\varepsilon) + C_4 MS(\hat{v}^\varepsilon, F^\varepsilon) + MS(\hat{v}^\varepsilon, E^\varepsilon \cup \varepsilon\mathcal{W}_*) \\
&\le k_0 MS(\tilde{u}, E^\varepsilon) \le k_0 \left( MS(u, \Omega(\varepsilon)) + \mathcal{H}^{n-1}(\partial\Omega) \right),
\end{aligned}$$

where

$$k_0 := c + (1 + C_1)\left( \frac{C_3}{\omega} + C_4 + 1 \right).$$

Therefore, the claim follows setting $T^\varepsilon u := v^\varepsilon{}_{|\Omega}$. □

## 7. Homogenization of Neumann problems

In this section we consider an application of the extension property to a non coercive homogenization problem. The starting point is the energy associated to a function $u \in SBV^2(\Omega) \cap L^2(\Omega)$, i.e.,

$$\mathcal{F}^\varepsilon(u) := \int_{\Omega(\varepsilon)} |\nabla u|^2 dx + \mathcal{H}^{n-1}(\Omega(\varepsilon) \cap S_u), \tag{7.1}$$

where $\Omega(\varepsilon) := \Omega \cap \varepsilon E$, and $E$ is an open connected periodic subset of $\mathbb{R}^n$ with Lipschitz boundary. Notice that we can rewrite the functional $\mathcal{F}^\varepsilon$ as

$$\mathcal{F}^\varepsilon(u) = \int_\Omega a\Big(\frac{x}{\varepsilon}\Big)|\nabla u|^2 dx + \int_{S_u} a\Big(\frac{x}{\varepsilon}\Big)\, d\mathcal{H}^{n-1}(x),$$

where $a$ is a $Q$-periodic function given by

$$a(y) = \begin{cases} 1 & \text{in } E, \\ 0 & \text{in } \mathbb{R}^n \setminus E. \end{cases}$$

*7.1. Compactness*

In this subsection we prove a compactness result for a sequence having equibounded energy $\mathcal{F}^\varepsilon$.

**Theorem 7.1.** *Let $(u^\varepsilon) \subset SBV^2(\Omega) \cap L^\infty(\Omega)$ be a sequence satisfying the following bounds:*

$$||u^\varepsilon||_{L^\infty(\Omega(\varepsilon))} \le c \quad \text{and} \quad \mathcal{F}^\varepsilon(u^\varepsilon) \le c < +\infty,$$

*where $c > 0$ is a constant independent of $\varepsilon$. Then there exist a sequence $(\tilde{u}^\varepsilon) \subset SBV^2(\Omega) \cap L^\infty(\Omega)$ and a function $u \in SBV^2(\Omega) \cap L^\infty(\Omega)$ such that $\tilde{u}^\varepsilon = u^\varepsilon$ a.e. in $\Omega(\varepsilon)$ for every $\varepsilon$ and $(\tilde{u}^\varepsilon)$ converges to $u$ weakly$^*$ in $BV(\Omega)$.*

*Proof.* Let us define $\tilde{u}^\varepsilon := T^\varepsilon u^\varepsilon$, where $T^\varepsilon$ is the extension operator defined in Theorem 1.3. Then, from the assumptions on the sequence $(u^\varepsilon)$ and using the properties of $T^\varepsilon$ we obtain

$$||\tilde{u}^\varepsilon||_{L^\infty(\Omega)} \le c \quad \text{and} \quad MS(\tilde{u}^\varepsilon, \Omega) \le c < +\infty.$$

Hence, by Ambrosio's compactness Theorem we have directly the claim. □

*7.2. Integral representation*

The present subsection is devoted to the identification of the Γ-limit of the sequence $(\mathcal{F}^\varepsilon)$ with respect to the strong convergence in $L^2(\Omega)$.

Let us define for $u \in SBV^2(\Omega) \cap L^2(\Omega)$ the functional $\mathcal{F}^{hom}$ as

$$\mathcal{F}^{hom}(u) := \int_\Omega f^{hom}(\nabla u)\,dx + \int_{S_u} \varphi(\nu_u)\,d\mathcal{H}^{n-1}. \tag{7.2}$$

The limit densities $f^{hom} : \mathbb{R}^n \to [0,+\infty]$ and $\varphi : S^{n-1} \to [0,+\infty]$ are characterised by means of homogenization formulas, as shown in the following lines. For the density of the volume term we have:

$$f^{hom}(\xi) := \min\Big\{ \int_Q a(y)|\,\xi + \nabla w(y)|^2 dy : w \in H^1_\#(Q)\Big\}, \tag{7.3}$$

where $H^1_\#(Q)$ denotes the space of $H^1(Q)$ functions with periodic boundary values on $\partial Q$. To characterise the density of the surface term in the functional we need some preliminary definitions. Let $Q_\nu$ be any unit cube in $\mathbb{R}^n$ with centre at the origin and one face orthogonal to $\nu$, and set

$$w_{1,\nu}(x) := \begin{cases} 1 & \text{if } \langle x,\nu\rangle \geq 0, \\ 0 & \text{if } \langle x,\nu\rangle < 0. \end{cases}$$

For every $\lambda > 0$ and $\nu \in \mathbb{S}^{n-1}$ we denote with $\mathcal{P}_{\lambda,\nu}$ the class of partitions of $\lambda Q_\nu$, i.e.,

$$\mathcal{P}_{\lambda,\nu} := \big\{ w \in SBV(\lambda Q_\nu) : \nabla w = 0 \,\text{a.e.}, w = w_{1,\nu} \text{ on } \partial\lambda Q_\nu \big\}. \tag{7.4}$$

The surface density $\varphi$ in (7.2) is characterised by the following minimisation problem:

$$\varphi(\nu) := \lim_{\lambda\to+\infty} \frac{1}{\lambda^{n-1}} \min\Big\{ \int_{S_w} a(y)\,d\mathcal{H}^{n-1} : w \in \mathcal{P}_{\lambda,\nu}\Big\}. \tag{7.5}$$

**Theorem 7.2.** *The family $(\mathcal{F}^\varepsilon)$ Γ-converges with respect to the strong topology of $L^2(\Omega)$ to the functional $\mathcal{F}^{hom}$ introduced in (7.2). More precisely for every $u \in SBV^2(\Omega) \cap L^2(\Omega)$ the following properties are satisfied:*

*(i) for every $(u^\varepsilon) \subset SBV^2(\Omega) \cap L^2(\Omega)$ converging to $u$ strongly in $L^2(\Omega)$*

$$\mathcal{F}^{hom}(u) \leq \liminf_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u^\varepsilon),$$

*(ii) there exists a sequence $(u^\varepsilon) \subset SBV^2(\Omega) \cap L^2(\Omega)$ converging to $u$ strongly in $L^2(\Omega)$ such that*

$$\mathcal{F}^{hom}(u) \geq \limsup_{\varepsilon\to 0} \mathcal{F}^\varepsilon(u^\varepsilon).$$

For the proof of Theorem 7.2 we rely on [8, Theorem 2.3]. Due to the lack of coerciveness, we cannot apply the results in [8] directly to the functionals $\mathcal{F}^\varepsilon$. So we first modify the sequence to get the coerciveness we need, and then we obtain the stated Γ-convergence by approximation.

Let us define for $\eta > 0$ the approximating functionals $\mathcal{F}^\varepsilon_\eta : SBV^2(\Omega) \cap L^2(\Omega) \to [0,+\infty)$ as

$$\mathcal{F}^\varepsilon_\eta(u) = \int_\Omega a_\eta\Big(\frac{x}{\varepsilon}\Big)|\nabla u|^2 dx + \int_{S_u} a_\eta\Big(\frac{x}{\varepsilon}\Big)\,d\mathcal{H}^{n-1},$$

where $a_\eta$ is a $Q$-periodic function given by

$$a_\eta(y) = \begin{cases} 1 & \text{if } y \in E, \\ \eta & \text{if } y \in \mathbb{R}^n \setminus E. \end{cases}$$

**Theorem 7.3.** *The family* $(\mathcal{F}^\varepsilon_\eta)$ $\Gamma$*-converges with respect to the strong topology of* $L^2(\Omega)$ *to the functional* $\mathcal{F}^{hom}_\eta : SBV^2(\Omega)\cap L^2(\Omega) \to [0,+\infty)$ *defined as*

$$\mathcal{F}^{hom}_\eta(u) := \int_\Omega f^{hom}_\eta(\nabla u)\,dx + \int_{S_u} \varphi_\eta(\nu_u)\,d\mathcal{H}^{n-1}.$$

*The limit densities* $f^{hom}_\eta : \mathbb{R}^n \to [0,+\infty]$ *and* $\varphi_\eta : S^{n-1} \to [0,+\infty]$ *are identified by means of the following homogenization formulas:*

$$f^{hom}_\eta(\xi) := \min\Big\{ \int_Q a_\eta(y)|\,\xi + \nabla w(y)|^2 dy : w \in H^1_\#(Q)\Big\}, \tag{7.6}$$

$$\varphi_\eta(\nu) := \lim_{\lambda\to+\infty} \frac{1}{\lambda^{n-1}} \min\Big\{ \int_{S_w} a_\eta(y)\,d\mathcal{H}^{n-1} : w \in \mathcal{P}_{\lambda,\nu}\Big\}, \tag{7.7}$$

*where* $H^1_\#(Q)$ *and* $\mathcal{P}_{\lambda,\nu}$ *are defined as above.*

*Proof.* The functionals $\mathcal{F}^\varepsilon_\eta$ satisfy all the assumptions required in order to apply [8, Theorem 2.3] and hence the thesis follows directly. □

Now we are ready to give the proof of Theorem 7.2.

*Proof of Theorem 7.2.* We split the proof into three steps.

*First step: approximation.* It turns out that for every $u \in SBV^2(\Omega)\cap L^2(\Omega)$

$$\mathcal{F}^{hom}(u) = \inf_{\eta>0} \mathcal{F}^{hom}_\eta(u) = \lim_{\eta\to0^+} \mathcal{F}^{hom}_\eta(u). \tag{7.8}$$

Indeed, since $a_\eta \downarrow a$ pointwise as $\eta \to 0^+$, one has

$$f^{hom}(\xi) = \inf_{\eta>0} f^{hom}_\eta(\xi) = \lim_{\eta\to0^+} f^{hom}_\eta(\xi). \tag{7.9}$$

For the surface integral one can proceed as follows. Since $(\varphi_\eta)$ is decreasing and $\varphi_\eta \ge \varphi$ for every $\eta > 0$, taking the limit as $\eta$ goes to zero we have directly

$$\varphi(\nu) \le \inf_{\eta>0} \varphi_\eta(\nu) = \lim_{\eta\to0^+} \varphi_\eta(\nu)$$

for every $\nu \in \mathbb{S}^{n-1}$.

On the other hand, for every $w \in \mathcal{P}_{\lambda,\nu}$ and for $\lambda > 0$ and $\nu \in \mathbb{S}^{n-1}$, the following estimate holds true:

$$\frac{1}{\lambda^{n-1}} \int_{S_w\cap\lambda Q_\nu} a_\eta(y)\,d\mathcal{H}^{n-1} \le \frac{1}{\lambda^{n-1}} \int_{S_w\cap\lambda Q_\nu} a(y)\,d\mathcal{H}^{n-1} + \frac{\eta}{\lambda^{n-1}}\mathcal{H}^{n-1}(S_w \cap \lambda Q_\nu). \tag{7.10}$$

Let $\widehat{w} \in \mathcal{P}_{\lambda,\nu}$ be a minimiser of the cell problem (7.5) for a fixed $\lambda$ (to shorten the notation we do not make explicit the dependence of $\widehat{w}$ on $\lambda$ and $\nu$). In virtue of the boundary conditions contained in the definition of $\mathcal{P}_{\lambda,\nu}$, we can assume $||\widehat{w}||_{L^\infty} \le 1$. Moreover, since the function $w_{1,\nu}$ is an admissible competitor in (7.5) the following bound is satisfied:

$$\frac{1}{\lambda^{n-1}} \int_{S_{\widehat{w}}\cap\lambda Q_\nu} a(y)\,d\mathcal{H}^{n-1} \le \frac{1}{\lambda^{n-1}} \int_{S_{w_{1,\nu}}\cap\lambda Q_\nu} a(y)\,d\mathcal{H}^{n-1} \le \frac{1}{\lambda^{n-1}}\mathcal{H}^{n-1}(S_{w_{1,\nu}} \cap \lambda Q_\nu) \le 1. \tag{7.11}$$

Moreover, by (7.10) we have in particular that

$$\frac{1}{\lambda^{n-1}} \int_{S_{\widehat{w}}\cap\lambda Q_\nu} a_\eta(y)\,d\mathcal{H}^{n-1} \le \frac{1}{\lambda^{n-1}} \int_{S_{\widehat{w}}\cap\lambda Q_\nu} a(y)\,d\mathcal{H}^{n-1} + \frac{\eta}{\lambda^{n-1}}\mathcal{H}^{n-1}(S_{\widehat{w}} \cap \lambda Q_\nu). \tag{7.12}$$

Notice that, from the definition of the class $\mathcal{P}_{\lambda,\nu}$ (see (7.4)), we can rewrite

$$\int_{S_{\widehat{w}}\cap\lambda Q_\nu} a(y)\,d\mathcal{H}^{n-1} = MS(\widehat{w},\lambda Q_\nu\cap E). \tag{7.13}$$

In order to estimate the right-hand side of (7.12) we apply Theorem 1.3 to the restriction of the function $\widehat{w}$ to $\lambda Q_\nu \cap E$ in the following way.

We define the function $w^\lambda(y) := (1/\sqrt{\lambda})\widehat{w}(\lambda y)$. By Theorem 1.3 the function $w^\lambda\,|_{Q_\nu\cap\left(\frac{1}{\lambda}E\right)}$ admits an extension $T^{\frac{1}{\lambda}}w^\lambda\,|_{Q_\nu\cap\left(\frac{1}{\lambda}E\right)}$ to the whole $Q_\nu$ satisfying

$$MS\left(T^{\frac{1}{\lambda}}w^\lambda\,|_{Q_\nu\cap\left(\frac{1}{\lambda}E\right)},Q_\nu\right) \le k_0\left(MS\left(w^\lambda,Q_\nu\cap\left(\tfrac{1}{\lambda}E\right)\right)+\mathcal{H}^{n-1}(\partial Q_\nu)\right). \tag{7.14}$$

At this point, for $x\in\lambda Q_\nu$, we can define $\widetilde{w}_\lambda$ as

$$\widetilde{w}_\lambda(x) := \sqrt{\lambda}\,(T^{\frac{1}{\lambda}}w^\lambda\,|_{Q_\nu\cap\left(\frac{1}{\lambda}E\right)})\left(\frac{x}{\lambda}\right),$$

and from (7.14) we have directly the estimate

$$\begin{aligned} MS(\widetilde{w}_\lambda,\lambda Q_\nu) &= \lambda^{n-1}MS(T^{\frac{1}{\lambda}}w^\lambda\,|_{Q_\nu\cap\left(\frac{1}{\lambda}E\right)},Q_\nu)\\ &\le \lambda^{n-1}k_0\left(MS\left(w^\lambda,Q_\nu\cap\left(\tfrac{1}{\lambda}E\right)\right)+\mathcal{H}^{n-1}(\partial Q_\nu)\right)\\ &= k_0\big(MS(\widehat{w},\lambda Q_\nu\cap E)+\mathcal{H}^{n-1}(\partial\lambda Q_\nu)\big). \end{aligned}$$

This implies in particular that

$$\mathcal{H}^{n-1}(S_{\widetilde{w}_\lambda}\cap\lambda Q_\nu) \le k_0\int_{S_{\widehat{w}}\cap\lambda Q_\nu} a(y)\,d\mathcal{H}^{n-1}+k_0\lambda^{n-1},$$

where we used (7.13). The previous estimate and (7.11) imply that

$$\frac{1}{\lambda^{n-1}}\mathcal{H}^{n-1}(S_{\widetilde{w}_\lambda}\cap\lambda Q_\nu)\le 2k_0. \tag{7.15}$$

Since $\widehat{w}=\widetilde{w}_\lambda$ a.e. in $\lambda Q_\nu\cap E$, it turns out that also the function $\widetilde{w}_\lambda$ is a minimiser of the cell problem (7.5). Therefore we can assume without loss of generality that (7.15) holds for the function $\widehat{w}$ and we obtain from (7.10) and (7.12)

$$\frac{1}{\lambda^{n-1}}\min_{w\in\mathcal{P}_{\lambda,\nu}}\int_{S_w\cap\lambda Q_\nu} a_\eta(y)\,d\mathcal{H}^{n-1}\le\frac{1}{\lambda^{n-1}}\min_{w\in\mathcal{P}_{\lambda,\nu}}\int_{S_w\cap\lambda Q_\nu} a(y)\,d\mathcal{H}^{n-1}+2k_0\eta.$$

If we let $\lambda\to+\infty$ and then $\eta\to 0^+$ we get

$$\varphi(\nu)=\inf_{\eta>0}\varphi_\eta(\nu)=\lim_{\eta\to0^+}\varphi_\eta(\nu). \tag{7.16}$$

Hence, from (7.9), (7.16) and monotone convergence we obtain (7.8).

*Second step: liminf inequality (i).* Let $u\in SBV^2(\Omega)\cap L^2(\Omega)$ and let $(u^\varepsilon)\subset SBV^2(\Omega)\cap L^2(\Omega)$ be a sequence converging to $u$ strongly in $L^2(\Omega)$ and such that $\mathcal{F}^\varepsilon(u^\varepsilon)\le c$, where $c$ is a constant independent of $\varepsilon$. Let $\ell>0$ and define the truncated functions $\mathcal{T}_\ell u := \big(u\wedge\ell\big)\vee(-\ell)$, and $\mathcal{T}_\ell u^\varepsilon$, for every $\varepsilon>0$. Then clearly $\mathcal{T}_\ell u^\varepsilon$ converges to $\mathcal{T}_\ell u$ strongly in $L^2$ and $\mathcal{F}^\varepsilon(\mathcal{T}_\ell u^\varepsilon)\le c$. For every $\varepsilon>0$ let us consider the restriction $\mathcal{T}_\ell u^\varepsilon\,|_{\Omega^\varepsilon}$ of the function $\mathcal{T}_\ell u^\varepsilon$ to the perforated set $\Omega^\varepsilon$. Let $T^\varepsilon\left(\mathcal{T}_\ell u^\varepsilon\,|_{\Omega^\varepsilon}\right)$ be the extension of $\mathcal{T}_\ell u^\varepsilon\,|_{\Omega^\varepsilon}$ to the set $\Omega$ provided by Theorem 4.1. By property (iii) of the quoted theorem it follows that

$$MS\left(T^\varepsilon\left(\mathcal{T}_\ell u^\varepsilon\,|_{\Omega^\varepsilon}\right),\Omega\right)\le k_0\left(\mathcal{F}^\varepsilon\left(\mathcal{T}_\ell u^\varepsilon\right)+\mathcal{H}^{n-1}(\partial\Omega)\right). \tag{7.17}$$

We also notice that

$$\begin{aligned}\mathcal{F}_{\eta}^{\varepsilon}\left(T^{\varepsilon}\left(\mathcal{T}_{\ell}u^{\varepsilon}\mid_{\Omega^{\varepsilon}}\right)\right)&\leq\mathcal{F}^{\varepsilon}(T^{\varepsilon}(\mathcal{T}_{\ell}u^{\varepsilon}\mid_{\Omega^{\varepsilon}}))+\eta\,MS\left(T^{\varepsilon}(\mathcal{T}_{\ell}u^{\varepsilon}\mid_{\Omega^{\varepsilon}}),\Omega\right)\\&=\mathcal{F}^{\varepsilon}\left(\mathcal{T}_{\ell}u^{\varepsilon}\right)+\eta\,MS\left(T^{\varepsilon}(\mathcal{T}_{\ell}u^{\varepsilon}\mid_{\Omega^{\varepsilon}}),\Omega\right)\end{aligned}$$

which implies, together with (7.17), that

$$\mathcal{F}_{\eta}^{\varepsilon}\left(T^{\varepsilon}\left(\mathcal{T}_{\ell}u^{\varepsilon}\mid_{\Omega^{\varepsilon}}\right)\right)\leq(1+\eta\,k_0)\mathcal{F}^{\varepsilon}(\mathcal{T}_{\ell}u^{\varepsilon})+\eta\,k_0\mathcal{H}^{n-1}(\partial\Omega).\tag{7.18}$$

We notice that the sequence $(T^{\varepsilon}\left(\mathcal{T}_{\ell}u^{\varepsilon}\mid_{\Omega^{\varepsilon}}\right))$ converges to $\mathcal{T}_{\ell}u$ strongly in $L^2$ as $\varepsilon\to 0$. Indeed, by Ambrosio's Compactness Theorem, there exists $w\in SBV^2(\Omega)\cap L^{\infty}(\Omega)$ such that $(T^{\varepsilon}\left(\mathcal{T}_{\ell}u^{\varepsilon}\mid_{\Omega^{\varepsilon}}\right))$ converges to $w$ weakly$^*$ in $BV(\Omega)$, and in particular strongly in $L^1(\Omega)$. Moreover, from the equiboundedness of the sequence $(T^{\varepsilon}\left(\mathcal{T}_{\ell}u^{\varepsilon}\mid_{\Omega^{\varepsilon}}\right))$ in $L^{\infty}$ we have the convergence in $L^2$. We claim that $w=\mathcal{T}_{\ell}u$ a.e. in $\Omega$. This follows by the Riemann-Lebesgue Lemma, as

$$0=\lim_{\varepsilon\to 0}\int_{\Omega}a\left(\frac{x}{\varepsilon}\right)\left|T^{\varepsilon}\left(\mathcal{T}_{\ell}u^{\varepsilon}\mid_{\Omega^{\varepsilon}}\right)-\mathcal{T}_{\ell}u^{\varepsilon}\right|dx=\vartheta\int_{\Omega}|w-\mathcal{T}_{\ell}u|\,dx,$$

where $\vartheta>0$ is the weak-$*$ limit of $a(\frac{\cdot}{\varepsilon})$ in $L^{\infty}(\Omega)$. From the previous expression we conclude immediately that $w=\mathcal{T}_{\ell}u$ a.e. on $\Omega$. Therefore, from (7.18) and from Theorem 7.3 we get

$$\mathcal{F}_{\eta}^{hom}(\mathcal{T}_{\ell}u)\leq\liminf_{\varepsilon\to 0}\mathcal{F}_{\eta}^{\varepsilon}\left(T^{\varepsilon}\left(\mathcal{T}_{\ell}u^{\varepsilon}\mid_{\Omega^{\varepsilon}}\right)\right)\leq(1+\eta\,k_0)\liminf_{\varepsilon\to 0}\mathcal{F}^{\varepsilon}(\mathcal{T}_{\ell}u^{\varepsilon})+\eta\,k_0\mathcal{H}^{n-1}(\partial\Omega),$$

that holds true for every $\eta>0$ and $\ell>0$. If we now let $\eta\to 0^+$ in the previous expression, recalling (7.8) we have

$$\mathcal{F}^{hom}(\mathcal{T}_{\ell}u)=\lim_{\eta\to 0^+}\mathcal{F}_{\eta}^{hom}(\mathcal{T}_{\ell}u)\leq\liminf_{\varepsilon\to 0}\mathcal{F}^{\varepsilon}(\mathcal{T}_{\ell}u^{\varepsilon}).\tag{7.19}$$

Moreover, since $\mathcal{F}^{\varepsilon}(\mathcal{T}_{\ell}u^{\varepsilon})\leq\mathcal{F}^{\varepsilon}(u^{\varepsilon})$ for every $\ell>0$ and $(\mathcal{T}_{\ell}u)$ converges to $u$ strongly in $L^2$ as $\ell\to+\infty$, (7.19) implies that

$$\mathcal{F}^{hom}(u)\leq\liminf_{\ell\to+\infty}\mathcal{F}^{hom}(\mathcal{T}_{\ell}u)\leq\liminf_{\varepsilon\to 0}\mathcal{F}^{\varepsilon}(u^{\varepsilon}),$$

where the first inequality follows by the lower semicontinuity of $\mathcal{F}^{hom}$ in $L^2$.

*Third step: limsup inequality (ii).* In this case we simply use the trivial estimate

$$\mathcal{F}_{\eta}^{\varepsilon}\geq\mathcal{F}^{\varepsilon}.\tag{7.20}$$

Indeed, let $u\in SBV^2(\Omega)\cap L^2(\Omega)$ and let $(u^{\varepsilon})\subset SBV^2(\Omega)\cap L^2(\Omega)$ be a recovery sequence for the functionals $\mathcal{F}_{\eta}^{\varepsilon}$. Then

$$\mathcal{F}_{\eta}^{hom}(u)=\limsup_{\varepsilon\to 0}\mathcal{F}_{\eta}^{\varepsilon}(u^{\varepsilon})\geq\limsup_{\varepsilon\to 0}\mathcal{F}^{\varepsilon}(u^{\varepsilon}).$$

This implies in particular that

$$\mathcal{F}^{hom}(u)=\inf_{\eta>0}\mathcal{F}_{\eta}^{hom}(u)\geq\limsup_{\varepsilon\to 0}\mathcal{F}^{\varepsilon}(u^{\varepsilon}),$$

and therefore the proof is concluded. □

*7.3. Γ-convergence under Dirichlet conditions*

This subsection is devoted to the proof of a result which is a version of Theorem 7.2 that takes into account boundary data.

We need a preliminary observation concerning the approximating coercive functionals $\mathcal{F}_{\eta}^{\varepsilon}$. First of all let us fix $\psi\in H^1(\mathbb{R}^n)$ and a set $\widetilde{\Omega}$ with $\Omega\subset\subset\widetilde{\Omega}$.

We define the densities $f^\varepsilon_\eta : \widetilde{\Omega} \times \mathbb{R}^n \to [0, +\infty]$ and $g^\varepsilon_\eta : \widetilde{\Omega} \to [0, +\infty]$ as

$$f^\varepsilon_\eta(x,\xi) := \begin{cases} a_\eta\Big(\dfrac{x}{\varepsilon}\Big)|\xi|^2 & \text{if } x \in \overline{\Omega}, \\ |\xi|^2 & \text{if } x \in \widetilde{\Omega} \setminus \overline{\Omega}, \end{cases} \qquad g^\varepsilon_\eta(x) := \begin{cases} a_\eta\Big(\dfrac{x}{\varepsilon}\Big) & \text{if } x \in \overline{\Omega}, \\ 2 & \text{if } x \in \widetilde{\Omega} \setminus \overline{\Omega}. \end{cases}$$

Therefore we define the sequence of functionals $\widetilde{\mathcal{F}}^\varepsilon_\eta$ on $SBV^2\big(\widetilde{\Omega}\big)$ as

$$\widetilde{\mathcal{F}}^\varepsilon_\eta(u) := \int_{\widetilde{\Omega}} f^\varepsilon_\eta(x, \nabla u)\, dx + \int_{S_u} g^\varepsilon_\eta(x)\, d\mathcal{H}^{n-1}.$$

Using Theorem 7.3, it is easy to verify that $\widetilde{\mathcal{F}}^\varepsilon_\eta$ $\Gamma$- converges with respect to the strong $L^2$ topology to $\widetilde{\mathcal{F}}_\eta$, where

$$\widetilde{\mathcal{F}}_\eta(u) := \int_{\widetilde{\Omega}} f_\eta(x, \nabla u)\, dx + \int_{S_u} g_\eta(x, \nu_u)\, d\mathcal{H}^{n-1},$$

and the limit densities $f_\eta$ and $g_\eta$ satisfy the relations

$$f_\eta(x,\xi) = \begin{cases} f^{hom}_\eta(\xi) & \text{if } x \in \overline{\Omega}, \\ |\xi|^2 & \text{if } x \in \widetilde{\Omega} \setminus \overline{\Omega}, \end{cases} \qquad g_\eta(x,\nu) = \begin{cases} \varphi_\eta(\nu) & \text{if } x \in \overline{\Omega}, \\ 2 & \text{if } x \in \widetilde{\Omega} \setminus \overline{\Omega}, \end{cases}$$

$f^{hom}_\eta$ and $\varphi_\eta$ being defined in (7.6) and (7.7), respectively.

**Lemma 7.4.** *The functionals $\widetilde{\mathcal{F}}^\varepsilon_{\eta,\psi}$ defined on $SBV^2\big(\widetilde{\Omega}\big)$ as*

$$\widetilde{\mathcal{F}}^\varepsilon_{\eta,\psi}(u) := \begin{cases} \widetilde{\mathcal{F}}^\varepsilon_\eta(u) & \textit{if } u = \psi \quad \textit{on } \widetilde{\Omega} \setminus \overline{\Omega}, \\ +\infty & \textit{otherwise} \end{cases}$$

*$\Gamma$-converge with respect to the strong $L^2$ topology to the functional $\widetilde{\mathcal{F}}_{\eta,\psi}$ given by*

$$\widetilde{\mathcal{F}}_{\eta,\psi}(u) := \begin{cases} \widetilde{\mathcal{F}}_\eta(u) & \textit{if } u = \psi \quad \textit{on } \widetilde{\Omega} \setminus \overline{\Omega}, \\ +\infty & \textit{otherwise.} \end{cases}$$

*Proof.* We omit the proof, which can be directly obtained by [18, Lemma 7.1]. □

Using the same notation adopted so far, we can define the functionals $\widetilde{\mathcal{F}}^\varepsilon$ on $SBV^2\big(\widetilde{\Omega}\big)$ as

$$\widetilde{\mathcal{F}}^\varepsilon(u) := \int_{\widetilde{\Omega}} f^\varepsilon(x, \nabla u)\, dx + \int_{S_u} g^\varepsilon(x)\, d\mathcal{H}^{n-1},$$

where $f^\varepsilon : \widetilde{\Omega} \times \mathbb{R}^n \to [0, +\infty]$, $g^\varepsilon : \widetilde{\Omega} \to [0, +\infty]$ are given by

$$f^\varepsilon(x,\xi) := \begin{cases} a\Big(\dfrac{x}{\varepsilon}\Big)|\xi|^2 & \text{if } x \in \overline{\Omega}, \\ |\xi|^2 & \text{if } x \in \widetilde{\Omega} \setminus \overline{\Omega}, \end{cases} \qquad g^\varepsilon(x) := \begin{cases} a\Big(\dfrac{x}{\varepsilon}\Big) & \text{if } x \in \overline{\Omega}, \\ 2 & \text{if } x \in \widetilde{\Omega} \setminus \overline{\Omega}. \end{cases}$$

We can finally state the $\Gamma$-convergence result for the functionals $\widetilde{\mathcal{F}}^\varepsilon$ under Dirichlet boundary conditions. Notice that $\widetilde{\mathcal{F}}^\varepsilon|_{SBV^2(\Omega)} = \mathcal{F}^\varepsilon|_{SBV^2(\Omega)}$, $\mathcal{F}^\varepsilon$ being defined in (7.1).

**Theorem 7.5.** *The functionals $\widetilde{\mathcal{F}}^\varepsilon_\psi$ defined on $SBV^2\big(\widetilde{\Omega}\big)$ as*

$$\widetilde{\mathcal{F}}^\varepsilon_\psi(u) := \begin{cases} \widetilde{\mathcal{F}}^\varepsilon(u) & \textit{if } u = \psi \quad \textit{on } \widetilde{\Omega} \setminus \overline{\Omega}, \\ +\infty & \textit{otherwise} \end{cases}$$

*Γ-converge with respect to the strong $L^2$ topology to the functional $\widetilde{\mathcal{F}}_\psi$ given by*

$$\widetilde{\mathcal{F}}_\psi(u) := \begin{cases} \widetilde{\mathcal{F}}(u) & \text{if } u = \psi \quad \text{on } \widetilde{\Omega} \setminus \overline{\Omega}, \\ +\infty & \text{otherwise.} \end{cases}$$

*The limit functional $\widetilde{\mathcal{F}}$ is defined as*

$$\widetilde{\mathcal{F}}(u) := \int_{\widetilde{\Omega}} f(x, \nabla u)\, dx + \int_{S_u} g(x, \nu_u)\, d\mathcal{H}^{n-1},$$

*where the limit densities $f : \widetilde{\Omega} \times \mathbb{R}^n \to [0, +\infty]$ and $g : \widetilde{\Omega} \times S^{n-1} \to [0, +\infty]$ satisfy*

$$f(x, \xi) = \begin{cases} f^{hom}(\xi) & \text{if } x \in \overline{\Omega}, \\ |\xi|^2 & \text{if } x \in \widetilde{\Omega} \setminus \overline{\Omega}, \end{cases} \qquad g(x, \nu) = \begin{cases} \varphi(\nu) & \text{if } x \in \overline{\Omega}, \\ 2 & \text{if } x \in \widetilde{\Omega} \setminus \overline{\Omega}, \end{cases}$$

*$f^{hom}$ and $\varphi$ being defined in (7.3) and (7.5), respectively.*

*Proof.* The convergence is a direct consequence of Lemma 7.4 and Theorem 7.2. □

## 8. Appendix

In this last section we prove some technical results that have been used in the paper. First, we show in a rigorous way an integral estimate for the composition of an $SBV$ function with a bilipschitz map. This provides a stability result for the Mumford-Shah functional under bilipschitz transformations of the domain. More precisely, we have the following theorem.

**Theorem 8.1.** *Let $W, W'$ be bounded open subsets of $\mathbb{R}^n$ with Lipschitz boundary, let $\phi : W' \to W$ be a bilipschitz function and let us set $\psi := \phi^{-1}$. For every $u \in SBV^2(W)$, let us define the function $v : W' \to \mathbb{R}$ as $v(x) := u(\phi(x))$. Then, for every $u \in SBV^2(W)$ we have that $v \in SBV^2(W')$ and*

$$\int_{W'} |\nabla v|^2 dx + \mathcal{H}^{n-1}(S_v) \leq C_1 \left( \int_W |\nabla u|^2 dx + \mathcal{H}^{n-1}(S_u) \right), \tag{8.1}$$

*where*

$$C_1 := \| \det \nabla\psi \, (\nabla\psi)^{-T} \|_{L^\infty(W;\mathbb{M}^n)}. \tag{8.2}$$

*Proof.* It is well known that the function $v$ belongs to $SBV(W')$ (see for example [5]). In order to prove the estimate (8.1), we split the proof into two steps.

*First step: approximation of $u$.*

As first step we approximate $u$ with more regular functions and we prove the claim for the approximating functions. More precisely, let $(u_h)$ be the sequence provided by Theorem 2.6, and set $v_h := u_h \circ \phi$. We claim that relation (8.1) holds true for the functions $v_h$, i.e. that

$$\int_{W'} |\nabla v_h|^2 dy + \mathcal{H}^{n-1}(S_{v_h}) \leq C_1 \left( \int_W |\nabla u_h|^2 dx + \mathcal{H}^{n-1}(S_{u_h}) \right) \qquad h \in \mathbb{N}, \tag{8.3}$$

where $C_1$ is defined in (8.2). Let us set $\psi := \phi^{-1}$. By property $(iii)$ of Theorem 2.6 we can apply the standard chain rule and we get

$$\nabla v_h = (\nabla\phi)^T (\nabla u_h \circ \phi) \quad \mathcal{L}^n\text{-a.e. on } W' \setminus \psi(\bar{S}_{u_h}),$$

that is, since $\psi$ maps $\mathcal{L}^n$-negligible sets into $\mathcal{L}^n$-negligible sets,

$$\nabla v_h = (\nabla\phi)^T (\nabla u_h \circ \phi) \quad \mathcal{L}^n\text{-a.e. on } W'. \tag{8.4}$$

Notice that, from the fact that $(\phi \circ \psi)(x) = x$ for every $x \in W$, we get

$$(\nabla\phi \circ \psi)\nabla\psi = Id \Longleftrightarrow (\nabla\phi \circ \psi) = (\nabla\psi)^{-1}.$$

Using last relation, (8.4) and the change of variables formula for integrals we have

$$\begin{aligned}\int_{W'} |\nabla v_h|^2 dy &= \int_{\psi(W)} |(\nabla\phi)^T(\nabla u_h \circ \phi)|^2 dy = \int_W |\det\nabla\psi\,(\nabla\phi \circ \psi)^T \nabla u_h|^2 dx \\ &= \int_W |\det\nabla\psi\,(\nabla\psi)^{-T}\,\nabla u_h|^2 dx \leq C_1 \int_W |\nabla u_h|^2 dx. \end{aligned} \tag{8.5}$$

To estimate the measure of the jump set of $v_h$, we use the generalized area formula (see [5, Theorem 2.91]). Since $S_{v_h} = \psi(S_{u_h})$, we obtain

$$\mathcal{H}^{n-1}(S_{v_h}) = \int_{\psi(S_{u_h})} 1\, d\mathcal{H}^{n-1} = \int_{S_{u_h}} |\det\nabla\psi\,(\nabla\psi)^{-T}[\nu_h]|\, d\mathcal{H}^{n-1} \leq C_1\mathcal{H}^{n-1}(S_{u_h}), \tag{8.6}$$

where $\nu_h$ denotes the normal to $S_{u_h}$. Therefore (8.3) follows from (8.5) and (8.6).

*Second step: limit estimate.*

It remains to pass to the limit in (8.3) as $h \to +\infty$. For the right-hand side the convergence is given by property $(v)$ of Theorem 2.6. So we reduced to prove the following result:

$$\int_{W'} |\nabla v|^2 dy + \mathcal{H}^{n-1}(S_v) \leq \liminf_{h\to+\infty} \left( \int_{W'} |\nabla v_h|^2 dy + \mathcal{H}^{n-1}(S_{v_h}) \right). \tag{8.7}$$

The lack of a uniform $L^\infty$ bound for the sequence $(v_h)$ forces us to use a truncation argument in order to apply Ambrosio's compactness theorem. Hence, let $M > 0$ and define $v_h^M := (v_h \wedge M) \vee (-M)$; clearly, $v_h^M \to v^M := (v \wedge M) \vee (-M)$ strongly in $L^2(W')$ as $h \to +\infty$. By Ambrosio's compactness theorem we have that $v_h^M \rightharpoonup v^M$ weakly* in $BV(W')$. At this point, by Ambrosio's lower semicontinuity theorem we obtain the following inequality:

$$\int_{W'} |\nabla v^M|^2 dy + \mathcal{H}^{n-1}(S_{v^M}) \leq \liminf_{h\to+\infty} \left( \int_{W'} |\nabla v_h^M|^2 dy + \mathcal{H}^{n-1}(S_{v_h^M}) \right). \tag{8.8}$$

It is immediate to notice that

$$\int_{W'} |\nabla v_h^M|^2 dy + \mathcal{H}^{n-1}(S_{v_h^M}) \leq \int_{W'} |\nabla v_h|^2 dy + \mathcal{H}^{n-1}(S_{v_h}).$$

Therefore, using last relation we can pass to the liminf as $h \to +\infty$ in (8.8) and

$$\int_{W'} |\nabla v^M|^2 dy + \mathcal{H}^{n-1}(S_{v^M}) \leq \liminf_{h\to+\infty} \left( \int_{W'} |\nabla v_h|^2 dy + \mathcal{H}^{n-1}(S_{v_h}) \right). \tag{8.9}$$

Now we let $M$ tend to $+\infty$ in order to pass from (8.9) to (8.7). We treat separately the volume term and the surface integral in the left-hand side of (8.9). For the jump set we simply notice that, being $M \mapsto S_{v^M}$ an increasing function and $S_v = \cup_M S_{v^M}$, we have the convergence

$$\mathcal{H}^{n-1}(S_v) = \lim_{M\to+\infty} \mathcal{H}^{n-1}(S_{v^M}).$$

For the volume integral we point out that, from the chain rule formula in $BV$, we can write the explicit expression of the absolutely continuous gradient of the truncated function $v^M$ as

$$\nabla v^M = \begin{cases} \nabla v & \text{if } |v| < M, \\ 0 & \text{otherwise.} \end{cases}$$

At this point, by Lebesgue dominated convergence theorem we get

$$\int_{W'} |\nabla v|^2 dy = \lim_{M\to+\infty} \int_{W'} |\nabla v^M|^2 dy,$$

and the proof is concluded. □

*Proof of Lemma 6.1.* Let $v : F \to \mathbb{R}$ be defined as $v(y) := w(\varepsilon y)$; we will prove the existence of an extension $\tilde{v} \in H^1(F \cup (\partial F \cap \partial G) \cup G)$ for the rescaled function $v$, satisfying

$$\int_{F \cup (\partial F \cap \partial G) \cup G} |\nabla \tilde{v}|^2 dx \leq C_4 \int_F |\nabla v|^2 dx, \tag{8.10}$$

where $C_4 = C_4(n, E)$ is a positive constant independent of $\varepsilon$ and $\Omega$. From this the conclusion will follow by rescaling back the function $\tilde{v}$, i.e., setting $(J^\varepsilon w)(x) := \tilde{v}(\varepsilon^{-1} x)$. Let us prove (8.10).

Without loss of generality, we assume that $F$ has only one connected component. In this case, also $G$ is connected. By definition of $F$ and $G$, up to a possible change in the enumeration of the vectors $h_k$'s, there exists an integer $\ell \in \mathbb{N}$ (depending on $\varepsilon$) such that

$$F = \bigcup_{k=1}^{\ell} \left( h_k + \Delta_{i_1(k)} \cup \ldots \cup \Delta_{i_{p_k}(k)} \right) \qquad \text{and} \qquad G = \bigcup_{k=1}^{\ell} \left( h_k + U''_{i_1(k)} \cup \ldots \cup U''_{i_{p_k}(k)} \right),$$

where $p_k \in \{1, \ldots, M\}$ and $1 \leq i_1(k) < \ldots < i_{p_k}(k) \leq M$ for every $k = 1, \ldots, \ell$.

We will adapt to the present situation the proof of [1, Lemma 2.7]. For every $i = 1, \ldots, M$, let us consider a nonnegative function $\varphi_i \in C_0^\infty(\mathbb{R}^n)$ such that

- supp $\varphi_i \subset\subset (\mathbf{k}+1)Q \setminus \left( \bigcup_{j \neq i} \overline{\Delta_j \cup U''_j} \right)$
- $\varphi_i > 0$ in $(\mathbf{k}+1)Q \cap \left( \Delta_i \cup (\partial \Delta \cap \partial U''_j) \cup U''_i \right)$.

We construct a partition of unity $\{\psi_j^k\}_{j=i_1(k),\ldots,i_{p_k}(k)}^{k=1,\ldots,\ell}$ associated to the family of open sets $\{h_k + (\mathbf{k}+1)Q \cap \left( \Delta_j \cup (\partial \Delta \cap \partial U''_j) \cup U''_j \right)\}_{j=i_1(k),\ldots,i_{p_k}(k)}^{k=1,\ldots,\ell}$ by defining

$$\psi_j^k(x) := \frac{\varphi_j(x - h_k)}{\displaystyle\sum_{r=1}^{\ell} \sum_{i=i_1(r)}^{i_{p_r}(r)} \varphi_i(x - h_r)}, \qquad \text{for every } x \in \mathbb{R}^n.$$

This implies in particular that

$$\sum_{k=1}^{\ell} \sum_{j=i_1(k)}^{i_{p_k}(k)} \psi_j^k(x) = 1 \qquad \text{for every } \; x \in F_1 \cup G_1. \tag{8.11}$$

Let $C_5 = C_5(n, E)$ be a positive constant such that

$$|\psi_j^k(x)| + |\nabla \psi_j^k(x)| \leq C_5, \quad \text{for every } k, j, \text{ for every } x \in \mathbb{R}^n.$$

For every $k = 1, \ldots, \ell$ and $j = i_1(k), \ldots, i_{p_k}(k)$, let us denote with $\tau_{j,k}$ the extension operator provided by Theorem 2.1 from $H^1(h_k + \Delta_j)$ to $H^1\left(h_k + \Delta_j \cup (\partial \Delta_j \cap \partial U''_j) \cup U''_j\right)$.

By (2.1), using the invariance of the constant $k_2$ under translations, we have that for every $k = 1, \ldots, \ell$ and $j = i_1(k), \ldots, i_{p_k}(k)$

$$\int_{h_k + \Delta_j \cup (\partial \Delta_j \cap \partial U''_j) \cup U''_j} |\nabla (\tau_{j,k} v)|^2 dx \leq K_2 \int_{h_k + \Delta_j} |\nabla v|^2 dx, \tag{8.12}$$

where, in analogy with (5.10), we set $K_2 := \max_{i=1,\ldots,M} \left\{ k_2(n, \Delta_j, U''_j) \right\}$. We define now

$$\tilde{v}(x) := \sum_{r=1}^{\ell} \sum_{j=i_1(r)}^{i_{p_r}(r)} \psi_j^r(x) (\tau_{j,r} v)(x) \quad \text{ for every } x \in F \cup G.$$

In order to show the estimate for the $L^2$-norm of the gradient, let us fix $s \in \{1, \dots, \ell\}$ (i.e., we fix a cube) and $k \in \{i_1(s), \dots, i_{p_s}(s)\}$ (i.e., we fix the connected component of $\Delta$ in the cube). Moreover, let $I(B)$ be defined as

$$I(B) := \{\alpha \in \mathbb{Z}^n : (\alpha + (\mathbf{k}+1)Q) \cap B \neq \emptyset\},$$

for every open set $B$. We have

$$\int_{h_s+(\Delta_k \cup U_k'')} |\nabla \tilde{v}|^2 dx \leq 2 \int_{h_s+(\Delta_k \cup U_k'')} \Bigg| \sum_{\substack{h_r \in I((\mathbf{k}+1)Q_s) \\ r=1,\dots,\ell}} \sum_{j=i_1(r)}^{i_{p_r}(r)} \psi_j^r \nabla(\tau_{j,r} v) \Bigg|^2 dx$$
$$+ 2 \int_{h_s+(\Delta_k \cup U_k'')} \Bigg| \sum_{\substack{h_r \in I((\mathbf{k}+1)Q_s) \\ r=1,\dots,\ell}} \sum_{j=i_1(r)}^{i_{p_r}(r)} \nabla \psi_j^r (\tau_{j,r} v) \Bigg|^2 dx, \tag{8.13}$$

where we used the fact that $h_s + (\Delta_k \cup U_k'') \subset (\mathbf{k}+1)Q_s$. Let $N$ denote the cardinality of the set $I((\mathbf{k}+1)Q_s)$. Concerning the first term in the right-hand side of (8.13) we have, using (8.12),

$$\int_{h_s+(\Delta_k \cup U_k'')} \Bigg| \sum_{\substack{h_r \in I((\mathbf{k}+1)Q_s) \\ r=1,\dots,\ell}} \sum_{j=i_1(r)}^{i_{p_r}(r)} \psi_j^r \nabla(\tau_{j,r} v) \Bigg|^2 dx \leq N \sum_{\substack{h_r \in I((\mathbf{k}+1)Q_s) \\ r=1,\dots,\ell}} \int_{h_s+(\Delta_k \cup U_k'')} \Bigg| \sum_{j=i_1(r)}^{i_{p_r}(r)} \psi_j^r \nabla(\tau_{j,r} v) \Bigg|^2 dx$$
$$\leq NM \sum_{\substack{h_r \in I((\mathbf{k}+1)Q_s) \\ r=1,\dots,\ell}} \sum_{j=i_1(r)}^{i_{p_r}(r)} \int_{(h_s+(\Delta_k \cup U_k''))\cap\left((\mathbf{k}+1)Q_r \setminus \bigcup_{i\neq j}(h_r + (\overline{\Delta_i \cup U_i''}))\right)} \left|\psi_j^r \nabla(\tau_{j,r} v)\right|^2 dx$$
$$= NM \sum_{\substack{h_r \in I((\mathbf{k}+1)Q_s) \\ r=1,\dots,\ell}} \sum_{j=i_1(r)}^{i_{p_r}(r)} \int_{(h_s+(\Delta_k \cup U_k''))\cap(h_r+(\Delta_j \cup U_j''))} \left|\psi_j^r \nabla(\tau_{j,r} v)\right|^2 dx,$$

where we used the definition of $\psi_j^r$ and the fact that

$$(h_s + (\Delta_k \cup U_k'')) \cap (\mathbf{k}+1)Q_r \subset (h_s + (\Delta_k \cup U_k'')) \cap \left( \bigcup_i \left( h_r + (\Delta_i \cup U_i'') \right) \right). \tag{8.14}$$

Now, applying (8.12), from the previous chain of inequalities we obtain

$$\int_{h_s+(\Delta_k \cup U_k'')} \Bigg| \sum_{\substack{h_r \in I((\mathbf{k}+1)Q_s) \\ r=1,\dots,\ell}} \sum_{j=i_1(r)}^{i_{p_r}(r)} \psi_j^r \nabla(\tau_{j,r} v) \Bigg|^2 dx \leq NMK_2 \sum_{\substack{r \in I((\mathbf{k}+1)Q_s) \\ r=1,\dots,\ell}} \sum_{j=i_1(r)}^{i_{p_r}(r)} \int_{h_r+\Delta_j} |\nabla v|^2 dx$$
$$\leq NM^2K_2 \sum_{\substack{h_r \in I((\mathbf{k}+1)Q_s) \\ r=1,\dots,\ell}} \int_{h_r+\Delta_{i_1(r)}\cup\dots\cup\Delta_{i_{p_r}(r)}} |\nabla v|^2 dx \leq NM^2K_2 \sum_{\substack{h_r \in I((\mathbf{k}+1)Q_s) \\ r=1,\dots,\ell}} \int_{\mathbf{p}Q_s \cap F} |\nabla v|^2 dx$$
$$\leq N^2M^2K_2 \int_{\mathbf{p}Q_s \cap F} |\nabla v|^2 dx, \tag{8.15}$$

where $\mathbf{p} = \mathbf{p}(n, \mathbf{k}) \in \mathbb{N}$ is the smallest integer such that $\bigcup_{h_r \in I((\mathbf{k}+1)Q_s)} (\mathbf{k}+1)Q_r \subseteq \mathbf{p}\, Q_s$. Summing up relation (8.15) with respect to $s$ and $k$:

$$\sum_{s=1}^{\ell} \sum_{k=i_1(s)}^{i_{p_s}(s)} \int_{h_s+(\Delta_k \cup U_k'')} \Bigg| \sum_{\substack{h_r \in I((\mathbf{k}+1)Q_s) \\ r=1,\dots,\ell}} \sum_{j=i_1(r)}^{i_{p_r}(r)} \psi_j^r \nabla(\tau_{j,r} v) \Bigg|^2 dx \leq N^2M^2K_2 \sum_{s=1}^{\ell} \sum_{k=i_1(s)}^{i_{p_s}(s)} \int_{\mathbf{p}Q_s \cap F} |\nabla v|^2 dx$$
$$\leq N^2M^3K_2 \sum_{s=1}^{\ell} \int_{\mathbf{p}Q_s \cap F} |\nabla v|^2 dx \leq C_6 N^2M^3K_2 \int_F |\nabla v|^2 dx, \tag{8.16}$$

where $C_6 = C_6(n, \mathbf{k})$ is a constant depending only on $\mathbf{k}$ and $n$, such that each point $x \in \mathbb{R}^n$ is contained in at most $C_6$ cubes of the form $(h + \mathbf{p}Q)_{h\in\mathbb{Z}^n}$.
Let now study the second term of (8.13). From the fact that, by (8.11),

$$\sum_{\substack{h_r \in I((\mathbf{k}+1)Q_s)\\ r=1,\dots,\ell}} \sum_{j=i_1(r)}^{i_{p_r}(r)} \nabla\psi_j^r(x) = 0 \quad \text{for every } x \in F \cup G,$$

we have

$$\sum_{\substack{h_r \in I((\mathbf{k}+1)Q_s)\\ r=1,\dots,\ell}} \sum_{j=i_1(r)}^{i_{p_r}(r)} \nabla\psi_j^r(\tau_{j,r}v) = \sum_{\substack{h_r \in I((\mathbf{k}+1)Q_s)\\ r=1,\dots,\ell}} \sum_{j=i_1(r)}^{i_{p_r}(r)} \nabla\psi_j^r(\tau_{j,r}v - \tau_{k,s}v) \quad \text{a.e. in } F \cup G.$$

Then last relation, together with (8.14), implies

$$\begin{aligned}
&\int_{h_s+(\Delta_k\cup U_k'')} \Bigg| \sum_{\substack{h_r \in I((\mathbf{k}+1)Q_s)\\ r=1,\dots,\ell}} \sum_{j=i_1(r)}^{i_{p_r}(r)} \nabla\psi_j^r(\tau_{j,r}v) \Bigg|^2 dx \\
&\le N \sum_{\substack{h_r \in I((\mathbf{k}+1)Q_s)\\ r=1,\dots,\ell}} \int_{h_s+(\Delta_k\cup U_k'')} \Bigg| \sum_{j=i_1(r)}^{i_{p_r}(r)} \nabla\psi_j^r(\tau_{j,r}v - \tau_{k,s}v) \Bigg|^2 dx \\
&\le NM \sum_{\substack{h_r \in I((\mathbf{k}+1)Q_s)\\ r=1,\dots,\ell}} \sum_{j=i_1(r)}^{i_{p_r}(r)} \int_{(h_s+(\Delta_k\cup U_k''))\cap(h_r+(\Delta_j\cup U_j''))} \left|\nabla\psi_j^r(\tau_{j,r}v - \tau_{k,s}v)\right|^2 dx \\
&\le NMC_5^2 \sum_{\substack{h_r \in I((\mathbf{k}+1)Q_s)\\ r=1,\dots,\ell}} \sum_{j=i_1(r)}^{i_{p_r}(r)} \int_{(h_s+(\Delta_k\cup U_k''))\cap(h_r+(\Delta_j\cup U_j''))} |\tau_{j,r}v - \tau_{k,s}v|^2 dx.
\end{aligned}$$

Notice that, if $(h_s + \Delta_k) \cap (h_r + \Delta_j) \neq \emptyset$, then $\tau_{j,r}v - \tau_{k,s}v = 0$ a.e. in $(h_s + \Delta_k) \cap (h_r + \Delta_j)$. Thus, by Poincaré inequality in $(h_s + (\Delta_k \cup U_k'')) \cap (h_r + (\Delta_j \cup U_j''))$, and summing up last relation with respect to $s$ and $k$, we get

$$\begin{aligned}
&\sum_{s=1}^{\ell} \sum_{k=i_1(s)}^{i_{p_s}(s)} \int_{h_s+(\Delta_k\cup U_k'')} \Bigg| \sum_{\substack{h_r \in I((\mathbf{k}+1)Q_s)\\ r=1,\dots,\ell}} \sum_{j=i_1(r)}^{i_{p_r}(r)} (\tau_{j,r}v)\nabla\psi_j^r \Bigg|^2 dx \\
&\le NMC_5^2 C_P \sum_{s=1}^{\ell} \sum_{k=i_1(s)}^{i_{p_s}(s)} \sum_{\substack{h_r \in I((\mathbf{k}+1)Q_s)\\ r=1,\dots,\ell}} \sum_{j=i_1(r)}^{i_{p_r}(r)} \int_{(h_s+(\Delta_k\cup U_k''))\cap(h_r+(\Delta_j\cup U_j''))} |\nabla(\tau_{j,r}v) - \nabla(\tau_{k,s}v)|^2 dx \\
&\le 2NMC_5^2 C_P K_2 \sum_{s=1}^{\ell} \sum_{k=i_1(s)}^{i_{p_s}(s)} \sum_{\substack{h_r \in I((\mathbf{k}+1)Q_s)\\ r=1,\dots,\ell}} \sum_{j=i_1(r)}^{i_{p_r}(r)} \left[ \int_{h_s+\Delta_k} |\nabla v|^2 \, dx + \int_{h_r+\Delta_j} |\nabla v|^2 \, dx \right],
\end{aligned} \tag{8.17}$$

where the constant $C_P = C_P(n, E)$ does not depend on $s, k, r$ and $j$.

Regarding the first term in the right-hand side of (8.17), we have

$$\begin{aligned}
&\sum_{s=1}^{\ell} \sum_{k=i_1(s)}^{i_{p_s}(s)} \sum_{\substack{h_r \in I((\mathbf{k}+1)Q_s)\\ r=1,\dots,\ell}} \sum_{j=i_1(r)}^{i_{p_r}(r)} \int_{h_s+\Delta_k} |\nabla v|^2 dx \le NM \sum_{s=1}^{\ell} \sum_{k=i_1(s)}^{i_{p_s}(s)} \int_{h_s+\Delta_k} |\nabla v|^2 dx \\
&\le NM^2 \sum_{s=1}^{\ell} \int_{h_s+\Delta_{i_1(s)}\cup\dots\cup\Delta_{i_{p_s}(s)}} |\nabla v|^2 dx \le NM^2 C_7 \int_F |\nabla v|^2 dx,
\end{aligned} \tag{8.18}$$

where $C_7 = C_7(n, \mathbf{k})$ is a constant depending only on $\mathbf{k}$ and $n$, such that each point $x \in \mathbb{R}^n$ is contained in at most $C_7$ different cubes of the form $(h + (\mathbf{k}+1)Q)_{h\in\mathbb{Z}^n}$.

Similarly, for the last term in the right-hand side of (8.17) we have

$$\sum_{s=1}^{\ell} \sum_{k=i_1(s)}^{i_{p_s}(s)} \sum_{\substack{h_r \in I((\mathbf{k}+1)Q_s) \\ r=1,\dots,\ell}} \sum_{j=i_1(r)}^{i_{p_r}(r)} \int_{h_r+\Delta_j} |\nabla v|^2 dx \le M \sum_{s=1}^{\ell} \sum_{\substack{h_r \in I((\mathbf{k}+1)Q_s) \\ r=1,\dots,\ell}} \sum_{j=i_1(r)}^{i_{p_r}(r)} \int_{h_r+\Delta_j} |\nabla v|^2 dx$$

$$\le M^2 \sum_{s=1}^{\ell} \sum_{\substack{h_r \in I((\mathbf{k}+1)Q_s) \\ r=1,\dots,\ell}} \int_{h_r+\Delta_{i_1(r)}\cup\ldots\cup\Delta_{i_{p_r}(r)}} |\nabla v|^2 dx \le M^2 \sum_{s=1}^{\ell} \sum_{\substack{h_r \in I((\mathbf{k}+1)Q_s) \\ r=1,\dots,\ell}} \int_{\mathbf{p}Q_s\cap F} |\nabla v|^2 dx$$

$$\le M^3 \sum_{s=1}^{\ell} \int_{\mathbf{p}Q_s\cap F} |\nabla v|^2 dx \le M^3 C_6 \int_F |\nabla v|^2 dx. \tag{8.19}$$

Collecting relations (8.13), (8.16), (8.18) and (8.19) we get the conclusion. □


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