Thin-walled beams: a derivation of Vlassov theory via Γ-convergence

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Abstract

This paper deals with the asymptotic analysis of the three-dimensional problem for a linearly elastic cantilever having an open cross-section which is the union of rectangles with sides of order ε and ε^2, as ε goes to zero. Under suitable assumptions on the given loads and for homogeneous and isotropic material, we show that the three-dimensional problem Γ-converges to the classical one-dimensional Vlassov model for thin-walled beams.

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1 Introduction

In this paper we continue a line of research initiated in [4] which aims at a rigorous variational deduction of the one-dimensional theory for thin-walled beams from the three-dimensional linear elasticity.

In [4] we considered a thin-walled cantilever $\Omega_\varepsilon = \omega_\varepsilon \times (0, \ell)$ of length ℓ, made of homogeneous linear isotropic material, with a rectangular cross-section ω_ε of

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sides ε and ε^2. By working in the framework of Γ-convergence, see, for example, [3] and [2], we proved that the three-dimensional elasticity problem converges in a suitable variational sense to a one-dimensional problem as ε goes to zero. The limit problem is defined by a functional which includes the extensional, the flexural and the torsional strain energies of the classical thin-walled model of beam, as they can be deduced from De Saint-Venant's theory. In particular, the strain energy density of the limit model can be written as a diagonal homogeneous quadratic form of the longitudinal strain, the curvatures of the beam axis evaluated with respect to the principal planes of bending, and the first derivative of the torsional twist. Therefore, the equations of equilibrium show a full decoupling between extensional, flexural and torsional effects.

The present paper extends the results obtained in [4] to the case of thin-walled beams with open (i.e., simply connected) multi-rectangular cross-section. More precisely, we consider a three-dimensional cylinder $\Omega_\varepsilon = \bigcup_{i=1}^{3} \Omega_\varepsilon^{(i)}$, where $\Omega_\varepsilon^{(i)} = \omega_\varepsilon^{(i)} \times (0, \ell)$ and $\omega_\varepsilon^{(i)}$ is a rectangle having sides of order ε and ε^2. The rectangles $\omega_\varepsilon^{(i)}$ partially overlap and are joined so as to achieve, for instance, cross-sections having a $\top$-like or $\sqsubset$-like shape.

Under the assumption that a end cross-section of the three-dimensional body is fixed and the material is homogeneous and isotropic, the theory of Γ-convergence is used to study the asymptotic behavior of the energy functional as ε goes to zero.

The limit functional strongly depends on the geometry of the cross-section. In particular, it is a homogeneous quadratic form of the longitudinal strain, of the two bending curvatures of the beam axis, of the first derivative of the twist and also of the square of the second derivative of the twist, in the case of a section composed by at least three not aligned rectangles. This last term is concerned with the so-called *nonuniform torsional effects*, which are responsible for the presence of normal stresses induced by torsional deformations and, as a consequence, for the coupling between extension, flexure and torsion. It should be remarked that the above mentioned effects are not included in the classical De Saint-Venant's theory and have proved to be important in several engineering fields, especially in aeronautical applications where the presence of open cross-sections formed by parts with dimension of different order of magnitude requires more refined mechanical beam models.

A direct analysis of the limit energy functional shows that the full decoupling between extensional, flexural and torsional problems can be obtained by choosing the axes of the reference system as the principal axes of the cross-section centered in his center of mass, and by determining the *sector coordinate function* - which is, roughly speaking, the limit of the warping function of the cross-section - with respect to the *shear center*.

The nonuniform torsion theory is well known in the engineering literature of thin-walled beams since the old paper by Timoshenko [11] and the fundamental contribution by Vlassov [12]. These approximate theories are usually based on some a-priori assumptions on the deformation of the body and on the induced stress field. Little attention has been given to the mathematical justification of Vlassov's theory; Rodriguez and Viaño, for example, presented in [10] an extension of Vlassov's theory for thin-walled beams as an asymptotic approximation of the three-dimensional model as the area and the thickness of the cross-section are assumed to tend to zero independently in a suitable order.

The paper is organized as follows. In Section 2 we introduce the three-dimensional problem and in Section 3 we rewrite it in a variational form on a fixed domain. The proof of some compactness results for a scaled displacement field is presented in Section 4. Section 5 is devoted to the establishment of the junction conditions and to use them to characterize the essential kinematic fields of the cross-section. Γ-convergence results are presented in Sections 6 and 7. The limit energy functional is investigated in Section 8 and some examples are discussed in Section 9. The contribution of the external loads and the strong convergence of minimizers are studied in Section 10.

Notation. Throughout this article, and unless otherwise stated, we use the Einstein summation convention and we index vector and tensor components as follows: Greek indices α, β and γ take values in the set $\{1,2\}$ and Latin indices i, j, h in the set $\{1,2,3\}$. Accordingly, $\alpha(i)$ will denote the parity of the index i, that is the function which takes the value 1 if i is odd and the value 2 if i is even. The component k of a vector $\mathbf{v}$ will be denoted either with $(\mathbf{v})_k$ or v_k and an analogous notation will be used to denote tensor components. $\mathcal{E}_{\alpha\beta}$ denotes the Ricci's symbol, that is $\mathcal{E}_{11} = \mathcal{E}_{22} = 0$, $\mathcal{E}_{12} = 1$ and $\mathcal{E}_{21} = -1$. $L^2(A;B)$ and $H^s(A;B)$ are the standard Lebesgue and Sobolev spaces of functions defined on the domain A and taking values in B, with the usual norms $\|\cdot\|_{L^2(A;B)}$ and $\|\cdot\|_{H^s(A;B)}$, respectively. When $B = \mathbb{R}$ or when the right set B is clear from the context, we will simply write $L^2(A)$ or $H^s(A)$, sometimes even in the notation used for norms. Convergence in the norm, that is the so called strong convergence, will be denoted by $\to$ while weak convergence is denoted with $\rightharpoonup$.

With a little abuse of notation, and because this is a common practice and does not give rise to any mistake, we use to call "sequences" even those families indicized by a continuous parameter ε which, throughout the whole paper, will be assumed to belong to the interval $(0,1]$.

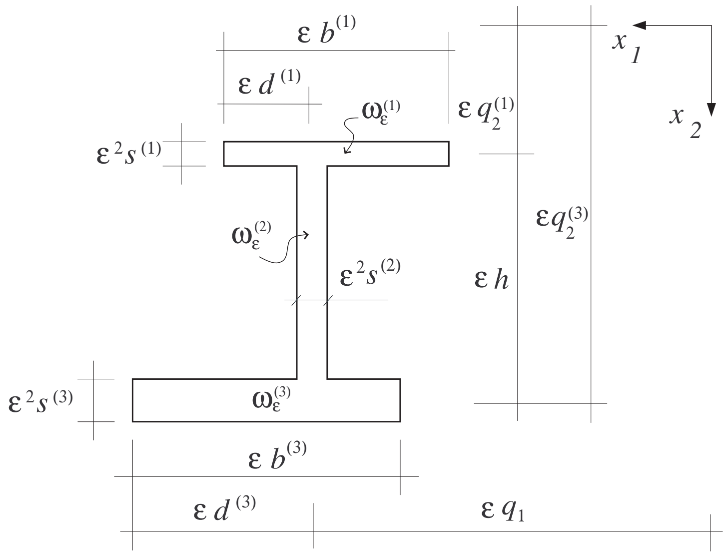
2 The 3-dimensional problem

We consider a three-dimensional body which is at rest in the placement $\Omega_\varepsilon \subset \mathbb{R}^3$, where $\Omega_\varepsilon := \omega_\varepsilon \times (0,\ell)$, $\omega_\varepsilon := \cup_{i=1}^3 \omega_\varepsilon^{(i)}$, and

$$\omega_\varepsilon^{(i)} := (\varepsilon q_1 + \varepsilon d^{(i)} - \varepsilon b^{(i)}, \varepsilon q_1 + \varepsilon d^{(i)}) \times (\varepsilon q_2^{(i)} - \varepsilon^2 \frac{s^{(i)}}{2}, \varepsilon q_2^{(i)} + \varepsilon^2 \frac{s^{(i)}}{2}), \quad i = 1,3$$

$$\omega_\varepsilon^{(2)} := (\varepsilon q_1 - \varepsilon^2 \frac{s^{(2)}}{2}, \varepsilon q_1 + \varepsilon^2 \frac{s^{(2)}}{2}) \times (\varepsilon q_2^{(1)}, \varepsilon q_2^{(3)}),$$

are three non-empty rectangles. Moreover we set $h := q_2^{(3)} - q_2^{(1)}$.



For later convenience we also set

$$\Omega_\varepsilon^{(i)} := \omega_\varepsilon^{(i)} \times (0,\ell), \quad i = 1,2,3,$$

and we note that $\Omega_\varepsilon = \Omega_\varepsilon^{(1)} \cup \Omega_\varepsilon^{(2)} \cup \Omega_\varepsilon^{(3)}$ and that they are not pairwise disjoint.

Remark 2.1 Within our framework we can cover also cross-sections having a ⊤-like shape by simply setting, for instance, $s^{(3)} = b^{(3)} = 0$ (of course, in this case the rectangle $\omega_\varepsilon^{(3)}$ is empty). A section having a ⊏-like shape can be achieved, for instance, by replacing the quantities $d^{(1)}$ and $d^{(3)}$ in the figure above with other two parameters $d_\varepsilon^{(1)}$ and $d_\varepsilon^{(3)}$ satisfying the properties $d_\varepsilon^{(1)}/\varepsilon \to d^{(1)}$ and $d_\varepsilon^{(3)}/\varepsilon \to d^{(3)}$, respectively, as ε goes to 0. The ⊏-like shape is reached then by making the choice

$d^{(1)} = d^{(3)} = s^{(2)}/2$ (and, for instance, $d^{(1)}_\varepsilon = d^{(3)}_\varepsilon = \varepsilon s^{(2)}/2$). Similarly one can consider also sections with a $\llcorner$ or $\urcorner$-like shape. Our analysis covers also these more general settings but, for simplicity, we will concentrate ourselves to the case displayed in the figure above.

In what follows we consider an homogeneous isotropic material, so that the elasticity tensor $\mathbb{C}$ writes as

$$\mathbb{C}\mathbf{A} = 2\mu\mathbf{A} + \lambda(\mathrm{tr}\mathbf{A})\,\mathbf{I}$$

for every symmetric matrix $\mathbf{A}$. Above, $\mathbf{I}$ denotes the 3×3 identity matrix. We assume $\mu > 0$ and $\lambda \geq 0$ so to have, for every symmetric tensor $\mathbf{A}$,

$$\mathbb{C}\mathbf{A} \cdot \mathbf{A} \geq \mu|\mathbf{A}|^2, \tag{1}$$

where $\cdot$ denotes the scalar product. We shall consider the spaces

$$H^1_\#(\Omega_\varepsilon;\mathbb{R}^3) := \big\{\mathbf{u} \in H^1(\Omega_\varepsilon;\mathbb{R}^3) : \mathbf{u} = \mathbf{0} \text{ on } \omega_\varepsilon \times \{0\}\big\}$$

and $H^1_\#(\Omega^{(i)}_\varepsilon;\mathbb{R}^3)$ defined in a similar way. We further denote by

$$\begin{aligned} \mathbf{E}\mathbf{u}(\mathbf{x}) &:= \mathrm{sym}(D\mathbf{u}(\mathbf{x})) := \frac{D\mathbf{u}(\mathbf{x}) + D\mathbf{u}^T(\mathbf{x})}{2}, \\ \mathbf{W}\mathbf{u}(\mathbf{x}) &:= \mathrm{skw}(D\mathbf{u}(\mathbf{x})) := \frac{D\mathbf{u}(\mathbf{x}) - D\mathbf{u}^T(\mathbf{x})}{2}, \end{aligned} \tag{2}$$

the strain of $\mathbf{u} : \Omega_\varepsilon \to \mathbb{R}^3$ and the skew symmetric part of the gradient $D\mathbf{u}$.

We consider the following total energy functionals

$$\mathcal{F}_\varepsilon(\mathbf{u}) := J_\varepsilon(\mathbf{u}) - \int_{\Omega_\varepsilon} \mathbf{b}^\varepsilon \cdot \mathbf{u}\, dx, \tag{3}$$

where

$$J_\varepsilon(\mathbf{u}) := \frac{1}{2}\int_{\Omega_\varepsilon} \mathbb{C}\mathbf{E}\mathbf{u} \cdot \mathbf{E}\mathbf{u}\, dx \tag{4}$$

are the bulk energies and $\mathbf{b}^\varepsilon \in L^2(\Omega_\varepsilon;\mathbb{R}^3)$ are the body forces.

Due to the coerciveness inequality (1) and the strict convexity of the integrand, the total energy functionals $\mathcal{F}_\varepsilon$ admit for every ε a unique minimizer among all competing displacements $\mathbf{u} \in H^1_\#(\Omega_\varepsilon;\mathbb{R}^3)$. As already explained in the introduction our aim is to study the asymptotic behavior of such minimizers as ε goes to 0, through the theory of Γ-convergence, for an account of it we refer to the books of Braides [2] and Dal Maso [3].

3 The rescaled problem

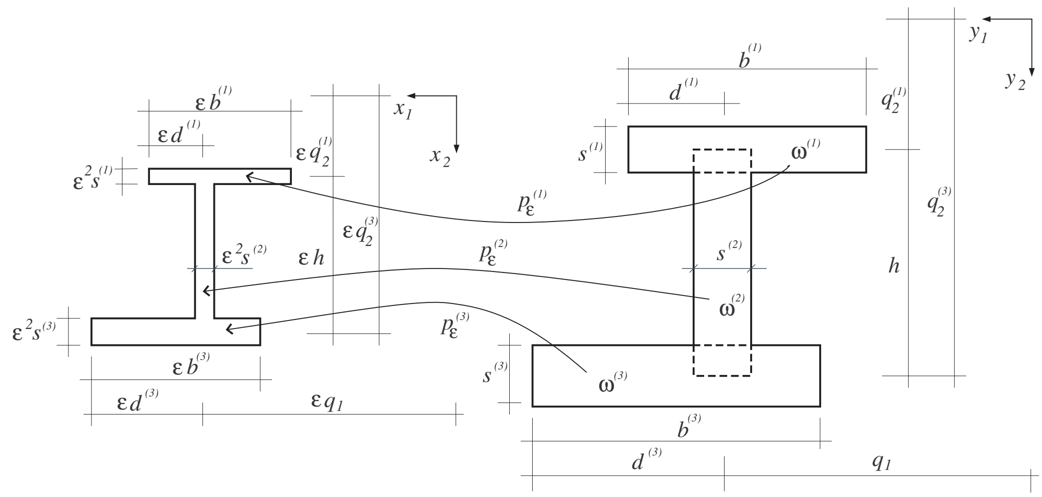
To state our results it is convenient to stretch the domains $\Omega_\varepsilon^{(i)}$ along the transverse directions x_1 and x_2 in suitable ways so that the transformed domains do not depend on ε. Hereafter we denote by $\omega^{(i)} := \omega_1^{(i)}$ and let

$$p_\varepsilon^{(i)} : \Omega^{(i)} \to \Omega_\varepsilon^{(i)},$$

be the (unique) affine invertible transformation between the sets $\Omega^{(i)}$ and $\Omega_\varepsilon^{(i)}$. It turns out to be defined by

$$p_\varepsilon^{(i)}(y_1, y_2, y_3) = \big(\varepsilon^{\alpha(i)} y_1 + (\varepsilon - \varepsilon^{\alpha(i)}) q_1, \varepsilon^{\alpha(i+1)} y_2 + (\varepsilon - \varepsilon^{\alpha(i+1)}) q_2^{(i)}, y_3\big)$$

where $\alpha(i)$ is the parity of i, that is the function which takes the value 1 if i is odd and the value 2 if i is even.



Given $\mathbf{u} \in H^1_\#(\Omega_\varepsilon; \mathbb{R}^3)$ we define three functions $\mathbf{u}^{(i)} \in H^1_\#(\Omega^{(i)}; \mathbb{R}^3)$ by

$$\mathbf{u}^{(i)} := \mathbf{u} \circ p_\varepsilon^{(i)}.$$

Of course, in the regions where the domains overlap the following two "junction conditions" must be satisfied

$$\mathbf{u}^{(i)} \circ p_\varepsilon^{(i)\,-1} = \mathbf{u}^{(2)} \circ p_\varepsilon^{(2)\,-1} \quad \text{in } \Omega_\varepsilon^{(i)} \cap \Omega_\varepsilon^{(2)}, \quad i = 1, 3. \tag{5}$$

Let us consider the following 3×3 matrix valued differential operators

$$\mathbf{H}_\varepsilon^{(i)} \mathbf{u} := D\mathbf{u} \circ Dp_\varepsilon^{(i)\,-1} = \left(\frac{D_1 \mathbf{u}}{\varepsilon^{\alpha(i)}}, \frac{D_2 \mathbf{u}}{\varepsilon^{\alpha(i+1)}}, D_3 \mathbf{u} \right)$$

where $D_i\mathbf{u}$ denotes the column vector of the partial derivatives of $\mathbf{u}$ with respect to y_i. We also set

$$\mathbf{E}^{(i)}_\varepsilon\mathbf{u} := \mathrm{sym}(\,\mathbf{H}^{(i)}_\varepsilon\mathbf{u}), \quad \mathbf{W}^{(i)}_\varepsilon\mathbf{u} := \mathrm{skw}(\,\mathbf{H}^{(i)}_\varepsilon\mathbf{u}). \tag{6}$$

Let us split the bulk energy J_ε, defined in (4), into the sum of three energies each one defined on a rectangular component $\Omega^{(i)}$; more precisely

$$J_\varepsilon(\mathbf{u}) = \sum_{i=1}^{3} J^{(i)}_\varepsilon(\mathbf{u}), \quad \text{where} \quad J^{(i)}_\varepsilon(\mathbf{u}) := \frac{1}{2}\int_{\Omega^{(i)}_\varepsilon} \chi_\varepsilon(x)\,\mathbb{C}\mathbf{E}\mathbf{u}\cdot\mathbf{E}\mathbf{u}\,dx,$$

and $\chi_\varepsilon : \Omega_\varepsilon \to \{1/2, 1\}$ is defined by

$$\chi_\varepsilon(x) := \begin{cases} 1/2 & \text{if } x \in (\Omega^{(1)}_\varepsilon \cup \Omega^{(3)}_\varepsilon) \cap \Omega^{(2)}_\varepsilon \\ 1 & \text{otherwise.} \end{cases}$$

Let $\mathcal{A}_\varepsilon := \big\{(\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \mathbf{u}^{(3)}) \in \times_{i=1}^{3} H^1_\#(\Omega^{(i)};\mathbb{R}^3) \;:\; \text{conditions (5) are satisfied}\big\}$; then we can consider the rescaled bulk energies $I_\varepsilon : \mathcal{A}_\varepsilon \to [0, +\infty)$ obtained by rescaling each term $I^{(i)}_\varepsilon$ of the sum on the corresponding domain $\Omega^{(i)}$ with the suitable change of variable, that is

$$I_\varepsilon(\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \mathbf{u}^{(3)}) := \sum_{i=1}^{3} I^{(i)}_\varepsilon(\mathbf{u}^{(i)}) \tag{7}$$

where

$$I^{(i)}_\varepsilon(\mathbf{u}^{(i)}) := \frac{1}{\varepsilon^3} J^{(i)}_\varepsilon(\mathbf{u}^{(i)} \circ p^{(i)}_\varepsilon{}^{-1}) = \frac{1}{2}\int_{\Omega^{(i)}} \chi^{(i)}_\varepsilon(y)\,\mathbb{C}\mathbf{E}^{(i)}_\varepsilon\mathbf{u}^{(i)}\cdot\mathbf{E}^{(i)}_\varepsilon\mathbf{u}^{(i)}\,dy, \tag{8}$$

and

$$\chi^{(i)}_\varepsilon := \chi_\varepsilon \circ p^{(i)}_\varepsilon.$$

Note that

$$I_\varepsilon(\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \mathbf{u}^{(3)}) = \frac{1}{\varepsilon^3} J_\varepsilon(\mathbf{u}).$$

4 Compactness lemmata

In this section we establish the compactness of appropriately rescaled sequences of displacements and prove that the limit functions are displacements of Bernoulli-Navier type. The proof of the next lemma follows immediately from (1).

Lemma 4.1 *Let* $(\mathbf{u}_\varepsilon^{(1)}, \mathbf{u}_\varepsilon^{(2)}, \mathbf{u}_\varepsilon^{(3)})$ *be a sequence in* $\mathcal{A}_\varepsilon$. *If*

$$\sup_{\varepsilon\in(0,1]} \frac{1}{\varepsilon^4} I_\varepsilon(\mathbf{u}_\varepsilon^{(1)}, \mathbf{u}_\varepsilon^{(2)}, \mathbf{u}_\varepsilon^{(3)}) < +\infty, \tag{9}$$

then there exists a constant $C > 0$ *such that*

$$\sum_{i=1}^{3} \|\mathbf{E}_\varepsilon^{(i)} \mathbf{u}_\varepsilon^{(i)}\|_{L^2(\Omega^{(i)};\mathbb{R}^{3\times 3})} \le C\varepsilon^2 \tag{10}$$

for every $\varepsilon \in (0, 1]$.

To prove the compactness of the displacements we need the following scaled Korn inequalities (already obtained in [4] in the particular case $i = 2$).

Theorem 4.2 *Let* $i \in \{1, 2, 3\}$. *There exists a constant* $C > 0$ *such that*

$$\int_{\Omega^{(i)}} \Big(|(\frac{u_1}{\varepsilon^{\alpha(i+1)-1}}, \frac{u_2}{\varepsilon^{\alpha(i)-1}}, \frac{u_3}{\varepsilon^2})|^2 + |\mathbf{H}_\varepsilon^{(i)}\mathbf{u}|^2\Big)\, dy \le \frac{C}{\varepsilon^4} \int_{\Omega^{(i)}} |\mathbf{E}_\varepsilon^{(i)}\mathbf{u}|^2\, dy \tag{11}$$

for every $\mathbf{u} \in H^1_\#(\Omega^{(i)};\mathbb{R}^3)$ *and every* $\varepsilon \in (0, 1]$.

Proof. Let us divide the section $\omega_\varepsilon^{(i)}$ in squares of size ε^2 and apply Korn's inequality (the one obtained by Anzellotti, Baldo and Percivale in [1]; see also [9] and Kondrat'ev and Oleinik [6], Theorem 2) to each beam of length ℓ and with section a square with side proportional to ε^2. Then, summing over all the obtained inequalities we have that there exists a constant $C > 0$ such that

$$\int_{\Omega_\varepsilon^{(i)}} \big(|\mathbf{u}|^2 + |D\mathbf{u}|^2\big)\, dx \le \frac{C}{\varepsilon^4} \int_{\Omega_\varepsilon^{(i)}} |\mathbf{E}\mathbf{u}|^2\, dx \tag{12}$$

for every $\mathbf{u} \in H^1_\#(\Omega_\varepsilon^{(i)};\mathbb{R}^3)$.

The inequality $\int_{\Omega^{(i)}} |\mathbf{H}_\varepsilon^{(i)}\mathbf{u}|^2\, dy \le \frac{C}{\varepsilon^4} \int_{\Omega^{(i)}} |\mathbf{E}_\varepsilon^{(i)}\mathbf{u}|^2\, dy$ is simply obtained by rescaling inequality (12). To show that

$$\int_{\Omega^{(i)}} |(\frac{u_1}{\varepsilon^{\alpha(i+1)-1}}, \frac{u_2}{\varepsilon^{\alpha(i)-1}}, \frac{u_3}{\varepsilon^2})|^2\, dy \le \frac{C}{\varepsilon^4} \int_{\Omega^{(i)}} |\mathbf{E}_\varepsilon^{(i)}\mathbf{u}|^2\, dy,$$

it suffices to set $\mathbf{v}^\varepsilon := (\frac{u_1}{\varepsilon^{\alpha(i+1)-1}}, \frac{u_2}{\varepsilon^{\alpha(i)-1}}, \frac{u_3}{\varepsilon^2})$, notice that $|\mathbf{E}^\varepsilon\mathbf{u}| \ge \varepsilon^2|\mathbf{E}\mathbf{v}^\varepsilon|$ and apply the standard Korn inequality to $\mathbf{v}^\varepsilon$ on the domain Ω (see for instance [8], Theorem 2.7). □

Let us consider the usual space of Bernoulli-Navier displacements on $\Omega^{(i)}$

$$H_{BN}(\Omega^{(i)};\mathbb{R}^3) := \Big\{\mathbf{v} \in H^1_{\#}(\Omega^{(i)};\mathbb{R}^3) : (\mathbf{E}\mathbf{v})_{j\alpha} = 0,\ j = 1,2,3,\ \alpha = 1,2\Big\} \quad (13)$$

and set

$$H_{BN} := \times_{i=1}^{3} H_{BN}(\Omega^{(i)};\mathbb{R}^3). \quad (14)$$

Analogously we denote by

$$L^2 := \times_{i=1}^{3} L^2(\Omega^{(i)}) \ \text{ and } \ H^1 := \times_{i=1}^{3} H^1(\Omega^{(i)};\mathbb{R}^3).$$

Inequality (11) motivates the introduction of the following scaling operators

$$\mathbf{S}^{(i)}_{\varepsilon}\mathbf{u} := \left(\frac{u_1}{\varepsilon^{\alpha(i+1)-1}}, \frac{u_2}{\varepsilon^{\alpha(i)-1}}, \frac{u_3}{\varepsilon^2}\right). \quad (15)$$

Lemma 4.3 *Let* $(\mathbf{u}^{(1)}_{\varepsilon}, \mathbf{u}^{(2)}_{\varepsilon}, \mathbf{u}^{(3)}_{\varepsilon})$ *be a sequence in* $\mathcal{A}_{\varepsilon}$ *which satisfies* (10). *Then, for any sequence of positive numbers* ε_n *converging to* 0 *there exist a subsequence (not relabeled) and functions* $(\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \mathbf{v}^{(3)}) \in H_{BN}$ *and* $(\vartheta^{(1)}, \vartheta^{(2)}, \vartheta^{(3)}) \in L^2$ *such that, as n goes to* ∞*,*

$$\mathbf{S}^{(i)}_{\varepsilon_n}\mathbf{u}^{(i)}_{\varepsilon_n} \rightharpoonup \mathbf{v}^{(i)} \quad \text{in } H^1(\Omega^{(i)};\mathbb{R}^3), \quad (16)$$

$$(\mathbf{W}^{(i)}_{\varepsilon_n}\mathbf{u}^{(i)}_{\varepsilon_n})_{12} \rightharpoonup -\vartheta^{(i)} \ \text{in } L^2(\Omega^{(i)}). \quad (17)$$

PROOF. It is convenient to set $\mathbf{v}^{(i)}_{\varepsilon} := \mathbf{S}^{(i)}_{\varepsilon}\mathbf{u}^{(i)}_{\varepsilon}$. It is easily checked that, for $\varepsilon \le 1$, $|\mathbf{E}^{(i)}_{\varepsilon}\mathbf{u}^{(i)}_{\varepsilon}| \ge \varepsilon^2|\mathbf{E}\mathbf{v}^{(i)}_{\varepsilon}|$, hence, by (10), $\mathbf{E}\mathbf{v}^{(i)}_{\varepsilon}$ is uniformly bounded in $L^2(\Omega^{(i)};\mathbb{R}^{3\times 3})$, and by Korn's inequality $\mathbf{v}^{(i)}_{\varepsilon}$ is uniformly bounded in $H^1(\Omega^{(i)};\mathbb{R}^3)$. It then exists a $\mathbf{v}^{(i)} \in H^1_{\#}(\Omega^{(i)};\mathbb{R}^3)$, and a subsequence (not relabeled) of ε_n such that $\mathbf{v}^{(i)}_{\varepsilon_n} \rightharpoonup \mathbf{v}^{(i)}$ in $H^1(\Omega^{(i)};\mathbb{R}^3)$. Again, it is easy to check that $|(\mathbf{E}^{(i)}_{\varepsilon}\mathbf{u}^{(i)}_{\varepsilon})_{j\alpha}| \ge \varepsilon|(\mathbf{E}\mathbf{v}^{(i)}_{\varepsilon})_{j\alpha}|$ for $j = 1,2,3$ and $\alpha = 1,2$, thus, using (10), we deduce that $C\varepsilon \ge \|(\mathbf{E}\mathbf{v}^{(i)}_{\varepsilon})_{j\alpha}\|_{L^2(\Omega^{(i)})}$ and consequently $(\mathbf{E}\mathbf{v}^{(i)})_{j\alpha} = 0$. Hence $\mathbf{v} \in H_{BN}(\Omega^{(i)};\mathbb{R}^3)$.

Using assumption (10) together with Theorem 4.2 we obtain that the sequence $\mathbf{H}^{(i)}_{\varepsilon_n}\mathbf{u}^{(i)}_{\varepsilon_n}$ is bounded in L^2 so that, up to subsequences, it weakly converges in $L^2(\Omega^{(i)};\mathbb{R}^{3\times 3})$ to some $\mathbf{H}^{(i)} \in L^2(\Omega^{(i)};\mathbb{R}^{3\times 3})$. Since, from (10), $\mathbf{E}^{(i)}_{\varepsilon_n}\mathbf{u}^{(i)}_{\varepsilon_n} \to \mathbf{0}$ in $L^2(\Omega^{(i)};\mathbb{R}^{3\times 3})$ we have $\mathbf{W}^{(i)}_{\varepsilon_n}\mathbf{u}^{(i)}_{\varepsilon_n} \to \mathbf{H}^{(i)}$ weakly in $L^2(\Omega^{(i)};\mathbb{R}^{3\times 3})$. In particular, $\mathbf{H}^{(i)}$ is, almost everywhere, a skew-symmetric matrix. Denoting $(\mathbf{H}^{(i)})_{12} = -\vartheta^{(i)}$ we obtain (17). □

Remark 4.4 Using the same notation as in the proof of Lemma 4.3, and since $(\mathbf{H}_\varepsilon^{(i)}\mathbf{u}_\varepsilon^{(i)})_{13} = D_3 u_{\varepsilon 1}^{(i)} = \varepsilon^{\alpha(i+1)-1} D_3 v_{\varepsilon 1}^{(i)}$, and $(\mathbf{H}_\varepsilon^{(i)}\mathbf{u}_\varepsilon^{(i)})_{23} = D_3 u_{\varepsilon 2}^{(i)} = \varepsilon^{\alpha(i)-1} D_3 v_{\varepsilon 2}^{(i)}$ we note that we have also

$$(\mathbf{H}^{(i)})_{13} = \big(\alpha(i) - 1\big) D_3 v_1^{(i)} \quad \text{and} \quad (\mathbf{H}^{(i)})_{23} = \big(\alpha(i+1) - 1\big) D_3 v_2^{(i)}.$$

By Lemma 4.3, the displacements $\mathbf{v}^{(i)}$ are of Bernoulli-Navier type and therefore they can be written as

$$v_\alpha^{(i)} = \xi_\alpha^{(i)}(y_3),\ \ \alpha = 1,2, \quad v_3^{(i)} = \xi_3^{(i)}(y_3) - y_\alpha \xi_\alpha^{(i)\prime}(y_3), \tag{18}$$

where

$$\xi_\alpha^{(i)} \in H^2_\#(0,\ell) := \{\xi \in H^2(0,\ell) : \xi(0) = \xi'(0) = 0\}$$

and

$$\xi_3^{(i)} \in H^1_\#(0,\ell).$$

The notation established in this section, in particular in Lemma 4.3 and in (18), will be used throughout the rest of the paper.

5 Junction conditions

The present section is devoted to establish the relationship existing between the limit fields $(\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \mathbf{v}^{(3)})$ and $(\vartheta^{(1)}, \vartheta^{(2)}, \vartheta^{(3)})$ introduced in Lemma 4.3. Hence, during the whole section, we assume that $(\mathbf{u}_\varepsilon^{(1)}, \mathbf{u}_\varepsilon^{(2)}, \mathbf{u}_\varepsilon^{(3)})$ be a sequence in $\mathcal{A}_\varepsilon$ which satisfies (10), that ε_n be a sequence of positive numbers converging to 0 and that a subsequence (not relabeled) and triples of functions $(\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \mathbf{v}^{(3)}) \in H_{BN}$ and $(\vartheta^{(1)}, \vartheta^{(2)}, \vartheta^{(3)}) \in L^2$ have been chosen in order to satisfy (16) and (17). Let us moreover use notation (18).

To find relations between the limit fields we study the junction conditions, that is the system (5), by adapting some inspiring ideas of Le Dret [7] and [5]. Since

$$p_\varepsilon^{(i)^{-1}}(x_1, x_2, x_3) = \Big(\frac{x_1 + (\varepsilon^{\alpha(i)} - \varepsilon) q_1}{\varepsilon^{\alpha(i)}}, \frac{x_2 + (\varepsilon^{\alpha(i+1)} - \varepsilon) q_2^{(i)}}{\varepsilon^{\alpha(i+1)}}, x_3\Big),$$

the two junction conditions can be equivalently written as

$$\begin{gathered}\mathbf{u}^{(i)}\big(\varepsilon(z_1 - q_1) + q_1, z_2, z_3\big) = \mathbf{u}^{(2)}\big(z_1, \varepsilon(z_2 - q_2^{(i)}) + q_2^{(i)}, z_3\big),\\ z \in \Omega^{(i)} \cap \Omega^{(2)}, \quad i = 1, 3,\end{gathered} \tag{19}$$

where

$$\Omega^{(1)} \cap \Omega^{(2)} = (q_1 - \frac{s^{(2)}}{2}, q_1 + \frac{s^{(2)}}{2}) \times (q_2^{(1)}, q_2^{(1)} + \frac{s^{(1)}}{2}) \times (0, \ell),$$

$$\Omega^{(3)} \cap \Omega^{(2)} = (q_1 - \frac{s^{(2)}}{2}, q_1 + \frac{s^{(2)}}{2}) \times (q_2^{(3)} - \frac{s^{(3)}}{2}, q_2^{(3)}) \times (0, \ell).$$

It is worth notice that in this way also the two junction regions, which originally depend on ε, have been trasformed into the fixed domains $\Omega^{(i)} \cap \Omega^{(2)}$, $i = 1, 3$.

The following lemma is stated for $\Omega^{(1)} \cap \Omega^{(2)}$ but, with straightforward adaptations, it holds also for $\Omega^{(3)} \cap \Omega^{(2)}$.

Lemma 5.1 *Let $w \in H^1(\Omega^{(1)} \cap \Omega^{(2)})$ and $w_\varepsilon \in H^1(\Omega^{(1)} \cap \Omega^{(2)})$ be a sequence such that*

$$w_\varepsilon \rightharpoonup w \text{ in } H^1(\Omega^{(1)} \cap \Omega^{(2)}).$$

Then the sequence of functions

$$(z_2, z_3) \mapsto \fint_{q_1 - s^{(2)}/2}^{q_1 + s^{(2)}/2} w_\varepsilon\big(\varepsilon(z_1 - q_1) + q_1, z_2, z_3\big)\, dz_1$$

converges in the norm of $L^2((q_2^{(1)}, q_2^{(1)} + s^{(1)}/2) \times (0, \ell))$ to the trace of the function w on $\{q_1\} \times (q_2^{(1)}, q_2^{(1)} + s^{(1)}/2) \times (0, \ell)$. We will denote such a trace simply by $w(q_1, z_2, z_3)$.

Proof. As

$$\begin{aligned}
&\int_0^\ell \int_{q_2^{(1)}}^{q_2^{(1)} + s^{(1)}/2} \Big| \fint_{q_1 - s^{(2)}/2}^{q_1 + s^{(2)}/2} w_\varepsilon\big(\varepsilon(z_1 - q_1) + q_1, z_2, z_3\big)\, dz_1 - w_\varepsilon(q_1, z_2, z_3) \Big|^2 dz_2 dz_3 = \\
&\quad = \int_0^\ell \int_{q_2^{(1)}}^{q_2^{(1)} + s^{(1)}/2} \Big| \fint_{q_1 - \varepsilon s^{(2)}/2}^{q_1 + \varepsilon s^{(2)}/2} w_\varepsilon(z_1, z_2, z_3) - w_\varepsilon(q_1, z_2, z_3)\, dz_1 \Big|^2 dz_2 dz_3 \\
&\quad = \int_0^\ell \int_{q_2^{(1)}}^{q_2^{(1)} + s^{(1)}/2} \Big| \fint_{q_1 - \varepsilon s^{(2)}/2}^{q_1 + \varepsilon s^{(2)}/2} \int_{q_1}^{z_1} D_1 w_\varepsilon(t, z_2, z_3)\, dt\, dz_1 \Big|^2 dz_2 dz_3 \\
&\quad \le s^{(2)} \varepsilon \int_0^\ell \int_{q_2^{(1)}}^{q_2^{(1)} + s^{(1)}/2} \int_{q_1 - \varepsilon s^{(2)}/2}^{q_1 + \varepsilon s^{(2)}/2} |D_1 w_\varepsilon(t, z_2, z_3)|^2\, dt dz_2 dz_3 \\
&\quad \le s^{(2)} \varepsilon \|D_1 w_\varepsilon\|^2_{L^2(\Omega^{(1)} \cap \Omega^{(2)})} \le C\varepsilon,
\end{aligned}$$

the claim follows by continuity of the trace. □

Lemma 5.2 *The following equalities hold for almost every $y_3 \in (0, \ell)$.*

1. $\xi_2^{(i)}(y_3) = 0, \quad i = 1, 3;$

2. $\xi_1^{(2)}(y_3) = 0;$

3. $\xi_3^{(i)}(y_3) - q_1 \xi_1^{(i)\prime}(y_3) = \xi_3^{(2)}(y_3) - q_2^{(i)} \xi_2^{(2)\prime}(y_3), \quad i = 1, 3.$

PROOF. As the pair $(\mathbf{u}_{\varepsilon_n}^{(1)}, \mathbf{u}_{\varepsilon_n}^{(2)})$ satisfies (19) with $i = 1$, averaging the second components with respect to z_1 we have

$$\fint_{q_1 - s^{(2)}/2}^{q_1 + s^{(2)}/2} u_{\varepsilon_n 2}^{(1)}\big(\varepsilon_n(z_1 - q_1) + q_1, z_2, z_3\big)\, dz_1 =$$
$$= \fint_{q_1 - s^{(2)}/2}^{q_1 + s^{(2)}/2} u_{\varepsilon_n 2}^{(2)}\big(z_1, \varepsilon_n(z_2 - q_2^{(1)}) + q_2^{(1)}, z_3\big)\, dz_1.$$

Applying Lemma 5.1 to the sequence $u_{\varepsilon_n 2}^{(1)}$ and using (16), we deduce that the left hand side of the equality above converges to $(z_2, z_3) \mapsto v_2^{(1)}(q_1, z_2, z_3)$ in $L^2((q_2^{(1)}, q_2^{(1)} + s^{(1)}/2) \times (0, \ell))$. On the other hand the right hand side converges to zero in the same space, indeed

$$\int_0^\ell \int_{q_2^{(1)}}^{q_2^{(1)} + s^{(1)}/2} \Big| \fint_{q_1 - s^{(2)}/2}^{q_1 + s^{(2)}/2} u_{\varepsilon_n 2}^{(2)}\big(z_1, \varepsilon_n(z_2 - q_2^{(1)}) + q_2^{(1)}, z_3\big)\, dz_1 \Big|^2 dz_2 dz_3 \leq$$
$$\leq \int_0^\ell \fint_{q_1 - s^{(2)}/2}^{q_1 + s^{(2)}/2} \int_{q_2^{(1)}}^{q_2^{(1)} + s^{(1)}/2} |u_{\varepsilon_n 2}^{(2)}\big(z_1, \varepsilon_n(z_2 - q_2^{(1)}) + q_2^{(1)}, z_3\big)|^2\, dz_2 dz_1 dz_3$$
$$\leq \int_0^\ell \fint_{q_1 - s^{(2)}/2}^{q_1 + s^{(2)}/2} \frac{1}{\varepsilon_n} \int_{q_2^{(1)}}^{q_2^{(1)} + \varepsilon_n s^{(1)}/2} |u_{\varepsilon_n 2}^{(2)}(z_1, z_2, z_3)|^2\, dz_2 dz_1 dz_3$$
$$\leq \frac{\varepsilon_n}{s^{(2)}} \int_{\Omega^{(2)}} \Big| \frac{u_{\varepsilon_n 2}^{(2)}}{\varepsilon_n} \Big|^2 dz$$

and the last term of the chain tends to zero due to (16). Taking into account (18), this proves *1* for $i = 1$. The case $i = 3$, follows by the same argument applied to the other junction condition. The item *2* of the statement, concerning $\xi_1^{(2)}$, can be proved similarly by considering the first component of (19) with $i = 1$.

In order to prove *3* we consider the following scaled average of the third component of (19) with $i = 1$

$$\fint_{q_2^{(1)}}^{q_2^{(1)}+s^{(1)}/2} \fint_{q_1-s^{(2)}/2}^{q_1+s^{(2)}/2} \frac{u^{(1)}_{\varepsilon_n 3}(\varepsilon_n(z_1-q_1)+q_1,z_2,z_3)}{\varepsilon_n^2}\,dz_1dz_2 =$$

$$= \fint_{q_1-s^{(2)}/2}^{q_1+s^{(2)}/2} \fint_{q_2^{(1)}}^{q_2^{(1)}+s^{(1)}/2} \frac{u^{(2)}_{\varepsilon_n 3}(z_1,\varepsilon_n(z_2-q_2^{(1)})+q_2^{(1)},z_3)}{\varepsilon_n^2}\,dz_2dz_1$$

and by applying twice Lemma 5.1 and (16) we deduce

$$\fint_{q_2^{(1)}}^{q_2^{(1)}+s^{(1)}/2} v_3^{(1)}(q_1,z_2,z_3)\,dz_2 = \fint_{q_1-s^{(2)}/2}^{q_1+s^{(2)}/2} v_3^{(2)}(z_1,q_2^{(1)},z_3)\,dz_1.$$

Taking into account (18) and the statements *1* and *2* of the present lemma we deduce item *3* for $i = 1$. The case $i = 3$ follows similarly by considering the second junction condition. □

To deduce further "limit" junction conditions we need the following two-dimensional Korn's inequality (see [8], Theorem 2.5)

$$\|\mathbf{w} - \wp\mathbf{w}\|_{H^1(\omega;\mathbb{R}^2)} \leq C\,\|\mathbf{E}\mathbf{w}\|_{L^2(\omega;\mathbb{R}^{2\times 2})}, \tag{20}$$

which holds for all $\mathbf{w} \in H^1(\omega;\mathbb{R}^2)$ and where ω is any Lipschitz bounded subset of $\mathbb{R}^2$. If we denote with $(y_1(G), y_2(G))$ the center of mass of ω, we have that the α-component of the projection of $\mathbf{w}$ on the space of "two-dimensional" infinitesimal rigid displacements, see [4], is (with the summation convention)

$$\wp\mathbf{w}_\alpha = t_\alpha(\mathbf{w}) + \mathcal{E}_{\beta\alpha}\,(y_\beta - y_\beta(G))\vartheta(\mathbf{w}) \tag{21}$$

where

$$\begin{aligned} \vartheta(\mathbf{w}) &= \frac{1}{I_G(\omega)}\int_\omega \mathcal{E}_{\gamma\delta}\,(y_\gamma - y_\gamma(G))w_\delta\,dy_1\,dy_2, \\ t_\alpha(\mathbf{w}) &= \frac{1}{|\omega|}\int_\omega w_\alpha\,dy_1\,dy_2. \end{aligned}$$

Above $I_G(\omega)$ denotes the polar moment of inertia of the section ω with respect to the center of mass,

$$I_G(\omega) := \int_\omega (y_1 - y_1(G))^2 + (y_2 - y_2(G))^2\,dy_1\,dy_2.$$

Analogously, $I_G(\omega^{(i)})$ will denote the polar moment of inertia with respect to the center of mass $(y_1(G^{(i)}), y_2(G^{(i)})$ of the section $\omega^{(i)}$. For later convenience we set

$$\mathbf{w}_\varepsilon^{(i)} := \frac{\mathbf{S}_\varepsilon^{(i)}\mathbf{u}_\varepsilon^{(i)}}{\varepsilon} \tag{22}$$

and

$$\vartheta_\varepsilon^{(i)} := \frac{1}{I_G(\omega^{(i)})} \int_{\omega^{(i)}} \mathcal{E}_{\gamma\delta}\,(y_\gamma - y_\gamma(G^{(i)}))w_{\varepsilon\delta}^{(i)}\,dy_1\,dy_2,$$

so that

$$(\wp \mathbf{w}_\varepsilon^{(i)})_\alpha = t_\alpha(\mathbf{w}_\varepsilon^{(i)}) + \mathcal{E}_{\beta\alpha}\,(y_\beta - y_\beta(G^{(i)}))\vartheta_\varepsilon^{(i)}. \tag{23}$$

Lemma 5.3 *There exists a constant $C > 0$ such that*

$$\|\mathbf{w}_\varepsilon^{(i)} - \wp \mathbf{w}_\varepsilon^{(i)}\|_{L^2(0,\ell;H^1(\omega^{(i)};\mathbb{R}^2))} \leq C\varepsilon,$$

for every $\varepsilon \in (0, 1]$.

Proof. Since $(\mathbf{E}\mathbf{w}_\varepsilon^{(i)})_{11} = \varepsilon^{(-1)^i}(\mathbf{E}_\varepsilon^{(i)}\mathbf{u}_\varepsilon^{(i)})_{11}$, $(\mathbf{E}\mathbf{w}_\varepsilon^{(i)})_{22} = \varepsilon^{(-1)^{i+1}}(\mathbf{E}_\varepsilon^{(i)}\mathbf{u}_\varepsilon^{(i)})_{22}$, and $(\mathbf{E}\mathbf{w}_\varepsilon^{(i)})_{12} = (\mathbf{E}_\varepsilon^{(i)}\mathbf{u}_\varepsilon^{(i)})_{12}$, we have

$$\|(\mathbf{E}\mathbf{w}_\varepsilon^{(i)})_{\alpha\beta}\|_{L^2(\Omega^{(i)})} \leq \frac{1}{\varepsilon}\|(\mathbf{E}_\varepsilon^{(i)}\mathbf{u}_\varepsilon^{(i)})_{\alpha\beta}\|_{L^2(\Omega^{(i)})}. \tag{24}$$

Hence, taking into account (20), (24) and (10), we have

$$\begin{aligned}\int_0^\ell \|\mathbf{w}_\varepsilon^{(i)} - \wp \mathbf{w}_\varepsilon^{(i)}\|^2_{H^1(\omega^{(i)};\mathbb{R}^2)} dy_3 &\leq C \int_0^\ell \sum_{\alpha\beta} \|(\mathbf{E}\mathbf{w}_\varepsilon^{(i)})_{\alpha\beta}\|^2_{L^2(\omega^{(i)})} dy_3 \\ &\leq \frac{C}{\varepsilon^2} \sum_{\alpha\beta} \|(\mathbf{E}_\varepsilon^{(i)}\mathbf{u}_\varepsilon^{(i)})_{\alpha\beta}\|^2_{L^2(\Omega^{(i)})} \leq C\varepsilon^2,\end{aligned}$$

which concludes the proof. □

Lemma 5.4 *We have*

1. *$\vartheta_{\varepsilon_n}^{(i)} \rightharpoonup \vartheta^{(i)}$ in $L^2(\Omega^{(i)})$. Therefore, $\vartheta^{(i)}$ does not depend on y_1 and y_2;*
2. *$\vartheta^{(i)} \in H^1_\#(0, \ell)$;*
3. *$\vartheta^{(1)}(y_3) = \vartheta^{(2)}(y_3) = \vartheta^{(3)}(y_3) =: \vartheta(y_3)$ for a.e. y_3 in $(0, \ell)$.*

Proof. Let's prove *1.* From Lemma 5.3 we have that

$$\|D_\alpha(\mathbf{w}_\varepsilon^{(i)} - \wp \mathbf{w}_\varepsilon^{(i)})\|_{L^2(\Omega^{(i)})} \leq C\varepsilon. \tag{25}$$

Since $(\mathbf{W}\wp \mathbf{w}_\varepsilon^{(i)})_{12} = -\vartheta_\varepsilon^{(i)}$ and $(\mathbf{W}\mathbf{w}_\varepsilon^{(i)})_{12} = (\mathbf{W}_\varepsilon^{(i)}\mathbf{u}_\varepsilon^{(i)})_{12}$, from the identity

$$\vartheta_\varepsilon^{(i)} = -(\mathbf{W}\wp \mathbf{w}_\varepsilon^{(i)})_{12} = -(\mathbf{W}_\varepsilon^{(i)}\mathbf{u}_\varepsilon^{(i)})_{12} + (\mathbf{W}(\mathbf{w}_\varepsilon^{(i)} - \wp \mathbf{w}_\varepsilon^{(i)}))_{12}, \tag{26}$$

and using (25), we get the following estimate

$$\|\vartheta_\varepsilon^{(i)} + (\mathbf{W}_\varepsilon^{(i)} \mathbf{u}_\varepsilon^{(i)})_{12}\|_{L^2(\Omega^{(i)})} = \|(\mathbf{W}(\mathbf{w}_\varepsilon^{(i)} - \wp \mathbf{w}_\varepsilon^{(i)}))_{12}\|_{L^2(\Omega^{(i)})} \leq C\varepsilon. \tag{27}$$

The claim *1* follows then by taking into account (17) and from the fact that, by definition, the functions $\vartheta_{\varepsilon_n}^{(i)}$ do not depend on y_1 and y_2.

Part *2* of the statement will be proven by following the same argument of Lemma 4.6 of [4]. Let $\xi \in C_0^\infty(\omega^{(i)})$ be such that

$$\int_{\omega^{(i)}} \xi \, dy_1 \, dy_2 = -\frac{I_G(\omega^{(i)})}{2}.$$

Then, taking into account (23), we have

$$\begin{aligned}
I_G(\omega^{(i)})\vartheta_\varepsilon^{(i)} &= -2\vartheta_\varepsilon^{(i)} \int_{\omega^{(i)}} \xi \, dy_1 \, dy_2 = -\vartheta_\varepsilon^{(i)} \int_{\omega^{(i)}} \xi D_\alpha y_\alpha \, dy_1 \, dy_2 \\
&= \vartheta_\varepsilon^{(i)} \int_{\omega^{(i)}} y_\alpha \, D_\alpha \xi \, dy_1 \, dy_2 = \vartheta_\varepsilon^{(i)} \int_{\omega^{(i)}} \mathcal{E}_{\alpha\gamma}\mathcal{E}_{\beta\gamma} y_\beta \, D_\alpha \xi \, dy_1 \, dy_2 \\
&= \int_{\omega^{(i)}} \mathcal{E}_{\alpha\gamma}(\mathcal{E}_{\beta\gamma} y_\beta \vartheta_\varepsilon^{(i)}) D_\alpha \xi \, dy_1 \, dy_2 \\
&= \int_{\omega^{(i)}} \mathcal{E}_{\alpha\gamma}\Big((\wp \mathbf{w}_\varepsilon^{(i)})_\gamma - \frac{1}{|\omega^{(i)}|}\int_{\omega^{(i)}} w_{\varepsilon\gamma}^{(i)} \, dy_1 \, dy_2\Big) D_\alpha \xi \, dy_1 \, dy_2 \\
&= \int_{\omega^{(i)}} \mathcal{E}_{\alpha\gamma}(\wp \mathbf{w}_\varepsilon^{(i)})_\gamma D_\alpha \xi \, dy_1 \, dy_2 \\
&= \int_{\omega^{(i)}} \mathcal{E}_{\alpha\gamma} w_{\varepsilon\gamma}^{(i)} D_\alpha \xi \, dy_1 \, dy_2 - \int_{\omega^{(i)}} \mathcal{E}_{\alpha\gamma}(\mathbf{w}_\varepsilon^{(i)} - \wp \mathbf{w}_\varepsilon^{(i)})_\gamma D_\alpha \xi \, dy_1 \, dy_2.
\end{aligned}$$

Hence, denoting by

$$\tilde{\vartheta}_\varepsilon^{(i)} := \frac{1}{I_G(\omega^{(i)})} \int_{\omega^{(i)}} \mathcal{E}_{\alpha\gamma} w_{\varepsilon\gamma}^{(i)} D_\alpha \xi \, dy_1 \, dy_2,$$

and recalling (25), we find

$$\vartheta_\varepsilon^{(i)} - \tilde{\vartheta}_\varepsilon^{(i)} \to 0 \text{ in } L^2(\Omega^{(i)}). \tag{28}$$

We now show that $D_3 \tilde{\vartheta}_\varepsilon^{(i)}$ is bounded in L^2. Since $\mathcal{E}_{\alpha\gamma} D_\alpha D_\gamma \xi = 0$ in $\omega^{(i)}$ and

$D_\alpha \xi = 0$ on $\partial\omega^{(i)}$, we have

$$
\begin{aligned}
I_G(\omega^{(i)}) D_3 \tilde{\vartheta}_\varepsilon^{(i)} &= \int_{\omega^{(i)}} \mathcal{E}_{\alpha\gamma} D_\alpha \xi \, D_3 w_{\varepsilon\gamma}^{(i)} \, dy_1 dy_2 \\
&= 2 \int_{\omega^{(i)}} \mathcal{E}_{\alpha\gamma} D_\alpha \xi \, (\mathbf{E}\mathbf{w}^\varepsilon)_{\gamma 3} \, dy_1 dy_2 \; - \int_{\omega^{(i)}} \mathcal{E}_{\alpha\gamma} D_\alpha \xi \, D_\gamma w_{\varepsilon 3}^{(i)} \, dy_1 dy_2 \\
&= 2 \int_{\omega^{(i)}} \mathcal{E}_{\alpha\gamma} D_\alpha \xi \, (\mathbf{E}\mathbf{w}_\varepsilon^{(i)})_{\gamma 3} \, dy_1 dy_2 - \int_{\omega^{(i)}} D_\gamma (\mathcal{E}_{\alpha\gamma} D_\alpha \xi \, w_{\varepsilon 3}^{(i)}) \, dy_1 dy_2 \\
&\qquad + \int_{\omega^{(i)}} \mathcal{E}_{\alpha\gamma} D_\alpha D_\gamma \xi w_{\varepsilon 3}^{(i)} \, dy_1 dy_2 \\
&= 2 \int_{\omega^{(i)}} \mathcal{E}_{\alpha\gamma} D_\alpha \xi \, (\mathbf{E}\mathbf{w}_\varepsilon^{(i)})_{\gamma 3} \, dy_1 dy_2,
\end{aligned}
$$

but $(\mathbf{E}\mathbf{w}^\varepsilon)_{13} = (\mathbf{E}^\varepsilon \mathbf{u}^\varepsilon)_{13}/\varepsilon^{\alpha(i+1)}$ and $(\mathbf{E}\mathbf{w}^\varepsilon)_{23} = (\mathbf{E}^\varepsilon \mathbf{u}^\varepsilon)_{23}/\varepsilon^{\alpha(i)}$, and therefore $D_3 \tilde{\vartheta}_\varepsilon^{(i)}$ is bounded in $L^2(0,\ell)$. Thus, from (28) and Lemma 5.4 we conclude that

$$
\tilde{\vartheta}_\varepsilon^{(i)} \rightharpoonup \vartheta^{(i)} \text{ in } H^1(\Omega^{(i)}).
$$

Therefore, since $\tilde{\vartheta}_\varepsilon^{(i)}(0) = 0$, we conclude that $\vartheta^{(i)} \in H^1_\#(0,\ell)$, that is *2*.

In order to prove *3*, let us prove the equality $\vartheta^{(i)}(y_3) = \vartheta^{(2)}(y_3)$ for $i = 1, 3$. By differentiating (5) we find

$$
\left.
\begin{aligned}
\varepsilon_n D_1 \mathbf{u}_{\varepsilon_n}^{(i)} \circ p_{\varepsilon_n}^{(i)^{-1}} &= D_1 \mathbf{u}_{\varepsilon_n}^{(2)} \circ p_{\varepsilon_n}^{(2)^{-1}} \\
D_2 \mathbf{u}_{\varepsilon_n}^{(i)} \circ p_{\varepsilon_n}^{(i)^{-1}} &= \varepsilon_n D_2 \mathbf{u}_{\varepsilon_n}^{(2)} \circ p_{\varepsilon_n}^{(2)^{-1}}
\end{aligned}
\right\} \quad \text{in } \Omega_{\varepsilon_n}^{(i)} \cap \Omega_{\varepsilon_n}^{(2)}
$$

which immediately lead to

$$
\left(\mathbf{W}_{\varepsilon_n}^{(i)} \mathbf{u}_{\varepsilon_n}^{(i)}\right)_{12} \circ p_{\varepsilon_n}^{(i)^{-1}} = \left(\mathbf{W}_{\varepsilon_n}^{(2)} \mathbf{u}_{\varepsilon_n}^{(2)}\right)_{12} \circ p_{\varepsilon_n}^{(2)^{-1}} \qquad \text{in } \Omega_{\varepsilon_n}^{(i)} \cap \Omega_{\varepsilon_n}^{(2)}.
$$

Then, from the equality

$$
\begin{aligned}
\vartheta_{\varepsilon_n}^{(2)}(z_3) - \vartheta_{\varepsilon_n}^{(i)}(z_3) = &\; \fint_{\omega_{\varepsilon_n}^{(i)} \cap \omega_{\varepsilon_n}^{(2)}} \left[\left(\mathbf{W}_{\varepsilon_n}^{(2)} \mathbf{u}_{\varepsilon_n}^{(2)}\right)_{12} \circ p_{\varepsilon_n}^{(2)^{-1}} + \vartheta_{\varepsilon_n}^{(2)}(z_3) \right] dx_1 dx_2 + \\
&- \fint_{\omega_{\varepsilon_n}^{(i)} \cap \omega_{\varepsilon_n}^{(2)}} \left[\left(\mathbf{W}_{\varepsilon_n}^{(i)} \mathbf{u}_{\varepsilon_n}^{(i)}\right)_{12} \circ p_{\varepsilon_n}^{(i)^{-1}} + \vartheta_{\varepsilon_n}^{(i)}(z_3) \right] dx_1 dx_2 \\
= &\; \frac{\varepsilon_n^3}{|\omega_{\varepsilon_n}^{(i)} \cap \omega_{\varepsilon_n}^{(2)}|} \int_{p_{\varepsilon_n}^{(2)^{-1}}(\omega_{\varepsilon_n}^{(i)}) \cap \omega^{(2)}} \left[\left(\mathbf{W}_{\varepsilon_n}^{(2)} \mathbf{u}_{\varepsilon_n}^{(2)}\right)_{12} + \vartheta_{\varepsilon_n}^{(2)}(z_3) \right] dy_1 dy_2 + \\
&- \frac{\varepsilon_n^3}{|\omega_{\varepsilon_n}^{(i)} \cap \omega_{\varepsilon_n}^{(2)}|} \int_{\omega^{(i)} \cap p_{\varepsilon_n}^{(i)^{-1}}(\omega_{\varepsilon_n}^{(2)})} \left[\left(\mathbf{W}_{\varepsilon_n}^{(i)} \mathbf{u}_{\varepsilon_n}^{(i)}\right)_{12} + \vartheta_{\varepsilon_n}^{(i)}(z_3) \right] dy_1 dy_2,
\end{aligned}
$$

where, with a small abuse of notation, ${p^{(i)}_{\varepsilon_n}}^{-1}(\omega^{(2)}_{\varepsilon_n})$ and ${p^{(2)}_{\varepsilon_n}}^{-1}(\omega^{(i)}_{\varepsilon_n})$ denote the inverse of the restriction to $\omega^{(i)}$ and $\omega^{(2)}$, respectively, of the projection on the first two factors of $p^{(i)}_{\varepsilon_n}$ and $p^{(2)}_{\varepsilon_n}$.

As $|\omega^{(i)}_{\varepsilon_n} \cap \omega^{(2)}_{\varepsilon_n}| = s^{(i)}s^{(2)}\varepsilon_n^4$, and using Hölder's inequality, we have

$$\begin{aligned}
&\int_0^\ell |\vartheta^{(i)}_{\varepsilon_n} - \vartheta^{(2)}_{\varepsilon_n}|\, dy_3 \le \\
&\le \frac{1}{s^{(i)}s^{(2)}\varepsilon_n} \int_0^\ell \int_{{p^{(2)}_{\varepsilon_n}}^{-1}(\omega^{(i)}_{\varepsilon_n})\cap\omega^{(2)}} \left| \left(\mathbf{W}^{(2)}_{\varepsilon_n}\mathbf{u}^{(2)}_{\varepsilon_n}\right)_{12} + \vartheta^{(2)}_{\varepsilon_n}(y_3) \right| dy_1 dy_2 dy_3 + \\
&\quad + \frac{1}{s^{(i)}s^{(2)}\varepsilon_n} \int_0^\ell \int_{\omega^{(i)}\cap{p^{(i)}_{\varepsilon_n}}^{-1}(\omega^{(2)}_{\varepsilon_n})} \left| \left(\mathbf{W}^{(i)}_{\varepsilon_n}\mathbf{u}^{(i)}_{\varepsilon_n}\right)_{12} + \vartheta^{(i)}_{\varepsilon_n}(y_3) \right| dy_1 dy_2 dy_3 \\
&\le C\frac{\varepsilon_n^{1/2}}{\varepsilon_n} \Big(\| \left(\mathbf{W}^{(2)}_{\varepsilon_n}\mathbf{u}^{(2)}_{\varepsilon_n}\right)_{12} + \vartheta^{(2)}_{\varepsilon_n} \|_{L^2(\Omega^{(2)})} + \| \left(\mathbf{W}^{(i)}_{\varepsilon_n}\mathbf{u}^{(i)}_{\varepsilon_n}\right)_{12} + \vartheta^{(i)}_{\varepsilon} \|_{L^2(\Omega^{(i)})} \Big) \\
&\le C\varepsilon_n^{1/2},
\end{aligned}$$

where the estimates (27) have been used to conclude the computation. Hence the right hand side goes to zero as $\varepsilon_n \to 0$ while the liminf of the left hand side is greater then the L^1-norm of $\vartheta^{(i)} - \vartheta^{(2)}$. Thus, $\vartheta^{(i)} = \vartheta^{(2)}$. □

5.1 The case without any junction

It is the simplest case of the rectangular cross-section which has been considered in detail in [4]. It can be obtained as a particular case of the present setting by taking $b^{(i)} = s^{(i)} = d^{(i)} = 0$ for $i = 1, 3$.

5.2 The case with only one junction

This case arises, for instance, when $b^{(3)} = s^{(3)} = d^{(3)} = 0$ and turns out to be simpler than the general case. Indeed the displacement fields $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$, by (18), can be written in terms of six fields $\xi^{(i)}_j$, $i = 1, 2$, $j = 1, 2, 3$, which depend only on the coordinate y_3. These six fields can be reduced, by using Lemma 5.2 to only three fields, which together with the rotation angle ϑ will fully describe the kinematics of the beam.

Lemma 5.5 *We have that*

$$\vartheta = \vartheta^{(1)} = \vartheta^{(2)} \in H^1_\#(0, \ell),$$

and there exist $\eta_1, \eta_2 \in H^2_\#(0,\ell)$ and $\eta_3 \in H^1_\#(0,\ell)$ such that

$$\begin{aligned} v_1^{(1)} &= \eta_1(y_3), & v_2^{(1)} &= 0, & v_3^{(1)} &= \eta_3(y_3) - y_1\eta_1'(y_3) - q_2^{(1)}\eta_2'(y_3), \\ v_1^{(2)} &= 0, & v_2^{(2)} &= \eta_2(y_3), & v_3^{(2)} &= \eta_3(y_3) - q_1\eta_1'(y_3) - y_2\eta_2'(y_3). \end{aligned} \tag{29}$$

for almost every $y_3 \in (0,\ell)$.

Proof. The first part of the statement has been already proven in Lemma 5.4. It remains to show that equalities (29) hold. That $v_1^{(2)} = v_2^{(1)} = 0$ follows from (18) and Lemma 5.2, and that $v_1^{(1)} = \eta_1(y_3)$ and $v_2^{(2)} = \eta_2(y_3)$ follows after setting $\eta_1 = \xi_1^{(1)}$ and $\eta_2 = \xi_2^{(2)}$. From *3* of Lemma 5.2 follows that

$$\xi_3^{(1)} - q_1\eta_1' = \xi_3^{(2)} - q_2^{(1)}\eta_2',$$

thus setting

$$\eta_3 := \xi_3^{(2)} + q_1\eta_1' = \xi_3^{(1)} + q_2^{(2)}\eta_2',$$

we deduce the desired expressions of $v_3^{(1)}$ and $v_3^{(2)}$. □

5.3 The case with two junctions

In the case with a beam section composed by three rectangles the displacement of the beam is described by the rotation angle ϑ and by nine fields $\xi_j^{(i)}$, $i, j = 1, 2, 3$, all depending only on the coordinate y_3. Lemma 5.2 gives us five conditions to which these fields have to satisfy. Thus, using these five conditions, we can reduce the ten original fields to only five fields and not to four as in the case of only one junction. This fact points out that we are still missing a junction condition.

The next lemma holds only if there are two distinct junctions.

Lemma 5.6 *We have that*

$$\xi_1^{(1)}(y_3) - \xi_1^{(3)}(y_3) = h\vartheta(y_3)$$

for almost every $y_3 \in (0,\ell)$. Moreover $\vartheta \in H^2_\#(0,\ell)$.

Proof. From the first component of (19), rescaling, integrating and making a change of variables, we obtain for a.e. $z_3 \in (0,\ell)$

$$\begin{aligned} \fint_{q_1-\varepsilon_n s^{(2)}/2}^{q_1+\varepsilon_n s^{(2)}/2} \fint_{q_2^{(1)}}^{q_2^{(1)}+s^{(1)}/2} \frac{u^{(1)}_{\varepsilon_n 1}}{\varepsilon_n}\, dz_2 dz_1 &= \fint_{q_1-s^{(2)}/2}^{q_1+s^{(2)}/2} \fint_{q_2^{(1)}}^{q_2^{(1)}+\varepsilon_n s^{(1)}/2} \frac{u^{(2)}_{\varepsilon_n 1}}{\varepsilon_n}\, dz_2 dz_1, \\ \fint_{q_1-\varepsilon_n s^{(2)}/2}^{q_1+\varepsilon_n s^{(2)}/2} \fint_{q_2^{(3)}-s^{(3)}/2}^{q_2^{(3)}} \frac{u^{(3)}_{\varepsilon_n 1}}{\varepsilon_n}\, dz_2 dz_1 &= \fint_{q_1-s^{(2)}/2}^{q_1+s^{(2)}/2} \fint_{q_2^{(3)}-\varepsilon_n s^{(3)}/2}^{q_2^{(3)}} \frac{u^{(2)}_{\varepsilon_n 1}}{\varepsilon_n}\, dz_2 dz_1. \end{aligned} \tag{30}$$

Recalling (22), the difference of the right hand sides of the two equations above can be rewritten as

$$\fint_{q_1-s^{(2)}/2}^{q_1+s^{(2)}/2} \Big(\fint_{q_2^{(1)}}^{q_2^{(1)}+\varepsilon_n s^{(1)}/2} w^{(2)}_{\varepsilon_n 1}\, dz_2 - \fint_{q_2^{(3)}-\varepsilon_n s^{(3)}/2}^{q_2^{(3)}} w^{(2)}_{\varepsilon_n 1}\, dz_2\Big) dz_1 =$$

$$= \fint_{q_1-s^{(2)}/2}^{q_1+s^{(2)}/2} \Big(\fint_{q_2^{(1)}}^{q_2^{(1)}+\varepsilon_n s^{(1)}/2} w^{(2)}_{\varepsilon_n 1} - (\wp \mathbf{w}^{(2)}_{\varepsilon_n})_1\, dz_2 +$$

$$- \fint_{q_2^{(3)}-\varepsilon_n s^{(3)}/2}^{q_2^{(3)}} w^{(2)}_{\varepsilon_n 1} - (\wp \mathbf{w}^{(2)}_{\varepsilon_n})_1\, dz_2\Big) dz_1 +$$

$$+ \fint_{q_1-s^{(2)}/2}^{q_1+s^{(2)}/2} \Big(\fint_{q_2^{(1)}}^{q_2^{(1)}+\varepsilon_n s^{(1)}/2} (\wp \mathbf{w}^{(2)}_{\varepsilon_n})_1\, dz_2 - \fint_{q_2^{(3)}-\varepsilon_n s^{(3)}/2}^{q_2^{(3)}} (\wp \mathbf{w}^{(2)}_{\varepsilon_n})_1\, dz_2\Big) dz_1,$$

but $(\wp \mathbf{w}^{(2)}_{\varepsilon_n})_1(z) = \big(\frac{q_2^{(1)} + q_2^{(3)}}{2} - z_2\big)\vartheta^{(2)}_{\varepsilon_n} + \fint_{\omega^{(2)}} w^{(2)}_{\varepsilon_n 1}\, dz_1 dz_2$ and therefore

$$\fint_{q_1-s^{(2)}/2}^{q_1+s^{(2)}/2} \Big(\fint_{q_2^{(1)}}^{q_2^{(1)}+\varepsilon_n s^{(1)}/2} (\wp \mathbf{w}^{(2)}_{\varepsilon_n})_1\, dz_2 - \fint_{q_2^{(3)}-\varepsilon_n s^{(3)}/2}^{q_2^{(3)}} (\wp \mathbf{w}^{(2)}_{\varepsilon_n})_1\, dz_2\Big) dz_1 =$$

$$= h\,\vartheta^{(2)}_{\varepsilon_n} - \varepsilon_n \frac{s^{(1)} + s^{(3)}}{2} \vartheta^{(2)}_{\varepsilon_n} \ .$$

Taking into account the identities above and using Lemma 5.3, from the difference of (30) we deduce

$$\lim_{n\to\infty} \int_0^\ell \Big| \fint_{q_1-\varepsilon_n s^{(2)}/2}^{q_1+\varepsilon_n s^{(2)}/2} \Big(\fint_{q_2^{(1)}}^{q_2^{(1)}+s^{(1)}/2} \frac{u^{(1)}_{\varepsilon_n 1}}{\varepsilon_n}\, dz_2 - \fint_{q_2^{(3)}-s^{(3)}/2}^{q_2^{(3)}} \frac{u^{(3)}_{\varepsilon_n 1}}{\varepsilon_n}\, dz_2\Big) dz_1 + $$

$$- h\,\vartheta^{(2)}_{\varepsilon_n}\Big|\, dz_3 = 0$$

and, applying Lemma 5.1, we get

$$\int_0^\ell \Big| \fint_{q_2^{(1)}}^{q_2^{(1)}+s^{(1)}/2} v^{(1)}_1(q_1, z_2, z_3)\, dz_2 - \fint_{q_2^{(3)}-\bar{s}/2}^{q_2^{(3)}} v^{(3)}_1(q_1, z_2, z_3)\, dz_2 - h\vartheta\Big|\, dy_3 = 0.$$

The statement of the lemma follows from (18). □

The next lemma, which summarizes the results for a beam with a section having two junctions, states that the transverse displacement of the cross section is described by a rigid translation, of components η_1 and η_2, and a rotation ϑ around a point $\mathbf{c}$.

Lemma 5.7 *For every* $\mathbf{c} = (c_1, c_2) \in \mathbb{R}^2$ *there exist* $\eta_1, \eta_2 \in H^2_\#(0,\ell)$ *and* $\eta_3 \in H^1_\#(0,\ell)$ *such that*

$$
\begin{aligned}
&v_1^{(i)} = \eta_1(y_3) - (q_2^{(i)} - c_2)\vartheta(y_3), \quad v_2^{(i)} = 0, && i = 1,3,\\
&v_1^{(2)} = 0, \qquad\qquad\qquad\qquad\qquad v_2^{(2)} = \eta_2(y_3) + (q_1 - c_1)\vartheta(y_3), &&
\end{aligned}
$$

$$
\begin{aligned}
v_3^{(i)} &= \eta_3(y_3) - y_1\eta_1'(y_3) - q_2^{(i)}\eta_2'(y_3) + \psi^{(i)}(y_1)\vartheta'(y_3), \quad i = 1,3,\\
v_3^{(2)} &= \eta_3(y_3) - q_1\eta_1'(y_3) - y_2\eta_2'(y_3) + \psi^{(2)}(y_2)\vartheta'(y_3),
\end{aligned}
\tag{31}
$$

where the so-called "sector coordinates"

$$
\begin{aligned}
\psi^{(i)}(y_1) &:= y_1(q_2^{(i)} - c_2) - q_1(q_2^{(i)} - c_2) - (q_1 - c_1)q_2^{(i)} + K, \quad i = 1,3,\\
\psi^{(2)}(y_2) &:= -(q_1 - c_1)y_2 + K,
\end{aligned}
$$

were $y_1 \in (q_1 + d^{(i)} - b^{(i)}, q_1 + d^{(i)})$ *and* $y_2 \in (q_2^{(1)}, q_2^{(3)})$*, are defined up to an additive constant* K*.*

Proof. The equalities $v_1^{(2)} = v_2^{(1)} = v_2^{(3)} = 0$ follow from (18) and Lemma 5.2. From (18) and Lemma 5.6, setting $\eta_1 := \xi_1^{(1)} + (q_2^{(1)} - c_2)\vartheta$, we deduce $v_1^{(1)} = \eta_1 - (q_2^{(1)} - c_2)\vartheta$ and $\xi_1^{(3)} = \eta_1 - (q_2^{(1)} + h - c_2)\vartheta = \eta_1 - (q_2^{(3)} - c_2)\vartheta$. Thus $v_1^{(3)} = \eta_1 - (q_2^{(3)} - c_2)\vartheta$. Setting $\eta_2 := \xi_2^{(2)} - (q_1 - c_1)\vartheta$, we have $v_2^{(2)} = \eta_2 + (q_1 - c_1)\vartheta$. Moreover, by definition, $\eta_1, \eta_2 \in H^2_\#(0,\ell)$.

Let K be any constant. Setting

$$
\eta_3 := \xi_3^{(2)} + q_1\eta_1' - K\vartheta',
$$

from (18) we immediately obtain the desired expression for $v_3^{(2)}$ and $\eta_3 \in H^1_\#(0,\ell)$.

From statement *3* of Lemma 5.2 we deduce

$$
\xi_3^{(i)} = \eta_3 - q_2^{(i)}\eta_2' + \big(- q_1(q_2^{(i)} - c_2) - (q_1 - c_1)q_2^{(i)} + K\big)\vartheta',
$$

from which follow the expressions for $v_3^{(i)}$, $i = 1,3$. □

We finally note that the sector coordinates depend on the coordinates of the point $\mathbf{c}$ and an additive constant K. For simplicity of notation, in the sequel we do not explicitly stress this dependence.

Remark 5.8 The main difference between Lemma 5.5 and 5.7, which respectively hold for one and two junction conditions, is that in the former ϑ is only once differentiable while in the latter it is twice differentiable. This higher regularity will be used to construct the recovery sequence in Section 7. On the other hand, the same procedure can be applied also in the case of only one junction by simply assuming ϑ smooth at the starting step (38).

We also note that formally we can deduce the displacements for a beam with only one junction from Lemma 5.7 by setting $c_1 = q_1$, $c_2 = q_2^{(1)}$ and $K = 0$.

Remark 5.9 The same technique can be used to treat hollow cross-sections. In this case it can be proved that the twist angle ϑ vanishes, which suggests that a different scaling of the rotation of the section, at level ε, is needed. This will require a further study.

6 A liminf inequality

In this section we deduce a lower bound for the limiting energy. We start by deducing the convergence of some of the components of the rescaled strain. The results of this section hold, with straightforward adaptations, also in the case of only one junction.

Lemma 6.1 *We have, up to subsequences,*

$$\begin{aligned}
&\frac{(\mathbf{E}^{(i)}_{\varepsilon_n}\mathbf{u}^{(i)}_{\varepsilon_n})_{33}}{\varepsilon_n^2} \rightharpoonup D_3 v_3^{(i)} = \eta_3' - y_1\eta_1'' - q_2^{(i)}\eta_2'' + \psi^{(i)}(y_1)\vartheta'' \quad \text{in } L^2(\Omega^{(i)}),\ i = 1,3,\\
&\frac{(\mathbf{E}^{(2)}_{\varepsilon_n}\mathbf{u}^{(2)}_{\varepsilon_n})_{33}}{\varepsilon_n^2} \rightharpoonup D_3 v_3^{(2)} = \eta_3' - q_1\eta_1'' - y_2\eta_2'' + \psi^{(2)}(y_2)\vartheta'' \quad \text{in } L^2(\Omega^{(2)}),\\
&\frac{(\mathbf{E}^{(i)}_{\varepsilon_n}\mathbf{u}^{(i)}_{\varepsilon_n})_{13}}{\varepsilon_n^2} \rightharpoonup -(y_2 - q_2^{(i)})\vartheta' + \eta^{(i)} \quad \text{in } L^2(\Omega^{(i)}),\ i = 1,3,\\
&\frac{(\mathbf{E}^{(2)}_{\varepsilon_n}\mathbf{u}^{(2)}_{\varepsilon_n})_{23}}{\varepsilon_n^2} \rightharpoonup +(y_1 - q_1)\vartheta' + \eta^{(2)} \quad \text{in } L^2(\Omega^{(2)}),
\end{aligned} \tag{32}$$

where $\eta^{(i)} \in L^2(\Omega^{(i)})$, $i = 1,3$, are independent of y_2, while $\eta^{(2)} \in L^2(\Omega^{(2)})$ is independent of y_1.

Proof. To prove the first and the second of the (32) it suffices to notice that $(\mathbf{E}^{(i)}_\varepsilon \mathbf{u}^{(i)}_\varepsilon)_{33} = D_3 u^{(i)}_{\varepsilon 3}$, divide by ε^2 and apply (16).

The statements concerning the quantities defined on $\Omega^{(2)}$ are proven in [4], Lemma 4.7. The similar statements on the quantities defined on $\Omega^{(i)}$, $i = 1,3$, can be obtained by carefully exchanging the exponents of ε.

From (10) we deduce that, up to subsequences, $\dfrac{(\mathbf{E}_\varepsilon^{(i)}\mathbf{u}_\varepsilon^{(i)})_{\alpha 3}}{\varepsilon^2} \rightharpoonup E_{\alpha 3}^{(i)}$ in $L^2(\Omega^{(i)})$. To characterize $E_{\alpha 3}^{(i)} \in L^2(\Omega^{(i)})$, note that

$$\begin{aligned}
2D_3(\mathbf{W}_\varepsilon^{(i)}\mathbf{u}_\varepsilon^{(i)})_{12} &= D_3\left(\tfrac{D_2 u_{\varepsilon 1}^{(i)}}{\varepsilon^{\alpha(i+1)}} - \tfrac{D_1 u_{\varepsilon 2}^{(i)}}{\varepsilon^{\alpha(i)}}\right)\\
&= D_2\left(\frac{D_3 u_{\varepsilon 1}^{(i)}}{\varepsilon^{\alpha(i+1)}} + \frac{D_1 u_{\varepsilon 3}^{(i)}}{\varepsilon^3}\right) - D_1\left(\frac{D_2 u_{\varepsilon 3}^{(i)}}{\varepsilon^3} + \frac{D_3 u_{\varepsilon 2}^{(i)}}{\varepsilon^{\alpha(i)}}\right)\\
&= 2D_2\frac{(\mathbf{E}_\varepsilon^{(i)}\mathbf{u}_\varepsilon^{(i)})_{13}}{\varepsilon^{\alpha(i+1)}} - 2D_1\frac{(\mathbf{E}_\varepsilon^{(i)}\mathbf{u}_\varepsilon^{(i)})_{23}}{\varepsilon^{\alpha(i)}},
\end{aligned}$$

in the sense of distributions. Hence for $\psi \in C_0^\infty(\Omega^{(i)})$ we have

$$\int_{\Omega^{(i)}} (\mathbf{W}_\varepsilon^{(i)}\mathbf{u}_\varepsilon^{(i)})_{12} D_3\psi\, dy = \int_{\Omega^{(i)}} \frac{(\mathbf{E}_\varepsilon^{(i)}\mathbf{u}_\varepsilon^{(i)})_{13}}{\varepsilon^{\alpha(i+1)}} D_2\psi\, dy - \int_{\Omega^{(i)}} \frac{(\mathbf{E}_\varepsilon^{(i)}\mathbf{u}_\varepsilon^{(i)})_{23}}{\varepsilon^{\alpha(i)}} D_1\psi\, dy.$$

On the other hand, using (10) we have that

$$\begin{aligned}
&\frac{(\mathbf{E}_\varepsilon^{(i)}\mathbf{u}_\varepsilon^{(i)})_{13}}{\varepsilon^{\alpha(i+1)}} \rightharpoonup 0 \text{ in } L^2(\Omega^{(i)}), \quad \text{if } i=2\\
&\frac{(\mathbf{E}_\varepsilon^{(i)}\mathbf{u}_\varepsilon^{(i)})_{23}}{\varepsilon^{\alpha(i)}} \rightharpoonup 0 \text{ in } L^2(\Omega^{(i)}), \quad \text{if } i=1,3.
\end{aligned}$$

Hence, passing to the limit in the previous equality we find

$$\begin{aligned}
\int_{\Omega^{(i)}} -\vartheta D_3\psi\, dy &= -\int_{\Omega^{(i)}} E_{23}^{(i)} D_1\psi\, dy, \quad &&\text{if } i=2,\\
\int_{\Omega^{(i)}} -\vartheta D_3\psi\, dy &= \int_{\Omega^{(i)}} E_{13}^{(i)} D_2\psi\, dy, &&\text{if } i=1,3.
\end{aligned}$$

Thus

$$\begin{aligned}
D_1 E_{23}^{(i)} &= D_3\vartheta, \qquad &&\text{if } i=2,\\
D_2 E_{13}^{(i)} &= -D_3\vartheta, &&\text{if } i=1,3.
\end{aligned}$$

in the sense of distributions, and therefore, taking into account that ϑ does not depend on y_1 and y_2 we have that

$$\begin{aligned}
E_{23}^{(i)} &= y_1 D_3\vartheta + \gamma^{(i)}, \qquad &&\text{if } i=2,\\
E_{13}^{(i)} &= -y_2 D_3\vartheta + \gamma^{(i)}, &&\text{if } i=1,3.
\end{aligned}$$

with $\gamma^{(i)}$ independent of y_2 if $i=1,3$ and of y_1 if $i=2$. The conclusion is obtained by setting $\eta^{(i)} := \gamma^{(i)} - q_2 D_3\vartheta$ if $i=1,3$ and $\eta^{(2)} := \gamma^{(2)} + q_1 D_3\vartheta$. □

Remark 6.2 In order to shorten notation, let us observe that the statement of Lemma 6.1 can be summarized as follows

$$\frac{(\mathbf{E}^{(i)}_{\varepsilon_n}\mathbf{u}^{(i)}_{\varepsilon_n})_{\alpha(i)3}}{\varepsilon_n^2} \rightharpoonup \mathcal{E}_{\beta\alpha(i)}(y_\beta - q^{(i)}_\beta)\vartheta' + \eta^{(i)} \text{ in } L^2(\Omega^{(i)}), \quad i = 1, 2, 3,$$

where we agree that $q^{(i)}_1 := q_1$ for any i, and $\alpha(i)$ is the parity of the index i.

As previously said, we consider homogeneous isotropic materials and denote by

$$f(\mathbf{A}) := \frac{1}{2}\mathbb{C}\mathbf{A}\cdot\mathbf{A} = \mu|\mathbf{A}|^2 + \frac{\lambda}{2}|\text{tr}\mathbf{A}|^2, \tag{33}$$

the stored energy density. Let

$$\begin{aligned} f_0(\alpha,\beta) : &= \min\{f(\mathbf{A}) \ : \ \mathbf{A} \in \text{Sym},\ A_{23} = \alpha,\ A_{33} = \beta\} \\ &= \min\{f(\mathbf{A}) \ : \ \mathbf{A} \in \text{Sym},\ A_{13} = \alpha,\ A_{33} = \beta\} \end{aligned}$$

Let us remark that, by isotropy, in the definition of f_0, A_{23} can be replaced with A_{13}. A simple computation shows that

$$f_0(\alpha,\beta) = 2\mu\alpha^2 + \frac{1}{2}E\beta^2 \tag{34}$$

where E is the Young modulus, which is given by

$$E = \frac{\mu(2\mu + 3\lambda)}{\mu + \lambda}.$$

Theorem 6.3 *For every sequence of positive numbers ε_n converging to* 0 *and for every sequence $(\mathbf{u}^{(1)}_{\varepsilon_n}, \mathbf{u}^{(2)}_{\varepsilon_n}, \mathbf{u}^{(3)}_{\varepsilon_n}) \in \mathcal{A}_{\varepsilon_n}$ which satisfies* (32), *we have*

$$\liminf_{n\to+\infty} \frac{1}{\varepsilon_n^4} I_{\varepsilon_n}(\mathbf{u}^{(1)}_{\varepsilon_n}, \mathbf{u}^{(2)}_{\varepsilon_n}, \mathbf{u}^{(3)}_{\varepsilon_n}) \geq \sum_{i=1}^{3} \int_{\Omega^{(i)}} f_0\big(\mathcal{E}_{\beta\alpha(i)}(y_\beta - q^{(i)}_\beta)\vartheta', D_3 v^{(i)}_3\big)\, dy,$$

where $q^{(i)}_1 := q_1$.

Proof. Taking into account the decomposition given in (7), it suffices to show that

$$\liminf_{n\to+\infty} \frac{1}{\varepsilon_n^4} I^{(i)}_{\varepsilon_n}(\mathbf{u}^{(i)}_{\varepsilon_n}) \geq \int_{\Omega^{(i)}} f_0\big(\mathcal{E}_{\beta\alpha(i)}(y_\beta - q^{(i)}_\beta)\vartheta', D_3 v^{(i)}_3\big)\, dy.$$

Looking at the expression (8) of the functional $I^{(i)}_\varepsilon$ and observing that, by the definitions of f and f_0 given in (33) and (34),

$$\frac{1}{2}\mathbb{C}\mathbf{A}\cdot\mathbf{A} \geq f_0(A_{\alpha(i)3}, A_{33}),$$

we have

$$\frac{1}{\varepsilon_n^4} I_{\varepsilon_n}^{(i)}(\mathbf{u}_{\varepsilon_n}^{(i)}) \geq \int_{\Omega^{(i)}} \chi_{\varepsilon_n}^{(i)} f_0\Big(\frac{(\mathbf{E}_{\varepsilon_n}^{(i)}\mathbf{u}_{\varepsilon_n}^{(i)})_{\alpha(i)3}}{\varepsilon_n^2}, \frac{(\mathbf{E}_{\varepsilon_n}^{(3)}\mathbf{u}_{\varepsilon_n}^{(3)})_{33}}{\varepsilon_n^2}\Big)\, dy.$$

Using the convexity of f_0,

$$\begin{aligned}
&\int_{\Omega^{(i)}} \chi_{\varepsilon_n}^{(i)} f_0\Big(\frac{(\mathbf{E}_{\varepsilon_n}^{(3)}\mathbf{u}_{\varepsilon_n}^{(i)})_{\alpha(i)3}}{\varepsilon_n^2}, \frac{(\mathbf{E}_{\varepsilon_n}^{(i)}\mathbf{u}_{\varepsilon_n}^{(i)})_{33}}{\varepsilon_n^2}\Big)\, dy \geq \\
&\qquad \geq \int_{\Omega^{(i)}} \chi_{\varepsilon_n}^{(i)} 4\mu \Big[\frac{(\mathbf{E}_{\varepsilon_n}^{(i)}\mathbf{u}_{\varepsilon_n}^{(i)})_{\alpha(i)3}}{\varepsilon_n^2} - \mathcal{E}_{\beta\alpha(i)}(y_\beta - q_\beta^{(i)})\vartheta' - \eta^{(i)}\Big] \cdot \\
&\qquad\qquad \cdot \big[\mathcal{E}_{\beta\alpha(i)}(y_\beta - q_\beta^{(i)})\vartheta' + \eta^{(i)}\big]\, dy \\
&\qquad + \int_{\Omega^{(i)}} \chi_{\varepsilon_n}^{(i)} E\Big[\frac{(\mathbf{E}_{\varepsilon_n}^{(i)}\mathbf{u}_{\varepsilon_n}^{(i)})_{33}}{\varepsilon_n^2} - D_3 v_3^{(i)}\Big] D_3 v_3^{(i)}\, dy \\
&\qquad + \int_{\Omega^{(i)}} \chi_{\varepsilon_n}^{(i)} f_0\big(\mathcal{E}_{\beta\alpha(i)}(y_\beta - q_\beta^{(i)})\vartheta' + \eta^{(i)}, D_3 v_3^{(i)}\big)\, dy
\end{aligned}$$

and by Lemma 6.1 we find

$$\liminf_{k\to+\infty} \frac{1}{\varepsilon_n^4} I_{\varepsilon_n}^{(i)}(\mathbf{u}_{\varepsilon_n}^{(i)}) \geq \int_{\Omega^{(i)}} f_0\big(\mathcal{E}_{\beta\alpha(i)}(y_\beta - q_\beta^{(i)})\vartheta' + \eta^{(i)}, D_3 v_3^{(i)}\big)\, dy.$$

On the other hand, by (34),

$$\begin{aligned}
&\int_{\Omega^{(i)}} f_0\big(\mathcal{E}_{\beta\alpha(i)}(y_\beta - q_\beta^{(i)})\vartheta' + \eta^{(i)}, D_3 v_3^{(i)}\big)\, dy = \\
&\quad = \int_{\Omega^{(i)}} f_0(\mathcal{E}_{\beta\alpha(i)}(y_\beta - q_\beta^{(i)})\vartheta', D_3 v_3^{(i)}) + \\
&\qquad + 4\mu \int_{\Omega^{(i)}} \mathcal{E}_{\beta\alpha(i)}(y_\beta - q_\beta^{(i)})\vartheta' \eta^{(i)}\, dy + 2\mu \int_{\Omega^{(i)}} {\eta^{(i)}}^2\, dy.
\end{aligned}$$

and the proof is concluded by observing that

$$\int_{\Omega^{(i)}} \mathcal{E}_{\beta\alpha(i)}(y_\beta - q_\beta^{(i)})\vartheta' \eta^{(i)}\, dy = 0,$$

because $\eta^{(i)}$ is independent of $y_{3-\alpha(i)}$, by Lemma 6.1. □

7 The limit energy

The content of this section refers to the case of two junctions, but it can be adapted with straightforward modifications to the case of a single junction.

Let

$$\mathcal{A} := \Big\{ (\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \mathbf{v}^{(3)}, \vartheta) \in H_{BN} \times H^2_{\#}(0,\ell) : \\ : \ \exists\, \eta_1, \eta_2 \in H^2_{\#}(0,\ell), \text{ and } \eta_3 \in H^1_{\#}(0,\ell) \text{ satisfying (31)} \Big\}. \tag{35}$$

Theorem 7.1 *Let $I : H^1 \times H^2(0,\ell) \to [0,+\infty)$ be defined by*

$$I(\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \mathbf{v}^{(3)}, \vartheta) := \begin{cases} \sum_{i=1}^{3} I^{(i)}(\mathbf{v}^{(i)}, \vartheta) & \text{if } (\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \mathbf{v}^{(3)}, \vartheta) \in \mathcal{A}, \\ +\infty & \text{otherwise,} \end{cases} \tag{36}$$

where

$$I^{(i)}(\mathbf{v}^{(i)}, \vartheta) = \int_{\Omega^{(i)}} f_0\big(\mathcal{E}_{\beta\alpha(i)}(y_\beta - q^{(i)}_\beta)\vartheta', D_3 v^{(i)}_3\big)\, dy.$$

As $\varepsilon \to 0^+$, the sequence of functionals $\frac{1}{\varepsilon^4} I_\varepsilon$ Γ-converges to the functional I, in the following sense:

1. *[liminf inequality] for every sequence of positive numbers ε_n converging to 0 and for every sequence $(\mathbf{u}^{(1)}_{\varepsilon_n}, \mathbf{u}^{(2)}_{\varepsilon_n}, \mathbf{u}^{(3)}_{\varepsilon_n}) \in \mathcal{A}_{\varepsilon_n}$ such that*

$$\mathbf{S}^{(i)}_{\varepsilon_n} \mathbf{u}^{(i)}_{\varepsilon_n} \rightharpoonup \mathbf{v}^{(i)} \text{ in } H^1(\Omega^{(i)}; \mathbb{R}^3),$$

$$(\mathbf{W}^{(i)}_{\varepsilon_n} \mathbf{u}^{(i)}_{\varepsilon_n})_{12} \rightharpoonup -\vartheta \text{ in } L^2(\Omega^{(i)}),$$

we have

$$\liminf_{n\to+\infty} \frac{1}{\varepsilon_n^4} I_{\varepsilon_n}(\mathbf{u}^{(1)}_{\varepsilon_n}, \mathbf{u}^{(2)}_{\varepsilon_n}, \mathbf{u}^{(3)}_{\varepsilon_n}) \geq I(\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \mathbf{v}^{(3)}, \vartheta);$$

2. *[recovery sequence] for every sequence of positive numbers ε_n converging to 0 and for every $(\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \mathbf{v}^{(3)}, \vartheta) \in \mathcal{A}$ there exists a sequence $(\mathbf{u}^{(1)}_n, \mathbf{u}^{(2)}_n, \mathbf{u}^{(3)}_n) \in \mathcal{A}_{\varepsilon_n}$ such that*

$$\mathbf{S}^{(i)}_{\varepsilon_n} \mathbf{u}^{(i)}_n \rightharpoonup \mathbf{v}^{(i)} \text{ in } H^1(\Omega^{(i)}; \mathbb{R}^3),$$

$$(\mathbf{W}^{(i)}_{\varepsilon_n} \mathbf{u}^{(i)}_n)_{12} \rightharpoonup -\vartheta \text{ in } L^2(\Omega^{(i)}),$$

and

$$\limsup_{n\to+\infty} \frac{1}{\varepsilon_n^4} I_{\varepsilon_n}(\mathbf{u}^{(1)}_n, \mathbf{u}^{(2)}_n, \mathbf{u}^{(3)}_n) \leq I(\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \mathbf{v}^{(3)}, \vartheta).$$

Proof. We start by proving the liminf inequality. Without loss of generality we may suppose that

$$\liminf_{n\to+\infty} \frac{1}{\varepsilon_n^4} I_{\varepsilon_n}(\mathbf{u}^{(1)}_{\varepsilon_n}, \mathbf{u}^{(2)}_{\varepsilon_n}, \mathbf{u}^{(3)}_{\varepsilon_n}) < +\infty,$$

and therefore that

$$\sup_n \frac{1}{\varepsilon_n^4} I_{\varepsilon_n}(\mathbf{u}^{(1)}_{\varepsilon_n}, \mathbf{u}^{(2)}_{\varepsilon_n}, \mathbf{u}^{(3)}_{\varepsilon_n}) < +\infty.$$

Thus the assumptions of Lemma 4.1 are satisfied and therefore Lemma 4.3, Lemma 5.7 and Lemma 6.1 hold. The liminf inequality then follows from Theorem 6.3.

We now find a recovery sequence. Let us first note that

$$f_0(\alpha,\beta) = f(\mathbf{\Lambda}^{(i)}(\alpha,\beta)),$$

where $\mathbf{\Lambda}^{(i)}$ are symmetric matrices with

$$\begin{aligned}
&\Lambda^{(i)}_{11}(\alpha,\beta) = \Lambda^{(i)}_{22}(\alpha,\beta) = -\nu\beta, \quad \Lambda^{(i)}_{12}(\alpha,\beta) = 0, \quad \Lambda^{(i)}_{33}(\alpha,\beta) = \beta, \quad i = 1,2,3,\\
&\Lambda^{(i)}_{23}(\alpha,\beta) = \Lambda^{(2)}_{13}(\alpha,\beta) = 0, \quad \Lambda^{(i)}_{13}(\alpha,\beta) = \Lambda^{(2)}_{23}(\alpha,\beta) = \alpha, \quad i = 1,3,
\end{aligned} \tag{37}$$

and ν denotes the Poisson's coefficient

$$\nu = \frac{\lambda}{2(\lambda+\mu)}.$$

Let us assume that $I(\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \mathbf{v}^{(3)}, \vartheta) < +\infty$, otherwise there is nothing to prove. Then $(\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \mathbf{v}^{(3)}, \vartheta) \in \mathcal{A}$ and therefore there exist $\eta_1, \eta_2 \in H^2_\#(0,\ell)$, and $\eta_3 \in H^1_\#(0,\ell)$ such that (31) hold.

To start, we further assume η_1, η_2, η_3 and ϑ to be equal to zero in a neighborhood of $y_3 = 0$. Let $\mathbf{u}_{0,\varepsilon} : \Omega_\varepsilon \to \mathbb{R}^3$, be the sequence of functions defined by

$$\mathbf{u}_{0,\varepsilon} := \mathbf{u}_{f,\varepsilon} + \mathbf{u}_{t,\varepsilon} \tag{38}$$

with

$$\begin{aligned}
(u_{f,\varepsilon})_1 &:= \varepsilon\eta_1 - \nu\big(\varepsilon^2 x_1\eta_3' + \frac{\varepsilon}{2}(x_2^2 - x_1^2)\eta_1'' - \varepsilon x_1 x_2 \eta_2''\big),\\
(u_{f,\varepsilon})_2 &:= \varepsilon\eta_2 - \nu\big(\varepsilon^2 x_2\eta_3' - \varepsilon x_1 x_2\eta_1'' + \frac{\varepsilon}{2}(x_1^2 - x_2^2)\eta_2''\big),\\
(u_{f,\varepsilon})_3 &:= \varepsilon^2\eta_3 - \varepsilon x_1\eta_1' - \varepsilon x_2\eta_2',
\end{aligned} \tag{39}$$

and

$$\begin{aligned}
(u_{t,\varepsilon})_1 &:= -(x_2 - \varepsilon c_2)\vartheta - \nu r_1^\varepsilon \vartheta'',\\
(u_{t,\varepsilon})_2 &:= (x_1 - \varepsilon c_1)\vartheta - \nu r_2^\varepsilon \vartheta'',\\
(u_{t,\varepsilon})_3 &:= \psi^\varepsilon \vartheta',
\end{aligned} \tag{40}$$

where

$$r_1^\varepsilon(x_1,x_2) := \begin{cases} \varepsilon^3 \Big(\displaystyle\int_{q_1}^{y_1} \psi^{(1)}(s)\,ds\,\kappa_\varepsilon^{(1)}\Big) \circ p_\varepsilon^{(1)^{-1}} & \text{in } \omega_\varepsilon^{(1)}, \\ \varepsilon^4 (\psi^{(2)}(y_2) y_1 \check{\kappa}_\varepsilon^{(2)} \hat{\kappa}_\varepsilon^{(2)}) \circ p_\varepsilon^{(2)^{-1}} & \text{in } \omega_\varepsilon^{(2)}, \\ \varepsilon^3 \Big(\displaystyle\int_{q_1}^{y_1} \psi^{(3)}(s)\,ds\,\kappa_\varepsilon^{(3)}\Big) \circ p_\varepsilon^{(3)^{-1}} & \text{in } \omega_\varepsilon^{(3)}, \end{cases}$$

$$r_2^\varepsilon(x_1,x_2) := \begin{cases} \varepsilon^4 (\psi^{(1)}(y_1) y_2\, \kappa_\varepsilon^{(1)}) \circ p_\varepsilon^{(1)^{-1}} & \text{in } \omega_\varepsilon^{(1)}, \\ \varepsilon^3 \Big(\displaystyle\int_{q_2^{(1)}}^{y_2} \psi^{(2)}(s)\,ds \frac{\hat{\kappa}_\varepsilon^{(2)} + \check{\kappa}_\varepsilon^{(2)}}{2}\Big) \circ p_\varepsilon^{(2)^{-1}} & \text{in } \omega_\varepsilon^{(2)}, \\ \varepsilon^4 (\psi^{(3)}(y_1) y_2\, \kappa_\varepsilon^{(3)}) \circ p_\varepsilon^{(3)^{-1}} & \text{in } \omega_\varepsilon^{(3)}, \end{cases}$$

and

$$\psi^\varepsilon(x_1,x_2) := \begin{cases} \varepsilon^2 \psi_\varepsilon^{(1)} \circ p_\varepsilon^{(1)^{-1}} & \text{in } \omega_\varepsilon^{(1)}, \\ \varepsilon^2 \psi_\varepsilon^{(2)} \circ p_\varepsilon^{(2)^{-1}} & \text{in } \omega_\varepsilon^{(2)}, \\ \varepsilon^2 \psi_\varepsilon^{(3)} \circ p_\varepsilon^{(3)^{-1}} & \text{in } \omega_\varepsilon^{(3)}, \end{cases}$$

where

$$\begin{aligned} \psi_\varepsilon^{(1)} &:= -\varepsilon(y_1-q_1)(y_2-q_2^{(1)})\kappa_\varepsilon^{(1)} + \psi^{(1)} - \varepsilon(q_1-c_1)(y_2-q_2^{(1)}), \\ \psi_\varepsilon^{(2)} &:= \check{\kappa}_\varepsilon^{(2)} \hat{\kappa}_\varepsilon^{(2)} (\psi^{(2)} + \varepsilon(y_1-q_1)(y_2-c_2)) + \\ &\quad + (1-\check{\kappa}_\varepsilon^{(2)})(\psi^{(2)} + \varepsilon(y_1-q_1)(q_2^{(3)}-c_2)) + \\ &\quad + (1-\hat{\kappa}_\varepsilon^{(2)})(\psi^{(2)} + \varepsilon(y_1-q_1)(q_2^{(1)}-c_2)), \\ \psi_\varepsilon^{(3)} &:= -\varepsilon(y_1-q_1)(y_2-q_2^{(3)})\kappa_\varepsilon^{(3)} + \psi^{(3)} - \varepsilon(q_1-c_1)(y_2-q_2^{(3)}). \end{aligned}$$

In the definitions above the cut-off functions $\kappa_\varepsilon^{(1)} : \omega^{(1)} \to [0,1]$, $\kappa_\varepsilon^{(3)} : \omega^{(3)} \to [0,1]$, and $\check{\kappa}_\varepsilon^{(2)}, \hat{\kappa}_\varepsilon^{(2)} : \omega^{(2)} \to [0,1]$, have the following properties

$$D_2\kappa_\varepsilon^{(1)} = D_2\kappa_\varepsilon^{(3)} = D_1\check{\kappa}_\varepsilon^{(2)} = D_1\hat{\kappa}_\varepsilon^{(2)} = 0,$$

$$|D_1\kappa_\varepsilon^{(1)}|, |D_1\kappa_\varepsilon^{(3)}|, |D_2\check{\kappa}_\varepsilon^{(2)}|, |D_2\hat{\kappa}_\varepsilon^{(2)}| \le \frac{2}{\varepsilon},$$

$$\kappa_\varepsilon^{(3)}(y_1,y_2) = \kappa_\varepsilon^{(1)}(y_1,y_2) = 0 \text{ for } q_1 - \varepsilon\frac{s^{(2)}}{2} \le y_1 \le q_1 + \varepsilon\frac{s^{(2)}}{2},$$

$$\kappa_\varepsilon^{(3)}(y_1,y_2) = \kappa_\varepsilon^{(1)}(y_1,y_2) = 1 \text{ for } y_1 \ge q_1 + \varepsilon\frac{s^{(2)}}{2} + \varepsilon \text{ and } y_1 \le q_1 - \varepsilon\frac{s^{(2)}}{2} - \varepsilon,$$

$$\hat{\kappa}^{(2)}_\varepsilon(y_1,y_2)=0 \text{ for } y_2\le q_2^{(1)}+\varepsilon\frac{s^{(1)}}{2},\quad \hat{\kappa}^{(2)}_\varepsilon(y_1,y_2)=1 \text{ for } y_2\ge q_2^{(1)}+\varepsilon\frac{s^{(1)}}{2}+\varepsilon,$$

$$\check{\kappa}^{(2)}_\varepsilon(y_1,y_2)=0 \text{ for } y_2\ge q_2^{(3)}-\varepsilon\frac{s^{(3)}}{2},\quad \check{\kappa}^{(2)}_\varepsilon(y_1,y_2)=1 \text{ for } y_2\le q_2^{(3)}-\varepsilon\frac{s^{(3)}}{2}-\varepsilon.$$

It can be easily checked that ψ^ε is well defined.

From the definitions above one can easily compute $\mathbf{u}^{(i)}_{0,\varepsilon}:=\mathbf{u}^{(i)}_{0,\varepsilon}\circ p^{(i)}_\varepsilon$ and verify that $\mathbf{u}^{(i)}_{0,\varepsilon}$ satisfies the following estimates

$$\begin{aligned}
&\|\frac{\mathbf{E}^{(i)}_\varepsilon\mathbf{u}^{(i)}_{0,\varepsilon}}{\varepsilon^2}-\mathbf{\Lambda}^{(i)}(\mathcal{E}_{\beta\alpha(i)}(y_\beta-q^{(i)}_\beta)\vartheta',D_3v^{(i)}_3)\|_{L^2(\Omega^{(i)})}\le\varepsilon C(\mathbf{v}^{(i)},\vartheta),\\
&\|(\mathbf{W}^{(i)}_\varepsilon\mathbf{u}^{(i)}_{0,\varepsilon})_{12}+\vartheta\|_{L^2(\Omega^{(i)})}\le\varepsilon C(\mathbf{v}^{(i)},\vartheta),\\
&\|\mathbf{S}^{(i)}_\varepsilon\mathbf{u}^{(i)}_{0,\varepsilon}-\mathbf{v}^{(i)}\|_{H^1(\Omega^{(i)})}\le\varepsilon C(\mathbf{v}^{(i)},\vartheta),
\end{aligned}\tag{41}$$

where $C(\mathbf{v}^{(i)},\vartheta)$ depends only on $\mathbf{v}^{(i)}$ and ϑ. Hence in this case $(\mathbf{u}_{0,\varepsilon_n})$ is a recovery sequence.

In the general case, i.e., $(\mathbf{v}^{(1)},\mathbf{v}^{(2)},\mathbf{v}^{(3)},\vartheta)\in\mathcal{A}$, we can find, by convolution, functions $(\mathbf{v}^{(1)}_k,\mathbf{v}^{(2)}_k,\mathbf{v}^{(3)}_k,\vartheta_k)\in\mathcal{A}$ which are smooth, equal to zero near by $y_3=0$ and such that

$$\begin{aligned}
\|\mathbf{\Lambda}^{(i)}(\mathcal{E}_{\beta\alpha(i)}(y_\beta-q^{(i)}_\beta)\vartheta'_k,D_3v^{(i)}_{3k})-\mathbf{\Lambda}^{(i)}(\mathcal{E}_{\beta\alpha(i)}(y_\beta-q^{(i)}_\beta)\vartheta',D_3v^{(i)}_3)\|_{L^2(\Omega^{(i)})}&\le 1/k,\\
\|\vartheta_k-\vartheta\|_{L^2(\Omega^{(i)})}&\le 1/k,\\
\|\mathbf{v}^{(i)}_k-\mathbf{v}^{(i)}\|_{H^1(\Omega^{(i)})}&\le 1/k,
\end{aligned}$$

for every k. Denoting by $\mathbf{u}_{k,\varepsilon}$ the sequence defined as $\mathbf{u}_{0,\varepsilon}$ in (38) but with $\eta^{(i)}$'s and ϑ replaced by the $\eta^{(i)}_k$'s and ϑ_k used in the definition of $(\mathbf{v}^{(1)}_k,\mathbf{v}^{(3)}_k,\mathbf{v}^{(2)}_k,\vartheta_k)\in\mathcal{A}$, given a sequence ε_n converging to zero, we can find an increasing sequence of integers (k_n) and therefore a diagonal $\mathbf{u}^{(i)}_n:=\mathbf{u}^{(i)}_{k_n,\varepsilon_n}$ such that the sequence $\mathbf{u}_n=(\mathbf{u}^{(1)}_n,\mathbf{u}^{(2)}_n,\mathbf{u}^{(3)}_n)$ satisfies the required recovery sequence conditions. □

8 The elastic energy and the shear center

The limit energy functional $I(\mathbf{v}^{(1)},\mathbf{v}^{(2)},\mathbf{v}^{(3)},\vartheta)$ can be written in a more explicit form by using Lemma 5.7 and the fact that η_1, η_2, η_3 and ϑ depend only on y_3.

Indeed, the limit strain energy rewrites as

$$I(\boldsymbol{\eta},\vartheta)=\int_0^\ell \frac{\mu}{2}J_t\vartheta'^2+\frac{E}{2}\begin{pmatrix} A & -S_2 & -S_1 & S_\psi \\ & I_{22} & I_{12} & -I_{\psi 2} \\ & & I_{11} & -I_{\psi 1} \\ \text{sym} & & & I_{\psi\psi}\end{pmatrix}\begin{pmatrix}\eta_3' \\ \eta_1'' \\ \eta_2'' \\ \vartheta''\end{pmatrix}\cdot\begin{pmatrix}\eta_3' \\ \eta_1'' \\ \eta_2'' \\ \vartheta''\end{pmatrix}\,dy_3 \tag{42}$$

where (denoting with $da=dy_1dy_2$)

$$A=\int_{\omega^{(1)}}da+\int_{\omega^{(3)}}da+\int_{\omega^{(2)}}da=b^{(1)}s^{(1)}+b^{(3)}s^{(3)}+hs^{(2)}$$

is the total area,

$$\begin{aligned}S_1&=\int_{\omega^{(1)}}q_2^{(1)}\,da+\int_{\omega^{(3)}}q_2^{(3)}\,da+\int_{\omega^{(2)}}y_2\,da,\\ S_2&=\int_{\omega^{(1)}}y_1\,da+\int_{\omega^{(3)}}y_1\,da+\int_{\omega^{(2)}}q_1\,da\end{aligned}$$

are the static moments,

$$\begin{aligned}J_t&=4\Big(\int_{\omega^{(1)}}(y_2-q_2^{(1)})^2\,da+\int_{\omega^{(3)}}(y_2-q_2^{(3)})^2\,da+\int_{\omega^{(2)}}(y_1-q_1)^2\,da\Big)\\ &=\frac{1}{3}(b^{(1)}{s^{(1)}}^3+b^{(3)}{s^{(3)}}^3+h{s^{(2)}}^3)\end{aligned}$$

is the "torsional" moment of inertia,

$$\begin{aligned}I_{11}&=\int_{\omega^{(1)}}{q_2^{(1)}}^2\,da+\int_{\omega^{(3)}}{q_2^{(3)}}^2\,da+\int_{\omega^{(2)}}y_2^2\,da,\\ I_{22}&=\int_{\omega^{(1)}}y_1^2\,da+\int_{\omega^{(3)}}y_1^2\,da+\int_{\omega^{(2)}}q_1^2\,da,\\ I_{12}&=\int_{\omega^{(1)}}y_1q_2^{(1)}\,da+\int_{\omega^{(3)}}y_1q_2^{(3)}\,da+\int_{\omega^{(2)}}q_1y_2\,da,\end{aligned}$$

are the moments of inertia, and

$$\begin{aligned}S_\psi&=\int_{\omega^{(1)}}\psi^{(1)}(y_1)\,da+\int_{\omega^{(3)}}\psi^{(3)}(y_1)\,da+\int_{\omega^{(2)}}\psi^{(2)}(y_2)\,da,\\ I_{\psi 1}&=\int_{\omega^{(1)}}q_2^{(1)}\psi^{(1)}(y_1)\,da+\int_{\omega^{(3)}}q_2^{(3)}\psi^{(3)}(y_1)da+\int_{\omega^{(2)}}y_2\psi^{(2)}(y_2)\,da,\\ I_{\psi 2}&=\int_{\omega^{(1)}}y_1\psi^{(1)}(y_1)\,da+\int_{\omega^{(3)}}y_1\psi^{(3)}(y_1)\,da+\int_{\omega^{(2)}}q_1\psi^{(2)}(y_2)\,da,\\ I_{\psi\psi}&=\int_{\omega^{(1)}}\psi^{(1)}(y_1)^2\,da+\int_{\omega^{(3)}}\psi^{(3)}(y_1)^2\,da+\int_{\omega^{(2)}}\psi^{(2)}(y_2)^2\,da,\end{aligned}$$

are the sectorial statical moment, the sectorial products of inertia and the sectorial moment of inertia, respectively.

We note that, as it stands, the extensional, flexural and torsional problems are all coupled together. If the axes are centered in the center of mass, i.e. $S_1 = S_2 = 0$, and are principal axes, i.e. $I_{12} = 0$, then there is coupling only between extension and torsion through the coupling term $S_\psi \eta_3' \vartheta''$, and flexure and torsion through the coupling terms $I_{\psi\alpha}\eta_\alpha''\vartheta''$. We will have full decoupling only in the case in which also $S_\psi = I_{\psi 1} = I_{\psi 2} = 0$. We recall (see Lemma 5.7) that the sector coordinates depend on three parameters, the coordinates of the point $\mathbf{c} = (c_1, c_2)$ and the additive constant K. Hence we can solve the system

$$\begin{cases} S_\psi = 0 \\ I_{\psi 1} = 0 \\ I_{\psi 2} = 0 \end{cases} \tag{43}$$

of three equations in the three unknowns c_1, c_2 and K. Roughly speaking we can say that we choose the constant K so that the sector coordinate will have null mean, i.e., $S_\psi = 0$, and then the point $\mathbf{c}$ so to make the sectorial products of inertia $I_{\psi 1}, I_{\psi 2}$ equal to zero. This latter point is usually called, in the engineering literature, the *shear center* and the resulting $I_{\psi\psi}$ is the *warping stiffness.*

Remark 8.1 When only one junction is present, which happens for instance if $b^{(3)} = s^{(3)} = d^{(3)} = 0$ (see Section 5.2) then, from Remark 5.8 follows that the sector coordinates can be chosen equal to zero and the limit strain energy functional takes the form

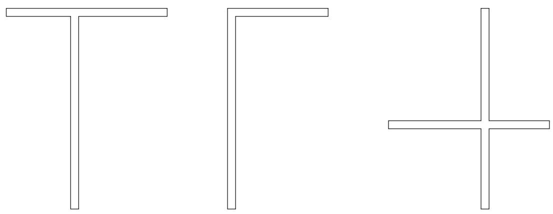
$$I(\boldsymbol{\eta}, \vartheta) = \int_0^\ell \frac{\mu}{2} J_t \vartheta'^2 + \frac{E}{2} \begin{pmatrix} A & -S_2 & -S_1 \\ & I_{22} & I_{12} \\ \text{sym} & & I_{11} \end{pmatrix} \begin{pmatrix} \eta_3' \\ \eta_1'' \\ \eta_2'' \end{pmatrix} \cdot \begin{pmatrix} \eta_3' \\ \eta_1'' \\ \eta_2'' \end{pmatrix} dy_3.$$

It is worth notice that this De Saint-Venant limit energy is formally the same which holds in the case when there are no junctions (see [4], Section 7).

9 Some examples

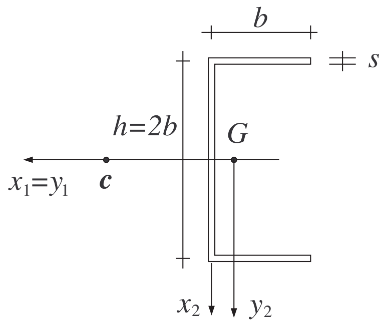
9.1 Single-junction sections

By Lemma 5.5, Lemma 5.7 and Remark 5.8 (see also Remark 8.1) follows that when the cross-section consists of straight segments connected by a unique common junction, that is when the one dimensional limit structure is star-shaped, then the warping stiffness is zero, the shear center is the common point of intersection. The figure below shows three examples of this kind of sections.



9.2 Channel-section

Consider the so-called Channel-section shown in the figure below.



The thickness s is much smaller then the height h. Let us follow the scheme of Section 8. The area is

$$A = 4bs.$$

The point G with coordinates $x_1(G) = -b/4$ and $x_2(G) = 0$ is the center of mass of the cross-section. Choosing a reference system Gy_1y_2 with origin in G we have that $q_1 = b/4$, $q_2^{(1)} = b$ and $q_2^{(1)} = -b$. Hence, from the integration formulae of Section 8, we easily get

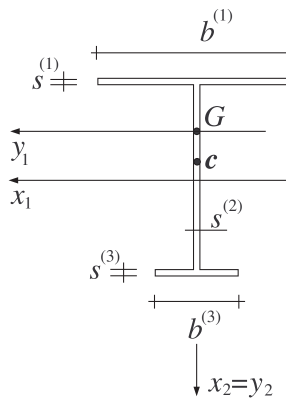
$$S_1 = 0,\ S_2 = 0,\ J_t = \frac{4}{3}bs^3,\ I_{11} = \frac{8}{3}b^3s,\ I_{22} = \frac{5}{12}b^3s,\ I_{12} = 0,$$

$$S_\psi = 4Kbs,\ I_{\psi 1} = -\frac{sb^3}{3}(5b - 8c_1),\ I_{\psi 2} = -\frac{5}{12}sb^3c_2.$$

Solving the system (43) we obtain that $K = 0$ and the shear center $\mathbf{c}$ has coordinates $c_1 = \frac{5}{8}b$ and $c_2 = 0$, in the reference Gy_1y_2, as shown in the figure above. The resulting warping stiffness is $I_{\psi\psi} = \frac{7}{24}b^5s$.

9.3 Symmetric double-T section

Consider the double-T section shown in the figure below.



The thicknesses $s^{(3)}$, $s^{(2)}$ and $s^{(1)}$ are much smaller then the height h. Let us follow the scheme of Section 8. It is useful to set $A^{(3)} := b^{(3)}s^{(3)}$, $A^{(1)} := b^{(1)}s^{(1)}$ and $A^{(2)} := hs^{(2)}$. Then the total area is

$$A = A^{(1)} + A^{(2)} + A^{(3)}.$$

The center of mass G of the cross-section has the following coordinates

$$x_1(G) = 0, \quad x_2(G) = \frac{h}{2}\frac{(A^{(3)} - A^{(1)})}{A}.$$

Choosing a reference system Gy_1y_2 we have that

$$q_1 = 0, \quad q_2^{(1)} = \frac{h}{2}(1 - \frac{A^{(3)} - A^{(1)}}{A}), \quad q_2^{(1)} = -\frac{h}{2}(1 + \frac{A^{(3)} - A^{(1)}}{A}).$$

Hence, from the integration formulae of Section 8, and setting also

$$I_2^{(1)} := \frac{A^{(1)}{b^{(1)}}^2}{12}, \quad I_2^{(3)} := \frac{A^{(3)}{b^{(3)}}^2}{12},$$

which are the axial moments of inertia of $\omega^{(1)}$ and $\omega^{(3)}$ with respect to the axis y_2, after some simple calculations we get

$$S_1 = 0, \quad S_2 = 0, \quad I_{12} = 0, \quad J_t = \frac{1}{3}(b^{(1)}{s^{(1)}}^3 + b^{(3)}{s^{(3)}}^3 + h{s^{(2)}}^3),$$

$$I_{11} = \frac{h^2}{4}[\frac{A^{(3)} - A^{(1)}}{A}(A^{(1)} - A^{(3)} + A^{(2)}) + A^{(1)} + A^{(3)} + \frac{A^{(2)}}{3}], \quad I_{22} = I_2^{(1)} + I_2^{(3)},$$

$$S_\psi = AK, \quad I_{\psi 1} = c_1 I_{11}, \quad I_{\psi 2} = \frac{h}{2}(I_2^{(3)} - I_2^{(1)}) - \Big(c_2 + \frac{h}{2}\frac{A^{(3)} - A^{(1)}}{A}\Big)(I_2^{(3)} + I_2^{(1)}).$$

Solving the system (43) we obtain that $K = 0$ and the shear center $\mathbf{c}$ has coordinates

$$c_1 = 0, \quad c_2 = \frac{h}{2}\Big(\frac{I_2^{(3)} - I_2^{(1)}}{I_2^{(3)} + I_2^{(1)}} - \frac{A^{(3)} - A^{(1)}}{A}\Big),$$

in the reference Gy_1y_2, as shown in the figure above. The resulting warping stiffness is

$$I_{\psi\psi} = h^2 \frac{I_2^{(3)} I_2^{(1)}}{I_2^{(3)} + I_2^{(1)}}.$$

Let us remark that from the expressions above one can deduce, in particular, that if $A^{(3)} = A^{(1)}$ and $I_2^{(3)} = I_2^{(1)}$, then $\mathbf{c} = (0, 0)$, that is $\mathbf{c} \equiv G$.

10 The loaded beam

As stated in (3), the total energy $\mathcal{F}_\varepsilon$ is equal to the elastic energy J_ε minus the work done by the external loads:

$$\mathcal{F}_\varepsilon(\mathbf{u}) = J_\varepsilon(\mathbf{u}) - \int_{\Omega_\varepsilon} \mathbf{b}^\varepsilon \cdot \mathbf{u}\, dx.$$

Let us denote by

$$\begin{aligned} \mathcal{L}_\varepsilon(\mathbf{u}) &:= \int_{\Omega_\varepsilon} \mathbf{b}^\varepsilon \cdot \mathbf{u}\, dx = \sum_{i=1}^{3} \int_{\Omega_\varepsilon^{(i)}} \chi_\varepsilon^{(i)}\, \mathbf{b}^\varepsilon \cdot \mathbf{u}\, dx \\ &=: \mathcal{L}_\varepsilon^{(1)}(\mathbf{u}) + \mathcal{L}_\varepsilon^{(2)}(\mathbf{u}) + \mathcal{L}_\varepsilon^{(3)}(\mathbf{u}) \end{aligned}$$

where $\chi_\varepsilon^{(i)}$ is defined in Section 3. Then we can write

$$\begin{aligned} \mathcal{F}_\varepsilon(\mathbf{u}) &= \mathcal{F}_\varepsilon^{(1)}(\mathbf{u}) + \mathcal{F}_\varepsilon^{(3)}(\mathbf{u}) + \mathcal{F}_\varepsilon^{(2)}(\mathbf{u}) \\ &:= J_\varepsilon^{(1)}(\mathbf{u}) - \mathcal{L}_\varepsilon^{(1)}(\mathbf{u}) + J_\varepsilon^{(3)}(\mathbf{u}) - \mathcal{L}_\varepsilon^{(3)}(\mathbf{u}) + J_\varepsilon^{(2)}(\mathbf{u}) - \mathcal{L}_\varepsilon^{(2)}(\mathbf{u}). \end{aligned}$$

With change of variables, as done in section 3, we deduce and set

$$\begin{aligned} F_\varepsilon^{(i)}(\mathbf{u}^{(i)}) &:= \frac{1}{\varepsilon^3} \mathcal{F}_\varepsilon^{(i)}(\mathbf{u}^{(i)} \circ {p_\varepsilon^{(i)}}^{-1}) \\ &= I_\varepsilon^{(i)}(\mathbf{u}^{(i)}) - \int_{\Omega^{(i)}} \mathbf{b}_\varepsilon^{(i)} \cdot \mathbf{u}^{(i)}\, dy =: I_\varepsilon^{(i)}(\mathbf{u}^{(i)}) - L_\varepsilon^{(i)}(\mathbf{u}^{(i)}), \end{aligned} \tag{44}$$

where

$$\mathbf{b}_\varepsilon^{(i)} = \chi_\varepsilon^{(i)} \mathbf{b}^\varepsilon \circ p_\varepsilon^{(i)}.$$

Let us assume that

$$\begin{aligned} b_{\varepsilon 1}^{(i)} &= \varepsilon^{\alpha(i)} \Big(\varepsilon^2 b_1^{(i)}(y) - \varepsilon \frac{m^{(i)}(y_3)}{I_G(\omega^{(i)})} (y_2 - y_2(G^{(i)})) \Big), \\ b_{\varepsilon 2}^{(i)} &= \varepsilon^{\alpha(i+1)} \Big(\varepsilon^2 b_2^{(i)}(y) + \varepsilon \frac{m^{(i)}(y_3)}{I_G(\omega^{(i)})} (y_1 - y_1(G^{(i)})) \Big), \\ b_{\varepsilon 3}^{(i)} &= \varepsilon^2 b_3^{(i)}(y), \end{aligned} \tag{45}$$

with $\mathbf{b}^{(i)} = (b_1^{(i)}, b_2^{(i)}, b_3^{(i)}) \in L^2(\Omega^{(i)}; \mathbb{R}^3)$ and $m^{(i)} \in L^2(0, \ell)$, while $I_G(\omega^{(i)})$ is always the moment of inertia with respect to the center of mass $(y_1(G^{(i)}), y_2(G^{(i)})$ of the section $\omega^{(i)}$. With this choice, the work of the external loads can be rewritten as

$$L_\varepsilon(\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \mathbf{u}^{(3)}) = \sum_{i=1}^{3} L_\varepsilon^{(i)}(\mathbf{u}^{(i)}) \tag{46}$$

where

$$L_\varepsilon^{(i)}(\mathbf{u}^{(i)}) = \varepsilon^4 \int_{\Omega^{(i)}} \mathbf{b}^{(i)} \cdot \mathbf{S}_\varepsilon^{(i)} \mathbf{u}^{(i)}\, dy + \varepsilon^4 \int_{\Omega^{(i)}} m^{(i)}\, \vartheta_\varepsilon^{(i)}(\mathbf{u}^{(i)})\, dy.$$

The sequence of functionals $L_\varepsilon/\varepsilon^4$ is continuously convergent with respect to the convergence used in Theorem 7.1, that is, for any sequence $(\mathbf{u}_\varepsilon^{(1)}, \mathbf{u}_\varepsilon^{(2)}, \mathbf{u}_\varepsilon^{(3)}) \in \mathcal{A}_\varepsilon$ such that

$$\begin{aligned} &\mathbf{S}_\varepsilon^{(i)} \mathbf{u}_\varepsilon^{(i)} \rightharpoonup \mathbf{v}^{(i)} \text{ in } H^1(\Omega^{(i)}; \mathbb{R}^3), \\ &(\mathbf{W}_\varepsilon^{(i)} \mathbf{u}^{(i)})_{12} \rightharpoonup -\vartheta \text{ in } L^2(\Omega^{(i)}), \end{aligned}$$

we have

$$\frac{1}{\varepsilon^4} L_\varepsilon^{(i)}(\mathbf{u}_\varepsilon^{(i)}) \to L^{(i)}(\mathbf{v}^{(i)}, \theta) := \int_{\Omega^{(i)}} \mathbf{b}^{(i)} \cdot \mathbf{v}^{(i)}\, dy + \int_0^\ell m^{(i)}\, \vartheta\, dy_3,$$

so that

$$\frac{1}{\varepsilon^4} L_\varepsilon(\mathbf{u}_\varepsilon^{(1)}, \mathbf{u}_\varepsilon^{(2)}, \mathbf{u}_\varepsilon^{(3)}) \to L(\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \mathbf{v}^{(3)}, \theta) \tag{47}$$

where

$$L(\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \mathbf{v}^{(3)}, \theta) := \sum_{i=1}^{3} L^{(i)}(\mathbf{v}^{(i)}, \theta) = \sum_{i=1}^{3} \Big(\int_{\Omega^{(i)}} \mathbf{b}^{(i)} \cdot \mathbf{v}^{(i)}\, dy + \int_0^\ell m^{(i)}\, \vartheta\, dy_3 \Big).$$

In terms of the variables $(\boldsymbol{\eta}, \vartheta)$ the limit load can be written in the form

$$L(\boldsymbol{\eta}, \vartheta) = \int_0^\ell l_1\eta_1 + l_2\eta_2 + l_3\eta_3 + m\vartheta - m_\alpha \eta'_\alpha + b\vartheta'\, dy_3 \tag{48}$$

where

$$\begin{aligned}
l_1 &= b_1^{(1)} + b_1^{(3)}, \qquad l_2 = b_2^{(2)}, \qquad l_3 = b_3^{(1)} + b_3^{(2)} + b_3^{(3)},\\
m &= \sum_{i=1}^{3} m^{(i)} - \int_{\omega^{(1)}} (q_2^{(1)} - c_2) b_1^{(1)}\, da - \int_{\omega^{(3)}} (q_2^{(3)} - c_2) b_1^{(3)}\, da +\\
&\qquad + \int_{\omega^{(2)}} (q_1 - c_1) b_2^{(2)}\, da,\\
m_1 &= \int_{\omega^{(1)}} y_1 b_3^{(1)}\, da + \int_{\omega^{(3)}} y_1 b_3^{(3)}\, da + \int_{\omega^{(2)}} q_1 b_3^{(2)}\, da,\\
m_2 &= \int_{\omega^{(1)}} q_2^{(1)} b_3^{(1)}\, da + \int_{\omega^{(3)}} q_2^{(3)} b_3^{(3)}\, da + \int_{\omega^{(2)}} y_2 b_3^{(2)}\, da,\\
b &= \int_{\omega^{(1)}} b_3^{(1)} \psi^{(1)}\, da + \int_{\omega^{(3)}} b_3^{(3)} \psi^{(3)}\, da + \int_{\omega^{(2)}} b_3^{(2)} \psi^{(2)}\, da,
\end{aligned}$$

and $da = dy_1 dy_2$.

We note that the contribution to the moments m of the loads $\mathbf{b}^{(i)}$ are moments evaluated with respect to the shear center and not with respect the center of mass of the section.

The theorem below follows from the stability of Γ-convergence with respect to continuously convergent, real valued, perturbations (see Dal Maso [3], Proposition 6.20).

Theorem 10.1 *Let the rescaled total energy F_ε be defined as*

$$F_\varepsilon(\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \mathbf{u}^{(3)}) = \sum_{i=1}^{3} F_\varepsilon^{(i)}(\mathbf{u}^{(i)}). \tag{49}$$

As $\varepsilon \to 0^+$, the sequence of functionals $\frac{1}{\varepsilon^4} F_\varepsilon$ Γ-converges to the limit functional $F := I - L$ in the sense specified in Theorem 7.1.

For every $\varepsilon \in (0, 1]$ let us denote by $\bar{\mathbf{u}}_\varepsilon$ the solution of the following minimization problem

$$\min\{\mathcal{F}_\varepsilon(\mathbf{u}) \ : \ \mathbf{u} \in H^1(\Omega_\varepsilon; \mathbb{R}^3),\, \mathbf{u} = \mathbf{0} \text{ on } S_\varepsilon(0)\}. \tag{50}$$

The existence of a solution can be proved by the direct method of the Calculus of Variations and the uniqueness follows by the strict convexity of the functionals F_ε. The next theorem describes the behaviour of the sequence of minimizers $\bar{\mathbf{u}}_\varepsilon$.

Theorem 10.2 *The following minimization problem for the* Γ*-limit functional* $F = I - L$

$$\min\{F(\boldsymbol{\eta}, \vartheta) \ : \ \eta_1, \eta_2 \in H^2_\#(0,\ell),\ \eta_3 \in H^1_\#(0,\ell) \text{ and } \vartheta \in H^2_\#(0,\ell)\} \tag{51}$$

admits a unique solution $(\bar{\boldsymbol{\eta}}, \bar{\vartheta})$. *Moreover, denoting with* $(\bar{\mathbf{v}}^{(1)}, \bar{\mathbf{v}}^{(2)}, \bar{\mathbf{v}}^{(3)})$ *a field related to* $(\bar{\boldsymbol{\eta}}, \bar{\vartheta})$ *throughout equations* (31) *we have that, as* $\varepsilon \to 0^+$,

1. $\mathbf{S}^{(i)}_\varepsilon \bar{\mathbf{u}}^{(i)}_\varepsilon \to \bar{\mathbf{v}}^{(i)}$ *strongly in* $H^i(\Omega^{(i)}; \mathbb{R}^3)$,
2. $(\mathbf{W}^{(i)}_\varepsilon \bar{\mathbf{u}}^{(i)}_\varepsilon)_{12} \to -\bar{\vartheta}$ *strongly in* $L^2(\Omega^{(i)})$,
3. $\dfrac{1}{\varepsilon^4} F_\varepsilon(\bar{\mathbf{u}}^{(1)}_\varepsilon, \bar{\mathbf{u}}^{(2)}_\varepsilon, \bar{\mathbf{u}}^{(3)}_\varepsilon)$ *converges to* $F(\bar{\boldsymbol{\eta}}, \bar{\vartheta})$.

Proof. From the Γ-convergence Theorem 10.1 and from well known properties of Γ-limits and in particular by putting together Propositions 6.8 and 8.16 (lower semicontinuity of sequential Γ-limits), Theorem 7.8 (coercivity of the Γ-limit) and Corollary 7.24 (convergence of minima and minimizers) of Dal Maso [3], follows that

1′. $\mathbf{S}^{(i)}_\varepsilon \bar{\mathbf{u}}^{(i)}_\varepsilon \rightharpoonup \bar{\mathbf{v}}^{(i)}$ in $H^i(\Omega^{(i)}; \mathbb{R}^3)$,

2′. $(\mathbf{W}^{(i)}_\varepsilon \bar{\mathbf{u}}^{(i)}_\varepsilon)_{12} \rightharpoonup -\bar{\vartheta}$ in $L^2(\Omega^{(i)})$,

3′. $\dfrac{1}{\varepsilon^4} F_\varepsilon(\bar{\mathbf{u}}^{(1)}_\varepsilon, \bar{\mathbf{u}}^{(2)}_\varepsilon, \bar{\mathbf{u}}^{(3)}_\varepsilon)$ converges to $F(\bar{\boldsymbol{\eta}}, \bar{\vartheta})$.

It remains to prove that in *1′* and *2′* the convergence is, in fact, strong.

Let us denote by $\boldsymbol{a}^\varepsilon$ the approximate minimizers defined as the sequence $\mathbf{u}_{0,\varepsilon}$ in (38), (39), (40) with $(\boldsymbol{\eta}, \vartheta)$ replaced by $(\bar{\boldsymbol{\eta}}, \bar{\vartheta})$. Even if we cannot say that it is a recovery sequence, for it does not satisfy the boundary conditions, the estimates (41) with $(\mathbf{v}, \vartheta)$ replaced by $(\bar{\mathbf{v}}, \bar{\vartheta})$ hold and therefore

$$\begin{aligned} \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^4} F_\varepsilon(\boldsymbol{a}^{(1)}_\varepsilon, \boldsymbol{a}^{(2)}_\varepsilon, \boldsymbol{a}^{(3)}_\varepsilon) &= F(\bar{\boldsymbol{\eta}}, \bar{\vartheta}), \\ \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^4} L_\varepsilon(\boldsymbol{a}^{(1)}_\varepsilon, \boldsymbol{a}^{(2)}_\varepsilon, \boldsymbol{a}^{(3)}_\varepsilon) &= L(\bar{\boldsymbol{\eta}}, \bar{\vartheta}). \end{aligned} \tag{52}$$

In particular we have that

$$\lim_{\varepsilon \to 0^+} \frac{F^{(i)}_\varepsilon(\bar{\mathbf{u}}^{(i)}_\varepsilon) - F^{(i)}_\varepsilon(\boldsymbol{a}^{(i)}_\varepsilon)}{\varepsilon^4} = 0, \qquad \lim_{\varepsilon \to 0^+} \frac{L^{(i)}_\varepsilon(\bar{\mathbf{u}}^{(i)}_\varepsilon - \boldsymbol{a}^{(i)}_\varepsilon)}{\varepsilon^4} = 0. \tag{53}$$

A key point in the proof of strong convergence of *1′* consists in showing that

$$\lim_{\varepsilon\to 0^+}\left\|\frac{\mathbf{E}_\varepsilon^{(i)}(\bar{\mathbf{u}}_\varepsilon^{(i)}-\boldsymbol{a}^\varepsilon)}{\varepsilon^2}\right\|_{L^2(\Omega^{(i)})}=0. \tag{54}$$

To start we observe that the quadratic form $f(\mathbf{A})$ defined in (33) satisfies the identity

$$f(\mathbf{U})=f(\mathbf{A})+\mathbb{C}\mathbf{A}\cdot(\mathbf{U}-\mathbf{A})+f(\mathbf{U}-\mathbf{A})$$

for every pair of 3×3 matrices $\mathbf{A}$ and $\mathbf{U}$. By (1) we thus obtain the inequality

$$f(\mathbf{U})\geq f(\mathbf{A})+\mathbb{C}\mathbf{A}\cdot(\mathbf{U}-\mathbf{A})+\mu|\mathbf{U}-\mathbf{A}|^2,$$

which can be used in the expression of the integral functional $F_\varepsilon^{(i)}$ to obtain that

$$\begin{aligned}F_\varepsilon^{(i)}(\bar{\mathbf{u}}_\varepsilon^{(i)})\geq\ & F_\varepsilon^{(i)}(\boldsymbol{a}_\varepsilon^{(i)})+\int_{\Omega^{(i)}}\chi_\varepsilon^{(i)}\,\mathbb{C}\mathbf{E}_\varepsilon^{(i)}\boldsymbol{a}_\varepsilon^{(i)}\cdot\mathbf{E}_\varepsilon^{(i)}(\bar{\mathbf{u}}_\varepsilon^{(i)}-\boldsymbol{a}_\varepsilon^{(i)})\,dy+\\ &+\mu\int_{\Omega^{(i)}}\chi_\varepsilon^{(i)}\,|\mathbf{E}_\varepsilon^{(i)}\bar{\mathbf{u}}_\varepsilon^{(i)}-\mathbf{E}_\varepsilon^{(i)}\boldsymbol{a}_\varepsilon^{(i)}|^2\,dy\,-\,L_\varepsilon^{(i)}(\bar{\mathbf{u}}_\varepsilon^{(i)}-\boldsymbol{a}_\varepsilon^{(i)}).\end{aligned}$$

Hence, taking also into account that $\chi_\varepsilon^{(i)}\geq 1/2$, we have

$$\begin{aligned}\frac{F_\varepsilon^{(i)}(\bar{\mathbf{u}}_\varepsilon^{(i)})-F_\varepsilon^{(i)}(\boldsymbol{a}_\varepsilon^{(i)})}{\varepsilon^4}&\geq\frac{1}{2}\int_{\Omega^{(i)}}\frac{\mathbb{C}\mathbf{E}_\varepsilon^{(i)}\boldsymbol{a}_\varepsilon^{(i)}\cdot\mathbf{E}_\varepsilon^{(i)}(\bar{\mathbf{u}}_\varepsilon^{(i)}-\boldsymbol{a}_\varepsilon^{(i)})}{\varepsilon^4}\,dy+\\ &+\frac{\mu}{2}\int_{\Omega^{(i)}}\frac{|\mathbf{E}_\varepsilon^{(i)}\bar{\mathbf{u}}_\varepsilon^{(i)}-\mathbf{E}_\varepsilon^{(i)}\boldsymbol{a}_\varepsilon^{(i)}|^2}{\varepsilon^4}\,dy\,-\,L_\varepsilon^{(i)}(\bar{\mathbf{u}}_\varepsilon^{(i)}-\boldsymbol{a}_\varepsilon^{(i)}).\end{aligned}\tag{55}$$

Let us then prove that

$$\lim_{\varepsilon\to 0^+}\int_{\Omega^{(i)}}\frac{\mathbb{C}\mathbf{E}_\varepsilon^{(i)}\boldsymbol{a}_\varepsilon^{(i)}\cdot\mathbf{E}_\varepsilon^{(i)}(\bar{\mathbf{u}}_\varepsilon^{(i)}-\boldsymbol{a}_\varepsilon^{(i)})}{\varepsilon^4}\,dy=0 \tag{56}$$

and (54) is obtained by passing to the upper limit as $\varepsilon\to 0^+$ in (55). The proof of (56) proceeds along the same lines of Theorem 8.1 of [4] but we give here full details for convenience of the reader.

In order to prove (56) we observe that for every pair of matrices $\mathbf{A}$ and $\mathbf{B}$

$$\mathbb{C}\mathbf{A}\cdot\mathbf{B}=2\mu\mathbf{A}\cdot\mathbf{B}+\lambda\,\mathrm{tr}(\mathbf{A})\,\mathrm{tr}(\mathbf{B}).$$

Then

$$\begin{aligned}\int_{\Omega^{(i)}}\frac{\mathbb{C}\mathbf{E}_\varepsilon^{(i)}\boldsymbol{a}_\varepsilon^{(i)}\cdot\mathbf{E}_\varepsilon^{(i)}(\bar{\mathbf{u}}_\varepsilon^{(i)}-\boldsymbol{a}_\varepsilon^{(i)})}{\varepsilon^4}dy=\ &2\mu\int_{\Omega^{(i)}}\frac{\mathbf{E}_\varepsilon^{(i)}\boldsymbol{a}_\varepsilon^{(i)}\cdot\mathbf{E}_\varepsilon^{(i)}(\bar{\mathbf{u}}_\varepsilon^{(i)}-\boldsymbol{a}_\varepsilon^{(i)})}{\varepsilon^4}\,dy+\\ &+\lambda\int_{\Omega^{(i)}}\frac{\mathrm{tr}(\mathbf{E}_\varepsilon^{(i)}\boldsymbol{a}_\varepsilon^{(i)})\,\mathrm{tr}(\mathbf{E}_\varepsilon^{(i)}(\bar{\mathbf{u}}_\varepsilon^{(i)}-\boldsymbol{a}_\varepsilon^{(i)}))}{\varepsilon^4}\,dy.\end{aligned}\tag{57}$$

In order to perform the computation, it is convenient to shorten the notation by setting

$$A^{\varepsilon}_{hj} := \frac{(\mathbf{E}^{(i)}_{\varepsilon}\boldsymbol{a}^{(i)}_{\varepsilon})_{hj}}{\varepsilon^2}, \quad U^{\varepsilon}_{hj} := \frac{(\mathbf{E}^{(i)}_{\varepsilon}\bar{\mathbf{u}}^{(i)}_{\varepsilon})_{hj}}{\varepsilon^2},$$

$$\Delta^{\varepsilon}_{hj} := U^{\varepsilon}_{hj} - A^{\varepsilon}_{hj} = \frac{(\mathbf{E}^{(i)}_{\varepsilon}(\bar{\mathbf{u}}^{(i)}_{\varepsilon} - \boldsymbol{a}^{(i)}_{\varepsilon}))_{hj}}{\varepsilon^2},$$

so that the integrand in (57) takes the following expression

$$2\mu(A^{\varepsilon}_{hj}\Delta^{\varepsilon}_{hj}) + \lambda A^{\varepsilon}_{hh}\Delta^{\varepsilon}_{jj}.$$

By (41) we have

$$\mathbf{A}^{\varepsilon} \to \boldsymbol{\Lambda}^{(i)}(\mathcal{E}_{\beta\alpha(i)}(y_\beta - q^{(i)}_\beta)\bar{\vartheta}', D_3\bar{v}^{(i)}_3) \quad \text{in } L^2(\Omega^{(i)}), \tag{58}$$

where $\boldsymbol{\Lambda}^{(i)}$ is defined in (37), and by the equation above and Lemma 4.1 it follows that $\boldsymbol{\Delta}^{\varepsilon}$ is bounded in $L^2(\Omega^{(i)})$. Thus, from (58) we immediately deduce that

$$\lim_{\varepsilon\to 0^+}\int_{\Omega^{(i)}} A^{\varepsilon}_{12}\Delta^{\varepsilon}_{12}\,dy = 0,$$

and

$$\lim_{\varepsilon\to 0^+}\int_{\Omega^{(i)}} A^{\varepsilon}_{\alpha(i+1)3}\Delta^{\varepsilon}_{\alpha(i+1)3}\,dy = 0. \tag{59}$$

From (58) and *1′* follows that $\Delta^{\varepsilon}_{33} \rightharpoonup 0$ in $L^2(\Omega^{(i)})$, and therefore

$$\lim_{\varepsilon\to 0^+}\int_{\Omega^{(i)}} A^{\varepsilon}_{hj}\Delta^{\varepsilon}_{33}\,dy = 0, \quad h,j = 1,2,3.$$

From *1′*, *2′*, *3′* and Lemma 6.1, follows that, up to subsequences, $U^{\varepsilon}_{\alpha(i)3}$ weakly converges in $L^2(\Omega^{(i)})$ to $\mathcal{E}_{\beta\alpha(i)}(y_\beta - q^{(i)}_\beta)\bar{\vartheta}' + \bar{\eta}^{(i)}$, for some $\bar{\eta}^{(i)}$ as specified in the statement of Lemma 6.1. By (58) we have that $A^{\varepsilon}_{\alpha(i)3} \to \mathcal{E}_{\beta\alpha(i)}(y_\beta - q^{(i)}_\beta)\bar{\vartheta}'$ strongly in $L^2(\Omega^{(i)})$ and hence, up to subsequences, $\Delta^{\varepsilon}_{\alpha(i)3} \to \bar{\eta}^{(i)}$ weakly in $L^2(\Omega^{(i)})$. Thus

$$\lim_{\varepsilon\to 0^+}\int_{\Omega^{(i)}} A^{\varepsilon}_{\alpha(i)3}\Delta^{\varepsilon}_{\alpha(i)3}\,dy = \int_{\Omega^{(i)}} \mathcal{E}_{\beta\alpha(i)}(y_\beta - q^{(i)}_\beta)\bar{\vartheta}'\bar{\eta}^{(i)}\,dy = 0.$$

Putting together with (59) we have that

$$\lim_{\varepsilon\to 0^+}\int_{\Omega^{(i)}} A^{\varepsilon}_{h3}\Delta^{\varepsilon}_{h3}\,dy = 0, \quad h,i = 1,2,3.$$

Let Δ_{11} and Δ_{22} be, up to subsequences, the weak limits in $L^2(\Omega^{(i)})$ of Δ^ε_{11} and Δ^ε_{22}, respectively. Summarizing and taking the limit as $\varepsilon \to 0^+$ in (57) we obtain (even for the whole sequence)

$$\lim_{\varepsilon\to 0^+} \int_{\Omega^{(i)}} \frac{\mathbb{C}\mathbf{E}^{(i)}_\varepsilon \boldsymbol{a}^{(i)}_\varepsilon \cdot \mathbf{E}^{(i)}_\varepsilon(\bar{\mathbf{u}}^{(i)}_\varepsilon - \boldsymbol{a}^{(i)}_\varepsilon)}{\varepsilon^4}\, dy =$$

$$= \lim_{\varepsilon\to 0^+} \int_{\Omega^{(i)}} 2\mu(A^\varepsilon_{11}\Delta^\varepsilon_{11} + A^\varepsilon_{22}\Delta^\varepsilon_{22}) + \lambda A^\varepsilon_{hh}\Delta^\varepsilon_{\alpha\alpha}\, dy$$

$$= \int_{\Omega^{(i)}} D_3\bar{v}^{(i)}_3(\Delta_{11} + \Delta_{22})[-2\nu(\mu+\lambda) + \lambda]\, dy = 0$$

because $-2\nu(\mu+\lambda)+\lambda = -\lambda+\lambda = 0$, and (56) and hence (54) are proven.

Setting $\mathbf{z}^{(i)}_\varepsilon = \mathbf{S}^{(i)}_\varepsilon(\bar{\mathbf{u}}^{(i)}_\varepsilon - \boldsymbol{a}^{(i)}_\varepsilon)$, we have

$$\|\mathbf{E}\mathbf{z}^{(i)}_\varepsilon\|_{L^2(\Omega^{(i)})} \leq \|\frac{\mathbf{E}^{(i)}_\varepsilon(\bar{\mathbf{u}}^{(i)}_\varepsilon - \boldsymbol{a}^{(i)}_\varepsilon)}{\varepsilon^2}\|_{L^2(\Omega^{(i)})}$$

and by the application of the standard Korn inequality to $\mathbf{z}^{(i)}_\varepsilon$, and using (54), we obtain that

$$\|\mathbf{z}^{(i)}_\varepsilon\|_{H^1(\Omega^{(i)})} \to 0.$$

From this fact and the third equation of (41) applied to the sequence $\boldsymbol{a}^\varepsilon$ we gain the strong convergence in *1′*. This proves *1*.

On the other hand, by (54) and inequality (11), we have that

$$\lim_{\varepsilon\to 0^+} \int_{\Omega^{(i)}} |\mathbf{H}^{(i)}_\varepsilon(\bar{\mathbf{u}}^{(i)}_\varepsilon - \boldsymbol{a}^{(i)}_\varepsilon)|^2\, dy = 0$$

and since the definition of $\mathbf{H}^{(i)}_\varepsilon$ we deduce from here that

$$\frac{D_1(\bar{u}^{(i)}_{\varepsilon 2} - a^{(i)}_{\varepsilon 2})}{\varepsilon^{\alpha(i)}} \to 0, \quad \frac{D_2(\bar{u}^{(i)}_{\varepsilon 1} - a^{(i)}_{\varepsilon 1})}{\varepsilon^{\alpha(i+1)}} \to 0, \quad \text{in } L^2(\Omega^{(i)}).$$

Thus, from the second equation of (41) applied to the sequence $\boldsymbol{a}^\varepsilon$ follows that

$$(\mathbf{W}^{(i)}_\varepsilon \bar{\mathbf{u}}^{(i)}_\varepsilon)_{12} = \frac{D_2(\bar{u}^{(i)}_{\varepsilon 1} - a^{(i)}_{\varepsilon 1})}{2\varepsilon^{\alpha(i+1)}} - \frac{D_1(\bar{u}^{(i)}_{\varepsilon 2} - a^{(i)}_{\varepsilon 2})}{2\varepsilon^{\alpha(i)}} + (\mathbf{W}^{(i)}_\varepsilon \boldsymbol{a}^{(i)}_\varepsilon)_{12} \to -\bar{\vartheta} \ \text{ in } L^2(\Omega^{(i)}),$$

which proves *2*. □

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