FINE PROPERTIES OF FUNCTIONS WITH BOUNDED DEFORMATION

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Abstract

The paper is concerned with the fine properties of functions u in BD, the space of functions with bounded deformation. We analyse the set of Lebesgue points and the set where these functions have one sided approximate limits. Moreover, following the analogy with BV, we decompose the symmetric distributional derivative Eu into an absolutely continuous part $E^a u = \mathcal{E}u\,\mathcal{L}^n$, a jump part $E^j u$, and a Cantor part $E^c u$. The main result of the paper is a structure theorem for BD functions, showing that these parts of the derivative can be recovered from the corresponding ones of the one dimensional sections. Moreover, we prove that BD functions are approximately differentiable in almost every point of their domain.

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1. Introduction

The space $BD(\Omega)$ of functions with *bounded deformation* in an open set $\Omega \subset \mathbf{R}^n$ is defined as the set of functions $u \in L^1(\Omega;\mathbf{R}^n)$ such that the symmetric distributional derivative

$$E_{ij}u := \frac{1}{2}\big(D_i u^j + D_j u^i\big)$$

is a (matrix valued) measure with finite total variation in Ω. This space, introduced in [22] and [33], has been widely considered in the literature (see [34], [6], [36], [35]) in connection with the mathematical theory of plasticity.

The goal of this paper is to study the fine properties of functions in $BD(\Omega)$. We will prove that many properties of $BV(\Omega;\mathbf{R}^n)$, the space of functions with *bounded variation*, can be suitably extended to $BD(\Omega)$. The crucial point is to choose a "correct" definition of jump set, consistent, but not necessarily equal, with the one usually adopted in the BV theory.

To illustrate our results, we briefly recall standard notations and definitions from that theory. Denoting, for any $u \in L^1(\Omega;\mathbf{R}^m)$, the complement of the Lebesgue set of u by S_u, it is well known that for any $u \in BV(\Omega;\mathbf{R}^m)$ the set S_u is countably $(\mathcal{H}^{n-1}, n-1)$ rectifiable and that for $\mathcal{H}^{n-1}$ almost every $x \in S_u$ the function u has one sided approximate limits $u^+(x) \neq u^-(x)$ with respect to a suitable direction $\nu_u(x)$ normal to S_u at x (see (2.6)). Moreover, the distributional derivative Du can be decomposed, with respect to the Lebesgue measure $\mathcal{L}^n$, into singular part $D^s u$ and absolutely continuous part $D^a u = \nabla u\,\mathcal{L}^n$, with density ∇u equal to the approximate differential of u. The singular part $D^s u$ can be written as the sum of $D^s u \llcorner S_u$ and of $D^s u \llcorner (\Omega \setminus S_u)$; the first part, called *jump part* and denoted by $D^j u$, can be represented by the integration of $(u^+ - u^-) \otimes \nu_u$ on S_u, while the second part, called *Cantor part* and denoted by $D^c u$, vanishes on any Borel set $B \subset \Omega$ which is σ-finite with respect to $\mathcal{H}^{n-1}$. The space $SBV(\Omega;\mathbf{R}^m)$ of *special functions* with bounded variation (i.e., functions whose derivative has no Cantor part), introduced in [14], has been widely studied in the last years in connection with the Mumford-Shah image segmentation problem and, more generally, in connection with free discontinuity problems (see [27], [13], [1], [2], [15], [28], [3], [29], [26], [25]).

In addition, it is well known that the space $BV(\Omega;\mathbf{R}^m)$ can be characterized by means of the one dimensional sections: a function u belongs to $BV(\Omega;\mathbf{R}^m)$ if and only if, for every direction $\xi \in \mathbf{S}^{n-1} := \{\zeta \in \mathbf{R}^n : |\zeta| = 1\}$, we have $u_{y,\xi} \in BV(\Omega^\xi_y;\mathbf{R}^m)$ for

$\mathcal{H}^{n-1}$ almost every $y \in \pi_\xi$ and

$$\int_{\pi_\xi} |Du_{y,\xi}|(\Omega_y^\xi)\, d\mathcal{H}^{n-1}(y) < +\infty\,,$$

where π_ξ is the hyperplane orthogonal to ξ passing through the origin, $\Omega_y^\xi = \{t \in \mathbf{R} : y + t\xi \in \Omega\}$, and $u_{y,\xi}(t) := u(y + t\xi)$ for every $t \in \Omega_y^\xi$. The *structure theorem* for BV functions states that the scalar products $(D^a u, \xi)$, $(D^j u, \xi)$, and $(D^c u, \xi)$ can be recovered from the corresponding parts of the derivative of $u_{y,\xi}$, i.e.,

$$(D^\sigma u, \xi) = \int_{\pi_\xi} D^\sigma u_{y,\xi}\, d\mathcal{H}^{n-1}(y) \qquad \text{for } \sigma = a, j, c \tag{1.1}$$

(see Section 2 for the main notations). The structure theorem has been used by the first author in [1] to prove the compactness properties of SBV, and later it has been used to get lower semicontinuity theorems in BV in the isotropic case (see [10]) and Γ-convergence results (see [4], [5], [11]).

For BD functions the *discontinuity* set S_u is not known to be σ-finite with respect to $\mathcal{H}^{n-1}$. However, we discovered that in the BD theory the rôle of S_u is played by the *jump set* J_u of u, defined as the set of discontinuity points x where u has one sided approximate limits $u^+(x) \neq u^-(x)$ with respect to a suitable direction $\nu_u(x)$ normal to J_u (see Section 2). Let us remark that for BV functions the set $S_u \setminus J_u$ is $\mathcal{H}^{n-1}$ negligible.

The set of jump points J_u turns out to be useful in the decomposition of Eu. If $E^a u = \mathcal{E}u\,\mathcal{L}^n$ and $E^s u$ are the absolutely continuous and the singular part of Eu with respect to the Lebesgue measure, we set

$$Eu = E^a u + E^s u = \mathcal{E}u\,\mathcal{L}^n + E^j u + E^c u\,,$$

where $E^j u$ and $E^c u$ are the restrictions of $E^s u$ to J_u and to the complement of J_u respectively. First of all we prove that the density $\mathcal{E}u$ of $E^a u$ with respect to $\mathcal{L}^n$ can still be interpreted as an approximate symmetric differential (Theorem 4.3). Moreover we prove that J_u is countably $(\mathcal{H}^{n-1}, n-1)$ rectifiable and that the *jump part* $E^j u$ can be represented on every Borel subset of Ω by the formula

$$E^j u(B) = \int_{B \cap J_u} \Big[\frac{(u^+ - u^-)\otimes\nu_u + \nu_u\otimes(u^+ - u^-)}{2}\Big]\, d\mathcal{H}^{n-1}\,,$$

where ν_u is the unit normal to J_u in an approximate sense. Let us remark that J_u coincides, up to an $\mathcal{H}^{n-1}$ negligible set, with the set Θ_u introduced by Kohn in [21] and defined as the set of points in Ω where the $n-1$ dimensional upper density of $|Eu|$ is positive, i.e.,

$$\Theta_u := \left\{ x \in \Omega : \limsup_{\varrho \to 0+} \frac{|Eu|(B_\varrho(x))}{\varrho^{n-1}} > 0 \right\}. \tag{1.2}$$

Finally we prove in Theorem 4.4 that the *Cantor part* $E^c u$ vanishes on any Borel set $B \subset \Omega$ which is σ-finite with respect to $\mathcal{H}^{n-1}$.

One of the main results of our paper is the extension of the structure theorem to $BD(\Omega)$. Let $u \in L^1(\Omega; \mathbf{R}^n)$, let $\xi \in \mathbf{S}^{n-1}$, and let $u_y^\xi(t) = (u(y+t\xi), \xi)$, where $(\cdot,\cdot)$ denotes the scalar product. Using a result of [23] it follows that $u \in BD(\Omega)$ if and only if, for every $\xi \in \mathbf{S}^{n-1}$, the function u_y^ξ belongs to $BV(\Omega_y^\xi)$ for $\mathcal{H}^{n-1}$ almost every $y \in \pi_\xi$ and

$$\int_{\pi_\xi} |Du_y^\xi|(\Omega_y^\xi)\, d\mathcal{H}^{n-1}(y) < +\infty$$

(see [7] and [8]). We prove that

$$(E^\sigma u\xi, \xi) = \int_{\pi_\xi} D^\sigma u_y^\xi \, d\mathcal{H}^{n-1}(y) \qquad \text{for } \sigma = a, j, c. \tag{1.3}$$

In particular, the equality for $\sigma = j$ implies that, generically, the section in the ξ direction of the jump set of u coincides with the jump set of u_y^ξ, i.e.,

$$\{t : y + t\xi \in J_u\} = J_{u_y^\xi} = S_{u_y^\xi} \tag{1.4}$$

for $\mathcal{H}^{n-1}$ almost every $\xi \in \mathbf{S}^{n-1}$ and for $\mathcal{H}^{n-1}$ almost every $y \in \pi_\xi$. More generally, using the structure theorem and the Lebesgue representative of u, we can relate $u^+(x)$ and $u^-(x)$ to left and right limits of the one dimensional sections u_y^ξ, and $(\mathcal{E}u\xi, \xi)$ to the classical derivative of u_y^ξ (Theorem 4.5).

Following the analogy with the theory of SBV functions, we can define the space of *special functions* with bounded deformation $SBD(\Omega)$ by

$$SBD(\Omega) := \{u \in BD(\Omega) : E^c u = 0\}.$$

As a consequence of the structure theorem, it turns out that a function $u \in BD(\Omega)$ belongs to $SBD(\Omega)$ if and only if for any $\xi \in \mathbf{S}^{n-1}$ the function u_y^ξ belongs to $SBV(\Omega_y^\xi)$ for $\mathcal{H}^{n-1}$ almost every $y \in \pi_\xi$ (Proposition 4.7). This condition has previously been used in [7] and [8] to define $SBD(\Omega)$. Moreover, using (1.4) we obtain in Theorem 4.10 that the integralgeometric measure λ_u, introduced in [8] to study compactness and semicontinuity properties in $SBD(\Omega)$, coincides with $\mathcal{H}^{n-1} \llcorner J_u$.

In general we have that $J_u \subset S_u$, but the inclusion can be strict for a function $u \in BD(\Omega)$. In Section 6 we study the set $S_u \setminus J_u$, using a "Poincaré type" inequality for BD functions (Theorem 6.5) that we found in the unpublished part of Kohn's Ph.D. thesis [21]. The main result, proved in Theorem 6.1, is that the set $S_u \setminus J_u$ of discontinuity points of u which are not jump points is $|Ev|$ negligible for every $v \in BD(\Omega)$, i.e., $|Ev|$ almost every x in Ω is either a Lebesgue point or a jump point for u.

Finally, in the last section of the paper we prove that any function $u \in BD(\Omega)$ is approximately differentiable in $\mathcal{L}^n$ almost every point of Ω. A similar result has been proved independently by Hajłasz in [20]. If either $\Omega = \mathbf{R}^n$ or Ω is bounded with Lipschitz boundary, the approximate differential ∇u satisfies the weak L^1 estimate

$$\mathcal{L}^n\big(\{x \in \Omega : |\nabla u(x)| > t\}\big) \leq \frac{c(n,\Omega)}{t}\Big[\,|Eu|(\Omega) + \int_\Omega |u(x)|\,dx\,\Big]$$

for every $t > 0$. In case $E^s u = 0$ and $\mathcal{E}u \in L^p(\Omega)$ for some $p > 1$, our proof of the approximate differentiability can also be used, with minor modifications, to obtain Korn's inequality (Remark 7.11).

2. Preliminaries and notation

Let Ω be an open subset of $\mathbf{R}^n$, $n \geq 1$. The σ-algebra of all Borel subsets of Ω is denoted by $\mathcal{B}(\Omega)$. The symbols $(\cdot,\cdot)$ and $|\cdot|$ stand for the euclidean scalar product and the euclidean norm. The Lebesgue measure in $\mathbf{R}^n$ is denoted by dx or by $\mathcal{L}^n$, and the k dimensional Hausdorff measure by $\mathcal{H}^k$ (see [17], 2.10.2). We recall that for any two vectors a and b in $\mathbf{R}^n$, the symmetric product $a \odot b$ is the symmetric $n \times n$ matrix defined by $a \odot b = (a \otimes b + b \otimes a)/2$, where $\otimes$ denotes the tensor product.

If μ is a Borel measure on Ω with values in $\mathbf{R}^m$, $m \geq 1$, its total variation is denoted by $|\mu|$. If F is a Borel subset of Ω, the measure $\mu \llcorner F$ is defined by $(\mu \llcorner F)(B) = \mu(B \cap F)$ for any $B \in \mathcal{B}(\Omega)$. For every μ integrable function f, the measure $f\mu$ is defined by $(f\mu)(B) = \int_B f\,d\mu$ for every $B \in \mathcal{B}(\Omega)$. The absolutely continuous and singular parts of μ with respect to the Lebesgue measure $\mathcal{L}^n$ are denoted by μ^a and μ^s respectively.

The k *dimensional density* of a non-negative measure μ at a point $x \in \Omega$ is defined as

$$\lim_{\varrho \to 0+} \frac{\mu(B_\varrho(x))}{\omega_k \varrho^k},$$

where $B_\varrho(x)$ is the open ball in $\mathbf{R}^n$ with centre x and radius ϱ, and ω_k is the k dimensional measure of the unit ball in $\mathbf{R}^k$.

If $\mu = \{\mu_{ij}\}$ is a Radon measure on Ω with values in the space of symmetric $n\times n$ matrices and $\xi = (\xi^1, \dots, \xi^n) \in \mathbf{R}^n$, then the real valued Radon measure $(\mu\xi, \xi)$ is defined by

$$(\mu\xi,\xi) := \sum_{i,j} \xi^i \xi^j \mu_{ij}\,, \tag{2.1}$$

so that $(\mu\xi,\xi)(B) = (\mu(B)\xi,\xi)$ for every $B \in \mathcal{B}(\Omega)$. It is easy to check that for any basis $\xi_1, \dots, \xi_n$ of $\mathbf{R}^n$ there exists a constant c, depending on the basis, such that

$$|\mu| \le c \sum_{i,j=1}^{n} \big|(\mu(\xi_i+\xi_j), \xi_i+\xi_j)\big|\,. \tag{2.2}$$

For any $y, \xi \in \mathbf{R}^n$, $\xi \neq 0$, and any $B \subset \mathbf{R}^n$ we define

$$\begin{aligned} \pi_\xi &:= \{y \in \mathbf{R}^n : (y,\xi) = 0\}\,, \\ B^\xi_y &:= \{t \in \mathbf{R} : y + t\xi \in B\}\,, \\ B^\xi &:= \{y \in \pi_\xi : B^\xi_y \neq \emptyset\}\,. \end{aligned} \tag{2.3}$$

In other words π_ξ is the hyperplane orthogonal to ξ passing through the origin and $B^\xi = p_\xi(B)$, where p_ξ denotes the orthogonal projection onto π_ξ. Moreover B^ξ_y is the one dimensional section of B on the straight line passing through y in the direction of ξ.

Let $\xi \in \mathbf{R}^n$ with $\xi \neq 0$. For every $y \in \Omega^\xi$ let $\nu_y : \mathcal{B}(\Omega^\xi_y) \to \mathbf{R}$ be a bounded measure. Suppose that for every $\varphi \in \mathcal{C}_c(\mathbf{R})$ the map

$$y \mapsto \int_{\Omega^\xi_y} \varphi(t)\, d\nu_y(t)$$

is Borel measurable on Ω^ξ and that

$$\int_{\Omega^\xi} |\nu_y|(\Omega^\xi_y)\, d\mathcal{H}^{n-1}(y) < +\infty\,.$$

Then for every $B \in \mathcal{B}(\Omega)$ the map $y \mapsto \nu_y(B^\xi_y)$ is Borel measurable and $\mathcal{H}^{n-1}$ integrable on Ω^ξ, and the set function $\lambda : \mathcal{B}(\Omega) \to \mathbf{R}$ defined by

$$\lambda(B) := \int_{\Omega^\xi} \nu_y(B^\xi_y)\, d\mathcal{H}^{n-1}(y) \qquad \forall B \in \mathcal{B}(\Omega)$$

is a measure, which will be denoted by

$$\int_{\Omega^\xi} \nu_y \, d\mathcal{H}^{n-1}(y)\,.$$

It is easy to see that its total variation $|\lambda|$ is given by

$$|\lambda| = \int_{\Omega^\xi} |\nu_y| \, d\mathcal{H}^{n-1}(y)\,. \tag{2.4}$$

If each measure ν_y is absolutely continuous (resp. singular) with respect to $\mathcal{L}^1$, then $\int_{\Omega^\xi} \nu_y \, d\mathcal{H}^{n-1}(y)$ is absolutely continuous (resp. singular) with respect to $\mathcal{L}^n$.

For simplicity we have considered only the case of measures obtained, by integration, from their one dimensional sections. In the proof of Proposition 3.4 and Theorem 4.5 we shall use similar constructions by slicing Ω by parallel k dimensional planes.

Lebesgue points, discontinuity points, and jump points. Given a function $u \in L^1(\Omega;\mathbf{R}^m)$, the set of Lebesgue points of u in Ω is denoted by Ω_u. In other words, $x \in \Omega_u$ if and only if there exists $\tilde{u}(x) \in \mathbf{R}^m$ such that

$$\lim_{\varrho \to 0^+} \frac{1}{\varrho^n} \int_{B_\varrho(x)} |u(y) - \tilde{u}(x)| \, dy \;=\; 0\,. \tag{2.5}$$

The *Lebesgue discontinuity* set S_u of u is defined as the set of points which are not Lebesgue points, i.e., $S_u := \Omega \setminus \Omega_u$. By Lebesgue's differentiation theorem the set S_u is negligible and the function $\tilde{u}\colon \Omega_u \to \mathbf{R}^m$ coincides with u $\mathcal{L}^n$ almost everywhere in $\Omega_u = \Omega \setminus S_u$. The function $\tilde{u}$ is the so-called *Lebesgue representative* of u. We say that u has *one sided Lebesgue limits* $u^+(x)$ and $u^-(x)$ at $x \in \Omega$, with respect to a suitable direction $\nu_u(x) \in \mathbf{S}^{n-1} := \{\xi \in \mathbf{R}^n : |\xi| = 1\}$, if

$$\lim_{\varrho \to 0^+} \frac{1}{\varrho^n} \int_{B^\pm_\varrho(x,\nu_u(x))} |u(y) - u^\pm(x)| \, dy \;=\; 0\,, \tag{2.6}$$

where $B^\pm_\varrho(x,\nu_u(x)) := \{y \in B_\varrho(x) : (y - x, \pm\nu_u(x)) > 0\}$. In this case the rescaled functions $u_\varrho(y) := u(x + \varrho y)$ converge in $L^1(B_1(0);\mathbf{R}^m)$, as $\varrho \to 0^+$, to the function

$$u_0(y) \;:=\; \begin{cases} u^+(x)\,, & \text{if } (y,\nu_u(x)) > 0\,, \\[2ex] u^-(x)\,, & \text{if } (y,\nu_u(x)) < 0\,. \end{cases} \tag{2.7}$$

This shows that, if $u^+(x) \neq u^-(x)$, then the triplet $(u^+(x), u^-(x), \nu_u(x))$ is uniquely determined up to a change of sign of $\nu_u(x)$ and a permutation of $(u^+(x), u^-(x))$. If

$u^+(x) = u^-(x)$, then $x \in \Omega_u$ and the one sided Lebesgue limits coincide with the Lebesgue representative. The *jump set* J_u of u is defined as the set of all points $x \in S_u$ such that u has one sided Lebesgue limits at x. Equivalently, a point $x \in \Omega$ belongs to J_u if and only if u has two different one sided Lebesgue limits at x. It is easy to prove that J_u and S_u are Borel sets and that $\tilde{u}$, u^+, and u^- are Borel functions.

Pointwise variation and distributional derivative. Let A be an open subset of $\mathbf{R}$, let E be a Borel subset of A such that $\mathcal{L}^1(A \setminus E) = 0$, and let $u\colon E \to \mathbf{R}^m$ be an integrable function. As u is defined $\mathcal{L}^1$ almost everywhere in A, it can be identified with a distribution on A, whose derivative is denoted by Du. For every open interval $I \subset A$, the *pointwise variation* $Vu(I)$ of u on I is defined by

$$Vu(I) := \sup\Big\{\sum_{i=1}^{k-1} |u(t_{i+1}) - u(t_i)| : t_1 < t_2 < \cdots < t_k,\ t_i \in E \cap I\Big\}. \tag{2.8}$$

Let us remark that Vu, unlike Du, is sensitive to changes of the values (or of the domain) of u on sets of Lebesgue measure zero. If $Vu(I) < +\infty$, then Du can be represented by a bounded measure on I, whose total variation satisfies

$$|Du|(I) \le Vu(I). \tag{2.9}$$

Moreover, if $Vu(I) < +\infty$ for every open interval $I \subset A$, then it is possible to extend Vu to a non-negative Radon measure defined on $\mathcal{B}(A)$, which will still be denoted by Vu. Indeed, given an open set $U \subset A$, it is enough to define $Vu(U)$ as the sum of $Vu(I)$ over all connected components I of U. It turns out that the set function $U \mapsto Vu(U)$ is increasing, subadditive, additive on disjoint open sets, and inner regular. These properties imply (see, e.g., [12], Theorem 14.23) that

$$Vu(B) := \inf\Big\{Vu(U) : U \text{ open },\ B \subset U \subset A\Big\} \qquad \forall B \in \mathcal{B}(A)$$

is a Radon measure on A which coincides with the function defined by (2.8) on all open intervals $I \subset A$. It is easy to prove that, if $Vu(I) < +\infty$ for every open interval $I \subset A$, then there exists an extension v of u, defined on A, such that $Vv = Vu$ as measures on A. Therefore, using the classical theory of functions of bounded variation on bounded intervals, that applies to v on each interval $I \subset A$, we obtain that u has left and right limits at every $t \in A$ and that these limits coincide except on a countable subset of A. Moreover the classical derivative

$$u'(t) = \lim_{\substack{s \to t \\ s \in E}} \frac{u(s) - u(t)}{s - t} \tag{2.10}$$

exists for $\mathcal{L}^1$ almost every $t \in E$ and coincides with the density of the absolutely continuous part of the measure Du with respect to $\mathcal{L}^1$.

It is well known that, if $E = A$ and $u \in \mathcal{C}^1(A)$, then

$$Vu = |Du| = |u'|\mathcal{L}^1 \tag{2.11}$$

as measures on A.

The spaces $BV(\Omega;\mathbf{R}^m)$ and $SBV(\Omega;\mathbf{R}^m)$. The space $BV(\Omega;\mathbf{R}^m)$, $m \geq 1$, of *functions of bounded variation* in Ω with values in $\mathbf{R}^m$ is defined as the set of all functions $u \in L^1(\Omega;\mathbf{R}^m)$ whose distributional gradient Du is a Radon measure (with values in the space of $m\times n$ matrices) with bounded total variation in Ω. We set $BV(\Omega) = BV(\Omega;\mathbf{R})$. It is well known that for every $u \in BV(\Omega;\mathbf{R}^n)$ the discontinuity set S_u is a countably $(\mathcal{H}^{n-1}, n-1)$ rectifiable Borel set, i.e., $\mathcal{H}^{n-1}$ almost all of S_u can be covered by a sequence of $\mathcal{C}^1$ hypersurfaces M_i (see, e.g., [17], 3.2.14). Hence, we can define $\mathcal{H}^{n-1}$ almost everywhere in S_u an approximate unit normal ν_u (see [31], Section 11), characterized by the fact that for every i

$$\nu_u(x) = \pm\nu_{M_i}(x) \qquad \text{for } \mathcal{H}^{n-1} \text{ a.e. } x \in S_u \cap M_i\,,$$

where $\nu_{M_i}(x)$ is a normal to M_i at the point x. Moreover, for $\mathcal{H}^{n-1}$ almost every $x \in S_u$, the function u has one sided Lebesgue limits $u^+(x) \neq u^-(x)$ with respect to the approximate unit normal $\nu_u(x)$, which is uniquely determined up to the sign. In particular, $\mathcal{H}^{n-1}(S_u \setminus J_u) = 0$.

If $u \in BV(\Omega;\mathbf{R}^m)$, then the Lebesgue decomposition of the measure Du can be written as

$$Du \;=\; D^a u + D^s u \;=\; \nabla u\, dx + D^s u\,,$$

where ∇u denotes the density of the absolutely continuous part of Du; the notation is motivated by the fact that ∇u can be interpreted as an approximate differential for u, i.e.,

$$\lim_{\varrho\to 0^+} \frac{1}{\varrho^n}\int_{B_\varrho(x)} \frac{|u(y)-u(x)-(\nabla u(x), y-x)|}{|y-x|}\, dy \;=\; 0 \tag{2.12}$$

for $\mathcal{L}^n$ almost every $x \in \Omega$. The singular part $D^s u$ can be further decomposed into two mutually singular measures as

$$D^s u \;=\; D^j u + D^c u \;=\; (u^+ - u^-)\otimes\nu_u\, \mathcal{H}^{n-1} \llcorner S_u + D^c u\,, \tag{2.13}$$

where $D^j u = D^s u \llcorner S_u = Du \llcorner S_u$ is the $n-1$ dimensional part of Du and $D^c u = D^s u \llcorner \Omega_u$ is the Cantor part of Du, vanishing on all Borel sets B with $\mathcal{H}^{n-1}(B) < +\infty$.

We say that u is a *special function of bounded variation* in Ω with values in $\mathbf{R}^m$, and we write $u \in SBV(\Omega; \mathbf{R}^m)$, if $u \in BV(\Omega; \mathbf{R}^m)$ and $D^c u = 0$ in Ω; again, we write $SBV(\Omega)$ for $SBV(\Omega; \mathbf{R})$. For a general exposition of the theory of functions of bounded variation we refer to [19], [37], [16]; for an introduction to the properties of the space $SBV(\Omega; \mathbf{R}^m)$ we refer to [14], [1], [2], [3].

3. The space $\mathbf{BD}(\mathbf{\Omega})$

Let $u = (u^1, \dots, u^n) \in L^1(\Omega; \mathbf{R}^n)$, and let Eu be the symmetric part of the distributional gradient of u, i.e.,

$$Eu := \{E_{ij} u\}, \qquad E_{ij} u := \frac{1}{2}\big(D_i u^j + D_j u^i\big).$$

The space $BD(\Omega)$ of vector fields of *bounded deformation* in Ω is defined as the set of all functions $u \in L^1(\Omega; \mathbf{R}^n)$ such that $E_{ij} u$ is a Radon measure with bounded total variation in Ω for any $i, j = 1, \dots, n$. The space $BD(\Omega)$ is a Banach space with the norm $\|u\|_{BD(\Omega)} = \|u\|_{L^1(\Omega)} + |Eu|(\Omega)$.

For a general exposition of the theory of functions of bounded deformation we refer to [33], [34], [21], [22], [6], [36], [35].

Now we summarize some results on functions of bounded deformation which will be useful in the sequel.

(i) For every $\xi = (\xi^1, \dots, \xi^n) \in \mathbf{R}^n$ let D_ξ be the distributional derivative in the direction ξ defined by $D_\xi v = \sum_j \xi^j D_j v$. For every function $u \colon \Omega \to \mathbf{R}^n$ let us define the function $u^\xi \colon \Omega \to \mathbf{R}$ by $u^\xi(x) = (u(x), \xi)$. If $u \in BD(\Omega)$, then $D_\xi u^\xi$ is a bounded Radon measure on Ω for every $\xi \in \mathbf{R}^n$ and

$$D_\xi u^\xi = (Eu\xi, \xi). \tag{3.1}$$

Conversely, let $\xi_1, \dots, \xi_n$ be a basis of $\mathbf{R}^n$ and let $u \in L^1(\Omega; \mathbf{R}^n)$; then $u \in BD(\Omega)$ if $D_\xi u^\xi$ is a bounded Radon measure on Ω for every ξ of the form $\xi_i + \xi_j$, $i, j = 1, \dots, n$ (see [35], Chapter II, Section 2.2).

(ii) Let $u \in BD(\Omega)$ and let M be a $\mathcal{C}^1$ hypersurface contained in Ω; it is well known (see [35], Chapter II) that the one sided Lebesgue limits $u^\pm(x)$ on both sides of M (i.e., with respect to $\nu_M(x)$) exist for $\mathcal{H}^{n-1}$ almost every $x \in M$ and satisfy

$$Eu \llcorner M = (u^+ - u^-) \odot \nu_M \, \mathcal{H}^{n-1} \llcorner M, \tag{3.2}$$

where ν_M is a unit normal to M and $\odot$ is the symmetric product between vectors (see Section 2). It is easy to extend these properties to any countably $(\mathcal{H}^{n-1}, n-1)$ rectifiable Borel set $M \subset \Omega$.

(iii) If Ω is a bounded open set with Lipschitz boundary $\partial\Omega$ and $u \in BD(\Omega)$, then the function

$$\bar{u}(x) := \begin{cases} u(x), & \text{if } x \in \Omega, \\ 0, & \text{if } x \notin \Omega, \end{cases} \tag{3.3}$$

belongs to $BD(\mathbf{R}^n)$ (see [35], Chapter II, Proposition 2.1) and, for $\mathcal{H}^{n-1}$ almost every $x \in \partial\Omega$, the one sided Lebesgue limits $\bar{u}^+(x)$ and $\bar{u}^-(x)$ with respect to the outer unit normal are respectively equal to 0 and to the trace of u on $\partial\Omega$, that will be denoted by u_*. Moreover we have the estimate $\|u_*\|_{L^1(\partial\Omega)} \le c(\Omega)\|u\|_{BD(\Omega)}$ for a suitable constant $c(\Omega)$ depending on Ω (see [35], Chapter II, Theorem 2.1, for the estimate of the trace, and [24], Theorem 1, for an argument which proves the existence of the one sided Lebesgue limits at the boundary). Given a smooth symmetric matrix valued function $\varphi = \{\varphi_{lm}\}$ with compact support in $\mathbf{R}^n$, applying (3.2) to $\bar{u}$ and using the coincidence of Eu and $E\bar{u}$ in Ω we obtain the *divergence theorem* in BD:

$$\sum_{l,m=1}^{n} \int_\Omega \varphi_{lm}\, dE_{lm}u + \int_\Omega (u, \mathrm{div}\varphi)\, dx \;=\; \int_{\partial\Omega} (u_* \odot \nu_\Omega, \varphi)\, d\mathcal{H}^{n-1}, \tag{3.4}$$

where ν_Ω is the outer unit normal to $\partial\Omega$ and $\mathrm{div}\varphi$ is the $\mathbf{R}^n$ valued function defined by

$$(\mathrm{div}\varphi)^l \;=\; \sum_{m=1}^{n} D_m\varphi_{lm} \qquad\qquad l = 1, \ldots, n.$$

(iv) Let $\mathcal{R}$ be the class of *rigid motions* in $\mathbf{R}^n$, i.e., affine maps of the form $Ax + d$ with A skew symmetric $n \times n$ matrix and $d \in \mathbf{R}^n$. The following "Poincaré type" inequality for BD functions follows from Proposition 2.2 and Remark 1.1 of Chapter II of [35]. For a complete proof see [21], Part II, Proposition 3.11.

Theorem 3.1. *Let Ω be a bounded connected open set with Lipschitz boundary and let $R: BD(\Omega) \to \mathcal{R}$ be a continuous linear map which leaves the elements of $\mathcal{R}$ fixed. Then there exists a constant $c(\Omega, R)$ such that*

$$\int_\Omega |u - R(u)|\, dx \;\le\; c(\Omega, R)|Eu|(\Omega) \qquad\qquad \forall u \in BD(\Omega).$$

Now we shall see that the space $BD(\Omega)$ can be characterized using suitable one dimensional sections (see [23], Theorem 3.3, and [17], Theorem 4.5.9, for the analogous property in BV; see also [8], Theorem 3.1 and Proposition 3.2, where a slightly weaker property is proved for BD). We shall use the notation introduced in (2.3). Given a function $u: B \to \mathbf{R}^n$, defined on a subset B of $\mathbf{R}^n$, for every $y, \xi \in \mathbf{R}^n$, $\xi \neq 0$, the function $u_y^\xi: B_y^\xi \to \mathbf{R}$ is defined by

$$u_y^\xi(t) = u^\xi(y + t\xi) = (u(y+t\xi), \xi) \qquad \forall t \in B_y^\xi . \tag{3.5}$$

When $\xi = e_i$, the i^{th} vector of the canonical basis, we write u_y^i instead of $u_y^{e_i}$. If $u \in L^1(\Omega; \mathbf{R}^n)$, then for every $\xi \in \mathbf{R}^n$, $\xi \neq 0$, and every $y \in \Omega^\xi$ the one dimensional section $\tilde{u}_y^\xi$ of the function $\tilde{u}$ introduced in (2.5) is defined for every $t \in (\Omega_u)_y^\xi$ and the corresponding pointwise variation $V\tilde{u}_y^\xi$ is evaluated using only points $t \in (\Omega_u)_y^\xi$.

Proposition 3.2. *Let $u \in BD(\Omega)$ and let $\xi \in \mathbf{R}^n$ with $\xi \neq 0$. Then the following two conditions hold for $\mathcal{H}^{n-1}$ almost every $y \in \Omega^\xi$:*

(i) *$\tilde{u}_y^\xi$ is defined and coincides with u_y^ξ $\mathcal{L}^1$ almost everywhere in Ω_y^ξ;*

(ii) *$u_y^\xi \in BV(\Omega_y^\xi)$ and the measures $|Du_y^\xi|$ and $V\tilde{u}_y^\xi$ coincide on Ω_y^ξ.*

Moreover

$$(Eu\xi, \xi) = \int_{\Omega^\xi} Du_y^\xi \, d\mathcal{H}^{n-1}(y) , \qquad |(Eu\xi,\xi)| = \int_{\Omega^\xi} |Du_y^\xi| \, d\mathcal{H}^{n-1}(y) \tag{3.6}$$

as measures in Ω.

Conversely, let $u \in L^1(\Omega; \mathbf{R}^n)$ and let $\xi_1, \ldots, \xi_n$ be a basis of $\mathbf{R}^n$. Assume that for every ξ of the form $\xi_i + \xi_j$

$$\begin{gathered} u_y^\xi \in BV(\Omega_y^\xi) \quad \text{for } \mathcal{H}^{n-1} \text{ a.e. } y \in \Omega^\xi , \\ \int_{\Omega^\xi} |Du_y^\xi|(\Omega_y^\xi) \, d\mathcal{H}^{n-1}(y) < +\infty . \end{gathered} \tag{3.7}$$

Then $u \in BD(\Omega)$.

Proof. Since all properties are local, by (3.3) we may assume $\Omega = \mathbf{R}^n$. Let $u \in BD(\mathbf{R}^n)$; as $\mathcal{L}^n(S_u) = 0$ and $u = \tilde{u}$ $\mathcal{L}^n$ almost everywhere in $\mathbf{R}^n$, Fubini's theorem implies (i).

Let $u_\epsilon := u \star \psi_\epsilon$ be a regularization of u; it is well known (see, e.g., [37], Theorem 1.6.1) that u_ϵ converges pointwise to $\tilde{u}$ at every Lebesgue point of u; moreover, since $(Eu_\epsilon\xi,\xi) = (Eu\xi,\xi)\star\psi_\epsilon$, Fubini's theorem and (2.11) imply that

$$|(Eu\xi,\xi)|(\mathbf{R}^n) \geq |(Eu_\epsilon\xi,\xi)|(\mathbf{R}^n) = \int_{\pi_\xi} V[(u_\epsilon)^\xi_y](\mathbf{R})\, d\mathcal{H}^{n-1}(y).$$

Now we pass to the limit as $\epsilon \to 0^+$, noticing that, by (2.8), $v \mapsto Vv(\mathbf{R})$ is lower semicontinuous with respect to pointwise convergence. By Fatou's lemma we get

$$|(Eu\xi,\xi)|(\mathbf{R}^n) \geq \int_{\pi_\xi} V\tilde{u}^\xi_y(\mathbf{R})\, d\mathcal{H}^{n-1}(y). \tag{3.8}$$

In particular, by (2.9) we infer that $u^\xi_y \in BV(\mathbf{R})$ for $\mathcal{H}^{n-1}$ almost every $y \in \pi_\xi$ and $\int_{\pi_\xi} |Du^\xi_y|(\mathbf{R})\, d\mathcal{H}^{n-1}(y) < +\infty$. For every function v defined on $\mathbf{R}^n$ we set $v_{y,\xi}(t) := v(y+t\xi)$ for every y, $\xi \in \mathbf{R}^n$ and for every $t \in \mathbf{R}$. Since $D_\xi\varphi(y+t\xi) = D\varphi_{y,\xi}(t)$, by Fubini's theorem we get

$$\begin{aligned} \int_{\mathbf{R}^n} \varphi\, d(Eu\xi,\xi) &= -\int_{\mathbf{R}^n} u^\xi D_\xi\varphi\, dx = -\int_{\pi_\xi} \Big[\int_{\mathbf{R}} u^\xi_y D\varphi_{y,\xi}\, dt\Big]\, d\mathcal{H}^{n-1}(y) \\ &= \int_{\pi_\xi} \Big[\int_{\mathbf{R}} \varphi_{y,\xi}\, d(Du^\xi_y)\Big]\, d\mathcal{H}^{n-1}(y) \end{aligned} \tag{3.9}$$

for any $\varphi \in \mathcal{C}^1_c(\mathbf{R}^n)$. Therefore, a density argument implies the first equality in (3.6); the second equality is a consequence of (2.4). From (3.6), (3.8), and (2.9) it follows that the measures $|Du^\xi_y|$ and $V\tilde{u}^\xi_y$ coincide for $\mathcal{H}^{n-1}$ almost every $y \in \pi_\xi$, and the proof of (ii) is complete.

Conversely, assume that $u \in L^1(\Omega;\mathbf{R}^n)$ and that (3.7) is satisfied for every ξ of the form $\xi_i + \xi_j$, where $\xi_1,\ldots,\xi_n$ be a basis of $\mathbf{R}^n$. By (3.9) for these vectors ξ there exist constants $C(\xi)$ such that

$$\int_{\mathbf{R}^n} u^\xi D_\xi\varphi\, dx \leq C(\xi)\|\varphi\|_{L^\infty(\mathbf{R}^n)} \qquad \forall\varphi \in \mathcal{C}^1_c(\mathbf{R}^n).$$

This implies that $D_\xi u^\xi$ is a bounded Radon measure on $\mathbf{R}^n$ and, consequently, u belongs to $BD(\mathbf{R}^n)$ (property (i) at the beginning of this section). $\square$

Remark 3.3. Using (3.6) we obtain that $|(Eu\xi,\xi)|(B)=0$ for every $B\in\mathcal{B}(\Omega)$ with $\mathcal{H}^{n-1}(B)=0$. Taking (2.2) into account we conclude that

$$\mathcal{H}^{n-1}(B)=0 \qquad\Longrightarrow\qquad |Eu|(B)=0 \tag{3.10}$$

for every $B\in\mathcal{B}(\Omega)$. More generally, $|Eu|(B)=0$ for every $B\in\mathcal{B}(\Omega)$ such that $\mathcal{H}^{n-1}\big(\pi_\xi(B)\big)=0$ for $\mathcal{H}^{n-1}$ almost every $\xi\in\mathbf{S}^{n-1}$ (see also [21], Part II, Proposition 4.16 and Lemma 4.2).

The next proposition partially extends Proposition 3.2 to a more general situation. For every linear subspace V of $\mathbf{R}^n$ we can define the space $BD(V)$ by introducing a suitable orthogonal coordinate system on V. This definition does not depend on the choice of the coordinate system (see property (i) at the beginning of this section).

Proposition 3.4. *Let V be a linear subspace of $\mathbf{R}^n$ of dimension k, let $V^\perp$ be the orthogonal subspace, and let p_V be the orthogonal projection onto V. Given a function $u\in BD(\mathbf{R}^n)$, for every $z\in V^\perp$ let $v_z\colon V\to V$ be the function defined by $v_z(y)=p_V u(y+z)$. Then $v_z\in BD(V)$ for $\mathcal{H}^{n-k}$ almost every $z\in V^\perp$, and for every $B\in\mathcal{B}(\mathbf{R}^n)$ and every $\xi\in v$ we have*

$$(Eu\xi,\xi)(B) \;=\; \int_{V^\perp}(Ev_z\xi,\xi)(B_z)\,d\mathcal{H}^{n-k}(z)\,, \tag{3.11}$$

$$|(Eu\xi,\xi)|(B) \;=\; \int_{V^\perp}|(Ev_z\xi,\xi)|(B_z)\,d\mathcal{H}^{n-k}(z)\,, \tag{3.12}$$

where $B_z=\{y\in V: y+z\in B\}$.

Proof. Let $\xi_1,\dots,\xi_k$ be a basis of V and let ξ be a vector of the form $\xi_i+\xi_j$. Recalling property (i) at the beginning of this section, it is enough to prove that $D_\xi(v_z)^\xi$ is a Radon measure on V for $\mathcal{H}^{n-k}$ almost every $z\in V^\perp$. This property follows from Theorem 3.3 of [23], which proves also that

$$|D_\xi u^\xi|(B) \;=\; \int_{V^\perp}|D_\xi(v_z)^\xi|(B_z)\,d\mathcal{H}^{n-k}(z)\,.$$

By (3.1) this formula is equivalent to (3.12). Now (3.11) can be obtained as in the proof of (3.6). $\square$

For every $u \in BD(\Omega)$ let us consider now the set Θ_u defined in (1.2) and introduced by Kohn in [21]. First of all we claim that $J_u \subset \Theta_u$. Indeed, let x be a point in the jump set J_u; using the lower semicontinuity of the map $v \mapsto |Ev|(B_1(0))$ with respect to the $L^1(B_1(0);\mathbf{R}^n)$ convergence, from (2.7) and (3.2) we infer

$$\begin{aligned} \liminf_{\varrho\to 0+} \frac{|Eu|(B_\varrho(x))}{\varrho^{n-1}} &= \liminf_{\varrho\to 0+} |Eu_\varrho|(B_1(0)) \ \geq\ |Eu_0|(B_1(0)) \\ &= \omega_{n-1}|(u^+(x)-u^-(x))\odot\nu(x)| \ \geq\ \frac{\omega_{n-1}}{\sqrt{2}}|u^+(x)-u^-(x)| \ >\ 0\,. \end{aligned}$$

Hence $x \in \Theta_u$ and the claim is proved.

The following proposition concerns the rectifiability of the sets Θ_u and J_u.

Proposition 3.5. *For every $u \in BD(\Omega)$ the jump set J_u is a countably $(\mathcal{H}^{n-1}, n-1)$ rectifiable Borel set. Moreover, $\mathcal{H}^{n-1}(\Theta_u \setminus J_u) = 0$.*

Proof. The fact that Θ_u is countably $(\mathcal{H}^{n-1}, n-1)$ rectifiable is proved in [21], Part II, Theorem 4.18. For the reader's convenience we repeat that proof here. For every $\epsilon > 0$ let us consider the set

$$\Theta_u^\epsilon := \{x \in \Omega : \limsup_{\varrho\to 0+} \frac{|Eu|(B_\varrho(x))}{\varrho^{n-1}} > \epsilon\}\,.$$

It is easy to see that Θ_u^ϵ is a Borel set for every $\epsilon > 0$. In order to show that Θ_u is countably $(\mathcal{H}^{n-1}, n-1)$ rectifiable it is enough to prove the same property for each Θ_u^ϵ. By Theorem 2.10.19 of [17] we have that

$$\epsilon\,\mathcal{H}^{n-1}(B) \leq \omega_{n-1}|Eu|(B) \qquad \forall B \in \mathcal{B}(\Omega),\ B \subset \Theta_u^\epsilon\,. \tag{3.13}$$

In particular, $\mathcal{H}^{n-1}(\Theta_u^\epsilon) < +\infty$ and we can apply Theorems 3.3.13 and 2.10.15 of [17] to obtain a countably $(\mathcal{H}^{n-1}, n-1)$ rectifiable Borel set $R^\epsilon \subset \Theta_u^\epsilon$ such that

$$\mathcal{H}^{n-1}(p_\xi(\Theta_u^\epsilon \setminus R^\epsilon)) = 0 \quad \text{for } \mathcal{H}^{n-1} \text{ a.e. } \xi \in \mathbf{S}^{n-1}\,.$$

Choosing a basis $\xi_1, \ldots, \xi_n$ of $\mathbf{R}^n$ satisfying $\mathcal{H}^{n-1}(p_{\xi_i}(\Theta_u^\epsilon \setminus R^\epsilon)) = 0$, by (3.6) we infer $|(Eu\xi_i, \xi_i)|(\Theta_u^\epsilon \setminus R^\epsilon) = 0$ for $i = 1, \ldots, n$. By (2.2) we obtain that $|Eu|(\Theta_u^\epsilon \setminus R^\epsilon) = 0$. Choosing $B = \Theta_u^\epsilon \setminus R^\epsilon$ in (3.13) we obtain that $\mathcal{H}^{n-1}(\Theta_u^\epsilon \setminus R^\epsilon) = 0$ and the claim is proved.

Since Θ_u is countably $(\mathcal{H}^{n-1}, n-1)$ rectifiable, the one sided Lebesgue limits $u^{\pm}(x)$ of u at x are well defined for $\mathcal{H}^{n-1}$ almost every $x \in \Theta_u$ (see (ii) at the beginning of the section). Now, for every $\epsilon > 0$ let us consider the Borel set $B^\epsilon = \{x \in \Theta_u^\epsilon : u^+(x) = u^-(x)\}$. By (3.13) and (3.2) we know that $\epsilon\mathcal{H}^{n-1}(B^\epsilon) \leq \omega_{n-1}|Eu|(B^\epsilon) = 0$, which implies that $\mathcal{H}^{n-1}(B^\epsilon) = 0$ for every $\epsilon > 0$. Therefore $\mathcal{H}^{n-1}(\{x \in \Theta_u : u^+(x) = u^-(x)\}) = 0$ and the proof is complete. □

Remark 3.6. By the definition of J_u for every $x \in J_u$ there exists a unit vector $\nu_u(x)$ which satisfies (2.6). Since J_u is countably $(\mathcal{H}^{n-1}, n-1)$ rectifiable, for $\mathcal{H}^{n-1}$ almost every $x \in J_u$ we can define a Borel measurable approximate unit normal ν_{J_u} characterized by the following property: for every $\mathcal{C}^1$ hypersurface M we have

$$\nu_{J_u}(x) = \pm\nu_M(x) \qquad \text{for } \mathcal{H}^{n-1} \text{ a.e. } x \in J_u \cap M\,.$$

By property (ii), stated at the beginning of this section, the function u has one sided Lebesgue limits $u^{\pm}(x)$ with respect to the normal $\nu_{J_u}(x)$ for $\mathcal{H}^{n-1}$ almost every $x \in J_u$. By the uniqueness of the one sided Lebesgue limits, this implies that $\nu_u(x) = \pm\nu_{J_u}(x)$ for $\mathcal{H}^{n-1}$ almost every $x \in J_u$. In other words, $\nu_u(x)$ is an approximate unit normal to J_u for $\mathcal{H}^{n-1}$ almost every $x \in J_u$.

4. The structure theorem

For every $u \in BD(\Omega)$ we shall consider the Radon-Nikodym decomposition $Eu = E^a u + E^s u$ of Eu, where $E^a u$ is absolutely continuous and $E^s u$ is singular with respect to $\mathcal{L}^n$.

Definition 4.1. Let $u \in BD(\Omega)$.

(i) The *approximate symmetric differential* of u is the density $\mathcal{E}u$ of $E^a u$ with respect to $\mathcal{L}^n$, i.e., $E^a u = \mathcal{E}u\,\mathcal{L}^n$.

(ii) The *jump part* of Eu is the restriction $E^j u$ of $E^s u$ to the jump set J_u, i.e., $E^j u := E^s u \,\llcorner\, J_u$.

(iii) The *Cantor part* of Eu is the restriction $E^c u$ of $E^s u$ to the complement of J_u, i.e., $E^c u := E^s u \,\llcorner\, (\Omega \setminus J_u)$.

Remark 4.2. Since $J_u \subset S_u$ is Lebesgue negligible, we have also $E^j u = Eu \llcorner J_u$. The definition above implies the following decomposition of Eu into three parts, analogous with the one adopted for BV functions:

$$Eu = \mathcal{E}u\,\mathcal{L}^n + E^j u + E^c u\,. \tag{4.1}$$

Since J_u is countably $(\mathcal{H}^{n-1}, n-1)$ rectifiable, from (3.2) with $M := J_u$ we infer the representation

$$E^j u \;=\; (u^+ - u^-) \odot \nu_u\,\mathcal{H}^{n-1} \llcorner J_u\,. \tag{4.2}$$

Since $\Theta_u \setminus J_u$ is $\mathcal{H}^{n-1}$ negligible, (3.10) shows that the measure $E^j u$ can also be obtained by restriction of Eu to the larger set Θ_u.

The following theorem justifies our terminology for the density $\mathcal{E}u$ of the absolutely continuous part of Eu (see also the Appendix of [8], where a weaker property of the same type is proved).

Theorem 4.3. *Let $u \in BD(\Omega)$. For $\mathcal{L}^n$ almost every $x \in \Omega$ we have*

$$\lim_{\varrho \to 0^+} \frac{1}{\varrho^n} \int_{B_\varrho(x)} \frac{|(u(y) - u(x) - \mathcal{E}u(x)(y-x), y-x)|}{|y-x|^2}\,dy \;=\; 0\,.$$

Proof. Let $B_\varrho(x)$ be a ball contained in Ω, let $\tau \in (0,1)$, and let $v \in \mathcal{C}^1(B_\varrho(x))$. Using the equality

$$(v(y) - v(x + \tau(y-x)), y-x) \;=\; \int_\tau^1 \big(\mathcal{E}v(x + t(y-x))(y-x), y-x\big)\,dt$$

and Fubini's theorem, a straightforward computation gives

$$\int_{B_\varrho(x)} \frac{|(v(y) - v(x + \tau(y-x)), y-x)|}{|y-x|^2}\,dy \le \int_\tau^1 \frac{|Ev|(B_{t\varrho}(x))}{t^n}\,dt\,. \tag{4.3}$$

Assume now that $v \in BD(\Omega)$ and let $v_\epsilon := v \star \psi_\epsilon$ be the corresponding regularization. Since $|Ev_\epsilon|(B_{t\varrho}(x))$ converges to $|Ev|(B_{t\varrho}(x))$ as $\epsilon \to 0^+$ for any $t \in (0,1)$ such that $|Ev|(\partial B_{t\varrho}(x)) = 0$, passing to the limit we can prove that (4.3) holds also for $v \in BD(\Omega)$. If $x \notin S_v$, letting $\tau \to 0^+$ and dividing both sides by ϱ^n we obtain

$$\frac{1}{\varrho^n} \int_{B_\varrho(x)} \frac{|(v(y) - \tilde{v}(x), y-x)|}{|y-x|^2}\,dy \le \int_0^1 \frac{|Ev|(B_{t\varrho}(x))}{(t\varrho)^n}\,dt\,. \tag{4.4}$$

Let us fix a Lebesgue point x for $\mathcal{E}u$ such that $\widetilde{\mathcal{E}u}(x) = \mathcal{E}u(x)$, $x \notin S_u$, $\tilde{u}(x) = u(x)$, and $|E^s u|$ has n dimensional density zero at x (all these conditions are satisfied $\mathcal{L}^n$ almost everywhere in Ω); applying (4.4) to $v(y) := u(y) - \mathcal{E}u(x)(y-x)$ we obtain

$$\begin{aligned}
&\frac{1}{\varrho^n}\int_{B_\varrho(x)} \frac{|(u(y)-u(x)-\mathcal{E}u(x)(y-x), y-x)|}{|y-x|^2}\,dy \\
&\leq \sup_{t\in(0,1)} \Big[\frac{|E^s u|(B_{t\varrho}(x))}{(t\varrho)^n} + \frac{1}{(t\varrho)^n}\int_{B_{t\varrho}(x)} |\mathcal{E}u(y) - \mathcal{E}u(x)|\,dy \Big].
\end{aligned}$$

By our choice of x, both terms in the right hand side vanish as $\varrho \to 0^+$. $\square$

The following proposition shows that, in (4.1), $E^j u$ is the only measure concentrated on $n-1$ dimensional sets.

Proposition 4.4. *For every $u \in BD(\Omega)$ the measure $E^c u$ vanishes on any Borel set $B \subset \Omega$ which is σ-finite with respect to $\mathcal{H}^{n-1}$.*

Proof. Let $B \subset \Omega$ be a Borel set such that $\mathcal{H}^{n-1}(B) < +\infty$. Since $\Theta_u \setminus J_u$ is $\mathcal{H}^{n-1}$ negligible (Proposition 3.5), by (3.10) the measure $|E^c u|$ is concentrated on $\Omega \setminus \Theta_u$. For this reason it is not restrictive to assume that $B \cap \Theta_u = \emptyset$; under this assumption, the $n-1$ dimensional density of $|E^c u|$ at any point of B is zero. A covering argument (see, e.g., [17], 2.10.19) shows that $|E^c u|(B) \leq \epsilon\mathcal{H}^{n-1}(B)$ for any $\epsilon > 0$, hence $|E^c u|(B) = 0$. $\square$

The main result of our paper is the following structure theorem for BD functions, that will be proved in Section 5. Properties (i) and (ii) are proved also in [8], Proposition 3.2 and Lemma 3.3. For every $\xi \in \mathbf{R}^n$ let us define

$$J_u^\xi := \{x \in J_u : (u^+(x) - u^-(x), \xi) \neq 0\}.$$

Since $\mathcal{H}^{n-1}(\{\xi \in \mathbf{S}^{n-1} : (u^+(x) - u^-(x), \xi) = 0\}) = 0$ for every $x \in J_u$, by Fubini's theorem we have

$$\mathcal{H}^{n-1}(J_u \setminus J_u^\xi) = 0 \qquad \text{for } \mathcal{H}^{n-1} \text{ a.e. } \xi \in \mathbf{S}^{n-1}. \tag{4.5}$$

We recall that u_y^ξ is the function defined in (3.5).

Theorem 4.5. *Let $u \in BD(\Omega)$ and let $\xi \in \mathbf{R}^n$ with $\xi \neq 0$. Then the following properties hold:*

(i) $$(E^a u\xi, \xi) = \int_{\Omega^\xi} D^a u^\xi_y \, d\mathcal{H}^{n-1}(y)\,, \qquad |(E^a u\xi, \xi)| = \int_{\Omega^\xi} |D^a u^\xi_y| \, d\mathcal{H}^{n-1}(y)\,;$$

(ii) *for $\mathcal{H}^{n-1}$ almost every $y \in \Omega^\xi$ the functions u^ξ_y and $\tilde{u}^\xi_y$ belong to $BV(\Omega^\xi_y)$ and coincide $\mathcal{L}^1$ almost everywhere on Ω^ξ_y, the measures $|Du^\xi_y|$ and $V\tilde{u}^\xi_y$ coincide on Ω^ξ_y, and $(\mathcal{E}u(y+t\xi)\xi, \xi) = \nabla u^\xi_y(t) = (\tilde{u}^\xi_y)'(t)$ for $\mathcal{L}^1$ almost every $t \in \Omega^\xi_y$;*

(iii) $$(E^j u\xi, \xi) = \int_{\Omega^\xi} D^j u^\xi_y \, d\mathcal{H}^{n-1}(y)\,, \qquad |(E^j u\xi, \xi)| = \int_{\Omega^\xi} |D^j u^\xi_y| \, d\mathcal{H}^{n-1}(y)\,;$$

(iv) *for $\mathcal{H}^{n-1}$ almost every $y \in \Omega^\xi$ we have $(J^\xi_u)^\xi_y = J_{u^\xi_y}$ and for every $t \in (J^\xi_u)^\xi_y$*

$$(u^+(y+t\xi), \xi) = (u^\xi_y)^+(t) = \lim_{s\to t^+} \tilde{u}^\xi_y(s)\,, \quad (u^-(y+t\xi), \xi) = (u^\xi_y)^-(t) = \lim_{s\to t^-} \tilde{u}^\xi_y(s)\,,$$

where the normals to J_u and $J_{u^\xi_y}$ are oriented so that $(\nu_u, \xi) \geq 0$ and $\nu_{u^\xi_y} = 1$;

(v) $$(E^c u\xi, \xi) = \int_{\Omega^\xi} D^c u^\xi_y \, d\mathcal{H}^{n-1}(y)\,, \qquad |(E^c u\xi, \xi)| = \int_{\Omega^\xi} |D^c u^\xi_y| \, d\mathcal{H}^{n-1}(y)\,.$$

Motivated by the structure theorem we can now define the class of special functions with bounded deformation.

Definition 4.6. We say that $u \in BD(\Omega)$ is a *special function with bounded deformation*, and we write $u \in SBD(\Omega)$, if the measure $E^c u$ in Definition 4.1 is zero, i.e., if $|E^s u|(\Omega \setminus J_u) = 0$.

The following proposition shows that the space $SBD(\Omega)$ of Definition 4.6 coincides with the one introduced in [8] and used to solve free discontinuity problems related to fracture mechanics.

Proposition 4.7. *Let $u \in BD(\Omega)$ and let $\xi_1, \ldots, \xi_n$ be a basis of $\mathbf{R}^n$. Then the following three conditions are equivalent:*

(i) $u \in SBD(\Omega)$;

(ii) *for every $\xi = \xi_i + \xi_j$ with $1 \leq i, j \leq n$, we have $u^\xi_y \in SBV(\Omega^\xi_y)$ for $\mathcal{H}^{n-1}$ almost every $y \in \Omega^\xi$;*

(iii) *the measure* $|E^s u|$ *is concentrated on a Borel set* $B \subset \Omega$ *which is* σ*-finite with respect to* $\mathcal{H}^{n-1}$.

Proof. The equivalence of (i) and (ii) follows easily from (v) of Theorem 4.5 and from (2.2). The implication (i)$\Rightarrow$(iii) follows from (4.2), taking into account that J_u is countably $(\mathcal{H}^{n-1}, n-1)$ rectifiable. Finally, the implication (iii)$\Rightarrow$(i) follows from Proposition 4.4. $\square$

Remark 4.8. It is easy to see that $SBV(\Omega;\mathbf{R}^n)$ is contained in $SBD(\Omega)$. We shall see in Example 7.7 that there are functions in $SBD(\Omega)$ which do not belong to $BV(\Omega;\mathbf{R}^n)$.

Following [8], we now introduce an integralgeometric measure associated with the jump sets of the one dimensional sections. As shown in [8], this measure is crucial in the proof of the compactness properties of $SBD(\Omega)$.

Definition 4.9. For any $u \in BD(\Omega)$ we define the non-negative Borel measure λ_u on Ω as

$$\lambda_u(B) := \kappa_n \int_{\mathbf{S}^{n-1}} \lambda_u^\xi(B)\, d\mathcal{H}^{n-1}(\xi) \qquad \forall B \in \mathcal{B}(\Omega), \tag{4.6}$$

where $\kappa_n := (2\omega_{n-1})^{-1}$ and for every $\xi \in \mathbf{S}^{n-1}$

$$\lambda_u^\xi(B) := \int_{\Omega^\xi} \mathcal{H}^0\big(J_{u_y^\xi} \cap B_y^\xi\big)\, d\mathcal{H}^{n-1}(y) \qquad \forall B \in \mathcal{B}(\Omega). \tag{4.7}$$

Another consequence of the structure theorem is the equality $\lambda_u = \mathcal{H}^{n-1} \llcorner J_u$.

Theorem 4.10. *For every* $u \in BD(\Omega)$ *and any* $\xi \in \mathbf{S}^{n-1}$ *we have*

$$\lambda_u^\xi(B) = \int_{J_u^\xi \cap B} |(\nu_u, \xi)|\, d\mathcal{H}^{n-1} \qquad \forall B \in \mathcal{B}(\Omega), \tag{4.8}$$

where ν_u *is the approximate unit normal to* J_u. *Moreover* $\lambda_u = \mathcal{H}^{n-1} \llcorner J_u$.

Proof. Using part (iv) of the structure Theorem 4.5, we have

$$\lambda_u^\xi(B) = \int_{\Omega^\xi} \mathcal{H}^0\big(\{t \in \mathbf{R} : y + t\xi \in B \cap J_u^\xi\}\big)\, d\mathcal{H}^{n-1}(y).$$

Let $p_\xi\colon J_u \to \pi_\xi$ be the orthogonal projection onto π_ξ. Equality (4.8) follows now from the area formula for maps defined on a countably $(\mathcal{H}^{n-1}, n-1)$ rectifiable Borel set (see [31], 12.4) using the fact that the $n-1$ dimensional area element of p_ξ on J_u is equal to $|(\nu_u, \xi)|$. Finally, since

$$\int_{\mathbf{S}^{n-1}} |(\nu, \xi)|\, d\mathcal{H}^{n-1}(\xi) \;=\; \kappa_n^{-1} = 2\omega_{n-1}$$

for any $\nu \in \mathbf{S}^{n-1}$, the equality $\lambda_u = \mathcal{H}^{n-1} \llcorner J_u$ can be obtained by integrating both sides in (4.8), taking (4.5) into account. $\square$

5. Proof of the structure theorem

The main ingredient in the proof of the structure Theorem 4.5 is the following result, which shows that, under some assumptions, points of $n-1$ dimensional density zero for $|Eu|$ are Lebesgue points for the restriction of the function $p_\eta u$ to π_η, where p_η is the orthogonal projection onto the hyperplane π_η orthogonal to $\eta \in \mathbf{S}^{n-1}$ and passing through the origin. For the sake of simplicity, since all our statements are local, here and in the rest of the section we assume that $\Omega = \mathbf{R}^n$.

Theorem 5.1. *Let $u \in BD(\mathbf{R}^n)$, $x \notin \Theta_u$, $\eta \in \mathbf{S}^{n-1}$. Assume that $\mathcal{H}^{n-1}$ almost every point of $x + \pi_\eta$ are Lebesgue points of u in $\mathbf{R}^n$. Let us consider the function $v\colon \pi_\eta \to \pi_\eta$ given by $v(y) := p_\eta \tilde{u}(x+y)$ and let us assume that v has one sided Lebesgue limits at 0 in π_η with respect to a suitable direction $\nu \in \mathbf{S}^{n-1} \cap \pi_\eta$. Then 0 is a Lebesgue point of v in π_η.*

Proof. Up to a rotation and a translation, it is not restrictive to assume that $\eta = e_n$ and $x = 0$. Then the function v is defined on the hyperplane $\pi := \pi_{e_n}$ by $v(y) := (\tilde{u}^1(y), \ldots, \tilde{u}^{n-1}(y), 0)$. Up to a rotation in π we can assume that $\nu^1 := (\nu, e_1) > 0$ and that the one sided Lebesgue limits v_0^+ and v_0^- of v at 0 in π satisfy $|(v_0^+ - v_0^-, e_1)| \geq \frac{1}{2}|v_0^+ - v_0^-|$. We have to prove that $v_0^+ = v_0^-$.

Let Y be the set of all real numbers $\varrho \geq 0$ such that both $y + \varrho e_n$ and $y - \varrho e_n$ are Lebesgue points of u in $\mathbf{R}^n$ for $\mathcal{H}^{n-1}$ almost every $y \in \pi$. We know that $\mathcal{L}^1(\mathbf{R}_+ \setminus Y) = 0$ and, by our assumption, $0 \in Y$.

For every $\varrho > 0$ let C_ϱ be the n dimensional closed cube in $\mathbf{R}^n$ with centre 0 and sides parallel to the axes with length 2ϱ. We set $Q_\varrho := C_\varrho \cap \pi$ and $Q_\varrho^\pm := \{y \in Q_\varrho :$

$(y, \pm\nu) \geq 0\}$. Let A_ϱ be the $n-1$ dimensional closed ball in π with centre 0 and radius $\nu^1 \varrho$, so that $A_\varrho + \varrho\, e_1 \subset Q^+_{2\varrho}$ and $A_\varrho - \varrho\, e_1 \subset Q^-_{2\varrho}$.

By our assumptions we have that

$$\lim_{\varrho \to 0^+} \frac{1}{\varrho^{n-1}} |Eu|(C_\varrho) \;=\; 0\,, \tag{5.1}$$

$$\lim_{\varrho \to 0^+} \frac{1}{\varrho^{n-1}} \int_{Q^\pm_\varrho} |v(y) - v^\pm_0|\, d\mathcal{H}^{n-1}(y) \;=\; 0\,. \tag{5.2}$$

As $|v^+_0 - v^-_0| \leq 2|(v^+_0 - v^-_0, e_1)|$, for every $\varrho > 0$ we have

$$\begin{aligned} |v^+_0 - v^-_0| \leq & \frac{2k}{\varrho^{n-1}} \Big[\int_{A_\varrho} |v(y + \varrho e_1) - v^+_0|\, d\mathcal{H}^{n-1}(y) + \int_{A_\varrho} |v(y - \varrho e_1) - v^-_0|\, d\mathcal{H}^{n-1}(y) \\ & + \int_{A_\varrho} |v^1(y + \varrho e_1) - v^1(y - \varrho e_1)|\, d\mathcal{H}^{n-1}(y) \Big]\,, \end{aligned}$$

where v^1 is the first component of v and $k = \big(\omega_{n-1}(\nu^1)^{n-1}\big)^{-1}$. By simple changes of variables we obtain

$$\begin{aligned} |v^+_0 - v^-_0| \leq & \; \frac{2k}{\varrho^{n-1}} \Big[\int_{A_\varrho + \varrho e_1} |v(y) - v^+_0|\, d\mathcal{H}^{n-1}(y) \; + \int_{A_\varrho - \varrho e_1} |v(y) - v^-_0|\, d\mathcal{H}^{n-1}(y) \\ & + \int_{A_\varrho} |v^1(y + \varrho e_1) - v^1(y - \varrho e_1)|\, d\mathcal{H}^{n-1}(y) \Big] \\ \leq & \; \frac{2k}{\varrho^{n-1}} \Big[\int_{Q^+_{2\varrho}} |v(y) - v^+_0|\, d\mathcal{H}^{n-1}(y) + \int_{Q^-_{2\varrho}} |v(y) - v^-_0|\, d\mathcal{H}^{n-1}(y) \Big] \\ & + \frac{2k}{\varrho^{n-1}} \int_{A_\varrho} |\tilde u^1(y + \varrho e_1) - \tilde u^1(y - \varrho e_1)|\, d\mathcal{H}^{n-1}(y)\,. \end{aligned} \tag{5.3}$$

By (5.2) the first part of the right hand side vanishes as $\varrho \to 0^+$. We now estimate the last term. For any $\varrho \in Y$ with $\varrho > 0$ we write

$$\begin{aligned} 2\big[\tilde u^1(y + \varrho e_1) - \tilde u^1(y - \varrho e_1)\big] = & \; \big[(\tilde u^1 - \tilde u^n)(y - \varrho e_n) - (\tilde u^1 - \tilde u^n)(y - \varrho e_1)\big] \\ & - \big[(\tilde u^1 - \tilde u^n)(y + \varrho e_n) - (\tilde u^1 - \tilde u^n)(y + \varrho e_1)\big] \\ & + \big[(\tilde u^1 + \tilde u^n)(y + \varrho e_n) - (\tilde u^1 + \tilde u^n)(y - \varrho e_1)\big] \\ & - \big[(\tilde u^1 + \tilde u^n)(y - \varrho e_n) - (\tilde u^1 + \tilde u^n)(y + \varrho e_1)\big] \\ & + 2[\tilde u^n(y - \varrho e_n) - \tilde u^n(y + \varrho e_n)] \end{aligned} \tag{5.4}$$

for $\mathcal{H}^{n-1}$ almost every $y \in \pi$ (see Figure 1).

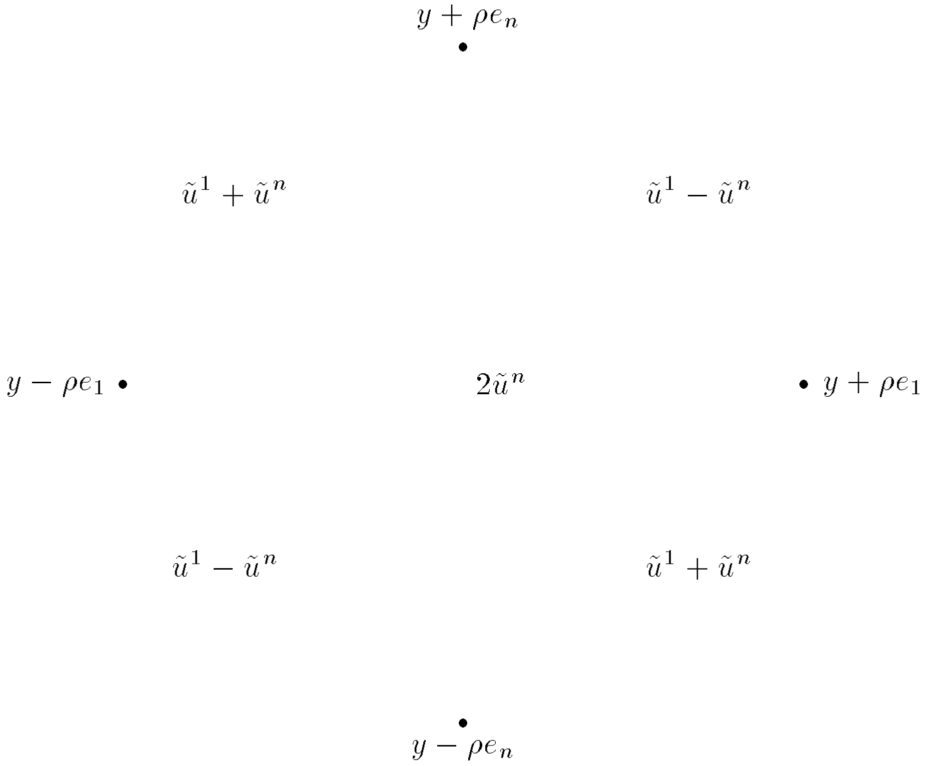

Figure 1. The arrow $a \xrightarrow{\varphi} b$ denotes the difference $\varphi(b) - \varphi(a)$.

The last term in (5.4) is an increment of the function $\tilde{u}_n$ in the direction e_n. Therefore, using (ii) of Proposition 3.2, we have that

$$|\tilde{u}^n(y - \varrho e_n) - \tilde{u}^n(y + \varrho e_n)| \leq V\tilde{u}^n_y([-\varrho, \varrho]) = |Du^n_y|([-\varrho, \varrho])$$

for $\mathcal{H}^{n-1}$ almost every $y \in A_\varrho$. Hence (3.6) implies that

$$\int_{A_\varrho} |\tilde{u}^n(y - \varrho e_n) - \tilde{u}^n(y + \varrho e_n)| \, d\mathcal{H}^{n-1}(y) \leq |(Eue_n, e_n)|(C_\varrho) \leq |Eu|(C_\varrho)$$

for any $\varrho \in Y$ with $\varrho > 0$. Analogously, the other terms in (5.4) are increments of the function $\tilde{u}^1 - \tilde{u}^n$ in the direction $e_1 - e_n$ or of the function $\tilde{u}^1 + \tilde{u}^n$ in the direction $e_1 + e_n$. In the same way they can be controlled by $|Eu|(C_{2\varrho})$ up to a multiplicative constant. For instance, setting $\xi = e_1 + e_n$, $p(y, \varrho) = p_\xi(y - \varrho e_1)$, and $\Sigma_\varrho = p_\xi(A_\varrho - \varrho e_1)$, we have

$$\int_{A_\varrho} |\tilde{u}^\xi(y+\varrho e_n) - \tilde{u}^\xi(y-\varrho e_1)|\, d\mathcal{H}^{n-1}(y) \le \int_{A_\varrho} |Du^\xi_{p(y,\varrho)}|([-\varrho,\varrho])\, d\mathcal{H}^{n-1}(y)$$

$$= \sqrt{2}\int_{\Sigma_\varrho} |Du^\xi_y|([-\varrho,\varrho])\, d\mathcal{H}^{n-1}(y) \le \sqrt{2}|(Eu\xi,\xi)|(C_{2\varrho}) \le 2\sqrt{2}|Eu|(C_{2\varrho}).$$

Therefore there exists a constant $c>0$ such that

$$\int_{A_\varrho} |\tilde{u}^1(y+\varrho e_1) - \tilde{u}^1(y-\varrho e_1)|\, d\mathcal{H}^{n-1}(y) \le c\,|Eu|(C_{2\varrho})$$

for any $\varrho \in Y$ with $\varrho > 0$. Thus, by (5.1), also the last term at the right hand side of (5.3) vanishes as $\varrho \to 0^+$, hence $v_0^+ = v_0^-$ and the proof is complete. $\square$

Proof of Theorem 4.5. Since the properties we have to prove are local, we may assume $\Omega = \mathbf{R}^n$. Let $u \in BD(\mathbf{R}^n)$ and let $\xi \in \mathbf{R}^n$ with $\xi \neq 0$. Let Y be the set of points $y \in \pi_\xi$ such that $u^\xi_y \in BD(\mathbf{R})$, $\tilde{u}^\xi_y$ is defined and agrees with u^ξ_y $\mathcal{L}^1$ almost everywhere in $\mathbf{R}$, and $V\tilde{u}^\xi_y = |Du^\xi_y|$ as measures on $\mathbf{R}$. By Proposition 3.2 we have $\mathcal{H}^{n-1}(\pi_\xi \setminus Y) = 0$.

We know from Proposition 3.2 that $(Eu\xi,\xi) = \int_{\pi_\xi} Du^\xi_y\, d\mathcal{H}^{n-1}(y)$, and we claim that this decomposition is still true for the absolutely continuous part, the jump part, and the Cantor part of Eu. We observe that

$$(E^a u\xi,\xi) - \int_{\pi_\xi} D^a u^\xi_y\, d\mathcal{H}^{n-1}(y) = \int_{\pi_\xi} D^s u^\xi_y\, d\mathcal{H}^{n-1}(y) - (E^s u\xi,\xi).$$

Since the left hand side is absolutely continuous with respect to $\mathcal{L}^n$, while the right hand side is singular, both sides are 0 and the first equality in (i) is proved. The second equality follows from (2.4).

Recalling that for $y \in Y$ we have $D^a u^\xi_y = \nabla u^\xi_y\, \mathcal{L}^1 = (\tilde{u}^\xi_y)'\mathcal{L}^1$, we get

$$(E^a u\xi,\xi) = \int_{\pi_\xi} \nabla u^\xi_y\, \mathcal{L}^1\, d\mathcal{H}^{n-1}(y) = \int_{\pi_\xi} (\tilde{u}^\xi_y)'\mathcal{L}^1\, d\mathcal{H}^{n-1}(y).$$

Since, by definition, the density of $E^a u$ is $\mathcal{E}u$, Fubini's theorem implies (ii).

Now we can further decompose $E^s u$ and $D^s u^\xi_y$, obtaining

$$(E^j u\xi,\xi) - \int_{\pi_\xi} D^j u^\xi_y\, d\mathcal{H}^{n-1}(y) = \int_{\pi_\xi} D^c u^\xi_y\, d\mathcal{H}^{n-1}(y) - (E^c u\xi,\xi). \tag{5.5}$$

We claim now that both sides are 0. Let us restrict both measures to J_u. We know, by definition, that $|E^c u|(J_u)=0$; moreover, we know (see [17], Corollary 2.10.11) that any set B which is σ-finite with respect to $\mathcal{H}^{n-1}$ has the property that, for $\mathcal{H}^{n-1}$ almost every $y\in\pi_\xi$, the line $y+\mathbf{R}\xi$ meets B in an at most countable set. Since the Cantor measures $D^c u^\xi_y$ in (5.5) are non atomic, applying this property to $B=J_u$ we obtain that the restriction to J_u of the measure $\int_{\pi_\xi} D^c u^\xi_y\, d\mathcal{H}^{n-1}(y)$ is zero. Hence, the restriction to J_u of the right hand side in (5.5) is 0. To prove the claim, we need only to show that the left hand side is concentrated on J_u.

By definition, $(E^j u\xi,\xi)$ is concentrated on J_u. The main difficulty is to prove that $\int_{\pi_\xi} D^j u^\xi_y\, d\mathcal{H}^{n-1}(y)$ is concentrated on J_u. Our proof is by induction on $n\geq 2$.

Assume $n=2$. The measure $\mu:=\int_{\pi_\xi} D^j u^\xi_y\, d\mathcal{H}^{n-1}(y)$ is concentrated on the set of points of the form $y+t\xi$ with $y\in Y$ and t in the jump set of u^ξ_y. By Theorem 5.1, these points belong to Θ_u, hence μ is concentrated on Θ_u. Since μ vanishes on $\mathcal{H}^{n-1}$ negligible sets and $\mathcal{H}^{n-1}(\Theta_u\setminus J_u)=0$ (Proposition 3.5), μ is also concentrated on J_u.

Assume the statement valid for $n-1\geq 2$. Choosing $\eta\in\mathbf{S}^{n-1}$ orthogonal to ξ, so that the hyperplane π_η contains 0 and ξ, for any $s\in\mathbf{R}$ we can consider the functions $v_s\colon\pi_\eta\to\pi_\eta$ defined by $v_s(y):=p_\eta u(s\eta+y)$. By Proposition 3.4 we have $v_s\in BD(\pi_\eta)$ for $\mathcal{L}^1$ almost every $s\in\mathbf{R}$. Moreover, since ξ belongs to π_η, for any $\omega\in\pi_{\eta\xi}:=\pi_\eta\cap\pi_\xi$ and any s, $t\in\mathbf{R}$ we have

$$(v_s)^\xi_\omega(t) \;=\; (v_s(t\xi+\omega),\xi) \;=\; (p_\eta u(t\xi+s\eta+\omega),\xi) \;=\; u^\xi_{s\eta+\omega}(t)\,.$$

For any $B\in\mathcal{B}(\mathbf{R}^n)$ and $s\in\mathbf{R}$ let $B_s=\{z\in\pi_\eta\ :s\eta+z\in B\}$. Since $\xi\in\pi_\eta$, for any $\omega\in\pi_{\eta\xi}$ we have $(B_s)^\xi_\omega=B^\xi_{s\eta+\omega}$. Therefore Fubini's theorem yields

$$\int_{\pi_\xi} D^j u^\xi_y(B^\xi_y)\, d\mathcal{H}^{n-1}(y)=\int_{\mathbf{R}}\Big[\int_{\pi_{\eta\xi}} D^j(v_s)^\xi_\omega((B_s)^\xi_\omega)\, d\mathcal{H}^{n-2}(\omega)\Big]\, ds\,. \tag{5.6}$$

Let S be the set of points $s\in\mathbf{R}$ such that $v_s\in BD(\pi_\eta)$ and $v_s(z)=p_\eta\tilde{u}(s\eta+z)$ for $\mathcal{H}^{n-1}$ almost every $z\in\pi_\eta$. By Fubini's theorem and by Proposition 3.4 we have $\mathcal{L}^1(\mathbf{R}\setminus S)=0$. By the induction assumption, for any $s\in S$ the measure

$$\mu_s(B)\;:=\int_{\pi_{\eta\xi}} D^j(v_s)^\xi_\omega((B_s)^\xi_\omega)\, d\mathcal{H}^{n-2}(\omega)\,,\qquad B\in\mathcal{B}(\pi_\eta)\,,$$

is concentrated on J_{v_s}.

Hence (5.6) implies that the measure $\int_{\pi_\xi} D^j u^\xi_y\, d\mathcal{H}^{n-1}(y)$ is concentrated on the set of points of the form $s\eta+y$, with $s\in S$ and $y\in J_{v_s}$. By Theorem 5.1 this set is contained in Θ_u, and this proves that $\int_{\pi_\xi} D^j u_y^\xi \, d\mathcal{H}^{n-1}(y)$ is concentrated on Θ_u. Arguing as in the case $n=2$ we obtain that this measure is also concentrated on J_u. This proves that both sides in (5.5) are 0, and concludes the proof of the first equalities in (iii) and (v). The second equalities follow from (2.4).

We conclude by showing the validity of (iv). Since J_u is countably $(\mathcal{H}^{n-1}, n-1)$ rectifiable and (ν_u, ξ) is the Jacobian of the map $p_\xi \colon J_u \to \pi_\xi$ (recall that $(\nu_u, \xi) \geq 0$), using (4.2) and the area formula for rectifiable sets (see, e.g., [31], Section 12) we get

$$\begin{aligned}(E^j u\xi, \xi)(B) &= \int_{B \cap J_u^\xi} (u^+(x) - u^-(x), \xi)(\nu_u(x), \xi)\, d\mathcal{H}^{n-1}(x) \\ &= \int_{\pi_\xi} \Big[\sum_{t \in (B \cap J_u^\xi)_y^\xi} (u^+(y+t\xi) - u^-(y+t\xi), \xi) \Big] \, d\mathcal{H}^{n-1}(y)\end{aligned}$$

for any Borel set $B \subset \mathbf{R}^n$. Hence, using (iii), we obtain the equality of measures

$$D^j u_y^\xi = \sum_{t \in (J_u^\xi)_y^\xi} (u^+(y+t\xi) - u^-(y+t\xi), \xi)\, \delta_t \tag{5.7}$$

for $\mathcal{H}^{n-1}$ almost every $y \in Y$. On the other hand by (2.13) we have also

$$D^j u_y^\xi = \sum_{t \in J_{u_y^\xi}} [(u_y^\xi)^+(t) - (u_y^\xi)^-(t)]\, \delta_t \tag{5.8}$$

for every $y \in Y$ (recall that $\nu_{u_y^\xi} = 1$). If $y \in Y$ satisfies (5.7), from (5.8) we obtain

$$(J_u^\xi)_y^\xi = J_{u_y^\xi}\,, \tag{5.9}$$

$$(u^+(y+t\xi) - u^-(y+t\xi), \xi) = (u_y^\xi)^+(t) - (u_y^\xi)^-(t) \qquad \forall t \in (J_u^\xi)_y^\xi\,. \tag{5.10}$$

In order to prove the full property (iv) we shall use a localization argument. Since J_u is countably $(\mathcal{H}^{n-1}, n-1)$ rectifiable, we can find a sequence of bounded open sets Ω_k with $\mathcal{C}^1$ boundary such that the union of $\partial\Omega_k$ covers $\mathcal{H}^{n-1}$ almost all of J_u. Let us fix k and let v be the function defined by $v(x) = u(x)$, if $x \in \Omega_k$, and by $v(x) = 0$ otherwise. By property (iii) stated at the beginning of Section 3 the function v belongs to $BD(\mathbf{R}^n)$. We choose the sign of ν_v so that $(\nu_v, \xi) \geq 0$ in J_v. We define

$$\partial_\xi^\pm \Omega_k := \{x \in \partial\Omega_k : \pm(\nu_{\Omega_k}(x), \xi) > 0\}\,, \qquad \partial_\xi^0 \Omega_k := \{x \in \partial\Omega_k : (\nu_{\Omega_k}(x), \xi) = 0\}\,,$$

where ν_{Ω_k} is the outer unit normal to $\partial\Omega_k$. Since $(\nu_u, \xi) \geq 0$ we have

$$\nu_u = \pm \nu_{\Omega_k} \qquad \mathcal{H}^{n-1} \text{ a.e. in } J_u \cap \partial_\xi^\pm \Omega_k \tag{5.11}$$

(see, e.g., [31], Remark 11.7).

We want to prove that for $\mathcal{H}^{n-1}$ almost every $y \in \pi_\xi$ we have

$$(u^-(y+t\xi), \xi) = (u^\xi_y)^-(t) \qquad \forall t \in (J^\xi_u \cap \partial^+_\xi \Omega_k)^\xi_y . \tag{5.12}$$

For every $x \in J^\xi_u \cap \partial^+_\xi \Omega_k$ with $\nu_u(x) = \nu_{\Omega_k}(x)$ the function v has one sided Lebesgue limits with respect to $\nu_u(x)$ and

$$v^+(x) = 0 , \qquad v^-(x) = u^-(x) . \tag{5.13}$$

If $y \in Y$ and $t \in (J^\xi_u \cap \partial^+_\xi \Omega_k)^\xi_y$, using the fact that $(\nu_{\Omega_k}(y+t\xi), \xi) > 0$, we can prove that the functions u^ξ_y and v^ξ_y have one sided Lebesgue limits with respect to $\nu = 1$ and

$$(v^\xi_y)^+(t) = 0 , \qquad (v^\xi_y)^-(t) = (u^\xi_y)^-(t) . \tag{5.14}$$

Let us consider separately the sets

$$M := \{x \in J^\xi_u \cap \partial^+_\xi \Omega_k : (u^-(x), \xi) \neq 0\} , \qquad N := \{x \in J^\xi_u \cap \partial^+_\xi \Omega_k : (u^-(x), \xi) = 0\} .$$

By (5.13) and (5.11) for $\mathcal{H}^{n-1}$ almost every $y \in \pi_\xi$ we have $M^\xi_y \subset (J^\xi_v)^\xi_y$ and $\nu_v(y+t\xi) = \nu_u(y+t\xi)$ for every $t \in M^\xi_y$ (recall that $(\nu_v, \xi) \geq 0$). If we apply (5.10) to the function v, taking (5.13) and (5.14) into account, for $\mathcal{H}^{n-1}$ almost every $y \in \pi_\xi$ we obtain

$$(u^-(y+t\xi), \xi) = (u^\xi_y)^-(t) \qquad \forall t \in M^\xi_y . \tag{5.15}$$

If we apply (5.9) to the function v, we obtain that $(J^\xi_v)^\xi_y = J_{v^\xi_y}$ for $\mathcal{H}^{n-1}$ almost every $y \in \pi_\xi$. As $(u^-(x), \xi) = 0$ for every $x \in N$, by (5.13) and (5.11) for $\mathcal{H}^{n-1}$ almost every $y \in \pi_\xi$ we have $N^\xi_y \cap J_{v^\xi_y} = N^\xi_y \cap (J^\xi_v)^\xi_y = \emptyset$, hence $(v^\xi_y)^+ = (v^\xi_y)^-$ in N^ξ_y. By (5.14) this implies that $(u^\xi_y)^- = 0$ in N^ξ_y for $\mathcal{H}^{n-1}$ almost every $y \in \pi_\xi$. As $(u^-(x), \xi) = 0$ for every $x \in N$, for $\mathcal{H}^{n-1}$ almost every $y \in \pi_\xi$ we get

$$(u^-(y+t\xi), \xi) = 0 = (u^\xi_y)^-(t) \qquad \forall t \in N^\xi_y . \tag{5.16}$$

From (5.15) and (5.16) we obtain (5.12), and from (5.12) and (5.10) for $\mathcal{H}^{n-1}$ almost every $y \in \pi_\xi$ we obtain that

$$(u^+(y+t\xi), \xi) = (u^\xi_y)^+(t) , \qquad (u^-(y+t\xi), \xi) = (u^\xi_y)^-(t) \tag{5.17}$$

for every $t \in (J_u^\xi \cap \partial_\xi^+ \Omega_k)_y^\xi$. In the same way we can prove that (5.17) holds for $\mathcal{H}^{n-1}$ almost every $y \in \pi_\xi$ and for every $t \in (J_u^\xi \cap \partial_\xi^- \Omega_k)_y^\xi$. Since, by the area formula, $\mathcal{H}^{n-1}(p_\xi(J_u^\xi \cap \partial_\xi^0 \Omega_k)) = 0$, (5.17) holds for $\mathcal{H}^{n-1}$ almost every $y \in \pi_\xi$ and for every $t \in (J_u^\xi \cap \partial\Omega_k)_y^\xi$. As the union of the sets $\partial\Omega_k$ covers $\mathcal{H}^{n-1}$ almost all of J_u, for $\mathcal{H}^{n-1}$ almost every $y \in \pi_\xi$ we have

$$(u^+(y+t\xi),\xi) = (u_y^\xi)^+(t), \qquad (u^-(y+t\xi),\xi) = (u_y^\xi)^-(t) \qquad \forall t \in (J_u^\xi)_y^\xi. \tag{5.18}$$

For every $y \in Y$ the function $\tilde{u}_y^\xi$ has bounded pointwise variation and coincides with u_y^ξ $\mathcal{L}^1$ almost everywhere on $\mathbf{R}$. Therefore

$$(u_y^\xi)^+(t) = \lim_{s \to t^+} \tilde{u}_y^\xi(s), \qquad (u_y^\xi)^-(t) = \lim_{s \to t^-} \tilde{u}_y^\xi(s) \tag{5.19}$$

for every $t \in J_{u_y^\xi}$. Together with (5.18), this concludes the proof of (iv). □

6. Discontinuity points which are not jump points

In this section we study the set $S_u \setminus J_u$ of Lebesgue discontinuity points which are not jump points. For BV functions this set is $\mathcal{H}^{n-1}$ negligible (see, e.g., [16], Section 5.9, Theorem 3); this property is not known for BD functions. The main result of this section is the following theorem, which shows that $S_u \setminus J_u$ is negligible for all measures related to BD functions.

Theorem 6.1. *If $u, v \in BD(\Omega)$, then $|Ev|(S_u \setminus J_u) = 0$. In other words, $|Ev|$ almost every point $x \in \Omega$ is either a Lebesgue point or a jump point of u.*

Remark 6.2. Among other things, Kohn proved in [21] that the set $S_u \setminus J_u$ has Hausdorff dimension at most $n-1$, i.e., $\mathcal{H}^{n-1+\epsilon}(S_u \setminus J_u) = 0$ for any $\epsilon > 0$. It is not known whether $\mathcal{H}^{n-1}(S_u \setminus J_u) = 0$ for every $u \in BD(\Omega)$ or whether there are examples of BD functions for which this set has positive $\mathcal{H}^{n-1}$ measure. In the latter case one can not obtain Theorem 6.1 from (3.10).

Remark 6.3. Using property (ii) of Section 3, it is easy to check that for every $u \in BD(\Omega)$ the set $S_u \setminus J_u$ is purely $(\mathcal{H}^{n-1}, n-1)$ unrectifiable, i.e., $\mathcal{H}^{n-1}\big((S_u \setminus J_u) \cap M\big) = 0$ for any $\mathcal{C}^1$ hypersurface $M \subset \mathbf{R}^n$ (see [21], Part II, Theorem 5.2). For a BV function the coarea formula implies that $|Du|$ vanishes on all purely $(\mathcal{H}^{n-1}, n-1)$ unrectifiable

Borel sets. For a BD function we can prove only that $|Eu|(B) = 0$ for every purely $(\mathcal{H}^{n-1}, n-1)$ unrectifiable Borel set B on which $\mathcal{H}^{n-1}$ is finite (or, more generally, σ-finite). Indeed, in this case $|Eu|(B) = |E^j u|(B) = |Eu|(B \cap J_u)$ by Proposition 4.4. Since J_u is countably $(\mathcal{H}^{n-1}, n-1)$ rectifiable, $\mathcal{H}^{n-1}(B \cap J_u) = 0$ and the conclusion follows from (3.10). The extension of this property to a more general class of unrectifiable sets is still an open problem. On the other hand for a function $u \in BD(\Omega)$ is not known whether the set $S_u \setminus J_u$ is σ-finite for the measure $\mathcal{H}^{n-1}$. Therefore these general arguments can not be used in the proof of Theorem 6.1.

Starting from the (trivial) one dimensional case, we will be able to prove Theorem 6.1 by induction on n. The induction step will be based on the following refinement of Theorem 5.1, which allows to prove that Lebesgue points for the restriction of u to a hyperplane π_η are Lebesgue points for $p_\eta u$ in $\mathbf{R}^n$.

Theorem 6.4. *Under the assumptions of Theorem 5.1, we have that x is a Lebesgue point for $p_\eta u$ in $\mathbf{R}^n$.*

Proof. We shall use the same notation adopted in the proof of Theorem 5.1, assuming $x = 0$ and π equal to the hyperplane orthogonal to e_n and passing through the origin. We already know that 0 is a Lebesgue point of the function $v(y) := (\tilde{u}^1(y), \ldots, \tilde{u}^{n-1}(y), 0)$ in π. The first component of the Lebesgue representative in π of v at 0 will be denoted by v_0^1. We shall prove that v_0^1 coincides with $\tilde{u}^1(0)$, the Lebesgue representative in $\mathbf{R}^n$ of u^1 at 0. A similar argument works for the other components.

Since $0 \notin \Theta_u$, Corollary 6.7 below yields that the quantity

$$\omega(\epsilon) := \frac{1}{\epsilon^n} \int_{Q_\epsilon} \Big[\int_{-\epsilon}^{\epsilon} |u^1(y + te_n) - d_\epsilon| \, dt \Big] \, d\mathcal{H}^{n-1}(y)$$

vanishes as $\epsilon \to 0^+$, for suitable real constants d_ϵ. We have to prove that d_ϵ converges to v_0^1 as $\epsilon \to 0^+$.

For any $\epsilon > 0$ we choose $\varrho = \varrho(\epsilon) \in (\epsilon/2, \epsilon) \cap Y$ such that

$$\frac{1}{\epsilon^{n-1}} \int_{Q_\epsilon} |\tilde{u}^1(y + \varrho e_n) - d_\epsilon| + |\tilde{u}^1(y - \varrho e_n) - d_\epsilon| \, d\mathcal{H}^{n-1}(y) \le 2\omega(\epsilon) \tag{6.1}$$

and we write

$$\begin{aligned} \tilde{u}^1(y + \varrho e_n) + \tilde{u}^1(y - \varrho e_n) - 2v_0^1 = {} & \big[(\tilde{u}^1 + \tilde{u}^n)(y + \varrho e_n) - (\tilde{u}^1 + \tilde{u}^n)(y - \varrho e_1)\big] \\ & + \big[(\tilde{u}^1 - \tilde{u}^n)(y - \varrho e_n) - (\tilde{u}^1 - \tilde{u}^n)(y - \varrho e_1)\big] \\ & + \big[\tilde{u}^n(y - \varrho e_n) - \tilde{u}^n(y + \varrho e_n)\big] + 2\big[v^1(y - \varrho e_1) - v_0^1\big] . \end{aligned}$$

If we integrate both sides of this equality on Q_ϵ, the same argument exploited in Theorem 5.1 and the Lebesgue continuity of v^1 at 0 in π show that

$$\lim_{\epsilon\to 0^+} \frac{1}{\epsilon^{n-1}} \int_{Q_\epsilon} |\tilde u^1(y+\varrho e_n) + \tilde u^1(y-\varrho e_n) - 2v_0^1|\, d\mathcal{H}^{n-1}(y) = 0\,.$$

Taking (6.1) into account, we obtain that d_ϵ converges to v_0^1 as $\epsilon \to 0^+$. □

The proof of Theorem 6.4 has been based on (6.7) of Corollary 6.7. We will obtain this and related formulas following the lines of Section 5 of Part II of Kohn's thesis [21], relying on an explicit version of Poincaré's inequality for BD functions. Given a matrix of measures $\{\mu_{lm}\}$ on $\mathbf{R}^n$, a tensor $\{T_{ij}^{lm}(x)\}$ of order 4, and a Borel set $B\subset \mathbf{R}^n$, we introduce the short notation

$$\alpha = \int_B T\, d\mu \qquad \text{for} \qquad \alpha_{ij} = \sum_{l,m=1}^n \int_B T_{ij}^{lm}\, d\mu_{lm}\,.$$

Let $\Gamma(x) = \{\Gamma_{ij}^{lm}(x)\}$ be the fourth order tensor defined by

$$\begin{cases} \Gamma_{ij}^{ii}(x) = -\Gamma_{ij}^{jj}(x) = 2x_ix_j\,, & \text{if } i\neq j,\\ \Gamma_{ij}^{ij}(x) = \Gamma_{ij}^{ji}(x) = x_j^2 - x_i^2\,, & \\ \Gamma_{ij}^{ir}(x) = \Gamma_{ij}^{ri}(x) = x_jx_r\,, & \text{if } i\neq j \text{ and } r\notin\{i,j\},\\ \Gamma_{ij}^{jr}(x) = \Gamma_{ij}^{rj}(x) = -x_ix_r\,, & \text{if } i\neq j \text{ and } r\notin\{i,j\},\\ \Gamma_{ij}^{lm}(x) = 0\,, & \text{if } i=j \text{ or } \{l,m\}\cap\{i,j\}=\emptyset. \end{cases}$$

Note that for every i, j, l, m we have

$$\Gamma_{ij}^{lm}(x) = \Gamma_{ij}^{ml}(x) = -\Gamma_{ji}^{lm}(x) \qquad \text{and} \qquad \int_{\mathbf{S}^{n-1}} \Gamma_{ij}^{lm}\, d\mathcal{H}^{n-1} = 0\,. \tag{6.2}$$

Theorem 6.5. *Let $u\in BD(\mathbf{R}^n)$, $n\geq 2$, and for any $x\in\mathbf{R}^n$ and $\varrho>0$ define*

$$\beta_\varrho(x) = -\frac{1}{2\omega_n}\int_{\{|y-x|\geq\varrho\}} \frac{\Gamma(y-x)}{|y-x|^{n+2}}\, dEu(y)\,. \tag{6.3}$$

Then, for suitable constants $d_\varrho(x)$, we have the estimate

$$\int_{B_\varrho(x)} |u(y) - d_\varrho(x) - \beta_\varrho(x)(y-x)|\, dy \leq c_n\varrho\, |Eu|(B_\varrho(x))\,, \tag{6.4}$$

where c_n depends only on n.

Proof. The proof can be obtained from Proposition 5.7 of Part II of [21] with obvious changes. For the reader's convenience we give the complete proof in the case $x = 0$. For every $\varrho > 0$ we set $B_\varrho = B_\varrho(0)$. Recalling that $\mathcal{R}$ is the class of rigid motions in $\mathbf{R}^n$ introduced in Section 3, we define a linear map $R_\varrho : BD(B_\varrho) \to \mathcal{R}$ as follows: $R_\varrho(v)(y) = \beta_\varrho y + d_\varrho$ with

$$(6.5) \qquad \beta_\varrho = \frac{1}{2\omega_n \varrho^n} \int_{\partial B_\varrho} (v_* \otimes \nu - \nu \otimes v_*) \, d\mathcal{H}^{n-1}, \qquad d_\varrho = \frac{1}{n\omega_n \varrho^{n-1}} \int_{\partial B_\varrho} v_* \, d\mathcal{H}^{n-1},$$

where v_* is the trace of v on ∂B_ϱ and ν is the exterior unit normal to ∂B_ϱ. It is easy to check that R_ϱ is the identity on $\mathcal{R}$, so that (6.4) follows from Theorem 3.1. The fact that the constant $c(B_\varrho, R_\varrho)$ has the special form $c_n \varrho$ can be easily proved by a rescaling argument.

We need only to prove that for $u \in BD(\mathbf{R}^n)$ the matrix β_ϱ in (6.5) can be computed by (6.3). To this aim, we fix two indices i, j, with $i \neq j$, and a constant $\tau > \varrho$. We apply the divergence formula (3.4) in $B_\tau \setminus \overline{B}_\varrho$ to the symmetric matrix valued function $\psi_{lm}(y) = \Gamma_{ij}^{lm}(y)/(2\omega_n |y|^{n+2})$. Since

$$\sum_{m=1}^n D_m \psi_{lm}(y) = 0 \qquad \text{and} \qquad \sum_{m=1}^n \psi_{lm}(y) \frac{y_m}{|y|} = \begin{cases} \dfrac{y_j}{2\omega_n |y|^{n+1}}, & \text{if } l = i, \\ -\dfrac{y_i}{2\omega_n |y|^{n+1}}, & \text{if } l = j, \\ 0, & \text{if } l \notin \{i, j\}, \end{cases}$$

using (3.4) (and also (3.2) if the inner and the outer trace of u on ∂B_ϱ are different) we obtain

$$(6.6) \qquad \int_{\{\tau > |y| \geq \varrho\}} \frac{\Gamma(y)}{2\omega_n |y|^{n+2}} \, dEu(y) = \frac{1}{2\omega_n \tau^n} \int_{\partial B_\tau} (u_* \otimes \nu - \nu \otimes u_*) \, d\mathcal{H}^{n-1} - \beta_\varrho,$$

where ν is the exterior unit normal to ∂B_τ. Using the integrability of u in $\mathbf{R}^n$, it is not hard to see that

$$\liminf_{\tau \to +\infty} \frac{1}{\tau^n} \int_{\partial B_\tau} |u_*| \, d\mathcal{H}^{n-1} = 0.$$

Hence, letting $\tau \to +\infty$ we obtain (6.3). □

Remark 6.6. By a similar argument it can be seen that a constant $d_\varrho(x)$ in (6.4) can be explicitly given by the formula

$$d_\varrho^i(x) = \sum_{l,m=1}^n \int_{\{|y-x| \geq \varrho\}} \frac{\Lambda_i^{lm}(y-x)}{n\omega_n |y-x|^n} \, dE_{lm}u(y),$$

with $\Lambda_i^{ri}(y) = \Lambda_i^{ir}(y) = y_r$ if $1 \leq r \leq n$, $\Lambda_i^{rr}(y) = -y_i$ if $r \neq i$ and $\Lambda_i^{lm}(y) = 0$ in all other cases.

Corollary 6.7. *Let* $u \in BD(\Omega)$. *Then*

$$\lim_{\varrho\to 0^+} \min_{d\in\mathbf{R}^n} \frac{1}{\varrho^n}\int_{B_\varrho(x)} |u(y)-d|\,dy = 0 \tag{6.7}$$

for every $x \in \Omega \setminus \Theta_u$.

Proof. The first proof of this property is given in Theorem 5.13 of Part II of [21]. For the reader's convenience we reproduce it with some variants.

Since the statement is local, we can assume $\Omega = \mathbf{R}^n$ and $x = 0$. By Theorem 6.5, we need only to prove that $\varrho\beta_\varrho$ converges to 0 as $\varrho \to 0^+$. Let us fix $\tau > 0$ and let $\varrho \in (0,\tau)$; using the identity (6.6) we find that the upper limit of $\varrho|\beta_\varrho|$ can be estimated, up to a multiplicative constant, by

$$\limsup_{\varrho\to 0^+} \varrho \int_{\{\tau>|y|\geq\varrho\}} \frac{1}{|y|^n}\,d|Eu|(y)\,.$$

Hence, we need only to show that the upper limit above tends to zero as $\tau \to 0^+$. Setting $\mu := |Eu| \llcorner \{\tau > |y| \geq \varrho\}$ and using the change of variables $t = 1/s^n$ we obtain

$$\begin{aligned}
\int_{\{\tau>|y|\geq\varrho\}} \frac{1}{|y|^n}\,d|Eu|(y) &= \int_0^{+\infty} \mu\big(\{\frac{1}{|y|^n} > t\}\big)\,dt \\
&= n\int_0^{+\infty} \frac{\mu(\{|y|<s\})}{s^{n+1}}\,ds \\
&= n\int_\varrho^\tau \frac{|Eu|(\{s>|y|\geq\varrho\})}{s^{n+1}}\,ds + \frac{|Eu|(\{\tau>|y|\geq\varrho\})}{\tau^n}\,.
\end{aligned}$$

Therefore, writing $s^{n+1} = s^{n-1}s^2$, we get the estimate

$$\limsup_{\varrho\to 0^+} \varrho \int_{\{\tau>|y|\geq\varrho\}} \frac{1}{|y|^n}\,d|Eu|(y) \leq n \sup_{t\in(0,\tau)} \frac{|Eu|(\{|y|<t\})}{t^{n-1}}\,.$$

By our assumption, the right hand side vanishes as $\tau \to 0^+$. □

We are now in a position to prove Theorem 6.1 in the special case $u = v$.

Proposition 6.8. *For every* $u \in BD(\Omega)$ *we have* $|Eu|(S_u \setminus J_u) = 0$.

Proof. Since the property to be proved is local, we may assume that $u \in BD(\mathbf{R}^n)$. The statement is trivial for $n = 1$; assume it valid for $n-1 \geq 1$ and let us prove it for n.

For any $\xi \in \mathbf{R}^n$ we claim that $|(Eu\xi, \xi)|$ almost every point of $\mathbf{R}^n \setminus J_u$ is a Lebesgue point of u^ξ. By Lemma 6.9 below with $\mu = Eu \llcorner (\mathbf{R}^n \setminus J_u)$, this property implies that $|Eu|$ almost every point in $\mathbf{R}^n \setminus J_u$ is a Lebesgue point for u.

Let $\eta \in \mathbf{S}^{n-1}$ be orthogonal to ξ, so that the hyperplane π_η contains 0 and ξ, and let S be the set of points $s \in \mathbf{R}$ such that the function $v_s(y) := p_\eta \tilde{u}(s\eta + y)$ is defined for $\mathcal{H}^{n-1}$ almost every $y \in \pi_\eta$ and belongs to $BD(\pi_\eta)$. By Proposition 3.4 we have $\mathcal{L}^1(\mathbf{R} \setminus S) = 0$. By the induction assumption v_s has one sided (possibly equal) Lebesgue limits v_s^+ and v_s^- in π_η at $|(Ev_s\xi, \xi)|$ almost every point of π_η. By Proposition 3.4, for every Borel set B of $\mathbf{R}^n$ we have

$$(Eu\xi, \xi)(B) = \int_{-\infty}^{+\infty} (Ev_s\xi, \xi)(B_s)\, ds\,,$$

where $B_s = \{y \in \pi_\eta \; : s\eta + y \in B\}$. Therefore $|(Eu\xi, \xi)|$ is concentrated on the set of all points x of the form $s\eta + y$, with $s \in S$ and y such that v_s^+ and v_s^- exist at y. By Theorem 6.4, for any point x of this type, either $x \in \Theta_u$ or x is a Lebesgue point of $p_\eta u$ in $\mathbf{R}^n$. As $\xi \in \pi_\eta$, this implies that $|(Eu\xi, \xi)|$ almost every point either belongs to Θ_u or is a Lebesgue point of u^ξ. Since $\mathcal{H}^{n-1}(\Theta_u \setminus J_u) = 0$, the claim follows from (3.10). $\square$

Lemma 6.9. *Let $u \in L^1(\mathbf{R}^n; \mathbf{R}^n)$, let $\mu = \{\mu_{ij}\}$ be a symmetric matrix of Radon measures in $\mathbf{R}^n$ and assume that, for any $\xi \in \mathbf{R}^n$, $|(\mu\xi, \xi)|$ almost every point of $\mathbf{R}^n$ is a Lebesgue point of u^ξ. Then $|\mu|$ almost every point of $\mathbf{R}^n$ is a Lebesgue point for u.*

Proof. Let $\eta \in \mathbf{R}^n$ be given. We have to prove that $|\mu|$ almost every point of $\mathbf{R}^n$ is a Lebesgue point of u^η. By (2.2), we need only to prove that for any choice of ξ in $\mathbf{R}^n$, $|(\mu\xi, \xi)|$ almost every point is a Lebesgue point for u^η.

Applying our hypothesis to the vectors ξ, η, $\xi \pm \eta$ and setting for brevity $\mu_{\xi\xi} := (\mu\xi, \xi)$, $\mu_{\eta\eta} := (\mu\eta, \eta)$, $\mu_{\xi\eta} := (\mu\xi, \eta)$, we obtain the following properties:

(a) $|\mu_{\xi\xi}|$ almost every $x \in \mathbf{R}^n$ is a Lebesgue point for u^ξ;

(b) $|\mu_{\eta\eta}|$ almost every $x \in \mathbf{R}^n$ is a Lebesgue point for u^η;

(c) $|\mu_{\xi\xi} \pm 2\mu_{\xi\eta} + \mu_{\eta\eta}|$ almost every $x \in \mathbf{R}^n$ is a Lebesgue point for $u^\xi \pm u^\eta$.

Let $\lambda := |\mu_{\xi\xi}| + |\mu_{\eta\eta}| + |\mu_{\xi\eta}|$. Since $\mu_{\xi\xi}$, $\mu_{\eta\eta}$, and $\mu_{\xi\eta}$ are absolutely continuous with respect to λ, there exist three locally λ integrable functions h_1, h_2, and h_3 such that $\mu_{\xi\xi} = h_1\lambda$, $\mu_{\eta\eta} = h_2\lambda$, and $\mu_{\xi\eta} = h_3\lambda$. Let us consider the mutually disjoint sets

$$A_1 := \big\{x \in \mathbf{R}^n : h_1(x) + h_2(x) = 0\big\}, \qquad A_2 := \big\{x \in \mathbf{R}^n : h_1(x) + h_2(x) \neq 0\big\},$$

whose union is $\mathbf{R}^n$. On A_1 we have $\mu_{\xi\xi} = -\mu_{\eta\eta}$, hence the total variations are the same and (b) implies that $|\mu_{\xi\xi}|$ almost every $x \in A_1$ is a Lebesgue point for u^η. On A_2, by condition (c), λ almost every x is a Lebesgue point either for $u^\xi + u^\eta$ or for $u^\xi - u^\eta$, since $h_1 + h_2 + 2h_3$ and $h_1 + h_2 - 2h_3$ do not vanish at the same points. Together with (a), this implies that $|\mu_{\xi\xi}|$ almost every $x \in A_2$ is a Lebesgue point for u^η. □

Using Proposition 6.8 we can now prove Theorem 6.1 in the general case.

Proof of Theorem 6.1. Let us fix u, $v \in BD(\Omega)$. By the Lebesgue decomposition theorem there exists a Borel set $N \subset \Omega$ such that $|Eu|(N) = 0$ and $|Ev| \llcorner (\Omega \setminus N)$ is absolutely continuous with respect to $|Eu|$. Since $|Eu|\big((S_u \setminus J_u) \setminus N\big) = 0$ by Proposition 6.8, we deduce that $|Ev|\big((S_u \setminus J_u) \setminus N\big) = 0$. It remains to prove that $|Ev|\big((S_u \setminus J_u) \cap N\big) = 0$. Since

$$S_u \setminus J_u \subset (S_v \setminus J_v) \cup [(S_u \setminus J_u) \setminus S_v] \cup [(S_u \setminus J_u) \cap J_v]$$

and $|Ev|(S_v \setminus J_v) = 0$ (Proposition 6.8), it is enough to show that

$$|Ev|\big(((S_u \setminus J_u) \setminus S_v) \cap N\big) = 0\,, \tag{6.8}$$

$$|Ev|\big((S_u \setminus J_u) \cap J_v \cap N\big) = 0\,. \tag{6.9}$$

It is easy to see that a Lebesgue point of v is a Lebesgue (resp. jump) point of u if and only if it is a Lebesgue (resp. jump) point of $u+v$; this shows that $(S_u \setminus J_u) \setminus S_v = (S_{u+v} \setminus J_{u+v}) \setminus S_v$, so that, by Proposition 6.8, $|E(u+v)|\big((S_u \setminus J_u) \setminus S_v\big) = 0$. As Ev and $E(u+v)$ coincide as measures on N, we obtain (6.8).

Let x be a jump point of v. If x is a Lebesgue point of $u+v$, then x is a jump point of u; if x is a jump point of $u+v$ and $\nu_{u+v}(x) = \pm\nu_v(x)$, then x is either a Lebesgue point or a jump point of u. As J_v and J_{u+v} are countably $(\mathcal{H}^{n-1}, n-1)$ rectifiable, the condition on ν_{u+v} and ν_v is satisfied $\mathcal{H}^{n-1}$ almost everywhere in $J_{u+v} \cap J_v$ (see Remark 3.6). This implies that $(S_u \setminus J_u) \cap J_v \subset S_{u+v} \setminus J_{u+v}$ up to an $\mathcal{H}^{n-1}$ negligible set. Consequently from Proposition 6.8 and Remark 3.3 we get $|E(u+v)|\big((S_u \setminus J_u) \cap J_v\big) = 0$. As Ev and $E(u+v)$ coincide as measures on N, we obtain (6.9). □

7. Approximate differentiability

In this section we prove that BD functions are approximately differentiable in $\mathcal{L}^n$ almost every point of their domain. We will also see that the same proof can be used to establish Korn's inequality. We start by recalling the definition of approximate limit (see [17], 2.9.19).

Definition 7.1. Let u be a real valued Borel function defined $\mathcal{L}^n$ almost everywhere in a neighbourhood of a point $x \in \mathbf{R}^n$. We say that $z \in \mathbf{R}$ is the *approximate limit* of u at x, and we write $z = \operatorname*{ap\,lim}_{y \to x} u(y)$, if, for every $\epsilon > 0$, the set $A_\epsilon := \{y : |u(y) - z| > \epsilon\}$ has $\mathcal{L}^n$ density zero at x, i.e.,

$$\lim_{\varrho \to 0+} \frac{\mathcal{L}^n(A_\epsilon \cap B_\varrho(x))}{\mathcal{L}^n(B_\varrho(x))} = 0 .$$

The *approximate upper limit* $\operatorname*{ap\,lim\,sup}_{y \to x} u(y)$ is defined as the infimum of the set of all numbers t such that the set $\{y : u(y) > t\}$ has $\mathcal{L}^n$ density zero at x.

If u is locally $\mathcal{L}^n$ integrable, it is easy to see that

$$\lim_{\varrho \to 0+} \frac{1}{\varrho^n} \int_{B_\varrho(x)} |u(y) - z| \, dy = 0 \qquad \Longrightarrow \qquad z = \operatorname*{ap\,lim}_{y \to x} u(y) ,$$

so that any Lebesgue point x where $\tilde{u}(x) = u(x)$ is an approximate continuity point for u. In general, the opposite implication does not hold, unless u is locally bounded.

We can now recall the definition of approximate differentiability.

Definition 7.2. Let $u : \Omega \to \mathbf{R}^m$ be a Borel function and let $x \in \Omega$. We say that u is *approximately differentiable* at x if there exists an $m \times n$ matrix L such that

$$\operatorname*{ap\,lim}_{y \to x} \frac{|u(y) - u(x) - L(y - x)|}{|y - x|} = 0 . \tag{7.1}$$

The matrix L, called *approximate differential* of u at x, is uniquely determined and will be denoted by $\nabla u(x)$.

We recall that functions which are approximately differentiable in $\mathcal{L}^n$ almost every point of their domain can be approximated, in the sense of Lusin, by Lipschitz functions. Moreover a necessary and sufficient condition for $\mathcal{L}^n$ almost everywhere approximate differentiability is the inequality

$$\operatorname*{ap\,lim\,sup}_{y \to x} \frac{|u(y) - u(x)|}{|y - x|} < +\infty \qquad \text{for } \mathcal{L}^n \text{ a.e. } x \in \mathbf{R}^n$$

(see [17], Theorem 3.1.8). A sufficient condition for the approximate differentiability, intermediate between the stronger condition (2.12) and (7.1), is given by the following proposition.

Proposition 7.3. *Let $u:\Omega \to \mathbf{R}^m$ be a locally $\mathcal{L}^n$ integrable function and assume that*

$$\lim_{\varrho\to 0^+} \frac{1}{\varrho^n}\int_{B_\varrho(x)} \frac{|u(y)-u(x)-L(y-x)|}{\varrho}\,dy = 0$$

for some $x \in \Omega$ and for some $m\times n$ matrix L. Then u is approximately differentiable at x and $\nabla u(x) = L$.

Proof. We will prove that for every $\epsilon > 0$ the set

$$A_\epsilon := \Big\{y\in\Omega : \frac{|u(y)-u(x)-L(y-x)|}{|y-x|} > \epsilon\Big\}$$

has $\mathcal{L}^n$ density 0 at x. For any $\varrho > 0$ and any $t \in (0,1)$ we can split $B_\varrho(x)$ into the union of the sets $B_\varrho(x)\setminus B_{t\varrho}(x)$ and $B_{t\varrho}(x)$. This leads to the estimate

$$\frac{1}{\varrho^n}\mathcal{L}^n\big(B_\varrho(x)\cap A_\epsilon\big) \le \frac{1}{t\epsilon\varrho^n}\int_{B_\varrho(x)} \frac{|u(y)-u(x)-L(y-x)|}{\varrho}\,dy + \omega_n t^n\,.$$

Taking the upper limit as $\varrho\to 0^+$ we obtain

$$\limsup_{\varrho\to 0^+}\frac{1}{\varrho^n}\mathcal{L}^n\big(B_\varrho(x)\cap A_\epsilon\big) \le \omega_n t^n\,.$$

Since $t\in(0,1)$ is arbitrary, the statement is proved. □

We can now state our approximate differentiability result.

Theorem 7.4. *Let $u\in BD(\mathbf{R}^n)$. Then for $\mathcal{L}^n$ almost every $x\in\mathbf{R}^n$ there exists a $n\times n$ matrix $\nabla u(x)$ such that*

$$\lim_{\varrho\to 0^+} \frac{1}{\varrho^n}\int_{B_\varrho(x)} \frac{|u(y)-u(x)-\nabla u(x)(y-x)|}{\varrho}\,dy = 0\,. \tag{7.2}$$

In particular, u is approximately differentiable $\mathcal{L}^n$ almost everywhere in $\mathbf{R}^n$. Moreover the function ∇u satisfies the weak L^1 estimate

$$\mathcal{L}^n\big(\{x\in\mathbf{R}^n : |\nabla u(x)| > t\}\big) \le \frac{c(n)}{t}|Eu|(\mathbf{R}^n) \qquad \forall t>0\,,$$

where $c(n)$ is a constant which depends only on n.

Remark 7.5. Property (7.2), together with Theorem 4.3, implies that $\mathcal{E}u(x)$ is equal to the symmetric part of the matrix $\nabla u(x)$ for $\mathcal{L}^n$ almost every $x \in \mathbf{R}^n$.

Remark 7.6. Property (7.2) implies that for $\mathcal{L}^n$ almost every $x \in \mathbf{R}^n$ the rescaled functions $v_\varrho(y) := [u(x+\varrho y) - u(x)]/\varrho$ converge in $L^1(B_1(0);\mathbf{R}^n)$, as $\varrho \to 0^+$, to the linear function $v_0(y) := \nabla u(x)\, y$. This information is very useful in connection with the so-called blow up method introduced in [18] in order to estimate the absolutely continuous part of relaxed functionals with linear growth. Under the weaker assumption (7.1) the convergence occurs only in measure.

Example 7.7. Assume that $n \geq 2$. Using an example of Ornstein [30] (see also [35], Chapter I, Section 1) we can construct a sequence of functions φ_k in $\mathcal{C}_c^\infty(\mathbf{R}^n;\mathbf{R}^n)$, with supports contained in $B := B_1(0)$, which do not satisfy Korn's inequality for the L^1 norm, i.e., after a suitable normalization, we have

$$2^k \int_B |\mathcal{E}\varphi_k|\, dx \;\leq\; \int_B |\nabla \varphi_k|\, dx \;=\; 1\,.$$

In particular, by Poincaré's inequality, the integrals $\int_B |\varphi_k|\, dx$ are bounded by a constant c independent of k. Let us consider now a sequence of pairwise disjoint balls $B_k := B_{r_k}(x_k)$ contained in B such that $\sum\limits_{k=1}^{\infty} r_k < +\infty$ and whose centres x_k converge to 0. Let $v_k(x) := r_k^{1-n}\varphi_k\big(r_k^{-1}(x-x_k)\big)$. By an elementary change of variables we obtain

$$2^k \int_{B_k} |\mathcal{E}v_k|\, dx \;\leq\; \int_{B_k} |\nabla v_k|\, dx \;=\; 1\,, \qquad \int_{B_k} |v_k|\, dx \;\leq c\, r_k\,.$$

Therefore the series $\sum\limits_{k=1}^{\infty} v_k$ is absolutely convergent in the BD norm and, consequently, its sum u belongs to $BD(\mathbf{R}^n)$ and satisfies $|Eu|(\mathbf{R}^n) \leq 1$; moreover, u has compact support in B and belongs to $\mathcal{C}^\infty(\mathbf{R}^n \setminus \{0\})$. By Proposition 4.7(iii) we have also $u \in SBD(\mathbf{R}^n)$. But

$$\int_B |\nabla u|\, dx \geq \sum_{k=1}^{\infty} \int_{B_k} |\nabla v_k|\, dx = +\infty\,.$$

This proves that $u \notin BV(\Omega;\mathbf{R}^n)$ and that, in general, no strong L^1 estimate can be obtained for ∇u.

The proof of Theorem 7.4 is mainly based on three tools: the first one is the explicit version of the "Poincaré type" inequality for BD functions stated in Theorem 6.5. Using that inequality we are able, in $\mathcal{L}^n$ almost every point, to estimate the difference quotient of u up to a constant vector $\xi_\varrho(x)$ and an affine map $\alpha_\varrho(x)(y-x)$. The vector $\xi_\varrho(x)$ can be easily estimated making use of Theorem 4.3, while the matrix $\alpha_\varrho(x)$ is represented by a singular integral. For this reason we will also need the following well known result.

Proposition 7.8. *Let $K\colon \mathbf{R}^n \to \mathbf{R}$ be a 0-homogeneous function, smooth and with mean value zero on the unit sphere $\mathbf{S}^{n-1}$. For any measure μ with finite total variation in $\mathbf{R}^n$ let us define the functions*

$$h_\varrho(x) := \int_{\{|y-x|\geq\varrho\}} \frac{K(y-x)}{|y-x|^n}\, d\mu(y), \qquad \varrho > 0.$$

Then the function $h(x) := \sup_{\varrho>0} |h_\varrho(x)|$ satisfies the weak L^1 estimate

$$\mathcal{L}^n\big(\{x\in\mathbf{R}^n : h(x) > t\}\big) \leq \frac{c(n,K)}{t}|\mu|(\mathbf{R}^n) \qquad \forall t > 0,$$

where $c(n,K)$ is a constant depending only on n and K.

Proof. If μ is absolutely continuous with respect to $\mathcal{L}^n$, the proof can be found, for instance, in Theorem 4 of Chapter II of [32]. In the general case the statement can be easily proved by approximating μ by $\mu_\epsilon := \mu \star \psi_\epsilon$, taking into account the left continuity of $\varrho \mapsto h_\varrho(x)$ and the continuity property

$$\lim_{\epsilon\to 0^+} \int_{\{|y-x|\geq\varrho\}} \frac{K(y-x)}{|y-x|^n}\, d\mu_\epsilon(y) = \int_{\{|y-x|\geq\varrho\}} \frac{K(y-x)}{|y-x|^n}\, d\mu(y)$$

for any ϱ such that $\mu(\partial B_\varrho(x)) = 0$. □

Remark 7.9. If $\mu = f\mathcal{L}^n$ for some $f \in L^p(\mathbf{R}^n)$ with $p > 1$, then we have a *strong* L^p estimate on h, namely $\|h\|_{L^p(\mathbf{R}^n)} \leq c(n,K)\|f\|_{L^p(\mathbf{R}^n)}$ (see, e.g., [32], Chapter II, Theorem 4).

Proof of Theorem 7.4. By Theorem 4.3 and Proposition 7.8 (that can be applied to each component of $\Gamma(x)/|x|^2$ thanks to (6.2)), $\mathcal{L}^n$ almost every $x \in \mathbf{R}^n$ satisfies the following properties:

(i) $$\lim_{\varrho\to 0^+} \frac{1}{\varrho^n} \int_{B_\varrho(x)} \frac{|(u(y)-u(x)-\mathcal{E}u(x)(y-x), y-x)|}{\varrho|y-x|}\, dy = 0;$$

(ii) x is a Lebesgue point for $\mathcal{E}u$ and $\widetilde{\mathcal{E}u}(x) = \mathcal{E}u(x)$;

(iii) $|E^s u|$ has n dimensional density zero at x;

(iv) the matrices $\beta_\varrho(x)$ in (6.3) are equi-bounded with respect to ϱ.

Step 1. Let us fix a point $x \in \mathbf{R}^n$ which satisfies (i)–(iv). We will prove that these properties imply

$$\lim_{\varrho\to 0^+} \frac{1}{\varrho^n} \int_{B_\varrho(x)} \frac{|u(y) - u(x) - \alpha_\varrho(x)(y-x)|}{\varrho}\, dy = 0 \tag{7.3}$$

for suitable $n\times n$ matrices $\alpha_\varrho(x)$ satisfying

$$\sup_{\varrho>0} |\alpha_\varrho(x)| \le k_n |\mathcal{E}u(x)| + \sup_{\varrho>0} |\beta_\varrho(x)|\,, \tag{7.4}$$

where k_n is a constant depending only on n. To this aim, given a function $\psi \in \mathcal{C}^1_c(\mathbf{R})$ such that $\psi(t) = 1$ for $|t| \le 1$, $\psi(t) = 0$ for $|t| \ge 2$, and $|\psi'(t)| \le 2$ for every $t \in \mathbf{R}$, we define $\varphi(y) := \psi(|y-x|)$, $w(y) := \varphi(y)\mathcal{E}u(x)(y-x)$, $v(y) := u(y) - w(y)$ for every $y \in \mathbf{R}^n$. Applying Theorem 6.5 to the function v, for any $\varrho \in (0,1)$ we obtain

$$\begin{aligned} &\int_{B_\varrho(x)} |v(y) - \gamma_\varrho(x)(y-x) - \xi_\varrho(x)|\, dy \\ &\le c_n \varrho \int_{B_\varrho(x)} |\mathcal{E}u(y) - \mathcal{E}u(x)|\, dy \;+\; c_n \varrho\, |E^s u|(B_\varrho(x)) \end{aligned}$$

for suitable constant vectors $\xi_\varrho(x)$, with

$$\gamma_\varrho(x) = \beta_\varrho(x) \;+\; \frac{1}{2\omega_n} \int_{\{|y-x|\ge\varrho\}} \frac{\Gamma(y-x)}{|y-x|^{n+2}}\, \mathcal{E}w(y)\, dy\,. \tag{7.5}$$

Setting $\alpha_\varrho(x) = \mathcal{E}u(x) + \gamma_\varrho(x)$, and dividing both sides by ϱ^{n+1}, properties (ii) and (iii) imply

$$\lim_{\varrho\to 0^+} \frac{1}{\varrho^n} \int_{B_\varrho(x)} \frac{|u(y) - \xi_\varrho(x) - \alpha_\varrho(x)(y-x)|}{\varrho}\, dy \;=\; 0. \tag{7.6}$$

Since $\gamma_\varrho(x)$ is skew symmetric by (6.2), property (i) is valid with $\mathcal{E}u(x)$ replaced by $\alpha_\varrho(x)$; using this fact together with (7.6) we infer

$$\lim_{\varrho\to 0^+} \frac{1}{\varrho^n} \int_{B_\varrho(x)} \frac{|(\xi_\varrho(x) - u(x), y-x)|}{\varrho|y-x|}\, dy \;=\; 0$$

which implies that $\xi_\varrho(x) = u(x) + o(\varrho)$. Inserting this into (7.6) we eventually obtain (7.3).

Since

$$\nabla w(y) = \psi(|y-x|)\mathcal{E}u(x) + [\mathcal{E}u(x)(y-x)]\otimes\nabla\varphi(y) \tag{7.7}$$

and, by (6.2), Γ has mean value zero on the unit sphere $\mathbf{S}^{n-1}$, the first term in the right hand side of (7.7) gives no contribution to the integral in (7.5). Consequently that integral can be estimated by

$$C_n|\mathcal{E}u(x)|\int_{B_2(x)}\frac{dy}{|y-x|^{n-1}} = 2n\omega_n C_n|\mathcal{E}u(x)|,$$

where C_n is a constant depending only on n. As $|\alpha_\varrho(x)| \le |\mathcal{E}u(x)| + |\gamma_\varrho(x)|$, inequality (7.4) follows now from (7.5).

Step 2. Using the same argument of Proposition 7.3, we obtain

$$\operatorname{ap}\limsup_{y\to x}\frac{|u(y)-u(x)|}{|y-x|} \le \sup_{\varrho>0}|\alpha_\varrho(x)| < +\infty$$

for any $x \in \mathbf{R}^n$ satisfying (i)–(iv), and hence (7.3) and (7.4). In particular, Theorem 3.1.8 in [17] implies that u is $\mathcal{L}^n$ almost everywhere approximately differentiable in $\mathbf{R}^n$ and that ∇u satisfies

$$|\nabla u(x)| \le \sup_{\varrho>0}|\alpha_\varrho(x)| \le k_n|\mathcal{E}u(x)| + \sup_{\varrho>0}|\beta_\varrho(x)|.$$

This pointwise estimate and the weak L^1 estimate of Proposition 7.8 imply the stated weak L^1 estimate for ∇u.

Step 3. Now we prove the stronger condition (7.2). Let x be an approximate differentiability point where (7.3) holds and the matrices $\alpha_\varrho(x)$ are equi-bounded. If L is the limit of $\alpha_{\varrho_k}(x)$ for some sequence ϱ_k converging to 0, we have

$$\lim_{h\to+\infty}\frac{1}{(\varrho_k)^n}\int_{B_{\varrho_k}(x)}\frac{|u(y)-u(x)-L(y-x)|}{\varrho_k}\,dy = 0.$$

This means that the rescaled functions $v_k(y) := [u(x+\varrho_k y) - u(x)]/\varrho_k$ converge to Ly in $L^1(B_1(0);\mathbf{R}^n)$; on the other hand, (7.1) implies that the same functions converge in measure in $B_1(0)$ to $\nabla u(x)y$, hence $L = \nabla u(x)$. Since the sequence ϱ_k is arbitrary, this proves that $\alpha_\varrho(x)$ converges to $\nabla u(x)$ as $\varrho \to 0^+$, so that (7.3) implies (7.2). $\square$

Remark 7.10. Since the approximate differentiability and the stronger condition (7.2) are local properties, they continue to hold, more generally, for any open set $\Omega \subset \mathbf{R}^n$ and any function $u \in BD(\Omega)$. If Ω is bounded with Lipschitz boundary, then, extending u by 0 outside Ω (see Section 3) and applying Theorem 7.4 to the extension we obtain the estimate

$$\mathcal{L}^n\big(\{x \in \Omega : |\nabla u(x)| > t\}\big) \leq \frac{c(n,\Omega)}{t}\Big[\,|Eu|(\Omega) + \int_\Omega |u(x)|\,dx\,\Big]\,.$$

Remark 7.11. We also notice that our argument provides a proof of Korn's inequality

$$\int_{\mathbf{R}^n} |\nabla u(x)|^p\,dx \leq k(n)\int_{\mathbf{R}^n} |\mathcal{E}u(x)|^p\,dx$$

in the case $E^s u = 0$ and $\mathcal{E}u \in L^p(\mathbf{R}^n)$ with $p > 1$. Indeed, using Remark 7.9, in this case we have a strong L^p estimate on $\sup\limits_{\varrho>0}|\beta_\varrho|$. Moreover, if we approximate u by convolutions and apply the estimate above to the regularized functions, we obtain also that u belongs to the Sobolev space $H^{1,p}(\mathbf{R}^n;\mathbf{R}^n)$.

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