On the interplay of two-scale convergence and translation

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Abstract

We study the effects of translation on two-scale convergence. Given a two-scale convergent sequence $(u_\varepsilon(x))_\varepsilon$ with two-scale limit $u(x,y)$, then in general the translated sequence $(u_\varepsilon(x+t))_\varepsilon$ is no longer two-scale convergent, even though it remains two-scale convergent along suitable subsequences. We prove that any two-scale cluster point of the translated sequence is a translation of the original limit and has the form $u(x+t,y+r)$ where the microscopic translation r belongs to a set that is determined solely by t and the vanishing sequence (ε). Finally, we apply this result to a novel homogenization problem that involves two different coordinate frames and yields a limiting behavior governed by emerging microscopic translations.

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1 Introduction

Two-scale convergence, as it was introduced by Nguetseng in [17] and elaborated by Allaire in [1], can loosely speaking be interpreted as an "intermediate" convergence between weak and strong convergence in $L^2(\Omega)$ with the capability to capture fine oscillation properties. The two-scale limit of a sequence $(u_{\varepsilon_j}(x))_j$ in $L^2(\Omega)$ with $\Omega \subset \mathbb{R}^N$ is a function $u(x,y) \in L^2(\Omega \times [0,1)^N)$ depending on an additional variable y that provides information on local oscillations. Starting with the seminal work by Nguetseng [17] and Allaire [1], the notion of two-scale convergence has greatly contributed to the understanding of a wide variety of problems in the context of homogenization. Recently, a reinvestigation of this notion — initiated by the dilation technique (see [4, 6]) — led to the periodic unfolding method (cf. [9, 10, 18]) and revealed that two-scale convergence can be equivalently defined by weak convergence in an appropriate larger space.

In this contribution we characterize, how a translation of the coordinate frame affects two-scale convergence of sequences in $L^2(\Omega)$. While the translation operator is continuous w.r.t. strong and weak convergence in $L^2(\Omega)$, its interplay with two-scale convergence is more subtle. As an illustration consider in one space dimension $\Omega = \mathbb{R}$ the sequence

$$u_\varepsilon : \mathbb{R} \to \mathbb{R}, \qquad u_\varepsilon(x) := u_{\text{macro}}(x)\, u_{\text{oscill}}(\tfrac{x}{\varepsilon}),$$

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parametrized by the small positive parameter ε, with u_{macro} being an arbitrary function in $L^2(\mathbb{R})$ and u_{oscill} a continuous, $[0,1)$-periodic function s.t. $\int_{(0,1)} u_{\mathrm{oscill}}(y)\,\mathrm{d}y = 1$. It is well-known that

$$u_{\varepsilon_j}(x) \rightharpoonup u_{\mathrm{macro}}(x) \qquad \text{weakly in } L^2(\mathbb{R})$$
$$u_{\varepsilon_j}(x) \xrightharpoonup{2} u_{\mathrm{macro}}(x)\, u_{\mathrm{oscill}}(y) \qquad \text{weakly two-scale in } L^2(\mathbb{R}\times[0,1))$$

for any vanishing sequence of positive numbers $(\varepsilon_j)_j$. Clearly, the translated sequence $(u_{\varepsilon_j}(x+t))_j$ converges to $u_{\mathrm{macro}}(x+t)$ weakly in $L^2(\mathbb{R})$. In contrast to this, $(u_{\varepsilon_j}(x+t))_j$ may not even be weakly two-scale convergent. Still, as it remains bounded in $L^2(\Omega)$, the translated sequence exhibits weakly two-scale convergent subsequences (by the classical two-scale compactness result in Proposition 2.3 below). For instance, if $\varepsilon_j = \frac{2}{j}$ and $t = 1$, then the translated sequence solely exhibits the weak two-scale cluster points $u_{\mathrm{macro}}(x+1)\, u_{\mathrm{oscill}}(y)$ and $u_{\mathrm{macro}}(x+1)\, u_{\mathrm{oscill}}\left(y+\frac{1}{2}\right)$. This suggests that for *any* sequence $(\varepsilon_j)_j$ and *any* translation t,

each weak two-scale cluster point of $(u_{\varepsilon_j}(x+t))_j$
is of the form $u_{\mathrm{macro}}(x+t)\, u_{\mathrm{oscill}}(y+r)$

for some $r \in [0,1]$, which we call a "microtranslation".

Indeed, we are going to prove the following: Let $(u_{\varepsilon_j})_j$ be a sequence in $L^2(\Omega)$ that weakly two-scale converges to $u \in L^2(\Omega\times[0,1)^N)$ w.r.t. the vanishing sequence $(\varepsilon_j)_j$ and consider an arbitrary translation $t \in \mathbb{R}^N$. Then each weak two-scale cluster point of the translated sequence $(u_{\varepsilon_j}(\cdot+t))_j$ has the form $u(\cdot+t, \cdot+r)$ for some microtranslation r. That is, the original weak two-scale limit not only shifted in its "macrovariable" by the "macrotranslation" t, but also in the "microvariable" by the microtranslation r. More precisely, we show that the set $\mathscr{C}_w$ of all weak two-scale cluster points of $(u(\cdot+t)_{\varepsilon_j})_j$ can be characterized according to

$$\mathscr{C}_w = \left\{ u(\cdot+t, \cdot+r) \;:\; r \in \mathscr{M} \right\}$$

where $\mathscr{M}$ denotes the set of all attainable microtranslations. It turns out, that $\mathscr{M}$ is completely determined by the translation t and the vanishing sequence $(\varepsilon_j)_j$. In the case where $(u_{\varepsilon_j})_j$ strongly two-scale converges to u, the very same representation holds true for the set $\mathscr{C}_s$ of all strong two-scale cluster points of the translated sequence. Thus translation of the coordinate frame considerably affects two-scale convergence in $L^2(\Omega)$, in a way that *only* depends on the translation t and the vanishing sequence $(\varepsilon_j)_j$ *but not* on the particular sequence of functions itself. Finally, we will show that $\mathscr{C}_w$, $\mathscr{C}_s$ and $\mathscr{M}$ are compact and illustrate the dependence of $\mathscr{M}$ on the choice of t and $(\varepsilon_j)_j$.

In the last section, we apply these results to a homogenization problem that involves two distinct coordinate frames being translated by a constant vector. Our goal is to characterize the asymptotic behavior of the oscillating functional

$$\mathscr{F}_\varepsilon : W_0^{1,2}(\Omega;\mathbb{R}^N) \to \mathbb{R}, \quad \mathscr{F}_\varepsilon(u) := \int_\Omega W\left(\tfrac{x}{\varepsilon}, \nabla u(x)\right)\mathrm{d}x + \int_{\Omega-t} W\left(\tfrac{x}{\varepsilon}, \nabla u(x+t)\right)\mathrm{d}x,$$

$t \in \mathbb{R}^N$ being an arbitrary translation, as the size ε of the oscillations tends to zero. Here, the energy density $W(y,F)$ shall be convex, continuous and of quadratic growth in F, and $[0,1)^N$-periodic in y. In the case $t = 0$, this kind of problem has been extensively studied, most notably by Marcellini in his fundamental contribution [14] and later on also in non-convex situations related to elasticity by Müller [16]. Since the early 1990s, many authors have revisited the above problem with trivial translation $t = 0$ employing methods related to two-scale convergence in order to elegantly prove Γ-convergence of $\mathscr{F}_\varepsilon$ to a homogenized limiting problem (see in particular [1, 10, 19], while we refer to [7] for the concept of Γ-convergence). In the new situation with nontrivial translation $t \neq 0$ on the other hand, Γ-convergence generally fails as we show in Example 4.1 at the end of this article. However, thanks to our characterization of weak two-scale cluster points of translated sequences we can explicitly state the lower and upper Γ-limits of $\mathscr{F}_\varepsilon$ (as ε vanishes) in terms of associated microtranslations.

2 Notation and Preliminaries

Let $p,q \in (1,\infty)$ with $\frac{1}{p}+\frac{1}{q} = 1$. Here and hereafter we will denote by Ω an open subset of $\mathbb{R}^N$. Given an arbitrary vector $t \in \mathbb{R}^N$, the set $\Omega - t$ is to be understood as the set $\{x - t \,:\, x \in \Omega\}$. The N-dimensional unit cube $[0,1)^N$ is abbreviated by Y and any function defined on Y in one of its variables is assumed to be extended to $\mathbb{R}^N$ by Y-periodicity. Furthermore, $C_{\text{per}}(Y)$ is defined as the space of Y-periodic, continuous functions from $\mathbb{R}^N$ to $\mathbb{R}$ (analogously we define $C^\infty_{\text{per}}(Y)$, $W^{1,p}_{\text{per}}(Y)$ etc.). Moreover, we will frequently encounter the sequence $(\varepsilon_j)_j$, which shall be an arbitrary but fixed vanishing sequence of positive real numbers. Following ideas from [9] and adopting the notation used in [18], we define $\mathscr{N} : \mathbb{R}^N \to \mathbb{Z}^N$ and $\mathscr{R} : \mathbb{R}^N \to Y$ by setting

$$\mathscr{N}(x) := \max\{z \in \mathbb{Z}^N : z_i \leq x_i,\ i = 1,\dots,N\}, \qquad \mathscr{R}(x) := x - \mathscr{N}(x)$$

wherein max is taken componentwise.

The classical definition of two-scale convergence by Nguetseng and Allaire (see [1, 17, 13]) is the following:

Definition 2.1. A sequence $(u_{\varepsilon_j})_j$ in $L^p(\Omega)$ is said to *weakly two-scale converge* in $L^p(\Omega\times Y)$ to a limit $u \in L^p(\Omega\times Y)$, if

$$\lim_{j\to\infty} \int_\Omega u_{\varepsilon_j}(x)\, \psi\left(x, \frac{x}{\varepsilon_j}\right) \mathrm{d}x = \int_\Omega \int_Y u(x,y)\, \psi(x,y)\, \mathrm{d}y\, \mathrm{d}x \tag{1}$$

holds for all $\psi \in L^q(\Omega; C_{\text{per}}(Y))$. In this case we write

$$u_{\varepsilon_j} \xrightarrow{2} u \quad \text{in } L^p(\Omega\times Y).$$

If in addition $(u_{\varepsilon_j})_j$ satisfies $\lim_j \|u_{\varepsilon_j}\|_{L^p(\Omega)} = \|u\|_{L^p(\Omega\times Y)}$, then the sequence is said to be *strongly two-scale convergent* in $L^p(\Omega\times Y)$ to u and we write

$$u_{\varepsilon_j} \xrightarrow{2} u \quad \text{in } L^p(\Omega\times Y).$$

In the proof of the main theorem and in its application to the homogenization problem, we will fall back to the following results.

Proposition 2.2. *Let $(u_{\varepsilon_j})_j$ be a sequence in $L^p(\Omega)$.*

1. *Weak and strong two-scale limits are unique.*

2. *If $(u_{\varepsilon_j})_j$ is bounded in $L^p(\Omega)$, then it is weakly two-scale convergent in $L^p(\Omega\times Y)$ if and only if* (1) *holds for any $\psi \in C_c^\infty(\Omega;C_{\mathrm{per}}^\infty(Y))$.*

3. *If $(u_{\varepsilon_j})_j$ is two-scale convergent in $L^p(\Omega\times Y)$, then it is bounded in $L^p(\Omega)$.*

Proposition 2.3. *1. Let $(u_{\varepsilon_j})_j$ be a bounded sequence in $L^p(\Omega)$. Then there exists a subsequence $(u_{\varepsilon_{j_k}})_k$ and a function $u \in L^p(\Omega\times Y)$ such that*

$$u_{\varepsilon_{j_k}} \xrightarrow{2} u \quad \text{in } L^p(\Omega\times Y).$$

2. *Let $(u_{\varepsilon_j})_j$ be a sequence which is weakly convergent to u in $W^{1,p}(\Omega;\mathbb{R}^M)$. Then there exists a subsequence $(u_{\varepsilon_{j_k}})_k$ and a function*

$$u_1 \in L^p(\Omega;W^{1,p}_{\mathrm{per}}(Y;\mathbb{R}^M))$$

such that

$$\nabla u_{\varepsilon_{j_k}} \xrightarrow{2} \nabla u + \nabla_y u_1 \quad \text{in } L^p(\Omega\times Y;\mathbb{R}^{M\times N}),$$

Herein, $\nabla_y u_1$ denotes the weak derivative of u_1 w.r.t. its second argument.

For proofs of these assertions, the reader is for instance referred to [1, 13, 19]. We conclude this section by recalling the notion of Γ-convergence, which we use in the last section of this contribution. A detailed exposition of Γ-convergence can be found in [7, 11].

Definition 2.4. Let $(X,\mathscr{T})$ be a topological space and $f_j : X \to [-\infty,+\infty]$, $j \in \mathbb{N}$. For all $x \in X$ we denote the set of all open neighborhoods of x by $\mathscr{U}(x)$.

The lower and upper Γ-limit of the sequence $(f_j)_j$ are the functions from X to $[-\infty,+\infty]$ defined by

$$\left(\Gamma\text{-}\liminf_{j\to\infty} f_j\right)(x) := \sup_{U\in\mathscr{U}(x)} \liminf_{j\to\infty} \inf_{y\in U} f_j(y),$$
$$\left(\Gamma\text{-}\limsup_{j\to\infty} f_j\right)(x) := \sup_{U\in\mathscr{U}(x)} \limsup_{j\to\infty} \inf_{y\in U} f_j(y).$$

If there exists a function $f_\infty : X \to [-\infty,+\infty]$ such that $\Gamma\text{-}\liminf_j f_j = f_\infty = \Gamma\text{-}\limsup_j f_j$, we say that $(f_j)_j$ Γ-converges to f_∞. In this case we write $\Gamma\text{-}\lim_j f_j = f_\infty$.

Under reasonable assumptions on the topology of X (cf. [11, Proposition 8.1]) one can give an equivalent, sequential characterization of the lower and upper Γ-limit, which turns out to be more convenient in applications. In particular, this equivalence is true (under suitable coercivity assumptions on the sequence) in the important case of X being a reflexive Banach space equipped with the weak topology (see [11, Proposition 8.16 and Example 1.14]):

Theorem 2.5. *Suppose X is a reflexive Banach space endowed with its weak topology. Moreover, let the sequence $(f_j)_j$ be equicoercive, i.e. there exists a lower semicontinuous function $\Psi : X \to [-\infty,+\infty]$ with $\Psi(x) \to +\infty$ as $\|x\| \to +\infty$, such that $f_j \geq \Psi$ for all $j \in \mathbb{N}$. In this situation the sequence $(f_j)_j$ Γ-converges to some function f_∞, if and only if*

1. *for every $x \in X$ and every sequence $(x_j)_j$ weakly converging to x it is*

$$f_\infty(x) \leq \liminf_{j\to\infty} f_j(x_j),$$

2. *for every $x \in X$ there exists a sequence $(x_j)_j$ weakly converging to x with*

$$f_\infty(x) = \lim_{j\to\infty} f_j(x_j).$$

Following the common convention, we call the sequence $(x_j)_j$ in 2 a recovery sequence and refer to 1 and 2 as the "liminf-inequality" and "the existence of recovery sequences".

3 Main Results

From now on, $(\varepsilon_j)_j$ denotes an arbitrary *but fixed* vanishing sequence of positive real numbers. As explained in the introduction, our intention is to understand the behavior of weakly or strongly two-scale convergent sequences under translation of the coordinate frame. To this end, it is convenient to define the following:

Definition 3.1. Let $(u_{\varepsilon_j})_j$ be a sequence in $L^p(\Omega)$ and $t \in \mathbb{R}^N$.

1. The set of all *weak two-scale cluster points* of the sequence $(u_{\varepsilon_j})_j$ translated by t is defined by

$$\mathscr{C}_w := \Big\{ v \in L^p((\Omega - t)\times Y) : \text{there exists a subsequence } (j_k)_k \text{ s.t. } u_{\varepsilon_{j_k}}(\cdot + t) \xrightarrow{2} v \quad \text{in } L^p((\Omega - t)\times Y) \Big\}.$$

The set of all *strong two-scale cluster points* of the sequence $(u_{\varepsilon_j})_j$ translated by t is defined by

$$\mathscr{C}_s := \Big\{ v \in L^p((\Omega - t)\times Y) : \text{there exists a subsequence } (j_k)_k \text{ s.t. } u_{\varepsilon_{j_k}}(\cdot + t) \xrightarrow{2} v \quad \text{in } L^p((\Omega - t)\times Y) \Big\}.$$

2. The set of all *microtranslations* emerging from the translation t and $(\varepsilon_j)_j$ is defined by

$$\mathscr{M} := \Big\{ r \in \overline{Y} : r \text{ is a cluster point of } \Big(\mathscr{R}(\tfrac{t}{\varepsilon_j})\Big)_j \text{ in } \mathbb{R}^N \Big\}.$$

Moreover, we call a subsequence $(j_k)_k$ a (t,r)*-subsequence*, if $\lim_{k\to\infty} \mathscr{R}(\frac{t}{\varepsilon_{j_k}}) = r$.

When considering a weakly or strongly two-scale convergent sequence $(u_{\varepsilon_j})_j$ in $L^p(\Omega)$, the reasoning in the introduction suggests that we cannot expect two-scale convergence of the translated sequence $(u_{\varepsilon_j}(\cdot + t))_j$. In particular, the sets $\mathscr{C}_w$ and $\mathscr{C}_s$ are going to be nontrivial. However, they can be characterized in a very precise manner:

Theorem 3.2. *Let $(u_{\varepsilon_j})_j$ be a sequence in $L^p(\Omega)$ and $u \in L^p(\Omega\times Y)$.*

1. *If $u_{\varepsilon_j} \xrightarrow{2} u$ in $L^p(\Omega\times Y)$, then*

$$\mathscr{C}_w = \Big\{ u(\cdot + t, \cdot + r) : r \in \mathscr{M} \Big\}.$$

2. *If $u_{\varepsilon_j} \xrightarrow{2} u$ in $L^p(\Omega\times Y)$, then*

$$\mathscr{C}_s = \Big\{ u(\cdot + t, \cdot + r) : r \in \mathscr{M} \Big\}.$$

Regarding this result, we urge the reader to notice that the set of all weak (strong) two-scale cluster points $\mathscr{C}_w$ ($\mathscr{C}_s$) of a two-scale convergent sequence $(u_{\varepsilon_j})_j$ translated by t does *not* depend on the sequence itself, but only on the vanishing sequence $(\varepsilon_j)_j$, the translation t and the original two-scale limit u. Indeed, since all two-scale cluster points can be obtained by doubly translating the original limit u by t in the macrovariable and by $r \in \mathscr{M}$ in the microvariable, we refer to $\mathscr{M}$ as the set of all microstranslations emerging from the translation t and $(\varepsilon_j)_j$.

The main ingredient in the proof of Theorem 3.2 will be the next lemma, which can be regarded as a result of its own interest.

Lemma 3.3. *Let $(j_k)_k$ be a (t,r)-subsequence and $(u_{\varepsilon_{j_k}})_k$ a weakly (respectively strongly) two-scale convergent sequence in $L^p(\Omega \times Y)$ with limit u. Then the sequence $(u_{\varepsilon_{j_k}}(\cdot + t))_k$ is weakly (respectively strongly) two-scale convergent in $L^p((\Omega - t) \times Y)$ to the function*

$$u(\cdot + t, \cdot + r) \in L^p((\Omega - t) \times Y).$$

Proof. The proof of this statement will be two-stage.

Step 1. We first consider the case of weak two-scale convergence $u_{\varepsilon_{j_k}} \xrightarrow{2} u$. Since the translated sequence remains bounded, it suffices to prove

$$\lim_{k\to\infty} \int_{\Omega - t} u_{\varepsilon_{j_k}}(x+t)\, \varphi(x, \tfrac{x}{\varepsilon_{j_k}})\, \mathrm{d}x = \int_{\Omega - t} \int_Y u(x+t, y+r)\, \varphi(x,y)\, \mathrm{d}y\, \mathrm{d}x \tag{2}$$

for all $\varphi \in C_c^\infty(\Omega - t; C_{\mathrm{per}}^\infty(Y))$. By the change of variables $x \mapsto x - t$ we can write the integral on the left hand side as

$$\begin{aligned} \int_{\Omega - t} u_{\varepsilon_{j_k}}(x+t)\, \varphi(x, \tfrac{x}{\varepsilon_{j_k}})\, \mathrm{d}x &= \int_\Omega u_{\varepsilon_{j_k}}(x)\, \varphi\left(x - t, \tfrac{x}{\varepsilon_{j_k}} - \tfrac{t}{\varepsilon_{j_k}}\right) \mathrm{d}x \\ = \int_\Omega u_{\varepsilon_{j_k}}(x) \varphi\left(x - t, \tfrac{x}{\varepsilon_{j_k}} - r\right) \mathrm{d}x &+ \int_\Omega u_{\varepsilon_{j_k}}(x) \varphi_{\varepsilon_{j_k}}(x)\, \mathrm{d}x, \end{aligned} \tag{3}$$

where

$$\varphi_{\varepsilon_{j_k}}(x) := \varphi\left(x - t, \tfrac{x}{\varepsilon_{j_k}} - \tfrac{t}{\varepsilon_{j_k}}\right) - \varphi\left(x - t, \tfrac{x}{\varepsilon_{j_k}} - r\right).$$

Since $\varphi(\cdot - t, \cdot - r) \in C_c^\infty(\Omega; C_{\mathrm{per}}^\infty(Y))$ is an admissible two-scale testfunction and $u_{\varepsilon_{j_k}} \xrightarrow{2} u$, we can pass to the limit in the first integral of (3) and obtain (after retransformation)

$$\lim_{k\to\infty} \int_\Omega u_{\varepsilon_{j_k}}(x) \varphi\left(x - t, \tfrac{x}{\varepsilon_{j_k}} - r\right) \mathrm{d}x = \int_{\Omega - t} \int_Y u(x+t, y+r) \varphi(x,y)\, \mathrm{d}y\, \mathrm{d}x. \tag{4}$$

Now, it remains to show that the second integral in (3) vanishes in the limit. Therefore, we prove that $\varphi_{\varepsilon_{j_k}} \to 0$ uniformly. Due to the decomposition

$$\tfrac{t}{\varepsilon_{j_k}} = \mathscr{N}\left(\tfrac{t}{\varepsilon_{j_k}}\right) + \mathscr{R}\left(\tfrac{t}{\varepsilon_{j_k}}\right)$$

and the Y-periodicity of φ in the second variable, we deduce

$$\varphi_{\varepsilon_{j_k}}(x) = \varphi\left(x - t, \tfrac{x}{\varepsilon_{j_k}} - r + r_k\right) - \varphi\left(x - t, \tfrac{x}{\varepsilon_{j_k}} - r\right),$$

where $r_k := r - \mathscr{R}\left(\tfrac{t}{\varepsilon_{j_k}}\right)$.

Now, the condition of $(j_k)_k$ being a (t,r)-subsequence and the smoothness of φ implies first that $r_k \to 0$ and secondly, that $\varphi_{\varepsilon_{j_k}}$ vanishes uniformly. Consequently, assertion (2) follows.

Step 2. Assume now strong two-scale convergence $u_{\varepsilon_{j_k}} \xrightarrow{2} u$. From Step 1 we already know that the translated sequence weakly two-scale converges to $u(\cdot+t,\cdot+r)$. However, this is already sufficient to infer strong two-scale convergence of the translated sequence, since $\|v\|_{L^p(\Omega)} = \|v(\cdot+t)\|_{L^p(\Omega-t)}$ for all $v \in L^p(\Omega)$ and thus $\|u_{\varepsilon_{j_k}}(\cdot+t)\|_{L^p(\Omega-t)} \to \|u(\cdot+t,\cdot+r)\|_{L^p((\Omega-t)\times Y)}$. □

Having this preparatory result at hand, the proof of the main Theorem 3.2 is no longer difficult:

Proof of Theorem 3.2. We confine ourselves to proving the statement for $\mathscr{C}_w$, since the proof for $\mathscr{C}_s$ is similar.

First, we show the inclusion $\{u(\cdot+t,\cdot+r) : r \in \mathscr{M}\} \subseteq \mathscr{C}_w$. Let $r \in \mathscr{M}$ and $(j_k)_k$ a corresponding (t,r)-subsequence. By Lemma 3.3 we obtain $u_{\varepsilon_{j_k}}(\cdot+t) \xrightharpoonup{2} u(\cdot+t,\cdot+r)$ and conclude $u(\cdot+t,\cdot+r) \in \mathscr{C}_w$.

In order to show the opposite inclusion, let $v \in \mathscr{C}_w$ and choose a subsequence $(j_k)_k$ such that $u_{\varepsilon_{j_k}}(\cdot+t) \xrightharpoonup{2} v$. By the compactness of $\overline{Y}$ we can assume without loss of generality that

$$\mathscr{R}\left(\tfrac{t}{\varepsilon_{j_k}}\right) \to r.$$

Hence, $(j_k)_k$ is a (t,r)-subsequence with $r \in \mathscr{M}$ and Lemma 3.3 implies that $v = u(\cdot+t,\cdot+r)$. □

The next result states that the set of all weak (respectively strong) two-scale cluster points $\mathscr{C}_w$ (respectively $\mathscr{C}_s$) of a translated weakly (respectively strongly) two-scale convergent sequence and the set of all microtranslations $\mathscr{M}$ are *compact*:

Proposition 3.4. *The set $\mathscr{M}$ is a compact subset of $\mathbb{R}^N$. In the situation of the first (respectively second) statement of Theorem 3.2, the set $\mathscr{C}_w$ (respectively $\mathscr{C}_s$) is compact w.r.t. strong convergence in $L^p((\Omega-t)\times Y)$.*

Proof. The proof is split into two parts, the first dealing with the compactness of $\mathscr{M}$, the second stating the compactness of $\mathscr{C}_w$. As the compactness of $\mathscr{C}_s$ can be proved similarly to $\mathscr{C}_w$, we do not go into the details of its proof.

Step 1. First, let us remark that

$$\mathscr{M} = \left\{ r \in \mathbb{R}^N : \liminf_{j\to\infty} \left| \mathscr{R}\left(\tfrac{t}{\varepsilon_j}\right) - r \right| = 0 \right\}.$$

Obviously $\mathscr{M}$ is bounded. We will see that it is also closed: Consider an arbitrary sequence $(r_k)_k$ in $\mathscr{M}$ that converges to some $r \in \mathbb{R}^N$. We define the quantity

$$c_{k,j} := \left| \mathscr{R}\left(\tfrac{t}{\varepsilon_j}\right) - r_k \right| + |r_k - r|.$$

Since $r_k \in \mathscr{M}$ there obviously holds

$$\limsup_{k\to\infty} \left(\liminf_{j\to\infty} c_{k,j} \right) = \limsup_{k\to\infty} |r_k - r| = 0. \tag{5}$$

Utilizing [5, Lemma 1.17], we can choose a subsequence $(k(j))_j$ such that

$$\liminf_{j\to\infty} c_{k(j),j} \le \limsup_{k\to\infty} \left(\liminf_{j\to\infty} c_{k,j} \right)$$

and deduce by (5) that the left hand side is equal to 0. A simple triangle inequality then leads to

$$\liminf_{j\to\infty}\left|\mathscr{R}\left(\tfrac{t}{\varepsilon_j}\right)-r\right| \le \liminf_{j\to\infty}\left(\left|\mathscr{R}\left(\tfrac{t}{\varepsilon_j}\right)-r_{k(j)}\right|+\left|r_{k(j)}-r\right|\right)$$
$$=\liminf_{j\to\infty} c_{k(j),j}=0;$$

hence, $r\in\mathscr{M}$.

Step 2. Like in the situation of Theorem 3.2, let u denote the weak two-scale limit of the sequence $(u_{\varepsilon_j})_j$. We introduce the mapping

$$\Phi:\mathscr{M}\to L^p((\Omega-t)\times Y),\quad r\mapsto u(\cdot+t,\cdot+r).$$

By Theorem 3.2 one can easily see that $\Phi(\mathscr{M})$ is equal to $\mathscr{C}_w$. Since $\mathscr{M}$ is compact by step 1, it is sufficient to prove that Φ is continuous. But this is true, since for every sequence $(r_j)_j$ in $\overline{Y}$ converging to some r we observe $u(\cdot+t,\cdot+r_j)\to u(\cdot+t,\cdot+r)$ in $L^p((\Omega-t)\times Y)$. □

We shall conclude this section by stating several explanatory comments.

Remark 3.1. Since $(\mathscr{R}(\frac{t}{\varepsilon_j}))_j$ is a sequence in the compact set $\overline{Y}$, we in particular conclude

$$\mathscr{M}\neq\emptyset \qquad\text{and thus}\qquad \mathscr{C}_w\neq\emptyset$$

for every weakly two-scale convergent sequence $(u_{\varepsilon_j})_j$ in $L^p(\Omega)$. The same observation is true for a strongly two-scale convergent sequence $(u_{\varepsilon_j})_j$ and $\mathscr{C}_s$.

Indeed, the sets $\mathscr{C}_w$ and $\mathscr{C}_s$ of all two-scale cluster points can be very rich, as it is revealed by the following example: We consider the case $N=1$. Let $t\neq 0$ be an arbitrary translation vector in $\mathbb{R}$ and define $\varepsilon_j:=\frac{|t|}{j+\operatorname{sgn}(t)\cdot q(j)}$, where $q:\mathbb{N}\to\mathbb{Q}\cap[0,1]$ is a surjective map and sgn returns the sign of its argument. In this case we obtain $\mathscr{M}=[0,1]=\overline{Y}$.

Remark 3.2. The result of this section can be easily extended to converging translations: Let $\Omega=\mathbb{R}^N$, $t_j\to t$ in $\mathbb{R}^N$ and consider a subsequence $(j_k)_k$ satisfying

$$\lim_{k\to\infty}\left|\mathscr{R}\left(\tfrac{t_{j_k}}{\varepsilon_{j_k}}\right)-r\right|=0$$

for some $r\in\overline{Y}$. Then for any weakly two-scale convergent sequence $(u_{\varepsilon_{j_k}})_k$ in $L^p(\mathbb{R}^N)$ with limit u we have

$$u_{\varepsilon_{j_k}}(\cdot+t_{j_k})\xrightarrow{2} u(\cdot+t,\cdot+r)\text{ in }L^p(\mathbb{R}^N\times Y).$$

This can be seen by modifying the definition of $\varphi_{\varepsilon_{j_k}}$ in equation (3) in the proof of Lemma 3.3:

$$\int_{\mathbb{R}^N}u_{\varepsilon_{j_k}}(x+t_{j_k})\varphi\left(x,\tfrac{x}{\varepsilon_{j_k}}\right)\mathrm{d}x=\int_{\mathbb{R}^N}u_{\varepsilon_{j_k}}(x)\varphi\left(x-t_{j_k},\tfrac{x}{\varepsilon_{j_k}}-\tfrac{t_{j_k}}{\varepsilon_{j_k}}\right)\mathrm{d}x$$
$$=\int_{\mathbb{R}^N}u_{\varepsilon_{j_k}}(x)\varphi\left(x-t,\tfrac{x}{\varepsilon_{j_k}}-r\right)\mathrm{d}x+\int_{\mathbb{R}^N}u_{\varepsilon_{j_k}}(x)\varphi_{\varepsilon_{j_k}}(x)\,\mathrm{d}x$$

with

$$\varphi_{\varepsilon_{j_k}}(x):=\varphi\left(x-t_{j_k},\tfrac{x}{\varepsilon_{j_k}}-\tfrac{t_{j_k}}{\varepsilon_{j_k}}\right)-\varphi\left(x-t,\tfrac{x}{\varepsilon_{j_k}}-r\right).$$

Since φ is smooth, $\varphi_{\varepsilon_{j_k}}$ vanishes uniformly.

Remark 3.3. In [15] Meunier and van Schaftingen introduce a modification of the periodic unfolding operator from [9] that features additive perturbations on the microscale ε, which they call microscopic translations. As concerns the objective of their contribution, Meunier and van Schaftingen prove that vanishing microscopic translations do not alter the two-scale limit behavior of a sequence of functions. Upon assuming that these microscopic translations originate from “very small macroscopic translations”, their insight corresponds to the previous remark for the case of macroscopic translations $(t_\varepsilon)_\varepsilon$ satisfying $t_\varepsilon/\varepsilon \to 0$ and consequently $\mathscr{M} = \{0\}$.

However, we would like to remark that in contrast to our analysis the subtle connection between macroscopic translations and the corresponding microscopic translations is not discussed in [15].

Remark 3.4. Translations occurring in the microscopic variable of homogenized quantities sometimes play a crucial role in the understanding of the underlying homogenization processes, see e.g. our results in Section 4 below. Let us remark that microtranslations may also originate from phenomena other than macroscopic translations of the coordinate frame. For instance, this is the case for the Bloch wave homogenization method due to Allaire and Conca [3]. The purpose of this method is to characterize the asymptotic behavior of the spectrum associated to linear second order elliptic PDEs with periodically oscillating coefficients as the size of the period tends to zero. Therein, microtranslations emerge from a Bloch wave decomposition (cf. [2]) and play a key role in the resulting cell problems (see [3, 8]).

4 Application to Homogenization

In this section we present an application of our previous insights to a novel homogenization problem. To this end, we will provide a convergence study for an oscillating convex integral functional, the oscillating arguments of which are translated by a nonzero macroscopic quantity. Whereas in the absence of translation one can prove Γ-convergence of the integral functionals, in its presence Γ-convergence no longer holds true in general. Nevertheless, the results of the Section 3 allow us to explicitly identify the lower and the upper Γ-limit.

For the notation used in the sequel we also refer to the preceding section. Let t be an arbitrary but fixed translation in $\mathbb{R}^N$ and Ω be an open and bounded subset of $\mathbb{R}^N$. We consider the functional

$$\mathscr{F}_\varepsilon : W_0^{1,2}(\Omega;\mathbb{R}^N) \to \mathbb{R}, \quad u \mapsto \int_\Omega W(\tfrac{x}{\varepsilon}, \nabla u(x))\,\mathrm{d}x + \int_{\Omega - t} W(\tfrac{x}{\varepsilon}, \nabla u(x+t))\,\mathrm{d}x$$

for a positive small parameter ε and seek to describe its behavior as ε vanishes. We assume that the integrand

$$W : Y \times \mathbb{R}^{N\times N} \to \mathbb{R}, \quad (y,F) \mapsto W(y,F)$$

satisfies the following properties:

$$\text{for all } F \in \mathbb{R}^{N\times N} \text{ the map } y \mapsto W(y,F) \text{ is measurable and } Y\text{-periodic}, \tag{W1}$$

$$\text{for a.e. } y \in Y \text{ the map } F \mapsto W(y,F) \text{ is convex and continuous}, \tag{W2}$$

$$\begin{cases} \text{there exist positive constants } c, C \text{ such that} \\ \qquad c(|F|^2 - 1) \le W(y,F) \le C(1+|F|^2) \\ \text{for a.e. } y \in Y \text{ and all } F \in \mathbb{R}^{N\times N}. \end{cases} \tag{W3}$$

Integral functionals with integrands satisfying the assumptions above are already well-studied. In the context of homogenization we refer e.g. to [14, 16, 1, 10, 19].

Prior to the statement of the homogenization result, we introduce the following objects, which depend on a microtranslation r and naturally arise in the subsequent analysis: For $r \in \overline{Y}$, $y \in Y$ and $F \in \mathbb{R}^{N\times N}$ we set

$$W_r(y,F) := W(y,F) + W(y-r,F)$$
$$W_{\mathrm{Hom},r}(F) := \inf\Big\{ \int_Y W_r(y, F + \nabla_y \varphi(y))\,\mathrm{d}y \,:\, \varphi \in W^{1,2}_{\mathrm{per}}(Y;\mathbb{R}^N),\, \int_Y \varphi(y)\,\mathrm{d}y = 0 \Big\}$$

and refer to W_r as the microtranslated energy density and to $W_{\mathrm{Hom},r}$ as its homogenization. Moreover, we define the corresponding two-scale energy functional $\overline{\mathscr{F}}_r$, as well as the homogenized microtranslated energy $\mathscr{F}_{\mathrm{Hom},r}$ according to

$$\overline{\mathscr{F}}_r(u,u_1) := \int_\Omega \int_Y W_r(y, \nabla u(x) + \nabla_y u_1(x,y))\,\mathrm{d}y\,\mathrm{d}x,$$
$$\mathscr{F}_{\mathrm{Hom},r}(u) := \int_\Omega W_{\mathrm{Hom},r}(\nabla u(x))\,\mathrm{d}x$$

for every $u \in W^{1,2}_0(\Omega;\mathbb{R}^N)$ and every $u_1 \in L^2(\Omega; W^{1,2}_{\mathrm{per}}(Y;\mathbb{R}^N))$.

Theorem 4.1. *Let $(\varepsilon_j)_j$ be an arbitrary vanishing sequence of positive real numbers. Then*

$$\Gamma\text{-}\liminf_{j\to\infty} \mathscr{F}_{\varepsilon_j} = \inf_{r\in\mathscr{M}} \mathscr{F}_{\mathrm{Hom},r}$$
$$\Gamma\text{-}\limsup_{j\to\infty} \mathscr{F}_{\varepsilon_j} = \sup_{r\in\mathscr{M}} \mathscr{F}_{\mathrm{Hom},r}$$

w.r.t. the weak topology in $W^{1,2}_0(\Omega;\mathbb{R}^N)$. Like in Definition 3.1, $\mathscr{M}$ denotes the set of all microtranslations emerging from the translation t and $(\varepsilon_j)_j$.

Theorem 4.1 reveals that in the presence of a nontrivial translation $t \neq 0$ *one can no longer expect* Γ*-convergence* of the sequence $(\mathscr{F}_{\varepsilon_j})_j$. This contrasts the classical homogenization problem for convex integral functionals, where there indeed holds Γ-convergence [1]. We remark that the classical homogenization problem corresponds in our exposition to the case of a trivial translation $t = 0$, in which the previous theorem recovers in fact Γ-convergence.

For the sake of a brief notation we set for every $u \in W^{1,2}_0(\Omega;\mathbb{R}^N)$

$$\mathscr{F}^-_{\mathrm{Hom}}(u) := \inf_{r\in\mathscr{M}} \mathscr{F}_{\mathrm{Hom},r}(u) \quad \text{and} \quad \mathscr{F}^+_{\mathrm{Hom}}(u) := \sup_{r\in\mathscr{M}} \mathscr{F}_{\mathrm{Hom},r}(u).$$

Regarding the proof of Theorem 4.1 we will rely on two main insights.

First, we observe that $(\mathscr{F}_{\varepsilon_j})_j$ is Γ-convergent along (t,r)-subsequences. To be more specific, for $r \in \mathscr{M}$ we will show that $\Gamma\text{-}\lim_k \mathscr{F}_{\varepsilon_{j_k}} = \mathscr{F}_{\mathrm{Hom},r}$ along every (t,r)-subsequence $(j_k)_k$. This will be done by combining Theorem 3.2 and general (lower semi-) continuity properties of convex integral functionals with oscillating integrands stated in Proposition 4.2 below. For a systematic investigation of the latter with methods related to two-scale convergence, we refer to [19].

Secondly, we will provide an abstract result, which allows us to identify the lower and upper Γ-limit of a sequence by falling back to Γ-convergent subsequences. Namely, if $(f_j)_j$ is a sequence of functions on a topological space X, such that the sequential characterization of Γ-convergence is valid for $(f_j)_j$ and the set $\mathscr{L}(x) := \{ f(x) : f \text{ is } \Gamma\text{-limit of a subsequence of } (f_j)_j \}$ attains its extrema for all $x \in X$, then the lower and upper Γ-limit of $(f_j)_j$ are given pointwise as $(\Gamma\text{-}\liminf_j f_j)(x) = \min \mathscr{L}(x)$ and $\big(\Gamma\text{-}\limsup_j f_j\big)(x) = \max \mathscr{L}(x)$, respectively. We will see that this observation together with the Γ-convergence of $(\mathscr{F}_{\varepsilon_j})_j$ along (t,r)-subsequences allows us to prove Theorem 4.1.

The following result and similar statements can be found in [19] and [1, 10, 9]. Its proof is omitted here.

Proposition 4.2. *Let W satisfy the conditions ($W1$),…,($W3$) and let U be an open and bounded subset of $\mathbb{R}^N$. For $\varepsilon > 0$ define the functionals*

$$\mathscr{J}_\varepsilon : L^2(U;\mathbb{R}^{N\times N}) \to \mathbb{R}, \quad F \mapsto \int_U W(\tfrac{x}{\varepsilon}, F(x))\,\mathrm{d}x,$$
$$\overline{\mathscr{J}} : L^2(U\times Y;\mathbb{R}^{N\times N}) \to \mathbb{R}, \quad F \mapsto \int_U \int_Y W(y, F(x,y))\,\mathrm{d}y\,\mathrm{d}x.$$

Then:

1. *The functionals $\mathscr{J}_\varepsilon$ and $\overline{\mathscr{J}}$ are continuous w.r.t. strong convergence and lower semi-continuous w.r.t. weak convergence. Moreover, the functionals are convex and finite.*
2. *Let $F \in L^2(U\times Y;\mathbb{R}^{N\times N})$ and $(F_{\varepsilon_j})_j$ be a sequence in $L^2(U;\mathbb{R}^{N\times N})$. If $F_{\varepsilon_j} \xrightharpoonup{2} F$ in $L^2(U\times Y;\mathbb{R}^{N\times N})$, then*
$$\overline{\mathscr{J}}(F) \le \liminf_{j\to\infty} \mathscr{J}_{\varepsilon_j}(F_{\varepsilon_j}).$$
Moreover, if $F_{\varepsilon_j} \xrightarrow{2} F$ in $L^2(U\times Y;\mathbb{R}^{N\times N})$, it is
$$\overline{\mathscr{J}}(F) = \lim_{j\to\infty} \mathscr{J}_{\varepsilon_j}(F_{\varepsilon_j}).$$

An immediate consequence of the foregoing proposition and Theorem 3.2 is the observation below.

Corollary 4.3. *Let $F \in L^2(\Omega\times Y;\mathbb{R}^{N\times N})$, be $(F_{\varepsilon_j})_j$ a sequence in $L^2(\Omega;\mathbb{R}^{N\times N})$, $r \in \mathscr{M}$ and $(j_k)_k$ a corresponding (t,r)-subsequence. Then*

1. *if $F_{\varepsilon_j} \xrightharpoonup{2} F$ in $L^2(\Omega\times Y;\mathbb{R}^{N\times N})$, then*
$$\int_\Omega \int_Y W_r(y, F(x,y))\,\mathrm{d}y\,\mathrm{d}x \le \liminf_{k\to\infty} \Big(\int_\Omega W(\tfrac{x}{\varepsilon_{j_k}}, F_{\varepsilon_{j_k}}(x))\,\mathrm{d}x + \int_{\Omega-t} W(\tfrac{x}{\varepsilon_{j_k}}, F_{\varepsilon_{j_k}}(x+t))\,\mathrm{d}x \Big),$$
2. *if $F_{\varepsilon_j} \xrightarrow{2} F$ in $L^2(\Omega\times Y;\mathbb{R}^{N\times N})$, then*
$$\int_\Omega \int_Y W_r(y, F(x,y))\,\mathrm{d}y\,\mathrm{d}x = \lim_{k\to\infty} \Big(\int_\Omega W(\tfrac{x}{\varepsilon_{j_k}}, F_{\varepsilon_{j_k}}(x))\,\mathrm{d}x + \int_{\Omega-t} W(\tfrac{x}{\varepsilon_{j_k}}, F_{\varepsilon_{j_k}}(x+t))\,\mathrm{d}x \Big).$$

With this result at hand, we can finally prove Γ-convergence of $(\mathscr{F}_{\varepsilon_j})_j$ along (t,r)-subsequences. Note that by the growth assumption ($W3$) and Poincaré's inequality the sequence $(\mathscr{F}_{\varepsilon_j})_j$ is equicoercive. Hence, by Theorem 2.5 Γ-convergence of $(\mathscr{F}_{\varepsilon_j})_j$ and its subsequences can equivalently be verified in terms of sequential Γ-convergence.

Proposition 4.4. *Let $r \in \mathscr{M}$ and $(j_k)_k$ be a corresponding (t,r)-subsequence. Then the following statements are valid.*

1. *For every $u \in W_0^{1,2}(\Omega;\mathbb{R}^N)$ and every sequence $(u_{\varepsilon_{j_k}})_k$ weakly converging to u in $W_0^{1,2}(\Omega;\mathbb{R}^N)$ we have*
$$\mathscr{F}_{\mathrm{Hom},r}(u) \le \liminf_{k\to\infty} \mathscr{F}_{\varepsilon_{j_k}}(u_{\varepsilon_{j_k}}).$$

2. *For every $u \in W_0^{1,2}(\Omega;\mathbb{R}^N)$ there is a sequence $(u_{\varepsilon_{j_k}})_k$ weakly converging to u in $W_0^{1,2}(\Omega;\mathbb{R}^N)$ such that*
$$\mathscr{F}_{\mathrm{Hom},r}(u) = \lim_{k\to\infty} \mathscr{F}_{\varepsilon_{j_k}}(u_{\varepsilon_{j_k}}).$$

Proof. The statements of the proposition will be proved separately in the following two steps.

Step 1. Let $(u_{\varepsilon_{j_k}})_k$ be a sequence that converges to u weakly in $W_0^{1,2}(\Omega;\mathbb{R}^N)$ and let $(\eta_\ell)_\ell$ be an arbitrary subsequence of $(\varepsilon_{j_k})_k$. In view of Proposition 2.3 we find another subsequence $(\eta_{\ell_m})_m$ and a function $u_1 \in L^2(\Omega;W^{1,2}_{\mathrm{per}}(Y;\mathbb{R}^N))$ with $\int_Y u_1(x,y)\,\mathrm{d}y = 0$ for a.e. $x \in \Omega$, such that
$$\nabla u_{\eta_{\ell_m}} \xrightharpoonup{2} \nabla u + \nabla_y u_1 \qquad \text{in } L^2(\Omega{\times}Y;\mathbb{R}^{N\times N}).$$
Upon recalling the definitions of $\mathscr{F}_{\mathrm{Hom},r}$ and $W_{\mathrm{Hom},r}$, an application of corollary 4.3 leads to
$$\mathscr{F}_{\mathrm{Hom},r}(u) \le \liminf_{m\to\infty} \mathscr{F}_{\eta_{\ell_m}}(u_{\eta_{\ell_m}}). \tag{6}$$
We see that every subsequence of $(\varepsilon_{j_k})_k$ has a further subsequence satisfying (6). Consequently, inequality (6) remains valid for the whole sequence $(\varepsilon_{j_k})_k$ and we infer the validity of the first assertion.

Step 2. Consider an arbitrary $u \in W_0^{1,2}(\Omega;\mathbb{R}^N)$. We start with the observation, that the functional $\mathscr{F}_{\mathrm{Hom},r}$ can be characterized by means of $\overline{\mathscr{F}}_r$ in the following way (see Remark 4.1 for details).
$$\mathscr{F}_{\mathrm{Hom},r}(u) = \overline{\mathscr{F}}_r(u,u_1) \quad \text{for some } u_1 \in L^2(\Omega;W^{1,2}_{\mathrm{per}}(Y;\mathbb{R}^N)) \quad \text{with} \int_Y u_1(x,y)\,\mathrm{d}y = 0 \text{ for a.e. } x\in\Omega. \tag{7}$$
By density we now find a sequence $(v_\ell)_\ell$ in $C_c^\infty(\Omega;C^\infty_{\mathrm{per}}(Y;\mathbb{R}^N))$ such that
$$v_\ell \to u_1 \quad \text{in } L^2(\Omega{\times}Y;\mathbb{R}^N),\ \nabla_y v_\ell \to \nabla_y u_1 \quad \text{in } L^2(\Omega{\times}Y;\mathbb{R}^{N\times N}) \tag{8}$$
subject to $\int_Y v_\ell(\cdot,y)\,\mathrm{d}y = 0$. We define the doubly indexed sequence of functions $(u_{\ell,k})_{\ell,k}$ in $W_0^{1,2}(\Omega;\mathbb{R}^N)$ by
$$u_{\ell,k}(x) := u(x) + \varepsilon_{j_k} v_\ell(x, \tfrac{x}{\varepsilon_{j_k}}).$$

Invoking the technique of two-scale decomposition (cf. e.g. [18]) we could easily find a diagonal sequence $u_{\varepsilon_{j_k}} := u_{\ell(k),k}$ such that $(u_{\varepsilon_{j_k}})_k \rightharpoonup u$ weakly in $W_0^{1,2}(\Omega;\mathbb{R}^N)$ and $(\nabla u_{\varepsilon_{j_k}})_k \xrightarrow{2} \nabla u + \nabla_y u_1$ in $L^2(\Omega{\times}Y;\mathbb{R}^{N\times N})$. In view of Corollary 4.3 this sequence would clearly recover the limit energy and prove the assertion. However, for the reader's convenience we will present a self-contained construction for the recovery sequence.

To this end we start with the observation, that for every $\ell \in \mathbb{N}$ we have
$$u_{\ell,k} \xrightharpoonup[k\to\infty]{} u \qquad \text{in } W_0^{1,2}(\Omega;\mathbb{R}^N), \tag{9}$$
$$\nabla u_{\ell,k} \xrightarrow[k\to\infty]{2} \nabla u + \nabla_y v_\ell \quad \text{in } L^2(\Omega{\times}Y;\mathbb{R}^{N\times N}), \tag{10}$$
where the latter follows from the smoothness of v_ℓ (see e.g. [18]). Hence, we can apply Corollary 4.3 and infer
$$\lim_{k\to\infty} \mathscr{F}_{\varepsilon_{j_k}}(u_{\ell,k}) = \overline{\mathscr{F}}_r(u,v_\ell).$$

Now, the strong convergence (8) and the assumptions ($W2$), ($W3$) imply

$$\lim_{\ell\to\infty}\left(\lim_{k\to\infty}\mathscr{F}_{\varepsilon_{j_k}}(u_{\ell,k})\right)=\overline{\mathscr{F}}_r(u,u_1)\underset{(7)}{=}\mathscr{F}_{\mathrm{Hom},r}(u). \tag{11}$$

The previous reasoning suggests that we may obtain the recovery sequence $(u_{\varepsilon_{j_k}})_{j_k}$ by carefully choosing a diagonal sequence of $(u_{\ell,k})_{\ell,k}$. With this intention in mind, we define the quantity

$$c_{\ell,k}:=\left|\mathscr{F}_{\varepsilon_{j_k}}(u_{\ell,k})-\mathscr{F}_{\mathrm{Hom},r}(u)\right|+\|u_{\ell,k}-u\|_{L^2(\Omega;\mathbb{R}^N)}.$$

By means of Rellich's compactness theorem, (9) and (11) imply

$$\lim_{\ell\to\infty}\left(\lim_{k\to\infty}c_{\ell,k}\right)=0.$$

Referring to [5, Corollary 1.18], we find a subsequence $(\ell(k))_k$ with $\lim_k c_{\ell(k),k}=0$ and consequently the sequence $u_{\varepsilon_{j_k}}:=u_{\ell(k),k}$ recovers the limit energy, i.e.

$$\lim_{k\to\infty}\mathscr{F}_{\varepsilon_{j_k}}(u_{\varepsilon_{j_k}})=\mathscr{F}_{\mathrm{Hom},r}(u),$$

and converges to u w.r.t. strong convergence in $L^2(\Omega;\mathbb{R}^N)$. It remains to prove

$$u_{\varepsilon_{j_k}}\xrightharpoonup[k\to\infty]{}u\quad\text{in } W_0^{1,2}(\Omega;\mathbb{R}^N). \tag{12}$$

First, we observe that the sequence $(u_{\varepsilon_{j_k}})_k$ must be norm bounded in $W_0^{1,2}(\Omega;\mathbb{R}^N)$ due to the growth condition ($W3$) and the boundedness of $(\mathscr{F}_{\varepsilon_{j_k}}(u_{\varepsilon_{j_k}}))_k$. Hence, we can extract from any subsequence of $(u_{\varepsilon_{j_k}})_k$ a further subsequence, which is weakly convergent in $W_0^{1,2}(\Omega;\mathbb{R}^N)$. But any weak limit of these subsequences must coincide with u, because of the strong convergence of $(u_{\varepsilon_{j_k}})_k$ to u in $L^2(\Omega;\mathbb{R}^N)$; thus, (12) is valid and $(u_{\varepsilon_{j_k}})_k$ is a recovery sequence. □

Before proceeding further, we analyze the dependence of the homogenized microtranslated energy $\mathscr{F}_{\mathrm{Hom},r}$ on the microtranslations r.

Lemma 4.5. *For all $F\in\mathbb{R}^{N\times N}$ and $u\in W_0^{1,2}(\Omega;\mathbb{R}^N)$ the maps*

$$r\mapsto W_{\mathrm{Hom},r}(F)\quad\textit{and}\quad r\mapsto\mathscr{F}_{\mathrm{Hom},r}(u)$$

are continuous.

Proof. We remark that due to the growth condition ($W3$) it is sufficient to prove the continuity of $r\mapsto W_{\mathrm{Hom},r}(F)$. Let $F\in\mathbb{R}^{N\times N}$ be arbitrary and define for all $r\in\overline{Y}$ the functional

$$\mathscr{E}_r:\left\{\varphi\in W^{1,2}_{\mathrm{per}}(Y;\mathbb{R}^N):\int_Y\varphi(y)\,\mathrm{d}y=0\right\}\to\mathbb{R},\quad\varphi\mapsto\int_Y W_r(y,F+\nabla_y\varphi(y))\,\mathrm{d}y.$$

By the assumptions on W and Poincaré's inequality, we see that $\varphi\mapsto\mathscr{E}_r(\varphi)$ is convex, coercive and continuous w.r.t. to strong convergence in $W^{1,2}_{\mathrm{per}}(Y;\mathbb{R}^N)$. Therefore, for all $r\in\overline{Y}$ the functional $\mathscr{E}_r$ admits a minimizer $\varphi_r\in W^{1,2}_{\mathrm{per}}(Y;\mathbb{R}^N)$ with $\int_Y\varphi_r(y)\,\mathrm{d}y=0$, thus

$$\mathscr{E}_r(\varphi_r)=W_{\mathrm{Hom},r}(F).$$

We recall the growth condition ($W3$) and infer that

$$2c\left(\left\|F+\nabla_y\varphi_r\right\|^2_{L^2(Y;\mathbb{R}^{N\times N})}-1\right)\le\mathscr{E}_r(\varphi_r)\le\int_Y 2W(y,F)\,\mathrm{d}y\le 2C(1+|F|^2).$$

As a consequence, Poincaré's inequality implies the boundedness of the sequence $(\varphi_r)_r$ in $W^{1,2}_{\text{per}}(Y;\mathbb{R}^N)$. Now, consider an arbitrary sequence $(r_\ell)_\ell$ in $\overline{Y}$ with $r_\ell\to r$. Due to the previous considerations, we can extract a subsequence $(r_{\ell_m})_m$ such that

$$\liminf_{\ell\to\infty}W_{\text{Hom},r_\ell}(F)=\lim_{m\to\infty}W_{\text{Hom},r_{\ell_m}}(F)$$

and in addition $\varphi_{r_{\ell_m}}\rightharpoonup\varphi_0$ weakly in $W^{1,2}_{\text{per}}(Y;\mathbb{R}^N)$ for a function $\varphi_0\in W^{1,2}_{\text{per}}(Y;\mathbb{R}^N)$ with $\int_Y\varphi_0(y)\,\mathrm{d}y=0$. As it is easily seen, we also have $\varphi_{r_{\ell_m}}(\cdot+r_{\ell_m})\rightharpoonup\varphi_0(\cdot+r)$ weakly in $W^{1,2}_{\text{per}}(Y;\mathbb{R}^N)$. Hence, we can exploit the weak lower semicontinuity of the functional $\varphi\mapsto\int_Y W(y,F+\nabla_y\varphi(y))\,\mathrm{d}y$ and obtain

$$\mathscr{E}_r(\varphi_0)\le\liminf_{m\to\infty}\mathscr{E}_{r_{\ell_m}}(\varphi_{r_{\ell_m}})=\liminf_{\ell\to\infty}W_{\text{Hom},r_\ell}(F).\tag{13}$$

On the other hand there holds

$$\limsup_{\ell\to\infty}W_{\text{Hom},r_\ell}(F)\le\limsup_{\ell\to\infty}\mathscr{E}_{r_\ell}(\varphi_r)=\mathscr{E}_r(\varphi_r).\tag{14}$$

Herein, the last identity can be inferred from the observation, that the functional $\varphi\mapsto\int_Y W(y,F+\nabla_y\varphi(y))\,\mathrm{d}y$ is continuous w.r.t. strong convergence and $\varphi_r(\cdot+r_\ell)\to\varphi_r(\cdot+r)$ strongly in $W^{1,2}_{\text{per}}(Y;\mathbb{R}^N)$ as $\ell\to\infty$. In view of (13) and (14), we have just shown that

$$\mathscr{E}_r(\varphi_0)\le\liminf_{\ell\to\infty}W_{\text{Hom},r_\ell}(F)\le\limsup_{\ell\to\infty}W_{\text{Hom},r_\ell}(F)\le W_{\text{Hom},r}(F).\tag{15}$$

Upon recalling $W_{\text{Hom},r}(F)=\inf_\varphi\mathscr{E}_r(\varphi)\le\mathscr{E}_r(\varphi_0)$, we realize that the inequalities in (15) are indeed equalities. This completes the proof. □

So far we have shown Γ-convergence of $(\mathscr{F}_{\varepsilon_j})_j$ to $\mathscr{F}_{\text{Hom},r}$ along (t,r)-subsequences and the continuous dependence of $\mathscr{F}_{\text{Hom},r}$ on the microtranslation r. As we will prove now, this already allows us to reduce the proof of Theorem 4.1 to the situation considered in Proposition 4.6, which states how lower and upper Γ-limits of a sequence can be characterized by means of its subsequences' Γ-limits.

Proof of Theorem 4.1. Define the set

$$\mathscr{L}:=\left\{\mathscr{F}\,:\,\mathscr{F}=\Gamma\text{-}\lim_{k\to\infty}\mathscr{F}_{\varepsilon_{j_k}}\text{ for a subsequence }(j_k)_k\right\}.$$

In view of Proposition 4.4 and the fact, that every subsequence $(j_k)_k$ contains a (t,r)-subsequence for a certain $r\in\mathscr{M}$, we immediately conclude that

$$\mathscr{L}=\{\mathscr{F}_{\text{Hom},r}\,:\,r\in\mathscr{M}\}.$$

Note that for any $u\in W^{1,2}_0(\Omega;\mathbb{R}^N)$ the map $r\mapsto\mathscr{F}_{\text{Hom},r}(u)$ is continuous due to Lemma 4.5. Since $\mathscr{M}$ is a compact subset of $\mathbb{R}^N$ by Proposition 3.4, there exist r^+ and r^- in $\mathscr{M}$ such that

$$\mathscr{F}^+_{\text{Hom}}(u)=\mathscr{F}_{\text{Hom},r^+}(u)\quad\text{and}\quad\mathscr{F}^-_{\text{Hom}}(u)=\mathscr{F}_{\text{Hom},r^-}(u).$$

This enables us to conclude the proof by applying the abstract proposition below, which might be regarded a result of its own interest. □

Proposition 4.6. *Let $(X,\mathscr{T})$ be a topological vector space and $(f_j)_j$ a sequence of functions from X into $[-\infty,+\infty]$. We define the set*

$$\mathscr{L} := \left\{ f \,:\, f = \Gamma\text{-}\lim_{k\to\infty} f_{j_k} \text{ for a subsequence } (j_k)_k \right\}$$

and assume that

1. *for all $x \in X$ there exists a sequence $(x_j)_j$ converging to x in X such that*

$$\left(\Gamma\text{-}\liminf_{j\to\infty} f_j\right)(x) = \liminf_{j\to\infty} f_j(x_j),$$

2. *any subsequence of $(f_j)_j$ has a Γ-convergent subsequence,*
3. *for all $x \in X$ there exist $f^-, f^+ \in \mathscr{L}$ such that*

$$f^-(x) \le f(x) \le f^+(x) \quad \text{for every } f \in \mathscr{L}.$$

Then there holds

$$\left(\Gamma\text{-}\liminf_{j\to\infty} f_j\right)(x) = \min_{f\in\mathscr{L}} f(x) \quad \text{and} \quad \left(\Gamma\text{-}\limsup_{j\to\infty} f_j\right)(x) = \max_{f\in\mathscr{L}} f(x).$$

for every $x \in X$.

Proof. Let $x \in X$ and $f^+, f^- \in \mathscr{L}$ according to assumption 3. We remind the reader of the definition of the lower and upper Γ-limit given in Section 2, which reads as

$$\left(\Gamma\text{-}\liminf_{j\to\infty} f_j\right)(x) := \sup_{U\in\mathscr{U}(x)} \liminf_{j\to\infty} \inf_{y\in U} f_j(y),$$
$$\left(\Gamma\text{-}\limsup_{j\to\infty} f_j\right)(x) := \sup_{U\in\mathscr{U}(x)} \limsup_{j\to\infty} \inf_{y\in U} f_j(y),$$

where $\mathscr{U}(x)$ denotes the set of all open neighborhoods of x. One immediately realizes that

$$\liminf_{j\to\infty} \inf_{y\in\widetilde{U}} f_j(y) \le \liminf_{k\to\infty} \inf_{y\in\widetilde{U}} f_{j_k}(y) \quad \text{and} \quad \limsup_{j\to\infty} \inf_{y\in\widetilde{U}} f_j(y) \ge \limsup_{k\to\infty} \inf_{y\in\widetilde{U}} f_{j_k}(y)$$

for any subsequence $(j_k)_k$ and any $\widetilde{U} \in \mathscr{U}(x)$. From this we easily infer the estimates

$$\left(\Gamma\text{-}\liminf_{j\to\infty} f_j\right)(x) \le f^-(x) \quad \text{and} \quad \left(\Gamma\text{-}\limsup_{j\to\infty} f_j\right)(x) \ge f^+(x)$$

by first considering particular subsequences satisfying $\Gamma\text{-}\lim_k f_{j_k} = f^-$ (respectively $\Gamma\text{-}\lim_k f_{j_k} = f^+$) and, secondly, taking the supremum over all $\widetilde{U} \in \mathscr{U}(x)$ on both sides.

In order to show the characterization of the lower Γ-limit of $(f_j)_j$ stated above, it remains to show $(\Gamma\text{-}\liminf_j f_j)(x) \ge f^-(x)$. By assumption 1 and 2 we can find a sequence $(x_j)_j$ converging to x in X, a subsequence $(j_k)_k$ and $f \in \mathscr{L}$ such that

$$\lim_{k\to\infty} f_{j_k}(x_{j_k}) = \left(\Gamma\text{-}\liminf_{j\to\infty} f_j\right)(x) \quad \text{and} \quad \Gamma\text{-}\lim_{k\to\infty} f_{j_k} = f.$$

Since Γ-limits naturally satisfy the 'liminf-inequality' (also in the case where the sequential characterization is not valid), we infer $\liminf_{k\to\infty} f_{j_k}(x_{j_k}) \geq f(x)$ and by applying condition 3 we finally obtain

$$\left(\Gamma\text{-}\liminf_{j\to\infty} f_j\right)(x) = \liminf_{k\to\infty} f_{j_k}(x_{j_k}) \geq f(x) \geq f^-(x).$$

For the characterization of the upper Γ-limit of $(f_j)_j$, we need to prove that

$$(\Gamma\text{-}\limsup_{j\to\infty} f_j)(x) \leq f^+(x).$$

Let $\widetilde{U} \in \mathscr{U}(x)$. Again due to assumption 2 we can switch to a subsequence $(j_k)_k$ with

$$\limsup_{j\to\infty} \inf_{y\in\widetilde{U}} f_j(y) = \lim_{k\to\infty} \inf_{y\in\widetilde{U}} f_{j_k}(y) \quad \text{and} \quad \Gamma\text{-}\lim_{k\to\infty} f_{j_k} = f,$$

where $f \in \mathscr{L}$. In passing we realize

$$\limsup_{j\to\infty} \inf_{y\in\widetilde{U}} f_j(y) \leq \sup_{U\in\mathscr{U}(x)} \limsup_{k\to\infty} \inf_{y\in U} f_{j_k}(y) = f(x) \leq f^+(x),$$

where the last inequality follows by assumption 3. Since this estimate is true for all $\widetilde{U} \in \mathscr{U}(x)$, it remains valid if we pass to the supremum over all $\widetilde{U} \in \mathscr{U}(x)$. This completes the proof. □

We remark that in the situation of Theorem 2.5 the sequential characterization of the lower and upper Γ-limit is valid and assumption 1 is fulfilled (see [11, Proposition 8.16] for details). Thus, Proposition 4.6 can indeed be applied in the proof of Theorem 4.1. Eventually, it remains to provide the technical detail, which we used in the proof of Proposition 4.4.

Remark 4.1. Let $r \in \overline{Y}$ and $F \in L^2(\Omega;\mathbb{R}^{N\times N})$. Then there exists a function $v \in L^2(\Omega;W^{1,2}_{\text{per}}(Y;\mathbb{R}^N))$ with $\int_Y v(x,y)\,\mathrm{d}y = 0$ for a.e. $x\in\Omega$, such that

$$\int_\Omega W_{\text{Hom},r}(F(x))\,\mathrm{d}x = \int_\Omega \int_Y W_r(F(x)+\nabla_y v(x,y))\,\mathrm{d}y\,\mathrm{d}x. \tag{16}$$

Due to the convexity and continuity ($W2$) and the growth ($W3$) of W, for a.e. $x\in\Omega$ there exists $\varphi \in W^{1,2}_{\text{per}}(Y;\mathbb{R}^N)$ with $\int_Y \varphi(y)\,\mathrm{d}y = 0$ and

$$W_{\text{Hom},r}(F(x)) = \int_Y W_r(y,F(x)+\nabla_y\varphi(y))\,\mathrm{d}y,$$

thus the multifunction

$$\Lambda(x) := \Big\{ \varphi \in W^{1,2}_{\text{per}}(Y;\mathbb{R}^N) : \int_Y W_r(y,F(x)+\nabla_y\varphi(y))\,\mathrm{d}y = W_{\text{Hom},r}(F(x)) \quad \text{and} \quad \int_Y \varphi(y)\,\mathrm{d}y = 0 \Big\}$$

is well-defined. One can now show (for instance by applying Proposition 6.3 and Theorem 6.5 of [12]) that Λ possesses a measurable selection $x \mapsto v(x) \in \Lambda(x)$. Moreover, the growth condition ($W3$) and Poincaré's inequality imply that $v \in L^2(\Omega;W^{1,2}_{\text{per}}(Y;\mathbb{R}^N))$. The definition of Λ then immediately implies the claimed identity (16). Note that (16) in particular establishes for every $u \in W^{1,2}_0(\Omega;\mathbb{R}^N)$ the existence of a $u_1 \in L^2(\Omega;W^{1,2}_{\text{per}}(Y;\mathbb{R}^N))$ with $\int_Y u_1(x,y)\,\mathrm{d}y = 0$ for a.e. $x\in\Omega$, such that $\mathscr{F}_{\text{Hom},r}(u) = \overline{\mathscr{F}}_r(u,u_1)$.

We conclude the discussion of Theorem 4.1 by commenting on a slight generalization of the result and presenting an illustrative example, explicitely showing the loss of Γ-convergence in the presence of a nontrivial macroscopic translation.

Remark 4.2. The restriction in Theorem 4.1 to the space $W_0^{1,2}(\Omega;\mathbb{R}^N)$ implies that the sequence $(\mathscr{F}_{\varepsilon_j})_j$ is equicoercive w.r.t. to weak convergence in $W_0^{1,2}(\Omega;\mathbb{R}^N)$ and therefore the sequential characterization of Γ-convergence (which we exploited in the proof of Theorem 4.1) is valid for $(\mathscr{F}_{\varepsilon_j})_j$. Likely, results similar to Theorem 4.1 hold in more general situations. For instance let $\mathscr{G}$ be a function from $W^{1,2}(\Omega;\mathbb{R}^N)$ into $\mathbb{R}$ and consider the functionals

$$\mathscr{F}_\varepsilon : W^{1,2}(\Omega;\mathbb{R}^N) \to \mathbb{R}, \quad u \mapsto \int_\Omega W(\tfrac{x}{\varepsilon}, \nabla u(x))\,\mathrm{d}x + \int_{\Omega - t} W(\tfrac{x}{\varepsilon}, \nabla u(x+t))\,\mathrm{d}x,$$

where $t \in \mathbb{R}^N$ is an arbitrary translation and W fulfills $(W1),\ldots,(W3)$. If $\mathscr{G}$ satisfies one of the following conditions

(G1) $\mathscr{G}$ is finite, continuous w.r.t. to weak convergence in $W^{1,2}(\Omega;\mathbb{R}^N)$ and coercive w.r.t. to the strong convergence in $L^2(\Omega;\mathbb{R}^N)$,

(G2) there exists $g \in W^{1,2}(\Omega;\mathbb{R}^N)$ such that $\mathscr{G}(u) = 0$ if $u - g \in W_0^{1,2}(\Omega;\mathbb{R}^N)$ and $+\infty$ otherwise,

then

$$\Gamma\text{-}\liminf_{j\to\infty}(\mathscr{F}_{\varepsilon_j} + \mathscr{G}) = \mathscr{F}^-_{\mathrm{Hom}} + \mathscr{G}$$
$$\Gamma\text{-}\limsup_{j\to\infty}(\mathscr{F}_{\varepsilon_j} + \mathscr{G}) = \mathscr{F}^+_{\mathrm{Hom}} + \mathscr{G}$$

with respect to weak convergence in $W^{1,2}(\Omega;\mathbb{R}^N)$. Herein, $\mathscr{F}^+_{\mathrm{Hom}}$ and $\mathscr{F}^-_{\mathrm{Hom}}$ are defined as before.

Example 4.1. We consider the special situation of quadratic energy functionals. To this end, let $\mathbb{L}$ be a Y-periodic function from $\mathbb{R}^N$ into the space of fourth order tensors over $\mathbb{R}$ and suppose that

$$\langle \mathbb{L}(y)F, G\rangle = \langle \mathbb{L}(y)G, F\rangle,$$
$$c|F|^2 \leq \langle \mathbb{L}(y)F, F\rangle \leq C|F|^2$$

for all $F, G \in \mathbb{R}^{N\times N}$, a.e. $y \in Y$ and some positive constants c, C. The energy density shall now be given as

$$W(y,F) := \langle \mathbb{L}(y)F, F\rangle,$$

which obviously complies with the properties $(W1),\ldots,(W3)$. Observe that

$$W_{\mathrm{Hom},r}(F) = \int_Y W(y, F + \nabla_y \varphi_{r,F}(y))\,\mathrm{d}y,$$

where $\varphi_{r,F} \in W^{1,2}_{\mathrm{per}}(Y;\mathbb{R}^N)$ is the unique solution of the linear problem

$$\int_Y \langle \mathbb{L}(y)(F + \nabla_y \varphi(y), \nabla_y \psi(y)\rangle\,\mathrm{d}y = 0 \quad \text{for all } \psi \in W^{1,2}_{\mathrm{per}}(Y;\mathbb{R}^N)$$

subject to $\int_Y \varphi(y)\,\mathrm{d}y = 0$.

In the particular situation where $N=1$, $\mathscr{M}=\{0,\frac{1}{2}\}$ and

$$\mathbb{L}(y)=\alpha(y)=\begin{cases}\alpha_1 & \text{if } y\in[k,k+\frac{1}{2}) \text{ for } k\in\mathbb{Z}\\ \alpha_2 & \text{else}\end{cases}$$

we can explicitely compute $\mathscr{F}^+_{\mathrm{Hom}}$ and $\mathscr{F}^-_{\mathrm{Hom}}$. The above set of microtranslations $\mathscr{M}=\{0,\frac{1}{2}\}$ occurs e.g. in the case of $\varepsilon_j=\frac{2}{j}$ and $t=1$. A calculations shows

$$W_{\mathrm{Hom},r}(F)=\alpha_{\mathrm{Hom},r}|F|^2,\quad \text{where}\quad \alpha_{\mathrm{Hom},r}:=\left(\int_Y\frac{1}{\alpha(y)+\alpha(y-r)}\,\mathrm{d}y\right)^{-1}.$$

In the case of $r\in\{0,\frac{1}{2}\}$ we infer

$$\alpha_{\mathrm{Hom},0}=\frac{4\alpha_1\alpha_2}{\alpha_1+\alpha_2},\quad \alpha_{\mathrm{Hom},\frac{1}{2}}=\alpha_1+\alpha_2\quad\text{and}\quad \alpha_{\mathrm{Hom},\frac{1}{2}}-\alpha_{\mathrm{Hom},0}=\frac{(\alpha_1-\alpha_2)^2}{\alpha_1+\alpha_2}\geq 0.$$

With the help of the homogenization result Theorem 4.1 one obtains the explicit representations

$$\left(\Gamma\text{-}\liminf_{j\to\infty}\mathscr{F}_{\varepsilon_j}\right)(u)=\frac{4\alpha_1\alpha_2}{\alpha_1+\alpha_2}\int_\Omega|u'(x)|^2\,\mathrm{d}x,$$
$$\left(\Gamma\text{-}\limsup_{j\to\infty}\mathscr{F}_{\varepsilon_j}\right)(u)=(\alpha_1+\alpha_2)\int_\Omega|u'(x)|^2\,\mathrm{d}x,$$

for all $u\in W^{1,2}_0(\Omega)$. Finally, we conclude that $(\mathscr{F}_{\varepsilon_j})_j$ is Γ-convergent, if and only if $\alpha_1=\alpha_2$.

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