UNIQUENESS, RENORMALIZATION AND SMOOTH APPROXIMATIONS FOR LINEAR TRANSPORT EQUATIONS

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ABSTRACT. Transport equations arise in various areas of fluid mechanics, but the precise conditions on the vector field for them to be well-posed are still not fully understood. The renormalized theory of DiPerna and Lions for linear transport equations with unsmooth coefficient uses the tools of approximation of an arbitrary weak solution by smooth functions, and the renormalization property, that is to say to write down an equation on a nonlinear function of the solution. Under some $W^{1,1}$ regularity assumption on the coefficient, well-posedness holds. In this paper, we establish that these properties are indeed equivalent to the uniqueness of weak solutions to the Cauchy problem, without any regularity assumption on the coefficient. Coefficients with unbounded divergence but with bounded compression are also considered.

Key-words: linear transport equations with unsmooth coefficient – renormalized solutions – approximation by smooth functions – coefficients of bounded compression

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1. Introduction

In this paper we consider linear transport equations

$$\partial_t u + \mathrm{div}(bu) = 0 \qquad \text{in } (0,T)\times\mathbb{R}^d, \tag{1}$$

where $b(t,x) \in \mathbb{R}^d$ is the coefficient, and u is scalar. Such equations arise in many areas of fluid mechanics, and a precise analysis of them is a key issue for the understanding of the particle flows in applications. In the present work, we give sharp results characterizing the well-posedness of transport equations. The question of well-posedness for the associated Cauchy problem for (1) has a well-known answer when b is continuous and Lipschitz continuous with respect to x, because of the Cauchy-Lipschitz theorem and the relation between (1) and the ordinary differential equation $dX/ds = b(s, X(s))$. When b is not smooth, the well-posedness is much more delicate. A general theory has been developed in [13] in the case where $b \in L^1((0,T), W^{1,1}_{loc}(\mathbb{R}^d))$, $\mathrm{div} b \in L^\infty$, and under some growth conditions on b. After some intermediate results (see in particular [5], [9] and [10]), the theory has been generalized in [2] to the case of only BV regularity for b instead of $W^{1,1}$. However, some recent counterexamples (as in [11] and [12], both inspired by [1]), show that there is not much room to weaken the regularity assumptions. Nevertheless, some questions remain open, as the case of BD regularity for b (the symmetric part of $\nabla_x b$ is a measure, instead of the full matrix as in the BV case), see [8] and [4] for some partial results in this direction. For a detailed exposition and for a wider bibliography, the reader is refered to [3].

In this paper, we intend to give results of a different type, that do not give directly the answer to the well-posedness problem, but rather give equivalent conditions for it to hold, without regularity assumptions on b. For simplicity we shall always assume that $b \in L^\infty((0,T)\times\mathbb{R}^d)$, and consider an L^2 framework. The approach of [13] and [2] rely on an approximation by convolution of a given weak solution to (1) and on the renormalized property, that is to say that if u solves (1) and if $\mathrm{div} b = 0$ (to simplify) then $\beta(u)$ also solves (1) for any suitable nonlinearity β. Theorem 2.1 states that such properties are indeed equivalent to the well-posedness of both forward and backward Cauchy problems, up to the fact that the smooth approximate solution (in the sense of the norm of the graph of the transport operator) is not necessarily given by convolution. Then, one can think to make the difference between forward and backward uniqueness. Theorem 3.1 states that a characterization of backward uniqueness is the existence of a solution to the forward Cauchy problem that is approximable by smooth functions in the sense of the norm of the graph of the transport operator. Finally, we also consider the case of a coefficient b with unbounded divergence, but with *bounded compression*. We show that the previous results extend naturally to this case.

2. Forward-backward formulation

Theorem 2.1. *Let $b \in L^\infty((0,T)\times\mathbb{R}^d;\mathbb{R}^d)$ such that $\mathrm{div} b = 0$. Then the following statements are equivalent:*

(i) *b has the uniqueness property for weak solutions in $C([0,T];L^2(\mathbb{R}^d)-w)$ for both the forward and the backward Cauchy problems starting respectively from 0 and T, i.e. the only solutions in $C([0,T];L^2(\mathbb{R}^d)-w)$ to the problems*

$$\begin{cases} \partial_t u_F + \mathrm{div}(b u_F) = 0, \\ u_F(0,\cdot) = 0, \end{cases} \qquad \textit{and} \qquad \begin{cases} \partial_t u_B + \mathrm{div}(b u_B) = 0, \\ u_B(T,\cdot) = 0, \end{cases}$$

are $u_F \equiv 0$ and $u_B \equiv 0$;

(ii) *the Banach space*

$$\mathscr{F} := \left\{ \begin{array}{c} u \in C([0,T];L^2(\mathbb{R}^d)-w) \ \textit{s.t.} \\ \partial_t u + \mathrm{div}(bu) \in L^2((0,T)\times\mathbb{R}^d) \end{array} \right\} \tag{2}$$

with norm

$$\|u\|_{\mathscr{F}} := \|u\|_{B([0,T];L^2(\mathbb{R}^d))} + \|\partial_t u + \mathrm{div}(bu)\|_{L^2((0,T)\times\mathbb{R}^d)} \tag{3}$$

has the property that the space of functions in $C^\infty([0,T]\times\mathbb{R}^d)$ with compact support in x is dense in $\mathscr{F}$;

(iii) *every weak solution in $C([0,T];L^2(\mathbb{R}^d)-w)$ of $\partial_t u + \mathrm{div}(bu) = 0$ lies in $C([0,T];L^2(\mathbb{R}^d)-s)$ and is a renormalized solution, i.e. for every function $\beta \in$*

$C^1(\mathbb{R};\mathbb{R})$ such that $|\beta'(s)| \le C(1+|s|)$ for some constant $C \ge 0$, one has $\partial_t\big(\beta(u)\big) + \mathrm{div}\big(b\beta(u)\big) = 0$ in $(0,T)\times\mathbb{R}^d$.

In the statement of the theorem we used the notations $C([0,T];L^2(\mathbb{R}^d)-w)$ and $C([0,T];L^2(\mathbb{R}^d)-s)$ for the spaces of maps which are continuous from $[0,T]$ into $L^2(\mathbb{R}^d)$, endowed with the weak or the strong topology respectively. We recall the classical fact that, up to a redefinition in a negligible set of times, every solution to (1) belongs to $C([0,T];L^2(\mathbb{R}^d)-w)$ (see for example Remark 3 in [3]).

Proof of Theorem 2.1. **(i) ⇒ (ii). Step 1. Cauchy problem in $\mathscr{F}$.** It is easy to check that $\mathscr{F}$ is a Banach space, since L^2 and $B([0,T];L^2(\mathbb{R}^d))$ are Banach spaces (the latter denotes the space of bounded functions, with the supremum norm). We preliminarily show that for any $f \in L^2((0,T)\times\mathbb{R}^d)$ and $u^0 \in L^2(\mathbb{R}^d)$, the Cauchy problem

$$\begin{cases} \partial_t u + \mathrm{div}(bu) = f, \\ u(0,\cdot) = u^0 \end{cases} \tag{4}$$

has a unique solution in $\mathscr{F}$. We proceed by regularization. Consider a sequence of smooth vector fields $\{b_n\}_n$, with $b_n \to b$ a.e., b_n is uniformly bounded in L^∞, and $\mathrm{div} b_n = 0$ for every n. Let u_n be the solution to the problem

$$\begin{cases} \partial_t u_n + \mathrm{div}(b_n u_n) = f, \\ u_n(0,\cdot) = u^0 \,. \end{cases}$$

Then, by standard results on the smooth theory of transport equations (see for example [6]), we know that the solution u_n is unique in $C([0,T];L^2(\mathbb{R}^d))$ and is given by

$$u_n(t,x) = u^0(X_n(0,t,x)) + \int_0^t f(\tau, X_n(\tau,t,x))\,d\tau\,,$$

where $X_n(s,t,x)$ is the flow of b_n at time s, starting at the point x at time t, i.e. the solution to the ordinary differential equation

$$\begin{cases} \dfrac{dX_n}{ds}(s,t,x) = b_n(s, X_n(s,t,x)), \\ X_n(t,t,x) = x\,. \end{cases}$$

Recalling that $\mathrm{div} b_n = 0$, so that $X_n(s,t,\cdot)_{\#}\mathscr{L}^d = \mathscr{L}^d$ for every s and t (we denote by $\mathscr{L}^d$ the d-dimensional Lebesgue measure on $\mathbb{R}^d$), we can estimate the L^2 norm of $u_n(t,\cdot)$ as follows,

$$\begin{aligned} \|u_n(t,\cdot)\|_{L^2} &\le \|u^0(X_n(0,t,\cdot))\|_{L^2} + \int_0^t \|f(\tau, X_n(\tau,t,.))\|_{L^2}\,d\tau \\ &\le \|u^0\|_{L^2} + \int_0^t \|f(\tau,\cdot)\|_{L^2}\,d\tau \\ &\le \|u^0\|_{L^2} + \sqrt{T}\|f\|_{L^2}\,. \end{aligned}$$

This implies that the sequence $\{u_n\}_n$ is equi-bounded in $C([0,T];L^2(\mathbb{R}^d))$. From the equation on u_n, we have also that for any $\varphi \in C_c^\infty(\mathbb{R}^d)$, $d/dt(\int u_n \varphi dx)$ is bounded in $L^2(0,T)$. We deduce that for any $\varphi \in L^2(\mathbb{R}^d)$, $\int u_n \varphi dx$ is uniformly in n equicontinuous in $[0,T]$. Thus, up to the passage to a subsequence (which does not depend on t), we can suppose that $u_n(t,\cdot) \rightharpoonup u(t,\cdot)$ in $L^2(\mathbb{R}^d)-w$, with $u \in C([0,T];L^2(\mathbb{R}^d)-w)$. By the semicontinuity of the norm with respect to weak convergence we also obtain that

$$\|u(t,\cdot)\|_{L^2} \leq \|u^0\|_{L^2} + \sqrt{T}\|f\|_{L^2}. \tag{5}$$

Passing to the limit in the transport equation, we obtain that u solves the Cauchy problem

$$\begin{cases} \partial_t u + \mathrm{div}(bu) = f, \\ u(0,\cdot) = u^0 \,. \end{cases}$$

Noticing that $\partial_t u + \mathrm{div}(bu) = f \in L^2$, we conclude that $u \in \mathscr{F}$. Uniqueness is clear: every solution to the Cauchy problem (4) is by definition a weak solution in $C([0,T];L^2(\mathbb{R}^d)-w)$ of the forward Cauchy problem with right-hand side, and thus by linearity, uniqueness is guaranteed by the forward part of assumption (i).

Step 2. Density of smooth functions. Define a linear operator

$$A: \begin{array}{ccc} \mathscr{F} & \to & L^2(\mathbb{R}^d) \times L^2((0,T)\times\mathbb{R}^d) \\ & & \\ u & \mapsto & \big(u(0,\cdot)\,,\ \partial_t u + \mathrm{div}(bu)\big)\,. \end{array}$$

This operator in clearly bounded by the definition of the norm we have taken on $\mathscr{F}$. It is also a bijection because of Step 1, with continuous inverse because of (5). This means that A is an isomorphism, and thus we can identify $\mathscr{F}$ with the space $L^2(\mathbb{R}^d) \times L^2((0,T)\times\mathbb{R}^d)$, and its dual $\mathscr{F}^*$ with $L^2(\mathbb{R}^d) \times L^2((0,T)\times\mathbb{R}^d)$. Therefore, for every functional $L \in \mathscr{F}^*$, we can uniquely define $v_0 \in L^2(\mathbb{R}^d)$ and $v \in L^2((0,T)\times\mathbb{R}^d)$ in such a way that

$$Lu = \int_{(0,T)\times\mathbb{R}^d} \big(\partial_t u + \mathrm{div}(bu)\big) v \, dtdx + \int_{\mathbb{R}^d} u(0,\cdot) v_0 \, dx \qquad \text{for every } u \in \mathscr{F}.$$

We recall the classical fact that a subspace of a Banach space is dense if and only if every functional which is zero on the subspace is in fact identically zero. Then the density of smooth functions is equivalent to the following implication:

$$\begin{array}{c} \int_{(0,T)\times\mathbb{R}^d} \big(\partial_t u + \mathrm{div}(bu)\big) v \, dtdx + \int_{\mathbb{R}^d} u(0,\cdot) v_0 \, dx = 0 \\ \text{for every } u \in C^\infty([0,T]\times\mathbb{R}^d) \text{ with compact support in } x \\ \Longrightarrow v_0 = 0 \text{ and } v = 0. \end{array} \tag{6}$$

If we first take u arbitrary but with compact support also in time, we obtain that

$$\int_{(0,T)\times\mathbb{R}^d} \big(\partial_t u + \mathrm{div}(bu)\big) v \, dtdx = 0\,,$$

and since $\mathrm{div} b = 0$ this is precisely the weak form of

$$\partial_t v + \mathrm{div}(bv) = 0\,.$$

This implies that $v \in C([0,T]; L^2(\mathbb{R}^d) - w)$. Now let χ be a cut-off function on $\mathbb{R}$, i.e. $\chi \in C_c^\infty(\mathbb{R})$, $\chi(z) = 1$ for $|z| \leq 1$ and $\chi(z) = 0$ for $|z| \geq 2$. For every function $\varphi \in C_c^\infty(\mathbb{R}^d)$, take a function $\widetilde{u} \in C^\infty([0,T] \times \mathbb{R}^d)$ with compact support in x such that $\widetilde{u}(T, \cdot) = \varphi$. Then, testing in (6) with $u(t,x) = \widetilde{u}(t,x)\chi\big((T-t)/\varepsilon\big)$, we obtain for $0 < \varepsilon < T/2$

$$\begin{aligned} 0 &= \int_{(0,T)\times\mathbb{R}^d} \left[\partial_t \left(\widetilde{u}(t,x)\chi\left(\frac{T-t}{\varepsilon}\right)\right) + \operatorname{div}\left(b(t,x)\widetilde{u}(t,x)\chi\left(\frac{T-t}{\varepsilon}\right)\right)\right] v(t,x)\, dtdx \\ &= \int_{(0,T)\times\mathbb{R}^d} \left[\partial_t\widetilde{u}(t,x) + \operatorname{div}(b(t,x)\widetilde{u}(t,x))\right] v(t,x)\chi\left(\frac{T-t}{\varepsilon}\right) dtdx \\ &\quad - \int_{(0,T)\times\mathbb{R}^d} \frac{1}{\varepsilon}\chi'\left(\frac{T-t}{\varepsilon}\right) \widetilde{u}(t,x)v(t,x)\, dtdx\,. \end{aligned} \tag{7}$$

Letting $\varepsilon \to 0$, we observe that the first integral clearly converges to 0 since $\operatorname{supp}\big(\chi\big((T-t)/\varepsilon\big)\big) \subset [T-2\varepsilon, T+2\varepsilon]$. The second integral can be rewritten as

$$-\int_0^T \frac{1}{\varepsilon}\chi'\left(\frac{T-t}{\varepsilon}\right)\left[\int_{\mathbb{R}^d} \widetilde{u}(t,x)v(t,x)\, dx\right] dt\,.$$

Now, since $\widetilde{u}$ is smooth and $v \in C([0,T]; L^2(\mathbb{R}^d) - w)$, the integral over $\mathbb{R}^d$ is a continuous function of t. Moreover, it is easy to check that

$$-\int_0^T \frac{1}{\varepsilon}\chi'\left(\frac{T-t}{\varepsilon}\right) dt = 1.$$

Therefore, coming back to (7) and letting $\varepsilon \to 0$ we get

$$0 = \int_{\mathbb{R}^d} \widetilde{u}(T,x)v(T,x)\, dx = \int_{\mathbb{R}^d} \varphi(x)v(T,x)\, dx\,.$$

Since $\varphi \in C_c^\infty(\mathbb{R}^d)$ is arbitrary, we obtain $v(T,.) = 0$. We conclude that $v \in C([0,T]; L^2(\mathbb{R}^d) - w)$ solves the Cauchy problem

$$\begin{cases} \partial_t v + \operatorname{div}(bv) = 0, \\ v(T,\cdot) = 0. \end{cases}$$

Thus, by the backward part of the uniqueness assumption (i), we get that $v = 0$. Substituting in (6), we get that $\int_{\mathbb{R}^d} u(0,\cdot)v_0\, dx = 0$ for every $u \in C^\infty([0,T] \times \mathbb{R}^d)$ with compact support in space, and this implies that $v_0 = 0$. This concludes the proof of the implication (6), that ensures that (ii) holds.

(ii) $\Rightarrow$ (iii). Let $u \in C([0,T], L^2(\mathbb{R}^d) - w)$ satisfy $\partial_t u + \operatorname{div}(bu) = 0$. Then by (ii), there exists a sequence $\{u_n\}$ of functions in $C^\infty([0,T] \times \mathbb{R}^d)$ with compact support in space such that $\|u_n - u\|_{\mathscr{F}} \to 0$. In particular this gives that $u_n \to u$ in $B([0,T]; L^2(\mathbb{R}^d))$, thus $u \in C([0,T], L^2(\mathbb{R}^d) - s)$. Then, define $f_n = \partial_t u_n + \operatorname{div}(bu_n) \in L^2((0,T) \times \mathbb{R}^d)$. By the definition of convergence in $\mathscr{F}$ we have that $f_n \to 0$ strongly in $L^2((0,T)\times\mathbb{R}^d)$. For every function β with the regularity stated we can apply the classical chain-rule giving

$$\partial_t(\beta(u_n)) + \mathrm{div}(b\beta(u_n)) = \beta'(u_n) f_n\,.$$

The left-hand side clearly converges to $\partial_t(\beta(u)) + \mathrm{div}(b\beta(u))$ in the sense of distributions. According to the assumed bound on β', we have that the sequence $\beta'(u_n)$ is equi-bounded in $L^2_{loc}((0,T)\times\mathbb{R}^d)$, hence with the strong convergence of f_n we deduce that the right-hand side converges strongly in $L^1_{loc}((0,T)\times\mathbb{R}^d)$ to zero. This implies that $\partial_t\big(\beta(u)\big) + \mathrm{div}\big(b\beta(u)\big) = 0$.

(iii) $\Rightarrow$ (i). This step is classical. Let $u \in C([0,T], L^2(\mathbb{R}^d)-w)$ satisfy $\partial_t u + \mathrm{div}(bu) = 0$. According to (iii), u lies in $C([0,T], L^2(\mathbb{R}^d)-s)$, and applying the renormalization property with $\beta(u) = u^2$, we get

$$\partial_t(u^2) + \mathrm{div}(bu^2) = 0\,,$$

with $u^2 \in C([0,T], L^1(\mathbb{R}^d)-s)$. Testing this equation against smooth functions of the form $\psi(t)\varphi_R(x)$, where $\psi \in C^\infty_c((0,T))$ and $\varphi_R(x) = \varphi(x/R)$ with $\varphi \in C^\infty_c(\mathbb{R}^d)$ is a cut-off function equal to 1 on the ball of radius 1 and equal to 0 outside the ball of radius 2, we get

$$\int_0^T \left[\int_{\mathbb{R}^d} u^2 \varphi\left(\frac{x}{R}\right)\,dx\right]\psi'(t)\,dt + \int_0^T \left[\int_{\mathbb{R}^d} bu^2 \frac{1}{R}\nabla\varphi\left(\frac{x}{R}\right)\,dx\right]\psi(t)\,dt = 0\,.$$

Thus, we get in the sense of distributions in $(0,T)$

$$\frac{d}{dt}\int_{\mathbb{R}^d} u^2\varphi\left(\frac{x}{R}\right)\,dx \;=\; \int_{\mathbb{R}^d} bu^2\frac{1}{R}\nabla\varphi\left(\frac{x}{R}\right)\,dx.$$

Since the right-hand side is in $L^\infty(0,T)$ and bounded in L^∞ by $\frac{1}{R}\|b\|_{L^\infty_{t,x}}\|\nabla\varphi\|_{L^\infty_x}\|u(t,\cdot)\|^2_{L^2_x}$, letting $R\to+\infty$ we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^d} u(t,x)^2\,dx = 0 \quad \text{in } (0,T).$$

Recalling that $u^2 \in C([0,T], L^1(\mathbb{R}^d)-s)$, this yields $\int u(t,x)^2 dx = cst$ on $[0,T]$, which implies uniqueness for both forward and backward Cauchy problems, proving (i). □

Remark 2.2 (Well-posedness)**.** The space $\mathscr{F}$ defined in (2) is a natural space for the study of the Cauchy problem (4). Whenever one of the statements of Theorem 2.1 is true, we have existence and uniqueness in $\mathscr{F}$ with the estimate (5), as shown in the proof. Moreover, every solution is renormalized and strongly continuous with respect to time, i.e. $u \in C([0,T]; L^2(\mathbb{R}^d)-s)$. Overall, the following weak stability holds: if $\{f_n\}_n$ is a bounded sequence in $L^2((0,T)\times\mathbb{R}^d)$ which converges weakly to f, $\{u^0_n\}_n$ is a bounded sequence in $L^2(\mathbb{R}^d)$ which converges weakly to u^0 and $\{b_n\}_n$ is a bounded sequence in $L^\infty((0,T)\times\mathbb{R}^d)$ which converges strongly in L^1_{loc} to b and such that $\mathrm{div} b_n = 0$ for every n, then the solutions $\{u_n\}_n$ to

$$\partial_t u_n + \mathrm{div}(b_n u_n) = f_n\,, \qquad u_n(0,\cdot) = u^0_n$$

converge in $C([0,T]; L^2(\mathbb{R}^d)-w)$ to the solution u to the Cauchy problem (4).

Remark 2.3 (L^p case)**.** We can modify the summability exponent in the definition of the space $\mathscr{F}$. For every $p \in]1,\infty[$, define $\mathscr{F}_p$ as the space containing those functions $u \in C([0,T];L^p(\mathbb{R}^d)-w)$ that satisfy $\partial_t u + \mathrm{div}(bu) \in L^p((0,T)\times\mathbb{R}^d)$ and define the norm $\|\cdot\|_{\mathscr{F}_p}$ in the obvious way, that makes $\mathscr{F}_p$ a Banach space. Denoting by p' the conjugate exponent of p, i.e. $\frac{1}{p}+\frac{1}{p'}=1$, the following statements are equivalent:

- Smooth functions with compact support in x are dense in $\mathscr{F}_p$ and in $\mathscr{F}_{p'}$;
- The vector field b has the forward and backward uniqueness property for weak solutions in $C([0,T];L^p(\mathbb{R}^d)-w)$ and in $C([0,T];L^{p'}(\mathbb{R}^d)-w)$.

Remark 2.4 (Equivalent norms)**.** According to the proof of Theorem 2.1, if one of the properties (i), (ii) or (iii) holds, then the norm of $\mathscr{F}$ is equivalent to the norm

$$\|u\|_{\mathscr{F},0} = \|u(0,\cdot)\|_{L^2(\mathbb{R}^d)} + \|\partial_t u + \mathrm{div}(bu)\|_{L^2((0,T)\times\mathbb{R}^d)}$$

(see the estimate (5)). In the same spirit, it is easy to prove that $\|\cdot\|_{\mathscr{F}}$ is in fact equivalent to every norm of the form

$$\|u\|_{\mathscr{F},s} = \|u(s,\cdot)\|_{L^2(\mathbb{R}^d)} + \|\partial_t u + \mathrm{div}(bu)\|_{L^2((0,T)\times\mathbb{R}^d)}\,,$$

for $s \in [0,T]$.

Remark 2.5 (Depauw's counterexample)**.** A simple modification (translation in time) of the counterexample constructed in [12] shows that the renormalization property is really linked to the uniqueness in *both* the forward and the backward Cauchy problems. In fact, we can construct a divergence free vector field $b \in L^\infty((0,1)\times\mathbb{R}^2;\mathbb{R}^2)$ and a function $\bar u \in L^\infty(\mathbb{R}^2)$ such that

- the backward Cauchy problem with datum $\bar u$ at time $t=1$ has a unique solution, which is however *not* renormalized and *not* strongly continuous with respect to time;
- the forward Cauchy problem with datum 0 at time $t=0$ has more than one solution;
- the unique solution $u(t,x)$ to the backward Cauchy problem with datum $\bar u$ at time $t=1$ satisfies
$$\begin{cases} |u(t,x)| = 0 & \text{for } 0 \le t \le 1/2,\\ |u(t,x)| = 1 & \text{for } 1/2 < t \le 1, \end{cases}$$
hence the equivalence of the norms in Remark 2.4 does *not* hold.

Remark 2.6 (The Sobolev and the BV cases)**.** In the case of a vector field with Sobolev regularity with respect to the space variable, $b \in L^1((0,T);W^{1,p'}_{loc}(\mathbb{R}^d))$ with $1<p<\infty$, it is almost possible to prove that the natural regularization by convolution with respect to the space variable of $u \in \mathscr{F}_p$ (see Remark 2.3) converges to u with respect to $\|\cdot\|_{\mathscr{F}_p}$. Indeed, let η_ε be a standard convolution kernel in $\mathbb{R}^d$ and set $u_\varepsilon = u*\eta_\varepsilon$. We can compute

$$\begin{aligned} &\partial_t u + \mathrm{div}(bu) - \partial_t u_\varepsilon - \mathrm{div}(bu_\varepsilon)\\ = \;&\big[\partial_t u + \mathrm{div}(bu)\big] - \big[\partial_t u + \mathrm{div}(bu)\big]*\eta_\varepsilon + \big[\mathrm{div}(bu)*\eta_\varepsilon - \mathrm{div}(bu_\varepsilon)\big]\,. \end{aligned}$$

Then the convergence of u_ε to u with respect to $\|\cdot\|_{\mathscr{F}_p}$ is equivalent to the strong convergence in $L^p((0,T)\times\mathbb{R}^d)$ to zero of the *commutator*

$$r_\varepsilon = \mathrm{div}(bu) * \eta_\varepsilon - \mathrm{div}(bu_\varepsilon)\,.$$

The results of [13] ensure this strong convergence for every convolution kernel η_ε, except that it holds in L^1_{loc} instead of L^p. We need also a regularization with respect to time and a cutoff in order to get the density property (ii), but this means that our strategy is more or less "equivalent" to the one of [13], in the framework of Sobolev vector fields. However, the situation is different in the BV case studied in [2]. In general, the commutator r_ε does not converge strongly to zero; our result shows that, even in this case, there exists some smooth approximation of the solution, but it less clear how to construct it in an explicit way.

Remark 2.7 (Strong continuity condition)**.** The condition of continuity with values in strong L^2 in (iii) cannot be removed, otherwise the equivalence with (i) fails. This can be seen again with Depauw's counterexample with singularity at time $t=0$. In this case all weak solutions are renormalized in $(0,T)\times\mathbb{R}^d$ since b is locally BV in x, but uniqueness of weak solutions does not hold. Another remark is that in general, a renormalized solution need not be continuous with values in strong L^2, even inside the interval, as the following counterexample shows. On the interval $(-1,1)$, take for b the one of Depauw's counterexample in $(0,1)$ (with singularity at 0), and define on $(-1,0)$ $b(t,x)=-b(-t,x)$. Consider then the weak solution u with value 0 at $t=0$, that we extend on $(-1,0)$ by $u(t,x)=u(-t,x)$. Then u is a renormalized solution on $(-1,1)$ but is not strongly continuous at $t=0$.

3. One-way formulation

Theorem 3.1. *Let $b\in L^\infty((0,T)\times\mathbb{R}^d;\mathbb{R}^d)$ such that $\mathrm{div}b\in L^\infty((0,T)\times\mathbb{R}^d)$, and let $c\in L^\infty((0,T)\times\mathbb{R}^d)$. Define the Banach space $\mathscr{F}$ and its norm $\|\cdot\|_{\mathscr{F}}$ as in (2)-(3). Moreover, define $\mathscr{F}^0\subset\mathscr{F}$ as the closure (with respect to $\|\cdot\|_{\mathscr{F}}$) of the subspace of functions in $C^\infty([0,T]\times\mathbb{R}^d)$ with compact support in x. Then the following statements are equivalent:*

(i) *for every $u^0\in L^2(\mathbb{R}^d)$ and every $f\in L^2((0,T)\times\mathbb{R}^d)$ there exist a solution $u\in\mathscr{F}^0$ to the Cauchy problem*

$$\begin{cases}\partial_t u + \mathrm{div}(bu) + cu = f, \\ u(0,\cdot) = u^0\,, \end{cases} \qquad u\in\mathscr{F}^0\,;$$

(ii) *there is uniqueness for weak solutions in $C([0,T];L^2(\mathbb{R}^d)-w)$ for the backward dual Cauchy problem starting from T, i.e. the only function $v\in C([0,T];L^2(\mathbb{R}^d)-w)$ which solves*

$$\begin{cases}\partial_t v + b\cdot\nabla v - cv = 0, \\ v(T,\cdot) = 0, \end{cases}$$

is $v\equiv 0$.

Here and further on, the advection term $b\cdot\nabla v$ is defined according to $b\cdot\nabla v\equiv\mathrm{div}(bv)-v\,\mathrm{div}b$, which makes sense since $\mathrm{div}b\in L^\infty$.

Remark 3.2. The two statements in Theorem 3.1 are really the "nontrivial" properties relative to the vector field b. In general, there is always uniqueness in $\mathscr{F}^0$ (see Step 1 in the proof) and there is always existence of weak solutions in $\mathscr{F}$ (this can be easily proved by regularization, as in the first step of the proof of Theorem 2.1).

Before proving the theorem, we recall the following standard result of functional analysis (see for example Theorem II.19 and Theorem II.20 of [7]).

Lemma 3.3. *Let E and F be Banach spaces and let $L : E \to F$ be a bounded linear operator. Denote by $L^* : F^* \to E^*$ the adjoint operator, defined by*

$$\langle v, Lu\rangle_{F^*,F} = \langle L^* v, u\rangle_{E^*,E} \qquad \text{for every } u \in E \text{ and } v \in F^*.$$

Then

(a) *L is surjective if and only if L^* is injective and with closed image;*
(b) *L^* is surjective if and only if L is injective and with closed image.*

Proof of Theorem 3.1. **Step 1. An energy estimate in $\mathscr{F}^0$.** In this first step we prove that for every $u \in \mathscr{F}^0$ the following energy estimate holds:

$$\|u(t,\cdot)\|_{L^2_x} \le \Big(\|u(0,\cdot)\|_{L^2_x} + \sqrt{T}\|\partial_t u + \mathrm{div}(bu) + cu\|_{L^2_{t,x}}\Big) \exp\left(T\|c + \frac{1}{2}\mathrm{div} b\|_{L^\infty_{t,x}}\right). \tag{8}$$

Let us first prove the estimate for u smooth with compact support in x. We define

$$f = \partial_t u + \mathrm{div}(bu) + cu,$$

and we multiply this relation by u, giving

$$\partial_t \frac{u^2}{2} + \mathrm{div}(b\frac{u^2}{2}) + (c + \frac{1}{2}\mathrm{div} b)u^2 = fu.$$

For justifying the previous identity, we used the Leibniz rule

$$\partial_i(H\psi) = \psi\partial_i H + H\partial_i\psi, \tag{9}$$

valid for $\psi \in C^\infty$ and H any distribution. Then, integrating over $x \in \mathbb{R}^d$ we get in the sense of distributions in $(0,T)$

$$\frac{d}{dt}\int_{\mathbb{R}^d} u(t,x)^2 dx = 2\int_{\mathbb{R}^d} fu\, dx - 2\int_{\mathbb{R}^d} (c + \frac{1}{2}\mathrm{div} b)\, u^2 dx.$$

Therefore, we get for a.e. $t \in (0,T)$

$$\left|\frac{d}{dt}\int_{\mathbb{R}^d} u(t,x)^2 dx\right| \le 2\|f(t,\cdot)\|_{L^2_x}\|u(t,\cdot)\|_{L^2_x} + 2\|(c + \frac{1}{2}\mathrm{div} b)(t,.)\|_{L^\infty_x}\|u(t,\cdot)\|^2_{L^2_x}.$$

This differential inequality can be easily integrated, obtaining

$$\begin{aligned}\|u(t,\cdot)\|_{L^2_x} \;&\le\; \|u(0,\cdot)\|_{L^2_x} \exp\left(\int_0^t \|(c + \frac{1}{2}\mathrm{div} b)(s,\cdot)\|_{L^\infty_x}\, ds\right)\\ &\quad + \int_0^t \|f(s,\cdot)\|_{L^2_x} \exp\left(\int_s^t \|(c + \frac{1}{2}\mathrm{div} b)(\tau,\cdot)\|_{L^\infty_x}\, d\tau\right) ds\,,\end{aligned}$$

which clearly implies (8). In the general case of $u \in \mathscr{F}^0$, we can find approximations u_n smooth with compact support such that $\|u_n - u\|_{\mathscr{F}} \to 0$, and we obtain the estimate (8) at the limit.

Step 2. The operator A^0. As in the proof of Theorem 2.1, we consider the linear operator

$$A^0 : \begin{array}{ccc} \mathscr{F}^0 & \to & L^2(\mathbb{R}^d) \times L^2((0,T)\times\mathbb{R}^d) \\ u & \mapsto & \big(u(0,\cdot)\ ,\ \partial_t u + \mathrm{div}(bu) + cu\big)\,. \end{array}$$

Since we can estimate

$$\begin{aligned} \|A^0 u\|_{L^2_x \times L^2_{t,x}} &= \|u(0,\cdot)\|_{L^2_x} + \|\partial_t u + \mathrm{div}(bu) + cu\|_{L^2_{t,x}} \\ &\leq \|u\|_{B_t(L^2_x)} + \|\partial_t u + \mathrm{div}(bu)\|_{L^2_{t,x}} + \|c\|_{L^\infty_{t,x}} \sqrt{T} \|u\|_{B_t(L^2_x)} \\ &\leq \big(1 + \|c\|_{L^\infty_{t,x}}\sqrt{T}\big)\|u\|_{\mathscr{F}}\,, \end{aligned}$$

we deduce that A^0 is a bounded operator. Next, the energy estimate established in the first step gives that for any $u \in \mathscr{F}^0$,

$$\|u\|_{B_t(L^2_x)} \leq \exp\left(T\|c + \frac{1}{2}\mathrm{div} b\|_{L^\infty_{t,x}}\right) \max(1,\sqrt{T})\|A^0 u\|_{L^2_x\times L^2_{t,x}}.$$

But we have

$$\|\partial_t u + \mathrm{div}(bu)\|_{L^2_{t,x}} \leq \|\partial_t u + \mathrm{div}(bu) + cu\|_{L^2_{t,x}} + \|c\|_{L^\infty_{t,x}}\sqrt{T}\|u\|_{B_t(L^2_x)},$$

and we conclude that

$$\|u\|_{\mathscr{F}} \leq C\,\|A^0 u\|_{L^2_x\times L^2_{t,x}}, \qquad u \in \mathscr{F}^0. \tag{10}$$

This means that A^0 is injective and with closed image. Notice that the injectivity of A^0 is equivalent to the fact that the only solution $u \in \mathscr{F}^0$ to

$$\begin{cases} \partial_t u + \mathrm{div}(bu) + cu = 0, \\ u(0,\cdot) = 0, \end{cases}$$

is $u \equiv 0$.

Step 3. Proof of the equivalence of the two statements. Since by Step 2, A^0 is injective with closed image, we can apply Lemma 3.3 (b) to get the surjectivity of the adjoint operator $(A^0)^* : L^2(\mathbb{R}^d) \times L^2((0,T)\times\mathbb{R}^d) \to (\mathscr{F}^0)^*$. We recall that the adjoint operator is characterized by the condition

$$\langle (A^0)^*(v_0,v), u\rangle = \langle (v_0,v), A^0 u\rangle = \int_{\mathbb{R}^d} v_0 u(0,\cdot)\,dx + \int_{(0,T)\times\mathbb{R}^d} v\big(\partial_t u + \mathrm{div}(bu) + cu\big)\,dtdx, \tag{11}$$

for $(v_0,v) \in L^2(\mathbb{R}^d) \times L^2((0,T)\times\mathbb{R}^d)$ and $u \in \mathscr{F}^0$. Since $(A^0)^*$ is surjective, in particular it has closed image. Therefore, applying Lemma 3.3 (a) we get the equivalence between surjectivity of A^0 and injectivity of $(A^0)^*$.

It is clear that the surjectivity of the operator A^0 is equivalent to the existence of solutions in $\mathscr{F}^0$ (statement (i)). Therefore, it only remains to characterize the injectivity of $(A^0)^*$.

Recalling the definition of $\mathscr{F}^0$ as the closure of the set of smooth functions with compact support in x and recalling the characterization of the adjoint operator given in (11), we obtain that the injectivity of $(A^0)^*$ is equivalent to the following implication:

$$\begin{aligned} &\textstyle\int_{(0,T)\times\mathbb{R}^d} \big(\partial_t u + \operatorname{div}(bu) + cu\big) v \, dtdx + \int_{\mathbb{R}^d} u(0,\cdot) v_0 \, dx = 0 \\ &\text{for every } u \in C^\infty([0,T]\times\mathbb{R}^d) \text{ with compact support in } x \\ &\qquad\qquad \Longrightarrow v_0 = 0 \text{ and } v = 0. \end{aligned} \tag{12}$$

Arguing as in Step 2 of the proof of Theorem 2.1, and eventually testing the integral condition with smooth functions of the form $u(t,x) = \chi(t/\varepsilon)\widetilde{u}(t,x)$ (using the same notations as in the proof of Theorem 2.1), we obtain that the following two properties are equivalent for given $v_0 \in L^2(\mathbb{R}^d)$ and $v \in L^2((0,T)\times\mathbb{R}^d)$:

(a) for every $u \in C^\infty([0,T]\times\mathbb{R}^d)$ with compact support in x we have

$$\int_{(0,T)\times\mathbb{R}^d} \big(\partial_t u + \operatorname{div}(bu) + cu\big) v \, dtdx + \int_{\mathbb{R}^d} u(0,\cdot) v_0 \, dx = 0\,,$$

(b) $v \in C([0,T]; L^2(\mathbb{R}^d) - w)$, $v_0 = v(0,\cdot)$ and v is a weak solution of the backward dual Cauchy problem

$$\begin{cases} \partial_t v + b\cdot\nabla v - cv = 0, \\ v(T,\cdot) = 0\,. \end{cases}$$

Therefore we deduce that the implication (12) is equivalent to the uniqueness of weak solutions in $C([0,T]; L^2(\mathbb{R}^d) - w)$ of the backward dual Cauchy problem, i.e. statement (ii). □

Remark 3.4 (Time inversion)**.** Just reversing the direction of time, there is existence for the backward Cauchy problem in $\mathscr{F}^0$ if and only if there is uniqueness for weak solutions to the forward dual Cauchy problem.

Remark 3.5 (Approximation by smooth functions and renormalization)**.** Solutions in $\mathscr{F}^0$ lie in $C([0,T], L^2(\mathbb{R}^d) - s)$ and are renormalized: this can be seen as in the proof of the implication (ii)⇒(iii) of Theorem 2.1, using the density of smooth functions in $\mathscr{F}^0$. Conversely, it is possible that some renormalized solutions do not belong to $\mathscr{F}^0$. This can be seen by noticing that one can have several renormalized solutions to the same Cauchy problem (see an example in [13]), while there is always uniqueness in $\mathscr{F}^0$. Another difference between the criterion of approximation by smooth functions and the renormalization property is that $\mathscr{F}^0$ is a vector space, while in general renormalized solutions are not a vector space.

Remark 3.6 (Depauw's example again)**.** We notice that forward and backward uniqueness of weak solutions are really distinct properties: the example described in Remark 2.5 shows how to construct bounded divergence free vector fields with backward uniqueness but not forward uniqueness, or vice-versa.

4. Vector fields of bounded compression

We shall say that a vector field $b \in L^\infty((0,T)\times\mathbb{R}^d;\mathbb{R}^d)$ has *bounded compression* if there exists a function $\rho \in C([0,T];L^\infty(\mathbb{R}^d)-w^*)$, with $0 < C^{-1} \le \rho \le C < \infty$ for some constant $C > 0$, such that the identity

$$\partial_t \rho + \mathrm{div}(b\rho) = 0 \tag{13}$$

holds in the sense of distributions in $(0,T)\times\mathbb{R}^d$. We remark that every vector field b with bounded divergence has bounded compression (if b is smooth, take for ρ the Jacobian determinant of the flow generated by b, $\rho(t,x) = \det\nabla_x X(0,t,x)$, which is bounded since $\rho(t,x) = \exp - \int_0^t (\mathrm{div} b)(\sigma, X(\sigma,t,x))d\sigma$, where $X(s,t,x)$ satisfies $dX(s,t,x)/ds = b(s,X(s,t,x))$, $X(t,t,x) = x$), but in general a vector field of bounded compression does not need to have absolutely continuous divergence.

Theorem 4.1. *Let $b \in L^\infty((0,T)\times\mathbb{R}^d;\mathbb{R}^d)$ be a vector field of bounded compression, and fix an associated function $\rho \in C([0,T];L^\infty(\mathbb{R}^d)-w^*)$. We define the Banach space $\mathscr{F}$ and its norm $\|\cdot\|_{\mathscr{F}}$ as in (2)-(3). Let $\mathscr{F}^1 \subset \mathscr{F}$ be the closure of*

$$\big\{\rho\varphi \ : \ \ \varphi \in C^\infty([0,T]\times\mathbb{R}^d) \text{ with compact support in } x\big\}$$

with respect to $\|\cdot\|_{\mathscr{F}}$. Then the following statements are equivalent:

(i) *for every $u^0 \in L^2$ and every $f \in L^2$ there exist a solution $u \in \mathscr{F}^1$ to the Cauchy problem*

$$\begin{cases} \partial_t u + \mathrm{div}(bu) = f, \\ u(0,\cdot) = u^0\,, \end{cases} \qquad u \in \mathscr{F}^1\,;$$

(ii) *there is uniqueness for weak solutions in $C([0,T];L^2(\mathbb{R}^d)-w)$ for the backward dual Cauchy problem starting from T, i.e. the only function $\rho v \in C([0,T];L^2(\mathbb{R}^d)-w)$ which solves*

$$\begin{cases} \partial_t(\rho v) + \mathrm{div}(b\rho v) = 0, \\ \rho(T,\cdot)v(T,\cdot) = 0, \end{cases}$$

is $\rho v \equiv 0$.

Remark 4.2. In this context, the equation $\partial_t(\rho v) + \mathrm{div}(b\rho v) = 0$ is dual to the equation $\partial_t u + \mathrm{div}(bu) = 0$, since we can write (formally, since it is not possible to give a meaning to the product $b\cdot\nabla v$ without a condition of absolute continuity of div$b$):

$$\partial_t(\rho v) + \mathrm{div}(b\rho v) = \rho\big(\partial_t v + b\cdot\nabla v\big)\,.$$

Proof of Theorem 4.1. The proof is very close to the one of Theorem 3.1, thus we shall sometimes omit the technical details.

Step 1. An energy estimate in $\mathscr{F}^1$. We preliminarily prove that for every $u \in \mathscr{F}^1$ the following estimate holds (C is the constant related to the function ρ):

$$\|u\|_{B_t(L^2_x)} \le C\|u(0,\cdot)\|_{L^2_x} + C\sqrt{T}\|\partial_t u + \mathrm{div}(bu)\|_{L^2_{t,x}}\,. \tag{14}$$

Fix a smooth function φ with compact support in $\mathbb{R}^d$, and define $f = \partial_t(\rho\varphi) + \mathrm{div}(b\rho\varphi) = \rho(\partial_t\varphi + b\cdot\nabla\varphi)$ (use the Leibnitz rule (9) and (13)). We deduce with the same argument that $2\varphi f = \rho(\partial_t\varphi^2 + b\cdot\nabla\varphi^2) = \partial_t(\rho\varphi^2) + \mathrm{div}(b\rho\varphi^2)$. Thus, we get the following estimate in the sense of distributions in $(0,T)$:

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d}\rho(t,x)\varphi(t,x)^2\,dx &= 2\int_{\mathbb{R}^d}\varphi(t,x)f(t,x)\,dx\\ &\le 2\|f(t,\cdot)\|_{L^2_x}\|\varphi(t,\cdot)\|_{L^2_x}\\ &\le 2\sqrt{C}\|f(t,\cdot)\|_{L^2_x}\left[\int_{\mathbb{R}^d}\rho(t,x)\varphi(t,x)^2\,dx\right]^{1/2}.\end{aligned}$$

By integration with respect to time this implies

$$\left[\int_{\mathbb{R}^d}\rho(t,x)\varphi(t,x)^2\,dx\right]^{1/2} \le \left[\int_{\mathbb{R}^d}\rho(0,x)\varphi(0,x)^2\,dx\right]^{1/2} + \sqrt{C}\int_0^t \|f(s,\cdot)\|_{L^2_x}\,ds\,.$$

Using the fact that $C^{-1}\le\rho\le C$ we deduce

$$\frac{1}{\sqrt{C}}\|\rho(t,\cdot)\varphi(t,\cdot)\|_{L^2_x} \le \sqrt{C}\|\rho(0,\cdot)\varphi(0,\cdot)\|_{L^2_x} + \sqrt{C}\int_0^t\|f(s,\cdot)\|_{L^2_x}\,ds\,,$$

and thus

$$\|\rho(t,\cdot)\varphi(t,\cdot)\|_{L^2_x} \le C\|\rho(0,\cdot)\varphi(0,\cdot)\|_{L^2_x} + C\sqrt{T}\|\partial_t(\rho\varphi)+\mathrm{div}(b\rho\varphi)\|_{L^2_{t,x}}\,. \tag{15}$$

But by definition of $\mathscr{F}^1$, the validity of (15) for every smooth function φ with compact support in x implies the validity of (14) for every function $u\in\mathscr{F}^1$.

Step 2. The operator A^1. We define the linear operator

$$A^1:\quad \begin{aligned}\mathscr{F}^1 &\to L^2(\mathbb{R}^d)\times L^2((0,T)\times\mathbb{R}^d)\\ u &\mapsto \big(u(0,\cdot)\,,\ \partial_t u + \mathrm{div}(bu)\big)\,.\end{aligned}$$

It is immediate to see that the operator A^1 is bounded. Using the energy estimate (14) it is also immediate to check that $\|u\|_{\mathscr{F}} \le \widetilde{C}\|A^1u\|$, and therefore that A^1 is injective with closed image. Applying Lemma 3.3(b) we obtain that the adjoint operator

$$(A^1)^*: L^2(\mathbb{R}^d)\times L^2((0,T)\times\mathbb{R}^d)\to(\mathscr{F}^1)^*$$

is surjective. The adjoint operator is characterized by the identity

$$\langle (A^1)^*(v_0,v),u\rangle = \langle (v_0,v),A^1u\rangle = \int_{\mathbb{R}^d} v_0u(0,\cdot)\,dx + \int_{(0,T)\times\mathbb{R}^d} v\big(\partial_t u+\mathrm{div}(bu)\big)\,dtdx, \tag{16}$$

for $(v_0,v)\in L^2(\mathbb{R}^d)\times L^2((0,T)\times\mathbb{R}^d)$ and $u\in\mathscr{F}^1$.

Step 3. Proof of the equivalence of the two statements. The statement (i) (existence of solutions in $\mathscr{F}^1$) is the surjectivity of the operator A^1, which is equivalent (applying Lemma 3.3(a) and using the surjectivity of $(A^1)^*$ proved in Step 2) to the injectivity of $(A^1)^*$. But recalling the characterization (16) and the definition of the space $\mathscr{F}^1$, the injectivity of $(A^1)^*$ is equivalent to the following implication for $v_0 \in L^2(\mathbb{R}^d)$ and $v \in L^2((0,T)\times\mathbb{R}^d)$:

$$\begin{gathered}\textstyle\int_{(0,T)\times\mathbb{R}^d} \big(\partial_t(\rho\varphi) + \mathrm{div}(b\rho\varphi)\big) v \, dtdx + \int_{\mathbb{R}^d} \rho(0,\cdot)\varphi(0,\cdot) v_0 \, dx = 0 \\ \text{for every } \varphi \in C^\infty([0,T]\times\mathbb{R}^d) \text{ with compact support in } x \\ \Longrightarrow v_0 = 0 \text{ and } v = 0.\end{gathered} \tag{17}$$

Arguing as in Step 3 of the proof of Theorem 3.1 we obtain that the following two properties are equivalent:

(a) for every $\varphi \in C^\infty([0,T]\times\mathbb{R}^d)$ with compact support in x we have

$$\int_{(0,T)\times\mathbb{R}^d} \big(\partial_t(\rho\varphi) + \mathrm{div}(b\rho\varphi)\big) v \, dtdx + \int_{\mathbb{R}^d} \rho(0,\cdot)\varphi(0,\cdot) v_0 \, dx = 0\,,$$

(b) $\rho v \in C([0,T]; L^2(\mathbb{R}^d) - w)$, $\rho(0,\cdot)v_0 = \rho(0,\cdot)v(0,\cdot)$, and ρv is a weak solution of the backward dual Cauchy problem

$$\begin{cases} \partial_t(\rho v) + \mathrm{div}(b\rho v) = 0, \\ \rho(T,\cdot)v(T,\cdot) = 0\,. \end{cases}$$

Then we deduce that implication (17) is equivalent to statement (ii), and this concludes the proof of the theorem. □

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