Gradient theory of phase transitions in composite media

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Synopsis We study the behaviour of non convex functionals singularly perturbed by a possibly oscillating inhomogeneous gradient term, in the spirit of the gradient theory of phase transitions. We show that a limit problem giving a sharp interface, as the perturbation vanishes, always exists, but may be inhomogeneous or anisotropic. We specialize this study when the perturbation oscillates periodically, highlighting three types of regimes, depending on the speed of the oscillations. In the two extreme cases a separation of scales effect is described.

1 Introduction

In the classical theory of phase transitions for mixtures of two immiscible fluids (or for two phases of the same fluid) it is assumed that, at equilibrium, the two fluids arrange themselves in such a way that the area of the interface which separates the regions occupied by the two phases is minimal. This 'minimal-interface criterion' can be interpreted in mathematical terms as an energy-minimization process. We can describe every configuration of the system by a function u defined on Ω (the 'container' of the fluids), taking the value 0 on the first phase and 1 on the second one. In addition u satisfies a 'volume constraint' $\int_\Omega u\,dx = V$, where V is the assigned total volume of the second fluid. The set of discontinuity points of u parametrizes the interface between the two fluids in the corresponding configuration and is denoted by $S(u)$. We then postulate that the energy of such a u is proportional to the area of the interfaces, i.e., it is given by

$$F(u) = \sigma_0 \, \mathcal{H}^2(S(u)),$$

where $\mathcal{H}^2$ denotes the 2-dimensional (Hausdorff) surface measure and σ_0 (the '*surface tension*') is a strictly positive constant, characteristic of the fluids. In such a way, the optimal configurations are obtained by minimizing this surface energy among all admissible configurations.

The 'gradient theory' of phase transitions is an alternative way to study these systems of fluids, by assuming that the transition between the phases is not concentrated on a interfacial surface, but takes place on a thin 'transition layer'. In this way, we allow fine mixtures of the two fluids, and an admissible configuration u will be a function taking its values in $[0, 1]$, so that $u(x)$ will be interpreted as a local average density or concentration of the second fluid. Following this model proposed by Cahn and Hilliard [11], to such a u, we associate the energy

$$E_\varepsilon(u) = \int_\Omega \Big(W(u) + \varepsilon^2 |Du|^2 \Big) dx,$$

where W is a 'double-well energy' with wells at 0 and 1 (i.e., a non-negative function vanishing only at 0 and 1), and ε is a small parameter linked to the width of the transition layer. In addition, the admissible configurations will always satisfy the same volume constraint as above. The competing effects of the two integrals in E_ε are to favour the configurations which take values close to 0 and 1 by the first term and at the same time to penalize spatial inhomogeneities of u (and hence the introduction of too many transition regions) by the second term.

The connection between these two standpoints had been conjectured by Gurtin [20], and was proved by L. Modica [23] (after an earlier work by Modica and Mortola [22]) by showing that minimum problems for the functional E_ε tend to minimum problems for F. More precisely, in terms of Γ-convergence he proved that the scaled functionals

$$\frac{1}{\varepsilon} E_\varepsilon(u) = \int_\Omega \Big(\frac{W(u)}{\varepsilon} + \varepsilon |Du|^2 \Big) dx$$

Γ-converge to F given above if the constant σ_0 is chosen as $\sigma_0 = 2 \int_0^1 \sqrt{W(s)} ds$. Loosely speaking, this convergence means that minimal configurations u_ε for E_ε will tend to have transition layers which 'concentrate' as $\varepsilon \to 0$ on the interface $S(u)$ of a minimizer u of F. Moreover, the scaled minimal values $\frac{1}{\varepsilon} E_\varepsilon(u_\varepsilon)$ will converge to the value $F(u)$. It is interesting to note that the proof of the Modica Mortola result is essentially one dimensional. The key point is to show that for minimizers of E_ε the profile of the transition layer approximately depends only on the direction orthogonal to $S(u)$ and is a scaling of an 'optimal profile'. After noticing this the convergence result can be proved first, with the due changes in the statement, if Ω is one-dimensional (in which case interfaces are points), and then the 3-dimensional case can be recovered by a 'slicing' argument (see e.g. [5], [1]).

In this paper we investigate the effect of the presence of small-scale heterogeneities on the passage to the limit described above. More precisely, we assume that the gradient term in the definition of E_ε may depend on the space variable x, so that we are lead to the study of the asymptotic behaviour of functionals of the form

$$F_\varepsilon(u) = \int_\Omega \Big(\frac{W(u)}{\varepsilon} + \varepsilon f_\varepsilon(x, Du)\Big) dx,$$

where f_ε are Borel functions convex and positively 2-homogeneous in the second variable. In this case, by a simple comparison argument with the case studied by Modica, we may see that the domain of the Γ-limit will be the same as that of the energy F above. However, the determination of the actual form of the limit is much more complex. By following the 'direct methods of Γ-convergence' (see [13], [9]) we have first given a general compactness result for Γ-limits of functionals F_ε as above, and then explicitly characterized the limit functional when f_ε is rapidly oscillating in the first variable. In our general framework (as it was already done by Modica and Mortola) we do not restrict to the case of space dimension $n = 3$.

Our compactness result (Theorems 3.3 and 3.5) shows that from every sequence (F_{ε_j}) of functionals as above, it is possible to extract a subsequence which converges to a functional F_0 of the form

$$F_0(u) = \int_{S(u)} \sigma(x, \nu_u)\, d\mathcal{H}^{n-1}$$

defined on functions $u : \Omega \to \{0, 1\}$ of bounded variation. In this case ν_u represents the measure-theoretical normal to $S(u)$. Note that in this case the limit may be *anisotropic and inhomogeneous*, but it is always in the same 'class' of the functional F above, which we recover when σ is a constant. To prove this result we follow a procedure which is by now customary in Γ-convergence, consisting in combining localization and integral representation arguments: first, we extend the definition of F_ε to every open set of $\mathbb{R}^n$ by

$$F_\varepsilon(u, A) = \int_A \Big(\frac{W(u)}{\varepsilon} + \varepsilon f_\varepsilon(x, Du)\Big) dx;$$

we then prove the existence of converging subsequences to an (abstract) functional $F_0(u, A)$, which is, among other things, (the restriction of) a measure in the second variable and by comparison we get $F_0 \le c\, F$ for some $c > 0$ We conclude then that

$$F_0(u, A) = \int_{S(u)\cap A} \sigma(x, \nu_u)\, d\mathcal{H}^{n-1}$$

for some Borel function σ by suitable representation results (see [8], [10]). This method is well established in the case of functionals defined on Sobolev spaces ([13],

[9]) and had been previously used within the framework of Caccioppoli partitions [2] or, in a way similar to the present paper, to characterize limits of non-local functionals [12].

It is interesting to note that the key point in the complex procedure above is proving that the set function $F_0(u,\cdot)$ is subadditive, and that this was the object of an early lemma by Dal Maso and Modica [14]. Their result was inspired by De Giorgi, clearly aiming to illustrate how the direct methods of Γ-convergence could be applied also outside the framework of Sobolev spaces. Only now we have at disposal powerful integral representation techniques for functionals defined on functions with bounded variation which allow to conclude this argument.

The main part of paper is Section 4, where we specialize the convergence result in the case of rapidly-oscillating perturbations. We fix a function $\delta=\delta(\varepsilon)$ such that $\delta\to 0$ as $\varepsilon\to 0$ and take

$$f_\varepsilon(x,z)=f\Big(\frac{x}{\delta},z\Big),$$

where f is periodic in the first variable; i.e.,

$$F_\varepsilon(u)=\int_\Omega\Big(\frac{W(u)}{\varepsilon}+\varepsilon f\Big(\frac{x}{\delta},Du\Big)\Big)dx.$$

We may interpret this situation as modelling the presence of *heterogeneities* at a scale δ, which locally favour or disfavour the onset of a transition layer. We show that the behaviour of the whole family (F_ε) can be completely described and depends on the mutual speed of convergence to 0 of δ and ε. The limit functional F_0 is homogeneous, but may be anisotropic:

$$F_0(u)=\int_{S(u)}\sigma(\nu_u)\,d\mathcal{H}^{n-1}.$$

In the first case, $\varepsilon<<\delta$ the final result is that we have a 'separation of scales' effect: we may first regard δ as fixed and let $\varepsilon\to 0$, and subsequently let $\delta\to 0$. In this way, we first obtain an inhomogeneous functional by applying the Modica Mortola procedure, which can be explicitly computed as

$$F^\delta(u)=\sigma_0\int_{S(u)}\sqrt{f\Big(\frac{x}{\delta},\nu_u\Big)}\,d\mathcal{H}^{n-1}$$

(for this anisotropic version see also [5] Chapter 4.3). The limit as $\delta\to 0$ of these types of functionals falls within the framework of Γ-convergence of functionals defined on Caccioppoli partitions [2] and can also be seen as a particular case of homogenization on BV spaces [4]. By applying either of these two procedures we obtain a formula for σ (see also [7]). A second case is when ε and δ are comparable (for simplicity, $\varepsilon=\delta$). In this case the two effects cannot be separated, and $\sigma(\nu)$ is described through an asymptotic formula which describes the optimal profile, which in this case is not depending only on the direction ν. Finally, when $\delta << \varepsilon$ we again find a separation of scales phenomenon: the total effect is as if first we freeze ε. In this case, letting $\delta \to 0$ we obtain a functional of the form

$$F^\varepsilon(u) = \int_\Omega \Big(\frac{W(u)}{\varepsilon} + \varepsilon f_{\rm hom}(Du)\Big)dx,$$

where $f_{\rm hom}$ is the *homogenized integrand* of f (see e.g. [9]). We eventually let $\varepsilon \to 0$, so that, by applying the Modica Mortola procedure, we have $\sigma(\nu) = \sigma_0\sqrt{f_{\rm hom}(\nu)}$. Note that by the inequality $w^2 + z^2 \geq 2wz$, we always have the estimate

$$F_\varepsilon(u) \geq \int_\Omega 2\sqrt{W(u)\, f\Big(\frac{x}{\delta}, Du\Big)}\, dx,$$

which turns out to be optimal if $\varepsilon << \delta$, but is not sharp in all other cases.

To briefly illustrate the difference in the separation of scales effect, as an example, we may consider the case of a simple inhomogeneous isotropic f_ε:

$$F_\varepsilon(u) = \int_\Omega \Big(\frac{W(u)}{\varepsilon} + \varepsilon a\Big(\frac{x}{\delta}\Big)|Du|^2\Big)dx,$$

where $n = 2$ and a for example is a 'chessboard coefficient' (taking the values α on 'white squares' and $\beta > \sqrt{2}\alpha$ on 'black squares'). If $\varepsilon << \delta$ then $F^\delta(u) = \sigma_0 \int_{S(u)} \sqrt{a\Big(\frac{x}{\delta}\Big)}\, d\mathcal{H}^1$, and

$$\sigma(\nu) = \sigma_0\alpha\Big((\sqrt{2}-1)|\nu_1| \wedge |\nu_2| + |\nu_1| \vee |\nu_2|\Big)$$

(see [7] Example 5.3). If $\delta << \varepsilon$ then we have by the classical Dychne formula (see [25]) $F^\varepsilon(u) = \int_\Omega \Big(W(u)/\varepsilon + \varepsilon\sqrt{\alpha\beta}|Du|^2\Big)dx$, and eventually

$$\sigma(\nu) = \sigma_0(\alpha\beta)^{1/4}.$$

We finally point out that throughout the paper we have chosen to make some continuity hypotheses on f in order to simplify formulas, which otherwise should take into account relaxation results in BV spaces. The reader interested in the problems connected to general Borel integrands is referred to [7], [8] and [10].

2 Notation and preliminary results

Let Ω be an open subset of $\mathbb{R}^n$. We denote by $\mathcal{A}$ and $\mathcal{B}$ the families of all bounded open and Borel subsets of $\mathbb{R}^n$, respectively. We denote by χ_E the *characteristic function* of E. We introduce the notation

$$Q(x,\rho) = x + \rho(-1/2, 1/2)^n$$

in particular $Q = Q(0,1)$; $Q^\nu_\rho(x)$ denotes an open cube of $\mathbb{R}^n$ centered at x, having side length ρ and one face orthogonal to ν; $Q^\nu_\rho = Q^\nu_\rho(0)$ and $Q^\nu = Q^\nu_1(0)$. By $[t]$ we denote the integer part of $t \in \mathbb{R}$.

Let U and U' be open sets with $U' \subset\subset U$. We say that $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a *cut-off function* related to U and U' if $\varphi \in \mathcal{C}^\infty_0(U')$ and $0 \le \varphi \le 1$ with $\varphi \equiv 1$ in a neighbourhood of $\overline{U}$.

Given a vector-valued measure μ on Ω, we adopt the notation $|\mu|$ for its total variation (see Federer [18], and $\mathcal{M}(\Omega)$ is the set of all signed measures on Ω with bounded total variation. The Lebesgue measure of a set E is denoted by $|E|$. The Hausdorff $(n-1)$-dimensional measure in $\mathbb{R}^n$ is denoted by $\mathcal{H}^{n-1}$.

We say that $u \in L^1(\Omega)$ is a *function of bounded variation*, and we write $u \in BV(\Omega)$, if its distributional first derivatives $D_i u$ belong to $\mathcal{M}(\Omega)$. We denote by Du the $\mathbb{R}^n$-valued measure whose components are $D_1 u, \cdots, D_n u$.

We will say that a set E is of *finite perimeter* in Ω, or a *Caccioppoli set*, if $\chi_E \in BV(\Omega)$, and for every open subset Ω of $\mathbb{R}^n$, we let

$$\mathcal{P}_\Omega(E) = |D\chi_E|(\Omega)$$

the *perimeter* of E in Ω. The family of Caccioppoli sets can be identified with the functions $u \in BV(\Omega; \{0,1\})$, the set of $BV(\Omega)$ functions which take almost everywhere the values 0 or 1.

In this case (if $u \in BV(\Omega; \{0,1\})$) the vector-valued measure Du can be represented as

$$Du(B) = \int_{B \cap S(u)} \nu_u \, d\mathcal{H}^{n-1}$$

for every Borel set $B \subseteq \Omega$, where $S(u)$ denotes the complement of the Lebesgue set of u, $\nu_u \in \mathbb{R}^n$ is a unit vector which is $\mathcal{H}^{n-1}$-a.e. defined in $S(u)$, interpreted as the *normal* to $S(u)$. Moreover, one can prove that, if $E = \{x : u(x) = 1\}$,

$$\mathcal{P}_\Omega(E) = |Du|(\Omega) = \mathcal{H}^{n-1}(S(u) \cap \Omega).$$

For the general exposition of the theory of functions of bounded variation we refer to Ambrosio, Fusco and Pallara [3], Federer [18], Giusti [19], Vol'pert [24] and Ziemer [26].

Since we will consider either functions in Sobolev spaces or characteristic functions of sets of finite perimeter, with a slight abuse we will use the notation $Du = (D_1 u, \cdots, D_n u)$ both for the gradient of a Sobolev function and for the distributional derivative of u, as no confusion may arise.

We recall the definition of Γ-*convergence* of a sequence of functionals F_j defined on $L^1(\Omega)$ (with respect to the $L^1(\Omega)$-convergence). We say that (F_j) Γ-converges to F_0 on $L^1(\Omega)$ if for all $u \in L^1(\Omega)$

(i) (Γ-*liminf inequality*) for all (u_j) sequences of functions in $L^1(\Omega)$ converging to u in $L^1(\Omega)$ we have

$$F_0(u) \le \liminf_j F_j(u_j);$$

(ii) (Γ-*limsup inequality*) there exists a sequence (u_j) of functions in $L^1(\Omega)$ converging to u in $L^1(\Omega)$ such that

$$F_0(u) \geq \limsup_j F_j(u_j).$$

We will say that a family (F_ε) Γ-converges to F_0 if for all sequences (ε_j) of positive numbers converging to 0 (i) and (ii) above are satisfied with F_{ε_j} in place of F_j. For a comprehensive study of Γ-convergence we refer to the book of Dal Maso [13] (for a simplified introduction see [6]), while a detailed analysis of some of its applications to homogenization theory can be found in [9].

The model example of Γ-convergence we have in mind is the following result (see [22] and [23]).

Theorem 2.1 *Let $W : \mathbb{R} \to [0, +\infty)$ be a continuous function such that*

$$\{z \in \mathbb{R} : W(z) = 0\} = \{0, 1\} \tag{2.1}$$

$$c_1(|z|^\gamma - 1) \leq W(z) \leq c_2(|z|^\gamma + 1) \qquad \text{for every } z \in \mathbb{R}, \tag{2.2}$$

with $\gamma \geq 2$.

Then, the functionals

$$E_\varepsilon(u, A) = \begin{cases} \displaystyle\int_A \Big(\frac{W(u)}{\varepsilon} + \varepsilon |Du|^2\Big)\, dx & \text{if } u \in W^{1,\gamma}(A) \\ +\infty & \text{otherwise} \end{cases}$$

Γ-converge as $\varepsilon \to 0$ to the functional

$$E(u, A) = c_0\, \Phi(u, A)$$

for every Lipschitz set $A \in \mathcal{A}$ and every function $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, where

$$\Phi(u, A) = \begin{cases} \mathcal{H}^{n-1}(S(u) \cap A) = |Du|(A) = \mathcal{P}_A(\{u = 1\}) & \text{if } u \in \mathrm{BV}(A; \{0, 1\}) \\ +\infty & \text{otherwise} \end{cases} \tag{2.3}$$

and

$$c_0 = 2 \int_0^1 \sqrt{W(z)\, dz}. \tag{2.4}$$

From this theorem and the properties of convergence of minima of Γ-limits the following corollary, which describes the limit behaviour of the gradient theory of phase transitions, holds (see [23], Proposition 3).

Corollary 2.2 *Let $0 \le V \le |\Omega|$. Let $\gamma > 2$, and let $u_\varepsilon \in W^{1,\gamma}(\Omega)$ be a solution of problem*

$$m_\varepsilon = \min\Big\{ \int_\Omega (W(u) + \varepsilon^2 |Du|^2)\, dx : \int_\Omega u\, dx = V \Big\}.$$

Then, upon extracting a subsequence, $u_\varepsilon \to u \in BV(\Omega; \{0,1\})$ in $L^1(\Omega)$, where u is a solution of problem

$$m = \min\Big\{ |Du|(\Omega) : u \in BV(\Omega;\{0,1\}), \int_\Omega u\, dx = V \Big\} = \min\Big\{ \mathcal{P}_\Omega(E) : |E| = V \Big\}$$

and $m_\varepsilon/\varepsilon \to c_0 m$.

3 A compactness result

For all $\varepsilon > 0$ let $f_\varepsilon : \mathbb{R}^n \times \mathbb{R}^n \to [0,+\infty)$ be a Borel function satisfying the following conditions:

$$f_\varepsilon(y,\cdot) \text{ is positively homogeneous of degree two for a.e. } y \in \mathbb{R}^n, \tag{3.1}$$

$$c_1|\xi|^2 \le f_\varepsilon(y,\xi) \le c_2|\xi|^2 \qquad \text{for a.e. } y \in \mathbb{R}^n, \qquad \text{for every } \xi \in \mathbb{R}^n, \tag{3.2}$$

with $0 < c_1 \le c_2$ independent of ε.

Let $W : \mathbb{R} \to [0,+\infty)$ be a continuous function satisfying (2.1), (2.2). We will consider the functionals $G_\varepsilon : L^1_{\rm loc}(\mathbb{R}^n) \times \mathcal{A} \to [0,+\infty]$ defined by

$$G_\varepsilon(u,A) = \begin{cases} \displaystyle\int_A \Big(\frac{W(u)}{\varepsilon} + \varepsilon f_\varepsilon(x, Du) \Big)\, dx & \text{if } u \in W^{1,\gamma}(A) \\ +\infty & \text{otherwise.} \end{cases} \tag{3.3}$$

Remark 3.1 By (3.2) it follows immediately that

$$\int_A \Big(\frac{W(u)}{\varepsilon} + \varepsilon c_1 |Du|^2 \Big)\, dx \le G_\varepsilon(u,A) \le \int_A \Big(\frac{W(u)}{\varepsilon} + \varepsilon c_2 |Du|^2 \Big)\, dx$$

for each $u \in W^{1,\gamma}(A)$ and hence, if we set

$$G'(u,A) = \Gamma\text{-}\liminf_{\varepsilon\to 0} G_\varepsilon(u,A),$$

$$G''(u,A) = \Gamma\text{-}\limsup_{\varepsilon\to 0} G_\varepsilon(u,A),$$

then by Theorem 2.1 $G'(u,A) = G''(u,A) = +\infty$ whenever $u \notin BV(A;\{0,1\})$. Moreover if $A \in \mathcal{A}$ is a Lipschitz set,

$$c_0\sqrt{c_1}\,\Phi(u,A) \le G'(u,A) \le G''(u,A) \le c_0\sqrt{c_2}\,\Phi(u,A) \tag{3.4}$$

where Φ is defined in (2.3).

The following lemma is crucial in the description of the behaviour of the Γ-limits with respect to the set variable.

Lemma 3.2 *(The fundamental estimate) Let G_ε be defined by* (3.3)*. Then for every $\varepsilon > 0$, for every bounded open sets U, U', V, with $U \subset\subset U'$, and for every u, $v \in L^1_{\rm loc}(\mathbb{R}^n)$ there exists a cut-off function φ related to U and U', that may depend on ε, U, U', V, u, v, such that*

$$G_\varepsilon(\varphi u + (1-\varphi)v, U \cup V) \leq G_\varepsilon(u, U') + G_\varepsilon(v, V) + \delta_\varepsilon(u, v, U, U', V)$$

where $\delta_\varepsilon : L^1_{\rm loc}(\mathbb{R}^n)^2 \times \mathcal{A}^3 \to [0, +\infty[$ are functions depending only on ε and G_ε, such that

$$\lim_{\varepsilon \to 0} \delta_\varepsilon(u_\varepsilon, v_\varepsilon, U, U', V) = 0$$

whenever $U, U', V \in \mathcal{A}$, $U \subset\subset U'$ and u_ε, $v_\varepsilon \in L^1_{\rm loc}(\mathbb{R}^n)$ have the same limit as $\varepsilon \to 0$ in $L^1((U' \setminus \overline{U}) \cap V)$ and satisfy $\sup\limits_{\varepsilon>0}(G_\varepsilon(u_\varepsilon, U') + G_\varepsilon(v_\varepsilon, V)) < +\infty$.

Proof. The proof follows the lines of that contained in the Appendix of [14], with slight modifications. However we include it, since the changes in the notation are heavy.

We fix ε, $U, U', V \in \mathcal{A}$ with $U \subset\subset U'$. Let k_ε denote the integer part of $1/\varepsilon$, let $d = \mathrm{dist}\,(U, \mathbb{R}^n \setminus U')$ and choose $k_\varepsilon + 1$ open sets $U_1, \cdots, U_{k_\varepsilon+1} \in \mathcal{A}$ such that

$$U \subset\subset U_1 \subset\subset \cdots \subset\subset U_{k_\varepsilon+1} \subset\subset U'$$

and

$$\mathrm{dist}\,(U_i, \mathbb{R}^n \setminus U_{i+1}) \geq \frac{d}{k_\varepsilon + 2} \qquad i = 1, 2, \cdots, k_\varepsilon\,.$$

For each $i = 1, \cdots, k_\varepsilon$ let φ_i be a cut-off function between U_i and U_{i+1} such that

$$\max |D\varphi_i| \leq \frac{2(k_\varepsilon + 2)}{d}. \tag{3.5}$$

We have for every $i = 1, \cdots, k_\varepsilon$ that

$$\begin{aligned}
& G_\varepsilon(\varphi_i u + (1-\varphi_i)v, U \cup V) \\
= \;& G_\varepsilon(\varphi_i u + (1-\varphi_i)v, (U \cup V) \cap \overline{U_i}) \\
& + G_\varepsilon(\varphi_i u + (1-\varphi_i)v, (U \cup V) \cap (\mathbb{R}^n \setminus U_{i+1})) \\
& + G_\varepsilon(\varphi_i u + (1-\varphi_i)v, (U \cup V) \cap (U_{i+1} \setminus \overline{U_i})) \\
= \;& G_\varepsilon(u, (U \cup V) \cap \overline{U_i}) + G_\varepsilon(v, V \cap (\mathbb{R}^n \setminus U_{i+1})) \\
& + G_\varepsilon(\varphi_i u + (1-\varphi_i)v, (U_{i+1} \setminus \overline{U_i}) \cap V) \\
\leq \;& G_\varepsilon(u, U') + G_\varepsilon(v, V) + G_\varepsilon(\varphi_i u + (1-\varphi_i)v, (U_{i+1} \setminus \overline{U_i}) \cap V)\,.
\end{aligned} \tag{3.6}$$

We now estimate the last term in (3.6). We denote $S_i = (U_{i+1} \setminus \overline{U_i}) \cap V$; by the growth conditions (3.2) and (3.5) we have that

$$\begin{aligned}
& G_\varepsilon(\varphi_i u + (1-\varphi_i)v, S_i) \\
= & \int_{S_i} \frac{1}{\varepsilon} W(\varphi_i u + (1-\varphi_i)v) + \varepsilon f(\frac{x}{\delta}, D(\varphi_i u + (1-\varphi_i)v))\, dx \\
\leq & \int_{S_i} \frac{1}{\varepsilon} W(\varphi_i u + (1-\varphi_i)v)\, dx + \int_{S_i} \varepsilon\, c_2\, |D(\varphi_i u + (1-\varphi_i)v)|^2\, dx \\
\leq & \int_{S_i} \frac{1}{\varepsilon} W(\varphi_i u + (1-\varphi_i)v)\, dx + \int_{S_i} \varepsilon\, c\, (|Du|^2 + |Dv|^2 + |D\varphi_i|^2 |u-v|^2)\, dx \\
\leq & \int_{S_i} \frac{1}{\varepsilon} W(\varphi_i u + (1-\varphi_i)v)\, dx + \varepsilon\, c \Big(\frac{2(k_\varepsilon+2)}{d}\Big)^2 \int_{S_i} |u-v|^2\, dx \\
& + c\, (G_\varepsilon(u, S_i) + G_\varepsilon(v, S_i)).
\end{aligned}$$

Summing on i we get

$$\begin{aligned}
& \sum_{i=1}^{k_\varepsilon} G_\varepsilon(\varphi_i u + (1-\varphi_i)v, S_i) \\
\leq & \sum_{i=1}^{k_\varepsilon} \int_{S_i} \frac{1}{\varepsilon} W(\varphi_i u + (1-\varphi_i)v)\, dx + \varepsilon\, c \Big(\frac{2(k_\varepsilon+2)}{d}\Big)^2 \int_S |u-v|^2\, dx \\
& + c\, (G_\varepsilon(u, S) + G_\varepsilon(v, S)),
\end{aligned}$$

where $S = (U' \setminus \overline{U}) \cap V$. Then there exists φ_h among $\varphi_1, \cdots, \varphi_{k_\varepsilon}$ such that

$$\begin{aligned}
& G_\varepsilon(\varphi_h u + (1-\varphi_h)v, S_h) \\
\leq & \frac{1}{\varepsilon k_\varepsilon} \Big(\sum_{i=1}^{k_\varepsilon} \int_{S_i} W(\varphi_i u + (1-\varphi_i)v)\, dx \Big) + c \frac{\varepsilon}{k_\varepsilon} \Big(\frac{2(k_\varepsilon+2)}{d}\Big)^2 \int_S |u-v|^2\, dx \\
& + \frac{c}{k_\varepsilon}\, (G_\varepsilon(u, S) + G_\varepsilon(v, S)).
\end{aligned}$$

If we define

$$\begin{aligned}
\delta_\varepsilon(u, v, U, U', V) = & \frac{1}{\varepsilon k_\varepsilon} \Big(\sum_{i=1}^{k_\varepsilon} \int_{S_i} W(\varphi_i u + (1-\varphi_i)v)\, dx \Big) \\
& + c \frac{\varepsilon}{k_\varepsilon} \Big(\frac{2(k_\varepsilon+2)}{d}\Big)^2 \int_S |u-v|^2\, dx \\
& + \frac{c}{k_\varepsilon}\, (G_\varepsilon(u, S) + G_\varepsilon(v, S))
\end{aligned} \tag{3.7}$$

and we choose $\varphi = \varphi_h$ cut-off function between U_h and U_{h+1}, by (3.6) we have that

$$G_\varepsilon(\varphi u + (1-\varphi)v, U \cup V) \leq G_\varepsilon(u, U') + G_\varepsilon(v, V) + \delta_\varepsilon(u, v, U, U', V)\,.$$

Let u_ε and v_ε be two sequences in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ with the same limit in $L^1(S)$ and with $\sup_{\varepsilon>0}(G_\varepsilon(u_\varepsilon,U')+G_\varepsilon(v_\varepsilon,V))\le M$. Under these conditions we can prove that the sequences u_ε and v_ε converge to the same limit also in $L^\gamma(S)$. In fact, let w be the common limit of u_ε and v_ε in $L^1(S)$ and let $r\in\mathbb{R}$ be such that

$$W(z)\ge \frac{c_1}{2}|z|^\gamma \qquad \text{if} \quad |z|>r.$$

We define

$$w^r(x)=-r\vee(r\wedge w(x)) \qquad x\in\mathbb{R}^n$$

and, analogously, u^r_ε and v^r_ε. It can be easily seen that u^r_ε and v^r_ε converge to w^r in $L^\gamma(S)$; moreover

$$\begin{aligned}\int_S |u_\varepsilon(x)-u^r_\varepsilon(x)|^\gamma\,dx &\le \int_{\{x\in S\,:\,|u_\varepsilon|>r\}}|u_\varepsilon(x)|^\gamma\,dx\\ &\le \frac{2}{c_1}\int_S W(u_\varepsilon(x))\,dx\\ &\le \frac{2}{c_1}\,\varepsilon\, G_\varepsilon(u_\varepsilon,S)\le \frac{2M}{c_1}\varepsilon.\end{aligned}$$

Hence, we can conclude that u_ε and v_ε converge to w^r in $L^\gamma(S)$. As they converge to w in $L^1(S)$ we have $w^r\equiv w$. To prove that

$$\lim_{\varepsilon\to0}\delta_\varepsilon(u_\varepsilon,v_\varepsilon,U,U',V)=0$$

it remains to study the convergence to zero of the first term in (3.7) since for the other ones is obvious.

Note that $\frac{1}{\varepsilon k_\varepsilon}$ is bounded; hence, it is sufficient to prove that

$$\lim_{\varepsilon\to0}\sum_{i=1}^{k_\varepsilon}\int_{S_i}W(\varphi_i u_\varepsilon+(1-\varphi_i)v_\varepsilon)\,dx=0\,.$$

We define for $x\in\mathbb{R}^n$

$$w_\varepsilon=\begin{cases}\varphi_i u_\varepsilon+(1-\varphi_i)v_\varepsilon & \text{if } x\in S_i \text{ for some } i=1,\cdots,k_\varepsilon\\[2ex] w(x) & \text{otherwise}\end{cases}$$

which converges to w in $L^\gamma(S)$. Since W is continuous, we have that

$$\begin{aligned}0 &\le \lim_{\varepsilon\to0}\sum_{i=1}^{k_\varepsilon}\int_{S_i}W(\varphi_i u_\varepsilon+(1-\varphi_i)v_\varepsilon)\,dx\\ &= \lim_{\varepsilon\to0}\int_S W(w_\varepsilon(x))\,dx=\lim_{\varepsilon\to0}\int_S W(u_\varepsilon(x))\,dx\\ &\le \lim_{\varepsilon\to0}\varepsilon\, G_\varepsilon(u_\varepsilon,S)=0\end{aligned}$$

which completes the proof. □

Theorem 3.3 *(Compactness by Γ-convergence) For every sequence $(\varepsilon_j)_j$ converging to 0, there exists a subsequence $(\varepsilon_{j_k})_k$ and a functional $G : L^1_{\rm loc}(\mathbb{R}^n) \times \mathcal{A} \to [0, +\infty]$, such that $(G_{\varepsilon_{j_k}})_k$ Γ-converges to G for every U bounded Lipschitz open set, and for every $u \in L^1_{\rm loc}(\mathbb{R}^n)$ such that $u \in BV(U; \{0, 1\})$, with respect to the strong topology of $L^1(U)$. Moreover, for every $u \in BV_{\rm loc}(\mathbb{R}^n; \{0, 1\})$ $G(u, \cdot)$ is the restriction to $\mathcal{A}$ of a regular Borel measure.*

Proof. By a standard compactness argument (see e.g. [9] Section 7.3) we can assume that $(G_{\varepsilon_{j_k}}(\cdot, R))_k$ Γ-converges to a functional $G_0(\cdot, R)$ with respect to the $L^1(R)$ convergence for all R belonging to the class $\mathcal{R}$ of all polyrectangle with rational vertices. If $u \in BV_{\rm loc}(\mathbb{R}^n; \{0, 1\})$ we define $G(u, A)$ on all open sets $A \in \mathcal{A}$ by setting

$$G(u, A) = \sup\{G_0(u, R) : R \subset\subset A, R \in \mathcal{R}\}.$$

For every $A, A' \in \mathcal{A}$ with $A' \subset\subset A$, there exists $R \in \mathcal{R}$ such that $A' \subset\subset R \subset\subset A$; hence, we get

$$\begin{aligned} G(u, A) &= \sup\{G'(u, A') : A' \subset\subset A,\ A' \in \mathcal{A}\} && (3.8)\\ &= \sup\{G''(u, A') : A' \subset\subset A,\ A' \in \mathcal{A}\} && (3.9) \end{aligned}$$

for all $A \in \mathcal{A}$; that is, G is the inner regular envelope of G' and of G''. Hence the set function $G(u, \cdot)$ is inner regular (see [13] Remark 16.3), superadditive (see [13] Proposition 16.12) and by using the fundamental estimate above we can prove that $G(u, \cdot)$ is also subadditive (see [13] Proposition 18.4); hence, by the measure property criterion of De Giorgi and Letta, $G(u, \cdot)$ is the restriction to $\mathcal{A}$ of a regular Borel measure (see [9] Chapter 10). Since by the fundamental estimate $G'(u, \cdot)$, $G''(u, \cdot)$ are themselves inner regular on the class of bounded Lipschitz open sets U (see [9] Propositions 11.5 and 11.6) then by (3.8) we deduce that

$$\begin{aligned} G'(u, U) &= \sup\{G'(u, A') : A' \subset\subset U,\ A' \in \mathcal{A}\}\\ &= G(u, U)\\ &= \sup\{G''(u, A') : A' \subset\subset U,\ A' \in \mathcal{A}\}\\ &= G''(u, U) \end{aligned}$$

for all such sets U; hence, G is the Γ-limit of $(G_{\varepsilon_{j_k}})_k$ for every U bounded Lipschitz open set and for every $u \in L^1_{\rm loc}(\mathbb{R}^n)$. Remark 3.1 completes the proof. □

In the sequel we still denote by G the extension of $G(u, \cdot)$ to the family $\mathcal{B}$ of all Borel subsets of $\mathbb{R}^n$.

Remark 3.4 G is a local functional on $\mathcal{A}$, i.e.,

$$G(u, A) = G(v, A)$$

for every set $A \in \mathcal{A}$ and every $u, v \in BV_{\rm loc}(\mathbb{R}^n; \{0, 1\})$ such that $u = v$ a.e. in A. This follows directly by applying the definition of Γ-convergence, being each

G_ε a local functional too. Moreover, by Remark 3.1 we can deduce the following estimate

$$G(u,U) \le c_0\sqrt{c_2}\Phi(u,U) = c_0\sqrt{c_2}\mathcal{H}^{n-1}(S_u \cap U) \tag{3.10}$$

for every Lipschitz set $U \in \mathcal{A}$ and every $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that $u \in BV(U;\{0,1\})$.

Theorem 3.5 *(Integral representation) There exists a Borel function $\varphi : \mathbb{R}^n \times S^{n-1} \to [0,+\infty[$ such that*

$$c_0\sqrt{c_1} \le \varphi(x,\nu) \le c_0\sqrt{c_2} \qquad \text{for a.e. } x \in \mathbb{R}^n, \nu \in S^{n-1} \tag{3.11}$$

$$G(u,B) = \begin{cases} \displaystyle\int_{S_u\cap B} \varphi(x,\nu_u)\, d\mathcal{H}^{n-1} & \textit{if } u \in BV(U;\{0,1\}) \\ +\infty & \textit{otherwise} \end{cases} \tag{3.12}$$

for every Lipschitz set $U \in \mathcal{A}$ and every Borel set $B \subseteq U$. Moreover φ satisfies the derivation formula

$$\varphi(x,\nu) = \limsup_{\rho\to 0+} \rho^{1-n} \inf\{G(u,\overline{Q^\nu_\rho(x)}) \;:\; u = u^\nu_x \text{ in } \mathbb{R}^n \setminus Q^\nu_\rho(x)\} \tag{3.13}$$

where u^ν_x is the characteristic function of the half-space $\{y \in \mathbb{R}^n : \langle y-x,\nu\rangle > 0\}$.

Proof. It suffices to notice that G as defined in Theorem 3.3 satisfies the hypotheses of Theorem 1.4 of [10] (a direct proof can be also obtained by following that of Lemma 3.5 in [8]). $\square$

Remark 3.6 If φ does not depend on x, then from (3.13)

$$\varphi(\nu) = \inf\{G(u,\overline{Q^\nu}) \;:\; u = u^\nu \text{ in } \mathbb{R}^n \setminus Q^\nu\}$$

where $u^\nu = u^\nu_0$. Moreover the one-homogeneous extension of φ to $\mathbb{R}^n$ is convex (see [2]), and in particular it is continuous. We will use this fact to identify φ by computing it on a dense set in S^{n-1}.

3.1 Boundary conditions

In this section we extend the preceding results to include the case of problems with some types of boundary conditions.

Let $w : \mathbb{R} \to [0,1]$ be such that

$$w(-\infty) := \lim_{t\to-\infty} w(t) = 0, \qquad w(+\infty) := \lim_{t\to+\infty} w(t) = 1$$

and

$$\int_{-\infty}^{+\infty} \Big(W(w) + |w'|^2\Big)\, dt = c < +\infty. \tag{3.14}$$

We define $w_\varepsilon(t) = w(\frac{t}{\varepsilon})$ and $v_\varepsilon(x) = w_\varepsilon(\langle x, \nu\rangle)$. We easily see that

$$v_\varepsilon \to u^\nu\,, \tag{3.15}$$

where $u^\nu = u^\nu_0$ is defined in Theorem 3.5.

With fixed $x \in \mathbb{R}^n$ and $\rho > 0$, we define

$$\widetilde{F}_\varepsilon(u, Q^\nu_\rho(x)) = \begin{cases} F_\varepsilon(u, Q^\nu_\rho(x)) & \text{if } u(y) = v_\varepsilon(y-x) \text{ on } \mathbb{R}^n \setminus Q^\nu_\rho(x), u \in H^1_{\mathrm{loc}}(\mathbb{R}^n) \\ \\ +\infty & \text{otherwise.} \end{cases}$$

Theorem 3.7 *Let F_ε be defined by* (3.3)*, and suppose that* (F_ε) Γ*-converges to F as in Theorem* 3.3 *(upon passing to a subsequence). Let the function φ given by the integral representation Theorem* 3.5 *be independent of x. Then*

$$\Gamma\text{-}\lim_{\varepsilon\to 0} \widetilde{F}_\varepsilon(u, Q^\nu_\rho(x)) = F(u, \overline{Q^\nu_\rho}(x)), \tag{3.16}$$

where we extend u by setting $u(y) = u^\nu(y-x)$ on $\mathbb{R}^n \setminus Q^\nu_\rho(x)$.

Proof. It clearly suffices to prove the theorem with $x = 0$. We begin by proving the Γ-liminf inequality.

Let $\rho_1 > \rho$; we then have

$$\widetilde{F}_\varepsilon(u, Q^\nu_\rho) = F_\varepsilon(u, Q^\nu_{\rho_1}) - F_\varepsilon(v_\varepsilon, Q^\nu_{\rho_1} \setminus Q^\nu_\rho)\,. \tag{3.17}$$

We define

$$Q^\nu_{\rho_1, n-1} = \{x \in Q^\nu_{\rho_1} \,:\, \langle x, \nu\rangle = 0\} \quad Q^\nu_{\rho, n-1} = \{x \in Q^\nu_\rho \,:\, \langle x, \nu\rangle = 0\}$$

and

$$\begin{aligned} A_1 &= Q^\nu_{\rho_1, n-1} \setminus Q^\nu_{\rho, n-1} \times (-\rho_1/2, \rho_1/2) \\ A_2 &= Q^\nu_{\rho, n-1} \times (-\rho_1/2, -\rho/2) \cup (\rho/2, \rho_1/2)\,. \end{aligned}$$

We now compute

$$\begin{aligned} F_\varepsilon(v_\varepsilon, Q^\nu_{\rho_1} \setminus Q^\nu_\rho) &= F_\varepsilon(v_\varepsilon, A_1) + F_\varepsilon(v_\varepsilon, A_2) \\ &\leq \bar{c}\,(\rho_1^{n-1} - \rho^{n-1}) \int_{-\infty}^{+\infty} (W(w) + |w'|^2)\,dt \qquad (3.18) \\ &\quad + \bar{c}\,\rho^{n-1}\Big(\int_{-\infty}^{\frac{-\rho}{2\varepsilon}} (W(w) + |w'|^2)\,dt + \int_{\frac{\rho}{2\varepsilon}}^{+\infty} (W(w) + |w'|^2)\,dt\Big) \end{aligned}$$

where $\bar{c} = \max\{1, c_2\}$. Hence, by (3.14) and (3.18), for every sequence u_ε converging to u such that $u_\varepsilon = v_\varepsilon$ on $\mathbb{R}^n \setminus Q^\nu_\rho$ and $\liminf_{\varepsilon\to 0} \widetilde{F}_\varepsilon(u_\varepsilon, Q^\nu_\rho) < +\infty$ we get that

$$\begin{aligned} \liminf_{\varepsilon\to 0} \widetilde{F}_\varepsilon(u_\varepsilon, Q^\nu_\rho) &\geq \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, Q^\nu_{\rho_1}) - O(\rho_1^{n-1} - \rho^{n-1}) \\ &\geq F(u, Q^\nu_{\rho_1}) - O(\rho_1^{n-1} - \rho^{n-1})\,. \qquad (3.19) \end{aligned}$$

Passing to the limit as ρ_1 tends to ρ we have the liminf inequality

$$F(u, \overline{Q^\nu_\rho}) \leq \liminf_{\varepsilon\to 0} \widetilde{F}_\varepsilon(u_\varepsilon, Q^\nu_\rho)\,. \tag{3.20}$$

We now prove the Γ-limsup inequality. Let $u \in BV(Q^\nu_\rho; \{0,1\})$ be such that $u = u^\nu$ on $\mathbb{R}^n \setminus Q^\nu_\rho$.

a) We first assume that $u = u^\nu$ on $\mathbb{R}^n \setminus Q^\nu_{\rho_1}$ with $\rho_1 < \rho$. Let u_ε be a sequence converging to u such that

$$F(u, Q^\nu_\rho) = \lim_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, Q^\nu_\rho);$$

in particular u_ε converges to u^ν on $\mathbb{R}^n \setminus Q^\nu_{\rho_1}$. Let φ_ε be a cut-off function between $U = Q^\nu_{\frac{\rho+\rho_1}{2}}$ and $U' = Q^\nu_\rho$ and let $V = Q^\nu_\rho \setminus \overline{Q^\nu_{\rho_1}}$; by the Fundamental Estimate

$$\begin{aligned} F_\varepsilon(u_\varepsilon\varphi_\varepsilon + (1-\varphi_\varepsilon)v_\varepsilon, Q^\nu_\rho) \quad &\leq \quad F_\varepsilon(u_\varepsilon, Q^\nu_\rho) + F_\varepsilon(v_\varepsilon, Q^\nu_\rho \setminus \overline{Q^\nu_{\rho_1}}) \\ &\quad\; +\delta_\varepsilon(u_\varepsilon, v_\varepsilon, U, U', V)\,. \end{aligned} \tag{3.21}$$

By the assumptions on u_ε and (3.15) we also have

$$u_\varepsilon \to u^\nu, \qquad\qquad v_\varepsilon \to u^\nu \qquad \text{on } V.$$

Hence we get

$$\lim_{\varepsilon\to 0} \delta_\varepsilon(u_\varepsilon, v_\varepsilon, U, U', V) = 0$$

and by (3.17), (3.18) and (3.21)

$$\Gamma\text{-}\limsup_{\varepsilon\to 0} \widetilde{F}_\varepsilon(u, Q^\nu_\rho) \leq F(u, Q^\nu_\rho)\,.$$

b) In the general case we consider $\rho_1 < \rho$ and we define $u_{\rho_1}(x) = u(\frac{\rho}{\rho_1}x)$. By the previous case (a) and (3.12)

$$\begin{aligned} \Gamma\text{-}\limsup_{\varepsilon\to 0} \widetilde{F}_\varepsilon(u_{\rho_1}, Q^\nu_\rho) \quad &\leq \quad F(u_{\rho_1}, Q^\nu_\rho) \\ &= \quad \int_{Q^\nu_\rho \cap S(u_{\rho_1})} \varphi(\nu_{u_{\rho_1}})\, d\mathcal{H}^{n-1} \\ &\leq \quad \int_{\overline{Q^\nu_\rho} \cap S(u)} \varphi(\nu_u)\, d\mathcal{H}^{n-1} + O(\rho^{n-1} - \rho_1^{n-1}) \\ &= \quad F(u, \overline{Q^\nu_\rho}) + O(\rho^{n-1} - \rho_1^{n-1}). \end{aligned} \tag{3.22}$$

Since u_{ρ_1} converges to u as ρ_1 tends to ρ, if we denote

$$\widetilde{F}''(u_{\rho_1}, Q^\nu_\rho) = \Gamma\text{-}\limsup_{\varepsilon\to 0} \widetilde{F}_\varepsilon(u_{\rho_1}, Q^\nu_\rho)$$

then by the lower semicontinuity of the Γ-upper limit (see e.g. [9] Remark 7.8) and (3.22)

$$\Gamma\text{-}\limsup_{\varepsilon\to 0} \widetilde{F}_\varepsilon(u, Q^\nu_\rho) \le \liminf_{\rho_1\to\rho} \widetilde{F}''(u_{\rho_1}, Q^\nu_\rho) \le F(u, \overline{Q^\nu_\rho})\,. \tag{3.23}$$

Hence by (3.23) and (3.20) we get the required equality (3.16). □

Corollary 3.8 Let the function φ given by the integral representation Theorem 3.5 be independent of x. Then

$$\begin{aligned}\varphi(\nu) &= \min\{F(u, \overline{Q^\nu})\ u = u^\nu \ \text{ on } \mathbb{R}^n \setminus Q^\nu\}\\ &= \lim_{j\to+\infty} \min\{\widetilde{F}_{\varepsilon_j}(u, Q^\nu)\ u = v_{\varepsilon_j} \ \text{ on } \partial Q^\nu\}\,.\end{aligned} \tag{3.24}$$

Proof. The first equality follows from Remark 3.6, while the convergence of minima comes from the Γ-convergence of $\widetilde{F}_{\varepsilon_j}$ and the fact that we may find a sequence of minimizers which is compact in $L^1(Q^\nu)$. This can be proved by following [23] Proposition 3, upon noticing that we may assume that minimizers take values in $[0, 1]$ by a truncation argument. □

Remark 3.9 We want to show by a simple example that if φ explicitly depends on x then Theorem 3.7 is not true. Consider

$$F_\varepsilon(u, U) = \int_U \Big(\frac{1}{\varepsilon}\, W(u) + \varepsilon\, a(x)\, |Du|^2\Big)\, dx$$

where

$$a(x) = \begin{cases} 1 & \text{if } x \in Q\\ \frac{1}{4} & \text{otherwise.}\end{cases}$$

It can be easily checked that

$$F(u, U) = \Gamma\text{-}\lim_{\varepsilon\to 0} F_\varepsilon(u, U) = c_0 \int_{S(u)\cap U} \sqrt{a(x)}\, d\mathcal{H}^{n-1}\,. \tag{3.25}$$

Now we want to show that there exists u such that

$$\Gamma\text{-}\liminf_{\varepsilon\to 0} \widetilde{F}_\varepsilon(u, Q) > F(u, \overline{Q})\,. \tag{3.26}$$

Such u can be chosen as

$$u = \begin{cases} 1 & \text{if } x_n > \frac{1}{2}\\ 0 & \text{otherwise.}\end{cases}$$

In fact, let u_ε be a sequence converging to u such that $u_\varepsilon = v_\varepsilon$ on ∂Q; then

$$\begin{aligned}&\widetilde{F}_\varepsilon(u_\varepsilon, Q) = \int_Q \Big(\frac{1}{\varepsilon} W(u_\varepsilon) + \varepsilon |Du_\varepsilon|^2\Big)\, dx\\ = \ &\int_{(1+\eta)Q} \Big(\frac{1}{\varepsilon} W(u_\varepsilon) + \varepsilon |Du_\varepsilon|^2\Big)\, dx - \int_{(1+\eta)Q\setminus Q} \Big(\frac{1}{\varepsilon} W(v_\varepsilon) + \varepsilon |Dv_\varepsilon|^2\Big)\, dx\end{aligned}$$

and

$$\liminf_{\varepsilon\to 0} \widetilde{F}_\varepsilon(u_\varepsilon, Q) \geq c_0\, \mathcal{H}^{n-1}(S(u)\cap(1+\eta)Q) - c\,((1+\eta)^{n-1}-1).$$

Passing to the limit as η tends to 0, we get

$$\begin{aligned}\liminf_{\varepsilon\to 0} \widetilde{F}_\varepsilon(u_\varepsilon, Q) &\geq c_0\, \mathcal{H}^{n-1}(S(u)\cap\overline{Q}) \\ &> c_0\, \mathcal{H}^{n-1}(S(u)\cap Q) + \frac{c_0}{2}\, \mathcal{H}^{n-1}(S(u)\cap\partial Q).\end{aligned}$$

By (3.25) we get the required inequality (3.26).

4 Homogenization

In this section we treat the case of highly-oscillating coefficients.

Let $f : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty)$ be a Borel function satisfying the conditions: there exist $0 < c_1 \leq c_2$ such that

$$c_1|\xi|^2 \leq f(y,\xi) \leq c_2|\xi|^2; \tag{4.1}$$

$$\text{for all } y \in \mathbb{R}^n \ f(y,\cdot) \text{ is positively homogeneous of degree two;} \tag{4.2}$$

$$\text{for all } \xi \in \mathbb{R}^n \ f(\cdot,\xi) \text{ is one-periodic,} \tag{4.3}$$

i.e., $f(x+e_i,\xi) = f(x,\xi)$ for all $x \in \mathbb{R}^n$, and $i = 1,\ldots,n$.

Let $\delta : (0,+\infty) \to (0,+\infty)$, and let W be as in Section 2. For all $\varepsilon > 0$ we consider the functional $F_\varepsilon : L^1_{\rm loc}(\mathbb{R}^n) \times \mathcal{A} \to [0,+\infty]$ defined by

$$F_\varepsilon(u,A) = \begin{cases} \displaystyle\int_A \Big(\frac{W(u)}{\varepsilon} + \varepsilon f\Big(\frac{x}{\delta(\varepsilon)}, Du\Big)\Big)\, dx & \text{if } u \in W^{1,\gamma}(A) \\ +\infty & \text{otherwise.}\end{cases} \tag{4.4}$$

With fixed a sequence (ε_j) of positive numbers converging to 0, by applying Theorem 3.3 with

$$f_\varepsilon(x,\xi) = f\Big(\frac{x}{\delta(\varepsilon)}, \xi\Big)$$

we conclude that, upon extracting a subsequence (not relabeled), the functionals F_{ε_j} Γ-converge on all Lipschitz bounded open subsets of $\mathbb{R}^n$. Their limit F can be represented as an integral by Theorem 3.5 with an energy density φ given by formula (3.13). In this section we will characterize this function φ and hence also F. We begin by remarking that φ is independent of x.

Proposition 4.1 *Let*

$$\lim_{\varepsilon\to 0} \delta(\varepsilon) = 0.$$

Then the function φ is independent of x.

Proof. To show that φ is independent of x we show that $\varphi(x,\nu)=\varphi(y,\nu)$ for all $x,y\in\mathbb{R}^n$; this will be done by using formula (3.13).

Fix $x,y\in\mathbb{R}^n$, $\rho>0$ and let $u^{x,\rho}$ be a minimizer of the problem

$$\inf\{F(u,\overline{Q^\nu_\rho(x)})\ :\ u=u^\nu_x \text{ in } \mathbb{R}^n\setminus Q^\nu_\rho(x)\}.$$

By Theorem 3.7 there exists a sequence $u^{x,\rho}_j$ converging to $u^{x,\rho}$ with $u^{x,\rho}_j(z)=v_{\varepsilon_j}(z-x)$ on $\mathbb{R}^n\setminus Q^\nu_\rho(x)$, such that

$$\lim_j F_{\varepsilon_j}(u^{x,\rho}_j,Q^\nu_\rho(x))=F(u^{x,\rho},\overline{Q^\nu_\rho(x)}).$$

We define $\tau_j\in\mathbb{Z}^n$ by

$$(\tau_j)_i=\Big[\frac{y_i-x_i}{\varepsilon_j}\Big],$$

and $u^{y,\rho}_j(z)=u^{x,\rho}_j(z-\varepsilon_j\tau_j)$. Note that $\lim_j\varepsilon_j\tau_j=y-x$, $u^{y,\rho}_j$ converges to $u^{y,\rho}$ given by $u^{y,\rho}(z)=u^{x,\rho}(z-y+x)$, and

$$u^{y,\rho}_j(z)=w_j(z)\qquad \text{on } \mathbb{R}^n\setminus\Big(\varepsilon_j\tau_j+Q^\nu_\rho(x)\Big),$$

where

$$w_j(z)=v_{\varepsilon_j}(z-x-\varepsilon_j\tau_j).$$

By plugging $u^{y,\rho}_j$ into F_{ε_j} we get

$$F_{\varepsilon_j}(u^{y,\rho}_j,\varepsilon_j\tau_j+Q^\nu_\rho(x))=F_{\varepsilon_j}(u^{x,\rho}_j,Q^\nu_\rho(x)),$$

so that, for fixed $r>1$ we get

$$\begin{aligned}
&F(u^{y,\rho},\overline{Q^\nu_\rho(y)})\\
\le\ & F(u^{y,\rho},Q^\nu_{r\rho}(y))\le\liminf_j F_{\varepsilon_j}(u^{y,\rho}_j,Q^\nu_{r\rho}(y))\\
=\ & \liminf_j\Big(F_{\varepsilon_j}(u^{y,\rho}_j,\varepsilon_j\tau_j+Q^\nu_\rho(x))+F_{\varepsilon_j}(u^{y,\rho}_j,Q^\nu_{r\rho}(y)\setminus(\varepsilon_j\tau_j+Q^\nu_\rho(x)))\Big)\\
=\ & \liminf_j\Big(F_{\varepsilon_j}(u^{x,\rho}_j,Q^\nu_\rho(x))+F_{\varepsilon_j}(w_j,Q^\nu_{r\rho}(y)\setminus(\varepsilon_j\tau_j+Q^\nu_\rho(x)))\Big)\\
=\ & \lim_j F_{\varepsilon_j}(u^{x,\rho}_j,Q^\nu_\rho(x))+\lim_j F_{\varepsilon_j}(w_j,Q^\nu_{r\rho}(y)\setminus(\varepsilon_j\tau_j+Q^\nu_\rho(x)))\\
\le\ & F(u^{x,\rho},\overline{Q^\nu_\rho(x)})+c\rho^{n-1}(r^{n-1}-1),
\end{aligned}$$

with

$$c=\int_{-\infty}^{+\infty}\Big(W(w)+|w'|^2\Big)\,dt.$$

By the arbitrariness of $r>1$ we get

$$F(u^{y,\rho},\overline{Q^\nu_\rho(y)})\le F(u^{x,\rho},\overline{Q^\nu_\rho(x)})$$

and, by symmetry, the equality, so that $\varphi(x,\nu)=\varphi(y,\nu)$ by letting $\rho\to0$, by formula (3.13). $\square$

Remark 4.2 The formula

$$\varphi(\nu) = \lim_{j\to+\infty} \min\Big\{\varepsilon_j^{n-1}\int_{\frac{1}{\varepsilon_j}Q^\nu}\Big(W(u)+f\Big(\frac{\varepsilon_j}{\delta(\varepsilon_j)}x, Du\Big)\Big)\,dx \;:\; u = v^\nu \text{ on } \partial\Big(\frac{1}{\varepsilon_j}Q^\nu\Big)\Big\} \tag{4.5}$$

holds, where $v^\nu(x) = w(\langle x,\nu\rangle)$. To check this it suffices to use the previous proposition and Corollary 3.8.

4.1 Oscillations on the scale of the transition layer

In this section we treat the case when the scale of oscillation δ and the scale of the transition layer ε are comparable.

Theorem 4.3 *Let F_ε be defined by* (4.4)*. Let $f(x,\xi)$ be a Borel function* 1*-periodic in x, positively homogeneous of degree two in ξ and satisfying the growth conditions* (4.1)*, and let $W(z)$ be a continuous function satisfying the conditions* (2.1) *and* (2.2)*. Let $\delta:(0,+\infty)\to(0,+\infty)$ be such that*

$$\lim_{\varepsilon\to 0}\frac{\varepsilon}{\delta(\varepsilon)} = c,$$

where c is a positive constant; then there exists the Γ-limit

$$\Gamma\text{-}\lim_{\varepsilon\to 0} F_\varepsilon(u,U) = \int_{S(u)\cap U}\varphi(\nu_u)\,d\mathcal{H}^{n-1}$$

for every $u\in BV(U;\{0,1\})$, where

$$\varphi(\nu) = \lim_{T\to+\infty}\frac{1}{T^{n-1}}\inf\Big\{\int_{TQ^\nu}\Big(W(u)+f(cx,Du)\Big)\,dx \;:\; u = v^\nu \text{ on } \partial(TQ^\nu)\Big\}. \tag{4.6}$$

Proof. First, we prove the theorem when $\delta(\varepsilon)=\varepsilon$.

Step 1 It is sufficient to prove the formula for a dense set Ξ of ν. In fact since φ is convex it is also continuous hence if the formula is true for every $\nu\in\Xi$ then φ is independent from ε_j for every $\nu\in\Xi$. By the continuity of φ, it is also independent from ε_j for every ν. Hence, we can conclude that there exists the Γ-limit of $F_{\varepsilon,\delta(\varepsilon)}$ and by the convergence of minima the formula is true for every ν.

Step 2 Let Ξ be the set of unit rational vectors, i.e.; $\Xi = \{\nu\in S^{n-1} : \exists\lambda\in\mathbb{R}, \lambda\nu\in\mathbb{Q}^n\}$. In can be easily seen that Ξ is dense in S^{n-1}. Now, for simplicity of notation, we develop the proof only in the case $\nu = e_n$, but the same arguments clearly work for any $\nu \in \Xi$, up to a change of variables and of the periodicity cell. We define for $T > 0$, $TQ = (-\frac{T}{2}, \frac{T}{2})^n$ and

$$g(T) = \inf\Big\{ \frac{1}{T^{n-1}} \int_{TQ} (W(u) + f(x, Du))\, dx \ : \ u = w(x_n) \text{ on } \partial TQ \Big\};$$

we have to prove that there exists the limit as T tends to $+\infty$. Let u_T be such that

$$\int_{TQ} (W(u_T) + f(x, Du_T))\, dx \leq T^{n-1} g(T) + 1\,.$$

Let $S > T$; we define $Q_{Tz} = z([T]+1) + TQ$ for $z \in \mathbb{Z}^{n-1} \times \{0\}$ and $I_S = \{z \in \mathbb{Z}^{n-1} \times \{0\} \ : \ Q_{Tz} \subseteq SQ\}$. We can construct

$$u_S(x) = \begin{cases} u_T(x - z([T]+1)) & \text{if } x \in Q_{Tz},\ z \in I_S \\ \\ w(x_n) & \text{otherwise}\ . \end{cases}$$

We can proceed as in the proof of [9] Proposition 14.4: plugging u_S into the definition of $g(S)$, we obtain the inequality

$$g(S) \leq g(T) + r(S, T),$$

with

$$\limsup_{T\to+\infty} \limsup_{S\to+\infty} r(S, T) = 0,$$

so that

$$\limsup_{S\to+\infty} g(S) \leq \liminf_{T\to+\infty} g(T).$$

Hence, we conclude the proof of the case $\delta(\varepsilon) = \varepsilon$.

If $\varepsilon = \delta c$ by a change of variables we can apply the previous case.

Finally if $\lim_{\varepsilon\to 0} \frac{\varepsilon}{\delta(\varepsilon)} = c$ then, by the change of variables $\frac{\varepsilon}{\delta(\varepsilon)} x = cy$, it can be easily checked that

$$\begin{aligned} &\varepsilon^{n-1} \int_{\frac{1}{\varepsilon}Q^\nu} \Big(W(u) + f\Big(\frac{\varepsilon}{\delta} x, Du\Big)\Big)\, dx \\ = \quad & \frac{1}{\varepsilon T} \frac{1}{T^{n-1}} \int_{TQ^\nu} \Big(W\Big(\Big(\frac{c\delta(\varepsilon)}{\varepsilon} y\Big)\Big) + \frac{1}{(\varepsilon T)^2} f\Big(cy, Du\Big(\frac{c\delta(\varepsilon)}{\varepsilon} y\Big)\Big)\, dy \quad (4.7) \end{aligned}$$

where $T = 1/\delta c$ and $\lim_{\varepsilon\to 0} 1/\varepsilon T = 1$. Hence, for every $\eta > 0$ there exists $\varepsilon_0 > 0$ such that for every $\varepsilon < \varepsilon_0$

$$\begin{aligned} &(1-\eta) \inf\Big\{ \frac{1}{T^{n-1}} \int_{TQ^\nu} \Big(W(u) + (1-\eta)^2 f\Big(cy, Du\Big)\Big)\, dy \ : \ u = v^\nu \text{ on } \partial TQ^\nu \Big\} \\ &\leq \inf\Big\{ \frac{1}{\varepsilon T} \frac{1}{T^{n-1}} \int_{TQ^\nu} \Big(W(u) + \frac{1}{(\varepsilon T)^2} f\Big(cy, Du\Big)\Big)\, dy \ : \ u = v^\nu \text{ on } \partial TQ^\nu \Big\} \quad (4.8) \\ &\leq (1+\eta) \inf\Big\{ \frac{1}{T^{n-1}} \int_{TQ^\nu} \Big(W(u) + (1+\eta)^2 f\Big(cy, Du\Big)\Big)\, dy \ : \ u = v^\nu \text{ on } \partial TQ^\nu \Big\}. \end{aligned}$$

By the previous case $\varepsilon = \delta c$, we can conclude that for every sequence ε_j converging to 0 there exists the limit

$$
\begin{aligned}
&\lim_{j\to+\infty} \inf\Big\{\varepsilon_j^{n-1}\int_{\frac{1}{\varepsilon_j}Q^\nu}\Big(W(u)+f\Big(\frac{\varepsilon_j}{\delta(\varepsilon_j)}x,Du\Big)\,dx \;:\; u=v^\nu \text{ on } \partial\Big(\frac{1}{\varepsilon_j}Q^\nu\Big)\Big)\Big\}\\
=&\lim_{T\to+\infty} \inf\Big\{\frac{1}{T^{n-1}}\int_{TQ^\nu}\Big(W(u)+f\Big(cy,Du\Big)\Big)\,dy \;:\; u=v^\nu \text{ on } \partial TQ^\nu\Big\}. \quad (4.9)
\end{aligned}
$$

Hence, by Remark 4.2, φ is independent from ε_j for every ν and satisfies formula (4.6). □

4.2 Oscillations on a larger scale than the transition layer

In this section we treat the case when the scale of oscillation δ is much larger that the scale of the transition layer ε.

Theorem 4.4 *Let F_ε be defined by (4.4). Let $f(x,\xi)$ be a continuous function 1-periodic in x, positively homogeneous of degree two and locally Lipschitz in ξ, satisfying the growth conditions (4.1), and let $W(z)$ be a continuous function satisfying conditions (2.1) and (2.2). Let $\delta : (0,+\infty) \to (0,+\infty)$ be such that*

$$
\lim_{\varepsilon\to 0}\delta(\varepsilon)=0 \qquad \lim_{\varepsilon\to 0}\frac{\varepsilon}{\delta(\varepsilon)}=0;
$$

then there exists the Γ-limit

$$
\Gamma\text{-}\lim_{\varepsilon\to 0} F_\varepsilon(u,U)=\int_{S(u)\cap U}\varphi(\nu_u)\,d\mathcal{H}^{n-1}
$$

for every Lipschitz set $U\in\mathcal{A}$ and every $u\in BV(U;\{0,1\})$, where

$$
\begin{aligned}
\varphi(\nu) \;=\; c_0 \inf_{T>0}\inf\Big\{&\frac{1}{T^{n-1}}\int_{\overline{TQ^\nu}\cap S(u)}\sqrt{f(x,\nu_u)}\,d\mathcal{H}^{n-1} \;:\\
&u\in BV(\Omega;\{0,1\})\; u=u^\nu \text{ on } \mathbb{R}^n\setminus TQ^\nu\Big\}.
\end{aligned}
$$

Proof. We recall that

$$
\begin{aligned}
c_0 &= 2\int_0^1\sqrt{W(z)}\,dz\\
&= \min\Big\{\int_{-\infty}^{+\infty}(W(v)+|v'|^2)\,dt \;:\; v(-\infty)=0,\; v(+\infty)=1\Big\} \quad (4.10)
\end{aligned}
$$

(see e.g. [1], [5]) and we denote

$$
\begin{aligned}
\psi_{\rm hom}(\nu) \;=\; \inf_{T>0}\inf\Big\{&\frac{1}{T^{n-1}}\int_{\overline{TQ^\nu}\cap S(u)}\psi(x,\nu_u)\,d\mathcal{H}^{n-1} \;:\\
&u\in BV(\Omega;\{0,1\})\; u=u^\nu \text{ on } \mathbb{R}^n\setminus TQ^\nu\Big\}, \quad (4.11)
\end{aligned}
$$

where $\psi(x,\xi)=\sqrt{f(x,\xi)}$.

Step 1 Γ-*liminf inequality* Let $u\in BV(\Omega;\{0,1\})$ and let u_ε be a sequence converging to u in $L^1(\Omega)$. We can always assume that $u_\varepsilon\in H^1(\Omega,[0,1])$. With fixed $N\in\mathbb{N}$, we divide $[0,1]$ in intervals of length $1/N$. If we define $I_k=\{x\in\Omega\ :\ \frac{k-1}{N}\le u_\varepsilon\le\frac{k}{N}\}$ and $u_\varepsilon^k=(u_\varepsilon\vee\frac{k-1}{N})\wedge\frac{k}{N}$ for $k=1,\cdots,N$, then u_ε^k converges to $u^k=(u\vee\frac{k-1}{N})\wedge\frac{k}{N}=\frac{k-1}{N}+\frac{u}{N}$ and

$$\begin{aligned}F_{\varepsilon,\delta(\varepsilon)}(u_\varepsilon,\Omega)&\ge 2\int_\Omega\sqrt{W(u_\varepsilon)\,f\Big(\frac{x}{\delta(\varepsilon)},Du_\varepsilon\Big)}\,dx\\&=2\sum_{k=1}^N\int_{I_k}\sqrt{W(u_\varepsilon)\,f\Big(\frac{x}{\delta(\varepsilon)},Du_\varepsilon\Big)}\,dx\\&=2\sum_{k=1}^N\int_\Omega\sqrt{W(u^k_\varepsilon)\,f\Big(\frac{x}{\delta(\varepsilon)},Du^k_\varepsilon\Big)}\,dx\\&\ge 2\sum_{k=1}^N\min_{z\in[\frac{k-1}{N},\frac{k}{N}]}\sqrt{W(z)}\int_\Omega\sqrt{f\Big(\frac{x}{\delta(\varepsilon)},Du^k_\varepsilon\Big)}\,dx\,.\end{aligned}\tag{4.12}$$

By [7] Theorem 5.1, we have that the Γ-limit as $\eta\to0$ of the functionals

$$u\mapsto\int_\Omega\sqrt{f\Big(\frac{x}{\eta},Du\Big)}\,dx$$

takes the value

$$\int_{S(u)\cap\Omega}\psi_{\rm hom}((u^+-u^-)\nu_u)\,d\mathcal{H}^{n-1}$$

if $u=u^k$. Then, since $\psi_{\rm hom}$ is a positively one-homogeneous function, we get that

$$\begin{aligned}\liminf_{\varepsilon\to0}\int_\Omega\sqrt{f\Big(\frac{x}{\delta(\varepsilon)},Du^k_\varepsilon\Big)}\,dx&\ge\int_{S(u^k)\cap\Omega}\psi_{\rm hom}((u^{k+}-u^{k-})\nu_{u^k})\,d\mathcal{H}^{n-1}\\&=\frac{1}{N}\int_{S(u)\cap\Omega}\psi_{\rm hom}(\nu_u)\,d\mathcal{H}^{n-1}\end{aligned}\tag{4.13}$$

so that

$$\liminf_{\varepsilon\to0}F_{\varepsilon,\delta(\varepsilon)}(u_\varepsilon,\Omega)\ge\sum_{k=1}^N\frac{2}{N}\min_{z\in[\frac{k-1}{N},\frac{k}{N}]}\sqrt{W(z)}\int_{S(u)\cap\Omega}\psi_{\rm hom}(\nu_u)\,d\mathcal{H}^{n-1}$$

and passing to the limit as N tends to $+\infty$, we get

$$\liminf_{\varepsilon\to0}F_{\varepsilon,\delta(\varepsilon)}(u_\varepsilon,\Omega)\ge2\int_0^1\sqrt{W(z)}\,dz\int_{S(u)\cap\Omega}\psi_{\rm hom}(\nu_u)\,d\mathcal{H}^{n-1}$$

(we have used the Riemann integrability of $\sqrt{W}$).

Step 2 Γ-*limsup inequality* We can consider the case $\nu = e_n$. By (4.11), if we fix $\eta > 0$ there exist $k > 0$ and $\bar u \in BV(kQ;\{0,1\})$ such that $\bar u = u^{e_n}$ on $\mathbb{R}^n \setminus kQ$ and

$$\frac{1}{k^{n-1}} \int_{S(\bar u)\cap \overline{kQ}} \psi(x,\nu_{\bar u})\, d\mathcal{H}^{n-1} \le \psi_{\rm hom}(e_n) + \eta\,. \tag{4.14}$$

We extend by periodicity $\bar u$ so that it is k-periodic in $(x_1,\cdots,x_{n-1})$ and $\bar u = u^{e_n}$ when $|x_n| > k/2$.

Let v such that

$$\begin{aligned} &\int_{-\infty}^{+\infty} (W(v) + |v'|^2)\, dt \\ = \quad &\min\Big\{\int_{-\infty}^{+\infty} (W(v) + |v'|^2)\, dt \ :\ v(-\infty) = 0,\ v(+\infty) = 1\Big\} \end{aligned}$$

if we define

$$v^\eta = 0 \vee \Big(((1+2\eta)v - \eta) \wedge 1\Big)$$

then there exists R such that $v^\eta(t) \in \{0,1\}$ if $|t| > R$, and

$$\int_{-\infty}^{+\infty} (W(v^\eta) + |Dv^\eta|^2)\, dt \to c_0 \qquad\qquad \text{as } \eta \to 0\,. \tag{4.15}$$

We can always assume that $\bar u$ is such that $S(\bar u)$ is of class C^2; hence, for $\alpha > 0$ small enough there exists a unique projection of class C^2

$$p : \{x \in \Omega\ :\ {\rm dist}(x, S(\bar u)) < \alpha\} \to S(\bar u)\,.$$

We set

$$\bar\nu(x) = \begin{cases} \nu(p(x)) & \text{if } {\rm dist}(x,S(\bar u)) < \alpha \\ \\ e_n & \text{otherwise} \end{cases}$$

and

$$d(x) = \begin{cases} {\rm dist}(x,\{u=0\}) & \text{if } u(x) = 1 \\ \\ -{\rm dist}(x,\{u=1\}) & \text{if } u(x) = 0\ . \end{cases}$$

We define

$$\bar u_{\varepsilon,\delta}(x) = v^\eta\Big(\frac{\delta d(x)}{\varepsilon\psi(x,\bar\nu(x))}\Big)$$

and

$$u_\varepsilon(x) = \bar u_{\varepsilon,\delta}\Big(\frac{x}{\delta}\Big) = v^\eta(d_\varepsilon(x))$$

where

$$d_\varepsilon(x) = \frac{\delta d(\frac{x}{\delta})}{\varepsilon\psi(\frac{x}{\delta},\bar\nu(\frac{x}{\delta}))}\,.$$

Hence

$$Du_\varepsilon(x) = Dv^\eta\Big(\frac{\delta Dd(\frac{x}{\delta})}{\varepsilon\psi(\frac{x}{\delta},\bar\nu(\frac{x}{\delta}))} - \frac{\delta d(\frac{x}{\delta})D\psi(\frac{x}{\delta},\bar\nu(\frac{x}{\delta}))}{\varepsilon\psi(\frac{x}{\delta},\bar\nu(\frac{x}{\delta}))^2}\Big)$$

but $Dv^\eta \neq 0$ on $D = \{x\in\Omega \;:\; |d(\frac{x}{\delta})| \le R\frac{\varepsilon}{\delta}\sqrt{c_1}\}$ and $D(\delta d(\frac{x}{\delta})) = \bar\nu(\frac{x}{\delta})$, so that

$$\begin{aligned}
& F_{\varepsilon,\delta(\varepsilon)}(u_\varepsilon, Q) \\
= \;& \int_{D\cap Q}\Bigg(\frac{1}{\varepsilon}W(u_\varepsilon) + \varepsilon f\Big(\frac{x}{\delta}, Dv^\eta\Big(\frac{\bar\nu(\frac{x}{\delta})}{\varepsilon\psi(\frac{x}{\delta},\bar\nu(\frac{x}{\delta}))} - \frac{\delta d(\frac{x}{\delta})D\psi(\frac{x}{\delta},\bar\nu(\frac{x}{\delta}))}{\varepsilon\psi(\frac{x}{\delta},\bar\nu(\frac{x}{\delta}))^2}\Big)\Bigg)\,dx \\
= \;& \frac{1}{\varepsilon}\int_{D\cap Q}\Bigg(W(u_\varepsilon) + \Big(\frac{Dv^\eta}{\psi(\frac{x}{\delta},\bar\nu(\frac{x}{\delta}))} - \frac{\delta d(\frac{x}{\delta})D\psi(\frac{x}{\delta},\bar\nu(\frac{x}{\delta}))}{\bar\nu(\frac{x}{\delta})\psi(\frac{x}{\delta},\bar\nu(\frac{x}{\delta}))^2}\Big)^2 f\Big(\frac{x}{\delta},\bar\nu\Big(\frac{x}{\delta}\Big)\Big)\Bigg)\,dx \\
= \;& \frac{1}{\varepsilon}\int_{D\cap Q}\Bigg(W(u_\varepsilon) + \Big(Dv^\eta - \frac{\delta d(\frac{x}{\delta})D\psi(\frac{x}{\delta},\bar\nu(\frac{x}{\delta}))}{\bar\nu(\frac{x}{\delta})\psi(\frac{x}{\delta},\bar\nu(\frac{x}{\delta}))}\Big)^2\Bigg)\,dx\,.
\end{aligned} \tag{4.16}$$

If we set $x = y + t\nu(y)$ with $t = \delta d(\frac{x}{\delta})$ and $y\in S(\bar u)$, then $\bar\nu(\frac{x}{\delta}) = \nu(\frac{y}{\delta})$ and by (4.16) and the coarea formula, using the fact that $|D(\delta d(\frac{x}{\delta}))| = 1$, we get

$$\begin{aligned}
& F_{\varepsilon,\delta(\varepsilon)}(u_\varepsilon, Q) \\
\le \;& \frac{1}{\varepsilon}\int_{-\varepsilon R\sqrt{c_1}}^{\varepsilon R\sqrt{c_1}}\int_{S(\bar u)\cap Q}\Bigg(W\Big(v^\eta\Big(\frac{t}{\varepsilon\psi\Big(\frac{y+t\nu(y)}{\delta},\nu(\frac{y}{\delta})\Big)}\Big)\Big) \\
& +\Bigg|Dv^\eta\Big(\frac{t}{\varepsilon\psi\Big(\frac{y+t\nu(y)}{\delta},\nu(\frac{y}{\delta})\Big)}\Big) - \frac{tD\psi\Big(\frac{y+t\nu(y)}{\delta},\nu(\frac{y}{\delta})\Big)}{\nu(\frac{y}{\delta})\psi\Big(\frac{y+t\nu(y)}{\delta},\nu(\frac{y}{\delta})\Big)}\Bigg|^2\Bigg)d\mathcal{H}^{n-1}(y)\,dt \\
= \;& \int_{-R\sqrt{c_1}}^{R\sqrt{c_1}}\int_{S(\bar u)\cap Q}\Bigg(W\Big(v^\eta\Big(\frac{s}{\psi\Big(\frac{y+\varepsilon s\nu(y)}{\delta},\nu(\frac{y}{\delta})\Big)}\Big)\Big) \qquad\qquad (4.17)\\
& +\Bigg|Dv^\eta\Big(\frac{s}{\psi\Big(\frac{y+\varepsilon s\nu(y)}{\delta},\nu(\frac{y}{\delta})\Big)}\Big) - \frac{\varepsilon s}{\nu(\frac{y}{\delta})}\frac{D\psi\Big(\frac{y+\varepsilon s\nu(y)}{\delta},\nu(\frac{y}{\delta})\Big)}{\psi\Big(\frac{y+t\nu(y)}{\delta},\nu(\frac{y}{\delta})\Big)}\Bigg|^2\Bigg)d\mathcal{H}^{n-1}(y)\,ds\,.
\end{aligned}$$

Since ψ is a Lipschitz function, by (4.17) we get

$$\begin{aligned}
& F_{\varepsilon,\delta(\varepsilon)}(u_\varepsilon, Q) \\
\le \;& \int_{-R\sqrt{c_1}}^{R\sqrt{c_1}}\int_{S(\bar u)\cap Q}\Bigg(W\Big(v^\eta\Big(\frac{s}{\psi}\Big)\Big) + \Big|Dv^\eta\Big(\frac{s}{\psi}\Big)\Big|^2 + (\varepsilon R)^2 c_1\Big|\frac{D\psi}{\psi}\Big|^2\Bigg)d\mathcal{H}^{n-1}(y)\,ds \\
& +\varepsilon c_2 \\
\le \;& \int_{-R\sqrt{c}_1}^{R\sqrt{c}_1}\int_{S(\bar u)\cap Q}\Bigg(W\Big(v^\eta\Big(\frac{s}{\psi}\Big)\Big) + \Big|Dv^\eta\Big(\frac{s}{\psi}\Big)\Big|^2\Bigg)d\mathcal{H}^{n-1}(y)\,ds + \varepsilon\tilde c\,.
\end{aligned} \tag{4.18}$$

By the change of variable

$$t = \frac{s}{\psi\Big(\frac{y+\varepsilon s\nu(y)}{\delta}, \nu(\frac{y}{\delta})\Big)}$$

we obtain

$$\begin{aligned}
& F_{\varepsilon,\delta(\varepsilon)}(u_\varepsilon, Q) \\
\leq \; & \int_{S(\bar{u})\cap Q} \int_{-\infty}^{+\infty} \Big(W(v^\eta(t)) + |Dv^\eta(t)|^2\Big)\, dt\, \frac{\psi\Big(\frac{y}{\delta}, \nu(\frac{y}{\delta})\Big) + O(\frac{\varepsilon}{\delta})}{1+O(\frac{\varepsilon}{\delta})} d\mathcal{H}^{n-1}(y) + \varepsilon\tilde{c} \\
= \; & \int_{-\infty}^{+\infty} \Big(W(v^\eta(t)) + |Dv^\eta(t)|^2\Big)\, dt\, \Big(\frac{\delta^{n-1}}{1+O(\frac{\varepsilon}{\delta})} \int_{S(\bar{u})\cap\frac{1}{\delta}Q} \psi(x,\nu(x))\, d\mathcal{H}^{n-1}(x) \\
& + \frac{O(\frac{\varepsilon}{\delta})}{1+O(\frac{\varepsilon}{\delta})}\, \delta^{n-1}\, \mathcal{H}^{n-1}(S(\bar{u}) \cap \frac{1}{\delta}Q)\Big) + \varepsilon\tilde{c}\,. \qquad (4.19)
\end{aligned}$$

By (4.19), (4.14) and (4.15) we get

$$\limsup_{\varepsilon\to 0} F_{\varepsilon,\delta(\varepsilon)}(u_\varepsilon, Q) \leq c_0\, \psi_{\rm hom}(e_n)$$

as desired. □

Remark 4.5 Note that the Γ-liminf inequality does not depend on the behaviour of δ with respect to ε and we do not use the assumption of f being locally Lipschitz.

4.3 Oscillations on a finer scale than the transition layer

Finally, in this section we treat the case when the scale of oscillation δ is much smaller that the scale of the transition layer ε.

In order to prove the liminf inequality, we make the following two additional technical hypotheses:

(H1) (*Lipschitz continuity* of W)

$$|W(u) - W(v)| \leq C|u - v|$$

if $0 \leq u, v, \leq 1$

(H2) we have

$$\delta << \varepsilon\sqrt{\varepsilon}\,.$$

These hypotheses will be used in the proof of Proposition 4.10 only, and will not be needed for the limsup inequality.

Theorem 4.6 *Let F_ε be defined by* (4.4)*. Let $f(x,\xi)$ be a Borel function* 1*-periodic in x, positively homogeneous of degree two in ξ and satisfying the growth conditions*

(4.1)*, and let* W *be a continuous function satisfying conditions* (H1), (2.1) *and* (2.2)*. Let* $\delta:(0,+\infty)\to(0,+\infty)$ *be such that*

$$\lim_{\varepsilon\to 0}\frac{\varepsilon\sqrt{\varepsilon}}{\delta(\varepsilon)}=+\infty\,;$$

then there exists the Γ*-limit*

$$\Gamma\text{-}\lim_{\varepsilon\to 0}F_\varepsilon(u,U)=\int_{S(u)\cap U}\varphi(\nu_u)\,d\mathcal{H}^{n-1}$$

for every Lipschitz set $U\in\mathcal{A}$ *and every* $u\in BV(U;\{0,1\})$*, where*

$$\varphi(\nu)=c_0\sqrt{f_{\rm hom}(\nu)}$$

and $f_{\rm hom}$ *is the* homogenized integrand *of* f *defined by*

$$f_{\rm hom}(\xi)=\inf\Big\{\int_{(0,1)^n}f(y,Du+\xi)\,dy: u\in H^1_{\rm loc}(\mathbb{R}^n),\, u\ \textit{one-periodic}\Big\}$$

for all $\xi\in\mathbb{R}^n$*.*

The proof of the theorem will be obtained from the results in the rest of the section.

The liminf inequality will be proved if we show that for every sequence (u_ε) such that

$$\sup_\varepsilon F_\varepsilon(u_\varepsilon)<+\infty,\qquad u_\varepsilon\to u$$

and for every $\eta>0$ there exists a sequence (u^η_ε) converging to u such that

$$\liminf_{\varepsilon\to 0}F_\varepsilon(u_\varepsilon)\geq\liminf_{\varepsilon\to 0}\int_\Omega\Big(\frac{W(u^\eta_\varepsilon)}{\varepsilon}+\varepsilon f_{\rm hom}(Du^\eta_\varepsilon)\Big)\,dx-\eta C. \tag{4.20}$$

The conclusion will then follow since we already know the Γ-limit of the functionals on the right hand side of (4.20) (see Proposition 4.11 below). Such (u^η_ε) will be obtained from (u_ε) by averaging on a intermediate scale between δ and ε. Before defining such functions we prove a preliminary proposition.

Proposition 4.7 *Let* U *be a connected bounded open set. For every* $\eta>0$ *there exists* $K\in\mathbb{N}$ *such that for all* $u\in H^1(U)$ *and for all* $h\geq K$*,* $h\in\mathbb{N}$*, we have*

$$⨍_U f(hx,Du)\,dx\geq f_{\rm hom}\Big(⨍_U Du\,dx\Big)-\eta\Big|⨍_U Du\,dx\Big|^2.$$

Proof. Suppose by contradiction that $\eta>0$, (h_k) an increasing sequence of integers and functions $u_k\in H^1(U)$ exist such that

$$\Big|⨍_U Du_k\,dx\Big|=1$$

and

$$\fint_U f(h_k x, Du_k)\,dx < f_{\rm hom}\Big(\fint_U Du_k\Big) - \eta$$

(we use a scaling argument by positive homogeneity). Upon a translation argument and a passage to a subsequence, we may assume that $u_k \rightharpoonup \overline{u}$ in $H^1(U)$. In particular we have

$$\fint_U Du_k\,dx \to \fint_U D\overline{u}\,dx,$$

and hence

$$\Big|\fint_U D\overline{u}\,dx\Big| = 1,$$

from which we obtain, by the classical Homogenization Theorem (see e.g. [9] Section 14) and Jensen's inequality,

$$\liminf_k \fint_U f(h_k x, Du_k)\,dx \geq \fint_U f_{\rm hom}(D\overline{u})\,dx \geq f_{\rm hom}\Big(\fint_U D\overline{u}\,dx\Big)$$

and a contradiction easily follows. □

Note preliminarily that by the compactness and representation theorem we may limit our analysis in (4.20) to the case $u = u^\nu$ with $\nu = e_n$, $\Omega = Q = (-1/2, 1/2)^n$. Moreover, we may suppose that $u_\varepsilon \in C^\infty(\mathbb{R}^n)$ and that

$$u_\varepsilon(x) = w(x_n/\varepsilon)$$

on $\mathbb{R}^n \setminus Q$ by Theorem 3.7.

With fixed $\eta > 0$ let K be given by Proposition 4.7. We define

$$u^\eta_\varepsilon(x) = \fint_{Q(x,K\delta)} u_\varepsilon(y)\,dy.$$

Note that $u^\eta_\varepsilon \in C^\infty(\mathbb{R}^n)$ and that

$$Du^\eta_\varepsilon(x) = \fint_{Q(x,K\delta)} Du_\varepsilon(y)\,dy.$$

Proposition 4.8 *Let $\varphi \in C^0(\mathbb{R}^n)$ and let η, K and u^η_ε be as above. Then there exists $y \in Q(0, K\delta)$ such that, if we set*

$$x^K_i = y + iK\delta, \qquad Q^K_i = Q(x^K_i, K\delta),$$

and

$$I^\delta_K = \{i \in \mathbb{Z}^n :\ Q^K_i \cap Q \neq \emptyset\},$$

we have

$$\int_Q \varphi(Du^\eta_\varepsilon)\,dx \leq \sum_{i\in I^\delta_K} (K\delta)^n \varphi(Du^\eta_\varepsilon(x^K_i)).$$

Proof. The thesis follows immediately from the Mean Value Theorem upon remarking that

$$\int_Q \varphi(Du_\varepsilon^\eta)\,dx \le \int_{Q(0,K\delta)} \sum_{i\in I_\delta^K} \varphi(Du_\varepsilon^\eta(z+iK\delta))\,dz. \qquad \square$$

Proposition 4.9 *Let (u_ε) and (u_ε^η) be as above. Then we have*

$$\liminf_{\varepsilon\to 0} \varepsilon \int_Q f\Big(\frac{x}{\delta}, Du_\varepsilon\Big)\,dx \ge \liminf_{\varepsilon\to 0} \varepsilon \int_Q f_{\rm hom}(Du_\varepsilon^\eta)\,dx - c\eta.$$

Proof. Let y be given by Proposition 4.8 with $\varphi(\xi) = f_{\rm hom}(\xi) - \eta|\xi|^2$. Then we have, using both propositions above (in addition to Proposition 4.7 we have to use a change of variable and the positive homogeneity of f),

$$\begin{aligned} & \varepsilon \int_Q f\Big(\frac{x}{\delta}, Du_\varepsilon\Big)\,dx + O(\frac{K\delta}{\varepsilon}) \\ = \;& \varepsilon \sum_{i\in I_\delta^K} \int_{Q_i^K} f\Big(\frac{x}{\delta}, Du_\varepsilon\Big)\,dx \\ \ge \;& \varepsilon \sum_{i\in I_\delta^K} (K\delta)^n \Big(f_{\rm hom}(Du_\varepsilon^\eta(x_i^K)) - \eta|Du_\varepsilon^\eta(x_i^K)|^2\Big) \\ \ge \;& \varepsilon \int_Q \Big(f_{\rm hom}(Du_\varepsilon^\eta) - \eta|Du_\varepsilon^\eta|^2\Big)\,dx. \end{aligned}$$

The thesis follows by remarking that we have

$$\sup_\varepsilon \varepsilon \int_Q |Du_\varepsilon^\eta|^2\,dx < +\infty. \qquad \square$$

Note that we do not have yet used the hypotheses (H1) and (H2).

Proposition 4.10 *Let (u_ε) and (u_ε^η) be as above. Then we have*

$$\liminf_{\varepsilon\to 0} \int_Q \frac{W(u_\varepsilon)}{\varepsilon}\,dx \ge \liminf_{\varepsilon\to 0} \int_Q \frac{W(u_\varepsilon^\eta)}{\varepsilon}\,dx$$

Proof. By Poincaré's inequality applied in each $Q(x,K\delta)$ and the Lipschitz continuity of translations in Sobolev spaces (see e.g. [26] Theorem 2.1.6), setting $x_i = K\delta i$,

$$Q_i = Q(x_i, K\delta) \qquad \text{and } I = \{i\in\mathbb{Z}^n : Q_i\cap Q \ne \emptyset\},$$

we have

$$\int_Q |u_\varepsilon(x) - u_\varepsilon^\eta(x)|\,dx \;\le\; \sum_{i\in I} \int_{Q_i} \Big|u_\varepsilon(x) - \fint_{Q_i} u_\varepsilon(y)\,dy\Big|\,dx$$

$$
\begin{aligned}
&+\sum_{i\in I}\int_{Q_i}\fint_{Q_i}|u_\varepsilon(y+(x-x_i))-u_\varepsilon(y)|\,dy\,dx\\
\leq\ & c\,K\delta\Big(\int_Q |Du_\varepsilon|^2\,dx\Big)^{1/2}.
\end{aligned}
$$

Hence, by (H1) we get

$$
\begin{aligned}
\int_Q W(u_\varepsilon)\,dx &\geq \int_Q W(u_\varepsilon^\eta)\,dx - C\int_Q |u_\varepsilon - u_\varepsilon^\eta|dx\\
&\geq \int_Q W(u_\varepsilon^\eta)\,dx - cK\delta\Big(\int_Q |Du_\varepsilon|^2 dx\Big)^{1/2}\\
&= \int_Q W(u_\varepsilon^\eta)\,dx - \varepsilon cK\frac{\delta}{\varepsilon\sqrt{\varepsilon}}\Big(\varepsilon\int_Q |Du_\varepsilon|^2 dx\Big)^{1/2},
\end{aligned}
$$

and the thesis follows by (H2). □

The Γ-liminf inequality reads as follows.

Proposition 4.11 *For all $u\in BV(U;\{0,1\})$ we have*

$$
\Gamma\text{-}\liminf_{\varepsilon\to 0} F_\varepsilon(u,U)\geq \int_{S(u)\cap U}\sqrt{f_{\rm hom}(\nu_u)}\,d\mathcal{H}^{n-1}.
$$

Proof. It suffices to use the two previous propositions and recall that the Γ-limit of the functionals

$$
u\mapsto \int_\Omega\Big(\frac{W(u)}{\varepsilon}+\varepsilon f_{\rm hom}(Du)\Big)\,dx
$$

is given by

$$
\int_{S(u)\cap\Omega}\sqrt{f_{\rm hom}(\nu_u)}\,d\mathcal{H}^{n-1}
$$

on $BV(\Omega;\{0,1\})$ (see [5], Section 4.2). □

It remains to prove the Γ-limsup inequality, which completes the proof of the Theorem 4.6.

Proposition 4.12 *For all $u\in BV(U;\{0,1\})$ we have*

$$
\Gamma\text{-}\limsup_{\varepsilon\to 0} F_\varepsilon(u,U)\leq \int_{S(u)\cap U}\sqrt{f_{\rm hom}(\nu_u)}\,d\mathcal{H}^{n-1}.
$$

Proof. We want to prove that there exists a sequence u_ε converging to u^ν such that

$$
\limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon,Q^\nu)\leq c_0\sqrt{f_{\rm hom}(\nu)}\,.
$$

By (4.10)

$$\begin{aligned}
& c_0\sqrt{f_{\rm hom}(\nu)} \\
= \;& \min\Big\{\int_{-\infty}^{+\infty}(W(v)+f_{\rm hom}(\nu)|v'|^2)\,dt \;:\; v(-\infty)=0,\; v(+\infty)=1\Big\} \\
= \;& \inf_{T\geq 0}\inf\Big\{\int_{-T}^{T}(W(v)+f_{\rm hom}(\nu)|v'|^2)\,dt \;: \\
& \qquad\qquad v(t)=0 \text{ if } t\leq -T,\; v(t)=1 \text{ if } t\geq T\Big\}; \qquad (4.21)
\end{aligned}$$

hence, fixed $\alpha>0$ there exist $T\geq 0$ and v_T such that

$$\int_{-T}^{T}(W(v_T)+f_{\rm hom}(\nu)|v_T'|^2)\,dt \leq c_0\sqrt{f_{\rm hom}(\nu)}+\alpha\,. \tag{4.22}$$

We define

$$c_T=\int_{-T}^{T}(W(v_T)+f_{\rm hom}(\nu)|v_T'|^2)\,dt$$

and $u^T(x)=v_T(\langle x,\nu\rangle)$. Then there exists a sequence u_η converging to u^T such that $u_\eta=u^T$ on $\partial\Big(Q^\nu_{n-1}\times(-T,T)\Big)$ and

$$\begin{aligned}
c_T \;&=\; \int_{Q^\nu_{n-1}\times(-T,T)}(W(u^T)+f_{\rm hom}(Du^T))\,dx \qquad (4.23)\\
&=\; \lim_{\eta\to 0}\int_{Q^\nu_{n-1}\times(-T,T)}\Big(W(u_\eta)+f\Big(\frac{x}{\eta},Du_\eta\Big)\Big)\,dx\,.
\end{aligned}$$

Let $\eta=\delta(\varepsilon)/\varepsilon$; we define a sequence u_ε on $([\frac{\varepsilon}{\delta}]+1)\delta Q^\nu_{n-1}\times\mathbb{R}$ as follow

$$u_\varepsilon(x)=\begin{cases}
u_\eta\Big(\dfrac{x}{\varepsilon}\Big) & \text{if } x\in\varepsilon Q^\nu_{n-1}\times(-\varepsilon T,\varepsilon T)\\
u^T\Big(\dfrac{x}{\varepsilon}\Big) & \text{if } x\in\Big(([\frac{\varepsilon}{\delta}]+1)\delta\, Q^\nu_{n-1}\setminus\varepsilon Q^\nu_{n-1}\Big)\times(-\varepsilon T,\varepsilon T)\\
1 & \text{if } x_n\geq\varepsilon T\\
0 & \text{if } x_n\leq-\varepsilon T
\end{cases} \tag{4.24}$$

and we extend it by periodicity so that u_ε is $([\frac{\varepsilon}{\delta}]+1)\delta$-periodic in the variables $(x_1,\cdots,x_{n-1})$. We define

$$I_\varepsilon=\Big\{i\in\mathbb{Z}^{n-1}\;:\;\varepsilon Q^\nu_{i,n-1}\times(-\varepsilon T,\varepsilon T)\cap Q^\nu\neq\emptyset\Big\},$$

where

$$Q^\nu_{i,n-1}=i\Big(\Big[\frac{\varepsilon}{\delta}\Big]+1\Big)\frac{\delta}{\varepsilon}+Q^\nu_{n-1}$$

and

$$J_\varepsilon = \Big\{ i \in \mathbb{Z}^{n-1} \ : \ i\Big(\Big[\frac{\varepsilon}{\delta}\Big]+1\Big)\delta Q^\nu_{n-1} \setminus \varepsilon Q^\nu_{i,n-1} \times (-\varepsilon T, \varepsilon T) \cap Q^\nu \neq \emptyset \Big\}.$$

We get

$$\begin{aligned} F_{\varepsilon,\delta(\varepsilon)}(u_\varepsilon, Q^\nu) &= \int_{Q^\nu} \Big(\frac{1}{\varepsilon}W(u_\varepsilon) + \varepsilon f\Big(\frac{x}{\delta(\varepsilon)}, Du_\varepsilon\Big)\Big)\,dx \\ &= \int_{Q^\nu_{n-1}\times(-\varepsilon T,\varepsilon T)} \Big(\frac{1}{\varepsilon}W(u_\varepsilon) + \varepsilon f\Big(\frac{x}{\delta(\varepsilon)}, Du_\varepsilon\Big)\Big)\,dx \\ &\le \sum_{i\in I_\varepsilon} \int_{\varepsilon Q^\nu_{i,n-1}\times(-\varepsilon T,\varepsilon T)} \Big(\frac{1}{\varepsilon}W(u_\varepsilon) + \varepsilon f\Big(\frac{x}{\delta(\varepsilon)}, Du_\varepsilon\Big)\Big)\,dx \\ &\quad + \sum_{i\in J_\varepsilon} \int_{i([\frac{\varepsilon}{\delta}]+1)\delta Q^\nu_{n-1}\setminus(\varepsilon Q^\nu_{i,n-1}\times(-\varepsilon T,\varepsilon T))} \Big(\frac{1}{\varepsilon}W(u_\varepsilon) \\ &\qquad\qquad + \varepsilon f\Big(\frac{x}{\delta(\varepsilon)}, Du_\varepsilon\Big)\Big)\,dx \qquad (4.25) \end{aligned}$$

in particular by (4.24)

$$\begin{aligned} &\sum_{i\in J_\varepsilon} \int_{i([\frac{\varepsilon}{\delta}]+1)\delta Q^\nu_{n-1}\setminus\varepsilon Q^\nu_{i,n-1}\times(-\varepsilon T,\varepsilon T)} \Big(\frac{1}{\varepsilon}W(u_\varepsilon) + \varepsilon f\Big(\frac{x}{\delta(\varepsilon)}, Du_\varepsilon\Big)\Big)\,dx \\ = &\sum_{i\in J_\varepsilon} \varepsilon^{n-1} \int_{i([\frac{\varepsilon}{\delta}]+1)\frac{\delta}{\varepsilon}Q^\nu_{n-1}\setminus Q^\nu_{i,n-1}\times(-T,T)} \Big(W(u^T) + f\Big(\frac{x}{\eta}, Du^T\Big)\Big)\,dx \\ \le &\sum_{i\in J_\varepsilon} \varepsilon^{n-1}\Big(\Big(\Big[\frac{\varepsilon}{\delta}\Big]\frac{\delta}{\varepsilon} + \frac{\delta}{\varepsilon}\Big)^{n-1} - 1\Big) \int_{-T}^{T} \Big(W(v_T) + c_2|v_T'|^2\Big)\,dt \\ \le &\sum_{i\in J_\varepsilon} \varepsilon^{n-1}\Big(\Big(\Big[\frac{\varepsilon}{\delta}\Big]\frac{\delta}{\varepsilon} + \frac{\delta}{\varepsilon}\Big)^{n-1} - 1\Big)\Big(c_T + c_2\int_{-T}^{T} |v_T'|^2\,dt\Big); \qquad (4.26) \end{aligned}$$

hence, by (4.26) we obtain

$$\limsup_{\varepsilon\to 0} \sum_{i\in J_\varepsilon} \int_{i([\frac{\varepsilon}{\delta}]+1)\delta Q^\nu_{n-1}\setminus Q^\nu_{i,n-1}\times(-\varepsilon T,\varepsilon T)} \Big(\frac{1}{\varepsilon}W(u_\varepsilon) + \varepsilon f\Big(\frac{x}{\delta(\varepsilon)}, Du_\varepsilon\Big)\Big)\,dx = 0\,. \qquad (4.27)$$

We now estimate the first term in (4.25) as

$$\begin{aligned} &\sum_{i\in I_\varepsilon} \int_{\varepsilon Q^\nu_{i,n-1}\times(-\varepsilon T,\varepsilon T)} \Big(\frac{1}{\varepsilon}W(u_\varepsilon) + \varepsilon f\Big(\frac{x}{\delta(\varepsilon)}, Du_\varepsilon\Big)\Big)\,dx \\ \le &\Big(\Big[\frac{1}{\varepsilon}\Big]+1\Big)^{n-1} \varepsilon^{n-1} \int_{Q^\nu_{n-1}\times(-T,T)} \Big(W(u_\eta) + f\Big(\frac{x}{\eta}, Du_\eta\Big)\Big)\,dx; \qquad (4.28) \end{aligned}$$

hence, by (4.25), (4.27), (4.28), (4.23) and (4.22) we get

$$\limsup_{\varepsilon\to 0} F_{\varepsilon,\delta(\varepsilon)}(u_\varepsilon, Q^\nu) \leq \lim_{\eta\to 0}\int_{Q^\nu_{n-1}\times(-T,T)} \Big(W(u_\eta)+f\Big(\frac{x}{\eta}, Du_\eta\Big)\Big)\,dx = c_T \leq c_0\sqrt{f_{\rm hom}(\nu)}+\alpha \tag{4.29}$$

and by the arbitrariness of α we obtain the Γ-limsup inequality. □

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