ON A BONNESEN TYPE INEQUALITY INVOLVING THE SPHERICAL DEVIATION

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1. Introduction

In recent years the stability of the isoperimetric and related inequalities has been the object of many investigations. Roughly speaking, given the well known isoperimetric property of balls, the question is how far a set $E \subset \mathbb{R}^n$ is from the unit ball B_1 if $|E| = |B_1|$ and its perimeter $P(E)$ is close to the perimeter of B_1.

The first results in this direction where obtained for planar sets by Bernstein in 1905 ([2]) and Bonnesen in 1924 ([3]). In particular in the latter paper it is proved that if $E \subset \mathbb{R}^2$ has the same area of the unit disk D and is bounded by a simple closed curve, then there exist two concentric disks $D_{r_1} \subset E \subset D_{r_2}$ of radii r_1, r_2 such that

$$(r_2 - r_1)^2 \leq \frac{P^2(E) - P^2(D)}{4\pi},$$

with equality holding if and only if E is a disk.

It took several years before this result was extended to higher dimension by Fuglede ([9]) who proved in particular that if $E \subset \mathbb{R}^n$ is a convex set with the same volume of the unit ball B_1, then, up to a translation, the Hausdorff distance from E to B_1 is controlled by a suitable power of its isoperimetric deficit $P(E) - P(B_1)$. This result is a consequence of an L^∞ estimate in terms of the isoperimetric deficit of E of the function u defined on the unit sphere and such that

$$E = \left\{ x \in \mathbb{R}^n : |x| \leq 1 + u\left(\frac{x}{|x|}\right) \right\},$$

under the assumption that the $W^{1,\infty}$-norm of u is smaller than a constant depending on the dimension n.

If E is not convex or nearly spherical, i.e., the function u is sufficiently close to 0 in $W^{1,\infty}$, one cannot expect such L^∞ estimate to hold. To this aim one may introduce the so called *Fraenkel asymmetry* of E defined as

$$\lambda(E) := \min\left\{ \frac{|E \,\triangle\, B_r(x)|}{r^n} : x \in \mathbb{R}^n, \ \ |E| = \omega_n r^n \right\}.$$

Then, it has been proved in [10] that for any set of finite perimeter $E \subset \mathbb{R}^n$ with finite measure

$$\lambda(E) \leq C\sqrt{D(E)}, \tag{1}$$

where C depends only on n and $D(E)$ stands for the *isoperimetric deficit*

$$D(E) := \frac{P(E) - P(B_r)}{P(B_r)}.$$

Note that the power $\frac{1}{2}$ on the right hand side of (1) is optimal as conjectured by Hall in [12] (see also [13, Section 4]), where a weaker estimate with the exponent $\frac{1}{4}$ was proved. We would like also to mention that in a recent paper by Figalli, Maggi and Pratelli ([8]), inequality (1) has been extended to the more general framework of anisotropic perimeters using a mass transportation approach (see [5] for a different proof in the euclidean case).

If one wants to improve (1) by replacing the Fraenkel asymmetry with a stronger notion of distance from a ball as the one considered by Fuglede in the convex case, it is clear that one has to require some special structure or regularity on the set E, which in particular avoids the presence of thin tentacles or tiny connected components. This is certainly the case if one imposes an a priori bound on the curvature of ∂E or some uniform interior ball condition.

Here, given $R > 0$, we consider the class $\mathcal{C}_R$ of all closed sets $E \subset \mathbb{R}^n$ satisfying at each point of the boundary a uniform cone condition with aperture equal to $\frac{\pi}{2}$ and height depending on R and $|E|$ (see Definition 2.7). Note that this is a quite mild regularity assumption on E. One can prove indeed, see Proposition 2.4, that if $E \in \mathcal{C}_R$, then its boundary ∂E has finite $\mathcal{H}^{n-1}$-measure and therefore E is of finite perimeter. Nevertheless, an example given in Section 2 shows that in general the $\mathcal{H}^{n-1}$-measure of the topological boundary ∂E can be strictly greater than the perimeter of E even if the cone condition is replaced by the stronger uniform interior ball condition.

In order to describe our result, we define the *deviation* from the spherical shape of a set $E \subset \mathbb{R}^n$ with finite measure as

$$\lambda_{\mathcal{H}}(E) := \min_{x \in \mathbb{R}^n} \left\{ \frac{d_{\mathcal{H}}(E, B_r(x))}{r} : |E| = \omega_n r^n \right\},$$

where $d_{\mathcal{H}}(\cdot,\cdot)$ denotes the Hausdorff distance between sets.

Then the main result of the paper reads as follow.

Theorem 1.1. *For any $R > 0$ there exist $0 < \delta_R < 1$ and a constant $C = C(R, n)$ depending only on R and n such that, for any $E \in \mathcal{C}_R$ with $D(E) < \delta_R$,*

$$\lambda_{\mathcal{H}}(E) \le C \begin{cases} D(E)^{\frac{1}{2}} & \text{for } n = 2 \\ D(E)^{\frac{1}{2}} \left(\log \frac{1}{D(E)}\right)^{\frac{1}{2}} & \text{for } n = 3 \\ D(E)^{\frac{1}{n-1}} & \text{for } n \ge 4. \end{cases} \tag{2}$$

We observe that the powers appearing in (2) are the same obtained by Fuglede in [9] for nearly spherical domains and by Rajala and Zhong in [15] for John domains whose complement with respect to a suitable ball is also a John domain. Note that though the sets considered in [15] do not necessarily belong to $\mathcal{C}_R$, they cannot have singularities such as inward cusps, which are, instead, admissible for sets in $\mathcal{C}_R$. Note also that the exponents appearing at the right hand-side of the inequality above are known to be optimal (see Example 3.1 in [9] for $n = 2, 3$ and [15] for $n \ge 4$).

The paper is organized as follows. In Section 2 we prove some preliminary facts on sets that satisfy the above mentioned cone property. In particular we prove that such sets have finite perimeter and that in the class $\mathcal{C}_R$ the spherical deviation $\lambda_{\mathcal{H}}(E)$ goes to zero if $D(E)$ tends to zero. This continuity property implies in turn that, if the isoperimetric deficit is sufficiently small, then the optimal balls for the

Fraenkel asymmetry and for the spherical deviation are close to each other, a fact that turns out to be an important tool in the proof of Theorem 1.1.

The proof of the main result is achieved in Section 3. The strategy is the following. First we prove a suitable variant of Fuglede's result stating that (2) holds if E is starshaped with respect to the center of an optimal ball B for $\lambda(E)$ and its boundary is a graph over ∂B with bounded $W^{1,\infty}(\partial B)$ norm.

Then, given any set $E \in \mathcal{C}_R$ with $D(E)$ sufficiently small, we analyze the two possible situations: either E contains a sufficiently large "hole" or not.

In the first case the perimeter of E is far from being optimal and indeed we easily prove that $\lambda_{\mathcal{H}}(E) \leq 2D(E)^{\frac{1}{n-1}}$, an inequality which is even better than (2) when $n = 2, 3$.

If E has no large holes, then the idea is to replace it by a set $\widetilde{E} \in \mathcal{C}_{R/2}$ with no holes, such that $|E| = |\widetilde{E}|$ and satisfying

$$\lambda_{\mathcal{H}}(E) \leq \lambda_{\mathcal{H}}(\widetilde{E}) + c\sqrt{D(E)} \quad \text{and} \quad D(\widetilde{E}) \leq D(E).$$

Thus, the proof of Theorem 1.1 is reduced to the case of a set E with small deficit, containing a ball of radius r close to 1. And this is the point where the interior cone condition with aperture $\pi/2$ comes into play. In fact, this assumption, via some careful geometric arguments, allows us first to show that E is starshaped with respect to the center of an optimal ball B for the Fraenkel asymmetry, then that its boundary is a graph over ∂B of a Lipschitz function with uniformly bounded L^∞ norm of the gradient. The conclusion then follows by using the above mentioned extension of Fuglede's result.

2. Preliminaries

Throughout the paper we will denote by $B_r(x)$ the **closed** ball centered in x with radius r. The volume of the unit ball will be denoted by ω_n.

If A, B are two subset of $\mathbb{R}^n$ the *Hausdorff distance* between A and B is defined as

$$d_{\mathcal{H}}(A, B) := \inf\{\varepsilon > 0 : \; B \subset A + B_\varepsilon \;, \; A \subset B + B_\varepsilon\}.$$

If A and B are compact, then $d_{\mathcal{H}}(A, B)$ can be computed as

$$d_{\mathcal{H}}(K, H) = \max\{\max_{x \in H} \operatorname{dist}(x, K), \max_{y \in K} \operatorname{dist}(y, H)\}.$$

Given a set $E \subset \mathbb{R}^n$ of finite perimeter we shall denote by $P(E)$ its perimeter and by $\partial^* E$ its reduced boundary. For the properties of sets of finite perimeter we refer to [1].

Les us now introduce a class of sets satisfying a suitable interior cone condition. To this aim, given $x \in \mathbb{R}^n$, $R > 0$, $\theta \in (0, \pi)$ and $\nu \in \mathbb{S}^{n-1}$, the *spherical sector* with vertex in x, axis of symmetry parallel to ν, radius R and aperture θ is defined as

$$S^{\theta,R}_{x,\nu} := \{y \in \mathbb{R}^n : \; |y - x| < R \;, \; \langle y - x, \nu\rangle > \cos(\theta/2)|y - x|\} \,.$$

Definition 2.1. *We say that a closed set E satisfies the* interior cone condition at the boundary *with radius $R > 0$ and aperture θ if for any $x \in \partial E$ there exists $\nu_x \in \mathbb{S}^{n-1}$ such that $S^{\theta,R}_{x,\nu_x} \subset E$.*

Remark 2.2. We point out that the property stated in Definition 2.1 is weaker than the classical interior cone condition which is imposed *at every point $x \in E$* and not just at the boundary points. In fact, if the aperture θ is strictly greater

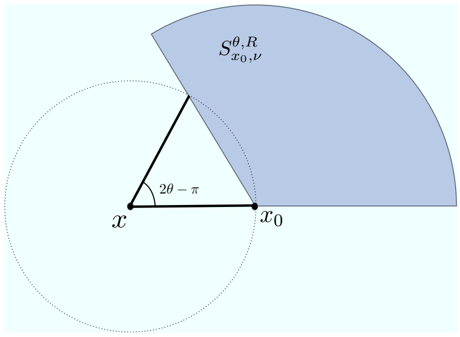

FIGURE 1. For $x \in \overset{\circ}{E}$ let $x_0 \in \partial E$ be the point of minimal distance, i.e. such that $\mathrm{dist}(x, \partial E) = |x - x_0|$, and $S^{\theta,R}_{x_0,\nu} \subset E$.

than $\pi/2$, one can prove that the classical interior cone condition is satisfied with aperture $2\theta - \pi$ and radius $R/4$ (the proof is obtained by showing that one can always reduce to the situation illustrated in Figure 1). If instead $\theta \leq \pi/2$, then, given any $0 < \phi \leq \theta$ and $0 < r \leq R$, one can always construct a set satisfying the cone condition at the boundary with radius R and aperture θ, but not satisfying the classical interior cone condition with radius r and aperture ϕ (see example below).

Example 2.3. Fix $R > 0$, $0 < \phi \leq \frac{\pi}{2}$ and $0 < r \leq R$. We are going to construct a closed set $\Omega \subset \mathbb{R}^2$ by removing from a ball of radius $2R$, centered at the origin, several holes having the shape of spherical sectors with vertices on the circle centered at the origin and with radius $r/2$.

In order to define precisely the holes, we fix $k \in \mathbb{N}$ such that $\frac{2\pi}{k} < \frac{\phi}{2}$ and set

$$x_h := \frac{r}{2} e^{i\frac{h2\pi}{k}}$$

for $h \in \{0, 1, \ldots, k\}$. The points x_h are the centers of the spherical sectors defining the holes of Ω. Indeed we set

$$\Omega := B_{2R} \setminus \bigcup_{h=0}^{k} S^{\beta,\overline{r}}_{x_h,\frac{x_h}{|x_h|}},$$

with β and $\overline{r}$ to be suitably chosen.

It is evident that in a neighborhood of the origin the interior cone condition with aperture ϕ and radius r is not satisfied.

On the other hand, if β and $\overline{r}$ are sufficiently small (depending on r and k), it is easy to check that the spherical sector with aperture $\frac{\pi}{2}$ and radius R, centered at x_h and lying on the side of $S^{\beta,\overline{r}}_{x_h,\frac{x_h}{|x_h|}}$, is contained in Ω (see Figure 2). Hence, Ω satisfies the interior cone condition with aperture $\frac{\pi}{2}$ and radius R at every point of its boundary.

Next proposition shows that the property stated in Definition 2.1 implies some mild regularity of the boundary.

Proposition 2.4. *Let K be a compact set satisfying the interior cone condition at the boundary with radius R and aperture θ. Then ∂K is contained in a finite*

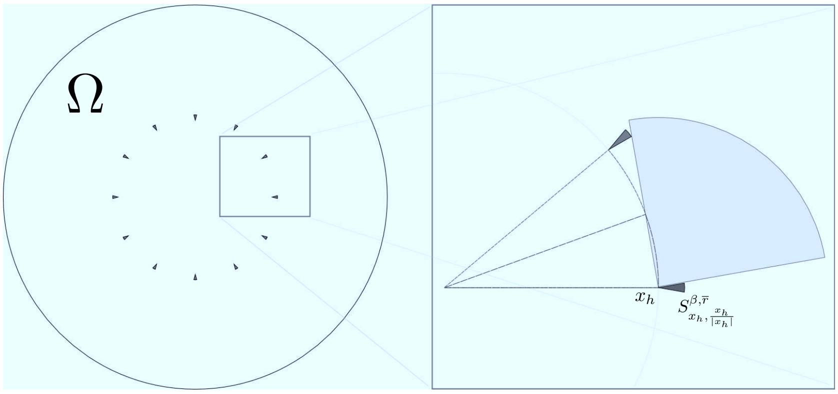

FIGURE 2. Construction of Ω

union of Lipschitz graphs. In particular ∂K is $(n-1)$-rectifiable and K has finite perimeter.

Proof. Note that there exist $\nu_1,\ldots,\nu_N \in \mathbb{S}^{n-1}$ (depending only on θ) such that for any $x \in \partial K$ there exists $i \in \{1,\ldots,N\}$ so that $S^{\theta/2,R}_{x,\nu_i} \subset K$. For $i = 1,\ldots,N$, set

$$S_i := \{x \in \partial K : S^{\theta/2,R}_{x,\nu_i} \subset K\}.$$

Clearly, $\partial K = \cup_{i=1}^N S_i$ and thus we are left with proving that each S_i can be covered by finitely many Lipschitz graphs. Fix i and assume that, up to a rotation, $\nu_i = -e_n$. Since S_i is compact, we can find a finite number of cubes Q^i_j with sides parallel to the coordinate directions and diameter strictly less than R such that $S_i \subset \cup_j Q^i_j$. We claim that for any $x, y \in S_i \cap Q^i_j$ with $x \neq y$, then $y \notin S^{\theta/2,R}_{x,-e_n} \cup S^{\theta/2,R}_{x,e_n}$. Indeed, since $|x-y| < R$, if $y \in S^{\theta/2,R}_{x,-e_n}$, then y would lie in $\overset{\circ}{K}$, thus contradicting the choice of x and y. Similarly, if $y \in S^{\theta/2,R}_{x,e_n}$, then x would belong to $S^{\theta/2,R}_{y,-e_n}$, that is again impossible. Thus we easily infer that if $x = (x', x_n)$, $y = (y', y_n)$, with $x', y' \in \Pi_{\mathbb{R}^{n-1}}(Q^i_j \cap S_i)$, then

$$|x_n - y_n| \leq \tan((2\pi - \theta)/4)|x' - y'|. \tag{3}$$

Define now $f^i_j : \Pi_{\mathbb{R}^{n-1}}(Q^i_j \cap S_i) \to \mathbb{R}$ by setting $f^i_j(x') := x_n$. From (3) we deduce that f^i_j is Lipschitz and that $\mathcal{H}^{n-1}(Q^i_j \cap S_i) \leq c(\theta, R)$. In particular ∂K is contained in a finite number of Lipschitz graphs and $\mathcal{H}^{n-1}(\partial K) < +\infty$. The last part of the statement follows from Theorem 4.5.11 in [7] (see also Proposition 3.62 in [1]). □

Remark 2.5. We point out that, despite the previous result, the *interior cone condition at the boundary* is a quite mild regularity assumption. Indeed, one can construct a compact set of finite perimeter $K \subset \mathbb{R}^2$ satisfying a uniform interior ball condition, such that $\mathcal{H}^1(\partial K \setminus \partial^* K) > 0$, as shown by the next example.

Example 2.6. This example is inspired to Example 4.1 in [6] (see also [14]). Let $C \subset \mathbb{S}^1$ be a compact set with $\mathcal{H}^1(C) > 0$ and empty interior. Set

$$K := B_1 \cup (B_4 \setminus \overset{\circ}{B}_2) \cup \bigcup_{x \in C} B_1(x).$$

Since C is closed it is easily checked that K is compact (recall that $B_r(x)$ denotes a closed ball). By construction K satisfies the interior ball condition, hence Proposition 2.4 implies that K is a set of finite perimeter and $\mathcal{H}^1(\partial K) < +\infty$.

Set $A = \mathbb{S}^1 \setminus C$ and observe that $A = \cup_{i=1}^{\infty} \Gamma_i$, with Γ_i connected open arcs such that $\Gamma_i \cap \Gamma_j = \emptyset$ if $i \neq j$. Let us denote by a_i, b_i the end points of Γ_i. We claim that for any point $x \in C$ we have $2x \in \partial K$. Indeed, thanks to the fact that C has empty interior there exists a sequence $\{x_h\} \subset A$ converging to x. For any h, $x_h \in \Gamma_{i_h}$ for some i_h. Thus we can find a point y_h lying in the interior of $B_2 \setminus (B_1(a_{i_h}) \cup B_1(b_{i_h}))$ such that $\pi(y_h) = x_h$, π being the standard projection on $\mathbb{S}^1$. By construction $y_h \notin K$ and the sequence y_h converges to $2x$. Hence $2x \in \partial K$. On the other hand, since $B_1(x) \subset K$ is tangent to ∂K at $2x$ and $B_1(3x) \subset K$ is also tangent to ∂K in $2x$, we get that the density of K at $2x$ is 1. Since K has density $1/2$ at each point of its reduced boundary (see Theorem 3.61 in [1]), we have that $2x \in \partial K \setminus \partial^* K$. Therefore $\mathcal{H}^1(\partial K \setminus \partial^* K) \geq 2\mathcal{H}^1(C) > 0$.

We now introduce the class of sets to which our main result Theorem 1.1 will apply. The reason of the choice $\theta = \pi/2$ will be clear in the next section.

Definition 2.7. *Given $R > 0$, we denote by $\mathcal{C}_R$ the family of closed sets E, with $|E| < \infty$, satisfying the interior cone condition at the boundary with radius $R|E|^{\frac{1}{n}}\omega_n^{-\frac{1}{n}}$ and aperture $\pi/2$. We set also $\mathcal{C}_R^1 := \{E \in \mathcal{C}_R : |E| = \omega_n\}$.*

Remark 2.8. Note that the family $\mathcal{C}_R$ is scale invariant and that a set in $\mathcal{C}_R^1$ satisfies the cone condition with radius R. Moreover, if $F \in \mathcal{C}_R$, setting $E = (\omega_n^{1/n}/|F|^{1/n})F$, then $E \in \mathcal{C}_R^1$, $D(E) = D(F)$ and $\lambda_{\mathcal{H}}(E) = \lambda_{\mathcal{H}}(F)$.

In the following lemmas we state some useful properties of the sets E satisfying an interior cone property for some $R, \theta > 0$. In particular we show that if the isoperimetric deficit is sufficiently small, then E is uniformly bounded and we prove the continuity at 0 of the function $\lambda_{\mathcal{H}}(E)$ with respect to the deficit $D(E)$.

Since the results proved in this section do not require the assumption $\theta = \pi/2$, in the following we denote by $\mathcal{C}_R^1(\theta)$ *the family of all closed sets of measure ω_n satisfying the interior cone property at the boundary with radius $R > 0$ and aperture $\theta \in (0, \pi)$.* Moreover, whenever the dependence on x and ν plays no role, we will use the simplified notation $S^{\theta,R}$ to denote a generic spherical sector $S^{\theta,R}_{x,\nu}$.

Lemma 2.9. *There exist δ and $L > 0$ such that for any $E \in \mathcal{C}_R^1(\theta)$ with $D(E) < \delta$ we have* $\operatorname{diam}(E) \leq L$.

Proof. Set $\delta := |S^{\theta,R}|^2/4C^2$, where C is the constant in (1), and $L := 2 + 2R$.

Assume by contradiction that $D(E) < \delta$ and $\operatorname{diam}(E) > L$. Then there exists $y \in \partial E$ with $\operatorname{dist}(y, B_1(x_1)) > R$, where $B_1(x_1)$ is such that $\lambda(E) = |E \,\triangle\, B_1(x_1)|$. Since $E \in \mathcal{C}_R^1(\theta)$, there exists $\nu \in \mathbb{S}^{n-1}$ such that $S^{\theta,R}_{y,\nu} \subset E$. By the choice of y we deduce that $S^{\theta,R}_{y,\nu} \subset E \setminus B_1(x_1)$ and this in turn gives

$$|E \,\triangle\, B_1(x_1)| \geq |S^{\theta,R}| \geq \delta^{1/2} 2C,$$

a contradiction to (1). Hence, the assertion follows. □

The following lemma asserts the continuity of the spherical deviation with respect to the perimeter deficit in the class $\mathcal{C}_R^1(\theta)$.

Lemma 2.10. *For any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $E \in \mathcal{C}_R^1(\theta)$ with $D(E) < \delta$ we have $\lambda_{\mathcal{H}}(E) \leq \varepsilon$.*

Proof. We argue by contradiction assuming that there exist $\varepsilon_0 > 0$ and a sequence $\{E_j\} \subset \mathcal{C}^1_R(\theta)$ such that $\lim_{j\to\infty} D(E_j) = 0$ and $\lambda_{\mathcal{H}}(E_j) > \varepsilon_0$.

From Lemma 2.9, by suitably translating the sets E_j, if needed, we deduce that there exists a large ball B such that $E_j \subset B$ for all j. Since all sets E_j have equibounded perimeters, by a well known compactness result (see Theorem 3.39 in [1]) we may assume that, up to a not relabeled subsequence, $\chi_{E_j} \to \chi_F$ in $L^1(\mathbb{R}^n)$ for a suitable measurable set F. Note that $|F| = \omega_n$ and, by the lower semicontinuity of the perimeter, $D(F) \le \liminf_{j\to\infty} D(E_j) = 0$. The isoperimetric inequality yields at once that F coincides a.e. with a unit ball, say B_1. Moreover by the compactness of the Hausdorff distance on equibounded sets, we may also assume that $E_j \to E_\infty$ in $\mathrm{d}_{\mathcal{H}}$.

We claim that $E_\infty = B_1$. Indeed, the inclusion $B_1 \subset E_\infty$ is straightforward, since a.e. $x \in B_1$ is the limit point of a sequence $\{x_j\}$ with $x_j \in E_j$. Assume by contradiction that $E_\infty \not\subset B_1$. Then there exists $\bar{x} \in \partial E_\infty$ with $\mathrm{dist}(\bar{x}, B_1) \ge r_0 > 0$. By the Hausdorff convergence of E_j to E_∞, $\bar{x}$ is the limit of a sequence $\bar{x}_j \in \partial E_j$, hence, for j large enough, we have that $\mathrm{dist}(\bar{x}_j, B_1) \ge r_0/2$. For any such j, let $\nu_j \in \mathbb{S}^{n-1}$ be such that $S^{\theta,R}_{\bar{x}_j,\nu_j} \subset E_j$. Then, $S^{\theta,R}_{\bar{x}_j,\nu_j} \cap B_{r_0/2}(\bar{x}_j) \cap B_1 = \emptyset$ and thus $|S^{\theta,R}_{\bar{x}_j,\nu_j} \setminus B_1| > |S^{\theta,r_0/2}|$. This in turn implies that $|E_j \setminus B_1| > |S^{\theta,r_0/2}|$, leading to a contradiction to the L^1 convergence of E_j to B_1 and thus proving the claim.

Finally, from the convergence of E_j to B_1 in the Hausdorff distance, we conclude that $\lim_{j\to\infty} \lambda_{\mathcal{H}}(E_j) = 0$, thus proving the assertion. □

As a corollary of Lemma 2.10 we have that, if the perimeter deficit is sufficiently small, any two optimal balls with respect to the L^1 and Hausdorff distance, respectively, are arbitrarily close.

Lemma 2.11. *For any $\varepsilon > 0$ there exists $\delta > 0$ such that for any set $E \in \mathcal{C}^1_R(\theta)$ with $D(E) < \delta$ we have*

$$\mathrm{d}_{\mathcal{H}}(B_1(x_1), B_1(x_\infty)) \le \varepsilon,$$

where $B_1(x_1)$, $B_1(x_\infty)$ are any two balls with the property that

$$\lambda(E) = |E \,\triangle\, B_1(x_1)|, \quad \lambda_{\mathcal{H}}(E) = \mathrm{d}_{\mathcal{H}}(E, B_1(x_\infty)).$$

Proof. We argue by contradiction. Assume that there exist $\varepsilon_0 > 0$ and a sequence $\{E_j\} \subset \mathcal{C}^1_R(\theta)$ such that $\lim_{j\to\infty} D(E_j) = 0$ and $\mathrm{d}_{\mathcal{H}}(B_1(x_1^j), B_1(x_\infty^j)) > \varepsilon_0$, where $\lambda(E_j) = |E_j \,\triangle\, B_1(x_1^j)|$ and $\lambda_{\mathcal{H}}(E_j) = \mathrm{d}_{\mathcal{H}}(E_j, B_1(x_\infty^j))$.

As in the proof of Lemma 2.10 we may assume with no loss of generality that E_j converge to a suitable ball $B_1(x_0)$ both in L^1 and in the Hausdorff distance. By compactness we may also assume that $B_1(x_1^j) \to B_1(y_1)$ and $B_1(x_\infty^j) \to B_1(y_\infty)$ both in L^1 and in Hausdorff distance. Hence

$$\mathrm{d}_{\mathcal{H}}(B_1(y_1), B_1(y_\infty)) \ge \varepsilon_0. \tag{4}$$

Since by Lemma 2.10 $\lambda_{\mathcal{H}}(E_j) \to 0$, while $\lambda(E_j) \to 0$ by (1), we conclude that $B_1(y_1) = B_1(x_0) = B_1(y_\infty)$. This gives a contradiction to (4). □

3. Proof of the main result

This section is devoted to the proof of Theorem 1.1. Since all the quantities considered are scaling invariant (see Remark 2.8), it is not restrictive to work in the class $\mathcal{C}^1_R$. Hence, from now on we tacitly assume $|E| = \omega_n$ for the set under consideration whenever the measure is not specified.

The proof of Theorem 1.1 is divided into several steps, each consisting of different types of results, some of them independent of the interior cone property. More precisely, in Proposition 3.3 we establish (2) under the assumption that the set E is starshaped with respect to the center of a ball realizing the minimum in the definition of $\lambda(E)$ and that its boundary is a Lipschitz graph over the boundary of this optimal ball (see Proposition 3.1 for the precise statement). As a second step, taking advantage of the results established in Section 2, in Proposition 3.4 we show that we can reduce the proof of (2) to sets containing a sufficiently large ball, provided that the set E satisfies the interior cone condition at the boundary for some $\theta > 0$ and contains an annulus with the same center of an optimal ball in the L^1-distance. Finally in Proposition 3.7, assuming that the aperture θ is equal to $\pi/2$, we prove that if the deficit is small enough such an annulus exists. The last step of the proof follows from Propositions 3.8 and 3.10 where we show that if $\theta = \pi/2$ and the deficit is small the boundary of E is the graph of a Lipschitz function defined on the boundary an L^1 optimal ball.

Note that the assumption $\theta = \pi/2$ plays a role only in Propositions 3.7 and 3.8.

Before proceeding with the proofs we fix some notations. For a measurable set E with $|E| = \omega_n$ we denote by $B_1(x_1)$ any ball realizing the Fraenkel asymmetry $\lambda(E)$. Note that for such a ball $|E \,\triangle\, B_1(x_1)| \le C\sqrt{D(E)}$. Similarly, we denote by $B_1(x_\infty)$ any ball such that $d_{\mathcal{H}}(E, B_1(x_\infty)) = \lambda_{\mathcal{H}}(E)$.

3.1. **Fuglede's result revisited.** In this subsection we prove (2) for sets E starshaped with respect to an optimal ball $B_1(x_1)$ and such that ∂E can be represented as the graph of a Lipschitz function defined on $\partial B_1(x_1)$. As the quantities involved in (2) are not affected by translations, for simplicity in this subsection we assume x_1 to be the origin in $\mathbb{R}^n$.

In a sense the following Proposition 3.3 generalizes Theorem 1.2 in [9]. As in that paper we prefer to state the result as a functional inequality for functions in $W^{1,\infty}(\Sigma)$, where Σ denotes the unit sphere in $\mathbb{R}^n$ equipped with the surface measure σ suitably normalized in order to have

$$\int_\Sigma d\sigma(x) = 1.$$

For a function $u : \Sigma \to (-1, 1)$ consider the associated set E defined as

$$E = \left\{ x \in \mathbb{R}^n : |x| \le 1 + u\left(\frac{x}{|x|}\right) \right\}. \tag{5}$$

Then the following formulas hold true:

$$\begin{aligned} \frac{P(E)}{n\omega_n} &= \int_\Sigma (1 + u(z))^{n-1} \sqrt{1 + (1 + u(z))^{-2} |\nabla u(z)|^2}\, d\sigma, \\ \frac{|E|}{\omega_n} &= \int_\Sigma (1 + u(z))^n\, d\sigma, \end{aligned}$$

where ∇ denotes the tangential gradient on Σ. Set

$$\Delta(u) := \frac{D(E)}{n\omega_n} = \int_\Sigma (1 + u)^{n-1} \sqrt{1 + (1 + u)^{-2} |\nabla u|^2}\, d\sigma - 1 \tag{6}$$

and observe that the condition $|E| = \omega_n$, i.e.,

$$\int_\Sigma (1 + u(z))^n \, d\sigma = 1, \tag{7}$$

entails

$$\|u\|_\infty = d_{\mathcal{H}}(E, B_1) \geq \lambda_{\mathcal{H}}(E).$$

The additional hypothesis that B_1 satisfies $\lambda(E) = |E \vartriangle B_1|$ immediately implies that

$$\int_\Sigma |u| \, d\sigma \leq C_0 \sqrt{\Delta(u)} \tag{8}$$

for some constant C_0 depending only on n.

Next proposition contains a key estimate on $\|\nabla u\|_{L^2(\Sigma)}$ that, together with Lemma 3.2, will allow us to prove Proposition 3.3.

Proposition 3.1. *For any $M > 0$ there exist constants $C_1, C_2 > 0$, depending only on M, n, such that if $u \in W^{1,\infty}(\Sigma)$ satisfies (7), (8),*

$$\|u\|_{L^\infty(\Sigma)} \leq \frac{1}{C_1} \qquad \textit{and} \qquad \|\nabla u\|_{L^\infty(\Sigma)} \leq M\,,$$

then

$$\int_\Sigma |\nabla u|^2 d\sigma \leq C_2 \Delta(u). \tag{9}$$

Proof. From assumption (7), expanding $(1+u)^n$, we obtain

$$n \int_\Sigma u \, d\sigma = -\sum_{h=2}^{n} \binom{n}{h} \int_\Sigma u^h \, d\sigma\,.$$

Hence, if $C_1 > 2$ and thus $\|u\|_\infty < 1/2$, we get

$$\Big| \int_\Sigma u \, d\sigma \Big| \leq c(n) \int_\Sigma u^2 \, d\sigma\,. \tag{10}$$

Since, $\|\nabla u\|_\infty \leq M$ and $\|u\|_\infty < 1/2$, from the concavity of the function $\sqrt{1+t}$, we deduce that there exists a constant $c(M) > 0$ such that

$$\frac{|\nabla u|^2}{c(M)} \leq \sqrt{1 + \frac{|\nabla u|^2}{(1+u)^2}} - 1 \leq 2|\nabla u|^2\,. \tag{11}$$

Recalling (6), we may rewrite

$$\begin{aligned}
\Delta(u) &= \int_\Sigma \Big(\sqrt{1 + \frac{|\nabla u|^2}{(1+u)^2}} - 1 \Big) d\sigma + \int_\Sigma \sqrt{1 + \frac{|\nabla u|^2}{(1+u)^2}} \Big((1+u)^{n-1} - 1 \Big) d\sigma \\
&= \int_\Sigma \Big(\sqrt{1 + \frac{|\nabla u|^2}{(1+u)^2}} - 1 \Big) d\sigma \\
&\quad + \int_\Sigma \sqrt{1 + \frac{|\nabla u|^2}{(1+u)^2}} \Big((n-1)u + \sum_{h=2}^{n-1} \binom{n-1}{h} u^h \Big) d\sigma.
\end{aligned}$$

Therefore, from (11) we conclude that there exist $c_1(M), c_2(M,n) > 0$ such that

$$\Delta(u) \geq \frac{1}{c_1}\int_\Sigma |\nabla u|^2\,d\sigma - c_2\int_\Sigma u^2\,d\sigma + \int_\Sigma (n-1)u\,d\sigma + \int_\Sigma \Big(\sqrt{1+\frac{|\nabla u|^2}{(1+u)^2}} - 1\Big)(n-1)u\,d\sigma.$$

Using (10) and (11) again, we get

$$\Delta(u) \geq \frac{1}{c_1}\int_\Sigma |\nabla u|^2\,d\sigma - c_2\int_\Sigma u^2\,d\sigma - 2(n-1)\|u\|_\infty\int_\Sigma |\nabla u|^2\,d\sigma\,,$$

for some possibly larger constant c_2, still depending only on M, n. Finally, choosing $C_1(M,n)$ sufficiently large, the previous inequality yields

$$\Delta(u) \geq \frac{1}{c_1}\int_\Sigma |\nabla u|^2\,d\sigma - c_2\int_\Sigma u^2\,d\sigma\,, \tag{12}$$

for some larger constant c_1 depending only on M.

We are now going to exploit assumption (8). To this aim, we need to introduce the spherical harmonics on Σ. For all integers $k \geq 0$, $i = 1, \ldots, G(k,n)$, let $Y_{k,i}$ denote the restriction to Σ of homogeneous polynomials of degree k on $\mathbb{R}^n$, normalized so that $\|Y_{k,i}\|_{L^2(\Sigma)} = 1$. Then,

$$u = \sum_{k=0}^{\infty}\sum_{i=1}^{G(k,n)} a_{k,i}Y_{k,i}, \quad \text{where} \quad a_{k,i} := \int_\Sigma uY_{k,i}\,d\sigma\,.$$

Since the functions $Y_{k,i}$ are all eigenfunctions of the Laplace–Beltrami operator $-\nabla^2$ on the sphere and

$$-\nabla^2 Y_{k,i} = k(k+n-2)Y_{k,i}\,,$$

we then get

$$\|u\|_2^2 = \sum_{k=0}^{\infty}\sum_{i=1}^{G(k,n)} a_{k,i}^2, \quad \|\nabla u\|_2^2 = \sum_{k=1}^{\infty}\sum_{i=1}^{G(k,n)} k(k+n-2)a_{k,i}^2\,. \tag{13}$$

From (8) we have that for any $k \geq 0$, $i = 1, \ldots, G(k,n)$,

$$|a_{k,i}| \leq C_0\|Y_{k,i}\|_\infty\sqrt{\Delta(u)}\,.$$

Therefore, for every $N \in \mathbb{N}$ there exists a constant $C(N)$ such that

$$\|u\|_2^2 \leq C(N)\Delta(u) + \sum_{k=N}^{+\infty}\sum_{i=1}^{G(k,n)} a_{k,i}^2. \tag{14}$$

Let us now choose N_0 such that

$$N_0(N_0+n-2) \geq 2c_1c_2,$$

where c_1, c_2 are as in (12). Then, plugging inequality (14) into (12) and using (13) we finally get

$$\begin{aligned}\Delta(u) \geq \frac{1}{2c_1}\int_\Sigma |\nabla u|^2\,d\sigma + \frac{1}{2c_1}\sum_{k=1}^{\infty}\sum_{i=1}^{G(k,n)} k(k+n-2)a_{k,i}^2 - \\ - c_2\Big[C(N_0)\Delta(u) + \sum_{k=N_0}^{\infty}\sum_{i=1}^{G(k,n)} a_{k,i}^2\Big] \\ \geq \frac{1}{2c_1}\int_\Sigma |\nabla u|^2\,d\sigma - c_2 C(N_0)\Delta(u).\end{aligned}$$

The assertion follows from this inequality. □

Next result has been proved in [9, Lemma 1.4].

Lemma 3.2. *For any $v \in W^{1,\infty}(\Sigma)$ such that $\int_\Sigma v\,d\sigma = 0$ we have*

$$\|v\|_\infty^{n-1} \leq \begin{cases} \pi\|\nabla v\|_2 & \text{for } n=2 \\ 4\|\nabla v\|_2^2 \log\left(\frac{8e\|\nabla v\|_\infty^2}{\|\nabla v\|_2^2}\right) & \text{for } n=3 \\ C\|\nabla v\|_2^2\|\nabla v\|_\infty^{n-3} & \text{for } n\geq 4 \end{cases} \tag{15}$$

for a constant $C = C(n)$ depending only on n.

We may now prove Theorem 1.1 for a set satisfying (5).

Proposition 3.3. *For any $M > 0$ there exist constants $C_1, C_2 > 0$, depending only on M, n, such that if $u \in W^{1,\infty}(\Sigma)$ satisfies (7), (8), $\Delta(u) < 1/2$,*

$$\|u\|_{L^\infty(\Sigma)} \leq \frac{1}{C_1} \quad \text{and} \quad \|\nabla u\|_{L^\infty(\Sigma)} \leq M,$$

then

$$\|u\|_\infty^{n-1} \leq C_2(M,n)\begin{cases} \Delta(u)^{\frac{1}{2}} & \text{for } n=2 \\ \Delta(u)\left(\log\frac{1}{\Delta(u)}\right) & \text{for } n=3 \\ \Delta(u) & \text{for } n\geq 4. \end{cases}$$

Proof. Define v as

$$v := \frac{1}{n}((1+u)^n - 1).$$

Since $\int_\Sigma v\,d\sigma = 0$, we may apply Lemma 3.2 to v to infer (15). Note that

$$v - u = \frac{1}{n}\sum_{h=2}^{n}\binom{n}{h}u^h.$$

Therefore if $\|u\|_\infty$ is chosen sufficiently small in dependence on n, we have

$$|u| \leq 2|v|, \quad \frac{|\nabla u|}{2} \leq |\nabla v| \leq 2|\nabla u|.$$

The assertion then follows by combining these two inequalities with (15) and (9). □

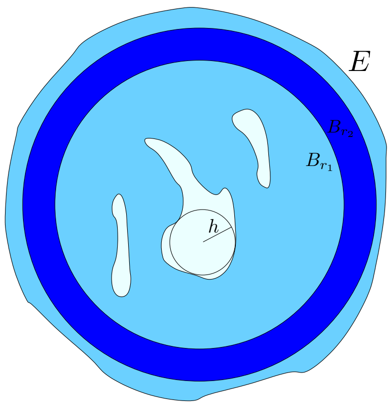

FIGURE 3. The “hole”, H, of E

3.2. **Reduction to sets containing a suitable ball.** In this subsection we are going to show that in order to prove (2) it is not restrictive to assume that the set E contains a suitable ball centered in x_1, the center of an optimal L^1 ball.

Proposition 3.4. *Let $0 < r_1 < r_2 < 1$ be fixed. There exists $\delta > 0$ such that for any $E \in \mathcal{C}^1_R$ with $D(E) < \delta$ and $B_{r_2}(x_1) \setminus B_{r_1}(x_1) \subset E$, where x_1 is the center of an optimal ball for $\lambda(E)$, then at least one of the following statements holds*

(*i*) $\lambda_{\mathcal{H}}(E) \leq 2D(E)^{\frac{1}{n-1}}$;

(*ii*) *there exists $\widetilde{E} \in \mathcal{C}^1_{R/2}$ with $B_{\frac{r_1+r_2}{2}}(\tilde{x}_1) \subset \widetilde{E}$, where $\tilde{x}_1$ is such that $\lambda(\widetilde{E}) = |\widetilde{E} \,\triangle\, B_1(\tilde{x}_1)|$, and satisfying*

$$\lambda_{\mathcal{H}}(E) \leq \lambda_{\mathcal{H}}(\widetilde{E}) + c\sqrt{D(E)}, \quad D(\widetilde{E}) \leq D(E), \tag{16}$$

for some constant c depending only on n.

Proof. By Lemma 2.10 we may choose $\delta > 0$ so that $\lambda_{\mathcal{H}}(E) \leq (r_2 - r_1)/2$ whenever $E \in \mathcal{C}^1_R$ and $D(E) \leq \delta$.

We define $H := B_{r_1}(x_1) \setminus E$, $\widetilde{E} := \alpha(E \cup H - x_1) + x_1$, with $\alpha := (\omega_n/(\omega_n + |H|))^{\frac{1}{n}}$, and

$$h := \sup\{r > 0 \ : \ B_r(x) \subset H \text{ for some } x \in H\}.$$

We will prove the validity of statement (*i*) or (*ii*) depending on the smallness of h defined above. Note that since $\partial B_{r_1}(x_1) \subset E$, $H = \mathring{B}_{r_1}(x_1) \setminus E$ is open and the very definition of h implies that $d(y, E) \leq h$ for any $y \in H$. Moreover, we may assume $h > 0$, otherwise (*ii*) trivially holds with $\widetilde{E} = E$.

Case I : Assume that $h \geq \frac{1}{2}\lambda_{\mathcal{H}}(E)$.

Since $|E \cap H| = 0$, from a well known property of the reduced boundary (see, for instance [8, Lemma 2.2], we have that $\partial^*(E \cup H)$ coincides $\mathcal{H}^{n-1}$-a.e. with $(\partial^* E \setminus \partial^* H) \cup (\partial^* H \setminus \partial^* E)$. Moreover, it is easily checked $\mathcal{H}^{n-1}(\partial^* H \setminus \partial^* E) = 0$.

Hence,

$$\begin{aligned} D(E) &= \frac{P(E) - n\omega_n}{n\omega_n} = \frac{P(E \cup H) + P(H) - n\omega_n}{n\omega_n} \\ &\geq \frac{P(E \cup H) + P(B_h) - n\omega_n}{n\omega_n} \\ &= \frac{P(E \cup H) + n\omega_n h^{n-1} - n\omega_n}{n\omega_n}, \end{aligned} \tag{17}$$

where the inequality follows from the isoperimetric inequality, since $|H| \geq |B_h|$. Similarly, since $|E \cup H| > |E| = \omega_n$, we have $P(E \cup H) \geq n\omega_n \geq 0$. Therefore, from (17) we conclude that

$$D(E) \geq h^{n-1} \geq \frac{1}{2^{n-1}} \lambda_{\mathcal{H}}(E)^{n-1}.$$

Hence, assertion (i) holds.

Case II : Assume that $0 < h < \frac{1}{2} \lambda_{\mathcal{H}}(E)$.

Observe first that $P(\widetilde{E}) = \alpha^{n-1} P(E \cup H) < P(E)$. Hence, $D(\widetilde{E}) < D(E)$, thus proving the second inequality in (16). Moreover, since $\alpha \to 1$ as $\delta \to 0$, we may always assume δ so small that $\widetilde{E} \in \mathcal{C}^1_{R/2}$.

Fix now an optimal ball $B_1(x_\infty)$ for $\lambda_{\mathcal{H}}(E)$. Since, by our choice of δ, $\lambda_{\mathcal{H}}(E) \leq (r_2 - r_1)/2$, we have that $B_{r_1}(x_1) \subset B_1(x_\infty)$. Moreover, choosing a smaller δ if needed, by Lemmas 2.10 and 2.11 we may also assume that $\max_{y \in E \cup H} |x_1 - y| \leq 2$.

We claim that

$$\lambda_{\mathcal{H}}(E) = \mathrm{d}_{\mathcal{H}}(E, B_1(x_\infty)) = \mathrm{d}_{\mathcal{H}}(E \cup H, B_1(x_\infty)). \tag{18}$$

Indeed, let $d = \mathrm{d}_{\mathcal{H}}(E, B_1(x_\infty))$. By the very definition of $\mathrm{d}_{\mathcal{H}}$ we have that $B_1(x_\infty) \subset E + B_d$, thus $B_1(x_\infty) \subset E \cup H + B_d$. On the other hand, $E \subset B_1(x_\infty) + B_d$. Therefore, thanks to the fact that $H \subset B_{r_1}(x_1) \subset B_1(x_\infty)$, we have also $E \cup H \subset B_1(x_\infty) + B_d$. Thus, we have shown that $\mathrm{d}_{\mathcal{H}}(E, B_1(x_\infty)) \geq \mathrm{d}_{\mathcal{H}}(E \cup H, B_1(x_\infty))$. To prove the opposite inequality recall that

$$d_{\mathcal{H}}(E, B_1(x_\infty)) = \max\left\{ \max_{x \in B_1(x_\infty)} \mathrm{dist}(x, E), \max_{y \in E} \mathrm{dist}(y, B_1(x_\infty)) \right\}.$$

If $\lambda_{\mathcal{H}}(E) = \mathrm{dist}(\bar{x}, B_1(\infty))$ for some $\bar{x} \in E$, the conclusion is trivial. Otherwise, we have that $\lambda_{\mathcal{H}}(E) = \mathrm{dist}(\bar{y}, E)$ for some $\bar{y} \in B_1(x_\infty)$. Then, $\bar{y} \notin H$ since by assumption $h < \frac{1}{2} \lambda_{\mathcal{H}}(E)$ and $h = \max_{y \in H} d(y, E)$. Hence, $\bar{y} \in B_1(x_\infty) \setminus B_{r_1}(x_1)$ and we may conclude that

$$\begin{aligned} \lambda_{\mathcal{H}}(E) &= \mathrm{dist}(\bar{y}, E) = \mathrm{dist}(\bar{y}, E \cup H) \\ &\leq \max_{y \in B_1(x_\infty)} \mathrm{dist}(y, E \cup H) \leq d_{\mathcal{H}}(E \cup H, B_1(x_\infty)), \end{aligned}$$

thus proving (18). Let us now show that, in addition to (18), we have also

$$\mathrm{d}_{\mathcal{H}}(E \cup H, B_1(x_\infty)) = \lambda_{\mathcal{H}}(E \cup H). \tag{19}$$

To prove this equality observe that if $B_1(x_0)$ is an optimal ball for $\lambda_{\mathcal{H}}(E \cup H)$, then $B_{r_1}(x_1) \subset B_1(x_0)$. In fact, if $B_{r_1}(x_1) \not\subset B_1(x_0)$ from (18) we have that $\mathrm{d}_{\mathcal{H}}(E \cup H, B_1(x_0)) \geq r_2 - r_1 > \lambda_{\mathcal{H}}(E) = \mathrm{d}_{\mathcal{H}}(E \cup H, B_1(x_\infty))$, thus contradicting the minimality of $B_1(x_0)$. Then, the inclusion $B_{r_1}(x_1) \subset B_1(x_0)$, together with the inequality $h < \frac{1}{2}\lambda_{\mathcal{H}}(E) \le \mathrm{d}_{\mathcal{H}}(E, B_1(x_0))$, imply, arguing as before,

$$\mathrm{d}_{\mathcal{H}}(E \cup H, B_1(x_0)) = \mathrm{d}_{\mathcal{H}}(E, B_1(x_0)) \ge \lambda_{\mathcal{H}}(E) = \mathrm{d}_{\mathcal{H}}(E \cup H, B_1(x_\infty)),$$

that is (19).

Let $B_1(\tilde{x}_\infty)$ be any unit ball such that $\lambda_{\mathcal{H}}(\widetilde{E}) = \mathrm{d}_{\mathcal{H}}(\widetilde{E}, B_1(\tilde{x}_\infty))$. From (19) we have

$$\begin{aligned}\lambda_{\mathcal{H}}(\widetilde{E}) = \mathrm{d}_{\mathcal{H}}(\widetilde{E}, B_1(\tilde{x}_\infty)) &\ge \mathrm{d}_{\mathcal{H}}(E \cup H, B_1(\tilde{x}_\infty)) - \mathrm{d}_{\mathcal{H}}(E \cup H, \widetilde{E}) \\ &\ge \mathrm{d}_{\mathcal{H}}(E \cup H, B_1(x_\infty)) - 2(1-\alpha),\end{aligned} \tag{20}$$

where the last inequality follows by observing that

$$\mathrm{d}_{\mathcal{H}}(E \cup H, \widetilde{E}) \le (1-\alpha) \max_{y \in E \cup H} |x_1 - y| \le 2(1-\alpha). \tag{21}$$

Then, an easy computation leads to

$$1 - \alpha = 1 - \left(\frac{\omega_n}{\omega_n + |H|}\right)^{\frac{1}{n}} \le c|H| \le c\lambda(E) \le cD(E)^{\frac{1}{2}}. \tag{22}$$

Combining this inequality with (20) and (18) yields the first inequality in (16).

To prove that $B_{\frac{r_1+r_2}{2}}(\tilde{x}_1) \subset \widetilde{E}$, where $\tilde{x}_1$ is the center of an optimal ball for $\lambda(\widetilde{E})$, we first observe that, since $D(\widetilde{E}) \le D(E)$, choosing δ sufficiently small (depending on r_2, r_1), by Lemma 2.11 it is sufficient to prove that

$$B_{\frac{r_1}{3}+\frac{2r_2}{3}}(\widetilde{x}_\infty) \subset \widetilde{E},$$

where $\widetilde{x}_\infty$ is the center of an optimal ball for the deviation $\lambda_{\mathcal{H}}(\widetilde{E})$. With this aim in mind we note that

$$\begin{aligned} d_{\mathcal{H}}(B_1(\widetilde{x}_\infty), B_1(x_\infty)) &\le \lambda_{\mathcal{H}}(\widetilde{E}) + d_{\mathcal{H}}(\widetilde{E}, E) + \lambda_{\mathcal{H}}(E) \\ &\le d_{\mathcal{H}}(\widetilde{E}, B_1(x_\infty)) + d_{\mathcal{H}}(\widetilde{E}, E) + \lambda_{\mathcal{H}}(E) \\ &\le 2\left[\lambda_{\mathcal{H}}(E) + d_{\mathcal{H}}(\widetilde{E}, E)\right]. \end{aligned}$$

Since from (21) we infer that

$$d_{\mathcal{H}}(\widetilde{E}, E) \le d_{\mathcal{H}}(\widetilde{E}, E \cup H) + d_{\mathcal{H}}(E \cup H, E) \le 2(1-\alpha) + h\,,$$

from the assumption $h < \frac{1}{2}\lambda_{\mathcal{H}}(E)$ and (22) we deduce that there exists a constant c, depending only on n such that

$$d_{\mathcal{H}}(B_1(\widetilde{x}_\infty), B_1(x_\infty)) \le c\left[\lambda_{\mathcal{H}}(E) + \sqrt{D(E)}\right].$$

Fix $\varepsilon > 0$. Choosing $\delta > 0$ sufficiently small, by Lemma 2.10, we deduce that

$$d_{\mathcal{H}}(B_1(\widetilde{x}_\infty), B_1(x_\infty)) < \varepsilon.$$

Note that by construction $B_{\alpha r_2}(x_1) \subset \widetilde{E}$. Moreover, since $\alpha \to 1$ and $|x_1 - x_\infty| \to 0$ as $\delta \to 0$, from the inequality above we may conclude that if δ is sufficiently small

$$B_{\frac{r_1}{3}+\frac{2r_2}{3}}(\widetilde{x}_\infty) \subset B_{\alpha r_2}(x_1) \subset \widetilde{E}.$$

This inclusion completes the proof of statement (*ii*). □

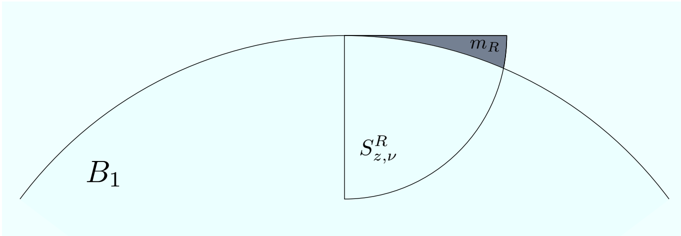

FIGURE 4. m_R is the measure of the shaded region

From now on we shall always assume $\theta = \pi/2$. Therefore, the explicit dependence on θ in the notation of sectors will be dropped. Moreover, we will denote by m_R the measure of the set obtained by subtracting the ball B_1 from a sector $S^R_{z,\nu}$ with vertex in $z \in \mathbb{S}^{n-1}$ and $\nu \in \mathbb{S}^{n-1}$ such that $\langle z, \nu\rangle = -1/\sqrt{2}$ (see Figure 4). Thus,

$$m_R = |S^R_{z,\nu} \setminus B_1|.$$

Next simple geometric lemma will be used in the proof of Proposition 3.7.

Lemma 3.5. *There exists* $\alpha \in (0,1)$ *with the property that for any* $r \in (\alpha, 1)$, $y \in \partial B_r$ *and* $z \in K_{y,\alpha,r}$, *with*

$$K_{y,\alpha,r} := \left\{ z \in B_1 \setminus B_r \ : \ \langle y, \frac{z-y}{|z-y|}\rangle \geq \alpha \right\},$$

we have

$$|S^R_{z,\nu} \setminus B_1| \geq \frac{m_R}{2} \quad \textit{for all} \ \ \nu \in \mathbb{S}^{n-1} \ \textit{with} \ \langle \nu, \frac{y-z}{|y-z|}\rangle \leq \frac{1}{\sqrt{2}}.$$

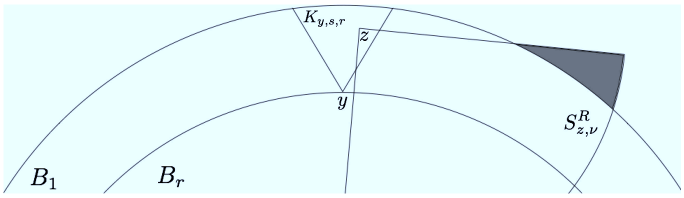

FIGURE 5. Graphic visualization of Lemma 3.5

Proof. We argue by contradiction, assuming that there exist sequences $\{r_j\}$, with $1 - \frac{1}{j} < r_j < 1$, $\{y_j\} \subset \partial B_{r_j}$, $\{z_j\} \subset B_1 \setminus B_{r_j}$ and $\{\nu_j\} \subset \mathbb{S}^{n-1}$, satisfying

$$\langle y_j, \frac{z_j - y_j}{|z_j - y_j|}\rangle \geq 1 - \frac{1}{j} \quad ; \quad \langle \nu_j, \frac{y_j - z_j}{|y_j - z_j|}\rangle \leq \frac{1}{\sqrt{2}} \tag{23}$$

and

$$|S^R_{z_j,\nu_j} \setminus B_1| < \frac{m_R}{2} \tag{24}$$

By a compactness argument, up to subsequences, we may assume $\nu_j \to \nu_0$, $z_j \to z_0$, $y_j \to y_0$ and $\frac{z_j - y_j}{|z_j - y_j|} \to \zeta_0$. Taking into account that $|y_j - z_j| \leq c/j$ we get that $z_0 = y_0$. Moreover, since

$$r_j \geq \langle y_j, \frac{z_j - y_j}{|z_j - y_j|} \rangle \geq 1 - \frac{1}{j},$$

passing to the limit we deduce $y_0 = \zeta_0$. Similarly from (23) we get $\langle -z_0, \nu_0 \rangle \leq 1/\sqrt{2}$. Finally, since $S^R_{z_j,\nu_j} \to S^R_{z_0,\nu_0}$ in the Hausdorff topology, passing to the limit in (24) we infer

$$|S^R_{z_0,\nu_0} \setminus B_1| \leq \frac{m_R}{2}.$$

Since $\langle -z_0, \nu_0 \rangle \leq 1/\sqrt{2}$ the last inequality contradicts the definition of m_R. □

Remark 3.6. Choosing $\overline{\nu} \in \mathbb{S}^{n-1}$ such that $|S^R_{e_n,\overline{\nu}} \setminus B_1| = \frac{2m_R}{3}$, defining $\beta := \arccos(\langle -e_n ; \overline{\nu} \rangle) < \frac{\pi}{4}$ and arguing exactly as in the previous lemma, one can prove that there exists a (possibly larger) number $\alpha \in (0,1)$ such that, for any $r \in (\alpha, 1)$ and $y \in \partial B_r$, we have also

$$|S^R_{y,\nu} \setminus B_1| \geq \frac{m_R}{2} \quad \text{for all} \;\; \nu \in \mathbb{S}^{n-1} \text{ with } \;\; \langle \nu, \frac{y}{|y|} \rangle \leq \cos(\beta). \tag{25}$$

Now we are ready to prove that, up to choosing the perimeter deficit small enough, there exists an annulus centered in x_1, with radii independent of E and δ, contained in E.

Let α be chosen so that the conclusions of Lemma 3.5 and (25) hold and set, for any $r < 1$, $k_r := |K_{y,\alpha,r}|$ with $y \in \partial B_r$. We define $r_0(R)$ as follows:

$$r_0(R) := \alpha \vee \left(1 - \frac{R}{4}\right). \tag{26}$$

Proposition 3.7. *Let $r_0 := r_0(R)$ be defined as in* (26) *and $r' \in (r_0, 1)$. There exists $\delta > 0$ such that for any $E \in \mathcal{C}^1_R$ with $D(E) < \delta$ we have*

$$B_{r'}(x_1) \setminus B_{r_0}(x_1) \subset E,$$

where x_1 is the center of an optimal ball for $\lambda(E)$.

Proof. Choose δ such that

$$D(E) \leq \delta \Rightarrow \lambda(E) < \frac{m_R}{2} \wedge k_{r'}.$$

Arguing by contradiction, assume that there exists $y \in \partial B_r(x_1) \setminus E$ for some $r \in (r_0, r']$. We claim that there exists $z \in K_{y,\alpha,r} \cap \partial E$. Indeed, if $K_{y,\alpha,r} \cap \partial E = \emptyset$ we would have

$$\lambda(E) \geq |K_{y,\alpha,r}| \geq k_r \geq k_{r'},$$

which is impossible. Let $S^R_{z,\nu} \subset E$ be an interior sector associated to z. Since $y \notin E$ we have that $\langle \nu, \frac{y-z}{|y-z|} \rangle < 1/\sqrt{2}$. Then by Lemma 3.5 we get the contradiction

$$\lambda(E) \geq |S^R_{z,\nu} \setminus B_1(x_1)| \geq \frac{m_R}{2}.$$

□

3.3. **Boundary of a set in $\mathcal{C}^1_R$ as a Lipschitz graph.** In this subsection we take advantage of the result established in the previous subsection to infer the existence of a Lipschitz map on the unit sphere parameterizing the boundary of a set $E \in \mathcal{C}^1_R$.

Proposition 3.8. *There exists $\delta > 0$ such that if $E \in \mathcal{C}^1_R$ with $D(E) < \delta$ and $B_r(x_1) \subset E$ for some $r > r_0(R)$, where $r_0(R)$ is defined as in (26) and x_1 is the center of an optimal ball for $\lambda(E)$, then for any $\xi \in \mathbb{S}^{n-1}$ there exists a unique $t > 0$ such that $x_1 + t\xi \in \partial E$.*

Proof. Let $\varepsilon := (1 - r_0(R))$. By Lemmas 2.10 and 2.11 we may choose δ such that $\lambda_{\mathcal{H}}(E) < \varepsilon$, $\lambda(E) < \frac{m_R}{2}$ and

$$\mathrm{d}_{\mathcal{H}}(B_1(x_\infty), B_1(x_1)) < \varepsilon.$$

This implies, since $1 - r_0(R) \leq \frac{R}{4}$, that

$$\mathrm{d}_{\mathcal{H}}(E, B_r(x_1)) \leq 3\varepsilon < R. \tag{27}$$

Indeed we can estimate

$$\begin{aligned}\mathrm{d}_{\mathcal{H}}(E, B_r(x_1)) &\leq \mathrm{d}_{\mathcal{H}}(E, B_1(x_\infty)) + \mathrm{d}_{\mathcal{H}}(B_1(x_\infty), B_1(x_1)) \\ &\quad + \mathrm{d}_{\mathcal{H}}(B_1(x_1), B_r(x_1)) < 3\varepsilon.\end{aligned}$$

Assume by contradiction that there exist $\xi \in \mathbb{S}^{n-1}$, $0 < t_1 < t_2$ such that $z_i = x_1 + t_i\xi \in \partial E$ for $i = 1, 2$. According to (27), we have that $d(z_2, z_1) < R$. Then, if $\nu \in \mathbb{S}^{n-1}$ is such that $S^R_{z_2,\nu} \subset E$, we have $\langle \nu, \frac{z_1 - z_2}{|z_1 - z_2|}\rangle \leq 1/\sqrt{2}$. Indeed, if this is not the case, z_1 would lie in $\overset{\circ}{E}$. If $z_2 \in K_{z_1,\alpha,r}$, we apply Lemma 3.5 to infer the contradiction $\lambda(E) \geq \frac{m_R}{2}$, while if $|z_2| \geq 1$, then $\lambda(E)$ is even bigger than m_R. □

Under the assumptions of Proposition 3.8, ∂E can be represented as the graph of a suitable function $\rho : \partial B_1(x_1) \to \mathbb{R}$. The regularity of ρ will be addressed in the next two propositions.

Proposition 3.9. *Assume that E satisfies the assumptions of Proposition 3.8 with $x_1 = 0$ and that*

$$\lambda(E) < \frac{m_R}{2}. \tag{28}$$

Then, the function ρ belongs to $W^{1,1}(\mathbb{S}^{n-1})$.

Proof. We start by proving that $\rho \in BV(\mathbb{S}^{n-1})$. We will argue locally using spherical coordinates. Let $J \subsetneqq \mathbb{S}^{n-1}$ be open and set

$$V := \left\{ x \in \mathbb{R}^n \setminus \{0\} : \frac{x}{|x|} \in J \right\}.$$

Let $\Phi : \mathbb{R}^{n-1} \times \mathbb{R}_+ \mapsto \mathbb{R}^n \setminus \{0\}$ be the map associating to $(\omega, t) \in \mathbb{R}^{n-1} \times \mathbb{R}_+$, the point in $\mathbb{R}^n$ having spherical coordinates (ω, t). Then, there exists an open set $I \subset \mathbb{R}^{n-1}$ such that $V = \Phi(I \times \mathbb{R}_+)$ and $\Phi_{|I\times\mathbb{R}_+}$ is a diffeomorphism. Then, the set $F := \Phi^{-1}(E) \cap (I \times \mathbb{R}_+)$ has finite perimeter in $I \times \mathbb{R}_+$. Moreover, $F = \{(\omega, t) \in I \times \mathbb{R}_+ : 0 < t < \sigma(\omega)\}$, where $\sigma : I \to \mathbb{R}$ is defined as $\sigma(\omega) := \rho(\Phi(\omega, 1))$, and since F has finite perimeter, σ is a function of bounded variation in I (see, for instance, Theorem B in [4]). The assertion will follow once we show that $\sigma \in W^{1,1}(I)$.

To this aim, let Γ^*_σ denote the extended graph of σ, i.e.,

$$\Gamma^*_\sigma := \{(\omega, t) \in I \times (0, \infty) \ : \ \sigma^-(\omega) \leq t \leq \sigma^+(\omega)\},$$

where $\sigma^{\pm}(\omega)$ is the approximate upper (lower) limit of σ at ω, respectively (see [1, Definiton 3.67]). Note that, by a well known result of the theory of BV functions (see [7, Theorem 4.5.9 (5)]), Γ^*_σ coincides $\mathcal{H}^{n-1}$-a.e. with the $\partial^* F \cap \big(I \times \mathbb{R}_+\big)$ and that, by another well known result (see [11, Theorem 5, Sect 4.1.5]), $\sigma \in W^{1,1}(I)$ if and only if $\langle \nu_\sigma(w), e_n\rangle \neq 0$ for $\mathcal{H}^{n-1}$-a.e. $w \in \Gamma^*_\sigma$, where $\nu_\sigma(w)$ is the exterior measure theoretic unit normal to F at w.

Assume by contradiction that there exists $w \in \Gamma^*_\sigma \cap \partial^* F$ such that $\langle \nu_\sigma(w), e_n\rangle = 0$ and set, for $x \in \mathbb{R}^n, \nu \in \mathbb{S}^{n-1}, r > 0$, $B^+_r(x,\nu) := \{y \in B_r(z) : \langle y - x, \nu\rangle \geq 0\}$. Since $w \in \partial^* F$, we have (see [1, Theorem 3.59])

$$\lim_{r\to 0^+} \frac{|F \cap B^+_r(w, \nu_\sigma(w))|}{r^n} = 0. \tag{29}$$

By the area formula

$$|F \cap B^+_r(w, \nu_\sigma(w))| = \int_{E\cap V\cap \Phi(B^+_r(w,\nu_\sigma(w)))} J\Phi^{-1}\, dy, \tag{30}$$

where $J\Phi^{-1}$ denotes the Jacobian of Φ^{-1}. Since $\Phi_{|I\times\mathbb{R}_+}$ is a diffeomorphism,

$$\Phi(B^+_r(w, \nu_\sigma(w))) \supset B^+_{cr}(\Phi(w), \xi),$$

for some $c > 0$ and $\xi \in \mathbb{S}^{n-1}$, with

$$\langle \xi, \Phi(w)\rangle = 0.$$

Therefore, from (29) and (30) we get

$$\lim_{r\to 0^+} \frac{|E \cap B^+_r(\Phi(w), \xi)|}{r^n} = 0.$$

Hence, if $S^R_{\Phi(w),\nu} \subset E$, we have $S^R_{\Phi(w),\nu} \cap B^+_r(\Phi(w),\xi) = \emptyset$ for all r. Therefore, by Lemma 3.5 we conclude that $|S^R_{\Phi(w),\nu} \setminus B_1| \geq m_R/2$, thus contradicting (28). □

Proposition 3.10. *Under the assumptions of Proposition 3.9, ρ is Lipschitz. Moreover, there exists a constant M depending only on δ, R and on the dimension such that*

$$|\nabla\rho(\xi)| \leq M.$$

Proof. Since $\rho \in W^{1,1}(\mathbb{S}^{n-1})$, it is easily checked that for $\mathcal{H}^{n-1}$-a.e. $z \in \mathbb{S}^{n-1}$ the exterior normal to E at $\rho(z)z$ is given by

$$\nu = \frac{\rho(z)z - \nabla\rho(z)}{\sqrt{\rho(z)^2 + |\nabla\rho(z)|^2}}\,,$$

where $\nabla\rho$ is the tangential gradient of ρ. Let $\theta > 0$ be the angle between ν and z. Recalling Remark 3.6 and (28), we have that θ has to be smaller than $\beta + \frac{\pi}{4}$. Therefore,

$$\cos(\theta) = |\langle \nu, z\rangle| = \frac{\rho(z)}{\sqrt{\rho(z)^2 + |\nabla\rho(z)|^2}} \geq \cos\big(\beta + \frac{\pi}{4}\big).$$

From this inequality the conclusion immediately follows. □

We are are now in position to prove Theorem 1.1.

Proof of Theorem 1.1. Let $E \in \mathcal{C}^1_R$ with $D(E) < \delta$. By taking δ sufficiently small, from Proposition 3.7 we get that $B_{r'}(x_1) \setminus B_{r_0(R)}(x_1) \subset E$, where $r_0(R)$ is defined as in (26) and r' is as close to 1 as we wish. Then, by Proposition 3.4, and provided δ is small enough, either

$$\lambda_{\mathcal{H}}(E) \leq 2D(E)^{\frac{1}{n-1}},$$

which in particular implies (2), or there exists $\widetilde{E} \in \mathcal{C}^1_{R/2}$ satisfying

$$\lambda_{\mathcal{H}}(E) \leq \lambda_{\mathcal{H}}(\widetilde{E}) + c\sqrt{D(E)} \quad \text{and} \quad D(\widetilde{E}) \leq D(E) \tag{31}$$

for some constant c depending only on n and such that, if $\lambda(\widetilde{E}) = |\widetilde{E} \,\triangle\, B_1(\tilde{x}_1)|$, we have $B_r(\tilde{x}_1) \subset \widetilde{E}$ for some r strictly larger than the radius $r_0(R/2)$ and independent on E. Then, we may apply Propositions 3.8 and 3.10 to infer that the boundary of $\widetilde{E}$ is the graph over the boundary of $B_1(\tilde{x}_1)$ of a function $\rho \in W^{1,\infty}(\partial B_1(\tilde{x}_1))$ with its tangential gradient $\nabla\rho$ uniformly bounded by a constant depending only on δ, R and n. Setting $u(x) = \rho(x) - 1$ and recalling that $|\tilde{E}| = \omega_n$, we have that u satisfies (7), (8) and the hypotheses of Proposition 3.3, provided δ is sufficiently small. The result then follows by combining Proposition 3.3 and (31).

□

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