

A VARIATIONAL PROBLEM IN THE MECHANICS OF COMPLEX MATERIALS

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Dedicated to Louis Nirenberg

ABSTRACT. We analyze the possible nucleation of cracked surfaces in materials in which changes in the material texture have a prominent influence on the macroscopic mechanical behavior. The geometry of crack patterns is described by means of stratified families of curvature varifolds with boundary. Possible non-local actions of the microstructures are accounted for. We prove existence of ground states of the energy in terms of deformation, descriptors of the microstructure and varifolds.

We discuss variational modeling tools for the mechanics of elastic-brittle complex materials. The adjective ‘complex’ is attributed to bodies characterized by a prominent influence of changes in the material texture on the macroscopic mechanical behaviour. Such an influence is exerted through actions that often can be hardly portrayed in terms of standard stresses, rather they require descriptions in terms of entities power-conjugated with the rates of appropriate geometrical descriptors of the material microstructure. The resulting framework, the one of the mechanics of complex bodies, implies ‘slidings’ in the paradigms of the traditional view on continuum mechanics. Richer is also the landscape when a body undergoes macroscopic mutations such as nucleation and eventual grow of fracture: the interplay between macroscopic and microscopic behaviour becomes more intricate.

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1. **An existence result.** The essential features of the point of view adopted here and the resulting variational problem are here briefly outlined up to the statement of the existence of minimizers for the energy. Details on the physical motivations leading to the adopted choices are given in the subsequent sections since the preliminary presentation of the analytical results furnishes a complete preliminary view on the structures used.

- The morphology of a generic complex body is described by selecting as reference macroscopic place a bounded connected open set \mathcal{B} in \mathbb{R}^n , $n \geq 2$, with Lipschitz boundary, and using a finite-dimensional differentiable manifold \mathcal{M} to collect the possible representations of the microstructure.
- Possible macroscopic crack patterns are described by rectifiable subsets of \mathcal{B} with Hausdorff dimension k , $1 \leq k \leq n - 1$. Rectifiability implies the existence of approximate tangent spaces indicating the local orientation of even complicated crack patterns. On these sets, a family of integer rectifiable k -varifolds, $\{V_k\}$, $k = 1, \dots, n - 1$, is defined. They are Radon measures on a fiber bundle $\pi : \mathcal{G}_k \rightarrow \mathcal{B}$, also indicated by $\mathcal{G}_k(\mathcal{B})$, with typical fiber the Grassmanian $\mathcal{G}_{k,n}$ of all planes of dimension $k = 1, \dots, n - 1$. The Grassmanian is in essence the collection of all possible directions of the fracture which can cross in principle a given point. Each member V_k of the varifold family is endowed with generalized curvature \mathbf{A}_k , boundary ∂V_k and, for $k = 2, \dots, n - 1$, it satisfies the condition

$$\pi_{\#} |\partial V_k| \leq \pi_{\#} |V_{k-1}|,$$

where $\pi_{\#}$ is the projection in terms of measures associated with π . Any family of varifolds with curvature is called **stratified** when it satisfies the last condition above. The geometry of crack patterns is described here by a **stratified family of varifolds**.

- Macroscopic **deformations** (or **transplacements**) are represented by approximately differentiable maps

$$u : \mathcal{B} \rightarrow \hat{\mathbb{R}}^n$$

which are also orientation preserving and such that their discontinuity set is ‘included’ – the sense is specified later – in the support of the varifolds.

- An approximately differentiable map $\nu : \mathcal{B} \rightarrow \mathcal{M}$ assigns to each material element at $x \in \mathcal{B}$ a **descriptor** $\nu(x)$ **of the inner microstructural morphology**. From a functional point of view, it is convenient sometimes to consider \mathcal{M} embedded in \mathbb{R}^N , N large enough, as a closed submanifold, and consider such maps as elements of a space $X \subset L^1(\mathcal{B}, \mathcal{M})$. Possible (even non-local) interactions among elements of the material microstructure and/or self-actions of a single microstructural element on itself are accounted by a functional

$$\mathfrak{F} : X \rightarrow \mathbb{R}$$

which is assumed to be lower semicontinuous in L^1 and such that, for every k , the set $\{\nu \in X \mid \mathfrak{F}(\nu) \leq k\}$ is compact for the L^1 convergence. Finally, a map $j : X \rightarrow L^1$ is selected. In typical examples, depending on the constitutive choices, $j(\nu)$ is either the approximate differential $D\nu$ of $\nu \in X$ or the n -vector $M(D\nu)$ including 1 and all the minors of $D\nu$. The map j is assumed to be such that $j(\nu_k) \rightharpoonup j(\nu)$ weakly in L^1 if $\sup_k \mathfrak{F}(\nu_k) < +\infty$ and $\nu_k \rightarrow \nu$ in L^1 .

The energy functional under scrutiny here is indicated by $\mathcal{E}(V, u, \nu, \mathcal{B})$ and has the following expression:

$$\begin{aligned} \mathcal{E}(V, u, \nu, \mathcal{B}) := & \int_{\mathcal{B}} e(x, u, Du, j(\nu)) \, dx + \sum_{k=1}^{n-1} \alpha_k \int_{\mathcal{G}_k(\mathcal{B})} |\mathbf{A}_{(k)}|^{p_k} \, dV_k + \\ & + \sum_{k=1}^{n-1} \beta_k \mathbf{M}(V_k) + \gamma \mathbf{M}(\partial V_1) + \mathfrak{F}(\nu), \end{aligned} \quad (1)$$

where $\alpha_k, \beta_k, \gamma$ are nonnegative constants. A modified version of it is also considered later.

We discuss conditions assuring the existence of triplets (V, u, ν) minimizing the energy functional $\mathcal{E}(u, \nu, V, \mathcal{B})$. The discussion requires the specifications of appropriate function spaces where the problem is set and classes of admissible stored energies. To this aim, further notions are necessary.

1.1. Curvature varifolds with boundary. The definition of varifolds relies on the construction of a fiber bundle over \mathcal{B} having as a typical fiber the Grassmanian of k -planes which can cross \mathcal{B} itself, $1 \leq k \leq n-1$. Consider projectors $\Pi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ onto k -planes, defined by $\Pi^2 = \Pi$, $\Pi^* = \Pi$, $\text{Rank } \Pi = k$. The set containing them is a compact subset of $\mathbb{R}^n \otimes \mathbb{R}^n$ and can be identified with the Grassmanian manifold, $\mathcal{G}_{k,n}$, of k -planes through the origin in \mathbb{R}^n . Such a manifold can be considered as the prototype fiber of a fiber bundle $\pi : \mathcal{G}_k(\mathcal{B}) \rightarrow \mathcal{B}$. A k -**varifold** on \mathcal{B} is a nonnegative Radon measure V over $\mathcal{G}_k(\mathcal{B})$, in short $V \in \mathbf{M}(\mathcal{G}_k(\mathcal{B}))$. A projection $\pi_{\#}$ of measures over \mathcal{B} is naturally associated with π and defines a Radon measure $\pi_{\#}V$ over \mathcal{B} . Its value on \mathcal{B} , namely $\Pi_{\#}V(\mathcal{B})$, is called the **mass** of the varifold and is indicated by $\mathbf{M}(V)$. Amid varifolds, the attention is focused here on that defined over \mathcal{H}^k -measurable, k -rectifiable subsets \mathfrak{b} of \mathcal{B} . These varifolds are absolutely continuous measures with respect to $\mathcal{H}^k \llcorner \mathfrak{b}$ with density $\theta \in L^1(\mathfrak{b}, \mathcal{H}^k)$ and are called the **rectifiable varifolds** to remind the rectifiability of \mathfrak{b} . The generic member of this class is indicated by $V_{\mathfrak{b},\theta}(\varphi)$ and defined by

$$V_{\mathfrak{b},\theta}(\varphi) := \int_{\mathcal{G}_k(\mathcal{B})} \varphi(x, \Pi) \, dV_{\mathfrak{b},\theta}(x, \Pi) := \int_{\mathfrak{b}} \theta(x) \varphi(x, \Pi(x)) \, d\mathcal{H}^k(x), \quad (2)$$

for any $\varphi \in C^0(\mathcal{G}_k(\mathcal{B}))$. In the previous formula, $\Pi(x)$ is the projector over the approximate tangent space $T_x \mathfrak{b}$ to \mathfrak{b} at x , which exists due to the rectifiability of \mathfrak{b} itself [1, 2, 3].

When the density θ is allowed to take only nonnegative integer values, $V_{\mathfrak{b},\theta}$ is called an **integer rectifiable k -varifold associated with $(\mathfrak{b}, \theta, \mathcal{H}^k)$** .

Generalized notions of curvature and boundary for \mathfrak{b} can be then defined (see [27], [28]). The curvature tensor is at each point of $\mathcal{G}_k(\mathcal{B})$ a third-rank tensor $A \in \mathbb{R}^{n*} \otimes \mathbb{R}^n \otimes \mathbb{R}^{n*}$, while the boundary ∂V of the k -varifold V on $\mathcal{G}_k(\mathcal{B})$ is a vector valued measure on $\mathcal{G}_k(\mathcal{B})$.

Definition 1.1. A varifold V is called an **integer rectifiable curvature k -varifold with boundary** if

1. V is an integer, rectifiable k -varifold $V_{\mathfrak{b},\theta}$ associated with the triple $(\mathfrak{b}, \theta, \mathcal{H}^k)$, i.e., V is defined by (2),
2. there exists a function $A \in L^1(\mathcal{G}_k(\mathcal{B}), \mathbb{R}^{n*} \otimes \mathbb{R}^n \otimes \mathbb{R}^{n*})$, in components A_j^{li} , and a vector Radon measure $\partial V \in \mathbf{M}(\mathcal{G}_k(\mathcal{B}), \mathbb{R}^n)$, called the **varifold boundary measure**, such that, for every $\varphi \in C_c^\infty(\mathcal{G}_k(\mathcal{B}))$, with values $\varphi(x, \Pi)$, the

following relation holds:

$$\int_{\mathcal{G}_k(\mathcal{B})} \left(\Pi_j^i D_{x_j} \varphi + A_j^{il} D_{\Pi_j^i} \varphi + A_j^{ij} \varphi \right) dV(x, \Pi) = - \int_{\mathcal{G}_k(\mathcal{B})} \varphi d\partial V^i(x, \Pi).$$

The class of integer rectifiable curvature k -varifolds with boundary for which the map $(x, \Pi) \mapsto A(x, \Pi)$ is in $L^p(\mathcal{G}_k(\mathcal{B}), \mathbb{R}^{n*} \otimes \mathbb{R}^n \otimes \mathbb{R}^{n*})$, with $p \geq 1$ will be denoted in the sequel by $CV_k^p(\mathcal{B})$. A compactness theorem proved in [28] is useful for subsequent developments and is recalled below.

Theorem 1.2 (Compactness). *For $1 < p < \infty$, consider a sequence $\{V_r\} \subset CV_k^p(\mathcal{B})$ of curvature varifolds with boundary and the corresponding sequences of curvatures $\{A_r\}$ and boundary measures $\{\partial V_r\}$. For every open set $\Omega \subset\subset \mathcal{B}$, assume the existence of a constant $c(\Omega)$, depending on Ω , such that*

$$\mu_{V_r}(\Omega) + |\partial V_r|(\mathcal{G}_k(\mathcal{B})) + \int_{\mathcal{G}_k(\mathcal{B})} |A^{(r)}|^p dV_r \leq c(\Omega)$$

for every r . Then, there exists a subsequence $\{V^{(r_s)}\}$ of $\{V^{(r)}\}$ and a k -varifold $V \in CV_k^p(\mathcal{B})$, with curvature A and boundary ∂V , such that

$$V_{r_s} \rightharpoonup V, \quad A_{r_s} dV_{r_s} \rightharpoonup A dV, \quad \partial V_{r_s} \rightharpoonup \partial V,$$

in the sense of measures, as $r_s \rightarrow \infty$. Moreover, for any convex and lower semi-continuous function $f : \mathbb{R}^{n*} \otimes \mathbb{R}^n \otimes \mathbb{R}^{n*} \rightarrow [0, +\infty]$, one gets

$$\int_{\mathcal{G}_k(\mathcal{B})} f(A) dV \leq \liminf_{r_s \rightarrow \infty} \int_{\mathcal{G}_k(\mathcal{B})} f(A_{r_s}) dV_{r_s}.$$

Of course, the compactness theorem extends verbatim to stratified curvature varifolds with boundary.

1.2. Functional choices for the transplacement field. The transplacement field $x \mapsto u(x)$ is assumed to be an orientation preserving approximately differentiable map with possible jumps – the ones due to crack opening – contained in the domain of integration of the varifolds that describe in \mathcal{B} the geometry of the fractures breaking $u(\mathcal{B})$. The transplacement – by assumption – will allow for self-contact along the deformation, but self-penetration of the matter is forbidden. Possibly, a quantitative control on the traces of the transplacement field on the crack margins in term of the mass of the varifolds is required. At least two classes of possible transplacements can be then defined. They differ essentially by the regularity imposed to the traces of the transplacement field (the deformation) along the crack. The first one assume essentially a L^1 bound of the traces while the second one is more stringent and implies the a.e. approximate differentiability of the traces.

The approximate gradient at x of the displacement field u is indicated by $Du(x)$. Denoting by $\Lambda_r(\mathbb{R}^n \times \mathbb{R}^n)$ the space of r -vectors over $\mathbb{R}^n \times \mathbb{R}^n$, an n -vector can be naturally associated with Du . It is indicated by $M(Du)$. As a member of $\Lambda_n(\mathbb{R}^n \times \mathbb{R}^n)$, it is defined by

$$M(Du) := (\mathbf{e}_1, (Du)\mathbf{e}_1) \wedge \cdots \wedge (\mathbf{e}_n, (Du)\mathbf{e}_n),$$

where $(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)$ is a basis in \mathbb{R}^n . $M(Du(x))$ has as components 1 and all the minors of $Du(x)$, the determinants of all $k \times k$ submatrices of $D\nu(x)$, varying $0 \leq k \leq n$, with a suitable sign. For the sake of convenience, fixed k , denote by

$M_k(Du)$ the collection of all $k \times k$ minors of Du , and by $|\cdot|$ the Euclidean norm. The norm of $M(Du)$ is then

$$|M(Du)|^2 = 1 + \sum_k |M_k(Du)|^2.$$

1.2.1. *SBV-diffeomorphisms.* Candidates to be representatives of transplacements with possible discontinuities are modeled first on the space of special bounded variation¹ functions.

Definition 1.3. Let $V = \{V_k\}$ be a stratified curvature varifold with boundary. A map $u : \mathcal{B} \rightarrow \hat{\mathbb{R}}^n$ is called an **SBV-diffeomorphism** of a body placed in the reference place at \mathcal{B} and with crack described by V if

1. $u \in SBV(\mathcal{B}, \hat{\mathbb{R}}^n)$,
2. $|M(Du)| \in L^1$,
3. $\det Du(x) > 0$ for a.e. $x \in \mathcal{B}$,
4. the following inequality holds

$$\int_{\mathcal{B}} f(x, u(x)) \det Du(x) \, dx \leq \int_{\hat{\mathbb{R}}^n} \sup_{x \in \mathcal{B}} f(x, y) \, dy,$$

- for every $f \in C_c^\infty(\hat{\mathbb{R}}^n)$,
5. $\mathcal{H}^{n-1} \llcorner J_u \leq \sum_k \pi_{\#} V_k$.

We denote the set of SBV-diffeomorphisms defined above by $SBV - \text{dif}(\mathcal{B}, V, \hat{\mathbb{R}}^n)$.

The assumption in (1) includes that u is a.e. approximately differentiable, with approximate differential $Du(x)$ at x . Condition (2) imposes regularity. The one in item (3) assures that the map is orientation preserving. The inequality in (4) permits self-contact along the transplacement (deformation) and excludes self-penetration of the matter (see [11], [25]). The inequality in (5) controls the jump set of u in terms of the crack V .

The following closure theorem easily follows from [20].

Theorem 1.4. Consider a sequence of stratified varifolds $\{V_j\}$, precisely $V_j = \{V_{k,j}\}$, on \mathcal{B} , with $V_{k,j} \in CV_k^p(\mathcal{B})$, $p > 1$, and equibounded total variations, i.e. $\sup_j \Pi_{\#} V_{k,j}(\mathcal{B}) < \infty$. Take a sequence $\{u_j\} \subset SBV - \text{dif}(\mathcal{B}, V, \hat{\mathbb{R}}^n)$ such that

$$\sup_k \sum_{k=1}^n |M_k(Du_j)|^{p_k} + \|u_j\|_{\infty} < \infty$$

with

$$p_1 \geq 2, \quad p_k \geq \frac{p_1}{p_1 - 1}, \quad 2 \leq k < n, \quad p_n > 1.$$

Assume that there exist $u \in L^1(\mathcal{B}, \hat{\mathbb{R}}^n)$, $v \in L^1(\mathcal{B}, \Lambda_n(\mathbb{R}^n \times \hat{\mathbb{R}}^n))$, and $V \in CV_1^p(\mathcal{B})$, $p > 1$, such that $u_k \rightharpoonup u$, $M(Du_k) \rightharpoonup v$ weakly in L^1 , and $V_k \rightharpoonup V$ as measures.

¹An $L^1(\mathcal{B})$ function u is said to be of bounded variation when its partial derivatives in the sense of distributions are signed Radon measures. Such functions constitute a subspace of $L^1(\mathcal{B})$, indicated by $BV(\mathcal{B})$. Their gradient measures admit additive decomposition into absolutely continuous (with respect to the volume measure) and singular components. The density of the absolutely continuous part is the approximate gradient of u . Moreover, the latter one is also decomposed into a jump part, restricted to the jump set J_u , a subset of the approximate discontinuity set S_u , and a (so-called) Cantor part restricted to $(\mathcal{B} \setminus S_u)$. The subspace of special bounded variation functions, indicated by SBV , collects the BV -functions having no Cantor part in the decomposition of their gradient (see [5]).

Then, $v = M(Du)$. Consequently, if $\det Du > 0$ a.e., it follows that $u \in SBV - \text{dif}(\mathcal{B}, V, \hat{\mathbb{R}}^n)$.

Proof. The assumptions imply that $\sup_j \mathcal{H}^{n-1}(J_{u_j}) < \infty$. Consequently, the claim $v = M(Du)$ follows from [20] (see Theorem 3.4 there). It is then easy to check that u satisfy (1), (2), (4) and (5) of Definition 1.3. \square

1.2.2. *Extended weak diffeomorphisms.* The definition of the second class of transplacements mentioned above is based on a more stringent control on the shape of the jumps of candidate transplacements across the crack pattern. The control is purely geometrical (kinematical in this sense) and avoids the recourse of subsequent constitutive ingredients like the lower bounds on the energy. As a tool the notion of current associated with the graph of a map is called upon here, in particular, currents that can be defined over the graphs of the transplacement maps $u : \mathcal{B} \rightarrow \hat{\mathbb{R}}^n$ which admit approximate derivatives.

Denote by $\mathcal{D}^n(\mathcal{B} \times \hat{\mathbb{R}}^n)$ the space of differential n -forms defined on $\mathbb{R}^n \times \hat{\mathbb{R}}^n$ with compact support on $\mathcal{B} \times \hat{\mathbb{R}}^n$ and by $\mathcal{D}_n(\mathcal{B} \times \hat{\mathbb{R}}^n)$ its dual². The n -**current integration over the graph**³ \mathcal{G}_u of u – also called **current of u** for short – is defined (see [25]) to be the linear functional

$$G_u(\omega) := \int_{\mathcal{B}} \langle \omega(x, u(x)), M(Du(x)) \rangle dx,$$

for any n -form in $\mathcal{D}^n(\mathcal{B} \times \hat{\mathbb{R}}^n)$.

²Members of $\mathcal{D}_n(\mathcal{B} \times \hat{\mathbb{R}}^n)$ are the so-called **currents**. A subclass of them can be defined on \mathcal{H}^n -measurable and countably n -rectifiable subsets U of \mathbb{R}^n . For a \mathcal{H}^n -measurable map $\xi : U \rightarrow \Lambda_n(\mathbb{R}^n \times \hat{\mathbb{R}}^n)$ with unit norm a.e. with respect to the Hausdorff measure \mathcal{H}^n , i.e. $\|\xi\| = 1\mathcal{H}^n \llcorner U$ – a.e., and a \mathcal{H}^n -measurable map $\vartheta : U \rightarrow \mathbb{R}$, a current $T \in \mathcal{D}_n(\mathcal{B} \times \hat{\mathbb{R}}^n)$ is called **rectifiable** when it is of the form

$$T = \int_U \langle \omega, \xi \rangle \vartheta d\mathcal{H}^n.$$

In particular, when ϑ is integer-valued, T is called a **integer rectifiable current** (or integer multiplicity rectifiable current, see [25], vol. 1, p. 140, [17]).

³A current can be associated with the graph of a map $x \mapsto u(x)$, see [25]. In fact, the map admits Lusin representative on the set $\tilde{\mathcal{B}} \subseteq \mathcal{B}$ of Lebesgue points of both u and Du , a set such that $|\mathcal{B} \setminus \tilde{\mathcal{B}}| = 0$. Denote by the same tilde the Lebesgue values of u and its derivatives, namely $\tilde{u}(x)$ and $\tilde{Du}(x)$. The **graph** \mathcal{G}_u of u is defined by

$$\mathcal{G}_u := \left\{ (x, y) \in \mathcal{B} \times \mathbb{R}^m \mid x \in \tilde{\mathcal{B}}, y = \tilde{u}(x) \right\}.$$

It turns out that \mathcal{G}_u is a n -rectifiable subset of $\mathcal{B} \times \hat{\mathbb{R}}^n$ with approximate tangent space generated at x by the vectors

$$\left(\left(\mathbf{e}_1, \tilde{Du}(x) \mathbf{e}_1 \right), \dots, \left(\mathbf{e}_n, \tilde{Du}(x) \mathbf{e}_n \right) \right).$$

The n -vector $M(Du)$ contains then the elements characterizing such an approximate tangent space. The area formula yields then

$$G_u(\omega) = \int \langle \omega, \xi \rangle d\mathcal{H}^n \llcorner \mathcal{G}_u,$$

where $\xi(x) := \frac{M(\tilde{Du}(x))}{|M(\tilde{Du}(x))|}$ is the unit n -vector that orients the approximate tangent n -space to \mathcal{G}_u at $(x, \tilde{u}(x))$, and $|M(\tilde{Du}(x))|$ is the modulus of $M(\tilde{Du}(x))$, given by the natural Euclidean norm in $\Lambda_n(\mathbb{R}^n \times \hat{\mathbb{R}}^n)$. Moreover, $M(\tilde{Du}(x))$ is also tangent at x to the graph of u . G_u is an integer rectifiable n -current (for further details, see [25]).

The scalar $\mathbf{M}(G_u)$, defined by

$$\mathbf{M}(G_u) = \int_{\mathcal{B}} |M(Du(x))| dx,$$

is called the **mass**⁴ of G_u .

The **boundary** of the current G_u is then defined by duality as the $(n-1)$ -current acting on compactly supported smooth $(n-1)$ -forms ω in $\mathcal{B} \times \hat{\mathbb{R}}^n$, namely

$$\partial G_u(\omega) := G_u(d\omega), \quad \omega \in \mathcal{D}^{n-1}(\mathcal{B} \times \hat{\mathbb{R}}^n),$$

where $d\omega$ is the differential of ω . Notice that $\partial G_u = 0$ on $\mathcal{D}^{n-1}(\mathcal{B} \times \hat{\mathbb{R}}^n)$ if u is of class C^2 , by Stokes theorem. Such a relation holds also for Sobolev maps $u \in W^{1,n}(\mathcal{B}, \hat{\mathbb{R}}^n)$ by approximation. In general ∂G_u does not vanish⁵ in $\mathcal{B} \times \hat{\mathbb{R}}^n$ if $u \in W^{1,p}(\mathcal{B}, \hat{\mathbb{R}}^n)$, $p < n$.

Definition 1.5. Assigned a stratified curvature varifold $V = \{V_k\}_{k=1}^{n-1}$ with boundary, i.e. $V_k \in CV_k^p$, a map $x \mapsto u(x)$ is called an **extended weak diffeomorphism** when

- (i) $u \in L^1(\mathcal{B})$ and u is a.e. approximately differentiable,
- (ii) $|M(Du)| \in L^1(\mathcal{B})$,
- (iii) $\det Du(x) > 0$ for almost every $x \in \mathcal{B}$,
- (iv) for any $f \in C_c^\infty(\mathcal{B} \times \mathbb{R}^m)$

$$\int_{\mathcal{B}} f(x, u(x)) \det Du(x) dx \leq \int_{\mathbb{R}^m} \sup_{x \in \mathcal{B}} f(x, w) dw,$$

- (v) $\pi_\# |\partial G_u| \leq \sum_{j=1}^{n-1} \pi_\# V_j + \pi_\# |\partial V_1|$.

Conditions (ii), (iii), (iv) have the same meaning of their counterparts (2), (3), (4) in the definition of SBV-diffeomorphisms. Condition (v) implies that the mass of the boundary current ∂G_u is finite, in particular that $u \in SBV$ and the traces of u along its jump are approximately differentiable (compare [4]).

The space of extended weak diffeomorphisms with boundary controlled by a varifold is indicated by

$$df^{1,1}(\mathcal{B}, V, \hat{\mathbb{R}}^n).$$

The following closure theorem holds.

Theorem 1.6. *Consider on \mathcal{B} a sequence of stratified curvature varifolds $\{V_j\}$ with boundary, $V_j = \{V_{k,j}\}$, with $V_{k,j} \in CV_k^p(\mathcal{B})$, $p > 1$, and equibounded variations, i.e., $\sup_j \pi_\# V_{k,j}(\mathcal{B}) < \infty$. Take a sequence $\{u_j\}$ such that $u_j \in df^{1,1}(\mathcal{B}, V, \hat{\mathbb{R}}^n)$. Assume that there exist $u \in L^1(\mathcal{B}, \hat{\mathbb{R}}^n)$, $v \in L^1(\mathcal{B}, \Lambda_n(\mathbb{R}^n \times \hat{\mathbb{R}}^n))$, and $V = \{V_k\}$, $V_k \in CV_k^p(\mathcal{B})$, $p > 1$, such that $u_k \rightarrow u$, $M(Du_k) \rightarrow v$, weakly in L^1 and $V_k \rightarrow V$ as measures. Then $v = M(Du)$. Moreover, if $\det Du > 0$ a.e., one also finds that $u \in df^{1,1}(\mathcal{B}, V, \hat{\mathbb{R}}^n)$.*

⁴In particular, the following relation holds:

$$\mathbf{M}(G_u) = \mathcal{H}^n(\mathcal{G}_u).$$

⁵A typical example is the map $u(x) := \frac{x}{|x|}$, that belongs to $W^{1,p}(B^3(0,1), \mathbb{R}^3)$ for every $p < 3$. One computes that $\partial G_u = -\delta_0 \times \llbracket S^2 \rrbracket$ on $\mathcal{D}^2(B^3(0,1) \times \mathbb{R}^3)$, with δ_0 the Dirac delta at 0.

Proof. The assumptions imply that $\mathbf{M}(G_{u_j}) + \mathbf{M}(\partial G_{u_j}) \leq C$ independently of j . In particular, the sequence $\{u_j\}$ is equibounded in $BV(\mathcal{B}, \hat{\mathbb{R}}^n)$ so that, by passing eventually to subsequences, $\{u_j\}$ converges strongly in L^1 and a.e. to u . Moreover, G_{u_j} converges to an integer multiplicity rectifiable current S . It then follows that $v = M(Du)$ a.e. and $S = G_u$ (see [25]). Properties (ii), (iv), and (v) in Definition 1 hold true with respect to V . Hence, if u satisfies also (i), then $u \in dif^{1,1}(\mathcal{B}, V, \hat{\mathbb{R}}^n)$. \square

1.3. On the microstructure descriptor field. The self-energy contribution of the microstructure – the one described by $\mathfrak{F}(\nu)$ – is left underspecified here. The analysis here concerns the interplay between gross deformation and microstructural behaviour on the possible nucleation and growth of cracks in classes of elastic-brittle complex bodies. However, just for illustration purposes, some specific choices of $\mathfrak{F}(\nu)$ are listed below.

1. \mathfrak{F} can be represented by a standard multiple integral in the calculus of variations, defined on $X = W^{1,p}(\mathcal{B}, \mathbb{R}^N)$, $p > 1$, $N \geq 1$, and satisfying the lower bound

$$\mathfrak{F}(\nu) \geq C_1 \|\nu\|_{W^{1,p}} - C_2, \quad C_1 > 0,$$

with C_1 and C_2 appropriate constants. More generally, $\mathfrak{F}(\nu)$ can be considered to be defined on $X = W^{1,p}(\mathcal{B}, \mathcal{M})$, $p > 1$, with the same growth condition just prescribed. In this case, a specific example is Dirichlet's integral

$$\mathcal{D}(\nu) = \frac{1}{2} \int_{\mathcal{B}} |D\nu|^2 dx$$

for mappings ν from \mathcal{B} into the unit sphere S^2 , or the corresponding nonlocal relaxed energy

$$\tilde{\mathcal{D}}(\nu) = \frac{1}{2} \int |D\nu|^2 dx + 4\pi L(\nu)$$

where $L(u)$ is the length of the minimal connection of the singularities of ν (see [6] and also [25]). The map j given by $j(\nu) := D\nu$ has the required continuity.

2. It is also possible to consider fields ν with jump set J_ν , by choosing

$$X = \left\{ \nu \in SBV(\mathcal{B}, \mathbb{R}^N) \mid \nu(x) \in \mathcal{M} \text{ for a.e. } x \in \mathcal{B} \right\} \quad (3)$$

and

$$\mathfrak{F}(\nu) = \int_{\mathcal{B}} g(D\nu) dx + c\mathcal{H}^{n-1}(J_\nu) + \|u\|_\infty, \quad p > 1, c > 0, \quad (4)$$

where $g = g(\xi)$ is convex and satisfies the inequality $g(\xi) \geq C_1|\xi|^p - C_2$, $p > 1$, $C_1 > 0$. The constant c has constitutive nature and adjusts the physical dimensions: it has the same role of the constants C_1 and C_2 the previous example. In this case, it is well-known that \mathfrak{F} is L^1 lower semicontinuous on X , also $D\nu_k \rightharpoonup D\nu$ if $\sup_k \mathfrak{F}(\nu_k) < \infty$, $\nu_k \in SBV$ and $\nu_k \rightarrow \nu$ (see [5]). Of course, it is possible to fix $K > 0$, to restrict X to $X \cap \{\nu \mid \|u\|_\infty \leq K\}$ and to drop the L^∞ norm from the specific expression (4) of $\mathfrak{F}(\nu)$. Here again, the map $j : X \rightarrow L^1$, defined by $j(\nu) := D\nu$ has the required continuity.

3. We can also consider functionals $\mathfrak{F} : X \rightarrow \mathbb{R}$ as in (3), (4), with polyconvex integrands. Preliminarily, we just remind that, for approximately differentiable maps $\nu : \mathcal{B} \rightarrow \mathcal{M}$, the n -vector $M(D\nu(x))$ can be naturally defined along the same path followed in discussing the transplacement field u . In this case, also,

the components of $M(D\nu(x))$ are 1 and all the minors of $D\nu(x)$, in this case the determinants of all $k \times k$ submatrices of $D\nu(x)$, varying $0 \leq k \leq n \wedge \dim \mathcal{M}$, with a suitable sign. For $\nu : \mathcal{B} \rightarrow \mathcal{M} \hookrightarrow \mathbb{R}^N$, if $k > \dim \mathcal{M}$, it follows that $M_k(D\nu) = 0$. If $\nu_k \rightarrow \nu$ in L^1 and

$$\sup_j \sum_k \int |M_k(D\nu_j)|^{p_k} + \mathcal{H}^{n-1}(J_{\nu_j}) + \|\nu_j\|_\infty < \infty,$$

we get $M_k(D\nu_j) \rightarrow M_k(D\nu)$ weakly in L^1 provided that

$$p_1 \geq 2, \quad p_i = \frac{p_1}{p_1 - 1}, \quad p_{\dim \mathcal{M} \wedge n} > 1$$

(see [20]). Consequently, functionals \mathfrak{F} as in (4), where $g(\xi)$ is polyconvex and bounded from below by

$$g(D\nu) \geq C_1 \sum_k |M_k(D\nu)|^{p_k},$$

are L^1 lower semicontinuous. Here the map j is given by $j(\nu) := M(D\nu)$.

Other examples are possible. The ones listed in the previous paragraphs are, however, sufficient to clarify the stage and to suggest significant analogies.

1.4. Assumptions on the structure of the energy and existence theorems.

We analyze three possible general structures of the energy and constitutive classes corresponding to them. Boundary conditions on the fields u and ν can be of various nature and the existence results obtained here also apply to different choices of boundary conditions that are left unspecified, leaving the reader free to imagine the appropriate choices, compatibly with the specification of the space X . Of course, some of the possible boundary conditions are physically significant, others are not.

Depending on constitutive requirements, at least three different problems can be considered.

1.4.1. *Case 1.* Consider first the functional in (5). The following assumptions on the bulk energy density $e(\cdot)$ apply:

(H1-1) Assume that $j : X \rightarrow L^1(\mathcal{B}, \mathbb{R}^X)$. Moreover, e is such that there exists a Borel function

$$Pe : \mathcal{B} \times \hat{\mathbb{R}}^n \times \mathcal{M} \times \Lambda_n(\mathbb{R}^n \times \hat{\mathbb{R}}^n) \times \mathbb{R}^X \rightarrow \bar{\mathbb{R}}^+,$$

with values $Pe(x, u, \nu, \xi, N)$, which is

- (a) l. s. c. in (u, ν, ξ, N) for a.e. $x \in \mathcal{B}$,
- (b) convex in (ξ, N) for any (x, u, ν) ,
- (c) such that $Pe(x, u, \nu, M(Du), j(\nu)) = e(x, u, \nu, Du, j(\nu))$ for any list of entries

$$(x, u, \nu, Du, j(\nu)) \text{ with } \det Du(x) > 0.$$

(H2-1) The energy density e satisfies the growth condition

$$e(x, u, \nu, Du, j(\nu)) \geq C_1 |M(Du)|^r + \vartheta(\det Du(x))$$

for any $(x, u, \nu, Du, j(\nu))$ with $\det Du(x) > 0$, $r > 1$, $C_1 > 0$ constants and $\vartheta : (0, +\infty) \rightarrow \mathbb{R}^+$ a convex function such that $\vartheta(t) \rightarrow +\infty$ as $t \rightarrow 0^+$.

Moreover, assume that there exists a (possibly empty) **comparison stratified varifold** $\tilde{V} = \{\tilde{V}_k\}$, $\tilde{V}_k \in CV_k^{p_k}(\mathcal{B})$ such that all competing stratified varifolds $V = \{V_k\}$ satisfy $\pi_\# \tilde{V}_k \leq \pi_\# V_k$ for every k . We write $\pi_\# \tilde{V} \leq \pi_\# V$ for short.

So, given a stratified \tilde{V} and a constant K , consider the space $\mathcal{A}_{r,p,X,K,\tilde{V}}(\mathcal{B})$ of triplets (V, u, ν) such that

1. V is a stratified varifold $V = \{V_k\}$, $V_k \in CV_k^p$, $p > 1$ such that $\pi_{\#}\tilde{V} \leq \pi_{\#}V$,
2. $u \in dif^{1,1}(\mathcal{B}, V, \hat{\mathbb{R}}^n)$, $\|u\|_{\infty} \leq K$,
3. $|M(Du)| \in L^r(\mathcal{B})$,
4. $\nu \in X$.

$\mathcal{A}_{r,p,X,K,\tilde{V}}(\mathcal{B})$ is then a natural ambient where the existence of minimizers of the functional $\mathcal{E}(u, \nu, V, \mathcal{B})$ in (5) can be investigated and, in fact, proven on the basis of the closure property in Theorem 1.7, by a standard lower semicontinuity argument.

Formally, the theorem reported below holds.

Theorem 1.7. *Take constants $r, p > 1$, $K > 0$, and fix $\tilde{V} = \{\tilde{V}_k\}$, $\tilde{V}_k \in CV_k^p(\mathcal{B})$ for $k = 1, \dots, n-1$. Assume the existence of $(V, u, \nu) \in \mathcal{A}_{r,p,X,K,\tilde{V}}(\mathcal{B})$ satisfying prescribed boundary conditions for u and ν , which are compatible with the specific choice of the function space X . The energy functional $\mathcal{E}(u, \nu, V, \mathcal{B})$ attains its minimum value in the class $\mathcal{A}_{r,p,X,K,\tilde{V}}(\mathcal{B})$ with the prescribed boundary conditions.*

1.4.2. *Case 2.* A similar result holds in the setting of SBV-diffeomorphisms. As before, fix constants $p > 1$, $K > 0$ and assign a (possibly empty) comparison stratified varifold $\tilde{V} = \{\tilde{V}_k\}$ with $\tilde{V}_k \in CV_k^p$. Consider the space \mathfrak{S} of triplets (V, u, ν) such that

1. V is a stratified varifold $\{V_k\}$, $V_k \in CV_k^p$, $p > 1$ such that $\pi_{\#}\tilde{V} \leq \pi_{\#}V$,
2. $u \in SBV - dif(\mathcal{B}, V, \hat{\mathbb{R}}^n)$, $\|u\|_{\infty} \leq K$,
3. $\sum_k |M_k(Du)|^{r_k} \in L^1(\mathcal{B})$, with

$$r_1 \leq 2, \quad r_j \geq \frac{r_1}{r_1 - 1}, \quad r_n > 1$$

4. $\nu \in X$.

Then, on account of the closure property in Theorem 1.7, the energy functional $\mathcal{E}(u, \nu, V, \mathcal{B})$ in (5) is L^1 -lower semicontinuous on \mathfrak{S} , provided the bound (H2-1) is enforced and reduces to

(H2-2)

$$e(x, u, \nu, Du, j(\nu)) \geq C_1 \sum_k |M_k(Du)|^{r_k} + \vartheta(\det Du(x))$$

for any $(x, u, \nu, Du, j(\nu))$ with $\det Du(x) > 0$, $C_1 > 0$ constant and $\vartheta : (0, +\infty) \rightarrow \mathbb{R}^+$ a convex function such that $\vartheta(t) \rightarrow +\infty$ as $t \rightarrow 0^+$.

Existence of minimizers of $\mathcal{E}(u, \nu, V, \mathcal{B})$ in the class \mathfrak{S} easily follows.

1.4.3. *Case 3.* Theorem 1.4 provides the required weak continuity of the minors of minimizing sequences assuming a bound for the \mathcal{H}^{n-1} measure of the jump set J_{ν} of ν , an assumption that can be evaluated component by component. This aspect allows one to discuss existence of minimizers in the case in which the interaction between the deformation and the microstructure depends on the whole set of minors of the couple $(u, \nu) : \mathcal{B} \rightarrow \hat{\mathbb{R}}^n \times \mathcal{M}$.

Consider the specific version of $\mathcal{E}(u, \nu, V, \mathcal{B})$ given by

$$\begin{aligned}
\mathcal{E}(u, \nu, V, \mathcal{B}) := & \int_{\mathcal{B}} e(x, u, \nu, Du, D\nu) \, dx + \sum_{k=1}^{n-1} \alpha_k \int_{\mathcal{G}_k(\mathcal{B})} |\mathbf{A}_{(k)}|^{p_k} \, dV_k + \\
& + \sum_{k=1}^{n-1} \beta_k \mathbf{M}(V_k) + \gamma \mathbf{M}(\partial V_1) + \delta \mathcal{H}^{n-1}(J_\nu)
\end{aligned} \tag{5}$$

where $\alpha_k, \beta_k, \gamma$ and δ are nonnegative constants. The following assumptions hold for $e(x, u, \nu, Du, D\nu)$:

(H1-3) e is such that there exists a Borel function

$$Pe : \mathcal{B} \times \hat{\mathbb{R}}^n \times \mathcal{M} \times \Lambda_n(\mathbb{R}^n \times \hat{\mathbb{R}}^n \times \mathbb{R}^N) \rightarrow \bar{\mathbb{R}}^+,$$

with values $Pe(x, u, \nu, \xi)$, which is

- (a) l. s. c. in (u, ν, ξ) for a.e. $x \in \mathcal{B}$,
- (b) convex in ξ for any (x, u, ν) ,
- (c) for any list of entries

$$(x, u, \nu, Du, D\nu) \text{ with } \det Du(x) > 0$$

we get $Pe(x, u, \nu, M(Du, \nu)) = e(x, u, \nu, Du, D\nu)$

(H2-3) we have

$$e(x, u, \nu, Du, D\nu) \geq C_1 \sum_k |M_k(D(u, \nu))|^{r_k} + \vartheta(\det Du(x))$$

for any $(x, u, \nu, Du, D\nu)$ with $\det Du(x) > 0$, where $C_1 > 0$ is constant, $\vartheta : (0, +\infty) \rightarrow \mathbb{R}^+$ a convex function such that $\vartheta(t) \rightarrow +\infty$ as $t \rightarrow 0^+$ and

$$r_1 \geq 2, \quad r_i > \frac{r_1}{r_1 - 1}, \quad r_n > 1.$$

Given constants $p > 1, K > 0$, and a stratified varifold $\tilde{V} = \{\tilde{V}_k\}$, $V_k \in CV_k^p$, $p > 1$, Theorem 1.4 implies that the functional \mathcal{E} is L^1 -lower semicontinuous on the space \mathcal{T} of triplets (V, u, ν) such that

1. V is a stratified varifold $\{V_k\}$, $V_k \in CV_k^p$, $p > 1$ such that $\pi_\# \tilde{V} \leq \pi_\# V$,
2. $u \in SBV - dif(\mathcal{B}, V, \hat{\mathbb{R}}^n)$, $\|u\|_\infty \leq K$,
3. $\nu \in SBV(\mathcal{B}, \mathbb{R}^N)$, $\nu(x) \in \mathcal{M}$ for a.e. $x \in \mathcal{B}$,
4. $\sum_k |M_k(D(u, \nu))|^{r_k} \in L^{p_k}(\mathcal{B})$.

Existence of minimizers then follows.

2. Bodies and their microscopic texture. The motivations of the functional choices made in the previous section have their start from the justification of the portrait of a generic complex body. In classical field theories, in a primitive way, a body is considered as an abstract set. Its elements, called the **material elements**, are vaguely considered as clusters of atoms characterizing the material texture of a body at a given length scale, the one selected as an ingredient of the modeling procedure. This notion of material element includes a number of modeling choices: the selection of the internal length scale, the characteristics of the texture that are considered peculiar for the whole material etc. The vagueness is however accepted in the traditional format of continuum mechanics because each material element is described only through the place in space of its centre of mass. Such a description is then irrespective of the morphology of the material texture. The entire body is only identified with the region \mathcal{B} that it occupies in ambient space – consider from now

on $n = 3$ for the sake of physical evidence. Geometry is then invoked essentially for describing crowding and shearing of material elements, i.e. the macroscopic deformation. In solids, material elements are linked one with the other by chemical bonds assuring the coherency of the matter, heaps of grains a part. To develop physical deformations, power needs to be expended. Such a power refers to the rate of change in the relative placements of neighbouring material elements. We then usually represent the local density of such a power through the action of a co-vector over the velocity. Such a co-vector represents contact actions amid neighbouring material elements. It is defined by the measures of velocity and power needed to develop an assigned deformation. The linear structure of the ambient space permits to represent easily the local contact actions in the matter in terms of standard stress tensor. In purely conservative setting and under appropriate growth conditions for the energy (see [25]), such a tensor is given by the derivative of the energy density e with respect to the gradient of deformation $Du(x)$.

Peculiar effects in condensed matter physics indicate often the presence of local contact actions that are not strictly power conjugated with the rate of change of relative placements amid neighbouring material elements. Rather they are generated by mutations in the inner structure of the material elements themselves. Moreover, local microscopic changes can generate (even non-local) microstructural self-actions. The description of such microstructural actions, or better of the effects they generate at macroscopic level, needs a preliminary description of the material microstructural morphology, even at a coarse level. The map $x \mapsto \nu(x) \in \mathcal{M}$ provides such a description: $\nu(x)$ collects information on the aspects of the microstructure that are considered prominent for the modelling purposes in special cases. It can be a scalar, a vector, a tensor, a list of various order parameters etc. To construct the essential aspects of the mechanics of complex bodies, it is only necessary to declare that \mathcal{M} is a differentiable manifold without any further geometric structure (see [8, 29]). The specific selection of \mathcal{M} is a structural component of the model. The geometric properties of \mathcal{M} reflect physical properties. For example, when the complex material under analysis admits microstructural kinetic energy given by a quadratic form of the time rate of ν – along a motion ν depends on time – the coefficients of such a form assign a metric on \mathcal{M} (see [9]). Conversely, when Riemannian structure can be attributed to \mathcal{M} for some physical reasons, the microstructural kinetic energy can be considered quadratic. The consequent Levi-Civita connection allows one to distinguish in invariant way microstructural self-actions from the contact ones [34]. Additionally, homology of \mathcal{M} plays a role in the evaluation of the minimizers of the energy (see [24, 32]).

Examples of material microstructures and connected choices of the manifold of microstructural shapes \mathcal{M} are of various nature.

- Microstructural self-action can be recognized clearly in ferroelectrics. They have complex crystalline structures that can be polarized as a consequence of external electric action, induced strain, temperature changes. Once polarization is created in the generic crystalline cell – a natural choice for the material element in this case – the polarization itself generates a local electromagnetic field (see [12, 33]). This field acts on the whole cell and may have non-local effects. Moreover, the polarization is commonly not homogeneous. Differences in polarization amid neighbouring material elements – say elements in ‘contact’ across a polarization domain wall – generate contact actions (stress-type)

power conjugated with the rate of the gradient of polarization. In this special case the natural geometrical descriptor of the material microstructure is a polarization vector ν , the manifold of microstructural shapes \mathcal{M} a ball with radius given by the saturation polarization of the specific class of ferroelectric analyzed in particular cases.

- Microstructural actions of ‘contact’ type occur also in nematic liquid crystals. Liquid crystals are made of stick molecules with end-to-tail symmetry. The generic material element can be considered as a family of stick molecules. The simplest geometrical descriptor of the microstructure can be thus a prevailing direction in the orientation of the molecules, as suggested in [14, 15]. \mathcal{M} is then the projective plane, a choice necessary to account for end-to-tail symmetry. The energy connected with microstructural actions of contact type is then the energy density

$$\frac{1}{2}c |D\nu|^2,$$

with c an appropriate constant. A more complete picture is given by the energy density

$$\frac{1}{2}c |D\nu|^2 + c_1 (\text{Div } \nu)^2 + c_2 (\nu \cdot \text{curl } \nu)^2 + c_3 |(D\nu)\nu|^2.$$

Interaction with external fields, such as the electric (or magnetic) one can be accounted for and produce peculiar behaviours such as Fréedericsz transition connected with metastable states, etc. Further interaction with the macroscopic deformation complicate the scenario. Moreover, although the choice of ν as a direction, i.e. as an element of the projective space, is natural, it is not unique for liquid crystals in nematic order. If a deeper description of the local distribution of stick molecules within each material element is expedient for the purposes at hand, ν can be considered as a second-rank symmetric tensor $\nu := \zeta (\varsigma \otimes \varsigma - \frac{1}{3}\mathbf{I})$, with $\varsigma \in S^2$ and $\zeta \in [-\frac{1}{2}, 1]$. In this way, one has two morphological descriptors: the vector ς representing the prevailing direction of stick molecules - really one selects $\varsigma \in P^2$ - and a scalar ζ indicating the *degree of orientation* (as defined in [16]). The choice approximates the distribution of stick molecules by an ellipsoid. We have then a first set of substructural interactions power conjugated with the rate of ς and a second set conjugated with the rate of ζ . Each set is balanced independently. The possible energy should take into account both ς and ζ , and their gradients. Optical biaxiality can emerge in nematic liquid crystals, reducing the symmetry to that of a rectangular box. In this case, besides ς and ζ , two other scalar descriptors are necessary: the *degree of prolation* d_g and the *degree of triaxiality* d_t defined in [10] in terms of the eigenvalues of ν when it is considered as a second-rank tensor as above. They add information on the structure of the ellipsoid.

These examples do not exhaust the list of possibilities. They just indicate that the choices of the descriptors of the material microstructure depend on modeling issues and, above all, they are not unique. In developing a general model-building framework for the mechanics of complex materials, it is then useful to consider the manifold \mathcal{M} as abstract as possible, in order to not link it to specific classes of materials.

Moreover, the choice of the function space X where the maps $x \mapsto \nu(x)$ are presumed to be hosted has constitutive nature. For example, *SBV*-maps may

describe the formation of domain patterns in the spatial distribution of microstructures – micromagnetics is the typical example. More regular maps may not allow such a description. The control imposed by the varifolds in the definitions of SBV -diffeomorphisms and extended weak diffeomorphism have also constitutive nature.

Boundary conditions have to be selected compatibly with the suggestion of the specific physical circumstances envisaged and the choice of X . Physics allows in appropriate circumstances the assignment of the values of ν along the boundary of \mathcal{B} but we do not know loading devices by which the assignment at the boundary of non-null microtractions is possible – when $j(\nu) = D\nu$, such microtractions can be regarded as the derivative of the energy density with respect to $D\nu$, applied to the normal to the boundary.

As mentioned in the early remarks, self-actions of the microstructure are possible, also they can be non-local in principle. Examples have been furnished. Others can be added. The functional $\mathfrak{F}(\nu)$ reminds this aspects, leaving essentially to the rest of the energy the description of the interaction with the macroscopic deformation. Hypotheses on the structure of $\mathfrak{F}(\nu)$ are motivated by the presumption that, in absence of macroscopic deformation, the microstructure can have at a given temperature a ground state. Care should be paid, however, in accepting such an assumption.

Consider, for example, the case of quasicrystals. They are $Al-Mn$ -based alloys made by atomic aggregates having point group symmetry which is inconsistent with lattice translation. These clusters are incompatible with any Bravais lattice and so with the tiling of the ambient space. They are characterized by quasi-periodicity determined by local rearrangements due to jumps of atoms between neighboring places and/or collective atomic modes generated for example by the flipping of crisscrossing alterations needed to maintain matching rules. The topological alterations determining quasiperiodicity do not have a precise location in space, a priori. “Even if a quasicrystal is energetically stabilized representing a ground state, it was shown numerically that above some critical temperature the system is in a random-tiling-like phase or unlocked phase” [13].

The example suggests us that the assumption that $\mathfrak{F}(\nu)$ can describe a ground state in absence of macroscopic deformation has to be supplemented by the implicit condition to be at a temperature at which such a ground state can be clearly recognizable.

3. Varifolds and crack patterns. Besides material complexity and deformation, the other ingredient appearing in the picture described by the energy $\mathcal{E}(u, \nu, V, \mathcal{B})$ is the possible formation of macroscopic fractures. Elastic-brittle complex bodies are, in fact, under analysis. The influence of microstructural changes on the formation and growth of fracture has been recognized experimentally in various classes of complex bodies (see remarks on what happens in ferroelectrics in [19] and materials displaying strain plasticity effects [35]). A general theoretical description of the essential features of this influence is in [30]: there the fields involved are considered smooth out of the crack, which is prescribed from the beginning. The variational view adopted here allows us to consider non-smooth fields and the possible nucleation of fracture. That essential aspects of fracture mechanics have variational nature is already suggested in the pioneer work by Griffith [26]. On the basis of

that initial indication, a model has been formulated in [18] – it has been motivation and source of various subsequent analytical work. It is based on a requirement of minimality of the overall energy \mathfrak{E} which accounts for the macroscopic deformation and the possible presence of cracks, and is defined by

$$\mathfrak{E}(\mathcal{C}, u) := \int_{\mathcal{B}} e(x, Du(x)) \, dx + \int_{\mathcal{C}} \phi \, d\mathcal{H}^2,$$

where \mathcal{C} is the representation in \mathcal{B} of a surface-like crack occurring in $u(\mathcal{B})$, ϕ a constant surface energy, $d\mathcal{H}^2$ the two-dimensional Hausdorff measure. A cracking process is considered in a time interval $[0, \bar{t}]$. The variational view when it is required that *at each instant $t \in [0, \bar{t}]$ of a cracking process the pair $(\mathcal{C}, u)(t)$ realizes a minimum of the global energy \mathfrak{E} with \mathcal{C} an admissible crack*. Admissibility is intended in the sense that \mathcal{C} is a rectifiable set with zero volume measure. Operationally, the interval of time is then discretized and minimality is required at time steps. Minimizers are pairs (\mathcal{C}, u) . In evaluating them, problems arise: the essential difficulty is the control in three dimensions of minimizing sequences of surfaces leading to the possible actual crack, or better to its picture \mathcal{C} in the reference place. A convenient simplification can be adopted when the entire crack is open, \mathcal{C} coincides with the jump set of the transplacement field $x \mapsto u(x)$. In this case, the crack pattern can be profitably identified with the jump sets of displacement fields. Special bounded variation functions are candidates to be minimizers of the energy considered as a functional of the sole u . Still open remains the problem of finding regularity results which select minimizers with discontinuity sets endowed with the geometry of compatible cracks. Additionally, the evaluation of balance equations arising from the first variation requires a number of regularity assumptions on the geometry of the crack pattern. An extended review of the state of the work along these guidelines can be found in [7].

The view that we follow here – a view that we have introduced in [21], and commented in [22, 31] – is essentially based on the premise that a body undergoing fracture is *mutant*. The statement needs clarifications. When we deform, in fact, a simple elastic body, described in the traditional paradigm dating back to Cauchy, admissible elastic deformations have all in common the same reference place \mathcal{B} . When the body is elastic-brittle, the occurrence of a fracture in $u(\mathcal{B})$ implies an alteration – tough virtual – of the reference place. The actual fractured configuration is no more in one-to-one correspondence with the original \mathcal{B} , rather with \mathcal{B} and a set \mathcal{C} which is the pre-image of the crack pattern, the region of \mathcal{B} which is candidate to host a fracture after a deformation (if you want, to be fractured). Assigned boundary conditions, above all if the body is uncracked, the placement and the shape of a possible fracture is not known a priori – the body could also remain uncracked. It depends not only on the boundary conditions but also on the shape, the nature of the body and its interactions with the rest of the environment. The nature of the body includes, obviously also the material threshold, behind which in some point the material is no more able to sustain stress, so it breaks. The formation of a fracture is the result of the competition between the bulk energy and the surface energy which must exist along the margins of a crack in order to assure the stability of the matter. Of course, such a competition can have some result and its opposite in different regions of the body under scrutiny, depending on the circumstances envisaged. A way to picture the situation is to imagine of having at disposal *a class of bodies*, all being able to occupy the region \mathcal{B} , in a macroscopic configuration that can be taken as reference. The class includes the uncracked body and bodies

that are the uncracked one *and* some possible compatible crack pattern, considered in the reference place as a structure over \mathcal{B} . In principle, at the sole geometrical level which is used to describe a morphology of a body, every point of \mathcal{B} can be in principle interested by the presence of a crack. To parametrize the class at hands, thus, it is only necessary to parametrize the possible crack patterns. At a given point x a fracture crossing it can have approximate tangent space (remind that any admissible crack is considered as a rectifiable set) oriented in principle along any direction, since no restriction is natural at the geometric level, when we do not know any information about the constitutive nature of the material, information that is furnished when the energy of the body is assigned. Such a remark leads naturally to the use of varifolds. In fact, when we construct over \mathcal{B} a bundle with typical fiber the Grassmanian of k -planes, as a first step in the definition of varifolds, we are assigning at each point x information on the set of all (in principle) potential directions that a crack could have at x (in this way we are following also here the spirit of the mechanics of complex bodies). The choice of k is then a choice of the type of crack we imagine to meet. Selection of some directions over this bundle is made by a Radon measure, a *varifold*. In this way, the parametrization of the class of bodies imagined in beginning this section, is naturally made by varifolds. In particular, we want to use varifolds supported by admissible crack patterns because our actual interest is just in describing such patterns. Since admissible cracks are rectifiable sets with zero volume measure, the notion of varifold with curvature appears to be natural to describe the situation. The availability of an extended notion of curvature permits also the control of sequences of minimizers, when the varifold is introduced in the expression of the energy, by modifying it from $\mathfrak{E}(\mathcal{C}, u)$ to $\mathcal{E}(u, V, \mathcal{B})$, presence of material complexity, a part. When such a complexity comes into play, in analyzing the structure of $\mathcal{E}(u, \nu, V, \mathcal{B})$, three essential ingredients are self-evident: (i) the self energy of the microstructure on itself, a term indicated by $\mathfrak{F}(\nu)$, (ii) the interaction between microstructure and the gross deformation, (iii) the surface energy along the margins of the crack. This energy is an evolution from what proposed in [26] and [18]: in fact it is the sum of Griffith's term – a constant time the area of the lateral margins of the crack – a term accounting for the curvature of the varifold and an energy prescribed along the tip of the crack. In three dimensions, the tip component of the energy considers in additive way energetic contributions associated with (a) the length of the tip, (b) the curvature of the tip itself, and it is possible to add energetic contributions due to corners of the tip. Not only the presence of the curvature in the structure of the energy has an obvious analytical advantage in the control of sequence of minimizers, also it carries a definite physical meaning. In breaking material bonds to determine a crack, local and/or collective bending effects can occur – there is no restriction a-priori forbidding them. Of course, bending occurs in the current configuration $u(\mathcal{B})$. The curvature of the varifold accounts at macroscopic level of local microstructural effects at low scale, effects occurring in the cracking process. However, when bending of material bonds in the actual configuration breaks the bonds themselves, such a bending has a configurational effect because it contributes to the mutation of \mathcal{B} due to the nucleation and possible propagation of a crack. The effect is thus configurational and – as it is accepted commonly – all configurational effects are measured in \mathcal{B} because they indicate a macroscopic irreversible mutation of a body. The nature of the surface contribution of the energy is then configurational, as it is the simpler surface energy suggested by Griffith.

Irreversibility in the nucleation and/or growth of cracks is here accounted for by the prescription of a *comparison varifold*. If the determination of minimizers is repeated at the time steps of a (discretized in time) loading program, at the step $n + 1$ the comparison varifold is the varifold determined by the minimization procedure at the step n . In this sense, the assignment of a comparison varifold corresponds to a requirement of monotonicity in the subsequent nucleation and/or growth of cracks. Of course, assigning it does not mean that we are treating only cases in which an initial crack is prescribed because the initial comparison varifold can be null.

The description of crack patterns in terms of varifolds furnishes us a tool which is also able to describe intricate defect patterns such as the presence of dislocations behind the tip or attached at the margins of the crack. The possibility is given by the stratified structure of the varifold family that we use. It is then possible to assign peculiar energy over low dimensional defects. Jump sets of the deformation are forced to be contained in the support of the varifold – partially closed cracks can be then described. Different is the case of the field of morphological descriptor of the microstructure. Domains can be formed in regions far from the crack – think, for example, of a crack running in a magnetoelastic solid: magnetic domains occur here and there throughout the body, naturally. The situation is described when X is a space of special bounded variation functions. Also, the morphological descriptor field can have in principle its own defects – for example, if it is a spin field (so $\mathcal{M} = S^2$), disclinations can occur independently of the crack pattern or in interaction with it [23]. The possibility of describing them depends on the choice of X . The latter case remind that the behaviour of the material microstructure at the margins of the crack has to be discussed with reference to the way the continuum modeling of complex bodies accounts for it. In fact, in principle, the field $x \mapsto \nu(x)$ may or may not be continuous across the crack pattern. Continuity or lack of it across the crack pattern depend on whether in cracking a body one alters along the margins of the crack the microstructure, possibly determining a new microstructure (even a new type), or, else, the microstructure remains the same across the margins of the crack. In the philosophy of continuum mechanics, the question is also whether a crack just divides neighboring material elements or breaks the material elements met in front of the tip. The answer is matter of modelling.

4. Additional remarks. The minimum problem discussed here involves three variables: (1) a stratified family of varifolds, (2) the deformation, (3) the field of descriptors of the microstructural morphology. Consider a loading program (that is, vary the boundary conditions in time) along time steps. It is possible that at the step n a triple of variables minimizing the energy includes a null varifold and the transplacement field $x \mapsto u(x)$ is a weak diffeomorphism. In this case the body behaves like a complex elastic body [32] (see also [25]), elasticity intended in the pure sense: no formation of ‘holes’ or cracks of any nature. Imagine then that the loading program be such that at the step $n + 1$ minimizing triple contains a non-null varifold. So, in going from the step n to the step $n + 1$ a material threshold is overcome. The point to be stressed is that in this process the threshold is not imposed a priori as an independent condition, as it is usual in mechanics of materials. Rather it is included implicitly in the structure of the energy and the properties of the function spaces in which we look for minimizers $x \mapsto u(x)$ and $x \mapsto \nu(x)$. Once more we can stress the concept that the choice of X and the

space of transplacements is constitutive, as the assignment of the energy density e and the prescription of the geometric properties of \mathcal{M} are.

Theorems allowing discrimination in terms of boundary conditions and structure of the energy are not at disposal actually.

The results presented previously furnishes a possible view on fracture mechanics.

- The description of the geometry of crack patterns in terms of curvature varifolds allows one to maintain distinction between cracks and jump set of the transplacement field along the minimizing process. Curvature play a role in controlling this process. Its presence has physical ground – as explained previously – and extends Griffith’s energetic description of cracks. A role is also played by the control of varifolds on the jumps of the deformation.
- Stratified varifolds permit the attribution of peculiar energy to the tip, in order to account for the peculiar state of material bonds in the region where the growth of the crack can start. The stratification of Radon measures on manifolds of different dimensions is also a tool for the representation of different low-dimensional defect patterns around the crack, as for example, nets of dislocations.
- The presence and the property of the microstructural self-energy $\mathfrak{F}(\nu)$ permits the prescription of condition for the energy density e – conditions assuring the existence of minimizers of the whole energy – that are analogous to the ones occurring in standard elasticity of simple bodies (see [32] for the case, in elasticity, in which $\mathfrak{F}(\nu)$ is not accounted for).
- The first variation of $\mathcal{E}(u, \nu, V, \mathcal{B})$ can be computed when the support of the varifold is a generic rectifiable set – that is a compatible crack – *without* further regularity conditions on its geometry. The balance at the tip involves the Hamilton-Eshelby tensor \mathbb{P} which is now influenced by the actions exerted by the material microstructure and takes the form

$$\mathbb{P} := eI - Du^* \partial_{Du} e - D\nu^* \partial_{D\nu} e,$$

(see also [30, 21]) contributions coming from the first variation of $\mathfrak{F}(\nu)$ a part. In the previous relation, I is the unit second-rank tensor, the star indicates the formal adjoint.

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