SPECTRAL RESOLUTION AND RIGIDITY OF SPHERICAL SPACE FORMS

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ABSTRACT. Let $n \geq 5$ be odd. We prove quantitative rigidity from scalar Laplace spectra for two classes of round n-manifolds: homogeneous spherical space forms of even order and scalar lens spaces of odd order at least five. At fixed sufficiently small normalized frequency tolerance, an explicit strict bound on the lower limit of the least total unmatched multiplicity divided by $R^{(n+1)/2}$, along prescribed cutoffs R, forces a closed connected Riemannian manifold to be isometric to the reference space. Nonisometric lens spaces show that this exponent is sharp in general. A Besse homotopy n-sphere is round if it admits a constant-displacement isometry of finite odd order at least five.

Let P be a self-adjoint classical elliptic operator of order one with positive scalar principal symbol, acting on a Hermitian bundle of positive rank over a closed connected d-manifold, $d \geq 2$. If its spectrum concentrates modulo one near s fixed phases, with vanishing tolerance and $o(R^d)$ exceptional eigenvalue copies along a sequence of cutoffs, then the time-2π Hamiltonian map has order at most s, and there is a commuting quantizer A. Finer spectral resolutions give sharp upper bounds on the order of $P - A$. If $P - A$ is not smoothing, signed displacement profiles and Weyl mass determine the Liouville eigenvalue distribution of its normalized leading symbol.

For finite free isometric quotients of closed connected Zoll manifolds of dimension at least two, the Poisson relation is an equality, and the scalar spectrum determines the maximal dimension of returning components at each length. Among closed connected Riemannian n-manifolds whose distinct scalar eigenvalues have multiplicities $m_j = cj^{n-1} + O(j^{n-2})$, the possible positive coefficients are exactly $c = 2^n/((n-1)!N)$, $N \in \mathbf{Z}_{>0}$. The largest value characterizes round real projective space up to scale.

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Introduction

Let $n = 2m + 1 \geq 5$, let $r > 0$, and let $Q = S^n(r)/\Gamma_0$ be a round spherical space form, with $q = |\Gamma_0|$. Let M be a smooth closed connected Riemannian manifold of positive dimension. Write

$$\mu_j = \sqrt{r^2\lambda_j(M) + m^2}, \qquad \beta_j = \sqrt{r^2\lambda_j(Q) + m^2},$$

where the scalar eigenvalues are repeated according to multiplicity. For $\eta \geq 0$ and $R > 0$, let $E_\eta(R)$ be the least total number of unpaired copies in the two frequency multisets restricted to $(-\infty, R]$, when a pair (μ, β) is allowed if $|\mu - \beta| \leq \eta$ (Definition 1.1). Exact scalar isospectrality is equivalent to $E_0(R) = 0$ for every R.

Rigidity at a fixed frequency tolerance. The reference frequencies belong to $m + \mathbb{Z}_{\geq 0}$; if q is even, they belong to $m + 2\mathbb{Z}_{\geq 0}$. For Riemannian manifolds, $\cong$ denotes isometry. Suppose first that q is even, and put $R_K = m + 2K + 1$. For $0 \leq \eta < 1/2$,

$$\liminf_{K\to\infty} \frac{E_\eta(R_K)}{R_K^{m+1}} < \frac{1}{q\,2^{m-1}(m+1)!} \quad \Longrightarrow \quad M \cong S^{2m+1}(r)/\Gamma, \qquad |\Gamma| = q.$$

If Q is homogeneous, then M is isometric to Q (Theorem 2.2).

Suppose instead that

$$Q = S^{2m+1}(r)/\langle e^{2\pi i/q} I_{\mathbb{C}^{m+1}}\rangle, \qquad q \geq 5 \text{ odd},$$

and put $R_K = m + K + 1/2$. For $0 \leq \eta < 1/4$,

$$\liminf_{K\to\infty} \frac{E_\eta(R_K)}{R_K^{m+1}} < c_{m,q} \quad \Longrightarrow \quad M \cong Q, \qquad c_{m,q} = \begin{cases} (48q)^{-1}, & m = 2, \\ \dfrac{2}{q|1 - e^{4\pi i/q}|^m (m+1)!}, & m \geq 3. \end{cases}$$

This is Theorem 2.4. Nonisometric round quotients with $E_\eta(R_K) \asymp R_K^{m+1}$ show that the exponent $(n+1)/2$ in the isometry conclusions cannot be increased (Theorem 2.7 and Proposition 22.2).

Ikeda's Problem (0.1) concerns determination by the scalar spectrum among spherical space forms. His Theorem III gives isometry when one of the two space forms is homogeneous. In dimension five, Theorem IV determines a homogeneous lens space among arbitrary closed Riemannian manifolds with the same scalar spectrum [39, Theorems III–IV]. Prüfer proves finite spectral determination in the round class: two unit round space forms of dimension $2m+1$ and fundamental-group order q are isospectral if their eigenvalues $\lambda \le m^2 q(q+2)$, with multiplicities, agree [56, Corollary 3, p. 293]. For $q \ge 3$, his Corollaries 4–6 characterize homogeneous lens spaces in this round class by the multiplicity $(m+1)^2-1$ of the eigenvalue $4(m+1)$ [56, pp. 308–310]. For finite subgroups of a fixed compact connected semisimple group G, a closed subgroup $K \subset G$, and a finite-dimensional representation τ of K, Lauret–Miatello determine all τ-spherical representation multiplicities from cofinite data. When the τ-spherical representations form a finite union of strings, suitable finite data suffice; sufficiently sparse exceptional indices are also allowed within each string [48, Theorems 1.1–1.2 and Corollary 3.2]. For an arbitrary comparison manifold, Theorems 2.2 and 2.4 deduce rigidity from a fixed frequency tolerance and a bound on unmatched eigenvalue copies, counted with multiplicity, along a sequence of cutoffs.

For homogeneous Q in the two classes, Corollary 2.5 gives

$$\limsup_{j\to\infty} r\left|\sqrt{\lambda_j(M)} - \sqrt{\lambda_j(Q)}\right| < \varepsilon_Q \quad \Longrightarrow \quad M \cong Q,$$

where $\varepsilon_Q = 1/2$ in the even case and $\varepsilon_Q = 1/4$ in the odd case. Hence

$$\operatorname{Spec}(\Delta_M + B_M) = \operatorname{Spec}(\Delta_Q + B_Q) \quad \Longrightarrow \quad M \cong Q$$

for arbitrary bounded self-adjoint operators B_M, B_Q on the respective scalar L^2 spaces (Corollary 2.6). The same conclusion holds for the self-adjoint form sums $\Delta_Z + b_Z$, $Z \in \{M, Q\}$, where b_Z is a symmetric sesquilinear form on $H^1(Z)$ satisfying

$$|b_Z[u]| \le C_Z \langle (1+\Delta_Z)^{\alpha} u, u\rangle, \qquad u \in H^1(Z), \qquad 0 \le \alpha < 1/2$$

(Proposition 19.1). For symmetric scalar $B_Z \in \Psi^1_{\mathrm{cl}}(Z)$, it suffices that the geodesic average of $\sigma_1(B_Q)/(2|\xi|_Q)$ over a common period be constant on S^*Q (Theorem 20.8). There are nonhomeomorphic manifolds whose scalar Laplacians become isospectral after symmetric first-order differential perturbations (Theorem 21.1). Isospectral connection Laplacians on line bundles over nonhomeomorphic manifolds also occur in the magnetic examples of Gordon–Kirwin–Schueth–Webb [24, Proposition 3.4 and Theorem 3.6].

Geodesic return and constant displacement. The matching bounds determine the dimension and volume by Weyl's law. Put $t = \pi$ in the even case and $t = 2\pi$ in the odd case. Along a subsequence of the stated cutoffs with $E_\eta(R) = O(R^{m+1})$, all but $o(R^n)$ phases $e^{it\mu_j}$, counted with multiplicity, lie in a fixed closed subarc of the open semicircle centered at e^{itm}. Positivity of localized traces gives the common geodesic period tr. For the scalar Laplacian, the principal return symbol is a fourth root of unity; the reference semicircle contains exactly one such root. A commuting logarithm gives a quantizer A on the reference lattice, with $P - A \in \Psi^{-1}_{\mathrm{cl}}$ for $P = \sqrt{r^2\Delta_M + m^2}$. The matching bound controls the difference between its cluster multiplicities and the reference multiplicities (Proposition 10.1). The return symbol is given by Duistermaat–Guillemin [17, proof of Theorem 3.1, pp. 53–54]; Colin de Verdière gives the scalar commuting normal form [12, Theorem 1.1, pp. 510–511].

A metric is Besse if all its geodesics are closed, and Zoll here if all unit tangent vectors have the same prime period. Under either matching hypothesis, M is Besse. Its universal Riemannian cover is a compact smooth homotopy sphere [4, Theorem 1.1 and the following paragraph]. All prime geodesics on that cover have the same length [57, Theorem A]. If the covering degree is even, the metric has constant sectional curvature [49, Theorem 2].

An isometry h has constant displacement if $d(x, hx)$ is independent of x. For odd covering degree, the following criterion applies: *a Besse homotopy sphere of odd dimension at least five is round if it admits a constant-displacement isometry of finite odd order at least five* (Theorem 14.2 and Corollary 14.3).

Berestovskii–Nikonorov classify simply connected Clifford–Wolf homogeneous manifolds, in which any two points are related by a constant-displacement isometry. On a compact homogeneous manifold, a sufficiently small such isometry is generated by a constant-length Killing field [6, Theorems 1 and 6]. The criterion above assumes a single isometry of the indicated finite order on a Besse homotopy sphere.

Let X be a closed connected Zoll manifold of prime period L, let a finite group Γ act freely by isometries, and put $Y = X/\Gamma$. Write G_t for the geodesic flow on SY. For every integer $s \geq 2$,

$$\operatorname{Fix} G_{L/s} \neq \emptyset \quad \Longleftrightarrow \quad \Gamma \text{ contains an element of order } s.$$

Every constant-speed closed geodesic of length L/s minimizes energy in its free homotopy class on a fixed parameter interval; its periodic Morse index is therefore zero (Lemma 15.1). At each nonempty reciprocal return L/s, the components of $\operatorname{Fix} G_{L/s}$ of maximal dimension have a common leading trace phase and positive densities (Lemma 15.2). For the odd scalar references, the trace bound gives an n-dimensional returning family on the cover whose evaluation map $(x, v) \mapsto x$ has nonzero degree (Proposition 13.6). Let L be the prime period on the cover. Surjectivity of evaluation gives $d(x, hx) \leq L/q$, while Lemma 15.1 gives $d(x, hx) \geq L/q$ for its deck transformation h of order q. Thus h has constant displacement. At order at least five, evaluation is a diffeomorphism and defines a geodesic circle action. The contact-volume inequality of Proposition 14.1, derived from [2, Theorem J.1(i)], and the volume identity of Proposition 12.3 make this action isometric. Its half-period is a free isometric involution; Lin–Schmidt's theorem gives roundness.

For an arbitrary odd-order reference $Q = S^n(r)/\Gamma_0$, put $R_K = m + K + 1/2$. If $E_\eta(R_{K_\nu}) = o(R_{K_\nu})$ along $K_\nu \to \infty$, with fixed $0 \leq \eta < 1/4$, the universal Riemannian cover X of M is a Zoll homotopy sphere with prime period $2\pi r$, and

$$|\pi_1(M)| = q, \qquad \operatorname{Vol}(X) = \operatorname{Vol}(S^n(r)).$$

The groups $\pi_1(M)$ and Γ_0 have the same sets of element orders (Theorem 15.6). If Γ_0 contains a Clifford translation of order at least five, then M is a round spherical space form of radius r and has the same complete scalar spectrum as Q (Theorem 15.7). For round space forms, the determination of the set of element orders is already contained in Ikeda's Corollary 2.8 [39], and its relation with the length set is given by Prüfer's Corollary 1 [55]. Colantonio–Lauret construct isospectral spherical space forms with nonisomorphic fundamental groups [14, Theorem 1.5].

Resolution and correction symbols. For a periodic quantizer A commuting with P, the eigenvalues of $P - A$ give the displacements within each A-cluster. Zelditch's Problem 1 asks for the relation between the geometry of a metric with periodic geodesic flow and the widths of its eigenvalue clusters [73].

Let E be a Hermitian bundle of positive rank over a closed connected n-manifold, $n \geq 2$, and let $P \in \Psi^1_{\mathrm{cl}}(M; E)$ be self-adjoint and elliptic, with positive scalar principal symbol. For $d \geq 1$, put

$$\Lambda = \bigcup_{j=1}^{d} (c_j + \mathbb{Z}), \qquad \mathcal{S} = \{e^{2\pi i c_1}, \dots, e^{2\pi i c_d}\},$$

where $c_1, \dots, c_d \in \mathbb{R}$ are distinct modulo $\mathbb{Z}$. Here $E_\eta(R)$ compares $\operatorname{Spec} P$ with a multiset supported on Λ. If, along $R_\nu \to \infty$,

$$\eta_\nu \longrightarrow 0, \qquad E_{\eta_\nu}(R_\nu) = o(R_\nu^n),$$

then the time-2π Hamiltonian map has finite order $L \leq d$. Thus $2\pi L$ is a common period; individual orbits may have smaller prime periods. The operator $e^{2\pi i L P}$ belongs to Ψ^0_{cl} and has unitary principal symbol H on $T^*M \setminus 0$. For every eigenvalue h of $H(\rho)$, all roots of $z^L = h$ belong to $\mathcal{S}$. There is a self-adjoint $A \in \Psi^1_{\mathrm{cl}}(M; E)$, with domain H^1 and strongly commuting with P, such that

$$\operatorname{Spec} A \subset \Lambda, \qquad P - A \in \Psi^{-1}_{\mathrm{cl}}$$

(Theorem 4.1).

If $s > 0$, $0 < \beta \le n$, and

$$\eta_\nu = O(R_\nu^{-s}), \qquad E_{\eta_\nu}(R_\nu) = o(R_\nu^{\beta}),$$

then

$$P - A \in \Psi_{\mathrm{cl}}^{-\lfloor ns/\beta \rfloor - 1}.$$

This upper bound on the order is sharp, even under the stronger hypothesis $\eta_\nu = o(R_\nu^{-s})$ (Theorem 5.1 and Proposition 5.3).

For positive self-adjoint scalar classical elliptic operators of order one on a closed connected manifold of dimension at least two, Duistermaat–Guillemin deduce periodicity from concentration near one arithmetic progression: there is a fixed interval length such that, for every $\varepsilon > 0$, at least a proportion $1 - \varepsilon$ of the frequencies in each sufficiently high interval of that length lie within distance ε of the progression; frequencies are counted with multiplicity [17, Theorem 3.2, p. 55]. Helton relates orbit periods to asymptotic eigenvalue differences. In particular, concentration of all sufficiently high scalar frequencies near one arithmetic progression, with errors tending to zero, implies periodicity [34, Propositions 4.1–4.2 and Theorem 4.4]. Humbert–Privat–Trélat obtain Besse periodicity from a uniform bound on the number of distinct scalar frequencies in intervals of a fixed length [38, Theorem 2(6)]. The finite-phase hypothesis above is imposed only along a sequence of cumulative cutoffs. It permits arbitrarily many distinct frequencies in a shrinking cluster and an exceptional set counted with multiplicity.

For the unit round reference $Q = L(p; 1, 1, a)$ in dimension five, where $p \ge 5$ is prime and $a \not\equiv 0, \pm 1 \pmod{p}$, the bounds $\eta_\nu = O(R_\nu^{-3/5})$ and $E_{\eta_\nu}(R_\nu) = o(R_\nu)$ give $P - A \in \Psi_{\mathrm{cl}}^{-4}$ for $P = \sqrt{\Delta_M + 4}$ by Theorem 5.1. The first three heat coefficients and the leading trace coefficient at time $2\pi/p$ then give isometry to Q (Theorem 23.2).

At a fixed negative order, the limiting displacement profiles are expressed by the Liouville eigenvalue measure of the leading correction symbol. Let $A \ge 1$ be a self-adjoint classical elliptic operator of order one on E, with positive scalar principal symbol and integer spectrum. Let $B = B^* \in \Psi_{\mathrm{cl}}^{-k}$, $k > 0$, strongly commute with A, and put $P = A + B$ on H^1. Write $c = \sigma_0(A^k B)$ and let ν_c be its Liouville eigenvalue measure, normalized as in (6.1). For joint eigenvalues (a_j, b_j) of (A, B), counted with multiplicity, put

$$D_\pm(R, \eta) = \#\{j \,:\, a_j \le R,\ \pm b_j > \eta\}.$$

For $t > 0$ and $R_N = N + 1/2$, Theorem 6.3 gives

$$\lim_{N\to\infty} R_N^{-n} D_\pm(R_N, t R_N^{-k}) = \int_{\mathbb{R}} \min\{1, ((\pm x)_+/t)^{n/k}\}\, d\nu_c(x),$$

$$\lim_{N\to\infty} R_N^{-n} E_{t R_N^{-k}}(R_N; P, A) = 2 \int_{\mathbb{R}} \min\{1, (|x|/t)^{n/k}\}\, d\nu_c(x),$$

where $y_+ = \max\{y, 0\}$. The second profile and the Weyl mass determine the pushforward of ν_c under $x \mapsto |x|$. The two signed profiles and that mass determine ν_c, including its atom at zero (Corollary 6.4). In the finite-phase setting, the signed profiles restricted to each spectral phase, together with its Weyl mass, determine the Liouville eigenvalue measure of the correction symbol on each eigensubbundle of H (Theorem 7.1). For scalar periodic operators, trace formulas for cluster distributions are given in [12, Theorem 3.1, pp. 518–519]. At coarser scales $\eta_N \downarrow 0$ with $\eta_N R_N^k \to \infty$, Theorem 6.5 is the Birman–Solomyak eigenvalue asymptotic for negative-order operators [54, Theorem 6.1 and Section 6.2].

Length spectra and maximal multiplicities. The cluster ranks determine the trace of the periodic quantizer. On a finite free quotient of a Zoll manifold, the scalar square root differs from a lattice quantizer by a constant and a commuting operator of order -1. The order -1 correction preserves the principal and subprincipal symbols, and hence the leading clean-trace coefficients [17, Theorem 4.5, equations (4.7)–(4.8)]. The length-spectrum problem asks whether the scalar spectrum determines

the lengths of all smoothly closed geodesics. Equality in the Poisson relation is sufficient; these are Problems 1 and 2 of Sutton [63], respectively, and the noncancellation question in [13, Section 13]. Write $\mathscr{L}(M)$ for the positive length set, including iterates. If M is a finite free isometric quotient of a closed connected Zoll manifold of dimension at least two, then

$$\operatorname{sing\,supp} \operatorname{Tr} e^{-it\sqrt{\Delta_M}} = \{0\} \cup \mathscr{L}(M) \cup (-\mathscr{L}(M)).$$

For every $\tau \in \mathscr{L}(M)$, the scalar spectrum also determines

$$D(\tau) = \max\{\dim C \,:\, C \text{ is a connected component of } \operatorname{Fix} G_\tau^M\},$$

where G_t^M is the geodesic flow on SM. The highest local frequency order of the wave trace at τ is $(D(\tau) - 1)/2$ (Theorem 24.1).

For unit round spherical space forms, Prüfer [55, Theorems 2–3 and Corollary 4, pp. 142, 150, 152] proves the corresponding equality for the shifted cosine trace

$$\operatorname{Tr} \cos\bigl(t\sqrt{\Delta + (n-1)^2/4}\bigr).$$

In the round case, the highest nonzero trace order also gives $D(\tau)$. In odd dimension, his proof combines a same-sign calculation at reciprocal periods with cyclotomic conjugacy, following Ikeda's argument for the poles of the harmonic generating function [39, Proposition 2.7, p. 83]. At lengths outside $\pi\mathbb{Z}$, Prüfer constructs local isometric immersions of spherical space forms carrying the returning geodesics, with totally geodesic image. He expresses their leading Poisson coefficients through volumes and transverse Poincaré determinants [56, Theorems 1–2, pp. 291–292]. His calculation gives the round Poincaré maps and their eigenvalues [56, Lemma 6 and its proof, p. 299]. For a unit round form of dimension $2m + 1$, the multiplicity of $k(k + 2m)$ is an exact quasipolynomial in the harmonic degree k for every $k \geq 0$ [56, Lemma 7, pp. 302–304]. Cianci gives direct clean-trace proofs for homogeneous lens spaces and three-dimensional lens spaces of prime order [11, Corollary 3 and Theorem 5].

At each nonempty reciprocal return of a finite free Zoll quotient, energy minimization gives a nonzero leading coefficient. Normalize the upstairs period to 2π. The eventual quasipolynomial multiplicities of the periodic quantizer give a generating function in $\mathbb{Q}(z)$ [12, Theorems 1.1 and 1.4, pp. 510–513]. The leading Weyl coefficient gives a pole of order $\dim M$ at 1. Cyclotomic conjugacy gives the same pole order at every primitive root of a given order. For $(a, s) = 1$, the fixed sets at $2\pi a/s$ and $2\pi/s$ coincide. Restoring the scalar phase gives a nonzero leading wave-trace coefficient at every length.

Fix an odd dimension $n \geq 5$, and let m_j denote the multiplicity of the jth distinct scalar eigenvalue. The asymptotic

$$m_j = cj^{n-1} + O(j^{n-2}), \qquad c > 0,$$

makes the geodesic flow periodic by Zelditch's Theorem A [73]. Each sufficiently high cluster then contains the same number of distinct eigenvalues (Lemma 25.1); the volume and covering identities restrict the leading coefficient. Among all closed connected Riemannian n-manifolds, its possible values are exactly

$$c = \frac{2^n}{(n-1)!\,N}, \qquad N \in \mathbb{Z}_{>0}.$$

Every value is realized by the round scalar lens space of order $2N$. The largest value occurs precisely on round real projective space, up to scale (Theorem 25.3 and Corollary 25.5). Within the round class, Prüfer's multiplicity formula and eventual spectral support already give this coefficient set and its maximal value [56, Lemma 7 and Corollary 1, pp. 302–305]. Zelditch's Theorem F(b) determines the round metric on an already specified real projective space when the complete multiplicity sequence, least geodesic-flow period, and volume agree with the round values [73].

Cyclic actions. For a finite free isometric quotient X/Γ, pullback identifies a scalar spectral cluster with the Γ-invariant subspace of the cluster on X in the same frequency interval. On a fixed closed manifold, Zoll contact forms with the same prime Reeb period are strictly contactomorphic when sufficiently close in C^2 [1, Proposition 3.2]. Let g_0 be the unit round metric on S^{2m+1}, $m \geq 2$. There is a C^3 neighborhood of g_0, independent of the finite group $H \subset O(2m+2)$, such that every H-invariant Zoll metric in this neighborhood with prime period 2π has the round H-representation in every sufficiently high spectral cluster (Theorem 28.1).

Let h generate a free isometric action of order three on a Zoll homotopy sphere of odd dimension at least five, with prime period L. Write G_t for its geodesic flow on the unit tangent bundle and $\widehat{h}$ for the tangent lift of h. The two families $\mathrm{Fix}(\widehat{h}^{-1}G_{L/3})$ and $\mathrm{Fix}(\widehat{h}^{-1}G_{-L/3})$ are homotopy spheres, and their evaluation maps are homotopy equivalences (Theorem 29.6). Roundness in this case is Question 29.9.

For a Zoll comparison action satisfying the prime two-weight hypotheses of Theorem 33.1, with $p \geq 5$, the four return families have dimensions $n-2, n-2, 1, 1$. Their finite symplectic return maps are symplectically conjugate to the corresponding round maps. If $p \geq 11$, the common evaluation image of the two large families and that of the two small families are disjoint embedded submanifolds with integral linking number of absolute value one (Theorem 34.1). Recognition of five-dimensional prime two-weight lens spaces at fixed positive tolerance is Question 23.3.

1. Spectral matching and operator conventions

All multisets are counted with multiplicity. For finite multisets $X, Y \subset \mathbb{R}$ and $\eta \geq 0$, an η-matching is a set of pairs of labelled copies such that each copy occurs in at most one pair and $|x-y| \leq \eta$ for every pair (x, y).

Definition 1.1. Let $b_\eta(X, Y)$ be the maximum cardinality of an η-matching, and put

$$E_\eta(X, Y) = |X| + |Y| - 2b_\eta(X, Y).$$

For multisets bounded below and finite on bounded intervals, put

$$E_\eta(R; X, Y) = E_\eta(X \cap (-\infty, R], Y \cap (-\infty, R]).$$

Intersections retain multiplicities. For operators, this notation refers to their spectral multisets. Limits are taken along the specified cutoff sequences.

Lemma 1.2. *For finite multisets, $E_\eta(X, Y) \geq \big||X| - |Y|\big|$. Inserting or deleting one copy changes E_η by at most one. A maximum matching can be chosen without crossings in the ordered lists.*

Proof. A matching has at most $\min(|X|, |Y|)$ pairs. Inserting one copy increases b_η by zero or one, so it changes E_η by one in absolute value. Deletion is the reverse operation. If $x \leq x'$ and $y \leq y'$ form the crossed pairs (x, y') and (x', y), then

$$x - \eta \leq x' - \eta \leq y \leq y' \leq x + \eta \leq x' + \eta.$$

Thus (x, y) and (x', y') are admissible. Among maximum matchings choose one with the fewest inversions of the matched right-hand indices. An inversion in two consecutive matched left-hand positions could be removed by this exchange, reducing the number of inversions by one. Hence there is none. □

$\Psi^a_{\mathrm{cl}}(M; E)$ denotes the classical pseudodifferential operators of order a on a Hermitian bundle E tensored with the half-density bundle. Unless stated otherwise, M is smooth, closed and connected, $\dim M = n \geq 2$, and E has positive rank. A self-adjoint elliptic operator $P \in \Psi^1_{\mathrm{cl}}(M; E)$ with positive scalar principal symbol qI_E has domain H^1, is bounded below, and has compact resolvent with smooth eigensections [23, Lemmas 1.3.5 and 1.6.3–1.6.4]. Strong commutation means commutation of all spectral projections. A self-adjoint elliptic operator of order one is a quantizer for a discrete set Λ if its spectrum is contained in Λ.

For a positive degree-one homogeneous symbol q, put $\Sigma_q = \{q = 1\}$. Write $\delta_s(x, \xi) = (x, s\xi)$ for fiberwise dilation. The radial Liouville measure $d\Sigma_q$ is characterized by

$$dx\, d\xi = s^{n-1}\, ds\, d\Sigma_q(\rho), \qquad (x, \xi) = \delta_s(\rho), \quad s > 0, \quad \rho \in \Sigma_q.$$

If $A \in \Psi^1_{\mathrm{cl}}(M; E)$ is self-adjoint with principal symbol qI_E, and $C \in \Psi^0_{\mathrm{cl}}(M; E)$ has principal symbol c, then

$$\operatorname{Tr}\big(C\chi(A/R)\big) = \frac{R^n}{(2\pi)^n} \int_{T^*M} \chi(q) \operatorname{tr}_E c\, dx\, d\xi + o(R^n), \qquad \chi \in C_c^\infty((0, \infty)). \tag{1.1}$$

Here degree-zero symbols are extended radially. The local semiclassical functional calculus has leading symbol $\chi(q)I_E$. Composition with C and integration of the diagonal fiber trace give (1.1); the scalar trace expansion is stated in [13, Section 5.4]. Since the fiber trace is invariant under changes of frame, the local formulas agree on overlaps.

Lemma 1.3. *Let P be self-adjoint and elliptic of order one with positive scalar principal symbol. Suppose that Z is unitary, strongly commutes with P, and satisfies $Z - I \in \Psi^{-1}_{\mathrm{cl}}$ and $\|Z - I\| < 1$. Then the principal logarithm $\log Z \in \Psi^{-1}_{\mathrm{cl}}$ is skew-adjoint and strongly commutes with P.*

Proof. Put $K = Z - I$ and let $f(z) = \log(1 + z)/z$, with $f(0) = 1$. This function is holomorphic on $|z| < 1$, which contains $\operatorname{Spec} K$. The holomorphic functional calculus for classical operators gives $f(K) \in \Psi^0_{\mathrm{cl}}$ [5, Example 2 and Theorem 5.4, arXiv version]. Indeed, on a contour $|z| = \rho$ with $\|K\| < \rho < 1$, the operators $zI - K$ are elliptic and invertible, so the cited spectral invariance theorem applies to their resolvents. For a nontrivial E, choose a complementary Hermitian bundle F with $E \oplus F$ trivial, apply that theorem to $(zI_E - K) \oplus I_F$, and restrict the inverse to E. Therefore

$$\log Z = K f(K) \in \Psi^{-1}_{\mathrm{cl}}.$$

The principal logarithm is purely imaginary on $\operatorname{Spec} Z$. The spectral theorem gives skew-adjointness and strong commutation with P. □

Lemma 1.4. *Let a finite group H act unitarily on a Hilbert space, and let Π_t, $0 \le t \le 1$, be a norm-continuous family of finite-rank orthogonal projections commuting with H. The representations $\operatorname{ran} \Pi_t$ are mutually isomorphic.*

Proof. If $\|\Pi_t - \Pi_s\| < 1$, Kato's intertwining operator [43, Chapter I, Section 4.6; Chapter III, Problem 3.21] belongs to the unital norm-closed algebra generated by Π_t and Π_s. It therefore commutes with H and identifies their ranges as H-modules. Norm continuity makes this isomorphism class locally constant; connectedness of $[0, 1]$ makes it constant. □

2. Recognition of spherical space forms

Throughout this section, M is a smooth closed connected Riemannian manifold of positive dimension. For such a manifold, let $\Delta_g = -\operatorname{div}_g \nabla_g$ be the scalar Laplacian, and write

$$0 = \lambda_0(g) < \lambda_1(g) \le \lambda_2(g) \le \cdots$$

for its eigenvalues, repeated according to multiplicity.

For a spherical reference Q of radius r, define

$$\delta_r(M, Q) = \limsup_{j \to \infty} r \left| \sqrt{\lambda_j(M)} - \sqrt{\lambda_j(Q)} \right|. \tag{2.1}$$

2.1. Reference metrics and matching conditions.

Put

$$n = 2m + 1 \ge 5, \qquad V_n = \operatorname{Vol}(\mathbb{S}^n(1)).$$

A spherical space form of radius $r > 0$ is a quotient $\mathbb{S}^n(r)/\Gamma_0$ by a finite orthogonal action that is free on the sphere. It is homogeneous when its isometry group acts transitively.

For integers $q \geq 2$ and $a_0, \dots, a_m$ prime to q, put

$$L(q; a_0, \dots, a_m) = \mathbb{S}^{2m+1}(1)/\langle \mathrm{diag}(\zeta^{a_0}, \dots, \zeta^{a_m}) \rangle, \qquad \zeta = e^{2\pi i/q}.$$

The metric is induced by the unit sphere. Multiplication of all weights by a unit modulo q, permutation of coordinates, and replacement of individual weights by their negatives preserve its isometry class.

The two reference classes are

$$\begin{aligned} &\text{(E)} \quad Q = \mathbb{S}^{2m+1}(r)/\Gamma_0, \qquad q = |\Gamma_0| \text{ even;} \\ &\text{(O)} \quad Q = Q_{q,n}(r) := \mathbb{S}^{2m+1}(r)/\langle e^{2\pi i/q} I_{\mathbb{C}^{m+1}} \rangle, \qquad q \geq 5 \text{ odd.} \end{aligned} \tag{2.2}$$

The dimension of M is not prescribed. Define the dimensionless frequencies

$$\mu_j = \sqrt{r^2 \lambda_j(M) + m^2}, \qquad \beta_j = \sqrt{r^2 \lambda_j(Q) + m^2}. \tag{2.3}$$

Put

$$a = 2 \quad \text{in case (E)}, \qquad a = 1 \quad \text{in case (O)}.$$

Set

$$R_K = m + aK + \frac{a}{2}, \qquad K \in \mathbb{Z}_{\geq 0}. \tag{2.4}$$

By Lemmas 8.1 and 8.3, the reference frequencies belong to $m + a\mathbb{Z}_{\geq 0}$.

Definition 2.1. With the convention of Definition 1.1, put

$$\mathcal{E}_\eta(K) = E_\eta(R_K; \{\mu_j\}_{j\geq 0}, \{\beta_j\}_{j \geq 0}), \qquad 0 \leq \eta < a/4. \tag{2.5}$$

Thus both frequency multisets are cut off at R_K, and $\mathcal{E}_\eta(K)$ counts the unmatched copies in both multisets.

The radius r, the reference Q, and the tolerance η are fixed as $K \to \infty$.

Theorem 2.2 (Rigidity for even-order spherical space forms). *Let Q be as in (E). If*

$$\liminf_{K\to\infty} \frac{\mathcal{E}_\eta(K)}{R_K^{m+1}} < c_{\mathrm{E}}(n, q) := \frac{1}{q 2^{m-1}(m+1)!}, \qquad 0 \leq \eta < \frac{1}{2}, \tag{2.6}$$

then M has dimension n and constant sectional curvature r^{-2}. More precisely,

$$M \cong \mathbb{S}^n(r)/\Gamma, \qquad |\Gamma| = q.$$

If Q is homogeneous, then $M \cong Q$ isometrically.

Theorem 2.3. *Let Q be as in (O). If*

$$\liminf_{K\to\infty} \frac{\mathcal{E}_\eta(K)}{R_K} = 0, \qquad 0 \leq \eta < \frac{1}{4}, \tag{2.7}$$

then $M \cong Q$ isometrically.

Theorem 2.4 (Rigidity for odd-order scalar lens spaces). *Let Q be as in (O). Put*

$$c_{\mathrm{O}}(n, q) = \begin{cases} \dfrac{1}{48q}, & n = 5, \\ \dfrac{2}{q|1 - e^{4\pi i/q}|^m (m+1)!}, & n \geq 7. \end{cases} \tag{2.8}$$

If

$$\liminf_{K\to\infty} \frac{\mathcal{E}_\eta(K)}{R_K^{m+1}} < c_{\mathrm{O}}(n, q), \qquad 0 \leq \eta < \frac{1}{4}, \tag{2.9}$$

then $M \cong Q$ isometrically.

Theorem 2.2 is proved in Sections 16–17; the two odd-order theorems are proved in Section 18. Condition (2.7) implies (2.9): along a sequence on which $\mathcal{E}_\eta(K)/R_K \to 0$, one has

$$\frac{\mathcal{E}_\eta(K)}{R_K^{m+1}} = \frac{\mathcal{E}_\eta(K)}{R_K}\, R_K^{-m} \longrightarrow 0.$$

Thus Theorem 2.3 is also a consequence of Theorem 2.4. Each hypothesis involves only the indicated lower limit; no bound at every sufficiently large cutoff is required.

Exact scalar isospectrality is the case $\eta = 0$ and $\mathcal{E}_0 = 0$.

Corollary 2.5. *Set $\varepsilon_Q = 1/2$ in case (E), and $\varepsilon_Q = 1/4$ in case (O). If*

$$\limsup_{j\to\infty} r\left|\sqrt{\lambda_j(M)} - \sqrt{\lambda_j(Q)}\right| < \varepsilon_Q, \tag{2.10}$$

then M is a spherical space form of radius r and is scalar-isospectral to Q. If Q is homogeneous, then $M \cong Q$. The latter conclusion always holds in case (O).

For a homogeneous reference this gives

$$M \not\cong Q \quad\Longrightarrow\quad \limsup_{j} r\left|\sqrt{\lambda_j(M)} - \sqrt{\lambda_j(Q)}\right| \geq \varepsilon_Q. \tag{2.11}$$

The constants ε_Q are sufficient bounds for the complete ordered spectra. In particular,

$$\lambda_j(M) - \lambda_j(Q) = O(1) \quad\Longrightarrow\quad \delta_r(M, Q) = 0.$$

For $\dim M = 5$ and $Q = \mathbb{RP}^5(r)$, Wu proved that $|\lambda_j(M) - \lambda_j(Q)| = O(j^{-4/5})$ implies $M \cong Q$ isometrically [71, Theorem 1].

Corollary 2.6. *Let Q be homogeneous in (E), or let Q belong to (O). If B_M and B_Q are bounded self-adjoint operators on the respective scalar L^2 spaces, then*

$$\mathrm{Spec}(\Delta_M + B_M) = \mathrm{Spec}(\Delta_Q + B_Q) \quad\Longrightarrow\quad M \cong Q. \tag{2.12}$$

In particular, this holds for arbitrary smooth real multiplication potentials. The same conclusion holds for the self-adjoint form sums associated with symmetric perturbations satisfying (19.1)*, with $0 \leq \alpha < 1/2$ (Proposition 19.1). This includes self-adjoint classical pseudodifferential perturbations of order strictly below one.*

Corollaries 2.5 and 2.6 are proved in Section 19.

Theorem 2.7 (Sharpness of the multiplicity exponent). *For $s \geq 3$, let*

$$Q = L(8; 1, \dots, 1), \qquad Q' = L(8; 1, \dots, 1, 3)$$

carry the unit round quotient metrics in dimension $2s - 1$. The first space is homogeneous, and the two spaces are not isometric. In Definition 2.1, take $M = Q'$, $r = 1$, and $m = s - 1$, so that $R_K = 2K + s$. For every $0 \leq \eta < 1/2$,

$$\mathcal{E}_\eta(K) = C_s R_K^s + O(R_K^{s-1}), \qquad C_s > 0. \tag{2.13}$$

In dimension seven, $C_4 = 1/192$. Thus an $O(R_K^{m+1})$ hypothesis with no restriction on its constant does not imply homogeneous quotient recognition. In particular, the exponent $m + 1$ in Theorem 2.2 cannot be increased.

The proof is given in Section 22.

Part 1. Spectral concentration and correction symbols

Let E be a positive-rank Hermitian bundle over a closed connected Riemannian n-manifold, $n \geq 2$. For $P = (\nabla^*\nabla + W)^{1/2} > 0$, with ∇ a unitary connection and W a smooth Hermitian endomorphism, the fixed-window hypothesis of Theorem 3.1 gives a common geodesic period and a pointwise constraint on the return symbol. For a general self-adjoint classical elliptic operator P of order one on E with positive scalar principal symbol, matching to a multiset supported on finitely many translates of $\mathbb{Z}$, with tolerance tending to zero and $o(R_\nu^n)$ unmatched copies along $R_\nu \to \infty$, gives a common Hamiltonian period and a strongly commuting quantizer A (Theorem 4.1). Polynomial bounds on the tolerance and unmatched count give an upper bound for the order of $P - A$ (Corollary 5.2). When $P - A$ has a first nonzero homogeneous term, signed phase profiles and phase Weyl masses determine the Liouville eigenvalue measures of its normalized leading symbol on each principal return eigensubbundle (Theorem 7.1).

3. Spectral phase concentration and bundle holonomy

Let $E \to M$ be a positive-rank Hermitian bundle over a smooth closed connected Riemannian manifold of dimension $n \geq 2$, with unitary connection ∇. For a smooth Hermitian endomorphism W such that $D = \nabla^*\nabla + W > 0$, put $P = D^{1/2}$. Its principal symbol is $|\xi|_g I_E$. Let G_t be the Hamiltonian flow of $|\xi|_g$ on $T^*M \setminus 0$, and put

$$\Pi_R = \mathbf{1}_{(0,R]}(P), \qquad N_P(R) = \operatorname{Tr} \Pi_R.$$

The propagator e^{itP} quantizes G_{-t}. For $v \in S^*M$ and $t \geq 0$, let $H_t(v)$ be parallel transport along $\tau \mapsto \pi G_{-\tau}(v)$, $0 \leq \tau \leq t$, where $\pi : T^*M \to M$ is the projection.

For scalar operators, concentration in shrinking neighborhoods of an arithmetic progression, with asymptotically full relative density uniformly in translated intervals of a fixed length, implies a common bicharacteristic period [17, Theorem 3.2]. For connection Laplacians on manifolds whose geodesics have a common least period, the return-symbol formula and the finite-cluster criterion are given in [45, Proposition 2.1 and Theorem 5.1]. For line bundles, see also [46, Proposition 3.1].

Theorem 3.1. *Fix $T > 0$, $\beta \in \mathbb{R}$, and $0 < \kappa \leq 1$. Suppose that along some $R_j \to \infty$, all but $o(N_P(R_j))$ frequency copies $\mu \leq R_j$ satisfy*

$$\operatorname{Re}(e^{-i\beta} e^{iT\mu}) \geq \kappa. \tag{3.1}$$

Then $G_T = I$. The unitary propagator $U = e^{iTP}$ belongs to $\Psi^0_{\mathrm{cl}}(M; E)$ and has unitary principal symbol u_T satisfying

$$\operatorname{Re}(e^{-i\beta} u_T(v)) \geq \kappa I_E \qquad (v \in S^*M). \tag{3.2}$$

Moreover

$$u_T(v) = \rho_T H_T(v), \qquad \rho_T \in \{1, -1, i, -i\}, \tag{3.3}$$

where ρ_T is the constant scalar full-return Maslov factor. The potential W does not enter the leading transport.

Proof. Let $\mathcal{H} = \operatorname{Re}(e^{-i\beta} U)$ and let E_{exc} be the spectral projection onto all frequencies violating (3.1). Spectral calculus gives the exact operator inequality

$$\mathcal{H} \geq \kappa I - (1 + \kappa) E_{\mathrm{exc}}. \tag{3.4}$$

Weyl's law gives $N_P(R) \asymp R^n$, so the number of exceptional copies below R_j is $o(R_j^n)$.

Suppose G_T moves a point of S^*M. By homogeneity and continuity, there is an open set $\Omega \Subset \{0 < |\xi|_g < 1\}$ such that $\overline{\Omega} \cap G_T(\overline{\Omega}) = \emptyset$. Choose $0 \neq b \in C_c^\infty(\Omega; \operatorname{End}(\pi^*E))$ and $\chi \in C_c^\infty((0,1))$ with $0 \leq \chi \leq 1$ and $\chi = 1$ on $\{|\xi|_g : (x,\xi) \in \operatorname{supp} b\}$. Put $h = R_j^{-1}$ and $B_h = \operatorname{Op}_h(b)$, with $\|B_h\| \leq C$ uniformly in h. Set

$$A_h = \chi(hP) B_h^* B_h \chi(hP).$$

Then

$$0 \le A_h \le C^2 \Pi_{1/h}, \qquad \operatorname{Tr} A_h = c_b h^{-n} + o(h^{-n}), \qquad c_b = (2\pi)^{-n} \int \operatorname{tr}(b^* b)\, dx\, d\xi > 0.$$

The first follows from $B_h^* B_h \le C^2 I$ and the exact spectral support of $\chi(hP)$; the second is the local Weyl formula. Taking the trace of (3.4) against A_h gives

$$\operatorname{Tr}(\mathcal{H} A_h) \ge \kappa \operatorname{Tr} A_h - (1+\kappa) C^2 \operatorname{Tr}(E_{\mathrm{exc}} \Pi_{1/h}) = \kappa c_b h^{-n} + o(h^{-n}).$$

Since $\operatorname{supp} b \cap G_{\pm T}(\operatorname{supp} b) = \emptyset$, the traces $\operatorname{Tr}(U A_h)$ and $\operatorname{Tr}(U^* A_h)$ therefore have no stationary point and are $O(h^N)$ for every N, a contradiction. Thus $G_T = I$. The canonical graph of U is now the diagonal, so $U \in \Psi^0_{\mathrm{cl}}$; unitarity makes its principal symbol unitary.

For arbitrary matrix symbols b in a local unitary trivialization, the principal symbol of A_h is $\chi(|\xi|)^2 b^* b$. Dividing by h^{-n} in the trace inequality and applying the matrix local Weyl formula yields

$$\int \operatorname{tr}\left[(\operatorname{Re}(e^{-i\beta} u_T) - \kappa I) \chi(|\xi|)^2 b^* b \right] dx\, d\xi \ge 0.$$

If (3.2) failed at one point, continuity would give a fixed local vector with negative expectation on a small neighborhood. Taking $b^* b$ to be a supported scalar multiple of its rank-one projection would make this integral negative. This proves (3.2).

On bundle-valued half-densities, the subprincipal partial connection of P is induced by ∇; see [42, Proposition 3.2]. Thus the bundle factor in the leading amplitude of U is H_T. The scalar factor is that of $P_0 = (\Delta_g + 1)^{1/2}$, whose subprincipal symbol on half-densities is zero. Since the Hamiltonian flow of $TP_0/(2\pi)$ has period 2π, this factor is the Maslov fourth root of [12, Theorem 1.1 and its proof, p. 510]. It is locally constant on S^*M, hence constant because M is connected and $n \ge 2$. The endomorphism W enters the symbol of P only at order -1. This gives (3.3). □

The adjoint action $a \mapsto H_T a H_T^{-1}$ is unchanged under $H_T \mapsto z H_T$, $z \in \mathbb{S}^1$. Thus the endomorphism transport alone does not determine the scalar factor in (3.3).

Theorem 3.2. *Let $K \ge 1$ be an integer. For $k = 0, \dots, K$, let $E_k = \operatorname{Sym}^k E$ have the induced connection, and put $P_k = (\nabla_k^* \nabla_k + W_k)^{1/2} > 0$ with smooth Hermitian W_k. Here E_0 is the trivial line. Suppose each P_k satisfies the hypotheses of Theorem 3.1 with the same T, β, κ; the subsequences may depend on k. Set*

$$a = \arccos \kappa < \pi/2, \qquad x_0 = \operatorname{Arg}(e^{-i\beta} \rho_T) \in [-a, a].$$

Every eigenvalue $e^{i\theta}$ of $H_T(v)$, with principal angle $\theta \in (-\pi, \pi]$, satisfies

$$|\theta| \le \frac{a - |x_0|}{K} \le \frac{a}{K}. \tag{3.5}$$

If the hypotheses hold for every $k \ge 0$, then $H_T(v) = I$ for every v. For the complexified tangent or cotangent bundle with its Levi-Civita connection, the full frame flow then has period T.

Proof. The scalar case $k = 0$ gives $G_T = I$ and the factor ρ_T. That factor is the same for all powers, since the scalar Hamiltonian and Maslov transport are the same. Their bundle factors are $\operatorname{Sym}^k H_T$. If z is a holonomy eigenvector with eigenvalue $e^{i\theta}$, its nonzero symmetric tensor $z^{\otimes k}$ has eigenvalue $e^{ik\theta}$. The pointwise matrix inequality gives

$$x_k = \operatorname{Arg}(e^{-i\beta} \rho_T e^{ik\theta}) \in [-a, a].$$

Each difference $x_{k+1} - x_k$ is congruent to θ modulo 2π and lies in $[-2a, 2a] \subset (-\pi, \pi)$. Thus every difference equals the same principal angle θ, and $x_K = x_0 + K\theta$.

At the opposite initial covector, the period-T path is traversed in reverse. Its holonomy is H_T^{-1}, while its scalar factor is still ρ_T. Applying $x_K = x_0 + K\theta$ to the inverse eigenvalue gives $x_0 - K\theta \in [-a, a]$. Hence

$$|x_0| + K|\theta| = \max\{|x_0 + K\theta|, |x_0 - K\theta|\} \le a,$$

which gives (3.5). If all powers satisfy the hypothesis, let $K \to \infty$. Every eigenangle is zero; unitary diagonalizability gives identity transport. □

The pure tensors $z^{\otimes k}$ give the same conclusion for tensor powers. The period T need not be prime.

For the trivial connection on the trivial line, the following theorem is [12, Theorem 1.1]. For connection Laplacians on manifolds with a common least geodesic period, the bundle logarithmic correction is constructed in [45, proof of Theorem 5.1, p. 255].

Theorem 3.3. *Under Theorem 3.1, suppose also $H_T(v) = I$ for every v. Put $b = 2\pi/T$ and choose the unique $\alpha \in (0, b]$ such that $e^{iT\alpha} = \rho_T$. There is a positive self-adjoint classical operator $A \in \Psi^1(M; E)$ with domain H^1, strongly commuting with P, such that*

$$A - P \in \Psi^{-1}_{\mathrm{cl}}, \qquad \operatorname{Spec}(A) \subset \alpha + b\mathbb{Z}_{\geq 0}.$$

At every sufficiently high P-eigenfrequency it takes the nearest lattice value and differs from P by $O((1+\mu)^{-1})$.

Proof. Put $\widetilde{P} = P/b$. The logarithm construction in the proof of Theorem 4.1(3) applies with $L = 1$, $H = \rho_T I$, and $\mathcal{S} = \mathcal{T} = \{\rho_T\}$: U is unitary, strongly commutes with $\widetilde{P}$, and $U - \rho_T I \in \Psi^{-1}_{\mathrm{cl}}$ by (3.3). After multiplication by b, it gives a self-adjoint $A_0 \in \Psi^1_{\mathrm{cl}}$ on H^1, strongly commuting with P, such that

$$A_0 - P \in \Psi^{-1}_{\mathrm{cl}}, \qquad \operatorname{Spec} A_0 \subset \alpha + b\mathbb{Z}.$$

Since $(A_0 - P)(1 + P)$ is bounded, each eigenvalue a of A_0 on the μ-eigenspace of P satisfies

$$|a - \mu| \leq C(1+\mu)^{-1}.$$

For large μ, this bound is less than $b/2$, so the lattice value is unique, nearest to μ, and belongs to $\alpha + b\mathbb{Z}_{\geq 0}$. Replace A_0 on the finitely many remaining P-eigenspaces by αI. This self-adjoint smoothing correction preserves strong commutation and the domain H^1. The resulting operator A is positive and has the required spectrum. □

Corollary 3.4. *Under Theorem 3.3, there is a bounded self-adjoint classical operator $C \in \Psi^0(M; E)$, commuting with D, such that*

$$\operatorname{Spec}(D + C) \subset \{(\alpha + b\,j)^2 \,:\, j \in \mathbb{Z}_{\geq 0}\}.$$

Proof. Put $B = A - P \in \Psi^{-1}$ and define $C = 2PB + B^2 \in \Psi^0$. The construction gives strong commutation of B with P. Thus C is symmetric on smooth sections and extends to a bounded self-adjoint operator commuting with $D = P^2$. Since $B : H^1 \to H^2$ and $\operatorname{Dom} A = H^1$, elliptic regularity gives

$$\operatorname{Dom} A^2 = \{u \in H^1 \,:\, Pu \in H^1\} = H^2.$$

On this domain, $D + C = A^2$. The spectral theorem gives $\operatorname{Spec}(D + C) = \{a^2 \,:\, a \in \operatorname{Spec}(A)\}$, with the multiplicities of A, since A is positive. □

3.1. **Connections with prescribed geodesic transport.** On the round $S^n \subset \mathbb{R}^{n+1}$, $n \geq 2$, put

$$\eta = x_3(x_1\,dx_2 - x_2\,dx_1), \qquad \nabla = d + i\varepsilon\eta \quad (\varepsilon \neq 0)$$

on the trivial complex line. Antipodal pullback sends η to $-\eta$. Every great circle has its second semicircle equal to the antipodal image of its first, so its integral of η is zero. All period-2π transports, in all symmetric powers, are the identity. Nevertheless, at e_3 the curvature is

$$i\varepsilon\, d\eta = 2i\varepsilon\; dx_1 \wedge dx_2 \neq 0.$$

Thus identity geodesic transport does not imply a flat connection. For the relation between antipodally odd curvature and spectral clusters on spheres, see [45, p. 258]. For transparent connections on negatively curved surfaces, see [53].

Finitely many powers do not force identity transport either. Take $\eta_0 = x_1 dx_2 - x_2 dx_1$ instead. If $\gamma(t) = u\cos t + v\sin t$, with u, v orthonormal, then

$$\int_\gamma \eta_0 = 2\pi(u_1v_2 - u_2v_1), \qquad |u_1v_2 - u_2v_1| \leq 1.$$

The integral is 2π for $u = e_1$, $v = e_2$. Given K and $0 < a < \pi/2$, choose $0 < |\varepsilon| < a/(2\pi K)$. All transports in powers $0, \dots, K$ then have eigenangles strictly inside $[-a, a]$, but some are nonidentity.

Choose smooth constant scalar shifts W_k so that $D_k = \nabla_k^* \nabla_k + W_k > 0$, and put $P_k = D_k^{1/2}$. Since $G_{2\pi} = I$, the propagators $e^{2\pi i P_k}$ belong to Ψ^0_{cl} and have principal symbols $\rho_{2\pi} H_{2\pi}^{\otimes k}$. Set $\beta = \operatorname{Arg} \rho_{2\pi}$, $\kappa = \cos a < 1$, and

$$\mathcal{H}_k = \operatorname{Re}(e^{-i\beta} e^{2\pi i P_k}), \qquad h_{0,k} = \sigma_0(\mathcal{H}_k).$$

For the first example, $h_{0,k} = 1$ for every $k \geq 0$. For the second, $h_{0,k} \geq \cos(2\pi K|\varepsilon|) > \kappa$ for $0 \leq k \leq K$. Thus in either case $h_{0,k} - \kappa \geq \delta > 0$ for the indicated powers. Gårding's inequality [65, Section I.4, equation (4.18), and Section I.8], with order zero and Sobolev exponent $-1/2$, gives

$$\langle (\mathcal{H}_k - \kappa I)u, u \rangle \geq \frac{\delta}{2} \|u\|^2 - C_k \|u\|^2_{H^{-1/2}}.$$

For $P_k u = \mu u$ and $\|u\| = 1$, elliptic Sobolev norm equivalence gives $\|u\|^2_{H^{-1/2}} \leq C_k'(1+\mu)^{-1}$. Consequently (3.1) holds apart from finitely many frequencies in each fixed power.

4. Finite spectral phases and Hamiltonian return

Let M be a closed connected manifold of dimension $n \geq 2$, let $E \to M$ be a Hermitian bundle of rank $r \geq 1$, and work on E-valued half-densities. Let $P \in \Psi^1_{\mathrm{cl}}(M; E)$ be self-adjoint and elliptic, with principal symbol $q(x, \xi) I_E$, where $q > 0$ is homogeneous of degree one. The unit q-cosphere $\Sigma = \{q = 1\}$ is compact and connected. Let G_t be the Hamiltonian flow of q, and put

$$U = e^{2\pi i P}, \qquad \Phi = G_{-2\pi}, \qquad N_P(R) = \operatorname{Tr} \mathbf{1}_{(-\infty, R]}(P).$$

Thus Φ is the canonical transformation of U. Changing finitely many eigenvalue copies changes each matching count by $O(1)$. All spectral lists and matchings retain multiplicities.

For a nonempty finite set $S \subset \mathbb{S}^1$, put

$$\Lambda(S) = \{x \in \mathbb{R} \,:\, e^{2\pi i x} \in S\}, \qquad F(z) = \prod_{s \in S} (z - s), \qquad d = |S|.$$

The phases need not be roots of unity. For a positive integer L, define

$$\mathcal{T}_L(S) = \{h \in \mathbb{S}^1 \,:\, \{z \,:\, z^L = h\} \subset S\}. \tag{4.1}$$

Thus $h \in \mathcal{T}_L(S)$ precisely when its Lth roots form a regular L-gon contained in S.

For scalar operators, concentration near one arithmetic progression in every interval of a fixed length implies periodicity, even when a proportion of eigenvalue copies tending to zero is exceptional [17, Theorem 3.2]. For a connection Laplacian on a manifold with a common least geodesic period, finite holonomy spectrum is equivalent to bounded-width clusters of Laplacian eigenvalues [45, Theorem 5.1].

Theorem 4.1 (Finite spectral phases and Hamiltonian return). *Suppose that, through cutoffs $R_\nu \to \infty$, the spectrum of P can be matched to a multiset supported on $\Lambda(S)$ with pairwise errors $\eta_\nu \to 0$ and total unmatched count $o(R_\nu^n)$.*

(1) The map Φ has finite order L, with $1 \leq L \leq d$. In particular, the Hamiltonian flow of q has period $2\pi L$. The period $2\pi L$ need not be the least period of every orbit.

(2) The operator $V = U^L$ is a classical order-zero pseudodifferential operator. Its principal symbol $H \in C^\infty(\Sigma; \operatorname{End}(\pi^* E))$ is unitary, and

$$\operatorname{Spec} H(\rho) \subset \mathcal{T}_L(S) \quad (\rho \in \Sigma).$$

The multiplicity of each $h \in \mathcal{T}_L(S)$ is constant on Σ.

(3) There is a self-adjoint $A \in \Psi^1_{\mathrm{cl}}(M;E)$ with domain H^1, strongly commuting with P, such that

$$A - P \in \Psi^{-1}_{\mathrm{cl}}(M;E), \qquad \operatorname{Spec} A \subset \Lambda(S).$$

It is unique up to an operator supported on a finite-dimensional P-spectral subspace among quantizers with these properties.

Proof. The wave parametrix for e^{-itP} has a local scalar homogeneous phase $\phi(t,x,\eta) - \langle y, \eta\rangle$, where

$$\partial_t \phi + q(x, \partial_x \phi) = 0, \qquad \phi(0,x,\eta) = \langle x, \eta\rangle$$

[36, Section 3]. In a bundle frame, the homogeneous amplitudes satisfy linear matrix transport equations

$$\left(\partial_t + \sum_\ell v_\ell \partial_{x_\ell} + b_0\right) a_{-j} = f_{-j},$$

where $v_\ell = \partial_{\xi_\ell} q(x, \partial_x \phi)$ is real and scalar, b_0 is a matrix of degree zero, and f_{-j} depends only on the preceding amplitudes; see [45, Section 2, equations (2.2)–(2.6)]. Along the scalar characteristics, these are linear ordinary differential equations whose solutions have degree $-j$. Asymptotic summation and Duhamel's formula identify the local parametrices with e^{-itP} modulo smoothing operators; hence they agree modulo smoothing on overlaps. Finite composition gives a classical order-zero Fourier integral operator with canonical relation $\operatorname{graph}(G_t)$ for every fixed t.

Choose a real nonzero $\psi \in C_c^\infty((0,1))$ and put $\chi = \psi^2$. A matched eigenvalue μ satisfies $|F(e^{2\pi i\mu})| \le C_F \eta_\nu$; the same expression is uniformly bounded on unmatched copies. The bundle Weyl law gives $O(R_\nu^n)$ copies in the cutoff. Therefore

$$\|F(U)\psi(P/R_\nu)\|^2_{\mathrm{HS}} \le C\eta_\nu^2 R_\nu^n + o(R_\nu^n) = o(R_\nu^n). \tag{4.2}$$

If $\operatorname{supp}\psi \subset [a,b] \subset (0,1)$, each eigenvalue selected by $\psi(P/R_\nu)$ satisfies $\mu \le bR_\nu < R_\nu$. For every fixed bounded operator D, $\|DF(U)\psi(P/R_\nu)\|_{\mathrm{HS}} \le \|D\|\|F(U)\psi(P/R_\nu)\|_{\mathrm{HS}}$.

Bounded pointwise periods. Write $F(z) = \sum_{j=0}^d f_j z^j$. Suppose that some $\rho \in \Sigma$ satisfies $\Phi^j(\rho) \ne \rho$ for every $1 \le j \le d$. Choose a small conic neighborhood W of ρ whose closure contains no fixed point of any of these powers, and a scalar order-zero cutoff D, acting as a multiple of I_E, with nonzero principal symbol a and all-orders microsupport in W, modulo smoothing terms.

Normality of $F(U)$, its commutation with $\chi(P/R)$, and cyclicity give the exact identity

$$\|DF(U)\psi(P/R)\|^2_{\mathrm{HS}} = \sum_{j,k=0}^d f_j\overline{f_k}\,\operatorname{Tr}\left(\chi(P/R)D^*D\,U^{j-k}\right). \tag{4.3}$$

For $j = k$ the positive Weyl leading coefficient is

$$\frac{r}{(2\pi)^n}\int_{T^*M} \chi(q)|a|^2\,dx\,d\xi.$$

For $j \ne k$, the canonical relation has no diagonal point on the microsupport of D^*D. In a local homogeneous wave parametrix, trace restriction has no stationary point on the compact rescaled frequency annulus. Repeated integration by parts makes the trace $O(R^{-N})$ for every N. The diagonal sum in (4.3) consequently has a strictly positive coefficient times R^n, contradicting (4.2). Thus

$$\Sigma = \bigcup_{j=1}^d \operatorname{Fix}(\Phi^j).$$

If $N = \operatorname{lcm}(1,\dots,d)$, every point is fixed by Φ^N, so $\Phi^N = I$. Homogeneity extends this identity to the whole punctured cotangent bundle.

The global order is at most d. Let L be the order of Φ on Σ. The cyclic group $C_L = \langle\Phi\rangle$ acts properly and effectively on Σ, and Σ/C_L is connected. By the principal orbit theorem [51, Theorem 1.32], the

principal stabilizer is contained in every stabilizer, since C_L is abelian. It is therefore contained in the kernel of the action and is trivial. Hence the set of points with orbit length L is open and dense. At any such point the preceding bound gives $L \le d$.

The return polynomial identities. At $t = -2\pi L$ the canonical relation is the diagonal. Thus $V = e^{2\pi i L P} \in \Psi^0_{\mathrm{cl}}(M; E)$. Unitarity gives $H^*H = I$. Decompose the polynomial uniquely as

$$F(z) = \sum_{j=0}^{L-1} z^j F_j(z^L). \tag{4.4}$$

In the open dense full-orbit locus, choose a cutoff D as above, now avoiding the fixed sets of Φ^j, $1 \le j < L$. Since all functions of U commute, the same exact norm identity becomes

$$\|DF(U)\psi(P/R)\|^2_{\mathrm{HS}} = \sum_{j,k=0}^{L-1} \mathrm{Tr}\big(\chi(P/R)D^*D\, U^{j-k} F_j(V) F_k(V)^*\big)\,.$$

The off-diagonal terms are rapidly decreasing. The coefficient of R^n in the diagonal sum is

$$\frac{1}{(2\pi)^n} \int_{T^*M} \chi(q)|a|^2 \sum_{j=0}^{L-1} \mathrm{tr}_E\big(F_j(H)F_j(H)^*\big)\; dx\, d\xi.$$

It is nonnegative and must vanish by (4.2). Arbitrary localization and homogeneity imply $F_j(H) = 0$ on the full-orbit locus; continuity extends this equality to all of Σ.

For a unit complex number h, the identities $F_j(h) = 0$ for all j are equivalent to F vanishing at every root of $z^L - h$. Indeed, reduction modulo $z^L - h$ gives the polynomial $\sum_{j<L} z^j F_j(h)$ of degree less than L; it vanishes at L distinct points if and only if it is zero. Applying this to each eigenvalue of the normal matrix H gives $\mathrm{Spec}\, H(\rho) \subset \mathcal{T}_L(\mathcal{S})$. Its spectral projectors are polynomial expressions in H: the set $\mathcal{T}_L(\mathcal{S}) \subset \{s^L : s \in \mathcal{S}\}$ is finite, and for each h in this set the projector is $\prod_{h' \ne h}(H - h'I)/(h - h')$. Their integer ranks are locally constant and hence constant on connected Σ.

Spectral projectors and the logarithmic correction. For scalar operators with zero subprincipal symbol and Hamiltonian period 2π, the commuting logarithmic correction is given in [12, Theorem 1.1]. The bundle construction of [45, proof of Theorem 5.1, p. 255] applies to $V = e^{2\pi i L P}$: it is unitary, belongs to Ψ^0_{cl}, strongly commutes with P, and has finite principal-symbol spectrum. Let $\mathcal{T}$ be the finite set of eigenvalues that occur in H, and put $G(z) = \prod_{h \in \mathcal{T}}(z - h)$. The principal symbol of $G(V)$ vanishes, so $G(V) \in \Psi^{-1}_{\mathrm{cl}}$. Since $G(V)$ commutes with P, boundedness of $G(V)P$ gives

$$|G(e^{2\pi i L\mu})| \le C/\mu$$

on every sufficiently high P-eigenvector. The roots of G are simple. On the unit circle this implies that $e^{2\pi i L\mu}$ lies within C'/μ of $\mathcal{T}$.

Change V on a fixed finite-dimensional P-spectral subspace to one chosen value in $\mathcal{T}$. The resulting unitary V' still strongly commutes with P, differs from V by a smoothing operator, and has its spectrum in arbitrarily small disjoint arcs about the points of $\mathcal{T}$. For each $h \in \mathcal{T}$, let Π_h be the Riesz projection onto the corresponding arc. Then

$$\Pi_h \in \Psi^0_{\mathrm{cl}}, \qquad \Pi_h^* = \Pi_h, \qquad \Pi_h \Pi_{h'} = 0\ (h \ne h'), \qquad \sum_h \Pi_h = I.$$

The defining contours lie in the resolvent set of V' and avoid the spectrum of H. Thus $zI - V'$ is elliptic and invertible there. Spectral invariance [5, Example 2 and Theorem 5.4, arXiv version], with the bundle reduction in the proof of Lemma 1.3, gives $(zI - V')^{-1} \in \Psi^0_{\mathrm{cl}}$. Contour integration gives $\Pi_h \in \Psi^0_{\mathrm{cl}}$, with principal symbol the h-eigenprojection of H.

Define

$$Z = \sum_{h \in \mathcal{T}} h^{-1} V' \Pi_h.$$

This operator is unitary, strongly commutes with P, and has principal symbol I. The arcs may be chosen so small that $\|Z - I\| < 1/2$. Lemma 1.3 gives

$$Q = \frac{1}{2\pi i L} \log Z \in \Psi^{-1}_{\mathrm{cl}}.$$

The operator Q is self-adjoint and strongly commutes with P.

On every sufficiently high P-eigenspace, $V' = e^{2\pi i L P}$ and

$$e^{2\pi i L(P-Q)} = VZ^{-1} = \sum_{h \in \mathcal{T}} h\Pi_h.$$

Every high eigenvalue of $P - Q$ consequently has its phase among the Lth roots of some $h \in \mathcal{T}$, all of which lie in $\mathcal{S}$. A final finite-rank self-adjoint correction in the remaining P-eigenspaces places each low eigenvalue in the root lattice belonging to its Π_h-block. This gives A and preserves the exact global identity

$$e^{2\pi i L A} = \sum_{h \in \mathcal{T}} h\Pi_h. \tag{4.5}$$

The correction to P is bounded and self-adjoint, so A has domain H^1 and is self-adjoint there.

Finally, let A_1, A_2 be two quantizers with the stated properties, and let $E_\mu = \ker(P - \mu)$. Strong commutation and boundedness of $(A_i - P)P$ give

$$\left\| (A_i - \mu I)|_{E_\mu} \right\| \le C_i/\mu \qquad (i = 1, 2)$$

for all sufficiently large μ. Since $\Lambda(\mathcal{S})$ has positive minimum separation, the interval $[\mu - C/\mu, \mu + C/\mu]$, $C = \max(C_1, C_2)$, contains at most one point of $\Lambda(\mathcal{S})$. Both restrictions are therefore the same scalar operator. Thus $A_1 = A_2$ outside a finite-dimensional P-spectral subspace. □

Corollary 4.2. *Under Theorem 4.1, write $h = e^{2\pi i \theta_h}$ with $0 \le \theta_h < 1$, and set $\Theta = \sum_h \theta_h \Pi_h$. There are strongly commuting self-adjoint classical operators K, Θ, Q such that*

$$LP = K + \Theta + Q, \qquad \operatorname{Spec} K \subset \mathbb{Z}, \qquad Q \in \Psi^{-1}_{\mathrm{cl}}, \qquad \Theta = \sum_h \theta_h \Pi_h \in \Psi^0_{\mathrm{cl}}.$$

Here K has scalar principal symbol $Lq I_E$. The principal symbols of Π_h are the spectral projections of H; their ranges are smooth subbundles of $\pi^* E$ over Σ.

Proof. Take $K = LA - \Theta$ and $Q = L(P - A)$. The projectors, A, and P strongly commute. Equation (4.5) gives $e^{2\pi i K} = I$, hence $\operatorname{Spec} K \subset \mathbb{Z}$. □

Corollary 4.3. *For $\mathcal{S} = \{1\}$ the conclusion is*

$$\Phi = I, \qquad e^{2\pi i P} - I \in \Psi^{-1}_{\mathrm{cl}}, \qquad \operatorname{Spec} A \subset \mathbb{Z}.$$

For a single arbitrary phase s, the principal return is sI_E and $\operatorname{Spec} A \subset \mathbb{Z} + \theta$, where $e^{2\pi i \theta} = s$. If $\mathcal{S}$ is contained in the set of q_0th roots of unity, then $L \mid q_0$ and

$$e^{2\pi i q_0 P} - I \in \Psi^{-1}_{\mathrm{cl}}.$$

Proof. If $|\mathcal{S}| = 1$, then $L = 1$ and $H = sI_E$. If every element of $\mathcal{S}$ is a q_0th root of unity, any nonempty polygon in $\mathcal{T}_L(\mathcal{S})$ has the quotient of consecutive vertices equal to a primitive Lth root of unity; both vertices are q_0th roots. Thus $L \mid q_0$. Each allowed h satisfies $h^{q_0/L} = 1$, so the principal symbol of $U^{q_0} = V^{q_0/L}$ is the identity. □

Proposition 4.4. *Assign a positive integer multiplicity to each member of a finite collection of pairwise disjoint regular L-gons in $\mathbb{S}^1$. If these multiplicities sum to r, the collection occurs in the bundle operator class of Theorem 4.1, with exact spectral inclusion and no correction term.*

Proof. On the round S^n, $n \geq 2$, let

$$K_0 = \sqrt{\Delta_{S^n} + (n-1)^2/4} - (n-1)/2 + 1. \tag{4.6}$$

It is a scalar classical integer-spectrum operator with principal symbol $|\xi|$, and its principal Hamiltonian flow has prime period 2π. For the prescribed polygons, write their Lth powers as $h = e^{2\pi i\theta_h}$. On the trivial rank-r bundle let Θ_0 be a constant Hermitian matrix with eigenvalue θ_h of the prescribed multiplicity and put

$$P = \frac{1}{L}(K_0 I_r + \Theta_0).$$

Its spectrum lies exactly in the union of the prescribed root lattices. The time-2π canonical map has order L, and $e^{2\pi iLP} = e^{2\pi i\Theta_0}$ has precisely the specified constant return eigenvalues. Taking the comparison multiset to be its own spectrum gives zero matching error and zero unmatched count. □

4.1. **The holonomy measure determined by scalar eigenvalues.** Let $d\Sigma$ be the radial Liouville measure on Σ from Section 1, and write $|\Sigma| = \int_\Sigma d\Sigma$. For connection Laplacians with common least geodesic period, Kuwabara's bandwise distribution formula [45, Theorem 3.1] assumes that the holonomy spectrum omits a point of $\mathbb{S}^1$. The following formula concerns cumulative counts for P.

Theorem 4.5. *Suppose that Φ has finite order L on Σ. Put $H = \sigma_0(e^{2\pi iLP})$. Then the probability measures*

$$\nu_R = \frac{1}{N_P(R)} \sum_{\mu_j(P) \leq R} \delta_{e^{2\pi i\mu_j(P)}}$$

converge weakly to the measure ν defined, for $f \in C(\mathbb{S}^1)$, by

$$\int_{\mathbb{S}^1} f \, d\nu = \frac{1}{rL|\Sigma|} \int_\Sigma \sum_{h \in \operatorname{Spec} H(\rho)} \sum_{z^L = h} f(z) \, d\Sigma(\rho). \tag{4.7}$$

The eigenvalues of H in the outer sum are counted with multiplicity.

Proof. It suffices first to compute the moments $f(z) = z^k$, $k \in \mathbb{Z}$. If $k = L\ell$, then U^k is a classical order-zero pseudodifferential operator with principal symbol H^ℓ. The bundle local Weyl formula gives

$$\frac{1}{N_P(R)} \operatorname{Tr}\big(U^{L\ell} \mathbf{1}_{(-\infty,R]}(P)\big) \longrightarrow \frac{1}{r|\Sigma|} \int_\Sigma \operatorname{tr}_E(H^\ell) \, d\Sigma.$$

One may obtain this sharp-cutoff formula from smooth annular cutoffs: the errors are bounded by the Weyl counts in arbitrarily small relative windows at zero and at the upper endpoint.

If $L \nmid k$, the subgroup $\langle \Phi^k \rangle$ is nontrivial. By the fixed-set theorem [51, Proposition 1.24], each component of $\mathcal{F} = \operatorname{Fix}(\Phi^k)$ is a closed embedded submanifold of Σ. The principal orbit theorem, applied to the effective cyclic action as above, makes the full-orbit locus dense. Thus every fixed component has positive codimension, and $\mathcal{F}$ has Liouville measure zero.

Fix $0 \leq \chi \in C_c^\infty((0,\infty))$. For $\varepsilon > 0$, choose $0 \leq a_\varepsilon \leq 1$, equal to one near $\mathcal{F}$, with $\int_\Sigma a_\varepsilon^2 \, d\Sigma < \varepsilon$. Choose $D_\varepsilon \in \Psi^0_{\mathrm{cl}}$ with scalar principal symbol $a_\varepsilon I_E$ and with $I - D_\varepsilon$ microsupported away from $\mathcal{F}$, modulo smoothing operators. The trace with insertion $I - D_\varepsilon$ is $O(R^{-N})$ for every N, by the nonstationary argument above. For the remaining term, the Hilbert–Schmidt inequality gives

$$\Big|\operatorname{Tr}\big(U^k D_\varepsilon \chi(P/R)\big)\Big| \leq \big(\operatorname{Tr} \chi(P/R)\big)^{1/2} \big(\operatorname{Tr}(D_\varepsilon^* D_\varepsilon \chi(P/R))\big)^{1/2}.$$

The local Weyl formula makes the upper limit after division by R^n at most $C_\chi \sqrt{\varepsilon}$. Letting $\varepsilon \downarrow 0$ proves that the trace is $o(R^n)$. For $T \in \Psi^0_{\mathrm{cl}}(M;E)$, insertion of T between U^k and D_ε multiplies the bound by $\|T\|$; the complementary trace still has no stationary point. Approximating the sharp cutoff as above proves that the normalized moment tends to zero.

The right side of (4.7) has exactly these moments: the sum of the kth powers of the Lth roots of h is zero unless $L \mid k$, and is $Lh^{k/L}$ otherwise. Trigonometric polynomials are uniformly dense in $C(\mathbb{S}^1)$. Since all measures in question have mass one, moment convergence proves weak convergence. □

Corollary 4.6. *Under Theorem 4.1, let r_h be the constant multiplicity of h in H. Then*

$$\nu = \sum_{h\in\mathcal{T}} \frac{r_h}{rL} \sum_{z^L=h} \delta_z, \qquad r_h \in \mathbb{Z}_{>0}, \qquad \sum_h r_h = r.$$

Thus supp ν *is a disjoint union of complete regular L-gons, and every vertex in one polygon has the same mass. For a scalar operator,* supp ν *is one regular L-gon, with mass $1/L$ at each vertex.*

Proof. The multiplicities are constant on connected Σ, so substitute them into (4.7). Different values of h give disjoint root sets. If $r = 1$, only one r_h can be nonzero. □

Remark 4.7 (The principal return measure). The principal return H is the product of its scalar Maslov factor and the unitary transport of the subprincipal partial connection around $t \mapsto G_{-t}(\rho)$, $0 \le t \le 2\pi L$; see [45, Section 2] and [42, Propositions 3.1 and 3.3]. For the square root of a connection Laplacian, the bundle factor is parallel transport for the original connection [42, Proposition 3.2]. Put

$$\overline{\nu}_H = \frac{1}{r|\Sigma|} \int_\Sigma \sum_{h\in\operatorname{Spec} H(\rho)} \delta_h \, d\Sigma(\rho),$$

where eigenvalues are counted with multiplicity. Then $(z \mapsto z^L)_* \nu = \overline{\nu}_H$.

Let P_1, P_2 be two such operators with finite-order canonical returns and matching errors $\eta_\nu \to 0$ such that $E_{\eta_\nu}(R_\nu; P_1, P_2) = o(R_\nu^n)$. Write $N_i(R) = \operatorname{Tr} \mathbf{1}_{(-\infty,R]}(P_i)$ and let $\nu_{i,R}$ be their normalized phase measures. For every trigonometric polynomial f, the matching inequality $|N_1(R_\nu) - N_2(R_\nu)| \le E_{\eta_\nu}(R_\nu; P_1, P_2)$ gives

$$\left| \int f \, d\nu_{1,R_\nu} - \int f \, d\nu_{2,R_\nu} \right| \le C_f \left(\eta_\nu + \frac{E_{\eta_\nu}(R_\nu; P_1, P_2)}{N_1(R_\nu)} \right) \longrightarrow 0.$$

Hence their limiting phase measures agree. If their return orders are equal to the same L, then $\overline{\nu}_{H_1} = \overline{\nu}_{H_2}$.

5. Spectral resolution and the order of correction

Theorem 5.1 (Resolution and order of the correction). *Let P and A be strongly commuting self-adjoint classical elliptic operators of order one on a Hermitian bundle over a closed n-manifold, $n \ge 2$. Suppose that their common principal symbol is qI_E, with $q > 0$, and that*

$$B = P - A \in \Psi_{\mathrm{cl}}^{-1}.$$

Let $\Lambda \subset \mathbb{R}$ have positive minimum separation, and suppose Spec $A \subset \Lambda$. *Compare P to any multiset supported on Λ, with the conventions of Definition 1.1. Suppose that, through $R_\nu \to \infty$, the matching error and total unmatched count satisfy*

$$\eta_\nu = O(R_\nu^{-s}), \qquad E_{\eta_\nu}(R_\nu) = o(R_\nu^\beta), \qquad s > 0, \quad 0 < \beta \le n.$$

Then, with $N = \lfloor ns/\beta \rfloor$,

$$P - A \in \Psi_{\mathrm{cl}}^{-N-1}. \tag{5.1}$$

For $N = 0$ this is the initial order assumption. For the same cutoffs and unmatched-count bound, one has $P - A \in \Psi^{-\infty}$ if $\eta_\nu = 0$ for all ν, or if $\eta_\nu = o(R_\nu^{-m})$ for every $m > 0$.

Proof. Strong commutation and compact resolvent give a common orthonormal eigenbasis. Write (μ, a) for the P- and A-eigenvalues of a basis vector, and put $b = \mu - a$. The operator $BP \in \Psi_{\mathrm{cl}}^0$ is bounded, so $|b| \le \|BP\|/\mu$ for $\mu > 0$. Let $\delta > 0$ be the minimum separation of Λ. For large μ, every A-eigenvalue in the μ-eigenspace lies within $\delta/4$ of μ. Hence A is scalar on that eigenspace. Since $\eta_\nu \to 0$, a reference value matched to such a copy must be its label $a \in \Lambda$.

There is nothing to prove if $N = 0$. Otherwise, suppose that $B \notin \Psi_{\mathrm{cl}}^{-N-1}$. Classicality gives a first nonzero homogeneous symbol b_{-k} of degree $-k$, with $1 \le k \le N$. For $t > 0$, put

$$n_B(t) = \operatorname{Tr} \mathbf{1}_{(t,\infty)}(|B|).$$

The Birman–Solomyak Weyl law for negative-order operators on Hermitian bundles [54, Theorem 6.1] gives

$$n_B(t) \sim C_B t^{-n/k} \qquad (t \downarrow 0).$$

For any auxiliary Riemannian metric g, its coefficient is

$$C_B = \frac{1}{n(2\pi)^n} \int_{S_g^*M} \operatorname{tr}_E |b_{-k}|^{n/k} \, d\Sigma_g > 0,$$

where $d\Sigma_g$ is the radial Liouville measure. Positivity follows from $b_{-k} \not\equiv 0$.

The operator $A^k B \in \Psi^0_{\mathrm{cl}}$ is bounded. On a joint eigenvector,

$$|a|^k |b| \le \|A^k B\|.$$

Set $T_\nu = R_\nu^{\beta/n}$. Since $k\beta/n \le s$,

$$L_\eta := \limsup_{\nu\to\infty} T_\nu^k \eta_\nu < \infty.$$

Choose $D > 1 + \max\{2^k \|A^k B\|, L_\eta\}$ and put $\tau_\nu = D T_\nu^{-k}$. Then $\eta_\nu < \tau_\nu$ for large ν, including when $k\beta/n = s$. If $|b| > \tau_\nu$, then $|a| < T_\nu/2$, and therefore

$$\mu = a + b \le T_\nu/2 + \|B\| < R_\nu$$

for large ν. Except for the fixed finite-dimensional subspace where μ is bounded, every copy counted by $n_B(\tau_\nu)$ is unmatched. Consequently,

$$E_{\eta_\nu}(R_\nu) \ge n_B(\tau_\nu) - O(1) = C_B D^{-n/k} R_\nu^\beta + o(R_\nu^\beta),$$

contrary to the hypothesis. This proves (5.1). For exact errors, or errors smaller than every inverse power, the conclusion holds for arbitrarily large s; hence $B \in \bigcap_{j\ge1} \Psi^{-j} = \Psi^{-\infty}$. □

Corollary 5.2. *In the setting of Theorem 4.1, assume*

$$\eta_\nu = O(R_\nu^{-s}), \qquad E_{\eta_\nu}(R_\nu) = o(R_\nu^\beta), \qquad s > 0, \quad 0 < \beta \le n.$$

Then its commuting phase quantizer satisfies

$$P - A \in \Psi^{-\lfloor ns/\beta \rfloor - 1}_{\mathrm{cl}}.$$

Proof. The hypotheses imply $\eta_\nu \to 0$ and unmatched $o(R_\nu^n)$. Theorem 4.1 gives a strongly commuting quantizer A with $P - A \in \Psi^{-1}_{\mathrm{cl}}$. The finite union of integer translates $\Lambda(S)$ has positive minimum separation, so Theorem 5.1 applies. □

Proposition 5.3. *For every $s > 0$ and $0 < \beta \le n$, there are operators satisfying Theorem 5.1 for which $P - A$ has exact order $-\lfloor ns/\beta \rfloor - 1$.*

Proof. Let K_0 be the round integer quantizer (4.6); thus $\operatorname{Spec} K_0 = \{1, 2, \ldots\}$. On the trivial rank-$r$ bundle over S^n put $A = K_0 I_r$. Let $k = \lfloor ns/\beta \rfloor + 1$ and $0 < c < 1/4$, and define $P = A + cA^{-k}$. These operators strongly commute, and the degree-$-k$ symbol of $P - A$ is $c|\xi|^{-k} I_r \ne 0$.

Compare P with $\operatorname{Spec} A$. At $R_\nu = \nu + 1/2$, both lists contain exactly the complete clusters with integer labels $a \le \nu$. Put

$$\eta_\nu = \frac{R_\nu^{-s}}{\log R_\nu}, \qquad X_\nu = (c/\eta_\nu)^{1/k}.$$

Since $k > ns/\beta \ge s$, one has $X_\nu = o(R_\nu)$. For large ν, a copy at $a + ca^{-k}$ can match only to the integer a. Match all copies with $ca^{-k} \le \eta_\nu$. The remaining copies occur in equal numbers on the two sides, so

$$\begin{aligned} E_{\eta_\nu}(R_\nu) &= 2\#\{A\text{-copies with } a < X_\nu\} \\ &\le 2N_A(X_\nu) = O\big(R_\nu^{ns/k} (\log R_\nu)^{n/k}\big) = o(R_\nu^\beta). \end{aligned}$$

The errors are $o(R_\nu^{-s})$, while the nonzero principal symbol gives $P - A \notin \Psi^{-k-\varepsilon}$ for every $\varepsilon > 0$. □

Remark 5.4 (Strong commutation). The strong commutation hypothesis cannot be omitted, even when the spectra are equal. Let K_0 be the round integer quantizer (4.6), let $q = |\xi|$, and let $f = x_1|_{S^n}$. Put

$$S = \tfrac{1}{2}(fK_0^{-1} + K_0^{-1}f), \qquad W = e^{iS}, \qquad P_2 = WK_0W^*.$$

Then $S = S^* \in \Psi_{\mathrm{cl}}^{-1}$ has symbol f/q. The holomorphic functional calculus gives $W = I + iS + \Psi_{\mathrm{cl}}^{-2}$ [5, Theorem 5.4, arXiv version]. Thus W is unitary, both W and W^* preserve H^1, and P_2 is self-adjoint on H^1, with

$$\operatorname{Spec} P_2 = \operatorname{Spec} K_0, \qquad \sigma_1(P_2) = \sigma_1(K_0) = q.$$

Since q is scalar, the commutator symbol formula gives

$$P_2 - K_0 = i[S, K_0] + \Psi_{\mathrm{cl}}^{-2}, \qquad \sigma_{-1}(P_2 - K_0) = -\frac{H_q f}{q} \neq 0.$$

Hence $P_2 - K_0 \in \Psi_{\mathrm{cl}}^{-1}$ but $P_2 - K_0 \notin \Psi^{-2}$. Moreover,

$$\sigma_{-1}([K_0, P_2]) = \frac{iH_q^2 f}{q} = -\frac{if}{q} \neq 0,$$

so K_0 and P_2 do not strongly commute.

6. Limiting distributions of correction symbols

Let M be a closed smooth manifold of dimension $n \geq 2$, and let $E \to M$ be a Hermitian vector bundle of rank r. Operators act on E-valued half-densities. Let $A \geq 1$ be a self-adjoint classical elliptic operator of order one, with scalar positive principal symbol $q\,\mathrm{Id}_E$, and suppose

$$\operatorname{Spec}(A) \subset \mathbb{N}.$$

Let $k > 0$, and let $B = B^* \in \Psi_{\mathrm{cl}}^{-k}(M; E)$ strongly commute with A. Put $P = A + B$, with domain H^1, and $C = A^k B$. The complex-power theorem gives $A^k \in \Psi_{\mathrm{cl}}^k$ with principal symbol $q^k I_E$ [5, Example 2 and Theorem 7.9, arXiv version]. Thus $C \in \Psi_{\mathrm{cl}}^0$ is bounded. Strong commutation makes C self-adjoint. Write c for its Hermitian degree-zero principal symbol, restricted to

$$\Sigma_q = \{(x, \xi) \,:\, q(x, \xi) = 1\}.$$

Let $d\Sigma_q$ denote the radial Liouville measure of Section 1.

For a Borel subset $J \subset \mathbb{R}$, define the finite measure

$$\nu_c(J) = \frac{1}{n(2\pi)^n} \int_{\Sigma_q} \operatorname{tr}_E\big(\mathbf{1}_J(c(\rho))\big)\, d\Sigma_q(\rho). \tag{6.1}$$

The ordered eigenvalues $c_1(\rho), \dots, c_r(\rho)$ are continuous, and the integrand equals $\sum_{\ell=1}^r \mathbf{1}_J(c_\ell(\rho))$. In particular,

$$\nu_c(\mathbb{R}) = \lim_{R \to \infty} R^{-n} N_A(R).$$

Take the matching of Definition 1.1 at the complete-cluster cutoffs $R_N = N + \frac{1}{2}$. Finite changes of the spectra alter the matching defect by $O(1)$, by Lemma 1.2.

6.1. **The matching reduction.** Strong commutation gives a common orthonormal eigenbasis. Write (a_j, b_j) for its A- and B-eigenvalues, so the corresponding P-eigenvalue is $a_j + b_j$. Since C is bounded,

$$|b_j| \leq \|C\| a_j^{-k}. \tag{6.2}$$

For all sufficiently large a_j, the nearest integer to $a_j + b_j$ is a_j. Hence the scalar spectral lists determine, up to $O(1)$, the signed counts

$$D_\pm(R, \eta) = \#\{j \,:\, a_j \leq R,\ \pm b_j > \eta\}.$$

Lemma 6.1. *Uniformly for $0 \le \eta < 1/4$ and all sufficiently large N,*

$$E_\eta(R_N; P, A) = 2D_+(R_N, \eta) + 2D_-(R_N, \eta) + O(1),$$

where the constant is independent of N and η.

Proof. Above a fixed A-level, (6.2) is less than $1/4$. A P-eigenvalue in such a cluster can match only to its own integer a_j: its distance to any other integer is greater than $3/4$. The A-cluster and the corresponding P-cluster have equal total multiplicity. Every copy with $|b_j| \le \eta$ can therefore be matched, and each copy with $|b_j| > \eta$ leaves one copy unmatched on each side.

At $R_N = N + 1/2$, every sufficiently high cluster is wholly included or wholly excluded on both sides. The finitely many remaining P- and A-eigenvalues are also included when N is large. Enlarge the fixed set of low clusters to include every integer within distance $1/4$ of one of those exceptional eigenvalues. Matchings outside this set are now independent cluster by cluster. Altering the matching inside the fixed set changes the unmatched count by a uniformly bounded number. This proves the formula. □

6.2. **A joint trace formula.** For operators on scalar functions, the cumulative moment formula is the weak Szegő law [27, §2, (2.1), p. 510]. For a positive self-adjoint scalar elliptic operator of order one with vanishing subprincipal symbol and 2π-periodic bicharacteristics, the cluster distributions have the expansions of [12, Theorem 3.1, pp. 518–519].

Lemma 6.2. *Let $F(s, z)$ be continuous, with compact support in $(0, \infty) \times \mathbb{R}$. Then*

$$\lim_{R\to\infty} R^{-n} \sum_j F(a_j/R, a_j^k b_j) = \frac{1}{(2\pi)^n} \int_{\Sigma_q} \int_0^\infty s^{n-1} \operatorname{tr}_E F(s, c(\rho))\, ds\, d\Sigma_q(\rho). \tag{6.3}$$

Proof. Choose a compact interval K containing Spec C and all eigenvalues of c. For $\chi \in C_c^\infty((0, \infty))$ and a polynomial p, strong commutation and (1.1) give

$$\lim_{R\to\infty} R^{-n} \sum_j \chi(a_j/R) p(a_j^k b_j) = \frac{1}{(2\pi)^n} \int_{\Sigma_q} \int_0^\infty s^{n-1} \chi(s) \operatorname{tr}_E p(c(\rho))\, ds\, d\Sigma_q.$$

Here $p(C) \in \Psi^0_{\mathrm{cl}}$ has principal symbol $p(c)$. Choose $0 < a < b$ and $\zeta \in C_c^\infty((a, b))$ equal to one on the radial support of F. Stone–Weierstrass on $[a, b] \times K$, followed by multiplication by ζ, gives uniform approximation on $(0, \infty) \times K$ by finite sums of the displayed products, all with radial support in $[a, b]$. An error ε changes the normalized sum by at most $\varepsilon R^{-n} N_A(bR) = O(\varepsilon)$; the corresponding integral has the same bound. Approximation proves (6.3). □

6.3. **The complete critical profile.** Put $\alpha = n/k$ and $x_+ = \max(x, 0)$.

Theorem 6.3 (The limiting spectral profile). *For every $t > 0$,*

$$\lim_{N\to\infty} R_N^{-n} D_\pm(R_N, t R_N^{-k}) = F_\pm(t) := \int_{\mathbb{R}} \min\{1, ((\pm x)_+/t)^\alpha\}\, d\nu_c(x), \tag{6.4}$$

$$\lim_{N\to\infty} R_N^{-n} E_{tR_N^{-k}}(R_N; P, A) = F(t) := 2 \int_{\mathbb{R}} \min\{1, (|x|/t)^\alpha\}\, d\nu_c(x). \tag{6.5}$$

The convergence is uniform for t in any compact subinterval of $(0, \infty)$. The formulas hold at every $t > 0$, including atoms of ν_c.

Proof. Fix $0 < \varepsilon < 1$ and choose

$$T > \max\{\|C\|, \sup_{\Sigma_q} \|c\|\}.$$

Restrict first to $\varepsilon R \le a_j \le R$. The condition $\pm b_j > tR^{-k}$ is equivalent to

$$\pm(a_j^k b_j) > t(a_j/R)^k.$$

Apply Lemma 6.2 to continuous upper and lower approximations of the indicator of

$$\{(s,z) : \varepsilon \le s \le 1,\ |z| \le T,\ \pm z > t s^k\}.$$

Its boundary has zero limiting measure. The surfaces $s = \varepsilon$ and $s = 1$ have zero radial measure. The faces $z = \pm T$ do not meet the support of the limiting measure. For fixed ρ, each eigenvalue x of $c(\rho)$ has at most one positive solution of $\pm x = t s^k$; when $\pm x \le 0$ it has none. Fubini's theorem thus gives zero measure for the remaining boundary. The joint limit therefore holds for this indicator.

The omitted eigenvalues $a_j < \varepsilon R$ contribute at most $N_A(\varepsilon R) = O(\varepsilon^n R^n) + o(R^n)$, with fixed ε. Letting $\varepsilon \downarrow 0$ leaves the radial integral

$$\int_0^1 s^{n-1} \mathbf{1}_{\{\pm x > t s^k\}}\, ds = \frac{1}{n} \min\{1, ((\pm x)_+/t)^\alpha\}.$$

This proves the signed limits, and Lemma 6.1 proves the unsigned limit.

The functions of t under the integral are bounded and continuous for $t > 0$. Dominated convergence makes $F_\pm$ and F continuous. For any partition $a = t_0 < \cdots < t_m = b$ and any one of the normalized counting functions G_N with limit G, monotonicity gives

$$\sup_{[a,b]} |G_N - G| \le \max_{0 \le i \le m} |G_N(t_i) - G(t_i)| + \max_{0 \le i < m} \big(G(t_i) - G(t_{i+1})\big).$$

First let $N \to \infty$, then let the mesh tend to zero. □

Corollary 6.4. *Together with the known Weyl mass, the signed profiles determine the measure ν_c. More precisely, their right derivatives exist for every $t > 0$ and*

$$\nu_c(\{x : \pm x > t\}) = F_\pm(t) + \frac{t}{\alpha} F'_{\pm,+}(t). \tag{6.6}$$

If $\nu_{|c|}$ is the pushforward of ν_c under $x \mapsto |x|$, then

$$\nu_{|c|}((t,\infty)) = \frac{1}{2}\left(F(t) + \frac{t}{\alpha} F'_+(t)\right),$$

and

$$\nu_c(\{0\}) = \nu_c(\mathbb{R}) - \frac{1}{2} \lim_{t \downarrow 0} F(t).$$

Proof. Put

$$M_\pm(t) = \int_{\{0 < \pm x \le t\}} (\pm x)^\alpha \, d\nu_c(x).$$

Then

$$F_\pm(t) = \nu_c(\{\pm x > t\}) + t^{-\alpha} M_\pm(t).$$

For $y \ge 0$, the right derivative of $\min(1, (y/t)^\alpha)$ is $-\alpha t^{-\alpha-1} y^\alpha \mathbf{1}_{\{0<y\le t\}}$. For $t \ge a > 0$, the corresponding right difference quotients are bounded in absolute value by α/a. Dominated convergence gives

$$F'_{\pm,+}(t) = -\alpha t^{-\alpha-1} M_\pm(t).$$

These identities give (6.6). The same calculation with $y = |x|$ gives the formula for $\nu_{|c|}$. Monotone convergence as $t \downarrow 0$ gives the atom at zero. The two strict tails and the zero atom determine ν_c. This recovers the distribution of the eigenvalues of c, not their locations in Σ_q or the pointwise conjugacy class of the matrix symbol. □

6.4. **Coarser shrinking scales and a noncommutative integral.**

Theorem 6.5. *Let $\eta_N \downarrow 0$ satisfy $\eta_N R_N^k \to \infty$. Then*

$$\lim_{N\to\infty} \eta_N^\alpha D_\pm(R_N, \eta_N) = \int ((\pm x)_+)^\alpha \, d\nu_c(x), \tag{6.7}$$

$$\lim_{N\to\infty} \eta_N^\alpha E_{\eta_N}(R_N; P, A) = 2\mathcal{C}_B, \qquad \mathcal{C}_B := \int |x|^\alpha \, d\nu_c(x). \tag{6.8}$$

If $c \not\equiv 0$, then $\mathcal{C}_B > 0$ and the second formula is the asymptotic equivalence $E_{\eta_N}(R_N; P, A) \sim 2\mathcal{C}_B \eta_N^{-\alpha}$.

Proof. Put

$$n_\pm(\eta) = \operatorname{Tr} \mathbf{1}_{(\eta,\infty)}(\pm B), \qquad \kappa_\pm = \int ((\pm x)_+)^\alpha \, d\nu_c(x).$$

The signed Birman–Solomyak law for classical negative-order operators on Hermitian bundles [54, Theorem 6.1 and Remark 5.4] gives

$$n_\pm(\eta) = \kappa_\pm \eta^{-\alpha} + o(\eta^{-\alpha}) \qquad (\eta \downarrow 0). \tag{6.9}$$

Indeed, if b_{-k} is the principal symbol of B, then $b_{-k} = q^{-k}c$. Since $k\alpha = n$, the homogeneous change of radial coordinate from a metric cosphere to Σ_q identifies the coefficient in that theorem with $\kappa_\pm$. No ellipticity of B is required.

By (6.2),

$$|b_j| > \eta_N \quad \Longrightarrow \quad a_j < \big(\|C\|/\eta_N\big)^{1/k} = o(R_N).$$

Thus $D_\pm(R_N, \eta_N) = n_\pm(\eta_N)$ for all sufficiently large N. Equation (6.9) and Lemma 6.1 give the two limits. Finally, $\kappa_+ + \kappa_- = \mathcal{C}_B$. If $c \not\equiv 0$, then $\operatorname{tr}_E |c|^\alpha$ is positive on a nonempty open subset of Σ_q, so $\mathcal{C}_B > 0$. □

Corollary 6.6. *Let $s_1(B) \geq s_2(B) \geq \cdots$ be the singular values of B, and let (η_N) satisfy the hypotheses of Theorem 6.5. Then*

$$\lim_{J\to\infty} \frac{1}{\log J} \sum_{j=1}^{J} s_j(B)^\alpha = \mathcal{C}_B = \frac{1}{2} \lim_{N\to\infty} \eta_N^\alpha E_{\eta_N}(R_N; P, A). \tag{6.10}$$

Thus $|B|^\alpha$ is measurable for Dixmier traces and every such trace equals $\mathcal{C}_B$. If B is elliptic, then $|B|^\alpha \in \Psi_{\mathrm{cl}}^{-n}$ and

$$\mathcal{C}_B = \frac{1}{n} \operatorname{Res}_W(|B|^\alpha),$$

where the residue is normalized by $(2\pi)^{-n}$.

Proof. The singular-value form of the same Weyl law [54, Theorem 6.1] gives

$$\lim_{j\to\infty} j \, s_j(B)^\alpha = \mathcal{C}_B.$$

Thus $|B|^\alpha$ is a positive Weyl operator in the weak trace class. Its logarithmic mean and all its Dixmier traces equal $\mathcal{C}_B$ by [54, Theorem 2.18 and Proposition 5.12]. The remaining equality in (6.10) follows from Theorem 6.5.

If B is elliptic, let F be the orthogonal projection onto $\ker B$. Elliptic regularity makes F smoothing and of finite rank. The operator $D = (B^2 + F)^{-1}$ is positive elliptic of order $2k$, with domain H^{2k}. The complex-power theorem gives

$$|B|^\alpha = D^{-\alpha/2} - F \in \Psi_{\mathrm{cl}}^{-n}, \qquad \sigma_{-n}(|B|^\alpha) = |b_{-k}|^\alpha.$$

The residue formula on Σ_q is therefore

$$\operatorname{Res}_W(|B|^\alpha) = \frac{1}{(2\pi)^n} \int_{\Sigma_q} \operatorname{tr}_E |c|^\alpha \, d\Sigma_q = n\mathcal{C}_B.$$

□

6.5. **Stability and the limits of reconstruction.**

Proposition 6.7. *Let $\mathcal{T}$ be any fixed discrete multiset supported on $\mathbb{Z}$ and bounded below. At complete-cluster cutoffs,*

$$\left|E_\eta(R_N; P, \mathcal{T}) - E_\eta(R_N; P, A)\right| \le E_0(R_N; A, \mathcal{T})$$

for every $\eta \ge 0$. Consequently the critical profile remains unchanged if $E_0(R_N; A, \mathcal{T}) = o(R_N^n)$. The coarse law remains unchanged if this integer multiplicity error is $o(\eta_N^{-\alpha})$.

Proof. Transform the integer list for A into that for $\mathcal{T}$ by deleting and inserting the excess copies at each integer. The number of changes is exactly $E_0(R_N; A, \mathcal{T})$. Lemma 1.2 bounds the change of the defect at each step by one, proving the estimate. Divide by the normalizations in the two theorems to obtain their conclusions. □

Proposition 6.8. *Suppose $c \not\equiv 0$. For each fixed sufficiently small $\eta > 0$, the limit*

$$E_\infty(\eta) := \lim_{N\to\infty} E_\eta(R_N; P, A)$$

exists, and

$$E_\infty(\eta) \sim 2C_B\eta^{-n/k} \quad (\eta \downarrow 0), \qquad \lim_{\eta\downarrow 0} \frac{\log E_\infty(\eta)}{\log(1/\eta)} = \frac{n}{k}.$$

For every $\gamma > 0$, the operator B belongs to the Schatten class $\mathcal{S}^\gamma$ if and only if $\gamma > n/k$.

Proof. Compactness and (6.2) imply that for a fixed η all sufficiently high clusters have no copies with $|b_j| > \eta$. Their copies match internally, so adding those clusters does not change the optimal defect. The fixed exceptional low clusters are already included. The limit therefore exists. Its difference from $2\#\{j : |b_j| > \eta\}$ is bounded independently of η by Lemma 6.1; the compact counting law proves the asymptotic and its logarithmic consequence. The singular-value asymptotic in Corollary 6.6, with $C_B > 0$, gives $s_j(B) \asymp j^{-k/n}$, so the series $\sum_j s_j(B)^\gamma$ converges exactly when $k\gamma/n > 1$. □

Remark 6.9 (Lower-order perturbations). If $B' = B'^* \in \Psi_{\mathrm{cl}}^{-k}$ strongly commutes with A and $B' - B \in \Psi^{-k-\varepsilon}$ for some $\varepsilon > 0$, then $\sigma_0(A^k B') = c$. The signed and unsigned profiles are therefore unchanged. For $t \ge \sup_{\Sigma_q} \|c\|$ and $t > 0$,

$$F(t) = \frac{2t^{-\alpha}}{n(2\pi)^n} \int_{\Sigma_q} \mathrm{tr}_E\, |c(\rho)|^\alpha \, d\Sigma_q.$$

Consequently $F \equiv 0$ if and only if $c \equiv 0$.

6.6. **An exact two-component example.** On S^5, let K_0 be the integer quantizer (4.6) and put

$$A = K_0 I_2, \qquad B = \begin{pmatrix} 1/8 & 0 \\ 0 & -1/16 \end{pmatrix} A^{-2}.$$

The levels of A are $a = 1, 2, \ldots$. Each scalar component has counting coefficient $1/60$, and

$$\nu_c = \frac{1}{60}\left(\delta_{1/8} + \delta_{-1/16}\right).$$

Thus $n = 5$, $k = 2$, and the unmatched fraction, normalized by the total number of copies in both lists, tends to

$$\frac{1}{2}\left[\min\{1, (1/(8t))^{5/2}\} + \min\{1, (1/(16t))^{5/2}\}\right].$$

Its two changes of derivative recover the two absolute symbol values. Signed counts distinguish their signs. By (8.1), the exact cumulative scalar multiplicity through level N is $\binom{N+4}{5} + \binom{N+3}{5}$.

7. Joint distributions of return phases and correction symbols

Retain the notation of Theorem 4.1. Let L be the order of the time-2π canonical map and let A be the strongly commuting quantizer satisfying (4.5). Increase each eigenvalue below one by an integer on its joint eigenspace. This changes A by a finite-rank smoothing operator, preserves $e^{2\pi iA}$, and gives $A \geq 1$. Put

$$U_A = e^{2\pi iA}, \qquad V_A = U_A^L, \qquad H = \sigma_0(V_A), \qquad \mathcal{H} = \bigcup_{\rho \in \Sigma_q} \operatorname{Spec} H(\rho).$$

The set $\mathcal{H}$ is finite, and

$$V_A = \sum_{h \in \mathcal{H}} h\Pi_h, \qquad \Pi_h \in \Psi^0_{\mathrm{cl}}, \qquad \pi_h = \sigma_0(\Pi_h) = \mathbf{1}_{\{h\}}(H).$$

These orthogonal projections commute with P and A. Every Lth root of each $h \in \mathcal{H}$ belongs to $\mathcal{S}$.

Suppose that $B = P - A \in \Psi^{-k}_{\mathrm{cl}}$, where $k > 0$, and put

$$C = A^k B \in \Psi^0_{\mathrm{cl}}, \qquad c = \sigma_0(C), \qquad \alpha = n/k.$$

The complex-power calculus of Section 6 applies to A. The order-zero operator C is bounded. Strong commutation gives $C = C^*$, $[\Pi_h, C] = 0$, and $[\pi_h, c] = 0$. For Borel sets $J \subset \mathbb{R}$, define

$$\nu_h(J) = \frac{1}{n(2\pi)^n} \int_{\Sigma_q} \operatorname{tr}_E\big(\pi_h(\rho)\mathbf{1}_J(c(\rho))\big)\, d\Sigma_q(\rho). \tag{7.1}$$

The commuting projections in the integrand have positive product. Thus ν_h is a finite positive measure, and

$$\nu_h(\mathbb{R}) = \frac{r_h}{n(2\pi)^n} \int_{\Sigma_q} d\Sigma_q, \qquad r_h = \operatorname{rank} \pi_h.$$

No choice of eigenvectors of c enters this definition.

In a common orthonormal eigenbasis, write $Ae_j = a_j e_j$, $Be_j = b_j e_j$, and $\mu_j = a_j + b_j$. Then

$$|b_j| \leq \|C\| a_j^{-k}.$$

For $s \in S$, let $\mathcal{A}_s$ and $\mathcal{P}_s$ be the multisets of a_j and μ_j, respectively, over the indices with $e^{2\pi i a_j} = s$. Define

$$D_{s,\pm}(R, \eta) = \#\{j \,:\, a_j \leq R,\ e^{2\pi i a_j} = s,\ \pm b_j > \eta\},$$
$$E_{s,\eta}(R) = E_\eta(R; \mathcal{P}_s, \mathcal{A}_s).$$

At sufficiently high energy, a_j is the unique nearest point of $\Lambda(\mathcal{S})$ to μ_j. Hence these multisets and signed offsets are determined by the scalar frequency lists up to finitely many copies.

Choose $\theta \in (0,1)$ with $\delta_\theta = \operatorname{dist}(\theta, \Lambda(\mathcal{S})) > 0$ and put $R_N = N + \theta$. Each cutoff has the same distance δ_θ from $\Lambda(\mathcal{S})$.

Theorem 7.1. *For $h \in \mathcal{H}$, $s^L = h$, and every $t > 0$,*

$$R_N^{-n} D_{s,\pm}(R_N, tR_N^{-k}) \longrightarrow \frac{1}{L} \int_{\mathbb{R}} \min\{1, ((\pm x)_+/t)^\alpha\}\, d\nu_h(x), \tag{7.2}$$

$$R_N^{-n} E_{s,tR_N^{-k}}(R_N) \longrightarrow \frac{2}{L} \int_{\mathbb{R}} \min\{1, (|x|/t)^\alpha\}\, d\nu_h(x). \tag{7.3}$$

The convergence is uniform on compact subintervals of $(0, \infty)$. Every vertex of a fixed holonomy polygon has the same limiting correction distribution. The signed phase profiles and the phase Weyl masses determine all measures ν_h.

Proof. The orthogonal projector onto the s phase of A is

$$E_s = \frac{1}{L}\Pi_h \sum_{\ell=0}^{L-1} s^{-\ell} U_A^{\ell}. \tag{7.4}$$

On the h block, $s^{-1}U_A$ has spectrum among the Lth roots of unity. Their character sum is L at 1 and zero elsewhere. The factor Π_h annihilates the other blocks.

Let $d \geq 0$ be an integer and let $\chi \in C_c^{\infty}((0,\infty))$. For $1 \leq \ell < L$, the trace estimate in Theorem 4.5, applied to A with the insertion $\Pi_h C^d$, gives

$$\operatorname{Tr}\big(U_A^{\ell}\Pi_h C^d \chi(A/R)\big) = o(R^n).$$

Indeed, A has the same scalar principal symbol as P, and Φ^{ℓ} is a nonidentity finite-order map; its fixed set has zero Liouville measure. The $\ell = 0$ term is given by (1.1). Therefore

$$R^{-n}\operatorname{Tr}(E_s C^d \chi(A/R)) \longrightarrow \frac{1}{L(2\pi)^n}\int_{\Sigma_q}\int_0^{\infty} u^{n-1}\chi(u)\operatorname{tr}_E(\pi_h c^d)\,du\,d\Sigma_q.$$

The approximation argument of Lemma 6.2 now gives, for $F \in C_c((0,\infty)\times\mathbb{R})$,

$$R^{-n}\sum_{e^{2\pi i a_j}=s} F(a_j/R, a_j^k b_j) \longrightarrow \frac{n}{L}\int_{\mathbb{R}}\int_0^{\infty} u^{n-1}F(u,x)\,du\,d\nu_h(x).$$

The approximation error is bounded by a constant times $\varepsilon R^{-n}N_A(KR)$ on every fixed radial interval $[K^{-1},K]$. The threshold argument of Theorem 6.3 therefore applies with ν_h/L in place of ν_c: the boundary $\pm x = tu^k$ has zero measure by radial integration, and $a_j < \varepsilon R$ contributes $O(\varepsilon^n)$ after normalization. This proves (7.2). The same joint limit gives

$$\lim_{R\to\infty} R^{-n}\#\{j : a_j \leq R,\ e^{2\pi i a_j} = s\} = \frac{1}{L}\nu_h(\mathbb{R}).$$

Let δ be the minimum separation of $\Lambda(\mathcal{S})$. For all sufficiently high copies, $|b_j| < \min(\delta_\theta/2, \delta/4)$. Then $a_j \leq R_N$ if and only if $\mu_j \leq R_N$, and for $a \in \Lambda(\mathcal{S})\setminus\{a_j\}$,

$$|\mu_j - a| \geq \delta - |b_j| > 3\delta/4.$$

For $0 \leq \eta < \delta/4$, the matching argument of Lemma 6.1 consequently gives

$$E_{s,\eta}(R_N) = 2D_{s,+}(R_N,\eta) + 2D_{s,-}(R_N,\eta) + O(1),$$

uniformly in N and η. This proves (7.3). The locally uniform convergence follows from monotonicity and continuity, as in Theorem 6.3.

Let $F_{s,\pm}$ denote the signed limits. Corollary 6.4, applied to ν_h/L, gives

$$\nu_h(\{x : \pm x > t\}) = L\left(F_{s,\pm}(t) + \frac{t}{\alpha}F'_{s,\pm,+}(t)\right),$$

where the derivative is taken from the right. Moreover,

$$\nu_h(\{0\}) = \nu_h(\mathbb{R}) - L\lim_{t\downarrow 0}\big(F_{s,+}(t) + F_{s,-}(t)\big).$$

The open tails and this atom determine ν_h. □

Simultaneous unitary conjugacy of the matrix symbols (H, c) leaves every measure ν_h unchanged. The rotation group of the principal phase measure can strictly contain the group of Lth roots of unity. Indeed, take $L = 1$ and the two phases $1, -1$, each with multiplicity one, in Proposition 4.4. Then $\Phi = I$, whereas

$$\nu = \tfrac{1}{2}(\delta_1 + \delta_{-1})$$

is invariant under $z \mapsto -z$.

Part 2. Geodesic return and spherical rigidity

Under the hypotheses of Theorem 2.2 or Theorem 2.4, fixed frequency windows yield a common geodesic period and a commuting operator whose spectrum lies on the reference lattice. The discrepancy from the reference multiplicities bounds the localized wave trace differences. At times L/s, where L is the prime geodesic period on the universal cover and $s \geq 2$ is an integer, the return components of maximal dimension have Morse index zero and a common leading trace phase. The resulting restrictions on the return families and their evaluation maps, together with the contact-volume identity, yield a round universal cover. For a homogeneous reference, the harmonic multiplicities further determine the quotient up to isometry.

8. Harmonic multiplicities and localized spectral traces

For a metric on a d-dimensional manifold, the replacement $g \mapsto r^{-2}g$ multiplies scalar eigenvalues by r^2 and volume by r^{-d}. Applied to both comparison metrics, it preserves the frequencies in (2.3). Rescale to $r = 1$ and put

$$P_Z = (\Delta_Z + m^2)^{1/2}, \qquad n = 2m + 1 \geq 5.$$

No equality between $\dim M$ and n is assumed.

Let $\mathcal{P}_k$ be the complex homogeneous polynomials of degree k on $\mathbb{R}^{n+1}$, and let $\mathcal{H}_k$ be their harmonic subspace. Put $\mathcal{P}_k = 0$ for $k < 0$, and set $\binom{b}{n} = 0$ for integers $b < n$. A finite free orthogonal group is a finite subgroup of $O(n+1)$ whose nonidentity elements have no fixed point on $\mathbb{S}^n(1)$.

Lemma 8.1. *For $k \geq 0$, restriction identifies $\mathcal{H}_k$ with the scalar eigenspace on $\mathbb{S}^n(1)$ of eigenvalue $k(k+n-1)$, and*

$$d_k := \dim \mathcal{H}_k = \binom{k+n}{n} - \binom{k+n-2}{n}. \tag{8.1}$$

For a finite free orthogonal group Γ, put $r_k = \dim \mathcal{H}_k^\Gamma$. These are the scalar multiplicities on $\mathbb{S}^n(1)/\Gamma$, and, for $|z| < 1$,

$$F_\Gamma(z) := \sum_{k\geq 0} r_k z^k = \frac{1-z^2}{|\Gamma|} \sum_{\gamma\in\Gamma} \frac{1}{\det(I - z\gamma)}. \tag{8.2}$$

Proof. The restriction and quotient identifications are [39, Lemma 1.3 and Proposition 1.5, pp. 78–79]. The decomposition

$$\mathcal{P}_k = \mathcal{H}_k \oplus |x|^2 \mathcal{P}_{k-2}$$

is [39, (1.1), p. 78]; taking dimensions gives (8.1). Since $n = 2m+1$ and Γ acts freely, [39, Lemma 1.1(ii) and Theorem 2.2, pp. 77–80] give $\Gamma \subset \mathrm{SO}(2m+2)$ and (8.2). For lens spaces the latter formula is also [40, Theorem 3.2, p. 452]. □

Lemma 8.2. *Let $\Gamma \subset O(2m+2)$ be finite and free on the unit sphere. Then $\Gamma \subset \mathrm{SO}(2m+2)$. Every eigenvalue of $\gamma \in \Gamma$ has multiplicative order exactly $\mathrm{ord}(\gamma)$. Every involution in Γ equals $-I$. Each element other than $I, -I$ has only nonreal eigenvalues, of multiplicity at most $m+1$.*

Proof. The inclusion in $\mathrm{SO}(2m+2)$ is [39, Lemma 1.1(ii)]. Let $\gamma \neq I$ and $b = \mathrm{ord}(\gamma)$. The subgroup $\langle\gamma\rangle$ acts freely on $\mathbb{S}^{2m+1}(1)$. By [39, §1, Example (ii), p. 77], γ is orthogonally conjugate to a direct sum of rotations through angles $2\pi p_j/b$, with $\gcd(p_j, b) = 1$ for every j. Its eigenvalues are therefore primitive b-th roots of unity. For $b = 2$ every rotation is $-I_2$, so $\gamma = -I$. For $b > 2$ the eigenvalues are nonreal. Complex conjugation preserves their multiplicities; each conjugate pair occupies twice the multiplicity of either eigenvalue in a space of dimension $2m+2$. Thus each multiplicity is at most $m+1$. □

Lemma 8.3. *In case (E),*

$$r_{2k} = \frac{2}{q} d_{2k} + O(k^m), \qquad r_{2k+1} = 0. \tag{8.3}$$

For every odd-order round spherical quotient with group of order q,

$$r_k = \frac{1}{q} d_k + O(k^m). \tag{8.4}$$

For $a = 2$ in (E) and $a = 1$ in (O), every sufficiently large level $m + ak$ has positive multiplicity. For each fixed integer $b \geq 1$ and $v \in \{0, \dots, b-1\}$,

$$\sum_{\substack{k \geq 0,\ m+ak \leq R \\ k \equiv v \pmod b}} r_{ak} = \frac{2}{qb\ n!} R^n + O(R^{n-1}) \qquad (R \to \infty).$$

Proof. In case (E), Cauchy's theorem gives an involution in Γ_0, which is $-I$ by Lemma 8.2. Its action on $\mathcal{H}_k$ is $(-1)^k I$, so $r_{2k+1} = 0$. The terms $I, -I$ in (8.2) contribute

$$\frac{1 + (-1)^k}{q} d_k.$$

For every other group element the poles have order at most $m + 1$. Since

$$[z^k](1 - z/\alpha)^{-j} = \binom{k+j-1}{j-1} \alpha^{-k} = O(k^{j-1}) \qquad (k \to \infty),$$

finite partial fractions give a total contribution $O(k^m)$. This proves (8.3). For an odd-order group, the identity is the only element with real eigenvalues. Its term contributes $q^{-1} d_k$, and the remaining coefficients are $O(k^m)$. Thus (8.4) holds.

Equation (8.1) gives

$$d_k = \frac{2}{(n-1)!} k^{n-1} + O(k^{n-2}), \qquad r_{ak} = \frac{2a^n}{q(n-1)!} k^{n-1} + O(k^{n-2}).$$

The second leading coefficient is positive. For fixed b, v and $X \to \infty$,

$$\sum_{\substack{0 \leq k \leq X \\ k \equiv v \pmod b}} k^{n-1} = \frac{X^n}{bn} + O(X^{n-1}).$$

Taking $X = (R - m)/a$ proves the last formula. □

Lemma 8.4. *Let $F(z) = \sum_{k \geq 0} a_k z^k$ be rational with poles at roots of unity. Let $s \geq 1$ be an integer. Suppose that $\xi = e^{it}$ is either a regular point or a pole of order at most s, and put*

$$L_\xi(F) = \lim_{z \to \xi} (1 - z/\xi)^s F(z).$$

For $\chi \in C_c^\infty((0,1))$,

$$\sum_{k \geq 0} a_k e^{it(k+m)} \chi((k+m)/R) = \frac{e^{itm} L_\xi(F)}{(s-1)!} R^s \int_0^\infty \chi(u) u^{s-1}\, du + O(R^{s-1}). \tag{8.5}$$

Proof. Write

$$F(z) = B(z) + \sum_\alpha \sum_{j=1}^{s_\alpha} c_{\alpha,j} (1 - z/\alpha)^{-j},$$

where B is a polynomial and α runs through the poles. The coefficients of B make no contribution for sufficiently large R. Put

$$p_j(x) = \frac{1}{(j-1)!} \prod_{\ell=1}^{j-1} (x + \ell), \qquad f_{j,R}(x) = p_j(x) \chi((x+m)/R),$$

with the empty product equal to one. For $k \geq 0$, $p_j(k) = \binom{k+j-1}{j-1}$. The contribution of $c_{\alpha,j}$ is therefore

$$e^{itm} c_{\alpha,j} \sum_{k \in \mathbb{Z}} (\xi/\alpha)^k f_{j,R}(k),$$

since $f_{j,R}$ has support in $(0,\infty)$ for large R.

If $\alpha = \xi$, put $g_{j,R}(u) = R^{1-j} f_{j,R}(Ru)$. Since

$$R^{1-j} p_j(Ru) = \frac{1}{(j-1)!} \prod_{\ell=1}^{j-1} (u + \ell/R),$$

these functions have common compact support and uniformly bounded derivatives. Put

$$\widehat{g}(\zeta) = \int_{\mathbb{R}} g(u) e^{-2\pi i u\zeta}\, du.$$

The estimates in the proof of [61, Chapter 5, Theorem 1.3, p. 137] give

$$|\widehat{g}_{j,R}(\zeta)| \leq C_N (1 + |\zeta|)^{-N} \qquad (N \geq 0),$$

uniformly for large R. Moreover,

$$\int_{\mathbb{R}} g_{j,R}(u)\, du = \frac{1}{(j-1)!} \int_{\mathbb{R}} \chi(v) \prod_{\ell=1}^{j-1} \left(v + \frac{\ell - m}{R} \right) dv.$$

Poisson summation [61, Chapter 5, Theorem 3.1, p. 154] gives, for every $A > 0$,

$$\begin{aligned} \sum_{k \in \mathbb{Z}} f_{j,R}(k) &= R^j \sum_{\ell \in \mathbb{Z}} \widehat{g}_{j,R}(R\ell) \\ &= R^j \int_{\mathbb{R}} g_{j,R}(u)\, du + O(R^{-A}) \\ &= \frac{R^j}{(j-1)!} \int_0^\infty \chi(u) u^{j-1}\, du + O(R^{j-1}). \end{aligned}$$

If $\alpha \neq \xi$, put $\omega = \xi/\alpha$ and $\Delta f(k) = f(k+1) - f(k)$. Iterated summation by parts [61, Chapter 2, Exercise 7(a), p. 60] gives

$$\left| \sum_{k \in \mathbb{Z}} \omega^k f_{j,R}(k) \right| \leq |1 - \omega|^{-N} \sum_{k \in \mathbb{Z}} |\Delta^N f_{j,R}(k)| = O(R^{j-N}).$$

Indeed, $\|f_{j,R}^{(N)}\|_\infty = O(R^{j-1-N})$, and the finite differences have support of length $O(R)$. Taking N arbitrarily large gives $O(R^{-A})$ for every $A > 0$. At $\alpha = \xi$, the coefficient $c_{\xi,s}$ is $L_\xi(F)$, with $c_{\xi,s} = 0$ if the pole has smaller order or is absent. The terms with $j < s$ are $O(R^{s-1})$. Summing proves (8.5). □

9. Scalar return and lattice quantization

Let M be closed and connected, with $\dim M \geq 2$, and put $P = (\Delta_M + c)^{1/2}$, $c > 0$, on scalar half-densities. Its subprincipal symbol is zero [17, p. 58]. If $G_T = I$, [12, Theorem 1.1 and its proof, p. 510], applied to $TP/(2\pi)$, gives

$$e^{iTP} - \rho_T I \in \Psi_{\mathrm{cl}}^{-1}, \qquad \rho_T \in \{1, -1, i, -i\}.$$

The Maslov factor ρ_T is constant because S^*M is connected.

Corollary 9.1. *Let $T > 0$, $\zeta \in \{1, -1, i, -i\}$, and $\kappa > 0$. Suppose that*

$$\operatorname{Re}(\bar{\zeta} e^{iT\mu}) \geq \kappa$$

for all but $o(N_P(R_j))$ frequency copies below R_j, along $R_j \to \infty$. Then

$$G_T = I, \qquad e^{iTP} - \zeta I \in \Psi_{\mathrm{cl}}^{-1}, \qquad |e^{iT\mu} - \zeta| \leq C(1+\mu)^{-1} \quad (\mu \in \operatorname{Spec} P).$$

Proof. The hypothesis implies $\kappa \le 1$. Apply Theorem 3.1 to the trivial line bundle with its flat connection and constant potential c. It gives $G_T = I$ and $\mathrm{Re}(\bar\zeta \rho_T) \ge \kappa$. Since $\bar\zeta \rho_T$ is a fourth root of unity with positive real part, it equals 1. Thus $\rho_T = \zeta$. The operator $(e^{iTP} - \zeta I)(1+P)$ has order zero and is bounded on L^2. Its restriction to the μ-eigenspace gives the stated estimate. □

Corollary 9.2. *Let $m > 0$, $a > 0$, $T = 2\pi/a$, and $\zeta = e^{iTm}$. If $e^{iTP} - \zeta I \in \Psi^{-1}_{\mathrm{cl}}$, there is a positive self-adjoint $A \in \Psi^1_{\mathrm{cl}}$ with domain H^1, strongly commuting with P, such that*

$$[A, P] = 0, \qquad A - P \in \Psi^{-1}_{\mathrm{cl}}, \qquad \mathrm{Spec}(A) \subset m + a\mathbb{Z}_{\ge 0}. \tag{9.1}$$

On every sufficiently high P-eigenspace, A equals the nearest point of $m + a\mathbb{Z}$.

Proof. Boundedness of $(e^{iTP} - \zeta I)(1+P)$ gives $|e^{iT\mu} - \zeta| = O((1+\mu)^{-1})$. Hence Theorem 3.1 applies with $\beta = Tm$ and $\kappa = 1/2$, apart from finitely many frequencies. The principal return symbol equals ζ, and the scalar holonomy is the identity. Let $\alpha \in (0, a]$ satisfy $m - \alpha \in a\mathbb{Z}$. Theorem 3.3 gives a positive self-adjoint $A_0 \in \Psi^1_{\mathrm{cl}}$ with domain H^1, strongly commuting with P, such that

$$A_0 - P \in \Psi^{-1}_{\mathrm{cl}}, \qquad \mathrm{Spec}(A_0) \subset \alpha + a\mathbb{Z}_{\ge 0},$$

and A_0 equals the nearest lattice point at high frequency. Since $m > 0$, one has $m = \alpha + a j_0$ for some $j_0 \in \mathbb{Z}_{\ge 0}$. The spectral projector $E = \mathbf{1}_{(-\infty, m)}(A_0)$ has finite rank and smooth kernel and strongly commutes with P. Therefore

$$A = A_0 + (m - A_0)E$$

has all the required properties. The correction is self-adjoint and smoothing and affects only finitely many P-eigenspaces. □

Remark 9.3. The condition $\kappa > 0$ cannot be replaced by $\kappa = 0$. On the unit sphere S^{2m+1}, with $m \in \mathbb{Z}_{\ge 2}$, Lemma 8.1 gives $\mathrm{Spec}\,\sqrt{\Delta + m^2} = m + \mathbb{Z}_{\ge 0}$. For $\zeta = i(-1)^m$ and every $k \ge 0$,

$$\mathrm{Re}\big(\bar\zeta e^{i\pi(k+m)}\big) = 0.$$

Under the metric identification of the unit tangent and cotangent bundles, $G_\pi(x, v) = (-x, -v) \ne (x, v)$.

10. Spectral matchings and common geodesic periods

Fix a reference in (E), or any odd-order round spherical reference, with spacing $a = 2$ or $a = 1$, respectively, and tolerance $0 \le \eta < a/4$. Suppose there are $K_j \to \infty$ such that, with $R_j = R_{K_j}$,

$$\mathcal{E}_\eta(K_j) = o(R_j^n). \tag{10.1}$$

The exponents in Theorems 2.2, 2.3, and 2.4 are strictly smaller than n.

Proposition 10.1. *The comparison manifold has dimension n, volume V_n/q, and least positive common geodesic period $T = 2\pi/a$. There is an operator A as in* (9.1) *for which*

$$\mathcal{E}_A(R_K) := \sum_{\omega \le R_K} |\,\mathrm{mult}_A(\omega) - \mathrm{mult}_{P_Q}(\omega)| \le \mathcal{E}_\eta(K) + C_0 \tag{10.2}$$

for all sufficiently large K. For every $t \in \mathbb{R}$, every $\chi \in C_c^\infty((0,1))$, and every such K,

$$\big|\mathrm{Tr}(e^{itA}\chi(A/R_K)) - \mathrm{Tr}(e^{itP_Q}\chi(P_Q/R_K))\big| \le \|\chi\|_\infty \mathcal{E}_A(R_K) \le \|\chi\|_\infty\big(\mathcal{E}_\eta(K) + C_0\big). \tag{10.3}$$

Proof. For every K, an admissible matching gives

$$|N_M(R_K) - N_Q(R_K)| \le \mathcal{E}_\eta(K).$$

Put $d = \dim M \ge 1$ and $c_d = (2\pi)^{-d}\,\mathrm{vol}(B^d)$. The scalar Weyl law [17, Corollary 2.5, equation (2.23)] gives

$$N_M(R) = c_d\,\mathrm{Vol}(M) R^d + o(R^d), \qquad N_Q(R) = c_n(V_n/q)R^n + o(R^n).$$

If $d < n$, divide the counting inequality along R_j by R_j^n; its left side tends to $c_n V_n/q > 0$, whereas its right side tends to zero by (10.1). If $d > n$, division by R_j^d gives the same contradiction with limit $c_d \operatorname{Vol}(M) > 0$. Thus $d = n$. Dividing by R_j^n now gives

$$\operatorname{Vol}(M) = V_n/q. \tag{10.4}$$

Put $T = 2\pi/a$ and $\zeta = e^{iTm} \in \{1, -1\}$. For every matched pair,

$$\operatorname{Re}(\bar{\zeta} e^{iT\mu}) = \cos(T(\mu - \beta)) \geq \cos(T\eta) > 0.$$

The number of competitor copies not accounted for by these pairs is at most $\mathcal{E}_\eta(K_j) = o(N_M(R_j))$. Corollary 9.1 and Corollary 9.2 give

$$G_T = I, \qquad e^{iTP_M} - e^{iTm} I \in \Psi^{-1}, \tag{10.5}$$

and an operator A as in (9.1).

Choose a common orthonormal eigenbasis of A and P_M. Choose μ_0 so large that the rounded value α of any $\mu > \mu_0$ satisfies $|\alpha - \mu| < a/4$. For a matched pair at such a frequency,

$$|\alpha - \beta| < a/4 + \eta < a/2.$$

Distinct points of $m + a\mathbb{Z}$ have separation at least a, so $\alpha = \beta$. Each R_K has distance $a/2$ from this lattice. Therefore μ and α lie on the same side of every R_K. Let B_0 be the total multiplicity of the P_M-frequencies at most μ_0. Fix a maximum admissible matching at R_K and discard its pairs involving these low-frequency copies. At most B_0 pairs are discarded. Every remaining pair transfers to an equal-frequency pair for A and P_Q, and distinct copies remain distinct under this transfer. If $b_A(R_K)$ denotes the maximum number of equal-frequency pairs for those two spectra, then

$$b_A(R_K) \geq b_\eta(K) - B_0.$$

For all sufficiently large K, the entire finite-dimensional subspace with P_M-frequencies at most μ_0 lies below R_K for both operators. The remaining copies lie on the same side of R_K. Hence

$$N_A(R_K) = N_M(R_K).$$

Since an equal-frequency matching leaves precisely the sum of the absolute multiplicity differences unmatched,

$$\begin{aligned} \mathcal{E}_A(R_K) &= N_A(R_K) + N_Q(R_K) - 2b_A(R_K) \\ &\leq N_M(R_K) + N_Q(R_K) - 2b_\eta(K) + 2B_0 \\ &= \mathcal{E}_\eta(K) + 2B_0. \end{aligned}$$

Thus (10.2) holds with $C_0 = 2B_0$. Expansion on the common lattice gives

$$\left| \sum_{\omega \leq R_K} e^{it\omega} \chi(\omega/R_K) \big(\operatorname{mult}_A(\omega) - \operatorname{mult}_{P_Q}(\omega) \big) \right| \leq \|\chi\|_\infty \mathcal{E}_A(R_K).$$

Thus (10.3) follows.

The period subgroup $\mathscr{P} = \{t \in \mathbb{R} : G_t = I\}$ is closed by continuity. The geodesic vector field on the compact manifold S^*M has no zero. Cover S^*M by finitely many smaller flow boxes whose closures lie inside flow boxes. There is $\epsilon > 0$ such that, for an initial point in any smaller box, the trajectory remains in the corresponding larger box for $|t| < \epsilon$. Its flow coordinate changes by t, so it cannot return when $0 < |t| < \epsilon$. Thus $\mathscr{P}$ is a nonzero discrete subgroup of $\mathbb{R}$, and $\mathscr{P} = T_0\mathbb{Z}$ for some $T_0 > 0$. Since $T \in \mathscr{P}$, write $T = bT_0$ with $b \in \mathbb{Z}_{\geq 1}$.

If $b > 1$, put $V = e^{iT_0(A-m)}$. Its canonical map is the identity, so $V \in \Psi^0_{\mathrm{cl}}$ [17, Theorem 1.1 and p. 44], and the arithmetic spectrum gives $V^b = I$. Its principal symbol v_0 has $v_0^b = 1$; continuity on connected S^*M makes it constant. Thus $V - v_0 I$ is compact. For $v \in \{0, \dots, b-1\}$, define

$$N_{A,v}(R) = \sum_{\substack{k \geq 0,\ m+ak \leq R \\ k \equiv v \pmod b}} \operatorname{mult}_A(m + ak),$$

and define $N_{Q,v}(R)$ by replacing A with P_Q. The triangle inequality and the definition of $\mathcal{E}_A$ give

$$|N_{A,v}(R_K) - N_{Q,v}(R_K)| \le \mathcal{E}_A(R_K).$$

Lemma 8.3 gives, for each v,

$$N_{Q,v}(R) = \frac{2}{qb\ n!} R^n + O(R^{n-1}).$$

Together with (10.2) and (10.1), this yields

$$N_{A,v}(R_j) = \frac{2}{qb\ n!} R_j^n + o(R_j^n) \longrightarrow \infty.$$

On the class $k \equiv v \pmod b$, the operator V acts by $e^{2\pi i v/b}$. Its eigenspace for this value therefore has infinite dimension. If $b > 1$, at least one of these values differs from v_0. This contradicts the finite multiplicity of every nonzero eigenvalue of the compact operator $V - v_0 I$ [43, Chapter III, Theorem 6.26]. Thus $b = 1$ and $T_0 = T$. □

Lemma 10.2. *Assume the conclusions of Proposition 10.1. Let $s \ge 1$ be an integer and let $R_j = R_{K_j} \to \infty$. Suppose $\mathcal{E}_\eta(K_j)/R_j^s \to \delta < \infty$. If, for a fixed time t and every nonnegative $\chi \in C_c^\infty((0,1))$, the trace difference in* (10.3) *has the expansion*

$$\frac{e^{i\theta} L}{(s-1)!} R_j^s \int_0^1 \chi(u) u^{s-1}\, du + o(R_j^s),$$

where $L \in \mathbb{C}$ and $\theta \in \mathbb{R}$ are fixed, then

$$\delta \ge \frac{|L|}{s!}. \tag{10.6}$$

Proof. Take $0 \le \chi \le 1$ in (10.3), divide by R_j^s, and pass to the limit. This gives

$$\delta \ge \frac{|L|}{(s-1)!} \int_0^1 \chi(u) u^{s-1}\, du.$$

Choose $\chi_\ell \in C_c^\infty((0,1))$ with $0 \le \chi_\ell \le 1$ and $\chi_\ell(u) \to 1$ for every $u \in (0,1)$. For each fixed ℓ, take the preceding limit in j. Then dominated convergence as $\ell \to \infty$ gives $\int_0^1 \chi_\ell(u) u^{s-1}\, du \to 1/s$. □

10.1. **Clean geodesic returns.** Suppose $G_T = I$. The period subgroup of each unit covector is $\tau\mathbb{Z}$, with $T = k\tau$ for an integer $k \ge 1$. The uniform flow-box bound in the preceding proof gives $\tau \ge \epsilon > 0$; hence $k \le T/\epsilon$. All positive return times therefore belong to

$$\bigcup_{1 \le k \le \lfloor T/\epsilon \rfloor} \frac{T}{k} \mathbb{Z}_{>0},$$

a locally finite subset of $(0,\infty)$. If t is a return time, then $t = jT/k$ for some positive integers j, k, so $G_t^k = G_{jT} = I$. The finite group $\langle G_t \rangle$ acts smoothly and properly on S^*M. The fixed-set theorem [51, Theorem 1.23 and Proposition 1.24] gives a closed smooth fixed locus and

$$T_v \operatorname{Fix}(G_t) = \ker(dG_t(v) - I).$$

Its connected components are open in the fixed locus; compactness therefore makes their number finite. In particular, the return is clean.

The degree-one homogeneity of $p(x,\xi) = |\xi|_g$ makes its Hamiltonian flow commute with positive fiber dilations. The fixed locus in $T^*M \setminus 0$ is consequently a cone. The finite cyclic group $\langle G_t \rangle$ acts symplectically there, so each component of this fixed cone is symplectic [19, Example 3.3, p. 18]. Let its dimension be $2h$. The Liouville vector field $Z = \xi \partial_\xi$ is tangent to the cone and satisfies $Zp = p$; thus it is transverse to the section $p = 1$. For $\omega = d\lambda$,

$$\iota_Z \omega^h = h\lambda \wedge \omega^{h-1}.$$

The left side restricts to a nonvanishing top-degree form on that section. Hence each component of $\mathrm{Fix}(G_t) \subset S^*M$ is a contact submanifold of dimension $2h-1$.

Let B be a positive self-adjoint scalar classical operator of order one on half-densities, with principal symbol $p = |\xi|_g$ and zero subprincipal symbol. This includes P_M and the operator A of Proposition 10.1, since $A - P_M \in \Psi^{-1}_{\mathrm{cl}}$. Fix a positive return time t. Let d be the largest dimension of a component of $\mathrm{Fix}(G_t)$, and suppose all components of dimension d have the same periodic Morse index σ. The fiberwise convexity condition in [17, pp. 69–70] holds:

$$d_\xi p(v) = 0 \quad \Longrightarrow \quad D^2_\xi p(v,v) = \frac{|v|^2_{g^{-1}}}{|\xi|_g} > 0 \qquad (v \neq 0).$$

Thus the integer in the trace-index phase is the Morse index for periodic boundary conditions.

Fix the Fourier convention

$$\widehat{f}(\rho) = \int_{\mathbb{R}} e^{-it\rho} f(t)\, dt, \qquad f(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{it\rho} \widehat{f}(\rho)\, d\rho.$$

Give every fixed component C its positive Duistermaat–Guillemin density ν_C, in the normalization of [17, pp. 60–61]. The clean trace formula [17, Theorem 4.5, equations (4.6)–(4.8)] gives a nonzero scalar $\gamma_{d,\sigma}$, depending only on d and σ, such that

$$\mathrm{Tr}(e^{itB}\chi(B/R)) = \gamma_{d,\sigma} R^{(d+1)/2} \left(\int_0^1 \chi(u) u^{(d-1)/2}\, du \right) \sum_{\substack{C \subset \mathrm{Fix}(G_t) \\ \dim C = d}} \int_C d\nu_C + o(R^{(d+1)/2}) \tag{10.7}$$

for every $\chi \in C_c^\infty((0,1))$.

Indeed, extend χ by zero and set

$$W_B(s) = \mathrm{Tr}(e^{isB}), \qquad \varphi_{R,t}(s) = \frac{R}{2\pi} \widehat{\chi}\big(R(s-t)\big).$$

Weyl's law makes W_B a tempered distribution. Fourier inversion, summed against the spectral multiplicities, gives

$$\mathrm{Tr}(e^{itB}\chi(B/R)) = \langle W_B, \varphi_{R,t} \rangle.$$

The sum converges because the Fourier transform of a Schwartz function decays faster than every power. Choose $\rho \in C_c^\infty(\mathbb{R})$ equal to one near t, with support containing no other return time. Since $\widehat{\chi}$ is Schwartz,

$$(1-\rho)\varphi_{R,t} = O(R^{-N}) \quad \text{in } \mathcal{S}(\mathbb{R}) \qquad \text{for every } N.$$

Consequently the pairing with this term is $O(R^{-N})$ for every N.

For the negative half-wave, the leading symbol in [17, equations (4.7)–(4.8)] is a positive density integral multiplied by a factor depending only on d and σ; the subprincipal factor is 1. Complex conjugation gives the positive half-wave formula. Pairing its leading homogeneous term with $\varphi_{R,t}$ gives

$$\int_0^\infty \chi(v/R) v^{(d-1)/2}\, dv = R^{(d+1)/2} \int_0^1 \chi(u) u^{(d-1)/2}\, du.$$

The next symbol term has degree $(d-3)/2$ and contributes $O(R^{(d-1)/2})$. Each smaller fixed component has odd dimension at most $d-2$, so its contribution has the same bound. A smooth local remainder, multiplied by ρ, has rapidly decreasing Fourier transform; its integral against $\chi(v/R)$ is $O(R^{-N})$ for every N, since $\mathrm{supp}\, \chi(v/R) \subset [cR, CR]$ for some $0 < c < C$. This gives (10.7).

If $0 \leq \chi \not\equiv 0$, its displayed moment and all the density integrals are positive. The leading coefficient is therefore nonzero. For two returns with the same d and σ, the ratio of their leading coefficients for the same such χ is positive real. In particular, A and P_M have identical leading coefficients. If t is not a return time, then W_B is smooth near t [17, Corollary 1.2]. The same localization gives

$$\mathrm{Tr}(e^{itB}\chi(B/R)) = O(R^{-N}) \qquad \text{for every } N.$$

Put $D = A - P_M$. It is bounded, self-adjoint, of classical order -1, and strongly commutes with P_M. Hence

$$e^{-itP_M} e^{itA} = e^{itD}.$$

For fixed t, the function $f_t(z) = (e^{itz} - 1)/z$, with $f_t(0) = it$, is entire. The holomorphic functional calculus [5, Example 2 and Theorem 5.4, arXiv version] gives $f_t(D) \in \Psi^0_{\mathrm{cl}}$: on a circle $|z| > \|D\|$, $zI - D$ is elliptic and invertible. Therefore

$$e^{itD} - I = D f_t(D) \in \Psi^{-1}_{\mathrm{cl}}. \tag{10.8}$$

11. Universal coverings and Morse indices

A Riemannian metric is Besse if every unit-speed geodesic is periodic. The prime length of a geodesic is the least positive period of its unit tangent vector. For smooth closed manifolds of dimension at least two, the following theorems hold.

(i) By the Bott–Samelson–McCleary theorem, a simply connected Besse manifold has the integral cohomology ring of a compact rank-one symmetric space. In odd dimension it is homeomorphic to a sphere. More generally, the universal cover of an odd-dimensional Besse manifold is homeomorphic to a sphere [4, Theorem 1.1 and the following paragraph].

(ii) On a topological sphere of dimension at least four, every Besse metric has one common prime geodesic length. On a simply connected Besse manifold, the free-loop energy is perfect for rational S^1-equivariant cohomology relative to the constant loops [57, Theorems A and D].

(iii) A Besse metric on a smooth n-manifold admitting an even-degree smooth covering by a smooth homotopy n-sphere has constant sectional curvature for $n \neq 3$ [49, Theorem 2].

Lemma 11.1. *Suppose the least common geodesic period on M^n, $n \geq 5$ odd, is T. Its universal Riemannian cover X is compact and is a smooth homotopy sphere. Its deck group Γ is finite. For one deck transformation D,*

$$\widetilde{G}_T = \widehat{D}, \qquad D^2 = I, \qquad \widetilde{G}_{2T} = I. \tag{11.1}$$

All prime geodesics of X have one length L, and

$$L \in \{T, 2T\}, \qquad D = I \iff L = T, \qquad \mathrm{Vol}(X) = |\Gamma| \,\mathrm{Vol}(M). \tag{11.2}$$

Proof. Let $p_X : X \to M$ be the universal Riemannian covering. Since M is a closed odd-dimensional Besse manifold, X is homeomorphic to $\mathbb{S}^n$ [4, Theorem 1.1 and the following paragraph]. Thus X is a compact smooth homotopy sphere. The fibers of p_X are compact and discrete, hence finite; consequently Γ is finite.

The induced covering $S^*X \to S^*M$ is connected and regular, with deck group given by the cotangent lifts of Γ. Since $G_T = I$, uniqueness of lifts gives $\widetilde{G}_T = \widehat{D}$ for one $D \in \Gamma$. Reversal $J(x, \xi) = (x, -\xi)$ commutes with every cotangent deck lift and satisfies

$$J \widetilde{G}_T J = \widetilde{G}_{-T}.$$

Hence $\widehat{D} = \widehat{D}^{-1}$, so $D^2 = I$ and $\widetilde{G}_{2T} = I$.

The lifted metric is Besse. Since X is homeomorphic to $\mathbb{S}^n$ and $n \geq 5$, [57, Theorem A] gives a common prime length L. The identity $\widetilde{G}_L = I$ descends to $G_L = I$, so $L = b\,T$ for some integer $b \geq 1$. Moreover, $2T = aL$ for some integer $a \geq 1$. Thus $ab = 2$, and $L \in \{T, 2T\}$. If $D = I$, then T is an upstairs period, whence $L = T$; the converse follows from $\widetilde{G}_L = I$. Since p_X is a $|\Gamma|$-sheeted local isometry, $\mathrm{Vol}(X) = |\Gamma| \,\mathrm{Vol}(M)$. □

Lemma 11.2. *Let X^n, $n \geq 5$ odd, be a Besse homotopy sphere whose prime geodesics all have length L. Their periodic Morse index is $n - 1$. In particular, their endpoint-fixed Morse index is at most $n - 1$.*

Proof. The critical manifold of prime geodesics in $\Lambda X = H^1(\mathbb{R}/\mathbb{Z}, X)$ is identified with SX by

$$v \longmapsto \big(t \longmapsto \gamma_v(Lt)\big).$$

The energy is Morse–Bott, and its index is constant on this connected critical manifold [57, Section 1, pp. 915–916]; denote the index by b. Every nonconstant critical loop is an iterate of a prime geodesic. Bott's iteration inequality [57, Section 6, p. 952] gives $\operatorname{ind}(\gamma^k) \geq \operatorname{ind}(\gamma)$ for $k \geq 1$. Thus b is the least index of a nonconstant critical loop. For a simply connected Besse manifold with the rational cohomology of $\mathbb{S}^n$, this least index is $n-1$ [57, Section 4, pp. 941–942]. These hypotheses hold for X, so $b = n-1$. The endpoint-fixed variation space is a subspace of the periodic variation space; its index is therefore at most $n-1$. □

Lemma 11.3. *Let X^n be a closed Riemannian manifold and let $L > 0$. Suppose the time-L geodesic map on SX is the identity. For every length-L unit-speed geodesic on X, the periodic and endpoint-fixed indices are equal.*

Proof. Fix such a geodesic γ, and put $V = \dot\gamma(0)^\perp$. The normal Poincaré map is $P_\gamma = dG_L|_{V\oplus V} = I$, where $V \oplus V$ is identified with the contact hyperplane by Jacobi initial data. The concavity formula [57, Section 4, p. 939] is

$$\operatorname{ind}_{\mathrm{per}}(\gamma) = \operatorname{ind}_D(\gamma) + \operatorname{ind}\widetilde{H} + \dim\ker\widetilde{H} - \dim\ker(P_\gamma - I),$$

where

$$\widetilde{H}(z,w) = -\omega((P_\gamma - I)z, w), \qquad z, w \in (P_\gamma - I)^{-1}(0 \oplus V).$$

Since $P_\gamma = I$, this domain is $V \oplus V$ and

$$\operatorname{ind}\widetilde{H} = 0, \qquad \dim\ker\widetilde{H} = \dim\ker(P_\gamma - I) = 2n-2.$$

The two indices are therefore equal. The tangential part of either index form is $\int_0^L |f'(t)|^2\,dt \geq 0$, so the normal and full indices agree. □

Lemma 11.4. *Under the hypotheses of Lemma 11.2, for any $c > 0$ the principal symbol of $e^{iL\sqrt{\Delta_X + c}}$ is 1.*

Proof. Put $P_X = \sqrt{\Delta_X + c}$. Since the time-$L$ geodesic map is the identity, the principal symbol of e^{iLP_X} is a constant u_0 of modulus one by Section 9. Fix $0 \leq \chi \in C_c^\infty((0,1))$, $\chi \neq 0$. By (1.1), the leading coefficient of $\operatorname{Tr}(e^{iLP_X}\chi(P_X/R))$ is $u_0 W_X$, where

$$W_X = (2\pi)^{-n}\int_{T^*X} \chi(|\xi|_g)\,dx\,d\xi > 0.$$

Let $S = \mathbb{S}^n(r_0)$, where $r_0 = L/(2\pi)$, and put $P_S = \sqrt{\Delta_S + c_0}$ with $c_0 = (n-1)^2/(4r_0^2)$. Lemma 8.1 and scaling give the frequencies $r_0^{-1}(k + (n-1)/2)$, $k \geq 0$. Hence

$$e^{iLP_S} = e^{i\pi(n-1)}I = I.$$

Its leading annular trace coefficient is therefore $W_S > 0$, defined by the same integral on T^*S.

Both full returns have fixed-set dimension $2n-1$ and periodic Morse index $n-1$ by Lemma 11.2. Both operators have zero subprincipal symbol on half-densities. Equation (10.7) therefore gives

$$\frac{u_0 W_X}{W_S} \in \mathbb{R}_{>0}.$$

Since $W_X, W_S > 0$ and $|u_0| = 1$, one has $u_0 = 1$. □

Lemma 11.5. *Let $X \to M$ be a finite regular Riemannian covering of closed connected manifolds, and let $c > 0$. If both time-T geodesic maps are the identity, the principal symbols of $e^{iT\sqrt{\Delta_M + c}}$ and $e^{iT\sqrt{\Delta_X + c}}$ agree under the covering map.*

Proof. Let $p_X : X \to M$ denote the covering, Γ its deck group, and $d = |\Gamma|$. The unitary map

$$\mathcal{U} : L^2(M) \longrightarrow L^2(X)^\Gamma, \qquad \mathcal{U} f = d^{-1/2} p_X^* f,$$

intertwines the Laplacians and their spectral functions. Let K_X and K_M be the propagator kernels with respect to Riemannian volume. On an evenly covered coordinate neighbourhood, integration over its lifts gives

$$K_M(T; p_X(x), p_X(y)) = \sum_{\gamma \in \Gamma} K_X(T; x, \gamma y).$$

Both propagators are pseudodifferential, since both time-T canonical maps are the identity. Near $x = y$ in one sheet, the terms with $\gamma \neq I$ are smooth: their arguments lie off the diagonal of $X \times X$. The diagonal singularities, and hence the principal symbols, agree under the local isometry p_X. The unitary identification with half-densities preserves this equality. □

12. Integral cohomology and contact volume

Let X^n, $n \geq 2$, be a closed smooth homotopy sphere with a Besse metric whose prime geodesics all have length L. Identify SX with S^*X and put $E = SX$. The geodesic flow defines a principal circle bundle

$$p : E \longrightarrow \mathcal{G} = E/(\mathbb{R}/L\mathbb{Z}).$$

The canonical contact form λ satisfies $\int_{p^{-1}(z)} \lambda = L$. Write $e \in H^2(\mathcal{G}; \mathbb{Z})$ for the Euler class with the convention

$$p^* \omega = d\lambda / L, \qquad [\omega] = e_{\mathbb{R}}$$

of [68, Section 2, p. 515]. The form ω is symplectic. Give the closed $(2n-2)$-manifold $\mathcal{G}$ its symplectic orientation.

Lemma 12.1 (Yang [72, Lemma 1, pp. 92–93]). *Suppose $n = 2m+1$, $m \geq 1$. The integral cohomology of E is free of rank one in degrees $0, 2m, 2m+1, 4m+1$ and zero in all other degrees. The cohomology of $\mathcal{G}$ is torsion-free, zero in odd degrees, of rank one in each even degree $0 \leq j \leq 4m$ other than $2m$, and of rank two in degree $2m$. Moreover,*

$$0 \longrightarrow \mathbb{Z} e^m \longrightarrow H^{2m}(\mathcal{G}; \mathbb{Z}) \xrightarrow{p^*} H^{2m}(E; \mathbb{Z}) \longrightarrow 0. \tag{12.1}$$

The class e^m is primitive and the middle intersection form is unimodular.

Proof. The cohomology groups and the middle Gysin sequence follow from Yang's lemma: X has the integral cohomology of $\mathbb{S}^{2m+1}$, and every orbit of its geodesic flow has least period L. Since $H^j(E; \mathbb{Z}) = 0$ for $0 < j < 2m$, the circle-bundle Gysin sequence gives

$$H^{2j}(\mathcal{G}; \mathbb{Z}) = \mathbb{Z} e^j \qquad (0 \leq j < m).$$

Thus its middle sequence is (12.1). The quotient is $\mathbb{Z}$, so e^m is primitive. Integral Poincaré duality on the closed oriented $4m$-manifold $\mathcal{G}$ gives unimodularity [31, Theorem 3.30]. □

Lemma 12.2 (Weinstein [68, Section 2, pp. 515–516]). *The positive integer $\mathfrak{a} = \langle e^{n-1}, [\mathcal{G}] \rangle$ satisfies*

$$\mathfrak{a} = 2 \frac{\mathrm{Vol}(X)}{V_n} \left(\frac{2\pi}{L} \right)^n. \tag{12.2}$$

Proof. In Weinstein's notation, X is a C_r-manifold with $r = L/(2\pi)$. With the Euler class and orientation fixed above, his formula is

$$L^n \langle e^{n-1}, [\mathcal{G}] \rangle = (n-1)! \, V_{n-1} \, \mathrm{Vol}(X).$$

The identity $(n-1)! \, V_{n-1} V_n = 2(2\pi)^n$ gives (12.2). □

Proposition 12.3 ([69, p. 282]). *Let X^n, $n \geq 5$, be a closed smooth homotopy sphere with a Besse metric g. If L is its common prime geodesic length, then*

$$\mathrm{Vol}(X) = V_n \left(\frac{L}{2\pi} \right)^n . \tag{12.3}$$

Proof. Smale's theorem [59, Theorem A] makes X homeomorphic to $\mathbb{S}^n$. By [57, Theorem A], its prime geodesics have a common length L. For $\widehat{g} = (\pi/L)^2 g$, their least period is π. The volume formula for C_π-manifolds homeomorphic to a compact rank-one symmetric space [69, p. 282] gives

$$\mathrm{Vol}_{\widehat{g}}(X) = 2^{-n} V_n.$$

Since $\mathrm{Vol}_{\widehat{g}}(X) = (\pi/L)^n \mathrm{Vol}_g(X)$, equation (12.3) follows. □

13. Cyclic actions and evaluation of returning geodesics

The tangent lift of an isometry h is denoted by $\widehat{h}$. If X has a Besse metric of common prime geodesic length 2π, write $E = SX$ and let $\varpi : E \to \mathcal{G} = E/S^1$ be its geodesic circle bundle. Write $\pi : E \to X$ for evaluation and $J(x, v) = (x, -v)$ for reversal.

Lemma 13.1. *Let X be a closed connected Riemannian manifold and let γ be a fixed-point-free isometry. Its minimum displacement b is positive and is attained. A minimizing segment realizing that minimum has matching endpoint velocities under $d\gamma$ and extends by γ to a complete invariant geodesic. If γ has finite order s, that geodesic closes after time sb. If, in addition, every unit-speed geodesic has least period 2π, then $sb \in 2\pi\mathbb{Z}_{>0}$.*

Proof. The fixed-point-free assumption makes the continuous function $x \mapsto d(x, \gamma x)$ strictly positive on compact X. Let $b > 0$ be its minimum and let $c : [0, b] \to X$ be a unit-speed minimizing geodesic attaining it. For $w \in T_{c(0)}X$, choose a smooth field W along c with $W(0) = w$ and $W(b) = d\gamma(w)$. For sufficiently small s, put $c_s(t) = \exp_{c(t)}(sW(t))$. Since γ is an isometry, $c_s(b) = \gamma c_s(0)$; hence $\mathrm{length}(c_s) \geq b$. The first variation formula gives

$$0 = \langle \dot{c}(b), d\gamma(w) \rangle - \langle \dot{c}(0), w \rangle.$$

Since $d\gamma$ is an isometry and w is arbitrary, $\dot{c}(b) = d\gamma \dot{c}(0)$. The curves $\gamma^j c$ have matching endpoints and velocities. Uniqueness for the geodesic differential equation makes their concatenation a smooth complete geodesic. If γ has order s, its unit tangent vector returns after sb. For common prime length 2π, the positive return time sb is an integral multiple of 2π. □

Lemma 13.2. *Let an odd-prime cyclic group C_r act smoothly on a closed manifold W, and put $\Bbbk = \mathbb{F}_r$ and $\mathbb{K} = \Bbbk(v)$, where v is the degree-two polynomial generator of $H^*(BC_r; \Bbbk) = \Bbbk[v] \otimes \Lambda(\epsilon)$, $|\epsilon| = 1$. Restriction to the fixed set induces an isomorphism*

$$H^*_{C_r}(W; \Bbbk) \otimes_{\Bbbk[v]} \mathbb{K} \cong H^*(W^{C_r}; \Bbbk) \otimes \Lambda(\epsilon) \otimes \mathbb{K}. \tag{13.1}$$

If the action on $H^*(W; \Bbbk)$ is trivial, then

$$\sum_j b_j(W^{C_r}; \Bbbk) \leq \sum_j b_j(W; \Bbbk). \tag{13.2}$$

More precisely, for every page E_s with $s \geq 2$ of the Borel spectral sequence,

$$2 \sum_j b_j(W^{C_r}; \Bbbk) \leq \dim_{\mathbb{K}} \big(E_s \otimes_{\Bbbk[v]} \mathbb{K} \big),$$

where the dimension is taken after summing all bidegrees.

Proof. Put $Z = W^{C_r}$ and $D = \dim W$. Apply the localization theorem [37, Chapter III, Theorem III.1, p. 40] with the multiplicative system $S = \{v^j : j \geq 0\}$. A stabilizer is either trivial or C_r. In the first case v restricts to zero; in the second no element of S does. Thus the set W^S in that theorem is Z. After extending scalars from $\Bbbk[v, v^{-1}]$ to $\mathbb{K}$, restriction is an isomorphism. The action on Z is trivial, so

$$H^*_{C_r}(Z;\Bbbk) = H^*(Z;\Bbbk) \otimes \Bbbk[v] \otimes \Lambda(\epsilon),$$

which gives (13.1).

All Borel differentials are $\Bbbk[v]$-linear. Localization is exact, so passing from a localized page to its homology cannot increase its $\mathbb{K}$-dimension. Only vertical degrees $0 \leq j \leq D$ occur; there are consequently no nonzero differentials after page $D+1$. For the rank of the abutment, consider the following finite filtration. If $F^pH^N_{C_r}(W;\Bbbk)$ is the Serre filtration, set

$$\mathcal{F}_jH^*_{C_r}(W;\Bbbk) = \bigoplus_{N\geq 0} F^{N-j}H^N_{C_r}(W;\Bbbk), \qquad -1 \leq j \leq D.$$

Here $F^pH^N = H^N$ for $p \leq 0$ and $F^pH^N = 0$ for $p > N$. Multiplication by v sends F^pH^N into $F^{p+2}H^{N+2}$; hence each $\mathcal{F}_j$ is a $\Bbbk[v]$-submodule. The vertical bound gives $\mathcal{F}_{-1} = 0$, $\mathcal{F}_D = H^*_{C_r}(W;\Bbbk)$, and

$$\mathcal{F}_j/\mathcal{F}_{j-1} \cong \bigoplus_{N\geq 0} E_\infty^{N-j,j}.$$

Exactness of localization and additivity of dimension in these finitely many short exact sequences show that the localized abutment and localized E_∞ have the same dimension. The localization isomorphism identifies this dimension with $2\sum_j b_j(Z;\Bbbk)$, proving the page bound.

If the action on $H^*(W;\Bbbk)$ is trivial, its second page is

$$E_2 = \Bbbk[v] \otimes \Lambda(\epsilon) \otimes H^*(W;\Bbbk),$$

whose localized dimension is $2\sum_j b_j(W;\Bbbk)$. The page bound at $s = 2$ proves (13.2). □

Lemma 13.3. *Let X^{2m+1}, $m \geq 2$, be a closed homotopy sphere with a Besse metric of common prime geodesic length 2π. An isometry of odd order acts trivially on $H^*(\mathcal{G};\mathbb{Z})$, where $\mathcal{G} = SX/S^1$. For an isometry a of odd prime order r,*

$$\sum_j b_j(\mathrm{Fix}(\bar{a});\mathbb{F}_r) \leq 2m+2.$$

Proof. An isometry preserves the canonical contact form and commutes with the geodesic circle, so it fixes the Euler class e. Every nonmiddle cohomology group of $\mathcal{G}$ is either zero or infinite cyclic by Lemma 12.1. Its automorphism has finite odd order and is therefore the identity. In the middle degree, choose an integral basis (e^m, z) by (12.1). The first vector is fixed. The action on the quotient by $\mathbb{Z}e^m$ is an automorphism of $\mathbb{Z}$ of odd order, hence is also the identity. Its matrix is consequently

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}.$$

Finite order gives $b = 0$. The cohomology is torsion-free, so the induced action on mod-r cohomology is trivial as well. Lemma 13.2 and the Betti numbers in Lemma 12.1 give

$$\sum_j b_j(\mathrm{Fix}(\bar{a});\mathbb{F}_r) \leq \sum_j b_j(\mathcal{G};\mathbb{F}_r) = 2m+2.$$

□

Lemma 13.4. *Let I be a symmetric bilinear form on a real vector space with $I(Y,Y) \geq 0$ for every Y. If $I(Y,Y) = 0$, then $I(Y,Z) = 0$ for every Z.*

Let $\gamma : [0,T] \to N$ be a unit-speed geodesic whose Dirichlet index form is nonnegative. For $0 < t_0 < T$, a Jacobi field along $\gamma|_{[0,t_0]}$ that vanishes at both endpoints is zero. In particular, this holds when γ is minimizing.

Proof. The first assertion follows from Cauchy–Schwarz for a positive semidefinite form: $|I(Y,Z)|^2 \le I(Y,Y)I(Z,Z)$. The nonnegative Dirichlet index form has index zero. The Morse index theorem [50, Theorem 15.1, p. 83] therefore excludes every conjugate instant in $(0,T)$. For a minimizing geodesic the Dirichlet index form is nonnegative by [50, Lemma 13.6, p. 76]. □

Lemma 13.5. *Let X^n, $n = 2m+1$ and $m \ge 2$, be a closed smooth homotopy sphere with a Besse metric of common prime geodesic length 2π and volume V_n. Let $x \in H^{2m}(E;\mathbb{Z})$ restrict to a generator on each unit tangent sphere, and let $y = \pi^*[X]^* \in H^{2m+1}(E;\mathbb{Z})$. In the integral Borel spectral sequence of the geodesic circle,*

$$d_2 y = \pm 2ux, \tag{13.3}$$

where $u \in H^2(BS^1;\mathbb{Z})$ is a generator.

Proof. The sphere-bundle Gysin sequence and the projection formula give

$$H^*(E;\mathbb{Z}) = \mathbb{Z}\langle 1, x, y, xy\rangle, \qquad |x| = 2m, \quad |y| = 2m+1, \quad x^2 = y^2 = 0.$$

Indeed, $\pi_! x = \pm 1$, so $\pi_!(xy) = (\pi_! x)[X]^*$ is a generator of $H^n(X;\mathbb{Z})$. The action is free, hence $E_{S^1} \simeq \mathcal{G}$ and the edge homomorphism sends u to e, up to sign. The second page is $\mathbb{Z}[u] \otimes H^*(E;\mathbb{Z})$. Write

$$d_2 y = d\, ux, \qquad d \in \mathbb{Z}.$$

At bidegree $(2, 2m)$ this is the only possible incoming differential: every later incoming differential has negative source column. The only possible outgoing differential has target $(2m+3, 0)$, which is zero. Thus

$$E_\infty^{2,2m} = \mathbb{Z}/d\mathbb{Z}.$$

In total degree $N = 2m+2$, the only possibly nonzero terms are $(N, 0)$ and $(2, 2m)$. The edge image is the last filtration term [37, Chapter III, Section 1(C), p. 37]; consequently

$$H^{2m+2}(\mathcal{G};\mathbb{Z})/\mathbb{Z}e^{m+1} \simeq \mathbb{Z}/d\mathbb{Z}.$$

By Lemma 12.1, e^{m-1} generates $H^{2m-2}(\mathcal{G};\mathbb{Z})$ and $H^{2m+2}(\mathcal{G};\mathbb{Z}) \cong \mathbb{Z}$. Integral Poincaré duality gives a generator z of the latter group with $\langle e^{m-1}z, [\mathcal{G}]\rangle = 1$. Lemma 12.2 gives $\langle e^{2m}, [\mathcal{G}]\rangle = 2$, so $e^{m+1} = 2z$. Therefore $\mathbb{Z}/d\mathbb{Z} \simeq \mathbb{Z}/2\mathbb{Z}$, and $d = \pm 2$. Reduction modulo an odd prime preserves the nonzero coefficient. □

Proposition 13.6. *Let X^n, $n = 2m+1$ and $m \ge 2$, be a closed smooth homotopy sphere with a Besse metric of common prime geodesic length 2π and volume V_n. Let $\langle h\rangle$ act freely by isometries, with odd order $q \ge 5$, and put $\ell = 2\pi/q$. If*

$$F \subset \mathrm{Fix}(\widehat{h}^{-1} G_\ell)$$

is a connected component of dimension $2m+1$, then F is a mod-p cohomology sphere for every prime $p \mid q$. The evaluation map $\pi_F : F \to X$ has degree nonzero modulo each such p and is surjective.

Proof. The map $\widehat{h}^{-1} G_\ell$ has order dividing q and preserves the canonical contact form. Its homogeneous extension to $T^*X \setminus 0$ is symplectic and has finite order. The fixed-cone argument of Section 10.1 therefore makes F a closed contact submanifold. The geodesic circle commutes with this map. Since the circle is connected, it preserves F. Put

$$B_+ = \varpi(F), \qquad B_- = \varpi(J(F)).$$

These are connected compact symplectic submanifolds of $\mathcal{G}$, each of dimension $2m$, and $F = \varpi^{-1}(B_+)$, $J(F) = \varpi^{-1}(B_-)$. If $B_+ \cap B_- \ne \emptyset$, a unit tangent vector on a common orbit would satisfy both $\widehat{h}v = G_\ell v$ and $\widehat{h}v = G_{-\ell} v$. It would return after $2\ell < 2\pi$, a contradiction.

Fix a prime $p \mid q$ and set $h_p = h^{q/p}$. On a connected component of $\mathrm{Fix}(\bar{h}_p)$, the time shift of h_p is continuous with values in $2\pi\mathbb{Z}/p$ modulo 2π, and is therefore constant. Its values on B_+ and B_- are

$2\pi/p$ and $-2\pi/p$. Thus these two submanifolds lie in distinct fixed components C_+ and C_-. Each $C_\pm$ is symplectic, of dimension $2d_\pm \geq 2m$ [19, Example 3.3, p. 18]. The powers of its symplectic class give at least $d_\pm + 1$ nonzero rational Betti numbers. The universal coefficient theorem and Lemma 13.3 give

$$2m + 2 \geq \sum_j b_j(\mathrm{Fix}(\bar{h}_p); \mathbb{F}_p) \geq \sum_j b_j(C_+; \mathbb{Q}) + \sum_j b_j(C_-; \mathbb{Q}) \geq 2m + 2.$$

Equality forces $d_+ = d_- = m$ and excludes every other fixed component. Each $B_\pm$ is then an embedded submanifold of the same dimension as $C_\pm$, and is both open and closed in it. Hence $C_\pm = B_\pm$. It follows that

$$\mathrm{Fix}(\widehat{h}_p^{-1} G_{2\pi/p}) = F \ :$$

above B_+ the shift is $2\pi/p$ and the full inverse image is F; above B_- the time difference is $4\pi/p$, which is not a return for odd p.

Let $\Bbbk = \mathbb{F}_p$. In the Borel spectral sequence for the action of $S^1 \times \langle h_p \rangle$ on E, write u, v for the two degree-two base generators. Both factors act trivially on $H^*(E; \Bbbk)$: the circle is connected and an odd-order action on each cyclic integral cohomology group is trivial. Hence $d_2 y = (au + bv)x$. Lemma 13.5 gives $a = \pm 2$. The evaluation $E \to X$ is equivariant for the deck factor and pulls back the top class to y. Its top class has zero d_2 because $H^{2m}(X; \Bbbk) = 0$. Naturality gives $b = 0$. For the diagonal subgroup generated by $(e^{2\pi i/p}, h_p^{-1})$ this becomes

$$d_2 y = \pm 2vx, \qquad d_2 x = 0.$$

After tensoring with $\Bbbk(v)$, multiplication by $2v$ is invertible. The differential sends y to $\pm 2vx$ and ϵy to $\mp 2v\epsilon x$. Since $x^2 = 0$, its homology is spanned by $1, xy, \epsilon, \epsilon xy$. The localized third page has dimension four. The page bound in Lemma 13.2 therefore gives

$$2 \sum_j b_j(F; \Bbbk) \leq 4.$$

The connected closed contact manifold F is oriented and has nonzero cohomology in degrees zero and $2m + 1$. All intermediate cohomology vanishes.

On F one has $\widehat{h}_p = G_{2\pi/p}$, so F is $\langle h_p \rangle$-invariant. This action is free on F and X, and evaluation is equivariant. For $Z = F, X$, the odd-order action preserves the integral orientation class. Its Borel fibration is therefore $\Bbbk$-orientable [41, Proposition 1.3]. Let $\eta_Z \in H^{n+1}(BC_p; \Bbbk)$ be its Euler class, and let D be the degree of evaluation modulo p. The equivariant degree formula [41, Theorem 1.9] gives

$$D\eta_F = \eta_X.$$

By [41, Theorem 2.3], η_X is the transgression of a top class. The Borel spectral sequence of X has only rows zero and n, and $X_{C_p} \simeq X/C_p$ has zero cohomology in degree $n + 1$. Thus this transgression is nonzero, so $\eta_X \neq 0$ and $D \neq 0$. The integral degree is nonzero as well. If evaluation omitted a point of X, its induced map on top cohomology would factor through the zero top cohomology of the punctured manifold. Its degree would then vanish. Hence evaluation is onto. □

14. Contact volume and Clifford translations

For a symplectic vector space $(\mathbb{R}^{2m}, \omega)$, write

$$\mathfrak{sp}^+(\omega) = \{C \ : \ C^t\omega + \omega C = 0, \quad \omega C \text{ symmetric positive definite}\}.$$

A smooth path U satisfying $U' = CU$ with $C(t) \in \mathfrak{sp}^+(\omega)$ is positive. Its Lorentz–Finsler length is

$$\mathsf{L}(U) = \int (\det C(t))^{1/(2m)} \, dt \tag{14.1}$$

[2, Proposition B.1]. The positively elliptic region consists of symplectic maps conjugate to direct sums of compatible planar rotations with angles in $(0, \pi)$. It is contractible [2, Proposition ii.2]. Its

distinguished lift to the universal covering group is represented by the corresponding rotation paths starting at the identity. If a positive path from the identity ends in this lift, with rotation angles $\theta_1, \dots, \theta_m$, then

$$\mathsf{L}(U) \le \frac{1}{m} \sum_{j=1}^{m} \theta_j \tag{14.2}$$

by [2, Theorem J.1(i)]. This bound applies to every positive path in the same fixed-endpoint homotopy class. Hedicke [33, Corollary 1.2, arXiv version] proves the sharper bound $\mathsf{L}(U) \le (\theta_1 \cdots \theta_m)^{1/m}$ in this lift.

Normalize the symplectic path index so that a positive planar rotation through an angle in $(0, 2\pi)$ has Conley–Zehnder index one. For a positive path $U : [0, T] \to \mathrm{Sp}(\omega)$ with $U' = CU$, the crossing form on $\ker(U(t) - I)$ is

$$v \longmapsto \omega(v, C(t)v) > 0 \qquad (v \neq 0).$$

Every crossing is regular and isolated, including one-sided isolation at an endpoint. Since the crossing set is closed, it is finite on a compact interval. A closed positive path has Robbin–Salamon index

$$\mu_{\mathrm{RS}}(U) = \frac{1}{2} \dim\ker(U(0) - I) + \sum_{0<t<T} \dim\ker(U(t) - I) + \frac{1}{2} \dim\ker(U(T) - I). \tag{14.3}$$

For a path from the identity with nondegenerate final endpoint, μ_{RS} agrees with the Conley–Zehnder index. Inserting a based loop of winding $b \in \pi_1(\mathrm{Sp}) \cong \mathbb{Z}$ changes that index by $2b$. These formulas and the direct-sum property are [29, Propositions 2.6, 4.6 and 4.7].

Proposition 14.1 (A contact-volume inequality). *Let (X^{2m+1}, g) be closed, with $m \ge 1$. Suppose a smooth unit field V generates a free circle action Φ_t of period 2π by geodesics, and suppose $\alpha = g(V, \cdot)$ is contact. Give X the contact orientation. Let $q \ge 5$ be odd, put $\ell = 2\pi/q$, and write $\gamma_x(t) = \Phi_t(x)$. Assume:*

(i) Φ_ℓ is an isometry;
(ii) the full geodesic flow satisfies $G_{2\pi} = I$;
(iii) the endpoint-fixed Morse index satisfies $\mathrm{ind}_D(\gamma_x|_{[0,2\pi]}) \le 2m$ for every $x \in X$;
(iv) for every $x \in X$, no point $\gamma_x(t)$ with $0 < t \le 2\ell$ is conjugate to x along γ_x.

Then

$$\mathrm{Vol}_g(X) \ge \int_X \frac{\alpha \wedge (d\alpha)^m}{2^m m!}. \tag{14.4}$$

If equality holds, $\mathcal{L}_V g = 0$.

Proof. Since $|V| = 1$ and $\nabla_V V = 0$, one has

$$\alpha(V) = 1, \qquad d\alpha(V, Y) = \langle \nabla_V V, Y\rangle - \langle \nabla_Y V, V\rangle = 0.$$

Cartan's formula gives $\mathcal{L}_V \alpha = 0$, hence $\Phi_t^* \alpha = \alpha$.

Fix $x \in X$ and a basis $E_1(0), \dots, E_{2m}(0)$ of $\ker \alpha_x$. Along γ_x, set

$$E_i(t) = d\Phi_t E_i(0), \quad H_{ij}(t) = g(E_i(t), E_j(t)), \quad \Omega_{ij} = d\alpha(E_i(t), E_j(t)), \quad D_t E = EB.$$

The fields E_i are Jacobi fields obtained by varying the geodesic orbits. They form a normal frame, since $d\Phi_t$ is invertible and $\Phi_t^* \alpha = \alpha$. Thus H is positive definite and Ω is a constant invertible skew matrix. Torsion-freeness gives $D_t E_i = \nabla_{E_i} V$, whence

$$H' = B^t H + HB, \qquad \Omega = B^t H - HB, \qquad 2HB = H' - \Omega. \tag{14.5}$$

For a normal Jacobi field $J = Ey$, the Jacobi equations for the columns of E give $y'' + 2By' = 0$. Therefore

$$(Hy')' = \Omega y', \qquad Hy' - \Omega y = c$$

for a constant vector c. Setting $u = y + \Omega^{-1}c$ gives

$$u' = C(t)u, \qquad C = H^{-1}\Omega. \tag{14.6}$$

Let $U' = CU$, $U(0) = I$. If $y(0) = 0$, then $c = H(0)y'(0)$ and

$$y(t) = (U(t) - I)\Omega^{-1}H(0)y'(0). \tag{14.7}$$

A tangential Jacobi field vanishing at both endpoints is zero. Since $\Omega^{-1}H(0)$ is invertible, the multiplicity of a conjugate time $t > 0$ is therefore $\dim\ker(U(t) - I)$.

Put $\omega = -\Omega$. Then

$$C^t\Omega + \Omega C = 0, \qquad \omega C = -\Omega H^{-1}\Omega > 0, \tag{14.8}$$

since $-v^t\Omega H^{-1}\Omega v = (\Omega v)^t H^{-1}(\Omega v)$. Thus $U(t) \in \mathrm{Sp}(\omega)$ and U is positive. The flow law and hypothesis (i) give

$$E_i(t + \ell) = d\Phi_\ell E_i(t), \qquad H(t + \ell) = H(t), \qquad C(t + \ell) = C(t).$$

Uniqueness for the matrix differential equation gives, with $S = U(\ell)$,

$$U(t + \ell) = U(t)S. \tag{14.9}$$

By hypothesis (ii), every Jacobi field obtained by varying the initial unit velocity at x vanishes at time 2π. Equation (14.7) therefore gives

$$U(2\pi) = I, \qquad S^q = I. \tag{14.10}$$

Average an inner product over $\langle S\rangle$ to obtain an S-invariant inner product h. The polar construction of [9, Proposition 12.3 and Remark 1, pp. 68–69, author's version] gives an ω-compatible complex structure commuting with S, since S preserves both h and ω. The unitary spectral theorem then gives a symplectically orthogonal decomposition into planes on which

$$S = \exp\big((2\pi r_j/q)J_j\big), \qquad 1 \le r_j < q, \quad 1 \le j \le m,$$

with J_j compatible. Here $r_j = 0$ is excluded by (iv) at $t = \ell$. Write $R(t) = \bigoplus_j \exp(t r_j J_j)$.

By (iv) and (14.7), the path $U|_{[0,\ell]}$ has no crossing after zero, so its Conley–Zehnder index is m. The same is true of $R|_{[0,\ell]}$, whose rotation angles lie in $(0, 2\pi)$. The based loop $t \mapsto U(t)R(t)^{-1}$ has winding zero by the loop formula. It is null-homotopic, so the two paths have the same lift.

Equation (14.9) identifies the lift of $U|_{[0,2\ell]}$ with the square of the lift of $U|_{[0,\ell]}$, represented by $R|_{[0,2\ell]}$. Since q is odd, its endpoint is nondegenerate. Its Conley–Zehnder index is

$$m + 2\sum_{j=1}^{m}\left\lfloor \frac{2r_j}{q}\right\rfloor.$$

By (iv), this index is m. Every floor is nonnegative, hence zero. Thus

$$0 < \theta_j := 2\pi r_j/q < \pi. \tag{14.11}$$

The path $R|_{[0,\ell]}$ lies in the positively elliptic region for every positive time. It therefore represents the distinguished lift required in (14.2).

Likewise, $U|_{[0,2\pi]}$ has the same lift as $R|_{[0,2\pi]}$, whose Robbin–Salamon index is $2\sum_j r_j$. The actual path has endpoint crossing spaces of dimension $2m$. By (14.7) and the endpoint-fixed Morse index theorem [50, Theorem 15.1, p. 83], its interior crossing dimensions sum to $\mathrm{ind}_D(\gamma_x|_{[0,2\pi]})$. Terminal conjugacy does not contribute to this negative index. Consequently, (iii) gives

$$2\sum_j r_j = 2m + \mathrm{ind}_D(\gamma_x|_{[0,2\pi]}) \le 4m. \tag{14.12}$$

Applying (14.2) in the distinguished lift yields

$$\int_0^\ell (\det C(t))^{1/(2m)}\,dt \le \frac{1}{m}\sum_{j=1}^{m}\theta_j \le 2\ell. \tag{14.13}$$

Write

$$d\mu_\alpha = \frac{\alpha \wedge (d\alpha)^m}{2^m m!}, \qquad dv_g = f\, d\mu_\alpha.$$

In the frame $(V, E_1, \dots, E_{2m})$, the Riemannian density is $\sqrt{\det H}$ and the contact density is $|\operatorname{Pf} \Omega|/2^m$. Hence

$$f = \frac{2^m \sqrt{\det H}}{|\operatorname{Pf} \Omega|}, \qquad (\det C)^{1/(2m)} = \left(\frac{\det \Omega}{\det H}\right)^{1/(2m)} = 2f^{-1/m}. \tag{14.14}$$

For every x, equation (14.13) and Jensen's inequality give

$$\left(\frac{1}{\ell}\int_0^\ell f(\Phi_t x)\, dt\right)^{-1/m} \le \frac{1}{\ell}\int_0^\ell f(\Phi_t x)^{-1/m}\, dt \le 1. \tag{14.15}$$

Thus the short average of f is at least one. Since μ_α is Φ_t-invariant, Fubini's theorem gives

$$\int_X f\, d\mu_\alpha = \int_X \frac{1}{\ell}\int_0^\ell f(\Phi_t x)\, dt\, d\mu_\alpha(x) \ge \int_X d\mu_\alpha.$$

This proves (14.4).

Suppose equality holds. The short average of f minus one is a continuous nonnegative function with zero integral against the positive smooth density μ_α. It vanishes everywhere. Both inequalities in (14.15) are therefore equalities. Strict convexity of $s^{-1/m}$ makes $t \mapsto f(\Phi_t x)$ constant on $[0, \ell]$, with value one. Hence

$$f \equiv 1, \qquad G(C) := (\det C)^{1/(2m)} \equiv 2. \tag{14.16}$$

The path $U|_{[0,\ell]}$ has length 2ℓ, so both bounds in (14.13) are equalities. By (14.2), it maximizes length among positive paths with the same endpoint lift. Small smooth variations with fixed endpoints preserve positivity and this lift. Thus U is stationary for the length functional. By [2, Proposition i.3, arXiv version], it is a reparametrized one-parameter subgroup. Since $G(C) \equiv 2$, the parameter has constant derivative, and C is constant. The identity $H = \Omega C^{-1}$ then gives $H' = 0$ on $[0, \ell]$, and periodicity gives $H' = 0$ for every t.

It follows that Φ_t preserves g on $\ker \alpha$. It also preserves V and α, with $|V| = 1$ and $\ker \alpha = V^\perp$. Hence $\Phi_t^* g = g$, and $\mathcal{L}_V g = 0$. □

14.1. **Clifford translations on Besse homotopy spheres.**

Theorem 14.2 (Clifford translations on Besse homotopy spheres). *Let (X, g) be a smooth homotopy sphere of odd dimension $n = 2m + 1 \ge 5$. Suppose that every unit-speed geodesic has prime period 2π. If h_0 is a nonidentity isometry of constant displacement and of finite odd order $q \ge 5$, then (X, g) is isometric to the unit round sphere. There is an isometry $X \to \mathbb{S}^{2m+1} \subset \mathbb{C}^{m+1}$ under which*

$$h_0(z) = e^{2\pi i k/q} z \qquad \text{for some } k \in \mathbb{Z} \text{ with } (k, q) = 1.$$

Proof. Write $E = SX$ and let G_t be its geodesic flow. The constant displacement c_0 of the nonidentity isometry h_0 is positive: if it were zero, h_0 would fix every point. Thus h_0 is fixed-point-free. Every minimizing segment from x to $h_0 x$ realizes the global minimum displacement. Lemma 13.1 shows that it extends to an oriented geodesic on which h_0 is the time-c_0 shift. Such a geodesic passes through every $x \in X$. Since $h_0^q = I$, its unit tangent vector returns after qc_0, whence

$$qc_0 = 2\pi k \qquad (k \in \mathbb{Z}_{>0}).$$

If $d = (k, q) > 1$, then on every one of these geodesics the isometry $h_0^{q/d}$ acts by time $2\pi k/d$ and fixes every point of X. This contradicts the exact order of h_0. Thus $(k, q) = 1$. Choose $a \in \mathbb{Z}$ with $ak \equiv 1 \pmod q$, and put

$$h = h_0^a, \qquad \ell = \frac{2\pi}{q}.$$

The isometries h and h_0 generate the same cyclic group and have the same fixed-point set. Hence h has no fixed point. On each h_0-invariant geodesic constructed above, h acts by time ℓ modulo 2π, so $d(x, hx) \le \ell$ for every x. If $b > 0$ is the minimum displacement of h, Lemma 13.1 and $h^q = I$ give $qb \in 2\pi\mathbb{Z}_{>0}$. Therefore

$$d(x, hx) = \ell \qquad (x \in X). \tag{14.17}$$

Let $p : E \to \mathcal{G} = E/S^1$ be the free geodesic circle bundle, and let $J(x, v) = (x, -v)$. Define

$$\mathcal{F} = \mathrm{Fix}(\widehat{h}^{-1} G_\ell).$$

The map $\widehat{h}^{-1} G_\ell$ has finite order dividing q, and its homogeneous extension to $T^*X \setminus 0$ is symplectic. The fixed-cone argument of Section 10.1 makes $\mathcal{F}$ a finite union of closed smooth contact submanifolds. The connected geodesic circle preserves each component. Equation (14.17) and Lemma 13.1 imply that evaluation $\mathcal{F} \to X$ is surjective. If all its components had dimension less than n, Sard's theorem would make the image of each a set of measure zero in X. Their finite union cannot be all of X. Thus some component $F_0 \subset \mathcal{F}$ has $\dim F_0 \ge n$. Set $B_0 = p(F_0)$. Since F_0 is circle-invariant, $F_0 = p^{-1}(B_0)$ and $B_0 = F_0/S^1$ is embedded in $\mathcal{G}$.

Every isometry b induces a map $\bar{b}$ on $\mathcal{G}$. Write J also for the induced reversal on $\mathcal{G}$. Fix an odd prime $r \mid q$, and put $h_r = h^{q/r}$. On the oriented geodesics in B_0 and $J(B_0)$, the time shifts of h_r are respectively $2\pi/r$ and $-2\pi/r$. On a connected component of $\mathrm{Fix}(\bar{h}_r : \mathcal{G} \to \mathcal{G})$, the shift is a continuous function with values in $\{2\pi j/r : 0 \le j < r\}$; it is therefore constant. Thus B_0 and $J(B_0)$ lie in distinct fixed components C_+ and C_-. These components are symplectic. Write $\dim C_\pm = 2d_\pm \ge 2m$. The powers of their symplectic classes give total rational Betti number at least $d_\pm + 1$. Lemma 13.3 and the universal coefficient theorem give

$$2m + 2 = \sum_j b_j(\mathcal{G}; \mathbb{F}_r) \ge \sum_j b_j(\mathrm{Fix}(\bar{h}_r); \mathbb{F}_r) \ge (d_+ + 1) + (d_- + 1) \ge 2m + 2.$$

All inequalities are equalities. Hence $\dim F_0 = n$, the only fixed components are $C_\pm$, and they have dimension $2m$. The compact embedded submanifolds $B_0, J(B_0)$ have that dimension, so they are open and closed in the respective components. Consequently

$$\mathrm{Fix}(\bar{h}_r) = B_0 \sqcup J(B_0).$$

The prime divisor r of q was arbitrary.

Every point of $\mathcal{F}$ projects into this union. Above B_0, $\widehat{h} = G_\ell$, whereas above $J(B_0)$, $\widehat{h} = G_{-\ell}$. The second possibility is incompatible with $\widehat{h} = G_\ell$, since $G_{2\ell}$ has no fixed unit tangent vector. Therefore $\mathcal{F} = F_0$. Write

$$F = \mathcal{F}, \qquad B_+ = F/S^1, \qquad B_- = J(F)/S^1.$$

In particular, F is connected, has dimension n, and evaluates onto X. Every nonidentity $b \in \langle h \rangle$ generates a subgroup containing a subgroup of prime order. Since both B_+ and B_- are fixed by all of $\langle h \rangle$, it follows that

$$\mathrm{Fix}(\bar{b}) = B_+ \sqcup B_- \qquad (b \in \langle h \rangle \setminus \{I\}). \tag{14.18}$$

The action of $\langle h \rangle$ on X is free. Indeed, if a nontrivial power fixed x, a prime-order subgroup $\langle h_r \rangle$ would also fix x. The orthogonal transformation $d(h_r)_x$ has odd order on the odd-dimensional real vector space T_xX and hence has eigenvalue 1. A corresponding unit vector defines an oriented geodesic fixed pointwise by h_r. Its shift is zero, contrary to (14.18), where the shifts are $\pm 2\pi/r$.

For $1 \le j < q$, surjectivity of $F \to X$ gives

$$d(x, h^j x) \le \ell \min\{j, q - j\}.$$

Put $b_j = \min_x d(x, h^j x) > 0$. Let $v \in SX$ be the initial unit vector of a segment attaining this minimum. Lemma 13.1 gives equality of unit tangent vectors

$$G_{b_j} v = \widehat{h}^j v.$$

Equation (14.18) places $p(v)$ in B_+ or B_-. On these two sets, respectively, $\widehat{h}^j v = G_{j\ell} v$ or $G_{-j\ell} v$. The prime period is 2π; hence

$$b_j \equiv j\ell \pmod{2\pi} \quad \text{or} \quad b_j \equiv -j\ell \pmod{2\pi}.$$

Since $b_j > 0$, either congruence implies $b_j \geq \ell \min\{j, q-j\}$. Together with the upper bound this gives

$$d(x, h^j x) = \ell \min\{j, q-j\} \qquad (x \in X, 1 \leq j < q). \tag{14.19}$$

Evaluation $\pi_F : F \to X$ is a diffeomorphism. Let $v, w \in F$ have the same base point x. Their first length-ℓ segments c_v, c_w satisfy

$$\operatorname{length}\big(c_v * (hc_w)\big) = 2\ell = d(x, h^2 x),$$

where the last equality requires $q \geq 5$. Varying the junction with velocity $z \in T_{hx}X$, while fixing the two outer endpoints, gives

$$0 = \langle dh(v) - dh(w), z \rangle.$$

Hence $dh(v) = dh(w)$ and $v = w$. If $\xi \in T_v F$ is vertical, differentiation of $G_\ell v = \widehat{h} v$ gives a normal Jacobi field with

$$Y(0) = Y(\ell) = 0.$$

The segment of length 2ℓ minimizes distance, and $0 < \ell < 2\ell$. Lemma 13.4 gives $Y = 0$, so its initial data, and therefore ξ, vanish. Thus $d\pi_F : T_v F \to T_x X$ is injective. Its domain and codomain both have dimension n, so it is an isomorphism. The inverse function theorem and bijectivity give a smooth global inverse.

The inverse graph defines a smooth unit field V on X. The invariance of F under geodesic flow gives

$$\nabla_V V = 0, \qquad \Phi_{2\pi} = I, \qquad \Phi_\ell = h.$$

The action Φ_t is free: a return of a base point on the graph is a return of its unit tangent vector. Let λ be the canonical contact form on SX. Then

$$\alpha = g(V, \cdot) = (\pi_F^{-1})^*(\lambda|_F)$$

is contact, and V is its Reeb field.

Set $T = 2\ell$ and let $\gamma(t) = \Phi_t(x)$ for $0 \leq t \leq T$. By (14.19), γ minimizes distance. Lemma 13.4 excludes conjugate times in $(0, T)$. For the endpoint, consider

$$\mathscr{H}_h = \{Z \in H^1(\gamma^* TX) : Z(T) = dh^2 Z(0)\}.$$

Every curve $c : [0, T] \to X$ with $c(T) = h^2 c(0)$ satisfies

$$E(c) := \frac{1}{2} \int_0^T |\dot{c}|^2 dt \geq \frac{\operatorname{length}(c)^2}{2T} \geq \frac{T}{2} = E(\gamma).$$

The graph of h^2 is totally geodesic: for every geodesic a in X, the curve $(a, h^2 a)$ is a product geodesic. The second variation therefore has no boundary curvature term and is

$$I(Y, Z) = \int_0^T \Big(\langle D_t Y, D_t Z \rangle - \langle R(Y, \dot{\gamma})\dot{\gamma}, Z \rangle \Big) dt, \qquad I(Z, Z) \geq 0 \quad (Z \in \mathscr{H}_h).$$

If a Jacobi field Y satisfies $Y(0) = Y(T) = 0$, integration by parts gives $I(Y, Y) = 0$. Lemma 13.4 then gives $I(Y, Z) = 0$ for every $Z \in \mathscr{H}_h$. A second integration by parts gives

$$0 = \langle dh^{-2} D_t Y(T) - D_t Y(0), Z(0) \rangle.$$

Every vector of $T_x X$ occurs as $Z(0)$ for some $Z \in \mathscr{H}_h$. Consequently

$$D_t Y(T) = dh^2 D_t Y(0).$$

The tangential part of Y is $f\dot\gamma$ with $f''=0$ and $f(0)=f(T)=0$; hence it vanishes. The normal initial data of Y define a vertical vector $\xi\in T_vSX$ fixed by $d(\widehat{h}^{-2}G_T)_v$. Put $A=\widehat{h}^{-2}G_T$. The commuting relations and $qT=4\pi$ give $A^q=I$. If $A(v)=v$, then $p(v)$ is fixed by $\overline{h^2}$, so (14.18) gives $p(v)\in B_+\sqcup B_-$. Above B_+ one has $A=I$; above B_- one has $A=G_{4\ell}$. The latter has no fixed tangent state, since $q\nmid 4$. Thus $\mathrm{Fix}(A)=F$. Section 10.1 gives

$$\xi\in\ker(dA_v-I)=T_vF.$$

The vertical kernel of $d\pi_F$ is zero. Hence $\xi=0$, and uniqueness for the Jacobi equation gives $Y=0$.

Put $B=X/S^1$ for the action Φ_t and let $p_B:X\to B$ be the quotient map. Since $i_V d\alpha=0$ and $\Phi_t^*d\alpha=d\alpha$, there is a two-form ω_B with $p_B^*\omega_B=d\alpha$. It is symplectic because α is contact. Orient B by ω_B^m and choose the Euler class $e_B\in H^2(B;\mathbb{Z})$ so that $(e_B)_{\mathbb{R}}=[\omega_B/(2\pi)]$. The homotopy-sphere cohomology gives $H^j(X;\mathbb{Z})=0$ for $0<j<2m+1$. The integral Gysin sequence [31, Section 4.D, p. 438] gives

$$H^1(B;\mathbb{Z})=0,\qquad \smile e_B:H^{j-2}(B;\mathbb{Z})\xrightarrow{\cong}H^j(B;\mathbb{Z})\quad(2\le j\le 2m).$$

Since $H^0(B;\mathbb{Z})=\mathbb{Z}$ and $\dim B=2m$, it follows that

$$H^*(B;\mathbb{Z})=\mathbb{Z}[e_B]/(e_B^{m+1}).$$

In particular e_B^m generates $H^{2m}(B;\mathbb{Z})$, so its pairing with $[B]$ is 1 or -1. Its representative $(\omega_B/(2\pi))^m$ is positive, so $\langle e_B^m,[B]\rangle=1$. Finally $\int_{p_B^{-1}(b)}\alpha=2\pi$, and integration over the fibers gives

$$\int_X\frac{\alpha\wedge(d\alpha)^m}{2^m m!}=\frac{(2\pi)^{m+1}}{2^m m!}=V_n. \tag{14.20}$$

Lemma 11.2 bounds the endpoint-fixed index on $[0,2\pi]$ by $2m$. Moreover, $\Phi_\ell=h$ is an isometry, $G_{2\pi}=I$, and no conjugate point occurs in $(0,2\ell]$. Proposition 14.1 and (14.20) give $\mathrm{Vol}_g(X)\ge V_n$. Proposition 12.3 gives equality. Hence every Φ_t is an isometry.

Its time-π map is a fixed-point-free isometric involution. The quotient is Besse and has a degree-two smooth covering by the homotopy sphere X. It has constant sectional curvature by Lin–Schmidt [49, Theorem 2]; so does X. Since X is compact and simply connected, it is a round sphere. Its prime period fixes its radius as one. On the unit sphere write $\Phi_t=e^{tA}$, where A is skew-adjoint. The equality $|V(x)|=|Ax|=1$ for every unit x gives $A^tA=I$, hence $A^2=-I$. Thus A is an orthogonal complex structure, and $h=\Phi_\ell$ is scalar rotation by $2\pi/q$. Finally $h_0=h^k$, proving the stated formula. □

Corollary 14.3. *Let X be a smooth homotopy sphere of odd dimension $n\ge 5$, with a Besse metric. If it admits a constant-displacement isometry of finite odd order at least five, then its sectional curvature is constant.*

Proof. By [59, Theorem A], X is homeomorphic to $\mathbb{S}^n$. Radeschi–Wilking [57, Theorem A] therefore give a common prime geodesic length L. The metric $(2\pi/L)^2g$ satisfies Theorem 14.2. □

Proposition 14.4. *Let X be a smooth homotopy sphere of dimension at least five with a Besse metric. If it admits a constant-displacement isometry of infinite order, then its sectional curvature is constant.*

Proof. Put $n=\dim X$. By [59, Theorem A], X is homeomorphic to $\mathbb{S}^n$. Radeschi–Wilking [57, Theorem A] give one common prime geodesic length L. Replace g by $(2\pi/L)^2g$. Let h have constant displacement $c>0$. For each x, Lemma 13.1 extends a unit-speed minimizing segment from x to hx to a complete geodesic γ_x satisfying $h\gamma_x(t)=\gamma_x(t+c)$, and hence $h^jx=\gamma_x(jc)$ for every $j\in\mathbb{Z}$. If $c/(2\pi)$ were rational, a positive power of h would fix every point. Hence $c/(2\pi)$ is irrational.

Choose $j\in\mathbb{Z}$ and $0<\delta<\min\{\mathrm{inj}(X),2\pi\}$ such that $jc\equiv\delta\pmod{2\pi}$. The field

$$V(x)=\delta^{-1}\exp_x^{-1}(h^jx)$$

is smooth and has unit length, since $h^jx=\gamma_x(\delta)$. For every t,

$$h^j\gamma_x(t)=\gamma_x(t+\delta),\qquad V(\gamma_x(t))=\delta^{-1}\exp_{\gamma_x(t)}^{-1}(\gamma_x(t+\delta))=\dot\gamma_x(t).$$

Thus its smooth flow satisfies

$$\Phi_t(x) = \gamma_x(t), \qquad \Phi_{2\pi} = I, \qquad \Phi_{kc} = h^k \quad (k \in \mathbb{Z}).$$

The times kc mod 2π are dense. Smooth dependence of Φ_t on t therefore gives $\Phi_t^* g = g$ for every t. A return of a base point also returns the generator, so the prime geodesic period makes this circle action free. Since V has no zero, $\chi(X) = 0$ and n is odd. The time-π map is a fixed-point-free isometric involution. The Besse quotient $X/\langle\Phi_\pi\rangle$ has a degree-two smooth covering by X. Lin–Schmidt [49, Theorem 2] gives constant sectional curvature on the quotient and hence on X. □

Remark 14.5. Let $m \geq 1$, let $q \geq 5$ be odd, and put $\zeta = e^{2\pi i/q}$. On the unit sphere in $\mathbb{C}^{m+1}$, the isometry

$$h = \operatorname{diag}(\zeta, \dots, \zeta, \zeta^2)$$

generates a free cyclic action: if h^j has eigenvalue one, then $q \mid j$ or $q \mid 2j$, and either condition gives $q \mid j$. For unit coordinate vectors e_1, e_{m+1},

$$d(e_1, he_1) = \frac{2\pi}{q}, \qquad d(e_{m+1}, he_{m+1}) = \frac{4\pi}{q}.$$

Thus the displacement of h is not constant.

15. Reciprocal times and orders of deck transformations

Write SY for the unit tangent bundle of a Riemannian manifold Y and G_t for its geodesic flow. The tangent lift of an isometry h is denoted by $\widehat{h}$.

Lemma 15.1. *Let X be a closed connected Riemannian manifold of positive dimension, and suppose every unit-speed geodesic has the same prime period L. Let a finite group Γ act freely by isometries and put $M = X/\Gamma$. For every integer $s \geq 2$,*

$$\operatorname{Fix}(G_{L/s} : SM \to SM) \neq \emptyset \quad \Longleftrightarrow \quad \Gamma \text{ contains an element of order } s.$$

Every constant-speed closed geodesic of length L/s on M minimizes energy in its free homotopy class, with the parameter interval fixed. In particular, its periodic Morse index is zero. If $\mathcal{O}(\Gamma) = \{\operatorname{ord}(h) : h \in \Gamma\}$ and $\mathscr{L}(M)$ is the set of lengths of nonconstant closed geodesics, including their iterates, then

$$\mathscr{L}(M) = \bigcup_{s \in \mathcal{O}(\Gamma)} \frac{L}{s}\mathbb{Z}_{>0}. \tag{15.1}$$

Proof. Let $h \in \Gamma$ have order $s \geq 2$ and put $b = \min_x d(x, hx) > 0$. Lemma 13.1 gives a complete unit-speed geodesic γ with $h\gamma(t) = \gamma(t + b)$. Since $h^s = I$, its unit tangent vector returns after sb, so $sb = kL$ for some $k \in \mathbb{Z}_{>0}$. If $d = \gcd(k, s) > 1$, the nonidentity element $h^{s/d}$ shifts γ by $(k/d)L$ and fixes a point of X, contrary to freeness. Thus k is a unit modulo s. Choose $a \in \mathbb{Z}$ with $ak \equiv 1 \pmod{s}$. Then h^a acts on γ by time L/s modulo L, so its projection to M returns after L/s.

Conversely, lift a returning tangent state to $v \in SX$ and write $G_{L/s}v = \widehat{h}v$ with $h \in \Gamma$. Isometries commute with geodesic flow. Hence $\widehat{h}^s v = G_L v = v$, and freeness gives $h^s = I$. If h had order $r < s$, then $G_{rL/s}v = v$, contrary to the prime period L. Thus $\operatorname{ord}(h) = s$.

The displacement argument also gives, for every element h of order $s \geq 2$,

$$s \min_x d(x, hx) \in L\mathbb{Z}_{>0}, \qquad \min_x d(x, hx) \geq L/s. \tag{15.2}$$

Let c be a quotient geodesic of length L/s, with deck monodromy h. Every loop freely homotopic to c has a lift from y to $h'y$, where h' is conjugate to h in Γ. Indeed, lifting a free homotopy preserves the endpoint deck transformation; changing its initial lift conjugates that transformation. Such a loop has length at least $d(y, h'y) \geq L/s$ by (15.2). On a fixed interval $[0, T]$, Cauchy–Schwarz gives

$$E(c') = \frac{1}{2}\int_0^T |\dot{c}'|^2 \, dt \geq \frac{\operatorname{length}(c')^2}{2T} \geq \frac{(L/s)^2}{2T} = E(c).$$

Every periodic variation of c is a free homotopy, so its second variation is nonnegative.

Finally, a return at time $t > 0$ lifts to $G_t v = \widehat{h} v$ for some $h \in \Gamma$. If $s = \operatorname{ord}(h)$, then $G_{st} v = v$, and hence $t \in (L/s)\mathbb{Z}_{>0}$. This proves one inclusion in (15.1). For every element order $s \geq 2$, the first paragraph produces a quotient geodesic of length L/s; its iterates give all positive multiples. A length-L geodesic on X supplies the case $s = 1$. □

Lemma 15.2. *In the scalar setting of Lemma 15.1, let $P = \sqrt{\Delta_M + c}$ be a positive self-adjoint square root. Suppose A is self-adjoint and $A - P \in \Psi_{\mathrm{cl}}^{-1}(M)$ commutes with P. Fix an integer $s \geq 2$ and put $t = L/s$. If $\operatorname{Fix}(G_t : SM \to SM)$ is nonempty and its maximum component dimension is d_s, then, for every nonzero $0 \leq \chi \in C_c^\infty((0,1))$,*

$$\operatorname{Tr}(e^{itA}\chi(A/R)) = C_{s,\chi} R^{(d_s+1)/2} + o(R^{(d_s+1)/2}), \qquad C_{s,\chi} \neq 0.$$

Here $d_s \geq 1$, so this trace is not $o(R)$. If the fixed set is empty, then, for every $\chi \in C_c^\infty((0,1))$, the trace is $O(R^{-N})$ for every N.

Proof. Since $G_L = I$ on SM, the time-t map has finite order. By Section 10.1, its fixed set is a finite union of closed clean contact submanifolds. Each nonempty component has positive odd dimension. By Lemma 15.1, every corresponding periodic Morse index is zero.

The operator $A = P + (A - P)$ is elliptic and bounded below. Its nonpositive spectral subspace is finite-dimensional and consists of smooth eigenvectors. Moving these eigenvalues to positive values gives a positive operator differing from A by a smoothing operator; for all sufficiently large R, its annular trace equals that of A. On scalar half-densities, P has principal symbol $|\xi|_g$ and zero subprincipal symbol [17, p. 58]; the same holds for A. The clean trace formula [17, Theorem 4.5, equations (4.7)–(4.8), and pp. 69–70], in the annular form (10.7), gives

$$C_{s,\chi} = \gamma_{d_s,0} \left(\int_0^1 \chi(u) u^{(d_s-1)/2} \, du \right) \sum_{\substack{C \subset \operatorname{Fix}(G_t) \\ \dim C = d_s}} \int_C dv_C.$$

The factor $\gamma_{d_s,0}$ is nonzero; the moment and the density integrals are positive [63, Section 1.2]. Components of smaller dimension contribute lower powers.

If the fixed set is empty, the wave trace is smooth near t [17, Corollary 1.2]. The annular localization in Section 10.1 then gives rapid decay. □

Theorem 15.3. *Let $M_i = X_i/\Gamma_i$, $i = 1, 2$, satisfy Lemma 15.1 with the same upstairs prime period L, and let A_i satisfy Lemma 15.2. Suppose that, along $R_j \to \infty$, for every fixed time t and every $\chi \in C_c^\infty((0,1))$,*

$$\operatorname{Tr}(e^{itA_1}\chi(A_1/R_j)) - \operatorname{Tr}(e^{itA_2}\chi(A_2/R_j)) = o(R_j).$$

Then Γ_1 and Γ_2 have the same sets of element orders.

Proof. Suppose precisely one group contains an element of order $s \geq 2$. By Lemma 15.1, precisely one quotient returns at $t = L/s$. Choose a nonzero $0 \leq \chi \in C_c^\infty((0,1))$. Lemma 15.2 makes the trace difference $CR_j^\alpha + o(R_j^\alpha)$, where $C \neq 0$ and $\alpha \geq 1$. Dividing by R_j^α contradicts the hypothesis. Both groups contain the identity. □

For $X = \mathbb{S}^n(1)$, $n \geq 2$, formula (15.1) is due to Prüfer [55, Corollary 1, p. 135]. For isospectral round spherical space forms, equality of the element-order sets is due to Ikeda [39, Corollary 2.8, p. 84].

Lemma 15.4. *Let a finite group Γ of odd order act freely and smoothly on a closed homotopy sphere X. Every Sylow subgroup of Γ is cyclic, and the exponent of Γ equals $|\Gamma|$.*

Proof. If $\dim X$ is even, the covering identity $2 = \chi(X) = |\Gamma|\chi(X/\Gamma)$ gives $|\Gamma| = 1$. Suppose $\dim X$ is odd. A finite CW structure on the smooth quotient X/Γ lifts to a finite free Γ-CW structure on X. Since X is an integral homology sphere, [8, Proposition VII.10.2, p. 180] gives periodic cohomology for Γ. By [8, Theorem VI.9.5, p. 157], its Sylow subgroups are cyclic or generalized quaternion. Their odd orders exclude the latter case.

A generator of each Sylow subgroup realizes the full prime-power factor of $|\Gamma|$. Hence $|\Gamma|$ divides the exponent. Lagrange's theorem gives the reverse divisibility. □

15.1. **The covering radius.**

Lemma 15.5. *Let $n = 2m + 1$, where $m \geq 2$ is even. Let $M = \mathbb{S}^n(2)/\Gamma$ and $Q = \mathbb{S}^n(1)/\Gamma_0$ be round spherical space forms with*

$$|\Gamma| = d = 2^{2m+1}q, \qquad |\Gamma_0| = q \text{ odd.}$$

Put

$$P = \sqrt{\Delta_M + m^2}, \qquad P_0 = \sqrt{\Delta_M + m^2/4}, \qquad P_Q = \sqrt{\Delta_Q + m^2}.$$

Only even harmonic degrees $2l$ occur on M. Their P-frequencies satisfy

$$\sqrt{l(l+m) + m^2} = l + \frac{m}{2} + O(l^{-1}), \tag{15.3}$$

and their P_0-frequencies are exactly $l + m/2$. For every $\chi \in C_c^\infty((0,1))$,

$$\operatorname{Tr} \chi(P_0/R) - \operatorname{Tr} \chi(P_Q/R) = \frac{m-1}{8q(2m-2)!} R^{2m-1} \int_0^1 \chi(u) u^{2m-2}\, du + O(R^{2m-2}). \tag{15.4}$$

Proof. Cauchy's theorem and Lemma 8.2 give $-I \in \Gamma$. It acts by $(-1)^k$ on degree-k harmonics, so the odd degrees have no invariants. At degree $2l$, the radius-two Laplace eigenvalue is $l(l+m)$. If $b = l + m/2$, then

$$\sqrt{l(l+m) + m^2} = \sqrt{b^2 + 3m^2/4} = b + \frac{3m^2}{8b} + O(b^{-3}), \qquad \sqrt{l(l+m) + m^2/4} = b.$$

In the variable $b = k + m$, the harmonic dimension formula (8.1) is

$$D_m(b) = \frac{2}{(2m)!} b^2 \prod_{j=1}^{m-1} (b^2 - j^2). \tag{15.5}$$

Put $c_m = 2/(2m)!$ and $S_m = \sum_{j=1}^{m-1} j^2 = m(m-1)(2m-1)/6$. Then

$$D_m(b) = c_m \big(b^{2m} - S_m b^{2m-2} + O(b^{2m-4}) \big).$$

Since m is even, the P_0- and P_Q-frequencies lie on the integer lattice. Apart from finitely many initial values of b, the identity contribution for Q at frequency b is $D_m(b)/q$, whereas the identity and antipodal contributions for M sum to $2D_m(2b)/d$. Hence

$$\frac{2}{d} D_m(2b) - \frac{1}{q} D_m(b) = \frac{3c_m S_m}{4q} b^{2m-2} + O(b^{2m-4}) = \frac{m-1}{8q(2m-2)!} b^{2m-2} + O(b^{2m-4}).$$

Every other Molien term is a finite sum of polynomial coefficients times root-of-unity phases. On M, restriction to degrees $2l$ can make such a phase constant only when the corresponding eigenvalue is 1 or -1. By Lemma 8.2, these occur only for I and $-I$. On Q, only the identity has eigenvalue 1. Thus the remaining annular sums are $O(R^{-N})$ for every N, by the finite-difference estimate in Lemma 8.4. The finitely many initial terms vanish for large R. Riemann summation of the displayed polynomial proves (15.4). □

Theorem 15.6. *Let $n = 2m + 1 \geq 5$ and let $Q = \mathbb{S}^n(1)/\Gamma_0$ be a round spherical space form with $q = |\Gamma_0|$ odd. Let M be a closed connected Riemannian manifold. Compare the scalar frequencies $\sqrt{\lambda + m^2}$ of M and Q, with cutoffs $R_K = m + K + \frac{1}{2}$. If, for some fixed $0 \leq \eta < 1/4$ and a sequence $K_j \to \infty$,*

$$E_\eta(R_{K_j}) = o(R_{K_j}),$$

then M has dimension n. Its universal Riemannian cover X is a smooth homotopy sphere on which all prime geodesics have length 2π, and

$$|\pi_1(M)| = q, \qquad \operatorname{Vol}(M) = V_n/q, \qquad \operatorname{Vol}(X) = V_n.$$

The groups $\pi_1(M)$ and Γ_0 have the same sets of element orders.

Proof. Put $R_j = R_{K_j}$ and $P_M = \sqrt{\Delta_M + m^2}$. A zero-dimensional M has finite spectrum, so the counting inequality $|N_M(R_j) - N_Q(R_j)| \le E_\eta(R_j)$ and Weyl's law for Q exclude this case. Proposition 10.1 gives dimension n, volume V_n/q, least common geodesic period 2π, and scalar return symbol 1. It also gives a positive self-adjoint quantizer A, strongly commuting with P_M, such that

$$A - P_M \in \Psi_{\mathrm{cl}}^{-1}, \qquad \mathrm{Spec}(A) \subset m + \mathbb{Z}_{\ge 0}.$$

The trace comparison (10.3) is $o(R_j)$ for every fixed time and every smooth annular cutoff. Lemma 11.1 gives a finite universal covering by a smooth homotopy sphere X, with degree d, common prime length $L \in \{2\pi, 4\pi\}$, and volume dV_n/q.

Suppose d is even. The Besse manifold M has an even-degree smooth covering by the homotopy n-sphere X, with $n \neq 3$. Lin–Schmidt rigidity [49, Theorem 2] therefore makes X round. Its prime length fixes its radius $R_0 \in \{1, 2\}$, and volume gives $d = qR_0^n$. Since d is even and q is odd, one has $R_0 = 2$ and $d = 2^n q$.

The deck group contains $-I$, so only degrees $2l$ occur and their P_M-frequencies satisfy

$$\mu_l = \sqrt{l(l+m) + m^2} = l + \frac{m}{2} + O(l^{-1}).$$

If m is odd, then $e^{2\pi i\mu} \to -1$ as $\mu \to \infty$ in $\mathrm{Spec}(P_M)$. Hence $e^{2\pi i P_M} + I$ is compact. But Proposition 10.1 gives $e^{2\pi i P_M} - I \in \Psi_{\mathrm{cl}}^{-1}$, which is also compact. Their difference is $2I$, a contradiction.

If m is even, $l + m/2$ is the nearest integer to μ_l for large l. The nearest-lattice property in Corollary 9.2 gives

$$A = P_0 := \sqrt{\Delta_M + m^2/4}$$

outside a finite sum of eigenspaces. These exceptional eigenspaces make no contribution to an annular trace for large R. For a nonnegative nonzero cutoff, Lemma 15.5 therefore gives a nonzero time-zero trace difference of order R_j^{2m-1}. Since $m \ge 2$, this contradicts the $o(R_j)$ trace comparison. Thus d is odd. The deck involution in Lemma 11.1 is then the identity, so $L = 2\pi$.

Theorem 15.3 applies to $M = X/\Gamma$ and $Q = \mathbb{S}^n(1)/\Gamma_0$, with common upstairs prime length 2π. Their sets of element orders coincide. Lemma 15.4 identifies the order of each odd-order group with its exponent. Hence $d = q$, and $\mathrm{Vol}(X) = dV_n/q = V_n$. □

Theorem 15.7 (Rigidity from a Clifford subgroup). *Under the hypotheses of Theorem 15.6, assume that Γ_0 contains a Clifford translation of odd order $r \ge 5$. Then M is a round unit spherical space form of degree q, and its complete scalar spectrum equals that of Q.*

Proof. Put $t = 2\pi/r$. A suitable power of the Clifford element has eigenvalues e^{it} and e^{-it}, each with complex multiplicity $m + 1$. For $g \in \Gamma_0$, let ρ_g be the complex multiplicity of e^{it}. If $\rho_g > 0$, Lemma 8.2 gives $\mathrm{ord}(g) = r$. Its free homotopy class contains no geodesic of length $k\pi$, $k \ge 1$: a lift would give $g^2 x = x$, whereas $g^2 \neq I$. By Prüfer's description [56, Theorem 1, p. 291, and its proof, p. 295], the lifted initial points of the time-t returns with monodromy g form a great sphere of dimension $2\rho_g - 1$; there are no such returns when $\rho_g = 0$. Since $\sin t \neq 0$, the initial velocity is uniquely determined by

$$v = \frac{gx - \cos(t)x}{\sin(t)}.$$

Thus the lifted returning states form a graph over this sphere. Freeness makes the graphs for distinct monodromies disjoint, and passage to the finite quotient preserves their dimensions. Since $\rho_g \le m + 1$, with equality for the chosen Clifford element, the largest return component on Q has dimension n. Lemma 15.2 gives a nonzero leading trace term of order R^{m+1}.

By Theorem 15.6, the comparison cover X has prime length 2π, volume V_n, and a deck element of order r. Its quotient therefore returns at t. If the largest return-component dimension on M differed from n, Lemma 15.2 would give a nonzero unmatched leading power in the $o(R_j)$ trace comparison. Hence this largest dimension is n.

Let $C \subset \mathrm{Fix}(G_t : SM \to SM)$ be a component of dimension n, and let F be a connected component of its inverse image in SX. The covering $SX \to SM$ is a local diffeomorphism, so $\dim F = n$. For each $v \in F$, there is a unique deck transformation h_v with $G_t v = \widehat{h_v} v$. The map $v \mapsto h_v$ is locally constant and therefore constant on F; write it as h. The set F is a connected component of $\mathrm{Fix}(\widehat{h}^{-1} G_t)$. Since $rt = 2\pi$, freeness gives $h^r = I$. An order smaller than r would give an upstairs return before 2π. Thus h has exact order r.

The subgroup $\langle h \rangle$ acts freely on X. Proposition 13.6, applied to this subgroup, makes evaluation $F \to X$ onto. It gives $d(x, hx) \le t$ for every x, while (15.2) gives the reverse inequality. Thus h has constant displacement $2\pi/r$. Theorem 14.2 makes X round of radius one.

Let a_k and b_k be the degree-k harmonic multiplicities on M and Q. Their shifted frequencies are $k + m$, so the tolerance $\eta < 1/4$ permits only equal-frequency pairs. Consequently

$$\sum_{k=0}^{K_j} |a_k - b_k| = E_\eta(R_{K_j}) = o(K_j).$$

Since the multiplicities are integers, there are only $o(K_j)$ indices $k \le K_j$ at which they differ. Each residue class modulo q therefore contains at least $K_j/q - o(K_j)$ indices of equality. The harmonic representations of $\mathrm{SO}(2m + 2)$ form the string with highest weights $k\varepsilon_1$. Both deck groups lie in this connected semisimple group and have order q. Its root system has $m(m + 1)$ positive roots, so every residue class eventually contains at least $m(m + 1) + 1$ such indices. The finite determination theorem [48, Proposition 3.1, arXiv version] gives $a_k = b_k$ for every $k \ge 0$. The complete scalar spectra agree. □

Corollary 15.8. *Let $m \ge 2$, let $q \ge 3$ be odd, and let $\gcd(a, q) = 1$. Theorem 15.7 applies to $L(q; 1, \dots, 1, a)$ whenever a divisor $r \mid q$, $r \ge 5$, satisfies $a \equiv \pm 1 \pmod r$. In particular, it applies to $L(25; 1, \dots, 1, 6)$ in every odd dimension at least five. It also applies to each member of the pair $L(49; 1, 6, 15)$, $L(49; 1, 6, 20)$.*

Proof. In the two-weight case, the generator's q/rth power has order r and rotation angles $\pm 2\pi/r$. Hence it is a Clifford translation. For the two order-49 triples, reduction modulo 7 gives $(1, -1, 1)$ and $(1, -1, -1)$, respectively. Their seventh powers are Clifford translations of order 7. □

For a lens space $L(q; 1, \dots, 1, a)$ of dimension $n = 2m + 1 \ge 5$, with q an odd prime and $a \not\equiv 0, \pm 1 \pmod q$, consider the return time $2\pi/q$. Write g for the deck generator. The powers $g^{\pm 1}$ give return graphs over the unit sphere in $\mathbb{C}^m$, whereas $g^{\pm a^{-1}}$ give return graphs over the unit sphere in $\mathbb{C}$; here a^{-1} is taken modulo q. These four powers are distinct, and no other power has eigenvalue $e^{\pm 2\pi i/q}$. Their images on the quotient have dimensions $2m - 1 = n - 2$ and 1. Thus the largest return component has dimension $n - 2$. Theorem 15.6 still determines the covering degree and volume.

16. Even-order spherical quotients

Assume (E) and (2.6). Choose $K_j \to \infty$ such that

$$\frac{\mathcal{E}_\eta(K_j)}{R_j^{m+1}} \longrightarrow \delta := \liminf_{K \to \infty} \frac{\mathcal{E}_\eta(K)}{R_K^{m+1}} < c_{\mathrm{E}}(n, q), \qquad R_j = R_{K_j}.$$

Since $m + 1 < n$, condition (10.1) holds. Proposition 10.1 gives common period $T = \pi$. Let $d = |\Gamma|$ for the universal covering group. Lemma 11.1 gives

$$L \in \{\pi, 2\pi\}, \qquad \mathrm{Vol}(X) = dV_n/q. \tag{16.1}$$

Lemma 16.1. *If $L = 2\pi$, then X is the round unit sphere and $d = q$. If $L = \pi$, then d is odd.*

Proof. If $L = 2\pi$, Lemma 11.1 gives a nonidentity deck involution D at time π. A deck transformation is an isometry, and a nonidentity deck transformation has no fixed point. The quotient $X/\langle D \rangle$ is therefore a smooth manifold with a degree-two Riemannian covering by the smooth homotopy sphere X. Its metric is Besse because projected geodesics are closed. By [49, Theorem 2], the quotient and X

have constant sectional curvature: the covering degree is two, X is a smooth homotopy n-sphere, and $n \geq 5$. The complete simply connected constant-curvature model is compact only in positive curvature, so X is a round sphere. Its prime length 2π fixes radius one. The equality $\mathrm{Vol}(X) = dV_n/q$ now gives $d = q$.

If $L = \pi$ and d were even, apply the same theorem directly to the covering $X \to M$. Its round covering radius would be $1/2$. The corresponding free orthogonal group has even order and contains $-I$ by Lemma 8.2 and Cauchy's theorem. On a round sphere of radius $1/2$, the cotangent lift of $-I$ is the time-$\pi/2$ geodesic map. It descends to the identity on S^*M. This contradicts the least common period π in Proposition 10.1. Hence d is odd. □

Lemma 16.2. *If m is odd, the alternative $L = \pi$ is impossible.*

Proof. Suppose $L = \pi$ and m is odd. Equation (10.5) says that the principal symbol of the downstairs scalar time-π propagator is $e^{i\pi m} = -1$. Both time-π geodesic maps are identities. Lemma 11.5 therefore gives the same symbol on X. On the other hand, π is the upstairs prime period, so Lemma 11.4 gives symbol $+1$. The two values are distinct. Thus $L = \pi$ is impossible. □

Lemma 16.3. *Under the hypotheses of this section, if $4 \mid q$, then M has a unit-speed closed geodesic of length $\pi/2$.*

Proof. Let S be a Sylow 2-subgroup of Γ_0. Its order is at least four. If it had no element of order four, every element of S would have order at most two: a power of an element of order 2^j, $j \geq 2$, has order four. Thus S would contain at least three involutions, contrary to Lemma 8.2. Let $b_4 > 0$ be the number of order-four elements in Γ_0.

By the calculation in [39, proof of Proposition 2.9, p. 84], their contribution to (8.2) is

$$\frac{b_4}{q}\frac{1-z^2}{(1+z^2)^{m+1}}.$$

No other element has eigenvalue i or $-i$, by Lemma 8.2. Hence the remaining terms are holomorphic at i. Since $1 + z^2 = (1 - z/i)(1 + z/i)$,

$$L_i(F_{\Gamma_0}) = \frac{b_4}{q2^m} \neq 0. \tag{16.2}$$

The order-four elements occur in distinct inverse pairs, so $b_4 \geq 2$. Suppose that $G_{\pi/2}$ has no fixed unit covector on M. For every $\chi \in C_c^\infty((0,1))$, the nonstationary trace for A is then $O(R^{-N})$ for every N. Lemma 8.4, with $s = m + 1$, therefore gives

$$\begin{aligned}&\mathrm{Tr}(e^{i\pi A/2}\chi(A/R)) - \mathrm{Tr}(e^{i\pi P_Q/2}\chi(P_Q/R))\\ &\quad= -\frac{e^{i\pi m/2}}{m!}\frac{b_4}{q2^m}R^{m+1}\int_0^1 \chi(u)u^m\,du + O(R^m).\end{aligned}$$

Apply Lemma 10.2 with $L = -b_4/(q2^m)$ and $\theta = \pi m/2$. It gives

$$\delta \geq \frac{b_4}{q2^m(m+1)!} \geq \frac{1}{q2^{m-1}(m+1)!} = c_{\mathrm{E}}(n,q),$$

contrary to (2.6). Hence $G_{\pi/2}$ has a fixed unit covector, which is a unit-speed closed geodesic of length $\pi/2$. □

Proof of the covering conclusion in Theorem 2.2. The conclusion follows from Lemma 16.1 if $L = 2\pi$. If $L = \pi$, Lemma 16.2 implies that m is even. In this case d is odd, and Proposition 12.3 gives

$$\frac{dV_n}{q} = 2^{-n}V_n.$$

Hence $q = 2^n d$, in particular $4 \mid q$. Lemma 16.3 gives a closed geodesic on M of length $\pi/2$. Since the upstairs prime length is π, Lemma 15.1 gives an element of order two in Γ, contrary to the oddness of d. Thus $L = 2\pi$. Lemma 16.1 gives a unit round cover and $d = q$. The scaling at the start of Section 8 restores radius r. □

17. Homogeneity and harmonic multiplicities

At radius one, the covering conclusion of Theorem 2.2 makes both metrics round quotients of even degree q. Their shifted spectra now belong to $m + 2\mathbb{Z}_{\geq 0}$. Since $\eta < 1/2$, an admissible matching pairs only equal frequencies. Thus

$$\mathcal{E}_\eta(K) = \sum_{\omega \leq R_K} |\operatorname{mult}_{P_M}(\omega) - \operatorname{mult}_{P_Q}(\omega)|. \tag{17.1}$$

Definition 17.1. An isometry of a round sphere is a Clifford translation if its displacement function is constant. For $\xi \in S^1 \setminus \{1, -1\}$, let $c_\Gamma(\xi)$ be the number of elements whose complex eigenvalues are exactly $\xi, \bar{\xi}$, each with multiplicity $m + 1$.

Lemma 17.2. *For a free orthogonal group $\Gamma \subset O(2m + 2)$ of order q and $\xi \in S^1 \setminus \{1, -1\}$,*

$$c_\Gamma(\xi) = q(1 - \xi^2)^m L_\xi(F_\Gamma), \tag{17.2}$$

where L_ξ is computed at pole order $m + 1$. The elements counted are precisely the Clifford translations other than $I, -I$ having that conjugate eigenvalue pair.

Proof. Put $s = m + 1$. Lemma 8.2 gives $\Gamma \subset \mathrm{SO}(2m + 2)$. By [39, Lemma 5.2, p. 90],

$$\lim_{z \to \xi} (z - \xi)^s F_\Gamma(z) = \frac{c_\Gamma(\xi)}{q} \frac{1 - \xi^2}{(\xi - \bar{\xi})^s}.$$

Since $1 - z/\xi = -\xi^{-1}(z - \xi)$ and $\xi - \bar{\xi} = -\xi^{-1}(1 - \xi^2)$, this is (17.2).

For an orthogonal map γ, $d(x, \gamma x) = \arccos\langle x, \gamma x\rangle$ on the unit sphere. Thus constant displacement is equivalent to $\gamma + \gamma^t = 2cI$ for some $c \in [-1, 1]$. Apart from $\gamma = \pm I$, this is exactly the stated conjugate-eigenvalue condition. □

Lemma 17.3. *Every deck transformation of a homogeneous spherical space form is a Clifford translation.*

Proof. Write the quotient as $\mathbb{S}^{2m+1}/\Gamma$. By Lemma 8.2, $\Gamma \subset \mathrm{SO}(2m + 2)$. The implication (1) ⇒ (2) in [39, Theorem 5.1, p. 90] gives the conjugate-eigenvalue condition of Lemma 17.2 for every element other than $I, -I$. These two elements also have constant displacement. □

Completion of Theorem 2.2. Assume Q is homogeneous. Let $\mathcal{R}$ be the finite set of nonreal eigenvalues of elements in $\Gamma \cup \Gamma_0$. Suppose that $c_\Gamma(\xi) \neq c_{\Gamma_0}(\xi)$ for some $\xi \in \mathcal{R}$. An element counted at $\xi \neq \pm 1$ is distinct from its inverse, and its inverse has the same unordered eigenvalue pair. Both counts are therefore even. Lemma 17.2 therefore gives

$$\left|L_\xi(F_\Gamma) - L_\xi(F_{\Gamma_0})\right| = \frac{|c_\Gamma(\xi) - c_{\Gamma_0}(\xi)|}{q|1 - \xi^2|^m} \geq \frac{1}{q2^{m-1}}.$$

At time t with $e^{it} = \xi$, Lemma 8.4 gives the corresponding trace difference of order R^{m+1}. The exact multiplicity discrepancy in (17.1) bounds this trace by $\|\chi\|_\infty \mathcal{E}_\eta(K_j)$. The proof of Lemma 10.2, now applied to the two metric operators, gives

$$\delta \geq \frac{1}{q2^{m-1}(m + 1)!} = c_{\mathrm{E}}(n, q),$$

a contradiction. Thus all Clifford counts agree.

Both groups contain $I, -I$. By Lemma 17.3, every other element of Γ_0 is counted once for each of its two conjugate eigenvalues. Consequently

$$2 + \frac{1}{2} \sum_{\xi \in \mathcal{R}} c_{\Gamma_0}(\xi) = q.$$

The corresponding expression for Γ counts a subset of its q elements. Equality of every count makes this expression equal to q. Hence every element of Γ is a Clifford translation. For each unordered pair

$\{\xi, \bar{\xi}\}$, the corresponding determinant is $(1-z\xi)^{m+1}(1-z\bar{\xi})^{m+1}$ and occurs $c_\Gamma(\xi)$ times. The remaining elements are I and $-I$ in both groups. Equality of the counts makes every contribution to (8.2) agree, so $F_\Gamma = F_{\Gamma_0}$. The complete scalar spectra agree. Ikeda's theorem [39, Theorem III, p. 76], also proved in [70, Theorem 8.2, p. 334], applies to these two spherical space forms, one homogeneous, and gives their isometry. □

Corollary 17.4. *Let Q be any reference in (E), and suppose $0 \le \eta < 1/2$ and*

$$\liminf_{K\to\infty} \frac{\mathcal{E}_\eta(K)}{R_K^n} = 0.$$

If $n \equiv 3 \pmod 4$, or if $n \equiv 1 \pmod 4$ and $v_2(q) \neq n$, then M is a round spherical space form with covering radius r and group order q. Here $v_2(q)$ is the exponent of 2 in q.

Proof. Normalize $r = 1$. Proposition 10.1 gives dimension n, volume V_n/q, least common period π, and scalar return symbol $e^{i\pi m}$. Lemma 11.1 gives

$$L \in \{\pi, 2\pi\}, \qquad \mathrm{Vol}(X) = dV_n/q.$$

By the proofs of Lemmas 16.1 and 16.2, the alternative $L = 2\pi$ gives a round unit cover of degree q, whereas $L = \pi$ forces d odd and m even. In the latter case, Proposition 12.3 gives $dV_n/q = 2^{-n}V_n$, hence $q = 2^n d$ and $v_2(q) = n$. The stated hypotheses therefore imply $L = 2\pi$. Rescaling gives covering radius r. □

18. Scalar quotients of odd order

For the scalar cyclic reference, put $\xi = e^{2\pi i/q}$. Only the generator and its inverse have eigenvalue ξ^{-1}, so $c_{\Gamma_0}(\xi) = 2$. Lemma 17.2 gives

$$L_\xi(F_Q) = \frac{2}{q}(1-\xi^2)^{-m}. \tag{18.1}$$

Proof of Theorem 2.4. Normalize $r = 1$, put $s = m + 1$, and choose $K_j \to \infty$ such that, with $R_j = R_{K_j}$,

$$\frac{\mathcal{E}_\eta(K_j)}{R_j^s} \longrightarrow \delta := \liminf_{K\to\infty} \frac{\mathcal{E}_\eta(K)}{R_K^s} < c_{\mathrm{O}}(n, q).$$

Since $s < n$, one has $\mathcal{E}_\eta(K_j) = o(R_j^\eta)$. Proposition 10.1 and Lemma 11.1 give a universal Riemannian covering $X \to M$ of finite degree d, where X is a smooth homotopy sphere of volume dV_n/q and common prime geodesic length $L \in \{2\pi, 4\pi\}$. They also give

$$e^{2\pi i P_M} - I \in \Psi_{\mathrm{cl}}^{-1}, \qquad A - P_M \in \Psi_{\mathrm{cl}}^{-1}, \qquad \mathrm{Spec}(A) \subset m + \mathbb{Z}_{\ge 0},$$

with A strongly commuting with P_M and satisfying (10.3).

Suppose d is even. Lin–Schmidt [49, Theorem 2] gives constant sectional curvature on M and hence on X. The radius of X is $L/(2\pi) \in \{1, 2\}$, and its volume gives $d = q(L/(2\pi))^n$. Since q is odd, the radius is two and $d = 2^n q$. The compactness argument in the proof of Theorem 15.6, applied to the displayed time-2π return, excludes odd m. For even m, (15.3) and the nearest-lattice property in Corollary 9.2 give

$$A = P_0 := \sqrt{\Delta_M + m^2/4}$$

outside finitely many eigenspaces. These eigenspaces contribute nothing to a sufficiently high annular trace. For every nonzero $0 \le \chi \le 1$ in $C_c^\infty((0,1))$, Lemma 15.5 and (10.3) therefore give

$$\frac{m-1}{8q(2m-2)!}\int_0^1 \chi(u)u^{2m-2}\,du \le \liminf_{j\to\infty} \frac{\mathcal{E}_\eta(K_j)}{R_j^{2m-1}}.$$

If $m \ge 4$, the right side is zero because $2m - 1 > s$, whereas the left side is positive. If $m = 2$, the right side is δ. Letting such cutoffs tend pointwise to 1 on $(0, 1)$ gives $\delta \ge 1/(48q)$, again a contradiction.

Thus d is odd. The time-2π deck involution in Lemma 11.1 is the identity, and $L = 2\pi$. Proposition 12.3 now gives $\mathrm{Vol}(X) = V_n$ and $d = q$.

Put $\ell = 2\pi/q$. For every $h \in \Gamma \setminus \{I\}$, Lemma 15.1 gives

$$\min_{x\in X} d(x, hx) \geq \frac{2\pi}{\mathrm{ord}(h)} \geq \ell,$$

since $\mathrm{ord}(h)$ divides q. By Lemma 15.2, if the time-ℓ return set on SM is nonempty and its largest component dimension is d_*, the annular trace for A has a nonzero leading term of order $R^{(d_*+1)/2}$. If the return set is empty, that trace is $O(R^{-N})$ for every N. Equation (18.1) and Lemma 8.4 give order R^s for the reference trace. Hence $d_* > n$ contradicts (10.3) and the finiteness of δ.

If the return set is empty or $d_* < n$, the trace for A is $o(R^s)$. Lemma 10.2 and (18.1) then imply

$$\delta \geq \frac{|L_\xi(F_Q)|}{(m+1)!} = \frac{2}{q|1 - e^{4\pi i/q}|^m (m+1)!}.$$

For $m = 2$, this is at least $1/(12q) > 1/(48q)$. For $m \geq 3$, it equals $c_{\mathrm{O}}(n, q)$. Both cases contradict the hypothesis. Thus the return set has a component C of dimension n.

Let F be a connected component of the inverse image of C in SX. The lifting and exact-order arguments in the proof of Theorem 15.7, with $r = q$, give $\dim F = n$ and a deck transformation h of order q such that F is a connected component of $\mathrm{Fix}(\widehat{h}^{-1} G_\ell)$. Since $|\Gamma| = q$, one has $\Gamma = \langle h \rangle$. Proposition 13.6 makes evaluation $F \to X$ surjective. Thus $d(x, hx) \leq \ell$ for every $x \in X$. The opposite inequality was proved above, so h has constant displacement ℓ. Theorem 14.2 identifies X with the unit round sphere and h with scalar multiplication by a primitive qth root of unity. Hence $M \cong Q_{q,n}(1)$. Rescaling restores radius r. □

Proof of Theorem 2.3. Condition (2.7) implies (2.9), as in Section 2. Theorem 2.4 therefore gives $M \cong Q$. □

Part 3. Perturbations and sharpness

For a homogeneous reference in (E) or a reference in (O), isospectrality after symmetric scalar classical perturbations of order less than one implies isometry. At the form bound $|b[u]| \leq C\langle (1+\Delta)^{1/2} u, u\rangle$, isospectral form perturbations exist between closed connected manifolds of positive dimension precisely when their dimensions and volumes agree (Theorem 20.1). For the same references, classical perturbations of order one retain rigidity under the constant orbit-average hypothesis of Theorem 20.8. Without this hypothesis, first-order differential perturbations can produce isospectral operators on nonhomeomorphic manifolds. Round lens-space pairs attain an unmatched count of order $R^{(n+1)/2}$, so this exponent cannot be increased for odd scalar references. The shrinking-resolution hypotheses of Theorem 23.2 give rigidity for prime two-weight lens spaces in dimension five.

19. Spectral separation and perturbations

Let M be a smooth closed connected Riemannian manifold of positive dimension.

Proof of Corollary 2.5. For $x > 0$,

$$0 \leq \sqrt{x^2 + m^2} - x = \frac{m^2}{\sqrt{x^2 + m^2} + x} \longrightarrow 0.$$

Since both Laplace eigenvalue sequences tend to infinity,

$$\limsup_j |\mu_j - \beta_j| = \limsup_j r \left| \sqrt{\lambda_j(M)} - \sqrt{\lambda_j(Q)} \right| < a/4.$$

Choose η strictly between this upper limit and $a/4$. For some J, one has $|\mu_j - \beta_j| \leq \eta$ whenever $j \geq J$. Every β_j belongs to $m + a\mathbb{Z}$, and $R_K = m + aK + a/2$. Hence $|\beta_j - R_K| \geq a/2$, so

$$\mu_j \leq R_K \quad \Longleftrightarrow \quad \beta_j \leq R_K \qquad (j \geq J).$$

Matching these equal indices leaves at most J copies in each list unmatched. Thus $\mathcal{E}_\eta(K) \le 2J$ for every K. Theorems 2.2 and 2.3 give a round cover of radius r and give isometry when Q is homogeneous.

In case (E), the covering degree is the even number q. By Lemma 8.3, both shifted spectra belong to $m + 2\mathbb{Z}_{\ge 0}$. Distinct elements of this lattice have distance at least two; hence $\mu_j = \beta_j$ for every $j \ge J$. Only finitely many harmonic multiplicities can differ. Both deck groups lie in $\mathrm{SO}(2m+2)$ by Lemma 8.2. The harmonic representations of this compact connected semisimple group have highest weights $k\varepsilon_1$, $k \ge 0$. Lauret–Miatello [48, Corollary 3.2, arXiv version] therefore gives equality of the multiplicities in every degree. The complete scalar spectra agree. □

Proposition 19.1. *Let Q be homogeneous in (E), or let Q belong to (O). For $Z \in \{M, Q\}$, let b_Z be a symmetric sesquilinear form on $H^1(Z)$. Suppose that, for constants $C_Z \ge 0$ and $0 \le \alpha < 1/2$,*

$$|b_Z[u]| \le C_Z \langle (1+\Delta_Z)^\alpha u, u\rangle, \qquad u \in H^1(Z), \tag{19.1}$$

where $b_Z[u] = b_Z(u,u)$. The form sum $\Delta_Z + b_Z$ defines a semibounded self-adjoint operator H_Z with compact resolvent. If $\mathrm{Spec}(H_M) = \mathrm{Spec}(H_Q)$, including multiplicities, then M and Q are isometric. This applies, in particular, to self-adjoint classical pseudodifferential perturbations of order $\rho < 1$.

Proof. Fix one of the manifolds and omit its subscript. Put $\mathfrak{q}_\Delta[u] = \int |du|^2\, dv_g$ on H^1. For every $\epsilon > 0$ there is $A_\epsilon \ge 0$ such that

$$C(1+t)^\alpha \le \epsilon t + A_\epsilon \qquad (t \ge 0).$$

Spectral calculus gives

$$|b[u]| \le \epsilon \mathfrak{q}_\Delta[u] + A_\epsilon \|u\|_2^2.$$

Thus b has relative form bound zero. Since Δ has compact resolvent, Kato's form perturbation and representation theorems [43, Chapter VI, Theorems 3.4 and 2.6] give a closed form $\mathfrak{q}_\Delta + b$ on H^1 and a semibounded self-adjoint operator H with compact resolvent.

Let λ_j and ν_j be the eigenvalues of Δ and H, respectively, in increasing order and with multiplicity, starting at $j = 0$. Put

$$f_\pm(t) = t \pm C(1+t)^\alpha.$$

The operators $f_\pm(\Delta)$ are semibounded and have compact resolvent and form domain H^1, since $f_\pm(t) \sim t$. Equation (19.1) gives, as forms on H^1,

$$f_-(\Delta) \le H \le f_+(\Delta).$$

The min–max theorem on the common form domain [66, Theorem 4.10 and the paragraph following it] therefore gives

$$\lambda_j(f_-(\Delta)) \le \nu_j \le \lambda_j(f_+(\Delta)).$$

The function f_+ is increasing on $[0,\infty)$. Choose $N \ge 1$ so that f_- is increasing on $[\lambda_N, \infty)$ and

$$f_-(\lambda_N) > \max_{i<N} f_-(\lambda_i).$$

Such an N exists because $f_-'(t) \to 1$ and $f_-(t) \to \infty$. For $j \ge N$, the ordered eigenvalues of $f_\pm(\Delta)$ are therefore $f_\pm(\lambda_j)$, and

$$\lambda_j - C(1+\lambda_j)^\alpha \le \nu_j \le \lambda_j + C(1+\lambda_j)^\alpha.$$

If the two perturbed spectra agree, let ν_j denote their common ordered list, including every negative eigenvalue. The preceding bounds and $\alpha < 1$ give $\lambda_j(M) \sim \nu_j \sim \lambda_j(Q)$. Consequently

$$\left|\sqrt{\lambda_j(M)} - \sqrt{\lambda_j(Q)}\right| = \frac{|\lambda_j(M) - \lambda_j(Q)|}{\sqrt{\lambda_j(M)} + \sqrt{\lambda_j(Q)}} = O(\nu_j^{\alpha - 1/2}) \longrightarrow 0.$$

Corollary 2.5 gives the isometry.

Let B be a symmetric classical pseudodifferential operator of order $0 \le \rho < 1$. By [5, Theorem 7.9, arXiv version], $(1+\Delta)^{-\rho/4}$ has order $-\rho/2$. The composition and boundedness theorems [65, Sections I.2, I.3, and I.8] give

$$T = (1+\Delta)^{-\rho/4} B (1+\Delta)^{-\rho/4} \in \Psi^0_{\mathrm{cl}}, \qquad T : L^2 \longrightarrow L^2 \text{ bounded}.$$

For $u \in C^\infty$,

$$|\langle Bu, u\rangle| = |\langle T(1+\Delta)^{\rho/4}u, (1+\Delta)^{\rho/4}u\rangle| \le \|T\| \langle (1+\Delta)^{\rho/2}u, u\rangle.$$

The form extends continuously to $H^{\rho/2}$, hence to H^1. Thus (19.1) holds with $\alpha = \rho/2 < 1/2$. If $\rho < 0$, then B is bounded on L^2, and $\alpha = 0$ suffices. □

Proof of Corollary 2.6. For $Z \in \{M, Q\}$, the form $b_Z[u] = \langle B_Z u, u\rangle$ satisfies

$$|b_Z[u]| \le \|B_Z\| \|u\|_2^2.$$

Its sum with the Laplace form is represented on $H^2(Z)$ by $\Delta_Z + B_Z$. This operator is self-adjoint by [43, Chapter V, Theorem 4.3], and hence is the operator associated with that form. Proposition 19.1 applies with $\alpha = 0$. □

Remark 19.2. For the form bound (19.1) with $\alpha = 1/2$, the same comparison gives only

$$\left| \sqrt{\lambda_j(M)} - \sqrt{\lambda_j(Q)} \right| = O(1).$$

The bound depends on C_M, C_Q and need not be smaller than ε_Q / r. The critical form class is determined in Theorem 20.1. For classical operators of order one, Theorem 20.8 gives rigidity under the constant reference orbit-average hypothesis; Theorem 21.1 gives two-sided differential counterexamples without that hypothesis.

Corollary 19.3. *For a homogeneous reference in (E) or a reference in (O), exact scalar isospectrality implies isometry. The same is true if the scalar spectral multisets agree outside a bounded interval.*

Proof. If the scalar spectral multisets agree outside $[0, C]$, only the finitely many copies with $\lambda \le C$ can fail to match. Thus $\mathcal{E}_0(K)$ is bounded independently of K, and its quotient by R_K^{m+1} tends to zero. Theorem 2.2 or 2.4 gives the isometry. □

Corollary 19.4. *Let Q be homogeneous in (E), or let Q belong to (O), and fix $0 \le \eta < a/4$. If $M \not\cong Q$, then*

$$\liminf_{K\to\infty} \frac{\mathcal{E}_\eta(K)}{R_K^{m+1}} \ge \begin{cases} c_{\mathrm{E}}(n,q), & \text{in case (E),} \\ c_{\mathrm{O}}(n,q), & \text{in case (O).} \end{cases}$$

Proof. If the lower bound failed, Theorem 2.2 or Theorem 2.4 would give $M \cong Q$. □

20. Perturbations of order one

The eigenvalues of a scalar Laplacian are numbered from zero and repeated according to multiplicity. All manifolds in this section are smooth, closed and connected. Forms are Hermitian sesquilinear forms.

Theorem 20.1. *Let M and N be Riemannian manifolds of positive dimension. The following conditions are equivalent.*

(i) $\dim M = \dim N$ *and* $\operatorname{Vol} M = \operatorname{Vol} N$.

(ii) *There is a form b on $H^1(M)$ such that*

$$|b[u]| \le C\langle (1+\Delta_M)^{1/2}u, u\rangle, \qquad \operatorname{Spec}(\Delta_M + b) = \operatorname{Spec}\Delta_N.$$

(iii) *There are forms b_M, b_N satisfying the corresponding bounds on M, N such that*

$$\operatorname{Spec}(\Delta_M + b_M) = \operatorname{Spec}(\Delta_N + b_N).$$

In (ii)*, b can be chosen diagonal in a Laplace eigenbasis. The corresponding diagonal operator B extends continuously from $H^s(M)$ to $H^{s-1}(M)$ for every $s \in \mathbb{R}$.*

Proof. For a d-dimensional closed manifold Z, the scalar counting function satisfies

$$\#\{j : \sqrt{\lambda_j(Z)} \le R\} = c_d \operatorname{Vol}(Z) R^d + O(R^{d-1}), \qquad c_d = (2\pi)^{-d} \operatorname{Vol}(B^d). \tag{20.1}$$

This follows by integrating [36, Theorem 5.1] for $1 + \Delta_Z$ and replacing its frequency parameter by $\sqrt{1+R^2}$. If $N_Z(R) = aR^d + O(R^{d-1})$, choose $A > 0$ so large that, for all sufficiently large x,

$$N_Z(x - A) < ax^d < N_Z(x + A).$$

Increasing A accommodates the bounded initial interval. Applied with $ax^d = j + 1$, these inequalities give

$$\sqrt{\lambda_j(Z)} = ((j+1)/a)^{1/d} + O(1). \tag{20.2}$$

Thus (i) implies $D := \sup_j |\sqrt{\lambda_j(M)} - \sqrt{\lambda_j(N)}| < \infty$. Put $\lambda_j = \lambda_j(M)$ and $\nu_j = \lambda_j(N)$. Then

$$|\nu_j - \lambda_j| \le D(2\sqrt{\lambda_j} + D) \le (2D + D^2)\sqrt{1 + \lambda_j}. \tag{20.3}$$

For an orthonormal eigenbasis (e_j) on M, define

$$b[u, v] = \sum_{j \ge 0} (\nu_j - \lambda_j) u_j \overline{v_j}, \qquad u_j = \langle u, e_j \rangle.$$

The series converges absolutely on $H^{1/2}(M) \times H^{1/2}(M)$, by Cauchy–Schwarz and (20.3). It satisfies the bound in (ii). Moreover,

$$\langle \Delta_M u, u \rangle + b[u] = \sum_{j \ge 0} \nu_j |u_j|^2.$$

Since $1 + \nu_j \asymp 1 + \lambda_j$, this is a closed nonnegative form with domain $H^1(M)$. Its operator has eigenvectors e_j and eigenvalues ν_j. If $Be_j = (\nu_j - \lambda_j)e_j$, then

$$\sum_j (1 + \lambda_j)^{s-1} |(Bu)_j|^2 \le (2D + D^2)^2 \sum_j (1 + \lambda_j)^s |u_j|^2.$$

This proves the stated Sobolev bounds and (ii).

Condition (ii) implies (iii) with $b_N = 0$. The form and eigenvalue estimates in the proof of Proposition 19.1 hold for $0 \le \alpha < 1$, on every closed manifold. At $\alpha = 1/2$ they give a semibounded self-adjoint form sum with compact resolvent, whose ordered eigenvalues satisfy

$$|\eta_j - \lambda_j| \le C\sqrt{1 + \lambda_j} \qquad (j \text{ sufficiently large}).$$

In particular, $\eta_j/\lambda_j \to 1$. If (iii) holds, let η_j be the common ordered list, including its negative eigenvalues. For $Z = M, N$, (20.2) gives

$$\eta_j \sim \lambda_j(Z) \sim \left(\frac{j+1}{c_{\dim Z} \operatorname{Vol} Z} \right)^{2/\dim Z}.$$

Equality of the powers of $j + 1$ gives $\dim M = \dim N$; equality of the leading coefficients then gives $\operatorname{Vol} M = \operatorname{Vol} N$. □

Example 20.2. Let $n \ge 2$, $a = 2^{-1/n}$, $M = \mathbb{S}^n(a)$ and $N = \mathbb{RP}^n(1)$. Then

$$\operatorname{Vol} M = \operatorname{Vol} N = \tfrac{1}{2} \operatorname{Vol} \mathbb{S}^n(1), \qquad \pi_1(M) = 0, \quad \pi_1(N) = \mathbb{Z}/2\mathbb{Z}.$$

The diagonal construction in Theorem 20.1 therefore gives nonisometric manifolds satisfying (ii).

20.1. **A constant translation of scalar frequencies.** For $c > 0$, write $P_{Z,c} = (\Delta_Z + c)^{1/2}$.

Lemma 20.3. *If* $\dim Z = n \geq 2$*, then, as* $t \downarrow 0$*,*

$$\operatorname{Tr} e^{-tP_{Z,c}} = \kappa_n \operatorname{Vol}(Z) t^{-n} + O(t^{2-n}), \qquad \kappa_n = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}}. \tag{20.4}$$

Proof. The scalar heat expansion gives

$$\operatorname{Tr} e^{-s(\Delta_Z + c)} = (4\pi s)^{-n/2} \operatorname{Vol}(Z) + O(s^{1-n/2}) \qquad (0 < s \leq 1).$$

The heat parametrix has only integral powers of s on a closed manifold; see [23, Lemmas 1.7.4 and 4.8.4]. Subordination gives

$$e^{-t\sqrt{x}} = \frac{t}{2\sqrt{\pi}} \int_0^\infty s^{-3/2} e^{-t^2/(4s)} e^{-sx}\, ds \qquad (x > 0).$$

All integrands in the spectral sum are nonnegative, so the same identity holds for the traces. The substitution $v = t^2/(4s)$ gives

$$\frac{t}{2\sqrt{\pi}} (4\pi)^{-n/2} \int_0^\infty s^{-(n+3)/2} e^{-t^2/(4s)}\, ds = \kappa_n t^{-n}.$$

The integral of the error on $0 < s \leq 1$ is bounded by

$$Ct \int_0^1 s^{-(n+1)/2} e^{-t^2/(4s)}\, ds \leq C' t^{2-n} \int_0^\infty v^{(n-3)/2} e^{-v}\, dv.$$

This integral is finite for $n \geq 2$. On $s \geq 1$, both the actual trace and the subtracted leading expression have integrable $s^{-3/2}$ multiples, and their contribution is $O(t)$. □

Lemma 20.4. *Let* M, N *have the same dimension* $n \geq 2$*. Let* μ_j, ν_j *be the eigenvalues of* $P_{M,c}, P_{N,c}$ *respectively. If*

$$\nu_j = \mu_j + d + O(\mu_j^{-1}),$$

then $\operatorname{Vol} M = \operatorname{Vol} N$ *and* $d = 0$*.*

Proof. The counting estimate $N_{P_{M,c}}(R) = O(R^n)$ and integration by parts give

$$\sum_j \mu_j^{-1} e^{-t\mu_j} = O(t^{1-n}) \qquad (0 < t \leq 1).$$

Writing $N(R) = \#\{j : \mu_j \leq R\}$, the tail is bounded by a constant multiple of

$$\int_{\sqrt{c}}^\infty R^n (R^{-2} + tR^{-1}) e^{-tR}\, dR = O(t^{1-n}).$$

The mean value theorem applied to e^{-tx} now gives

$$\sum_j e^{-t\nu_j} = e^{-td} \sum_j e^{-t\mu_j} + O(t^{2-n}).$$

The finitely many exceptional indices contribute $O(t)$ after subtraction. Substitution of (20.4) yields

$$\kappa_n \operatorname{Vol}(N) t^{-n} = \kappa_n \operatorname{Vol}(M) t^{-n} - d\kappa_n \operatorname{Vol}(M) t^{1-n} + O(t^{2-n}).$$

Multiplication first by t^n and then, after subtracting the leading term, by t^{n-1} gives the two conclusions. □

Lemma 20.5. *Let* $P_0 = P_{M,c}$ *and let* $R \in \Psi_{\mathrm{cl}}^{-1}(M)$ *be self-adjoint. If* μ_j *are the eigenvalues of* P_0*, then the ordered eigenvalues of* $P_0 + d + R$ *satisfy*

$$\eta_j = \mu_j + d + O(\mu_j^{-1}).$$

Proof. The operator $P_0^{1/2} R P_0^{1/2}$ has order zero and is bounded. Thus, for some $C > 0$,

$$P_0 + d - CP_0^{-1} \le P_0 + d + R \le P_0 + d + CP_0^{-1}$$

as forms. The functions $x + d \pm C/x$ are increasing for all sufficiently large x. Their values on the bounded initial part of the spectrum lie below every sufficiently large value. Consequently, for all large j, the jth ordered eigenvalues of the two bounding operators are exactly $\mu_j + d \pm C/\mu_j$. The variational inequalities give the result. □

20.2. **Scalar return and conjugation.** Operators in this subsection act on half-densities. The Hamiltonian flow of $p(x,\xi) = |\xi|_g$ is denoted by G_t. The scalar subprincipal symbol of $(\Delta_g + c)^{1/2}$ is zero. Normalize the Poisson bracket so that $H_p f = \{p, f\}$ and $\sigma(i[F,P]) = \{f, p\}$ when F has order zero and principal symbol f.

Lemma 20.6. *Let $A \in \Psi^2_{\mathrm{cl}}(M)$ be a strictly positive self-adjoint scalar operator with positive principal symbol a_2. Then $A^{1/2} \in \Psi^1_{\mathrm{cl}}(M)$, and*

$$\sigma_1(A^{1/2}) = \sqrt{a_2}, \qquad \operatorname{sub}(A^{1/2}) = \frac{\operatorname{sub}(A)}{2\sqrt{a_2}}.$$

In particular, if $B \in \Psi^1_{\mathrm{cl}}(M)$ is symmetric and $\Delta_g + B + c > 0$, then

$$(\Delta_g + B + c)^{1/2} - (\Delta_g + c)^{1/2} \in \Psi^0_{\mathrm{cl}}(M),$$

with principal symbol $\sigma_1(B)/(2|\xi|_g)$.

Proof. The complex-powers theorem [5, Theorem 7.9, arXiv version], applied to the classical calculus on the closed manifold M, gives $P = A^{1/2} \in \Psi^1_{\mathrm{cl}}$ and $\sigma_1(P) = \sqrt{a_2}$. The subprincipal product formula gives

$$\operatorname{sub}(A) = \operatorname{sub}(P^2) = 2\sqrt{a_2}\operatorname{sub}(P).$$

For $P_0 = (\Delta_g + c)^{1/2}$ on half-densities, $\operatorname{sub}(P_0) = 0$; see [17, p. 61]. Hence $\operatorname{sub}(\Delta_g + c) = 0$ and $\operatorname{sub}(\Delta_g + B + c) = \sigma_1(B)$. The two square roots have the same principal symbol $|\xi|_g$; the principal symbol of their difference is therefore $\sigma_1(B)/(2|\xi|_g)$. □

Lemma 20.7. *Let $\dim M \ge 2$, $P_0 = (\Delta_g + c)^{1/2}$, and let $P = P_0 + B_0$ be self-adjoint, with $B_0 \in \Psi^0_{\mathrm{cl}}(M)$. Write b for the principal symbol of B_0.*

(i) *If $e^{iTP} - \zeta I$ is compact, where $T > 0$ and $|\zeta| = 1$, then $G_T = I$, and*

$$\overline{b} := T^{-1} \int_0^T b \circ G_t \, dt$$

is a real constant.

(ii) *If $G_T = I$ and $\overline{b} = d$ is constant, there is a unitary $U \in \Psi^0_{\mathrm{cl}}(M)$ such that*

$$UPU^* = P_0 + d + R, \qquad R \in \Psi^{-1}_{\mathrm{cl}}(M).$$

Proof. Put $P' = TP/(2\pi)$ and choose $\beta \in \mathbb{R}$ such that $e^{2\pi i\beta} = \zeta$. If λ'_j are the eigenvalues of P', compactness gives

$$\operatorname{dist}(\lambda'_j, \beta + \mathbb{Z}) \longrightarrow 0.$$

Pair each λ'_j with a nearest point of $\beta + \mathbb{Z}$, retaining one copy for each index. Choose J_ν so that the pairing errors are less than $\min(1/\nu, 1/4)$ for $j \ge J_\nu$, and choose cutoffs

$$R_\nu \in \beta + \tfrac{1}{2} + \mathbb{Z}, \qquad R_\nu \longrightarrow \infty, \qquad J_\nu = o(R_\nu^n), \qquad n = \dim M.$$

No pair with $j \ge J_\nu$ straddles a cutoff. Thus at most $2J_\nu = o(R_\nu^n)$ copies remain unmatched. Theorem 4.1, applied to P' and $\mathcal{S} = \{\zeta\}$, gives

$$G_T = I, \qquad e^{iTP} \in \Psi^0_{\mathrm{cl}}(M), \qquad \sigma(e^{iTP}) = \zeta.$$

The principal amplitude at a full return is

$$\sigma(e^{iTP})(\rho) = \rho_T \exp\left(i\int_0^T b(G_t\rho)\,dt\right), \qquad \rho_T \in \{1,-1,i,-i\}. \tag{20.5}$$

For the zero-subprincipal operator P_0, the full-return formula gives $e^{iTP_0} = \rho_T I \pmod{\Psi_{\mathrm{cl}}^{-1}}$; see [12, Theorem 1.1 and its proof], applied to $TP_0/(2\pi)$. The Maslov factor is locally constant and hence constant on the connected cosphere bundle. For the remaining factor put $W(t) = e^{-itP_0}e^{itP}$. Then

$$W'(t) = iB(t)W(t), \quad W(0) = I, \qquad B(t) = e^{-itP_0}B_0e^{itP_0}.$$

Egorov's theorem [65, Theorem I.6.1] gives $B(t) \in \Psi_{\mathrm{cl}}^0$ with principal symbol $b \circ G_{-t}$. The composition $W(t)$ has the identity canonical transformation and belongs to Ψ_{cl}^0. Its principal symbol satisfies $\partial_t w = i(b \circ G_{-t})w$, $w(0) = 1$. Hence

$$\sigma(W(T)) = \exp\left(i\int_0^T b \circ G_{-t}\,dt\right).$$

Because $G_T = I$, the integral equals $\int_0^T b \circ G_t\,dt$. Multiplication by the return symbol of P_0 proves (20.5). The real function $\int_0^T b \circ G_t\,dt$ is continuous. Its exponential is the fixed number ζ/ρ_T. Its values therefore lie in one discrete translate of $2\pi\mathbb{Z}$, and connectedness makes it constant. This proves (i).

For (ii), put

$$f = \frac{1}{T}\int_0^T t\,(b-d) \circ G_t\,dt.$$

Differentiating along the flow and integrating by parts gives

$$H_p f = \frac{1}{T}\left[t(b-d) \circ G_t\right]_0^T - \frac{1}{T}\int_0^T (b-d) \circ G_t\,dt = b - d.$$

Choose a self-adjoint $F \in \Psi_{\mathrm{cl}}^0$ with principal symbol f. Let γ be the positively oriented circle $|z| = A$, where $A > \max(\|F\|, \|f\|_\infty)$. For $z \in \gamma$, the operator $z - F$ is elliptic and invertible. Spectral invariance and continuity of inversion [5, Theorem 5.4 and proof of Theorem 7.6, arXiv version], together with the contour formula

$$U_s = e^{isF} = \frac{1}{2\pi i}\int_\gamma e^{isz}(z-F)^{-1}\,dz \qquad (0 \le s \le 1)$$

give $U_s \in \Psi_{\mathrm{cl}}^0$, with principal symbol e^{isf}. Each U_s is unitary. Put $U = U_1$. The commutator formula gives

$$\sigma(i[F,P]) = \{f,p\} = -(b-d), \qquad [F,[F,P]] \in \Psi_{\mathrm{cl}}^{-1}.$$

The identity

$$UPU^* = P + i[F,P] - \int_0^1 (1-s)U_s[F,[F,P]]U_s^*\,ds$$

therefore gives $UPU^* = P_0 + d \pmod{\Psi_{\mathrm{cl}}^{-1}}$. The unitary U and its inverse preserve H^1. Thus the conjugated operator has domain H^1, and R is bounded and self-adjoint. □

20.3. **The spherical reference.** Let $n = 2m+1 \ge 5$. Let Q be either a homogeneous unit spherical space form of even group order, or the scalar lens space

$$Q = \mathbb{S}^{2m+1}(1)/\langle e^{2\pi i/q}I\rangle, \qquad q \ge 5 \text{ odd}.$$

Put $T = \pi$ in the first case and $T = 2\pi$ in the second. In both cases $G_T^Q = I$: in the even-order case, the deck group contains $-I$ by Lemma 8.2. These references satisfy the ordered scalar isolation statement in Corollary 2.5: if

$$\sqrt{\lambda_j(M)} - \sqrt{\lambda_j(Q)} \longrightarrow 0,$$

then M is isometric to Q.

Theorem 20.8 (Perturbations of order one). *Let $B_M \in \Psi^1_{\mathrm{cl}}(M)$ and $B_Q \in \Psi^1_{\mathrm{cl}}(Q)$ be symmetric scalar operators. Let b_Q be the degree-zero symbol*

$$b_Q(x,\xi) = \frac{\sigma_1(B_Q)(x,\xi)}{2|\xi|_Q}.$$

Suppose that

$$\frac{1}{T}\int_0^T b_Q(G_t^Q \rho)\, dt \tag{20.6}$$

is independent of $\rho \in S^*Q$. Then

$$\mathrm{Spec}(\Delta_M + B_M) = \mathrm{Spec}(\Delta_Q + B_Q) \quad \Longrightarrow \quad M \cong Q.$$

In particular, the conclusion holds for arbitrary symmetric $B_M \in \Psi^1_{\mathrm{cl}}(M)$ when $B_Q = 0$.

Proof. For $Z = M, Q$, the operator $(1+\Delta_Z)^{-1/4} B_Z (1+\Delta_Z)^{-1/4}$ has order zero and is bounded. Hence B_Z satisfies the form bound in Theorem 20.1. The form sums are semibounded and self-adjoint, with compact resolvent. Their equal spectra and Theorem 20.1 give $\dim M = \dim Q = n$ and $\mathrm{Vol}\, M = \mathrm{Vol}\, Q$. Choose $c > 0$ so large that both $\Delta_Z + B_Z + c$ are positive. For $Z = M, Q$, put

$$P_Z = (\Delta_Z + B_Z + c)^{1/2}, \qquad P_Z^0 = (\Delta_Z + c)^{1/2}.$$

Lemma 20.6 gives $P_Z - P_Z^0 \in \Psi^0_{\mathrm{cl}}$ with principal symbol $b_Z = \sigma_1(B_Z)/(2|\xi|_Z)$. Let d_Q denote (20.6). Lemma 20.7(ii) gives

$$U_Q P_Q U_Q^* = P_Q^0 + d_Q + R_Q, \qquad R_Q \in \Psi^{-1}_{\mathrm{cl}}.$$

The reference scalar frequencies satisfy

$$\sqrt{\lambda_j(Q) + m^2} \in \begin{cases} m + 2\mathbb{Z}_{\geq 0}, & T = \pi, \\ m + \mathbb{Z}_{\geq 0}, & T = 2\pi. \end{cases}$$

Write μ_j^Z for the eigenvalues of P_Z^0 and η_j for the common eigenvalues of P_M, P_Q. Lemma 20.5 gives

$$\eta_j = \mu_j^Q + d_Q + O((\mu_j^Q)^{-1}).$$

Moreover,

$$\mu_j^Q - \sqrt{\lambda_j(Q) + m^2} = \frac{c - m^2}{\mu_j^Q + \sqrt{\lambda_j(Q) + m^2}} = O((\mu_j^Q)^{-1}).$$

The displayed lattice therefore implies $e^{iT\eta_j} \to e^{iT(m+d_Q)}$. In an eigenbasis of either P_Z, the diagonal entries of $e^{iTP_Z} - e^{iT(m+d_Q)} I$ tend to zero. Both operators are compact. Lemma 20.7 gives a constant d_M and a unitary U_M such that

$$U_M P_M U_M^* = P_M^0 + d_M + R_M, \qquad R_M \in \Psi^{-1}_{\mathrm{cl}}.$$

The common leading Weyl term gives $\mu_j^M \sim \mu_j^Q$. Lemma 20.5 and equality of the spectra give

$$\mu_j^Q = \mu_j^M + d_M - d_Q + O((\mu_j^M)^{-1}).$$

Lemma 20.4 gives $d_M = d_Q$. Hence $\mu_j^M - \mu_j^Q \to 0$. Since

$$\sqrt{\lambda + c} - \sqrt{\lambda} = \frac{c}{\sqrt{\lambda + c} + \sqrt{\lambda}} \longrightarrow 0,$$

the unshifted scalar frequencies also have difference tending to zero. Ordered scalar isolation gives $M \cong Q$. □

Remark 20.9. In Example 20.2, the diagonal operator B cannot belong to $\Psi^1_{\mathrm{cl}}(M)$. If it did, the equality of its perturbed spectrum with that of $\mathbb{RP}^n(1)$ would give

$$e^{i\pi(\Delta_M + B + (n-1)^2/4)^{1/2}} = e^{i\pi(n-1)/2} I.$$

Lemma 20.7(i) would force $G_\pi^M = I$. On $\mathbb{S}^n(a)$ the prime geodesic period is $2\pi a$, whereas $0 < \pi/(2\pi a) = 2^{1/n-1} < 1$. This is impossible.

21. First-order differential perturbations on different manifolds

All operators act on complex-valued functions. The Laplacian is nonnegative. If V is a Killing field, put $D_V = -i\mathcal{L}_V$.

Theorem 21.1. *Let $m, p \geq 1$ be integers. Put*

$$Q_p = L(p; 1, \dots, 1) = \mathbb{S}^{2m+1}(1)/\langle e^{2\pi i/p} I \rangle, \qquad M_p = \left(\prod_{j=1}^{m} \mathbb{S}^2\left(\frac{1}{2\sqrt{j}}\right)\right) \times \mathbb{S}^1\left(\frac{1}{p}\right).$$

Let K be the vector field on Q_p induced by

$$(z_0, \dots, z_m) \longmapsto (e^{imt} z_0, e^{i(m-2)t} z_1, \dots, e^{-imt} z_m) \quad \text{on } \mathbb{S}^{2m+1} \subset \mathbb{C}^{m+1}.$$

For $1 \leq j \leq m$, let V_j be the rotation field of angular speed one about the third coordinate axis of the jth sphere factor of M_p. Set

$$B_Q = 2D_K, \qquad B_M = 4\sum_{j=1}^{m} jD_{V_j}.$$

These are symmetric scalar differential operators of order one. The operators $\Delta_{Q_p} + B_Q$ and $\Delta_{M_p} + B_M$ are nonnegative, self-adjoint on H^2, and have the same spectrum, with multiplicities. More precisely, their common spectrum is

$$\coprod_{\substack{a\in\mathbb{Z}\\ n_1,\dots,n_m\geq 0}} \left\{ p^2a^2 + 4\sum_{j=1}^{m} jn_j \right\}, \tag{21.1}$$

with one copy for each displayed tuple. Thus, for every $t > 0$,

$$\operatorname{Tr} e^{-t(\Delta_{Q_p}+B_Q)} = \operatorname{Tr} e^{-t(\Delta_{M_p}+B_M)} = \frac{\displaystyle\sum_{a\in\mathbb{Z}} e^{-tp^2a^2}}{\displaystyle\prod_{j=1}^{m}(1-e^{-4jt})}. \tag{21.2}$$

Nevertheless,

$$\pi_1(Q_p) = \mathbb{Z}/p\mathbb{Z}, \qquad \pi_1(M_p) = \mathbb{Z}.$$

In particular, M_p and Q_p are not isometric. Both have volume $2\pi^{m+1}/(p\,m!)$.

Proof. Harmonic decompositions.

For $d \geq 3$, let $\mathcal{P}_k(\mathbb{R}^d)$ be the complex homogeneous polynomials of degree k, and let $\mathcal{H}_k(\mathbb{R}^d)$ be their harmonic subspace. The Fischer decomposition

$$\mathcal{P}_k = \mathcal{H}_k \oplus |x|^2 \mathcal{P}_{k-2}$$

is $O(d)$-equivariant [15, Theorems 1.3 and 1.7]. The restrictions of the spaces $\mathcal{H}_k$ give the complete orthogonal decomposition into scalar eigenspaces on $\mathbb{S}^{d-1}(1)$, with eigenvalues $k(k+d-2)$ [39, Lemma 1.3 and Proposition 1.5, pp. 78–79].

On $\mathbb{C}^{m+1}$, let $\mathcal{P}_{r,s}$ be the polynomials of holomorphic degree r and antiholomorphic degree s, and let $\mathcal{H}_{r,s}$ be their harmonic subspace. Put $\mathcal{P}_{r,s} = 0$ when either index is negative. In total degree $r+s$, Hopf weight $r-s$ is precisely bidegree (r, s). Restricting the Fischer decomposition to this weight gives

$$\mathcal{P}_{r,s} = \mathcal{H}_{r,s} \oplus |z|^2 \mathcal{P}_{r-1,s-1}. \tag{21.3}$$

The summands are invariant under the coordinate torus. The restrictions of $\mathcal{H}_{r,s}$ have Laplace eigenvalue $(r+s)(r+s+2m)$ and Hopf weight $r-s$. Consequently, the spaces with $r - s \in p\mathbb{Z}$ give the complete orthogonal decomposition on Q_p.

The product.

On $\mathbb{S}^2(1)$, put $D = -i\partial_\phi$ in spherical coordinates. The standard orthogonal basis $Y_{k,a}(\theta,\phi)$, $-k \le a \le k$, of $\mathcal{H}_k$ satisfies

$$\Delta Y_{k,a} = k(k+1)Y_{k,a}, \qquad DY_{k,a} = aY_{k,a};$$

see [15, Theorem 4.5 and equation (6.6)]. Thus

$$\operatorname{Spec}(\Delta_{\mathbb{S}^2(1)} + D) = \coprod_{k\ge 0}\{k(k+1) + a \,:\, -k \le a \le k\} = \{0,1,2,\ldots\},$$

with multiplicity one: the kth interval consists of the integers from k^2 through $(k+1)^2 - 1$.

In the product coordinates of the theorem,

$$\Delta_{M_p} + B_M = 4\sum_{j=1}^{m} j\big(\Delta^{(j)}_{\mathbb{S}^2(1)} + D_{V_j}\big) + p^2(-\partial_\theta^2).$$

The product eigenfunctions therefore have eigenvalues

$$p^2r^2 + 4\sum_{j=1}^{m} jn_j, \qquad r \in \mathbb{Z}, \quad n_1,\ldots,n_m \in \mathbb{Z}_{\ge 0},$$

with one copy for each displayed tuple. For $0 < q < 1$, this gives

$$\operatorname{Tr} q^{(\Delta_{M_p}+B_M)/4} = \frac{\sum_{r\in\mathbb{Z}} q^{p^2r^2/4}}{\prod_{j=1}^{m}(1-q^j)}. \tag{21.4}$$

The joint character on the sphere.

Put $w_j = m - 2j$ for $0 \le j \le m$. The coordinates $z_j, \overline{z}_j$ have D_K-weights $w_j, -w_j$. The multiset of the numbers $w_j/2$ is $\{-m/2, -m/2+1, \ldots, m/2\}$, as is the multiset of their negatives. Thus

$$\operatorname{Tr}_{\mathcal{P}_{r,s}} q^{D_K/2} = q^{-m(r+s)/2}[u^r v^s]\prod_{j=0}^{m}(1-q^j u)^{-1}(1-q^j v)^{-1}.$$

The polynomial $|z|^2$ has bidegree $(1,1)$ and D_K-weight zero. Equation (21.3) gives

$$\operatorname{Tr}_{\mathcal{H}_{r,s}} q^{D_K/2} = q^{-m(r+s)/2}[u^r v^s]F_m(u,v), \tag{21.5}$$

where

$$F_m(u,v) = \frac{1-q^m uv}{\prod_{j=0}^{m}(1-q^j u)(1-q^j v)}.$$

On $\mathcal{H}_{r,s}$, $\Delta/4$ has eigenvalue $k(k+2m)/4$, where $k = r+s$. Hence

$$\operatorname{Tr}_c q^{(\Delta+2D_K)/4} = \sum_{\substack{r,s\ge 0\\ r-s=c}} q^{(r+s)^2/4}[u^r v^s]F_m(u,v). \tag{21.6}$$

The subscript c denotes the Hopf weight-c subspace. On total degree k, every D_K-weight has absolute value at most mk. Thus every eigenvalue of $\Delta + 2D_K$ is at least k^2, and all the displayed traces converge. For a monomial $z^\alpha \overline{z}^\beta$ of bidegree (r,s), put $c = r - s$ and $d = \sum_{j=0}^{m} w_j(\alpha_j - \beta_j)$. Then

$$\frac{k(k+2m) + 2d - c^2}{4} = rs + \sum_{j=0}^{m}(m-j)\alpha_j + \sum_{j=0}^{m} j\beta_j \in \mathbb{Z}_{\ge 0}.$$

Every weight of the harmonic subspace is a polynomial weight. Thus each fixed-charge trace, after division by $q^{c^2/4}$, is a power series with nonnegative integral exponents.

A power-series identity.

Put

$$(a;q)_j=\prod_{h=0}^{j-1}(1-aq^h),\qquad (a;q)_\infty=\prod_{h=0}^{\infty}(1-aq^h),$$

with $(a;q)_0=1$. The q-binomial theorem [10, p. 4, following equation (3.5)] gives

$$A_j:=[u^j]\prod_{h=0}^{m}(1-q^hu)^{-1}=\frac{(q^{m+1};q)_j}{(q;q)_j}\quad(j\ge 0);\qquad A_{-1}=0.$$

Write

$$Z_{m,c}=\sum_{\substack{r,s\ge 0\\ r-s=c}} q^{(r+s)^2/4}[u^rv^s]F_m(u,v).$$

Since $F_m(u,v)=F_m(v,u)$, it suffices to take $c\ge 0$. For $j\ge 0$, subtraction of the two coefficients gives

$$\begin{aligned}[u^{j+c}v^j]F_m&=A_{j+c}A_j-q^mA_{j+c-1}A_{j-1}\\&=C_{m,c}\,\frac{(q^{m+c};q)_j(q^m;q)_j}{(q^{c+1};q)_j(q;q)_j}(1-q^{m+c+2j}),\\C_{m,c}&=\frac{(q;q)_{m+c-1}}{(q;q)_m(q;q)_c}.\end{aligned}$$

Apply the Rogers–Fine identity [10, Corollary 1, equation (2.6), p. 3] with $a=q^{m+c}$ and $b=q^{c+1}$. Its hypotheses hold since $0<q<1$ and $b\ne q^{-h}$ for every $h\ge 0$. As $b^jq^{j^2-j}=q^{j^2+cj}$, it gives

$$Z_{m,c}=q^{c^2/4}C_{m,c}\frac{(q^{m+c};q)_\infty}{(q^{c+1};q)_\infty}=\frac{q^{c^2/4}}{\prod_{j=1}^{m}(1-q^j)}.\tag{21.7}$$

Equations (21.6) and (21.7) show that the spectrum of $\Delta+2D_K$ on the Hopf weight-c subspace is

$$\coprod_{n_1,\dots,n_m\ge 0}\left\{c^2+4\sum_{j=1}^{m}jn_j\right\}.$$

The equality of multiplicities follows by comparing the coefficients of the convergent power series after division by $q^{c^2/4}$. Retaining $c=pa$ for $a\in\mathbb{Z}$ proves (21.1) for Q_p, and hence equality of all eigenvalues and multiplicities in the theorem.

Domains and geometry.

Each displayed vector field is Killing, so its flow preserves Riemannian measure and D_V is symmetric on smooth functions. For $Z=Q_p,M_p$, the first-order differential operator B_Z satisfies, for every $\varepsilon>0$,

$$\|B_Zu\|_{L^2}\le C_Z\|u\|_{H^1}\le\varepsilon\|\Delta_Zu\|_{L^2}+C_{Z,\varepsilon}\|u\|_{L^2},\qquad u\in H^2(Z).$$

Thus B_Z has relative operator bound zero, and [43, Chapter V, Theorem 4.3, p. 287] gives self-adjointness of Δ_Z+B_Z on $H^2(Z)$. The same estimate makes its graph norm equivalent to the H^2 norm; hence it is the closure of its restriction to $C^\infty(Z)$.

The isometric flows commute with the corresponding Laplacian. Their commuting infinitesimal generators are Hermitian on each Laplace eigenspace, so the simultaneous eigenfunctions above form a complete orthogonal basis. The displayed eigenvalues are nonnegative. Every bounded interval contains finitely many tuples in (21.1), and summing their exponentials gives (21.2).

The torus action defining K commutes with scalar multiplication, so K descends to every Q_p. All the sphere factors of M_p are simply connected, while its circle factor has fundamental group $\mathbb{Z}$. The universal cover of Q_p is $\mathbb{S}^{2m+1}$, with deck group $\mathbb{Z}/p\mathbb{Z}$. These fundamental groups differ.

Finally,

$$\mathrm{Vol}(Q_p) = \frac{2\pi^{m+1}}{p\,m!}, \qquad \mathrm{Vol}(M_p) = \left(\prod_{j=1}^{m} \frac{\pi}{j}\right)\frac{2\pi}{p} = \frac{2\pi^{m+1}}{p\,m!}.$$

□

Remark 21.2. For $m \geq 2$, the theorem applies in every odd dimension at least five. The case $p = 2$ gives a homogeneous even-order spherical reference; every odd $p \geq 5$ gives precisely a scalar odd-order lens reference. Theorem 21.1 therefore excludes a two-sided extension of Theorem 20.8 without its orbit-average hypothesis. The perturbations are scalar and self-adjoint on complex L^2; they do not commute with complex conjugation.

The square-root subprincipal symbol for the reference perturbation is

$$b_Q(x,\xi) = \frac{\langle K(x),\xi\rangle}{|\xi|}.$$

For a unit-speed geodesic γ,

$$\frac{d}{dt}\langle K(\gamma(t)),\dot\gamma(t)\rangle = \langle \nabla_{\dot\gamma} K,\dot\gamma\rangle = \tfrac{1}{2}(\mathcal{L}_K g)(\dot\gamma,\dot\gamma) = 0.$$

Thus b_Q is constant along each geodesic. It is not constant on the unit cosphere bundle: on the two orientations of the first coordinate circle it equals m and $-m$. Thus these examples do not satisfy the constant orbit-average hypothesis.

22. Sharpness of the multiplicity exponent

Lemma 22.1. *Let $q \geq 1$ and $d \geq 2$ be integers, and let $F(z) = \sum_{k\geq 0} c_k z^k$ be a rational function analytic at zero, with real Taylor coefficients. Suppose all its poles are qth roots of unity, their orders are at most d, and at least one has order d. Then there are real numbers $b_0,\dots,b_{q-1}$, not all zero, such that*

$$c_k = b_v k^{d-1} + O(k^{d-2}) \qquad (k \equiv v \pmod q).$$

Consequently

$$\sum_{0\leq k\leq H} |c_k| = \frac{1}{qd}\left(\sum_{v=0}^{q-1}|b_v|\right)H^d + O(H^{d-1}).$$

Proof. Subtracting the polynomial part of F changes only finitely many coefficients. By [60, Proposition 4.4.1], its remaining coefficients have the form

$$c_k = \sum_{\xi^q=1} Q_\xi(k)\xi^{-k}, \qquad \deg Q_\xi \leq d-1,$$

for all sufficiently large k, and at least one Q_ξ has degree $d-1$. Put

$$a_\xi = [X^{d-1}]Q_\xi(X), \qquad b_v = \sum_{\xi^q=1} a_\xi \xi^{-v}.$$

Then $c_k = b_v k^{d-1} + O(k^{d-2})$ for $k \equiv v \pmod q$. The coefficient b_v is real, being the limit of c_k/k^{d-1} on that residue class. Fourier inversion gives

$$a_\xi = \frac{1}{q}\sum_{v=0}^{q-1} b_v \xi^v,$$

so the b_v are not all zero. Moreover,

$$|c_k| = |b_v|k^{d-1} + O(k^{d-2}) \qquad (k \equiv v \pmod q),$$

and

$$\sum_{\substack{0\leq k\leq H\\ k\equiv v \pmod q}} k^{d-1} = \frac{H^d}{qd} + O(H^{d-1}).$$

Summing over v proves the second formula; the accumulated error is $O(H^{d-1})$. □

22.1. **Even-order lens spaces.** For a cyclic space group of order at least three, let $\rho_1, \dots, \rho_k$ be the complex multiplicities of the distinct conjugate eigenvalue pairs of a generator. The invariant harmonic degree-two multiplicity is $\sum_{\nu=1}^{k} \rho_\nu^2 - 1$ [56, Corollary 4, p. 308].

Proof of Theorem 2.7. Put $n = 2s - 1$ and $\zeta = e^{\pi i/4}$. The two generating matrices are

$$\operatorname{diag}(\zeta, \dots, \zeta), \qquad \operatorname{diag}(\zeta, \dots, \zeta, \zeta^3).$$

Every weight is a unit modulo eight, so no nonidentity power has eigenvalue one. Thus both actions are free. Their fourth powers are $-I$; hence, by Lemma 8.1, only even harmonic degrees occur, and both shifted spectra lie in $s - 1 + 2\mathbb{Z}_{\geq 0}$. Since $\eta < 1/2$, a matching pairs only equal frequencies.

The conjugate-eigenvalue multiplicity lists are (s) and $(s-1, 1)$. The degree-two formula above gives $s^2 - 1$ and $(s-1)^2$, respectively. Their difference is $2(s-1) > 0$, so the quotients are not isometric. The scalar action commutes with the transitive group $U(s)$, which descends to Q. Thus Q is homogeneous.

For the group powers with exponents $0, 2, 4, 6$, the determinant polynomials agree in the two representations. Put

$$a(z) = 1 - \sqrt{2}z + z^2, \qquad b(z) = 1 + \sqrt{2}z + z^2, \qquad a(z)b(z) = 1 + z^4.$$

The remaining exponents $1, 3, 5, 7$ occur in conjugate pairs. Subtracting their four terms in (8.2) gives

$$F_Q - F_{Q'} = \frac{1 - z^2}{4} \left(a^{-s} + b^{-s} - \frac{1}{a^{s-1} b} - \frac{1}{b^{s-1} a} \right). \tag{22.1}$$

At a root of a, the factor b and the numerator $1 - z^2$ are nonzero. The pole from a^{-s} has order s, while every other term has order at most $s - 1$. The same holds at roots of b. These are precisely the primitive eighth roots.

Let $c_k = [z^k](F_Q - F_{Q'})$. By Lemma 22.1,

$$\sum_{k \leq H} |c_k| = C_s H^s + O(H^{s-1}), \qquad C_s = \frac{1}{8s} \sum_{v=0}^{7} |b_v| > 0,$$

where b_v is the leading coefficient on the residue class v modulo eight. At $R_K = 2K + s$, the possible harmonic degrees are even and at most $2K$. At each such degree the largest matching has the smaller of the two multiplicities. Hence

$$\mathcal{E}_\eta(K) = \sum_{k=0}^{2K} |c_k|.$$

Replacing $H = 2K$ by R_K changes only the order-R_K^{s-1} term. This gives (2.13).

For $s = 4$, multiply (22.1) by $(ab)^4$. The numerator before its factor $(1 - z^2)/4$ is

$$a^4 + b^4 - ab(a^2 + b^2) = (a - b)^2 (a^2 + ab + b^2) = 8z^2 (3 + 8z^2 + 3z^4).$$

Therefore

$$F_Q - F_{Q'} = \frac{6z^2 + 10z^4 - 10z^6 - 6z^8}{(1 + z^4)^4}. \tag{22.2}$$

Since $(1 + z^4)^{-4} = \sum_{j \geq 0} (-1)^j \binom{j+3}{3} z^{4j}$, odd coefficients vanish. For $j \geq 1$,

$$c_{4j} = (-1)^{j-1} \left(10 \binom{j+2}{3} + 6 \binom{j+1}{3} \right),$$

and for $j \geq 0$,

$$c_{4j+2} = (-1)^j \left(6 \binom{j+3}{3} + 10 \binom{j+2}{3} \right).$$

Consequently

$$|c_k| = \begin{cases} k(k+4)(2k+7)/48, & k \equiv 0 \pmod 4, \\ (k+2)(k+6)(2k+5)/48, & k \equiv 2 \pmod 4, \end{cases}$$

with $c_0 = 0$. Thus $|c_k| = k^3/24 + O(k^2)$ on even indices, and

$$\sum_{\substack{k \le H \\ k \text{ even}}} \frac{k^3}{24} = \frac{1}{24}\, 8 \sum_{j \le H/2} j^3 = \frac{H^4}{192} + O(H^3).$$

This proves $C_4 = 1/192$. □

22.2. **Sharpness for odd-order scalar lens spaces.**

Proposition 22.2. *Let $q \ge 5$ be odd and let $n = 2s - 1$ with $s \ge 3$. The unit round lens spaces*

$$Q = L(q; 1, \dots, 1), \qquad Q' = L(q; 1, \dots, 1, 2)$$

are not isometric. For every $0 \le \eta < 1/4$, their unmatched multiplicity satisfies

$$\mathcal{E}_\eta(K) = C_{q,s} R_K^s + O(R_K^{s-1}), \qquad C_{q,s} > 0. \tag{22.3}$$

For $q = 5$ and $s = 3$, one has $C_{5,3} = 4/75$.

Proof. Both actions are free because 1 and 2 are units modulo q. Since $2 \not\equiv \pm 1 \pmod q$, their conjugate-eigenvalue multiplicity lists are (s) and $(s-1, 1)$. The degree-two formula above gives a multiplicity difference $2(s-1) > 0$. Hence the quotients are not isometric.

Write

$$D(z) = F_Q(z) - F_{Q'}(z) = \sum_{k \ge 0} c_k z^k.$$

The identity terms in the Molien formula cancel. Every other pole is a nonidentity qth root of unity and has order at most s. At a primitive qth root ξ, exactly two terms of the scalar Molien sum have a pole. Their common denominator is $(1 - z/\xi)^s (1 - \xi z)^s$, and their sum has coefficient

$$\frac{2}{q}(1 - \xi^2)^{1-s} \ne 0$$

in front of $(1 - z/\xi)^{-s}$. For a term of the mixed action to have an order-s pole there, its weight-one and weight-two blocks would both have to contain ξ^{-1} as an eigenvalue. Writing the corresponding group element as a power with exponent j gives $\zeta^{\pm j} = \xi^{-1} = \zeta^{\pm 2j}$, where $\zeta = e^{2\pi i/q}$. Since ξ is primitive, j is a unit modulo q; hence this equality implies $2 \equiv \pm 1 \pmod q$. Neither congruence holds for odd $q \ge 5$. The mixed sum therefore has pole order at most $s - 1$ at ξ, so D has a pole of order s at ξ.

By Lemma 22.1,

$$\sum_{k \le H} |c_k| = C_{q,s} H^s + O(H^{s-1}), \qquad C_{q,s} = \frac{1}{qs} \sum_{v=0}^{q-1} |b_v| > 0,$$

where b_v is the leading coefficient on the residue class v modulo q. Both shifted spectra have frequency $k + s - 1$ in harmonic degree k. Since $\eta < 1/4$, a matching pairs only equal frequencies. Here (2.4) gives $R_K = s - 1 + K + 1/2$. Matching the smaller multiplicity at each frequency therefore gives

$$\mathcal{E}_\eta(K) = \sum_{k=0}^{K} |c_k|.$$

Since $K = R_K + O(1)$, the preceding asymptotic formula proves (22.3).

For $q = 5$ and $s = 3$, put

$$a(z) = 1 - \frac{\sqrt{5}-1}{2}z + z^2, \qquad b(z) = 1 + \frac{\sqrt{5}+1}{2}z + z^2.$$

The two nonidentity conjugate pairs give

$$\begin{aligned} D(z) &= \frac{2(1-z^2)}{5}\left(\frac{1}{a^3} + \frac{1}{b^3} - \frac{1}{a^2 b} - \frac{1}{ab^2}\right) \\ &= \frac{2(1-z^2)(a+b)(a-b)^2}{5a^3b^3}. \end{aligned}$$

Since

$$ab = 1 + z + z^2 + z^3 + z^4, \qquad a + b = 2 + z + 2z^2, \qquad (a-b)^2 = 5z^2,$$

one obtains the exact identity

$$D(z) = \frac{4z^2 - 10z^3 + 6z^4 + 6z^7 - 10z^8 + 4z^9}{(1-z^5)^3}. \tag{22.4}$$

The expansion $(1-z^5)^{-3} = \sum_{j\ge 0}\binom{j+2}{2}z^{5j}$ gives $c_{5j} = c_{5j+1} = 0$ and

$$\begin{aligned} c_{5j+2} &= 4\binom{j+2}{2} + 6\binom{j+1}{2}, \\ c_{5j+3} &= -10\binom{j+2}{2} - 10\binom{j+1}{2}, \\ c_{5j+4} &= 6\binom{j+2}{2} + 4\binom{j+1}{2} \qquad (j \ge 0). \end{aligned}$$

The coefficients of k^2 on the five residue classes are therefore

$$(b_0, b_1, b_2, b_3, b_4) = \left(0, 0, \frac{1}{5}, -\frac{2}{5}, \frac{1}{5}\right),$$

and hence $C_{5,3} = (4/5)/15 = 4/75$. □

Proposition 22.2 shows that the exponent $(n+1)/2$ in Theorem 2.4 cannot be increased. Its positive-threshold hypothesis cannot be replaced by $\mathcal{E}_\eta(K) = O(R_K^{(n+1)/2})$ with unrestricted constant.

23. Five-dimensional lens spaces at shrinking resolution

Lemma 23.1. *Let Y be a closed Riemannian 5-manifold, put $P_Y = (\Delta_Y + 4)^{1/2}$, and normalize its heat coefficients by*

$$\operatorname{Tr}(e^{-t\Delta_Y}) \sim (4\pi t)^{-5/2}\sum_{j\ge 0} a_j(Y)\,t^j.$$

For $\chi \in C_c^\infty((0,\infty))$,

$$\begin{aligned} \operatorname{Tr}\chi(P_Y/R) = {} & \frac{a_0(Y)}{12\pi^3}R^5\int_0^\infty \chi(u)u^4\,du \\ & + \frac{a_1(Y) - 4a_0(Y)}{8\pi^3}R^3\int_0^\infty \chi(u)u^2\,du \\ & + \frac{a_2(Y) - 4a_1(Y) + 8a_0(Y)}{16\pi^3}R\int_0^\infty \chi(u)\,du + O_\chi(R^{-1}). \end{aligned} \tag{23.1}$$

Moreover, with Riemannian volume densities,

$$\begin{aligned} a_0(Y) &= \operatorname{Vol}(Y), \qquad a_1(Y) = \frac{1}{6}\int_Y \operatorname{Scal}, \\ 360\,a_2(Y) &= \int_Y \left(2|\operatorname{Rm}|^2 - 2|\operatorname{Ric}|^2 + 5\operatorname{Scal}^2\right). \end{aligned} \tag{23.2}$$

Proof. Put $L = \Delta_Y + 4$, $h = R^{-1}$, and $f(s) = \chi(\sqrt{s})$ for $s > 0$, extended by zero. Then $f \in C_c^\infty(\mathbb{R})$. The integrated functional calculus gives

$$\operatorname{Tr} f(h^2 L) \sim \sum_{j \ge 0} h^{-5+2j} C_j(f),$$

where the C_j are distributions supported in $[0, \infty)$; the odd terms vanish by the parity of the Laplace symbol [13, Section 5.4 and Remark 2, p. 2445]. For $\tau > 0$, comparison with the same expansion at $\sqrt{\tau} h$ gives

$$C_j(f(\tau\,\cdot)) = \tau^{-5/2+j} C_j(f).$$

For $j = 0, 1, 2$, these homogeneities give $C_j = \gamma_j s_+^{3/2-j}\, ds$; no distribution supported at zero has the required homogeneity. The heat specialization of the same functional calculus [13, Remark 2, p. 2445] gives

$$\gamma_j \Gamma(5/2 - j) = (4\pi)^{-5/2} \widetilde{a}_j(Y), \qquad j = 0, 1, 2.$$

Here $e^{-tL} = e^{-4t} e^{-t\Delta_Y}$, so

$$\widetilde{a}_0 = a_0, \qquad \widetilde{a}_1 = a_1 - 4a_0, \qquad \widetilde{a}_2 = a_2 - 4a_1 + 8a_0.$$

Substitution $s = u^2$ and

$$\Gamma(5/2) = \frac{3\sqrt{\pi}}{4}, \qquad \Gamma(3/2) = \frac{\sqrt{\pi}}{2}, \qquad \Gamma(1/2) = \sqrt{\pi}$$

give (23.1), with the next term of order $h = R^{-1}$. Formulas (23.2) are the scalar heat-kernel coefficients, with the integral of the scalar Laplacian term equal to zero on a closed manifold [23, Theorem 4.8.18(d)]. □

Theorem 23.2 (Prime two-weight lens spaces in dimension five). *Let $p \ge 5$ be prime and $a \not\equiv 0, 1, -1 \pmod p$. Let $Q = L(p; 1, 1, a)$ with sectional curvature one, and let M be a smooth closed connected Riemannian manifold. Put*

$$P_M = (\Delta_M + 4)^{1/2}, \qquad P_Q = (\Delta_Q + 4)^{1/2}.$$

If $R_\nu \to \infty$ and $\eta_\nu \ge 0$ satisfy

$$\eta_\nu = O(R_\nu^{-3/5}), \qquad E_{\eta_\nu}(R_\nu; P_M, P_Q) = o(R_\nu), \tag{23.3}$$

then M is isometric to Q.

Proof. Write $E_\nu = E_{\eta_\nu}(R_\nu; P_M, P_Q)$. The inequality $|N_{P_M}(R_\nu) - N_{P_Q}(R_\nu)| \le E_\nu$ and Weyl's law give

$$\dim M = 5, \qquad \operatorname{Vol}(M) = V_5/p. \tag{23.4}$$

By Lemma 8.1, the frequency of P_Q in harmonic degree k is $k + 2$. Theorem 4.1, with one phase and period 2π, gives a strongly commuting self-adjoint classical quantizer A with

$$\operatorname{Spec}(A) \subset \mathbb{Z}, \qquad P_M - A \in \Psi_{\mathrm{cl}}^{-1}.$$

Theorem 5.1, with $n = 5$, $s = 3/5$, and $\beta = 1$, has $\lfloor ns/\beta \rfloor = 3$ and gives

$$P_M - A \in \Psi_{\mathrm{cl}}^{-4}. \tag{23.5}$$

On a joint high-frequency eigenvector, let μ and $k \in \mathbb{Z}$ be the eigenvalues of P_M and A. Boundedness of $(P_M - A)A^4$ gives $|\mu - k| \le Ck^{-4}$. For large ν, every sufficiently high reference frequency matched to μ is therefore k: both integers are within $1/4$ of μ.

Put $\ell = 2\pi/p$. For $t = 0, \ell$ and $\chi \in C_c^\infty((0,1))$, the matched copies of A and P_Q make equal contributions to $e^{itk}\chi(k/R_\nu)$. Each remaining copy has absolute contribution at most $\|\chi\|_\infty$. Every

contributing A-copy corresponds to a P_M-copy below R_ν for large ν, since $\sup \operatorname{supp} \chi < 1$. The finitely many low frequencies have zero weight. Thus

$$\operatorname{Tr}\big(e^{itA}\chi(A/R_\nu)\big) - \operatorname{Tr}\big(e^{itP_Q}\chi(P_Q/R_\nu)\big) = o(R_\nu), \qquad t = 0, \ell. \tag{23.6}$$

On the union of the two contributing annuli, $k \asymp R$ and $|\mu - k| = O(R^{-4})$. Weyl's law gives $O(R^5)$ joint eigenvectors there, counted with multiplicity. At $t = 0$ the derivative of $u \mapsto e^{itu}\chi(u/R)$ is $O(R^{-1})$; at $t = \ell$ it is $O(1)$. Equation (23.5) therefore gives

$$\operatorname{Tr} \chi(P_M/R) - \operatorname{Tr} \chi(A/R) = O(1),$$

and

$$\operatorname{Tr}\big(e^{i\ell P_M}\chi(P_M/R)\big) - \operatorname{Tr}\big(e^{i\ell A}\chi(A/R)\big) = O(R).$$

Consequently

$$\operatorname{Tr} \chi(P_M/R_\nu) - \operatorname{Tr} \chi(P_Q/R_\nu) = o(R_\nu), \tag{23.7}$$

$$\operatorname{Tr}\big(e^{i\ell P_M}\chi(P_M/R_\nu)\big) - \operatorname{Tr}\big(e^{i\ell P_Q}\chi(P_Q/R_\nu)\big) = o(R_\nu^2). \tag{23.8}$$

Choose $\chi \geq 0$, $\chi \not\equiv 0$. All three moments in (23.1) are positive. Subtract that expansion for M and Q and apply (23.7). Successive division by R_ν^5, R_ν^3, and R_ν shows that

$$a_j(M) = a_j(Q), \qquad j = 0, 1, 2. \tag{23.9}$$

Let W and Ric_0 be the Weyl and trace-free Ricci tensors. The curvature identity in the proof of [64, Theorem A′(1), equation (3.1), pp. 394–395], together with (23.9), gives

$$0 = \int_M \left(2|W|^2 + \frac{2}{3}|\operatorname{Ric}_0|^2 + \frac{24}{5}(\operatorname{Scal} - 20)^2 \right).$$

Indeed, that identity has coefficients 2, 2/3, and 14/3 on $|W|^2$, $|\operatorname{Ric}|^2$, and Scal^2 in dimension five; here $|\operatorname{Ric}|^2 = |\operatorname{Ric}_0|^2 + \operatorname{Scal}^2/5$ and $\int_M \operatorname{Scal} = 20 \operatorname{Vol}(M)$. All these integrals are taken against Riemannian volume densities, so orientability is unnecessary. The integrand is nonnegative and continuous. Hence $W = 0$, $\mathrm{Ric}_0 = 0$, and $\operatorname{Scal} = 20$ everywhere, and M has sectional curvature one.

The complete simply connected cover of M is the unit sphere. Write $M = \mathbb{S}^5/\Gamma$ for a finite free orthogonal group Γ [39, Section 1, p. 77]. Equation (23.4) gives $|\Gamma| = p$, so Γ is cyclic. A generator has three nonzero rotation weights modulo p [39, Section 1, Example (ii), p. 77]. After grouping weights modulo sign, their multiplicities have one of the forms (3), (2, 1), or (1, 1, 1).

Set $\zeta = e^{2\pi i/p}$ and let F_M be the harmonic series (8.2). At $z = \zeta$, each Molien summand has pole order at most the largest weight multiplicity. Indeed a factor $1 - \zeta^j z$ vanishes there precisely when $j = -1$ modulo p, and $1 - \zeta^2 \neq 0$. The nonzero coefficients below give equality for the partitions (3) and (2, 1). For partition (3), change the generator so that the common weight is one. Only the generator and its inverse have eigenvalue ζ^{-1}, and their sum gives

$$L_\zeta(F_M) = \frac{2}{p(1 - \zeta^2)^2} \neq 0$$

at pole order three. Lemma 8.4, with the shift $k \mapsto k + 2$, then gives a nonzero term of order R^3 in the time-ℓ annular trace. For partition (1, 1, 1) every pole there is simple, so the trace is $O(R)$. The reference has partition (2, 1) and a nonzero term of order R^2, computed below. Both other partitions contradict (23.8). Thus M has partition (2, 1).

Change the generator and conjugate coordinates to write $M = L(p; 1, 1, b)$ with $2 \leq b \leq (p - 1)/2$. Make the same normalization for a. The double poles at $z = \zeta$ come only from the generator and its inverse. Removing the two factors $1 - z/\zeta$ from their denominators gives

$$L_\zeta(F_M) = \frac{2}{p(1 - \zeta^2)(1 - \zeta^{b+1})(1 - \zeta^{1-b})}.$$

This is nonzero. Lemma 8.4 shows that the coefficient of $R^2 \int_0^1 \chi(u)u\,du$ in the shifted trace is

$$t_b = \frac{2\zeta^2}{p(1-\zeta^2)(1-\zeta^{b+1})(1-\zeta^{1-b})}. \tag{23.10}$$

Equation (23.8) gives $t_b = t_a$. Since

$$|t_b| = \frac{2}{p\,[2\sin(2\pi/p)]\,4[\sin^2(\pi b/p) - \sin^2(\pi/p)]},$$

and $\sin^2(\pi b/p)$ is strictly increasing for $2 \le b \le (p-1)/2$, it follows that $b = a$. The two quotients are isometric. □

Question 23.3. Let $p \ge 5$ be prime, let $a \not\equiv 0, 1, -1 \pmod p$, and put $Q = L(p; 1, 1, a)$. Let M be a smooth closed connected Riemannian manifold. For a fixed $0 < \eta < 1/4$, does

$$E_\eta\Big(R_v; \sqrt{\Delta_M + 4}, \sqrt{\Delta_Q + 4}\Big) = o(R_v) \qquad (R_v \longrightarrow \infty)$$

imply that M and Q are isometric?

Part 4. Closed geodesics and cyclic actions

On a finite free isometric quotient of a closed connected Zoll manifold of dimension at least two, the scalar spectrum determines the closed-geodesic lengths and the maximal dimension of a returning component at each length. For cyclic deck groups, the cyclotomic parts of the normalized cluster character determine the eventual cluster ranks of intermediate covers and rational holonomy bundles (Section 26). Under the prime two-weight comparison hypotheses of Section 31, trace coefficients and Morse indices determine the marked phases, signed normal weights and homotopy types of the return families. For $p \ge 11$, the large and small families have two disjoint embedded images with linking number of absolute value one; these images are the minimum and maximum sets of a suitable deck transformation's displacement.

24. Periods and the wave trace

For a closed connected Riemannian manifold Y, let SY denote its unit tangent bundle and G_t^Y its geodesic flow. Write $\mathscr{L}(Y)$ for the set of positive lengths of smoothly closed geodesics, including all iterates. A Zoll manifold has a common prime geodesic period; simplicity of the geodesics is not assumed. The Fourier convention is $\widehat{u}(\lambda) = \int e^{it\lambda} u(t)\,dt$. Frequency orders refer to the Fourier transform of a time-localized wave trace.

Theorem 24.1 (The Poisson relation). *Let X be a closed connected Zoll manifold of dimension $n \ge 2$ and common prime period L. Let a finite group Γ act freely by isometries on X, and put $M = X/\Gamma$. Then*

$$\operatorname{sing\,supp} \operatorname{Tr} e^{-it\sqrt{\Delta_M}} = \{0\} \cup \mathscr{L}(M) \cup (-\mathscr{L}(M)).$$

For $\tau \in \mathscr{L}(M)$, put

$$D(\tau) = \max\{\dim C \,:\, C \text{ is a connected component of } \operatorname{Fix} G_\tau^M\}.$$

The coefficient of frequency order $(D(\tau)-1)/2$ in the clean wave trace at τ is nonzero. Consequently the scalar spectrum determines $\mathscr{L}(M)$ and $D(\tau)$ for every such τ.

For a Zoll manifold itself, the Poisson equality follows from the clean trace formula at a full return; see [63, Section 1.3]. Prüfer proved the Poisson equality for spherical space forms [55, Theorem 2 and Corollary 4, pp. 142 and 152]. In odd dimension $n = 2m+1$, his formula concerns $\operatorname{Tr}\cos(t\sqrt{\Delta + m^2})$ and its highest coefficient is nonzero at every return time. His proof first treats $2\pi/s$ by positivity and then transfers the conclusion by automorphisms of the sth cyclotomic field. Theorem 24.1 applies this argument to finite free quotients of arbitrary Zoll manifolds. The required rational function is the cluster character of Colin de Verdière's periodic quantizer [12, Theorems 1.1 and 1.4]; the minimizing property is Lemma 15.1.

Lemma 24.2. *Every returning time in Theorem 24.1 is clean. Each connected component of its fixed set has positive odd dimension. After the normalization $L = 2\pi$, all returning times lie in $(2\pi/|\Gamma|)\mathbb{Z}$.*

Proof. Equation (15.1) and Lagrange's theorem place every return time in $(L/|\Gamma|)\mathbb{Z}$. Since $G_L^M = I$, every returning time map has finite order. Section 10.1 then gives a finite union of clean contact components. Their dimensions are positive and odd. Set $L = 2\pi$ for the stated grid. □

Lemma 24.3. *Normalize $L = 2\pi$ and put $q = |\Gamma|$ and $P = \sqrt{\Delta_M}$. There are a real number β and a self-adjoint $Q_{-1} \in \Psi_{\mathrm{cl}}^{-1}(M)$, strongly commuting with P, such that the self-adjoint operator*

$$A = P + Q_{-1} - \beta \qquad \textit{satisfies} \qquad \operatorname{Spec} A \subset \mathbb{Z}_{\geq 0}.$$

If $a_k = \dim\ker(A - k)$, then a_k is eventually a quasipolynomial of period dividing q and degree at most $n - 1$. In particular,

$$F_A(z) := \sum_{k\geq 0} a_k z^k \in \mathbb{Q}(z), \qquad (1 - z^q)^n F_A(z) \in \mathbb{Z}[z]. \tag{24.1}$$

Changing finitely many eigenvalue labels within $\mathbb{Z}_{\geq 0}$ changes F_A by a polynomial.

For odd-dimensional spherical space forms, an exact quasipolynomial multiplicity formula is given in [56, Lemma 7, pp. 302–303].

Proof. Let Π_0 be the orthogonal projection onto $\ker \Delta_M$. The positive operator $P + \Pi_0$ has principal symbol $|\xi|_g$ and zero subprincipal symbol on scalar half-densities. Its Hamiltonian flow has period 2π, and S^*M is connected since $n \geq 2$. Theorem 1.1 of [12], with its spectral construction, gives a commuting correction of order -1 and a single Maslov shift β. Absorb Π_0 into the correction. Any negative integer labels can be moved into $\mathbb{Z}_{\geq 0}$ by finite-rank smoothing operators commuting with P.

Theorem 1.4 of [12] expresses a_k, for sufficiently large k, as a finite sum of terms $\zeta^k p_\zeta(k)$, where p_ζ has degree at most $n - 1$ and each ζ is a root of unity associated with a geodesic return time. Lemma 24.2 makes every such root a qth root of unity. Thus a_k is polynomial on each residue class modulo q. These polynomials have rational coefficients, by interpolation at n sufficiently large integers in each class. Their nth differences of step q vanish. The coefficients of $(1 - z^q)^n F_A(z)$ are integer differences of this kind and hence vanish for all sufficiently large degrees. This proves (24.1). □

Proof of Theorem 24.1. Rescale the metric by $(2\pi/L)^2$, so that $L = 2\pi$. Take A from Lemma 24.3 and put

$$B = A + \beta = \sqrt{\Delta_M} + Q_{-1}.$$

The operator $B - \sqrt{\Delta_M + 1}$ has order -1 and commutes with $\sqrt{\Delta_M + 1}$. Thus Lemma 15.2 applies to B. Moreover, B and $\sqrt{\Delta_M}$ have the same principal and subprincipal symbols. Their highest clean coefficients agree [17, Theorem 4.5, equations (4.7)–(4.8)]. Finitely many nonpositive eigenvalues do not affect these coefficients or the annular traces below.

Fix $0 \leq \chi \in C_c^\infty((0,1))$, $\chi \neq 0$. For an element order $s \geq 2$, put $t_s = 2\pi/s$ and $D_s = D(t_s)$. Lemma 15.2 gives

$$\operatorname{Tr}(e^{it_s B}\chi(B/R)) = C_{s,\chi} R^{(D_s+1)/2} + o(R^{(D_s+1)/2}), \qquad C_{s,\chi} \neq 0. \tag{24.2}$$

The nonvanishing follows from the zero periodic Morse indices in Lemma 15.1.

Apply the calculation in Lemma 8.4 to F_A, with m replaced by the fixed real number β. A pole of order $r \geq 1$ at $\xi = e^{it}$ gives

$$\operatorname{Tr}(e^{itB}\chi(B/R)) = \frac{e^{it\beta}}{(r-1)!}\left(\lim_{z\to\xi}(1 - z/\xi)^r F_A(z)\right) R^r \int_0^1 \chi(u) u^{r-1}\, du + O(R^{r-1}).$$

The displayed coefficient is nonzero. If ξ is a regular point, the proof of that lemma gives $O(R^{-N})$ for every N. Comparison with (24.2) shows that F_A has a pole of order $(D_s + 1)/2$ at $e^{2\pi i/s}$.

Write $F_A = N/D$ in lowest terms over $\mathbb{Q}$. The irreducibility of Φ_s over $\mathbb{Q}$ implies that F_A has the same pole order at every primitive sth root. Moreover, for $(a, s) = 1$,

$$\operatorname{Fix} G^M_{2\pi a/s} = \operatorname{Fix} G^M_{2\pi/s}.$$

Indeed, if $ua + vs = 1$, then $G_{2\pi/s} = (G_{2\pi a/s})^u (G_{2\pi})^v$; the reverse inclusion follows by taking the ath power. The annular trace at $2\pi a/s$ therefore has a nonzero coefficient of order $R^{(D_s+1)/2}$. If the clean frequency coefficient of order $(D_s - 1)/2$ vanished, the annular trace would be $o(R^{(D_s+1)/2})$, a contradiction. Complex conjugation gives the corresponding nonzero coefficient for $\operatorname{Tr} e^{-itB}$. Its highest coefficient agrees with that of $\operatorname{Tr} e^{-it\sqrt{\Delta_M}}$.

At $t = 0$ the positive leading Weyl coefficient gives a pole of order n at 1. Since $\operatorname{Spec} A \subset \mathbb{Z}$, one has $e^{i(t+2\pi j)B} = e^{2\pi i j\beta} e^{itB}$. This gives a nonzero coefficient of frequency order $n - 1$ at every $2\pi j$ and extends the preceding coefficients to all translates by $2\pi\mathbb{Z}$. Every positive geometric length has the form $2\pi k/s$ with $s \in \mathcal{O}(\Gamma)$ by (15.1). Reducing k/s gives a denominator d dividing s. A power of an element of order s has order d, so this length is among the times just treated. Negative times follow by complex conjugation. The opposite inclusion is [17, Corollary 1.2]. The length set is discrete by Lemma 24.2.

The spectrum determines the wave trace and hence its singular support and highest local frequency order. The latter equals $(D(\tau) - 1)/2$. Scaling back gives both assertions for the original metric. □

Corollary 24.4. *In the normalization $L = 2\pi$, put $D_1 = 2n - 1$ and $D_s = D(2\pi/s)$ for $s \in \mathcal{O}(\Gamma)$, $s > 1$. The reduced denominator of F_A, up to a nonzero rational scalar, is*

$$\prod_{s \in \mathcal{O}(\Gamma)} \Phi_s(z)^{(D_s+1)/2}. \tag{24.3}$$

Proof. The proof of Theorem 24.1 gives these orders at all primitive roots whose orders lie in $\mathcal{O}(\Gamma)$. At every other qth root there is no geometric return, by (15.1) and closure of $\mathcal{O}(\Gamma)$ under divisors. The Poisson inclusion makes the trace of $B = A + \beta$ smooth there. Its annular trace then decays faster than every power, whereas Lemma 8.4 gives a nonzero leading term at a pole. Thus there is no pole there. Equation (24.1) excludes poles away from the qth roots of unity. □

Corollary 24.5. *The conclusion of Theorem 24.1 holds on every closed connected Besse manifold of odd dimension $n \geq 5$.*

Proof. Wadsley's period theorem, in the formulation of [4, Theorem 4.6], gives a global geodesic period. A flow box excludes sufficiently small nonzero global periods, so there is a least positive global period. Lemma 11.1 then gives a compact Zoll homotopy-sphere universal cover with finite deck group. Theorem 24.1 applies. □

Proposition 24.6. *Under the hypotheses of Theorem 24.1, let T be the least positive global period of G^M_t. Then $T = L$ or $L/2$. If $|\Gamma|$ is odd, then $T = L$, and the scalar spectrum determines L and $\mathcal{O}(\Gamma)$.*

Proof. The global periods form a closed subgroup of $\mathbb{R}$ containing $L\mathbb{Z}$. A flow box excludes sufficiently small nonzero global periods, since the geodesic vector field never vanishes. Thus this subgroup has a positive generator $T \leq L$. For each $v \in SX$, there is a unique $h_v \in \Gamma$ such that $G^X_T v = \widehat{h_v} v$. The map $v \mapsto h_v$ is locally constant. Since SX is connected, $h_v = h$ is independent of v. Velocity reversal $R(x, v) = (x, -v)$ commutes with $\widehat{h}$ and satisfies $RG^X_T R = G^X_{-T}$. Hence

$$G^X_{-T} = \widehat{h} = G^X_T, \qquad G^X_{2T} = I, \qquad h^2 = 1.$$

Every upstairs geodesic has prime period L, so $2T \in L\mathbb{N}$. Together with $T \leq L$, this gives $T = L$ or $L/2$. If $|\Gamma|$ is odd, then $h = 1$; hence T is an upstairs period and $T = L$.

A fixed component has dimension $2n - 1$ if and only if it is open in SM. Such a component is also closed, and SM is connected; hence it is all of SM. By Theorem 24.1, T is therefore the least positive time with a nonzero wave coefficient of frequency order $n - 1$. The Weyl law determines n, so this describes T spectrally. In the odd-order case it gives L, and Lemma 15.1 gives

$$\mathcal{O}(\Gamma) = \{s \in \mathbb{N} \,:\, L/s \in \mathscr{L}(M)\}.$$

□

25. Arithmetic of eigenvalue multiplicities

Throughout this section,

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots, \qquad m_j = \dim\ker(\Delta - \lambda_j),$$

are the distinct scalar eigenvalues and their multiplicities. When the geodesic flow is periodic, let T be its least positive global period. After multiplication of frequencies by $T/(2\pi)$, the high T-clusters lie in the disjoint intervals $[k + \sigma - C/k, k + \sigma + C/k]$ of periodic quantization. If the number of distinct eigenvalues in each cluster is eventually constant, denote its value by R_{cl}.

Lemma 25.1. *Let M be a closed connected Besse manifold of dimension $n \geq 2$ and suppose*

$$m_j = cj^{n-1} + O(j^{n-2}), \qquad c > 0. \tag{25.1}$$

Then each sufficiently high T-cluster contains the same positive integer R_{cl} of distinct eigenvalues. Its total rank d_k satisfies

$$d_k = cR_{\mathrm{cl}}^n k^{n-1} + O(k^{n-2}). \tag{25.2}$$

If m_j is eventually quasipolynomial of period d, with the same positive leading coefficient on every residue class, then d_k is eventually quasipolynomial of period dividing

$$q_d = \frac{d}{\gcd(R_{\mathrm{cl}}, d)}. \tag{25.3}$$

Proof. Put $P = (T/2\pi)\sqrt{\Delta}$. On scalar half-densities, P has zero subprincipal symbol; its Hamiltonian flow has period 2π, and S^*M is connected. Move its zero eigenvalue to a positive number below its first positive eigenvalue and apply [12, Theorem 1.1]. Absorbing this commuting finite-rank smoothing change into the correction gives $P = A + \sigma + Q_{-1}$, where A is self-adjoint, $\operatorname{Spec} A \subset \mathbb{Z}_{\geq 0}$, and $Q_{-1} \in \Psi_{\mathrm{cl}}^{-1}(M)$ is self-adjoint and commutes with A. For all large k, $d_k = \dim\ker(A - k)$. Since T is least, the circle action on S^*M is effective. By [12, Proposition 1.3], the fixed components of a nonidentity circle element have odd dimension; effectiveness bounds their dimensions by $2n - 3$. The corresponding terms in [12, Theorem 1.4] therefore have degree at most $n - 2$. The identity term has positive leading coefficient, so

$$d_k = ak^{n-1} + O(k^{n-2}), \qquad a > 0, \tag{25.4}$$

with one leading coefficient for all residue classes. For real-projective multiplicities, the cluster-count argument below appears in [73, Section 6, pp. 583–584].

Let J_k be the last distinct-eigenvalue index in cluster k. All high clusters are nonempty by (25.4). The Weyl law at the normalized frequency $k + \sigma + 1/2$ gives

$$\sum_{j=0}^{J_k} m_j = wk^n + O(k^{n-1}), \qquad w = \frac{\omega_n \operatorname{Vol}(M)}{T^n},$$

where ω_n is the Euclidean unit-ball volume. On the other hand, (25.1) gives

$$\sum_{j=0}^{J} m_j = \frac{c}{n}J^n + O(J^{n-1}).$$

Comparison gives $J_k/k \to \gamma = (nw/c)^{1/n}$ and then $J_k^n - \gamma^n k^n = O(k^{n-1})$. Factoring this difference yields $J_k = \gamma k + O(1)$. Thus $R_k = J_k - J_{k-1}$ is bounded, and, uniformly over the indices in cluster k,

$$m_j = c\gamma^{n-1}k^{n-1} + O(k^{n-2}).$$

Summing over that cluster and using (25.4) gives

$$R_k = \frac{a}{c\gamma^{n-1}} + O(k^{-1}).$$

This convergent sequence of positive integers is eventually constant, equal to R_{cl}. Summing (25.4) and comparing with the Weyl law gives $a = nw$. Hence $R_{\mathrm{cl}} = \gamma$ and $a = cR_{\mathrm{cl}}^n$.

For some integer b and all large k,

$$J_{k-1} = R_{\mathrm{cl}}k + b, \qquad d_k = \sum_{i=1}^{R_{\mathrm{cl}}} m_{R_{\mathrm{cl}}k+b+i}. \tag{25.5}$$

Since $d \mid R_{\mathrm{cl}}q_d$, each index on the right has a fixed residue modulo d when k varies in a residue class modulo q_d. The sum is therefore polynomial on each such class. □

Theorem 25.2. *Let M be closed and connected of odd dimension $n \geq 5$, and assume* (25.1)*. Its universal Riemannian cover X is a compact Zoll homotopy sphere. Let L be its prime period, $\Gamma = \pi_1(M)$, and T the least global period on M. Then $\kappa = L/T \in \{1, 2\}$ and*

$$(n-1)!\, c\, |\Gamma| R_{\mathrm{cl}}^n = 2\kappa^n. \tag{25.6}$$

Moreover, $\kappa = 2$ if and only if $|\Gamma|$ is even. In this case M has constant positive curvature and $R_{\mathrm{cl}} = 1$.

The number $(n-1)!c$ is rational. Write it as A/B in lowest terms. If

$$v_\ell(2B) < n \qquad \text{for every prime } \ell, \tag{25.7}$$

then $R_{\mathrm{cl}} = 1$.

Proof. Zelditch's Theorem A [73] applies to the distinct eigenvalues and the asymptotic (25.1); it makes the geodesic flow periodic. Lemma 11.1 gives the compact Zoll homotopy-sphere cover X and $\kappa \in \{1, 2\}$. If $\kappa = 2$, the lifted time-T map is a nonidentity deck involution, so $|\Gamma|$ is even. Conversely, if $|\Gamma|$ is even, $X \to M$ is an even-degree smooth covering by a smooth homotopy sphere. Since M is Besse and $n \neq 3$, Lin–Schmidt's theorem [49, Theorem 2] gives constant curvature. Its compact simply connected cover is a round sphere. Cauchy's theorem and Lemma 8.2 give $-I \in \Gamma$. Thus $L/2$ is a global period downstairs, and $\kappa = 2$.

Proposition 12.3 gives

$$\mathrm{Vol}(X) = \mathrm{Vol}(S^n)\left(\frac{L}{2\pi}\right)^n, \qquad \mathrm{Vol}(M) = \frac{\mathrm{Vol}(X)}{|\Gamma|}.$$

In Lemma 25.1, $a = n\omega_n \mathrm{Vol}(M)/T^n$. The identity

$$\frac{n\omega_n \mathrm{Vol}(S^n)}{(2\pi)^n} = \frac{2}{(n-1)!}$$

therefore gives (25.6) and the rationality of $(n-1)!c$.

Suppose $\kappa = 2$ and scale the universal round sphere to radius one. Then $L = 2\pi$, $T = \pi$, and $-I \in \Gamma$. Lemma 8.3, specifically (8.3), gives zero multiplicity in odd harmonic degrees and, in even degrees,

$$\dim \mathcal{H}_l^\Gamma = \frac{2}{|\Gamma|}\dim \mathcal{H}_l + O(l^{n-2}). \tag{25.8}$$

Its leading coefficient is positive, so every sufficiently high even degree $l = 2k$ occurs. The normalized frequency is

$$\frac{1}{2}\sqrt{2k(2k+n-1)} = k + \frac{n-1}{4} + O(k^{-1}).$$

Their consecutive gaps tend to one, whereas the cluster widths tend to zero. Every high cluster is nonempty, so $R_{\mathrm{cl}} = 1$.

If $\kappa = 1$, equation (25.6) becomes $A|\Gamma|R_{\mathrm{cl}}^n = 2B$. A prime divisor ℓ of R_{cl} would give $v_\ell(2B) \geq n$, contrary to (25.7). □

Theorem 25.3 (Leading multiplicity coefficients). *Fix an odd integer $n \geq 5$. The possible positive coefficients c in* (25.1)*, among all closed connected Riemannian n-manifolds, are exactly*

$$\left\{ \frac{2^n}{(n-1)!N} \,:\, N \in \mathbb{Z}_{>0} \right\}. \tag{25.9}$$

Every value is realized by the round scalar lens space S^n/C_{2N}, with generator $z \mapsto e^{\pi i/N} z$.

Proof. For $\kappa = 1$, equation (25.6) gives

$$(n-1)!c = \frac{2}{|\Gamma| R_{\mathrm{cl}}^n} = \frac{2^n}{2^{n-1}|\Gamma| R_{\mathrm{cl}}^n}.$$

For $\kappa = 2$, it gives $R_{\mathrm{cl}} = 1$ and $(n-1)!c = 2^n/(|\Gamma|/2)$. Both have the form in (25.9).

The stated scalar action of C_{2N} is free and contains $-I$. Equations (25.8) and (8.1) give

$$\dim \mathcal{H}_{2k}^{C_{2N}} = \frac{1}{N} \dim \mathcal{H}_{2k} + O(k^{n-2}) = \frac{2^n}{N(n-1)!} k^{n-1} + O(k^{n-2}).$$

All high even degrees occur and odd degrees do not. The index of the corresponding distinct eigenvalue differs from k by a fixed integer, which does not change this asymptotic. □

Corollary 25.4. *Write $c = 2^n/((n-1)!N)$ as in* (25.9)*. If $v_2(N) \not\equiv n-1 \pmod{n}$, then M has constant positive curvature, $R_{\mathrm{cl}} = 1$, and $|\pi_1(M)| = 2N$. In particular this holds if $2^{n-1} \nmid N$, or if $c > 2/(n-1)!$.*

Proof. If $|\Gamma|$ is odd, then $\kappa = 1$ and $N = 2^{n-1}|\Gamma| R_{\mathrm{cl}}^n$. Hence $v_2(N) = n - 1 + n v_2(R_{\mathrm{cl}})$, contrary to the hypothesis. Theorem 25.2 gives constant curvature, $\kappa = 2$, and $R_{\mathrm{cl}} = 1$, whence $|\Gamma| = 2N$. If $2^{n-1} \nmid N$, then $v_2(N) < n-1$, so the stated congruence cannot hold. If $c > 2/(n-1)!$, the alternative $\kappa = 1$ is excluded by (25.6). □

Corollary 25.5. *The largest possible coefficient in* (25.1) *is $2^n/(n-1)!$. It occurs precisely on round real projective space, up to scale.*

Proof. The largest value in (25.9) has $N = 1$. Corollary 25.4 gives a round universal cover and a deck group of order two. By Lemma 8.2, this group is $\{I, -I\}$. The converse is the case $N = 1$ of Theorem 25.3. □

Proposition 25.6. *Under* (25.1) *in odd dimension $n \geq 5$, suppose m_j is eventually quasipolynomial of period d, with common leading coefficient c. Put $q_d = d/\gcd(R_{\mathrm{cl}}, d)$ and $\kappa = L/T$. Then*

$$\exp \pi_1(M) \mid \kappa q_d. \tag{25.10}$$

Furthermore,

$$v_d := (n-1)! d^{n-1} c \in \mathbb{Z}_{>0}, \qquad v_d |\Gamma| R_{\mathrm{cl}}^n = 2\kappa^n d^{n-1}. \tag{25.11}$$

Proof. Rescale the metric so that $T = 2\pi$; the multiplicities, R_{cl}, and κ are unchanged. Lemma 25.1 makes the cluster ranks quasipolynomial of period dividing q_d. By [60, Proposition 4.4.1], $F_A(z) = \sum_{k \geq 0} d_k z^k$ is rational, with poles only at q_dth roots of unity. Finite changes of the labels add a polynomial and do not affect these poles.

Put $B = A + \sigma$, so $B - (\Delta + 1)^{1/2} \in \Psi_{\mathrm{cl}}^{-1}(M)$ and the two operators commute. The trace of e^{-itB} is the Abel boundary value of $e^{-it\sigma} F_A(e^{-\epsilon - it})$; it is smooth unless $q_d t/(2\pi) \in \mathbb{Z}$. For $s \in \mathcal{O}(\Gamma)$, $s > 1$, Lemma 15.1 supplies a return at L/s. Lemma 15.2, applied to B and $(\Delta + 1)^{1/2}$, gives a nonzero leading annular coefficient there for e^{itB}. Complex conjugation gives the same nonsmoothness for e^{-itB}. Consequently $q_d L/(2\pi s) = \kappa q_d/s$ is an integer. Taking the least common multiple of the element orders gives (25.10); the order $s = 1$ imposes no condition.

On each residue progression of m_j, the $(n-1)$st forward difference of step d is $(n-1)! d^{n-1} c$. It is an integer because the multiplicities are integers, and it is positive since $c > 0$. Multiplying (25.6) by d^{n-1} proves the second identity in (25.11). □

Corollary 25.7. *If the quasiperiod d in Proposition 25.6 is odd and squarefree, then $R_{\mathrm{cl}} = 1$.*

Proof. Write $(n-1)!c = A/B$ in lowest terms. Equation (25.11) gives $B \mid d^{n-1}$. Thus B is odd, $v_2(2B) = 1 < n$, and $v_\ell(2B) \le n-1$ for every odd prime ℓ. Apply Theorem 25.2. □

Corollary 25.8. *Let M be closed and connected of odd dimension $n \ge 5$. Suppose $m_j = P(j)$ for every sufficiently large j, where P is a polynomial of degree $n-1$ and positive leading coefficient. Then every geodesic has the same prime period and $R_{\mathrm{cl}} = 1$. Exactly one of the following holds:*

(1) M is a Zoll smooth homotopy sphere;

(2) M is round real projective space, up to scale.

Proof. Take $d = 1$. Corollary 25.7 gives $R_{\mathrm{cl}} = 1$, and (25.10) gives $\exp\Gamma \mid \kappa$. If $\kappa = 1$, then $\Gamma = 1$, so $M = X$ is a Zoll smooth homotopy sphere. If $\kappa = 2$, Theorem 25.2 makes M round and Γ nontrivial. Every nonidentity element of Γ is an involution, hence equals $-I$ by Lemma 8.2. Thus $\Gamma = C_2$. Both alternatives have a common prime geodesic period, and they are disjoint because their fundamental groups differ. □

Corollary 25.9. *Suppose* (25.1) *holds in odd dimension $n \ge 5$ and $\pi_1(M) = C_p$ for an odd prime p. If m_j is eventually quasipolynomial of period p with common leading coefficient, then*

$$R_{\mathrm{cl}} = 1, \qquad c = \frac{2}{p(n-1)!}.$$

The same conclusion $R_{\mathrm{cl}} = 1$ follows from (25.1) *alone when its coefficient equals $2/(p(n-1)!)$.*

Proof. Since $|\Gamma| = p$ is odd, $\kappa = 1$. In the quasipolynomial case, Corollary 25.7, with $d = p$, gives $R_{\mathrm{cl}} = 1$; equation (25.6) then determines c. If $c = 2/(p(n-1)!)$, the same identity gives $R_{\mathrm{cl}}^n = 1$ directly. □

26. Covering clusters and rational holonomy

Let X be a closed connected Zoll manifold of dimension $n \ge 2$ and common prime period L, and let Γ be a finite free isometry group of order q. Put $M = X/\Gamma$ and $\mathcal{O}(\Gamma) = \{\mathrm{ord}(g) : g \in \Gamma\}$. For an intermediate quotient $Y = X/H$, set $P_Y = (L/2\pi)\sqrt{\Delta_Y}$. Let $\beta \in [0,1)$ be the scalar quantizer shift on X, modulo $\mathbb{Z}$. For every sufficiently large integer k, put

$$b_Y(k) = \#\{\nu \in \operatorname{Spec} P_Y : |\nu - (k+\beta)| < 1/2\}, \tag{26.1}$$

counting multiplicities. Assigning the finitely many remaining frequencies to nonnegative integer labels defines $F_Y(z) = \sum_{k\ge 0} b_Y(k) z^k$ modulo $\mathbb{Q}[z]$. For a rational function with poles at roots of unity, $\mathrm{PP}_d F$ denotes the sum of its proper partial-fraction terms at the primitive dth roots. If $F \in \mathbb{Q}(z)$, then $\mathrm{PP}_d F \in \mathbb{Q}(z)$, since the primitive dth roots form a Galois-invariant set. This operation vanishes on polynomials.

Lemma 26.1. *There is a self-adjoint quantizer*

$$A_X = P_X + Q_X - \beta, \qquad Q_X \in \Psi_{\mathrm{cl}}^{-1}(X), \qquad \operatorname{Spec} A_X \subset \mathbb{Z}_{\ge 0},$$

which commutes with P_X and every isometry of X. It descends, with the same shift β, to every X/H and every flat bundle associated with a finite-dimensional unitary representation of Γ. One threshold, independent of H and of the representation, makes its labels agree with the frequency intervals in (26.1).

Proof. On scalar half-densities, P_X has zero subprincipal symbol. Its Hamiltonian flow has period 2π, and S^*X is connected. Theorem 1.1 of [12] gives a commuting self-adjoint quantizer. A smoothing change on the zero eigenspace permits the positivity hypothesis in that theorem; it is absorbed into Q_X. Subtracting an integer from the shift gives the chosen β, and finitely many negative labels can be changed.

The operator $Q_X P_X$ has order zero and is bounded. Thus, on a P_X-eigenspace of eigenvalue $\nu > 0$,

$$\|Q_X u\| \le C\nu^{-1}\|u\|.$$

Every eigenvalue of A_X on that space is an integer within C/ν of $\nu-\beta$. For large ν, this interval contains at most one integer. Since A_X is self-adjoint, its restriction is the scalar nearest integer to $\nu-\beta$. Replace its restriction on each remaining P_X-eigenspace by a nonnegative scalar integer. These eigenspaces are finite-dimensional and smooth, so the change is finite-rank and smoothing. Every isometry preserves the P_X-eigenspaces and hence commutes with the resulting A_X.

Restrict A_X to H-invariant functions, or restrict $A_X\otimes 1$ to $(L^2(X)\otimes V^*)^\Gamma$. The corresponding Laplacian is the restriction of Δ_X or $\Delta_X\otimes 1$, so functional calculus gives the corresponding restriction of P_X. Choose a covering chart whose distinct translates are disjoint. In its descended correction kernel, the identity sheet contributes Q_X or $Q_X\otimes 1$; the other sheets contribute smooth kernels. The correction therefore has order -1 on the quotient or flat bundle. The restrictions remain self-adjoint and commute with the corresponding P. Tensoring by 1_{V^*} preserves the bound C, and restriction cannot increase it. The nearest-integer rule, with the finitely many modified labels excluded, proves the common threshold. □

The scalar quantizer on X/H supplied by [12, Theorem 1.1] has the same shift modulo $\mathbb{Z}$. Indeed, if its shift is $\beta_H\in[0,1)$, a common high frequency ν gives $\operatorname{dist}(\beta_H-\beta,\mathbb{Z})=O(\nu^{-1})$; hence $\beta_H=\beta$. Its high labels coincide with the descended labels. Every return on X/H lies in $(L/|H|)\mathbb{Z}$, by lifting the returning tangent vector and applying Lagrange's theorem. The multiplicity formula [12, Theorem 1.4] consequently gives, as in Lemma 24.3,

$$(1-z^{|H|})^n F_{X/H}(z)\in\mathbb{Z}[z].$$

In particular these characters are rational, with poles only at $|H|$th roots of unity.

Put $E_k=\ker(A_X-k)$ and $\chi_k(g)=\operatorname{Tr}(R_g|E_k)$, where $R_g f=f\circ g^{-1}$. All representations and function spaces are complex. For a finite-dimensional unitary representation $\rho_V:\Gamma\to U(V)$, put

$$\rho_V^*(g)\ell=\ell\circ\rho_V(g)^{-1},\qquad \mathcal{E}_V=(X\times V^*)/\Gamma,\qquad g\cdot(x,\ell)=(gx,\rho_V^*(g)\ell).$$

Sections of $\mathcal{E}_V$ are maps $s:X\to V^*$ satisfying $s(gx)=\rho_V^*(g)s(x)$. The product connection descends to a flat unitary connection, whose Laplacian has kth cluster $(E_k\otimes V^*)^\Gamma$. Choose all low labels by restriction from A_X. The character inner-product formula [20, Theorem 4.5.1] gives

$$b_{X/H}(k)=\frac{1}{|H|}\sum_{g\in H}\chi_k(g), \tag{26.2}$$

$$b_V(k)=\frac{1}{q}\sum_{g\in\Gamma}\overline{\chi_V(g)}\chi_k(g). \tag{26.3}$$

Here $b_V(k)=\dim\operatorname{Hom}_\Gamma(V,E_k)$ and $F_V(z)=\sum_{k\ge0}b_V(k)z^k$. For irreducible V, $b_V(k)$ is its multiplicity in E_k. The conjugate in (26.3) is the character of the dual representation used in $\mathcal{E}_V$.

Lemma 26.2. *For $d\in\mathcal{O}(\Gamma)$, put*

$$r_d(k)=\sum_{\operatorname{ord}(g)=d}\chi_k(g),\qquad R_d(z)=\sum_{k\ge0}r_d(k)z^k.$$

Then $r_d(k)\in\mathbb{Z}$, $R_d\in\mathbb{Q}(z)$, and every pole of R_d is a primitive dth root of unity.

Proof. Let $C\le\Gamma$ be cyclic of order d, and let C_e be its subgroup of order $e\mid d$. Since $\sum_{g\in C_e}\chi_k(g)=e\dim E_k^{C_e}$, Möbius inversion on the divisors of d [60, Proposition 3.7.1 and Example 3.8.4] gives

$$\sum_{\substack{g\in C\\ \operatorname{ord}(g)=d}}\chi_k(g)=\sum_{e\mid d}\mu(d/e)e\dim E_k^{C_e}.$$

Each element of order d generates exactly one cyclic subgroup of order d. Summing over these subgroups proves integrality and expresses R_d as an integer linear combination of F_{X/C_e}. Thus $R_d\in\mathbb{Q}(z)$ and its poles are dth roots.

Write $\Phi_t = G^X_{Lt/(2\pi)}$. The coefficients of $T_g(z) = \sum_{k\ge 0} \chi_k(g) z^k$ have polynomial growth, since $|\chi_k(g)| \le \dim E_k$. Its Abel boundary, parametrized by $z = e^{-it}$, is the distribution $\mathrm{Tr}(R_g e^{-itA_X})$. The canonical relation of $R_g e^{-itA_X}$ can contribute to its trace only if

$$\Phi_t v = \widehat{g}^{-1} v \qquad \text{for some } v \in SX;$$

see [17, Theorem 1.1 and equations (1.20)–(1.23)]. If g has order d, this equality implies $dt = 2\pi a$ for some $a \in \mathbb{Z}$. If $r = \gcd(a, d) > 1$, then $\widehat{g}^{\,d/r} v = v$, contradicting freeness. Thus, for $d > 1$, $t/(2\pi) = a/d$ in lowest terms; for $d = 1$, $t \in 2\pi\mathbb{Z}$. The twisted trace is smooth at every other time. The Abel boundary of R_d is the sum of these traces over the elements of order d. It is smooth at every nonprimitive dth root. A pole on the unit circle has a singular Abel boundary, by the calculation in Lemma 8.4. Such poles are therefore absent. □

Theorem 26.3. *Let $H \le \Gamma$ be a union of complete element-order classes, and write*

$$H = \{g \in \Gamma \,:\, \mathrm{ord}(g) \in \mathcal{D}_H\}.$$

Then

$$F_{X/H} \equiv \frac{q}{|H|} \sum_{d \in \mathcal{D}_H} \mathrm{PP}_d\, F_M \pmod{\mathbb{Q}[z]}. \tag{26.4}$$

In particular $F_X \equiv q\,\mathrm{PP}_1\, F_M$. For $\Gamma = C_q$ and its subgroup H_h of order $h \mid q$,

$$F_{X/H_h} \equiv \frac{q}{h} \sum_{d \mid h} \mathrm{PP}_d\, F_M \pmod{\mathbb{Q}[z]}. \tag{26.5}$$

Proof. Equation (26.2) gives $F_M = q^{-1} \sum_{d \in \mathcal{O}(\Gamma)} R_d$. The pole sets in Lemma 26.2 are pairwise disjoint, so

$$R_d \equiv q\,\mathrm{PP}_d\, F_M \pmod{\mathbb{Q}[z]}.$$

Since H contains every element of each order in $\mathcal{D}_H$, averaging over H gives (26.4). For $H = \{1\}$, one has $\mathcal{D}_H = \{1\}$. In C_q, the subgroup of order h consists exactly of the elements whose orders divide h, giving (26.5). □

Theorem 26.4. *Let V be a finite-dimensional unitary representation of Γ whose character is constant on each complete element-order class: $\chi_V(g) = v_d$ when $\mathrm{ord}(g) = d$. Then $v_d \in \mathbb{Z}$, and the cluster character of the flat-bundle Laplacian on $\mathcal{E}_V$ satisfies*

$$F_V \equiv \sum_{d \in \mathcal{O}(\Gamma)} v_d\, \mathrm{PP}_d\, F_M \pmod{\mathbb{Q}[z]}. \tag{26.6}$$

For $\Gamma = C_q = \langle g_0 \rangle$ and $e \mid q$, put

$$W_e = \bigoplus_{\substack{a \bmod e \\ (a,e)=1}} \chi_{a/e}, \qquad \chi_{a/e}(g_0) = e^{2\pi i a/e}.$$

Then

$$F_{W_e} \equiv \sum_{d \mid q} c_e(q/d)\, \mathrm{PP}_d\, F_M \pmod{\mathbb{Q}[z]}, \qquad c_e(j) = \sum_{\substack{a \bmod e \\ (a,e)=1}} e^{2\pi i a j/e}. \tag{26.7}$$

Proof. Let N be the exponent of Γ. The eigenvalues of $\rho_V(g)$ are Nth roots of unity, so $\chi_V(g)$ is an algebraic integer in $\mathbb{Q}(\zeta_N)$. For $(a, N) = 1$, the Galois automorphism $\zeta_N \mapsto \zeta_N^a$ sends $\chi_V(g)$ to $\chi_V(g^a)$. The elements g and g^a have the same order, so every v_d is fixed by the Galois group. Hence $v_d \in \mathbb{Q}$, and a rational algebraic integer is an integer [20, Propositions 5.2.4–5.2.5]. In particular $\overline{v_d} = v_d$. Grouping (26.3) by element order gives $F_V = q^{-1} \sum_d v_d R_d$, and the preceding partial-fraction identity proves (26.6).

For an element g_0^j of order d, $\gcd(j,q) = q/d$. Since $e \mid q$, $\gcd(j,e) = \gcd(\gcd(j,q),e) = \gcd(q/d,e)$. The Ramanujan-sum identity [58, Section 2, equations (2.3) and (2.7)]

$$c_e(j) = \sum_{r \mid \gcd(e,j)} r\mu(e/r)$$

therefore gives $\chi_{W_e}(g_0^j) = c_e(q/d)$. This proves (26.7). Changing the generator g_0 permutes the primitive characters modulo e, so the isomorphism class of W_e is unchanged. □

Corollary 26.5. *Within the class of free cyclic Zoll quotients of odd order, the scalar spectrum determines L, q, and β, and therefore the eventual cluster ranks of every corresponding intermediate cover and every representation W_e in Theorem 26.4.*

Proof. Proposition 24.6 determines L. Since $\mathcal{O}(C_q)$ is the set of divisors of q, equation (15.1) gives $\mathscr{L}(M) = (L/q)\mathbb{Z}_{>0}$. Theorem 24.1 determines its least positive element $\ell_{\min}$ and hence $q = L/\ell_{\min}$. For any sequence of scaled scalar frequencies $\nu_j \to \infty$, the common quantizer gives

$$e^{2\pi i \nu_j} \longrightarrow e^{2\pi i \beta}.$$

This determines $\beta \in [0,1)$ and the eventual coefficients of F_M. Apply Theorems 26.3 and 26.4. □

For round lens spaces, Lauret [47, Theorem 4.3 and its proof] proves that scalar isospectrality passes to corresponding intermediate covers. His proof separates the generating function by primitive orders of poles. Equations (26.4) and (26.6) give this decomposition for eventual cluster ranks on free Zoll quotients, for the subgroups and characters specified above. The polynomial ambiguity leaves finitely many initial ranks undetermined; the eigenvalue locations inside each cluster are not determined by these formulas.

27. Exact deficits in normalized prime quotients

Fix $n = 2m+1 \geq 5$, a prime $p \geq 5$, and the unit round reference $Q = L(p; 1^m, a)$ with $a \not\equiv 0, 1, -1 \pmod{p}$. Suppose

$$M = X/C_p, \tag{27.1}$$

where C_p acts freely by isometries on a smooth homotopy n-sphere X with common prime period 2π and $\mathrm{Vol}(X) = \mathrm{Vol}(S^n)$. In addition, suppose

$$P_M = \sqrt{\Delta_M + m^2} = A + B, \quad \operatorname{Spec} A \subset \mathbb{Z}, \quad B \in \Psi^{-1}(M), \tag{27.2}$$

where A is self-adjoint and strongly commutes with P_M. Let $P_Q = \sqrt{\Delta_Q + m^2}$ and put

$$a_k = \dim\ker(A-k), \qquad q_k = \dim\ker(P_Q - k), \qquad d_k = a_k - q_k.$$

All frequency lists are counted with multiplicity. Put $E_\eta(R) = E_\eta(R; \operatorname{Spec} P_M, \operatorname{Spec} P_Q)$, with the notation of Definition 1.1, and take $R_K = m + K + 1/2$.

Theorem 27.1. *Under (27.1)–(27.2), d_k is eventually quasipolynomial of period dividing p and degree at most $n-1$. For every fixed $0 < \eta < 1$, the sequence $E_\eta(R_K)$ is eventually quasipolynomial in K, of period dividing p. For $0 \leq r < p$, let $D_r \in \mathbb{Q}[t]$ satisfy $d_k = D_r(k)$ for all sufficiently large $k \equiv r \pmod{p}$. Precisely one of the following holds:*

(1) Every D_r is zero, and $E_\eta(R_K)$ is eventually constant.

(2) Some D_r is nonzero. Put $D = \max_{D_r \neq 0} \deg D_r$, and let $c_r = \Delta_p^D D_r \in \mathbb{Z}$ for constituents of degree D, with $c_r = 0$ for the others. Then

$$E_\eta(R_K) = C R_K^{D+1} + O(R_K^D), \qquad C = \frac{\sum_{r=0}^{p-1} |c_r|}{p^{D+1}(D+1)!} > 0. \tag{27.3}$$

In this case $\liminf_K E_\eta(R_K)/R_K \geq 1/p$.

Here $\Delta_p f(k) = f(k+p) - f(k)$.

For finite subgroups of a compact connected semisimple group, Lauret–Miatello prove rationality and finite determination of multiplicities along representation strings in [48, arXiv version 2, Propositions 2.2 and 3.1]. Their Corollary 3.2 gives equality of these multiplicities when the set of unequal labels has upper density less than $1/q$, where q is the least common multiple of the two group orders.

Proof. Lemma 24.3, applied to M, gives an integer quantizer

$$\widetilde{A} = P_M + \widetilde{Q} - \beta, \qquad \widetilde{Q} \in \Psi^{-1}_{\mathrm{cl}}(M),$$

whose multiplicities are eventually quasipolynomial of period dividing p and degree at most $n-1$. Indeed,

$$P_M - \sqrt{\Delta_M} = m^2 (P_M + \sqrt{\Delta_M})^{-1} \in \Psi^{-1}_{\mathrm{cl}}(M),$$

and all these operators commute with P_M. On a P_M-eigenspace of eigenvalue ν, boundedness of BP_M and $\widetilde{Q}P_M$ gives corrections of norm $O(\nu^{-1})$. The nearest-integer argument in the proof of Lemma 26.1 makes both A and $\widetilde{A}$ scalar on every sufficiently high such eigenspace. Their integer labels u_ν, v_ν satisfy

$$u_\nu - v_\nu = \beta + O(\nu^{-1}).$$

Hence $\beta \in \mathbb{Z}$ and $u_\nu = v_\nu + \beta$ for all large ν. Thus $a_k = \dim\ker(\widetilde{A} - (k-\beta))$ for all large k. The finitely many low or negative labels have no effect on this conclusion. For Q, Lemma 8.1 makes P_Q an integer quantizer; the same argument applies with $B = 0$. Subtracting the two multiplicity sequences proves the assertion about d_k.

Since $BA \in \Psi^0(M)$ is bounded and B commutes with A,

$$\mathrm{Spec}(P_M|_{\ker(A-k)}) \subset [k - C_0/k,\, k + C_0/k]$$

for large positive k. Fix an integer k_0 so that $C_0/k < \min(\eta, 1-\eta, 1/4)$ for $k \geq k_0$. Each such cluster can match only the reference integer k, and every copy in it can match every reference copy at k. Since B is bounded, every frequency with A-label below k_0 is less than $k_0 + \|B\|$. These frequencies form a finite set. Choose an integer $k_1 > \max\{k_0, m\}$ larger than every one of these frequencies by more than $\eta + 1$. No matching pair joins a label below k_1 to a reference label at least k_1, or conversely: for labels at least k_0 this follows from the preceding separation, and for the remaining frequencies it follows from the choice of k_1. Moreover, every cluster with label $k \geq k_1$ lies wholly below R_K if and only if $k \leq m + K$. The finite part below this boundary has a fixed deficit $e_{\eta,0} \in \mathbb{Z}_{\geq 0}$. Each later cluster has deficit $a_k + q_k - 2\min(a_k, q_k) = |d_k|$. Therefore

$$E_\eta(R_K) = e_{\eta,0} + \sum_{k=k_1}^{m+K} |d_k| \tag{27.4}$$

for all sufficiently large K.

Each nonzero D_r has a constant sign on a sufficiently large positive tail. Hence $|d_k|$ is eventually quasipolynomial of period dividing p. By [60, Proposition 4.4.1], its generating function has poles only at pth roots of unity. Multiplication by $(1-z)^{-1}$ gives the generating function of the partial sums. The same proposition, after removal of the polynomial part, proves eventual quasipolynomiality in (27.4).

If every D_r vanishes, the sum is eventually constant. Otherwise let γ_r be the coefficient of degree D in D_r, with $\gamma_r = 0$ for lower degrees. Taking the Dth difference along its integer-valued residue tail gives

$$c_r = D!\, p^D \gamma_r \in \mathbb{Z}.$$

For each residue r,

$$\sum_{\substack{0 \leq k \leq R \\ k \equiv r\ (p)}} k^D = \frac{R^{D+1}}{p(D+1)} + O(R^D).$$

Since $m + K = R_K - 1/2$, substitution in (27.4) gives (27.3). At least one c_r is a nonzero integer, so $C > 0$. For $D \geq 1$, the quotient by R_K tends to infinity. For $D = 0$, it tends to $\sum_r |c_r|/p \geq 1/p$. □

Corollary 27.2. *Within the class* (27.1)–(27.2)*, the following are equivalent:*

(1) for one $0 < \eta < 1$ *and a sequence* $K_\nu \to \infty$, $E_\eta(R_{K_\nu}) = o(R_{K_\nu})$;

(2) $a_k = q_k$ *for all sufficiently large* k;

(3) for every fixed $\epsilon > 0$, $E_\epsilon(R_K) = O_\epsilon(1)$.

The first condition can equivalently be replaced by $\liminf_K E_\eta(R_K)/R_K < 1/p$ *for one* $0 < \eta < 1$.

Proof. Theorem 27.1 gives the second condition from the first and from the displayed liminf bound. Under the second, match every high copy within its cluster once $C_0/k < \epsilon$. Only finitely many copies remain unmatched, which proves the third. Taking $\epsilon = 1/2$ gives the first. Under the second, the displayed liminf is zero. □

Let $b_{k,r}$ be the eigenvalues of B on $\ker(A - k)$. Then $b_{k,r} = O(k^{-1})$, and the corresponding scalar Laplace eigenvalues satisfy

$$\lambda_{k,r} = k^2 - m^2 + 2kb_{k,r} + b_{k,r}^2, \qquad \Delta_M - (A^2 - m^2) = 2AB + B^2 \in \Psi^0(M).$$

Changing finitely many integer labels in a common eigenbasis of A and P_M leaves the condition $a_k = q_k$ for all large k unchanged and alters B only by a finite-rank smoothing operator.

28. Equivariant spectral comparison near the round metric

Let g_0 be the metric of curvature one on S^n, where $n = 2m + 1$ and $m \geq 2$. A normalized Zoll metric is a smooth Riemannian metric whose unit-speed geodesics have prime period 2π. Operators in this section act on complex half-densities. Put

$$P_g = (\Delta_g + m^2)^{1/2}, \qquad P_0 = (\Delta_{g_0} + m^2)^{1/2}.$$

Their principal symbols are

$$p_g(x, \xi) = \sqrt{g_x^{-1}(\xi, \xi)}, \qquad p_0 = p_{g_0}.$$

The spectrum of P_0 is $m + \mathbb{Z}_{\geq 0}$, with the harmonic multiplicities of Lemma 8.1. The canonical and Fourier integral comparison for Zoll metrics on S^2 appears in [73, Section 3, equations (3.6)–(3.8)].

Theorem 28.1 (Equivariant Zoll spectra). *There is a* C^3 *neighborhood* $\mathcal{U}_n$ *of* g_0 *with the following property. Let* $g \in \mathcal{U}_n$ *be a normalized Zoll metric, and let* $H \subset O(n + 1)$ *be a finite group preserving* g*. There is an exactly unitary* H*-equivariant Fourier integral operator* U *such that*

$$U^* P_g U = P_0 + K, \qquad K = K^* \in \Psi_{\mathrm{cl}}^{-1}(S^n). \tag{28.1}$$

There are constants C_g, k_g *such that, for every integer* $k \geq k_g$*, the spectral subspace of* P_g *in* $(k - 1/4, k + 1/4)$ *has the same* H*-representation as* $\ker(P_0 - k)$*, and its eigenvalues belong to* $[k - C_g/k, k + C_g/k]$*. These intervals contain every sufficiently large eigenvalue of* P_g*.*

If H *acts freely, write* $M_g = (S^n, g)/H$ *and* $M_0 = (S^n, g_0)/H$*. For every fixed* $\eta > 0$*,*

$$E_\eta\big(m + N + \tfrac{1}{2}; (\Delta_{M_g} + m^2)^{1/2}, (\Delta_{M_0} + m^2)^{1/2}\big) = O_{g,\eta,H}(1) \qquad (N \to \infty). \tag{28.2}$$

The neighborhood $\mathcal{U}_n$ *is independent of* H*. For fixed* g*, the operator* U *and the constants* C_g, k_g *may also be chosen independently of* H*.*

28.1. **The geodesic circle action.** Identify the round unit tangent bundle with

$$Y = \{Z = [\, x\ v\,] \in \mathbb{R}^{(n+1)\times 2} \,:\, Z^{\mathsf{T}}Z = I_2\}.$$

Put

$$R_t = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}, \quad D = \operatorname{diag}(1,-1), \quad \rho(Z) = ZD,$$

and write $\widehat{h}(Z) = hZ$ for $h \in O(n+1)$. Identify the two unit tangent bundles by

$$\nu_g(x,v) = \left(x, \frac{v}{\sqrt{g_x(v,v)}}\right), \qquad \Gamma_g^t = \nu_g^{-1} G_g^t \nu_g.$$

Thus $\Gamma_0^t(Z) = ZR_t$.

Lemma 28.2. *For a normalized Zoll metric g sufficiently close to g_0 in C^2, the map*

$$B_g(Z) = \frac{1}{2\pi}\int_0^{2\pi} \Gamma_g^t(Z) R_{-t}\, dt, \qquad R_g(Z) = B_g(Z)(B_g(Z)^{\mathsf{T}}B_g(Z))^{-1/2} \tag{28.3}$$

is a diffeomorphism of Y, and

$$R_g\Gamma_g^t = \Gamma_0^t R_g, \qquad R_g\rho = \rho R_g, \qquad R_g\widehat{h} = \widehat{h}R_g \quad (h \in O(n+1),\ h^*g = g).$$

Both R_g and R_g^{-1} tend to the identity in C^1 as $g \to g_0$ in C^2.

Proof. On the compact unit tangent bundle, the coefficients of the geodesic vector field depend on the metric and its first derivatives. The first variational equation depends on its second derivatives. The integral equations for the flow and its differential, followed by Gronwall's inequality on $[0, 2\pi]$, give

$$\sup_{0\le t\le 2\pi} \|\Gamma_g^t - \Gamma_0^t\|_{C^1} \le C_n \|g - g_0\|_{C^2}$$

in a fixed neighborhood. Hence $B_g \to \mathrm{Id}$ in C^1. The polar retraction $r(Z) = Z(Z^{\mathsf{T}}Z)^{-1/2}$ is smooth on the full-rank matrices and satisfies

$$r(AZQ) = Ar(Z)Q \qquad (A \in O(n+1),\ Q \in O(2)).$$

It follows that R_g is C^1-close to the identity.

Since Y is compact and has no boundary, the diffeomorphisms form an open subset of $C^1(Y,Y)$ [35, Chapter 2, Theorem 1.6, p. 38]. Thus R_g is a diffeomorphism for g sufficiently close to g_0. Its inverse tends to the identity in C^1, by $D(R_g^{-1}) = (DR_g \circ R_g^{-1})^{-1}$.

Periodicity gives

$$B_g(\Gamma_g^s Z) = \frac{1}{2\pi}\int_s^{2\pi+s} \Gamma_g^u(Z) R_{-u}\, du\, R_s = B_g(Z)R_s.$$

Reversibility gives $\Gamma_g^t\rho = \rho\Gamma_g^{-t}$. Since $DR_{-t} = R_t D$, changing t to $-t$ in the circle integral gives $B_g(\rho Z) = B_g(Z)D$. An orthogonal isometry common to g and g_0 commutes with ν_g and Γ_g^t; hence $B_g(\widehat{h}Z) = hB_g(Z)$. The polar retraction proves all three identities. None of the estimates depends on the common isometry. □

28.2. **Strict contact and homogeneous Hamiltonian conjugacy.** Let θ be the tautological form on T^*S^n, and take $\iota_{X_a} d\theta = -da$. Define

$$\iota_g(x,v) = \left(x, \frac{g_x(v,\cdot)}{\sqrt{g_x(v,v)}}\right), \qquad \alpha_g = \iota_g^*\theta.$$

The Reeb flow of α_g is Γ_g: on $\{p_g = 1\}$, Euler's identity gives $\theta(X_{p_g}) = 1$, while dp_g vanishes on the tangent space of this hypersurface.

Lemma 28.3. *After restricting $\mathcal{U}_n$, there is a homogeneous Hamiltonian isotopy κ_t on $T^*S^n \setminus 0$ such that*

$$\kappa_0 = I, \qquad p_g \circ \kappa_1 = p_0, \qquad \kappa_t^* \theta = \theta.$$

It commutes with all common orthogonal isometries and with $\jmath(x,\xi) = (x,-\xi)$. Its generators are real, homogeneous of degree one, invariant under the common isometries, and odd under $\jmath$.

Proof. Put $\beta = (R_g^{-1})^* \alpha_g$ and $\eta = \beta - \alpha_0$. The two forms have the same Reeb field R_0. Thus

$$\eta(R_0) = 0, \qquad \iota_{R_0} d\eta = 0, \qquad \mathcal{L}_{R_0} \eta = 0.$$

They are invariant under the common isometries and change sign under ρ. For $0 \le s \le 1$ the forms $\alpha_s = \alpha_0 + s\eta$ are contact in a sufficiently small neighborhood and satisfy

$$\alpha_s(R_0) = 1, \qquad \iota_{R_0} d\alpha_s = 0.$$

The Moser isotopy in the proof of [1, Proposition 3.2], with equal periods, is the flow B_s of the uniquely determined vector field

$$\alpha_s(Y_s) = 0, \qquad \iota_{Y_s} d\alpha_s = -\eta.$$

It satisfies $B_s^* \alpha_s = \alpha_0$. Uniqueness makes Y_s, and therefore B_s, commute with the round flow, the common isometries, and ρ; for the last symmetry both $d\alpha_s$ and η change sign. The map $\psi = R_g^{-1} B_1$ satisfies $\psi^* \alpha_g = \alpha_0$. Define

$$\kappa(r\iota_0(z)) = r\iota_g(\psi z), \qquad r > 0.$$

The tautological forms in these coordinates are $r\alpha_0$ and $r\alpha_g$, respectively. Thus $\kappa^*\theta = \theta$ and $p_g \circ \kappa = p_0$. The second variational equation contains derivatives of the geodesic field of order at most two, hence metric derivatives of order at most three. The C^3 metric bound therefore gives C^2 closeness of R_g. Substituting $d\beta = (R_g^{-1})^* d\alpha_g$ in the defining linear system for Y_s gives C^1 closeness of Y_s to zero, uniformly in s. Hence κ is C^1-close to the identity on compact cotangent annuli.

The isotopy can be chosen equivariantly as follows. For nearby $x, y \in S^n$, there is a unique midpoint q and a small $u \in T_q S^n$ such that

$$x = \exp_q(-u), \qquad y = \exp_q(u).$$

This is an $O(n+1)$-equivariant coordinate map near the diagonal. The graph of κ, regarded in $T^*(S^n \times S^n)$ with covectors $(-\xi, \upsilon)$, is Lagrangian. Pull its tautological form back to these midpoint coordinates and write it as $A_i\, dq^i + B_i\, du^i$. Set $\zeta_i = B_i/2$. Subtraction of $d(2\zeta_i u^i)$ gives

$$A_i\, dq^i - 2u^i\, d\zeta_i.$$

This is the canonical form on $T^*(T^*S^n)$ with base (q,ζ) and fiber $(A,-2u)$. Both the original tautological form and the differential of the pairing $2\zeta_i u^i$ are intrinsic, so this symplectic map is independent of coordinates. The diagonal corresponds to the zero section. The transformed graph of κ is the graph of a small closed one-form σ on $T^*S^n \setminus 0$. Projection to (q,ζ) is a local diffeomorphism by the inverse function theorem. On the unit cosphere, its angular part is a C^1-small perturbation of the identity and hence a diffeomorphism. Its radial part is multiplication by the positive function $|\zeta|/|\xi|$. Homogeneity therefore makes the full projection a diffeomorphism of the punctured cotangent bundle.

For fiber dilation δ_r and its generator E, homogeneity implies $\delta_r^* \sigma = r\sigma$. Hence

$$\sigma = d(\sigma(E))$$

by Cartan's formula. Replace σ by $t\sigma$ and invert the midpoint construction. Projection of each inverse image to the input covector (x,ξ) again has angular part C^1-close to the identity and positive radial factor, uniformly for $0 \le t \le 1$. Thus these Lagrangians are graphs of homogeneous symplectic diffeomorphisms κ_t. Every such map satisfies

$$\kappa_t^* \theta = \kappa_t^*(\iota_E d\theta) = \iota_E \kappa_t^* d\theta = \theta.$$

All maps are equivariant under the common orthogonal group. Simultaneous reversal of ξ, υ leaves q, u unchanged and negates A, B, so $\jmath^* \sigma = -\sigma$; this remains true for $t\sigma$. Thus $\kappa_t \jmath = \jmath \kappa_t$. Finally, for $W_t = \dot{\kappa}_t \kappa_t^{-1}$, set $G_t = \theta(W_t)$. Differentiation of $\kappa_t^* \theta = \theta$ gives $\iota_{W_t} d\theta = -dG_t$. Homogeneity gives degree one; the stated invariance and oddness follow from the symmetries of W_t and θ. □

28.3. **Quantization through the subprincipal term.**

Proposition 28.4. *Let $H \subset O(n+1)$ be finite and preserve g. Suppose a homogeneous Hamiltonian isotopy κ_t, with $\kappa_0 = I$, is generated by real degree-one functions G_t, commutes with H, and satisfies $p_g \circ \kappa_1 = p_0$. Then* (28.1) *holds.*

Proof. Since G_t has degree one, $G_t = \theta(\dot\kappa_t \kappa_t^{-1})$ by Euler's identity. The cotangent lifts of H preserve θ, so G_t is H-invariant. Choose symmetric operators $A_t \in \Psi^1_{\mathrm{cl}}$ with principal symbol G_t and subprincipal symbol zero. In half-density coordinates this is real Weyl quantization. A real partition $\sum_j \chi_j^2 = 1$ and the sum $\sum_j \chi_j A_{t,j} \chi_j$ preserve both symbols: the two Poisson terms involving χ_j cancel. Let $\mathcal{R}_h = (h^{-1})^*$ on half-densities. Averaging by $|H|^{-1} \sum_h \mathcal{R}_h(\cdot)\mathcal{R}_h^{-1}$ makes the operators H-equivariant without changing those symbols. The same construction applies to a compact group of common orthogonal isometries, with Haar probability measure in place of the finite average. Conjugation by an orthogonal map is continuous in each classical symbol seminorm. Compactness of $O(n+1)$ therefore bounds these seminorms before and after averaging, independently of the subgroup.

Solve $\partial_t U_t = -iA_t U_t$, $U_0 = I$. The first-order evolution and Sobolev energy estimates of [65, Section I.5, equations (5.1)–(5.8)] apply on the compact time interval. The homogeneous flow is complete there: $|X_{G_t} p_0| \le C p_0$ gives $e^{-Ct} p_0 \le p_0 \circ \kappa_t \le e^{Ct} p_0$, and the angular variables lie in the compact cosphere. Differentiating $\langle U_t u, U_t v\rangle$ gives zero. The backwards equation supplies the inverse, so U_t is exactly unitary. Uniqueness makes it commute with every $\mathcal{R}_h$.

The scalar Cauchy propagator construction [22, Section 2.1, equation (6)], with $\lambda = -G_t$, gives a classical Fourier integral operator of order zero associated with κ_t. The spacetime symbol $\tau + G_t$ satisfies $X_{\tau+G_t} t = 1$, so the initial hypersurface is noncharacteristic for the transport equations [18, Theorems 5.3.1–5.3.2]. The local propagators compose over $[0,1]$; the Sobolev estimates and the Duhamel formula identify their composition with U_t modulo a smoothing operator.

For a scalar order-one operator with principal and subprincipal symbols b, s, respectively, the scalar Weyl product gives

$$\sigma_1(i[A_t, B]) = \{G_t, b\}, \qquad \mathrm{sub}(i[A_t, B]) = \{G_t, s\}.$$

The next odd Weyl-product term has degree $1 + 1 - 3 = -1$. Egorov's theorem [65, Theorem I.6.1] gives $U_t B U_t^* \in \Psi^1_{\mathrm{cl}}$. Since $\partial_t(U_t B U_t^*) = -i[A_t, U_t B U_t^*]$, the displayed identities give $\partial_t a_t + \{G_t, a_t\} = 0$ for its principal and subprincipal symbols. Their initial values are b and s; hence they are $b \circ \kappa_t^{-1}$ and $s \circ \kappa_t^{-1}$. Equivalently the inverse conjugation has symbols $b \circ \kappa_t$ and $s \circ \kappa_t$. Lemma 20.6 gives $\mathrm{sub}(P_g) = \mathrm{sub}(P_0) = 0$ on half-densities. Hence $U_1^* P_g U_1 - P_0$ has order -1. The propagators preserve every Sobolev space, so the two order-one operators have the same domain H^1. Their bounded difference is self-adjoint. Take $U = U_1$. □

28.4. **High eigenspaces and quotient multiplicities.**

Proof of Theorem 28.1. The common orthogonal isometry group

$$\mathcal{I}_g = \{h \in O(n+1) : h^* g = g\}$$

is closed in $O(n+1)$ and hence compact. Lemma 28.3 gives an $\mathcal{I}_g$-equivariant isotopy. In the construction of Proposition 28.4, average over $\mathcal{I}_g$ with its Haar probability measure. This gives one operator U and one correction K for g, commuting with every subgroup $H \le \mathcal{I}_g$. The neighborhood and these operators are independent of the chosen H. Put $B_t = P_0 + tK$ and $C = \|P_0^{1/2} K P_0^{1/2}\|$. Then

$$P_0 - C P_0^{-1} \le B_t \le P_0 + C P_0^{-1} \qquad (0 \le t \le 1).$$

Lemma 20.5, with the displayed common form bound, gives uniformly in t the ordered estimate

$$|\lambda_j(B_t) - \lambda_j(P_0)| \le C/\lambda_j(P_0)$$

for all large j, without an index shift. Thus the high spectrum lies in disjoint intervals $[k - C/k, k + C/k] \subset (k - 1/8, k + 1/8)$. The resolvent identity makes the projections $\Pi_{t,k}$ onto $(k - 1/4, k + 1/4)$ norm-continuous in t.

Each $\Pi_{t,k}$ commutes with H. Lemma 1.4 identifies its representation with that of $\Pi_{0,k} = \mathbf{1}_{\{k\}}(P_0)$. The unitary U transfers this identification to P_g.

For a free action, pullback from the quotient, multiplied by $|H|^{-1/2}$, identifies quotient eigenspaces with invariant subspaces upstairs. Choose k_η so that $C/k \le \eta$ for $k \ge k_\eta$. Equal invariant multiplicities match every copy in such a cluster to its round integer center. The cutoffs $m + N + 1/2$ contain whole high clusters. The two finite spectra below $k_\eta + 1/4$ contain only finitely many copies, independent of N. This proves (28.2). □

Remark 28.5. The functions $p_t = p_0 \circ \kappa_t^{-1}$ associated with Lemma 28.3 are positive and homogeneous of degree one. They need not be Riemannian dual norms, and their Hamiltonian generators G_t need not be positive. Theorem 28.1 holds for each normalized Zoll metric $g \in \mathcal{U}_n$. For the invariant family of Proposition 28.6, the cluster bounds are uniform on $[0,1]$; the metrics need not belong to $\mathcal{U}_n$.

28.5. **Invariant families of Zoll metrics.** Let p and a be as in Section 27.

Proposition 28.6. *Fix a free smooth C_p-action on S^{2m+1}, $m \ge 2$, which is linear with weights $(1, \dots, 1, a)$ for the unit round metric. Suppose g_s, $s \in [0,1]$, is a smooth family of invariant Riemannian metrics with g_0 the unit round metric, such that every prime geodesic of every g_s has length 2π. Put $M_s = (S^{2m+1}, g_s)/C_p$ and $P_s = \sqrt{\Delta_{g_s} + m^2}$. The kth cluster is the spectrum in $(k - 1/4, k + 1/4)$. There are K_0 and C, independent of s, such that for every integer $k \ge K_0$:*

(1) every frequency of P_s in the kth cluster differs from k by at most C/k;

(2) the C_p-representation carried by that cluster is isomorphic to its representation for g_0;

(3) the quotient cluster multiplicity on M_s equals that on M_0.

These clusters contain every frequency above $K_0 - 1/2$. Consequently, for each fixed $\eta > 0$, the quotient frequencies admit matchings at the whole-cluster cutoffs $R_K = m + K + \frac{1}{2}$ satisfying

$$E_\eta(R_K; M_s, M_0) = O_\eta(1),$$

uniformly in $s \in [0,1]$.

Proof. Identify the scalar L^2 spaces by multiplication with the square root of the smoothly varying density, or work directly with half-densities. This identification commutes with the fixed C_p-action. The scalar subprincipal symbol of P_s vanishes, and its Hamiltonian flow has period 2π. Formula (20.5), with $B_0 = 0$, gives $\sigma_0(e^{2\pi i P_s}) = \rho_s$, where ρ_s is the Maslov factor. Under radial identification with $S^*_{g_0}S^n$, this fourth root of unity is locally constant on the connected space $[0,1] \times S^*_{g_0}S^n$. At $s = 0$ the frequencies are the integers $k + m$, so $\rho_s = \rho_0 = 1$. In the Fourier integral construction [18, Theorems 5.3.1–5.3.2], the phase and transport equations have coefficients smooth in s. Their solutions and asymptotic sums are smooth in s on a finite cover of $[0,1] \times [0, 2\pi] \times S^*_{g_0}S^n$. The Sobolev estimates for the remainders are uniform on this compact set. The return graphs are the diagonal for every s; therefore

$$e^{2\pi i P_s} = I + R_s, \qquad R_s \in \Psi^{-1},$$

as a smooth family of classical pseudodifferential operators.

The order-zero family $R_s P_s$ has uniformly bounded L^2 operator norm, say at most C_0. For an eigenvector with frequency μ,

$$|e^{2\pi i\mu} - 1| \le C_0/\mu.$$

If $d = \operatorname{dist}(\mu, \mathbb{Z}) \le 1/2$, then $|e^{2\pi i\mu} - 1| = 2\sin(\pi d) \ge 4d$. Hence $d \le C_0/(4\mu)$. Choose K_0 large enough that all frequencies above $K_0 - 1/2$ lie in disjoint intervals $(k - 1/4, k + 1/4)$ and satisfy $|\mu - k| \le C/k$. Increase K_0 so that $C/K_0 < 1/8$.

For each such k, let $\Pi_{s,k}$ be the Riesz projector onto $(k - 1/4, k + 1/4)$. Its contour is uniformly separated from the spectrum. The family P_s is smooth as a map $H^1 \to L^2$; elliptic estimates and the resolvent identity therefore make $\Pi_{s,k}$ norm-continuous in s. It commutes with the fixed C_p-action. Lemma 1.4 proves equality of the cluster representations and of their invariant multiplicities.

Match all copies in each quotient cluster with the copies of the round frequency k. If $k \geq \max\{K_0, C/\eta\}$, all match errors are at most η. The whole-cluster cutoffs include the same labels on both sides. Compactness of the metric family gives uniform upper and lower bounds for the metric tensors and their volume densities. The Rayleigh quotients and the min–max principle therefore give

$$\lambda_j(\Delta_{g_s}) \geq c\lambda_j(\Delta_{g_0}) \qquad (s \in [0,1],\ j \geq 0)$$

with $c > 0$ independent of s, j. The number of frequencies below any fixed bound is consequently bounded uniformly in s. The quotient multiplicity is at most the corresponding multiplicity upstairs. Leaving the remaining low frequencies unmatched gives $E_\eta(R_K; M_s, M_0) = O_\eta(1)$ uniformly in s. □

29. Integral topology of order-three returns

29.1. **The geodesic circle.** Let X^{2m+1} be a closed smooth homotopy sphere, where $m \geq 2$. Suppose that every unit-speed geodesic has prime period 2π. Write $E = SX$, let G_t denote geodesic flow, and put

$$p : E \longrightarrow \mathcal{G} = E/S^1, \qquad \pi : E \longrightarrow X.$$

The action of $\mathbb{R}/(2\pi\mathbb{Z})$ on E is free. The canonical contact form is

$$\lambda_{(x,v)}(\xi) = g_x(v, d\pi\xi).$$

There is a symplectic form ω on $\mathcal{G}$ with $p^*\omega = d\lambda$. Give $\mathcal{G}$ its symplectic orientation. Let $e \in H^2(\mathcal{G};\mathbb{Z})$ be the Euler class with the positive convention of Section 12; thus $e_{\mathbb{R}} = [\omega/(2\pi)]$. Proposition 12.3 gives $\mathrm{Vol}(X) = \mathrm{Vol}(S^{2m+1})$.

Lemma 29.1. *There are integral classes x, y such that*

$$H^*(E;\mathbb{Z}) = \mathbb{Z}\langle 1, x, y, xy\rangle, \qquad |x| = 2m, \quad |y| = 2m+1, \quad x^2 = y^2 = 0,$$

where $y = \pi^* u_X$ for an oriented generator u_X of $H^{2m+1}(X;\mathbb{Z})$. The integral cohomology of $\mathcal{G}$ is torsion-free. Its odd groups vanish. For $0 \leq j \leq 2m$, the group $H^{2j}(\mathcal{G};\mathbb{Z})$ has rank one, except when $j = m$, when it has rank two. The class e^m is primitive, and

$$\langle e^{2m}, [\mathcal{G}]\rangle = 2. \tag{29.1}$$

Every odd-order isometry of X acts trivially on $H^*(\mathcal{G};\mathbb{Z})$. In the Borel spectral sequence of the geodesic circle action,

$$d_2 y = \pm 2ux, \qquad u \in H^2(BS^1;\mathbb{Z}) \text{ a generator.} \tag{29.2}$$

Proof. The sphere-bundle computation in Lemma 13.5 gives the ring $H^*(E;\mathbb{Z})$ and the stated class y. Lemma 12.1 gives the cohomology of $\mathcal{G}$ and the primitivity of e^m. Proposition 12.3 and (12.2) give (29.1). The odd-order action is determined in Lemma 13.3, and the integral transgression is Lemma 13.5. □

Lemma 29.2. *Let q be an odd prime and let h generate a free isometric action of C_q on X. Denote its tangent lift by $\widehat{h}$ and its induced map on $\mathcal{G}$ by $\bar{h}$. For $c \in \mathbb{F}_q^*$, set*

$$F_c = \mathrm{Fix}(\widehat{h}^{-1} G_{2\pi c/q}), \qquad B_c = F_c/S^1.$$

Every nonempty F_c is a connected closed contact manifold and a mod-q cohomology sphere. If $\dim F_c = 2d_c - 1$, then

$$H^*(B_c;\mathbb{F}_q) = \mathbb{F}_q[e_c]/(e_c^{d_c}),$$

where e_c is the restriction of e reduced modulo q. Moreover,

$$\mathrm{Fix}(\bar{h} : \mathcal{G} \to \mathcal{G}) = \coprod_{c \neq 0} B_c.$$

Reversal interchanges F_c and F_{-c}, and

$$\sum_{\{c,-c\}:\, F_c \neq \emptyset} d_c = m + 1. \tag{29.3}$$

Proof. The lift $\widehat{h}$ preserves λ and commutes with G_t. Both factors of $S^1 \times C_q$ act trivially on $H^*(E;\mathbb{Z})$: the circle is connected, and an odd-order automorphism of each cyclic group is the identity. In its mod-q Borel sequence, write u, v for the degree-two generators of the circle and deck factors. Degree considerations give

$$d_2 y = (au + bv)x, \qquad d_2 x = 0.$$

Circle restriction and Lemma 29.1 give $a = \pm 2$. Evaluation is deck equivariant and pulls the top class of X back to y. The d_2 target for that class on X is zero, so deck restriction gives $b = 0$.

Restrict to the subgroup of $S^1 \times C_q$ generated by $(e^{2\pi i c/q}, h^{-1})$, whose action is generated by $K_c = \widehat{h}^{-1} G_{2\pi c/q}$. The circle class restricts to cv, so $d_2 y = \pm 2cvx$. Over $\mathbb{F}_q(v)$ this coefficient is invertible, x is a boundary, and xy is a cycle because $x^2 = 0$. The third page therefore has basis

$$1, \quad xy, \quad \epsilon, \quad \epsilon xy$$

where $|\epsilon| = 1$ and $\epsilon^2 = 0$. Lemma 13.2 therefore gives $2\sum_j b_j(F_c;\mathbb{F}_q) \le 4$.

For a K_c-invariant auxiliary metric, the fixed set is a closed submanifold with tangent space $\ker(dK_c - I)$ [44, Chapter II, Theorem 5.1]. This space contains the Reeb direction. Its intersection with the contact hyperplane is symplectic: for a fixed vector v, averaging a second vector w over the finite group does not change $d\lambda(v, w)$. Nondegeneracy of $d\lambda$ on the contact hyperplane gives nondegeneracy on its fixed subspace. Thus every nonempty component of F_c is a closed oriented contact manifold of positive odd dimension. Each contributes its degree-zero and top cohomology to the preceding bound. Hence F_c is connected and has no intermediate mod-q cohomology. The Gysin sequence of its free geodesic circle action gives the ring $H^*(B_c;\mathbb{F}_q) = \mathbb{F}_q[e_c]/(e_c^{d_c})$, including $d_c = 1$ when B_c is a point.

On a circle fiber above $\mathrm{Fix}(\bar{h})$, the map $\widehat{h}$ is a rotation. Its angle depends continuously on the base point in a local bundle trivialization. Since $\widehat{h}^{\,q} = I$, that angle belongs to $\{2\pi c/q : c \in \mathbb{F}_q\}$ and is locally constant. The value $c = 0$ would give a fixed point of h on X. Thus $\mathrm{Fix}(\bar{h}) = \coprod_{c\neq 0} B_c$. The identities $JG_t = G_{-t}J$ and $J\widehat{h} = \widehat{h}J$, for $J(x, v) = (x, -v)$, give $J(F_c) = F_{-c}$ and $d_c = d_{-c}$.

Give $\mathcal{G}$ a $\bar{h}$-invariant Riemannian metric by averaging. The Lefschetz formula for an isometry [44, Chapter II, §5, p. 63] gives $L(\bar{h}) = \chi(\mathrm{Fix}(\bar{h}))$. Lemma 29.1 and the cohomology of B_c therefore give

$$2m + 2 = \sum_{c\neq 0} \chi(B_c) = \sum_{c\neq 0} d_c.$$

Division into reversal pairs proves (29.3). □

29.2. **The short return families.** Let h be a fixed-point-free isometry of exact order three, and put

$$\ell = \frac{2\pi}{3}, \qquad F_+ = \mathrm{Fix}(\widehat{h}^{-1} G_\ell), \qquad F_- = \mathrm{Fix}(\widehat{h}^{-1} G_{-\ell}), \qquad B_\pm = F_\pm / S^1.$$

Here $\widehat{h}$ is the tangent lift. Since h has prime order, its action on X is free.

Theorem 29.3. *The sets F_+ and F_- are connected closed contact manifolds of dimension $2m + 1$. They are interchanged by $(x, v) \mapsto (x, -v)$, and*

$$H^*(F_\pm;\mathbb{F}_3) \cong H^*(S^{2m+1};\mathbb{F}_3), \qquad H^*(B_\pm;\mathbb{F}_3) \cong \mathbb{F}_3[e]/(e^{m+1}).$$

Moreover,

$$\mathrm{Fix}(\bar{h} : \mathcal{G} \to \mathcal{G}) = B_+ \sqcup B_-. \tag{29.4}$$

Each evaluation map $F_\pm \to X$ has integral degree prime to three. In particular it is onto, and

$$d(x, hx) = d(x, h^2 x) = \frac{2\pi}{3} \qquad (x \in X). \tag{29.5}$$

Proof. For $q = 3$ the only possible nonzero phases are 1 and -1. Lemma 29.2 gives precisely one reversal pair, and (29.3) gives $d_1 = m + 1$. This proves the statements about dimensions, cohomology, and the fixed set.

The deck action on F_+ is free, since evaluation maps it equivariantly to the free action on X. On F_+ it equals the contact flow G_ℓ; on X its degree has odd order in $\{1, -1\}$. Both actions therefore preserve orientation. For $Z = F_+$ and $Z = X$, the mod-three Borel spectral sequence has

$$E_2 = \mathbb{F}_3[v] \otimes \Lambda(\epsilon) \otimes H^*(Z; \mathbb{F}_3), \qquad |v| = 2, \quad |\epsilon| = 1,$$

and only the rows zero and $2m + 1$ are nonzero. If u_Z is the reduction of an oriented integral top class, the only differential that can kill v^{m+1} is

$$d_{2m+2} u_Z = a_Z v^{m+1}.$$

The Borel construction is homotopy equivalent to the closed $(2m+1)$-manifold Z/C_3. Its cohomology in degree $2m + 2$ vanishes, so $a_Z \neq 0$. If D is the evaluation degree modulo three, naturality gives $a_X = D a_{F_+}$. Hence $D \neq 0$. The integral degree is prime to three. Evaluation is onto, since $H_{2m+1}(X \setminus \{x\}; \mathbb{F}_3) = 0$ for every $x \in X$ [31, Proposition 3.29]. Reversal commutes with evaluation and identifies F_+ with F_-. Their evaluation degrees agree up to sign, so the assertion holds for F_- as well.

For every x, choose $v \in F_+$ based at x. Its length-ℓ geodesic ends at hx, so $d(x, hx) \leq \ell$. Equation (15.2), applied with $s = 3$ and $L = 2\pi$, gives $d(x, hx) \geq \ell$. Hence $d(x, hx) = \ell$ for every x. Finally, $d(x, h^2 x) = d(hx, h^3 x) = d(x, hx)$. □

Corollary 29.4. *For $M = X/\langle h \rangle$, every prime geodesic has length $2\pi/3$ or 2π. The prime geodesics of length $2\pi/3$ are exactly the noncontractible prime geodesics. One passes through every point of M.*

Proof. Extend a unit-speed lift to a geodesic on X, of prime period 2π. The stabilizer of its oriented geodesic in C_3 is either trivial or all of C_3. In the first case its projection has prime period 2π and trivial monodromy. In the second, Theorem 29.3 places the oriented geodesic in B_+ or B_-; its prime projected period is $2\pi/3$ and its monodromy is h or h^2. Since X is simply connected, a quotient loop is contractible exactly when its monodromy is trivial. Surjectivity of $F_+ \to X$ gives a noncontractible prime geodesic through every point of the quotient. □

Corollary 29.5. *For odd-dimensional Besse homotopy spheres of dimension at least five, the following conditions are equivalent:*

(1) every sphere admitting a nonidentity Clifford translation of order three has constant sectional curvature;

(2) every sphere admitting a free isometric C_3 action has constant sectional curvature.

Proof. Smale's theorem [59, Theorem A] makes each such homotopy sphere a topological sphere. By [57, Theorem A], its prime geodesics have one common length. Rescale that length to 2π. A nonidentity constant-displacement map has positive displacement and no fixed point. Conversely, Theorem 29.3 makes every generator of a free isometric C_3 action a Clifford translation. □

29.3. **Twisted indices and integral homotopy type.**

Theorem 29.6. *Under the hypotheses of Theorem 29.3, both F_+ and F_- are homotopy $(2m+1)$-spheres. Each evaluation $F_\pm \to X$ has degree ± 1 and is a homotopy equivalence. Put $e_\pm = e|_{B_\pm} \in H^2(B_\pm; \mathbb{Z})$. With the symplectic orientation and the positive Euler-class convention,*

$$H^*(B_\pm; \mathbb{Z}) = \mathbb{Z}[e_\pm]/(e_\pm^{m+1}), \qquad \langle e_\pm^m, [B_\pm] \rangle = 1, \qquad B_\pm \simeq \mathbb{CP}^m.$$

Moreover,

$$X/\langle h \rangle \simeq L(3; 1, \dots, 1).$$

The critical manifolds of the energy on the h-twisted path space, in increasing order of length, have dimension $2m + 1$ and indices $0, 2m, 4m, 6m, \dots$.

For an isometry f of a closed manifold X, put

$$\Lambda_f X = \{c \in H^1([0,1], X) \,:\, c(1) = f\,c(0)\}, \qquad E(c) = \frac{1}{2}\int_0^1 |\dot{c}|^2\, dt.$$

A path is critical if and only if it is a smooth geodesic satisfying $\dot{c}(1) = df\,\dot{c}(0)$ [25, Theorem 1.6(2)].

Lemma 29.7. *The energy on $\Lambda_f X$ satisfies the Palais–Smale condition. Suppose its critical set at a value c is a compact submanifold whose Hessian kernel is its tangent bundle. Then c is isolated among the critical values. Up to homotopy, crossing c attaches the negative disk bundle along its sphere bundle. Suppose the full minimum set C is such a submanifold, every higher critical level has the same property, and every negative bundle above C has rank at least $r \geq 1$. Then*

$$\pi_j(\Lambda_f X, C) = 0 \qquad (j < r).$$

If X is a homotopy n-sphere, $n \geq 5$, and f has degree one, then

$$\pi_j(\Lambda_f X) = 0 \quad (1 \leq j \leq n-2), \qquad (\mathrm{ev}_0)_* \,:\, H_n(\Lambda_f X; \mathbb{Z}) \xrightarrow{\cong} H_n(X; \mathbb{Z}).$$

Proof. Take $N = \mathrm{graph}(f) \subset X \times X$. This is a closed submanifold with compact projections, and X is complete. Grove's theorem therefore gives the Palais–Smale condition [25, Theorem 2.4]. The graph of f is totally geodesic. At a critical path, the Hessian, represented for the H^1 inner product $\int_0^1 (\langle D_t Y, D_t Z\rangle + \langle Y, Z\rangle)\, dt$, is identity minus a compact self-adjoint operator: its lower-order terms factor through the compact inclusion $H^1 \hookrightarrow L^2$. It has finite negative index and is invertible on the orthogonal complement of its kernel. The Morse–Bott attachment theorem applies [3, Appendix B, Theorems B.1–B.3]. An accumulation of critical values in a bounded interval would, by Palais–Smale, yield a critical point with distinct critical values in every neighborhood, contrary to its Morse–Bott normal form. Thus only finitely many attachments occur in each bounded interval. A CW decomposition of each compact critical manifold gives relative cells of dimension at least the negative rank. Cellular approximation gives the asserted relative connectivity on each bounded sublevel. Every map from a compact pair into $(\Lambda_f X, C)$ has bounded energy, so the same conclusion holds for the full path space.

The inclusion into the continuous twisted path space is a homotopy equivalence [25, Theorem 1.3]. Evaluation is therefore a fibration up to homotopy with fiber ΩX. Its homotopy sequence gives $\pi_j(\Lambda_f X) = 0$ for $1 \leq j \leq n-2$. The James reduced-product model $J(S^{n-1})$ for ΩS^n has cells only in dimensions divisible by $n-1$ [31, Theorem 4J.1]. Since $n \geq 5$, it gives $H_n(\Omega X; \mathbb{Z}) = 0$. Since f has degree one, it is smoothly homotopic to id_X. Such a homotopy gives a section of evaluation. In its integral homology spectral sequence,

$$E^2_{a,b} = H_a(X; H_b(\Omega X; \mathbb{Z})),$$

the only nonzero term of total degree n is $E^2_{n,0} = \mathbb{Z}$. The section makes this term permanent. The edge homomorphism is $(\mathrm{ev}_0)_*$, which is consequently an isomorphism. □

Lemma 29.8. *Let γ be a unit-speed geodesic with $(\gamma(0), \dot{\gamma}(0)) \in F_+$. For $|z| = 1$, let $j(z)$ be the complex negative index of*

$$I_z(J) = \int_0^\ell \left(|D_t J|^2 - \langle R(J, \dot{\gamma})\dot{\gamma}, J\rangle\right) dt$$

on the complex normal H^1 fields satisfying $J(\ell) = z\, dh\, J(0)$. Then

$$j(1) = 0, \qquad j(e^{i\theta}) = \begin{cases} m, & 0 < \theta \leq 2\pi/3 \text{ or } 4\pi/3 \leq \theta < 2\pi, \\ 2m, & 2\pi/3 < \theta < 4\pi/3. \end{cases}$$

For an integer $r \geq 1$, let i_r be the real Morse index of the arc $\gamma|_{[0,r\ell]}$ with endpoint twist h^r. One has

$$i_{3k+1} = 4km, \qquad i_{3k+2} = (4k+2)m, \qquad i_{3k} = (4k-2)m, \tag{29.6}$$

where $k \geq 0$ in the first two formulas and $k \geq 1$ in the last.

Proof. Theorem 29.3 gives $d(x, hx) = \ell$ for every x. Thus the short arc minimizes the twisted energy and $j(1) = 0$. The endpoint terms in its Hessian cancel, since h is an isometry and $\dot\gamma(\ell) = dh\,\dot\gamma(0)$.

Write P_ℓ for Jacobi transport on normal initial data and set

$$T = (dh^{-1} \oplus dh^{-1})P_\ell.$$

This map is symplectic on a real vector space of dimension $4m$. The identity $G_{3\ell} = I$ and commutation with $\widehat{h}$ give $T^3 = I$. Its fixed space is the contact tangent space of F_+, of dimension $2m$. Hence T is semisimple, and its complex eigenspaces at $\zeta = e^{2\pi i/3}$ and ζ^2 each have dimension m.

Extend a field on the r-fold arc by $J(t + r\ell) = dh^r J(t)$. Translation gives the unitary operator

$$(S_r J)(t) = dh^{-1} J(t + \ell), \qquad S_r^r = I.$$

It preserves the index form. Restriction to $[0, \ell]$ identifies its z-eigenspace with the domain of I_z, and the restricted form is rI_z. The character decomposition [26, Lemma 2.3 and (2.4)] gives

$$i_r = \sum_{z^r=1} j(z).$$

Here complexification preserves the negative index, and the tangential form $\int_0^{r\ell} |f'|^2\,dt$ is nonnegative. In particular, Lemma 11.2 gives

$$2m = j(1) + j(\zeta) + j(\zeta^2).$$

The kernel of I_z consists of Jacobi fields satisfying also $D_t J(\ell) = z\,dh\,D_t J(0)$; it is therefore the z-eigenspace of T. Let the Hermitian inner product be linear in its first argument. For Jacobi fields, the Wronskian

$$\Omega(Y, \overline{Z}) = \langle D_t Y, Z\rangle - \langle Y, D_t Z\rangle$$

is constant. At a phase θ_0, the gauge $J(t) = e^{i(\theta-\theta_0)t/\ell} Y(t)$ fixes the form domain, and its derivative on the kernel is

$$Q_{\theta_0}(Y, Z) = \frac{i}{\ell}\int_0^\ell \big(\langle Y, D_t Z\rangle - \langle D_t Y, Z\rangle\big)\,dt = -i\Omega(Y, \overline{Z}).$$

If E_z is an eigenspace of T, symplectic invariance gives $\Omega(E_z, E_w) = 0$ for $zw \neq 1$. Semisimplicity makes the pairing of E_z with $E_{z^{-1}}$ nondegenerate. Thus Q_{θ_0} is nondegenerate at each of the three eigenphases.

Under the same gauge, the Jacobi operators have a fixed H^2 domain with twisted value and derivative conditions. They form a self-adjoint holomorphic family of type (A), with compact resolvent. The derivatives of the eigenvalues through zero are the eigenvalues of Q_{θ_0} relative to the L^2 inner product [43, Chapter VII, Theorem 3.9; Chapter II, Theorem 5.4]. The index is constant off the three eigenphases. At $z = 1$, the real fixed symplectic space has dimension $2m$, so Q_0 has inertia (m, m). Since $j(1) = 0$, it follows that $j(e^{i\theta}) = m$ for $0 < \theta < 2\pi/3$.

Complex conjugation gives $j(\zeta) = j(\zeta^2)$, so both equal m. At $\theta = 2\pi/3$, the index immediately before and at the crossing is m, whereas the kernel has dimension m. Every crossing eigenvalue is therefore positive on the left and negative on the right. The index is $2m$ on $(2\pi/3, 4\pi/3)$. Conjugation gives its values on the remaining arc and at $4\pi/3$.

Finally,

$$i_r = m\big(r - 1 + \#\{k \in \mathbb{Z} : r/3 < k < 2r/3\}\big).$$

The number in braces is k, $k + 1$, or $k - 1$ when $r = 3k + 1$, $3k + 2$, or $3k$, respectively. This gives (29.6). □

Proof of Theorem 29.6. Put $n = 2m + 1$. A critical path in $\Lambda_h X$ extends by h to a closed geodesic. If its length is L, its unit tangent state returns after $3L$; hence $L = r\ell$ for an integer $r \geq 1$. The integer r is not divisible by three, since h has no fixed point. For $r = 3k + 1$ the initial unit state belongs to F_+; for $r = 3k + 2$ it belongs to F_-. Conversely, each of these states determines a critical path of

the indicated length. Uniqueness for the geodesic and Jacobi equations makes the initial-state map an injective immersion. Compactness makes it an embedding, so each critical manifold is compact and has dimension n. If $r \equiv 2 \pmod 3$, time reversal gives an r-fold arc in F_+ with endpoint twist $h^{-1} = h^r$ and preserves the index form. Formula (29.6) therefore gives the successive indices

$$0,\ n-1,\ 2(n-1),\ 3(n-1), \dots .$$

The normal Hessian kernel consists of the fixed Jacobi data of the relative return and has dimension $2m$. The tangential Jacobi fields are $(A + Bt)\dot\gamma$; the twisted value condition gives $B = 0$. Thus the full kernel has dimension n. The differential of the initial-state parametrization identifies the tangent space of the critical manifold with this kernel: the fixed set of the finite-order return has tangent space equal to its fixed subspace. All critical manifolds are consequently Morse–Bott. Every $c \in \Lambda_h X$ satisfies

$$E(c) \geq \tfrac{1}{2} L(c)^2 \geq \tfrac{1}{2} d(c(0), hc(0))^2 = \tfrac{1}{2}\ell^2.$$

The family F_+ attains equality; the classification of critical paths makes it the full minimum set. The map h has degree one, since its order is odd. Lemma 29.7 gives

$$\pi_j(\Lambda_h X, F_+) = 0 \quad (j < n-1), \qquad \pi_j(F_+) = 0 \quad (1 \leq j \leq n-3).$$

For the second vanishing statement, apply the relative homotopy sequence and the vanishing for $\Lambda_h X$ in that lemma.

As $n \geq 5$, F_+ is simply connected and $H_j(F_+; \mathbb{Z}) = 0$ for $1 \leq j \leq n-3$ by Hurewicz. The universal coefficient theorem gives $H^1(F_+; \mathbb{Z}) = H^2(F_+; \mathbb{Z}) = 0$. Integral Poincaré duality then gives $H_{n-1}(F_+; \mathbb{Z}) = H_{n-2}(F_+; \mathbb{Z}) = 0$. Thus F_+ is a simply connected integral homology sphere. Hurewicz and the homological Whitehead theorem give $F_+ \simeq S^n$ [31, Corollary 4.33]. Reversal gives $F_- \simeq S^n$.

The first attachment above F_+ has rank $n-1$ over F_-. Its negative bundle is orientable because F_- is simply connected. The Thom isomorphism gives relative homology in degrees $n, n+1$ equal to $H_1(F_-; \mathbb{Z}), H_2(F_-; \mathbb{Z})$, respectively; both vanish. Every later negative rank is at least $2(n-1) > n+1$. The exact sequences of the sublevel pairs therefore give

$$H_n(F_+; \mathbb{Z}) \xrightarrow{\cong} H_n(\Lambda_h X; \mathbb{Z}).$$

Compose with the evaluation isomorphism of Lemma 29.7. Evaluation $F_+ \to X$ has degree ± 1 and is a homotopy equivalence. Reversal gives the same conclusion for F_-.

The integral Gysin sequence of $F_\pm \to B_\pm$ gives

$$H^*(B_\pm; \mathbb{Z}) = \mathbb{Z}[e_\pm]/(e_\pm^{m+1}).$$

In particular $e_\pm^m$ generates the top cohomology. Its symplectic representative has positive integral, so $\langle e_\pm^m, [B_\pm] \rangle = 1$. The homotopy sequence of the circle bundle gives $\pi_1(B_\pm) = 0$. Since $\dim B_\pm = 2m$, cellular approximation gives a classifying map for $e_\pm$ with values in $\mathbb{CP}^m$. It induces an integral cohomology isomorphism; the homological Whitehead theorem gives $B_\pm \simeq \mathbb{CP}^m$.

The quotient $X/\langle h \rangle$ is a finite CW complex with fundamental group C_3 and universal cover homotopy equivalent to S^n. By the homotopy classification [67, Proposition 1, p. 117], its polarized homotopy type is determined by a generator of $H^{n+1}(C_3; \mathbb{Z}) \cong \mathbb{Z}/3$. Postcomposing the chosen equivalence of the universal cover with a self-map of S^n of degree -1 negates this generator. The two generators therefore give the same unpolarized homotopy type, represented by $L(3; 1, \dots, 1)$. □

Question 29.9. Let X^{2m+1}, $m \geq 2$, be a normalized Zoll homotopy sphere with a free isometric action of order three. Must its sectional curvature be one?

30. Prime cyclic actions and their geodesic return families

Let $n = 2m+1 \geq 5$, let $p \geq 5$ be prime, and put $\ell = 2\pi/p$. Let X be a closed smooth homotopy n-sphere with a Riemannian metric such that

$$\text{every unit-speed geodesic has prime period } 2\pi, \qquad \mathrm{Vol}(X) = V_n. \tag{30.1}$$

The volume equality also follows from Proposition 12.3. Let h generate a free isometric action of C_p on X, let $\widehat{h}$ be its tangent lift, and put

$$E = SX, \qquad \mathcal{G} = E/S^1, \qquad M = X/C_p.$$

The circle acts by the geodesic flow G_t, with $t \in \mathbb{R}/(2\pi\mathbb{Z})$. Write $\pi : E \to X$ for evaluation and $p_E : E \to \mathcal{G}$ for the circle projection. Let λ be the canonical contact form, and let ω be the symplectic form on $\mathcal{G}$ with $p_E^*\omega = d\lambda$. The positive Euler class $e \in H^2(\mathcal{G};\mathbb{Z})$ has real image $e_{\mathbb{R}} = [\omega/(2\pi)]$, as in Section 12. For $b \in \mathbb{F}_p$, put

$$F_b = \{v \in E : \widehat{h}v = G_{b\ell}v\}, \qquad B_b = F_b/S^1.$$

Freeness gives $F_0 = \emptyset$. Reversal $J(x,v) = (x,-v)$ sends F_b to F_{-b}. Empty families are allowed.

30.1. **Cohomology and the partition of return dimensions.**

Lemma 30.1. *The integral cohomology of E has additive basis $1, x, y, xy$, where $|x| = 2m$, $|y| = 2m+1$, $x^2 = y^2 = 0$, and $y = \pi^* u_X$ for an oriented generator $u_X \in H^{2m+1}(X;\mathbb{Z})$. The geodesic circle action has Borel differential*

$$d_2 y = \pm 2ux, \qquad u \in H^2(BS^1;\mathbb{Z}) \text{ a generator.}$$

Every nonempty F_b, $b \neq 0$, is a connected closed contact manifold which is a mod-p cohomology sphere. If $\dim F_b = 2d_b - 1$, then

$$H^*(B_b;\mathbb{F}_p) \cong \mathbb{F}_p[e_b]/(e_b^{d_b}), \qquad |e_b| = 2,$$

where e_b is the restriction of e reduced modulo p.

Proof. The integral cohomology and circle differential are those of Lemma 29.1. Apply Lemma 29.2 with $q = p$ to the return families F_b. □

Theorem 30.2. *Under* (30.1)*, the nonempty F_b occur in pairs F_b, F_{-b} of the same dimension. Choosing one representative from each pair, their integers $d_b = (\dim F_b + 1)/2$ satisfy*

$$\sum_{\{b,-b\}:\, F_b \neq \emptyset} d_b = m + 1.$$

In particular, a free order-p isometry determines a partition of $m+1$ with at most $(p-1)/2$ parts, together with nonzero phase labels modulo sign.

Proof. This is (29.3). There are $(p-1)/2$ nonzero phase pairs, and every nonempty pair contributes a positive integer. □

30.2. **Two return dimensions.**

Theorem 30.3. *Assume* (30.1)*, and suppose the largest component of* $\mathrm{Fix}(G_\ell : S(X/C_p) \to S(X/C_p))$ *has dimension $n-2$. After replacing h by a generator of the same group, there is $b \in \mathbb{F}_p^* \setminus \{1,-1\}$ such that the only nonempty phase families are*

$$F_1, F_{-1}, F_b, F_{-b},$$

where

$$\dim F_1 = \dim F_{-1} = n-2, \qquad \dim F_b = \dim F_{-b} = 1.$$

The first two are mod-p cohomology $(n-2)$-spheres, the last two are circles. Their quotients have respectively the mod-p cohomology of $\mathbb{CP}^{m-1}$ and of a point.

Proof. Let $\rho : E \to SM$ be the tangent covering. For $a \in \mathbb{F}_p^*$,

$$\operatorname{Fix}(\widehat{h}^{-a} G_\ell) = F_{a^{-1}}, \qquad \rho^{-1} \operatorname{Fix}(G_\ell : SM \to SM) = \coprod_{c \neq 0} F_c.$$

Indeed, $G_\ell v = \widehat{h}^{\,a} v$ is equivalent to $G_{a^{-1}\ell} v = \widehat{h} v$, since $G_{2\pi} = \widehat{h}^{\,p} = I$. The case $a = 0$ is excluded by the prime period 2π. The deck group preserves each F_c. Since these manifolds are connected, their quotients are precisely the connected components of the fixed set in SM, with the same dimensions. The largest part of the partition in Theorem 30.2 is therefore m. Its parts are positive and sum to $m + 1$, so they are m and 1. If c labels a large family, replacing h by $h^{c^{-1}}$ multiplies every phase by c^{-1}. The two large labels become $1, -1$, and the remaining labels are $b, -b$ with $b \neq \pm 1$. Lemma 30.1 gives the cohomology statements. A connected closed one-dimensional manifold is a circle, and its free circle quotient is a point. □

Theorem 30.4. *In the setting of Theorem 30.3, put* $|c|_p = \min\{r, p - r\}$ *for* $r \in \{0, \dots, p - 1\}$ *representing* c. *Then for every* $j \in \{1, \dots, p - 1\}$,

$$\min_{x \in X} d(x, h^j x) = \ell \min\{|j|_p, |jb|_p\}.$$

Proof. Put $a_j = \min\{|j|_p, |jb|_p\}$. For $c \in \{1, b\}$, an arc on a geodesic in F_c or F_{-c} joins a point x to $h^j x$ with length $\ell |jc|_p$. Thus the minimum displacement L_j satisfies $L_j \leq \ell a_j$. The isometry h^j is fixed-point-free. Lemma 13.1 gives $L_j > 0$ and extends every minimizing segment attaining L_j to an h^j-invariant oriented geodesic. Its initial unit state v satisfies $\widehat{h}^{\,j} v = G_{L_j} v$. Since j is a unit modulo p, this geodesic is also h-invariant. Thus $v \in F_c$ for some $c \in \{1, -1, b, -b\}$, and

$$L_j \equiv jc\ell \pmod{2\pi}.$$

Every positive representative of this class is at least $\ell |jc|_p \geq \ell a_j$. Hence $L_j = \ell a_j$. □

Corollary 30.5. *Let* $N_0 = \pi(F_1) = \pi(F_{-1})$ *and* $C_0 = \pi(F_b) = \pi(F_{-b})$, *where* $\pi : SX \to X$ *is evaluation. For every* $j \in \mathbb{F}_p^*$, *the two numbers* $|j|_p$ *and* $|jb|_p$ *are distinct, and*

$$\operatorname*{argmin}_{x \in X} d(x, h^j x) = \begin{cases} N_0, & |j|_p < |jb|_p, \\ C_0, & |jb|_p < |j|_p. \end{cases}$$

In particular, no nonidentity element of the deck group is a Clifford translation. These identities hold for every $p \geq 5$.

Proof. Equality of the two least absolute residues would give $jb \equiv \pm j \pmod{p}$, contrary to $b \not\equiv \pm 1$. Put $a_j = \min\{|j|_p, |jb|_p\}$ and let A_j be the indicated set N_0 or C_0. Every $x \in A_j$ lies on an invariant geodesic whose shorter arc from x to $h^j x$ has length ℓa_j. By Theorem 30.4, this is the global minimum. Conversely, the phase calculation in that theorem's proof places the initial state of any globally minimizing segment in some F_c and gives its length at least $\ell |jc|_p$. Since ℓa_j is the minimum and the two residues are distinct, c belongs to the pair defining A_j. Its initial point therefore lies in A_j.

Each A_j is the image of a smooth map from a compact manifold of dimension at most $n - 2$. Every point of the domain is critical, so Sard's theorem [35, Chapter 3, Theorem 1.3] gives that A_j has zero n-dimensional measure. Thus $A_j \neq X$. A constant-displacement isometry has all of X as its minimum set, so no h^j, $j \neq 0$, is a Clifford translation. □

30.3. **Embedded minimum sets.**

Lemma 30.6. *Let* $b \in \mathbb{F}_p^* \setminus \{1, -1\}$. *If* $p \geq 11$, *there is* $k \in \{2, 3\}$ *such that*

$$k \leq (p - 1)/2, \qquad |kb|_p \geq k.$$

One can take $k = 2$ *unless* $b = \pm 2^{-1}$; *in that case one can take* $k = 3$. *For* $p = 7$, $k = 2$ *satisfies these inequalities when* $b = \pm 2$, *whereas neither* $k = 2$ *nor* $k = 3$ *does when* $b = \pm 3$. *For* $p = 5$ *and* $b = \pm 2$, *no such integer* k *exists.*

Proof. Since $2b \neq 0$, the inequality $|2b|_p < 2$ is equivalent to $2b = \pm 1$. In this case

$$|3b|_p = (p-3)/2 \geq 3 \qquad (p \geq 11).$$

Both 2 and 3 are at most $(p-1)/2$ for $p \geq 11$. For the remaining primes,

$$|2 \cdot 2|_5 = 1, \qquad |2 \cdot 2|_7 = 3, \qquad |2 \cdot 3|_7 = 1, \qquad |3 \cdot 3|_7 = 2.$$

□

Theorem 30.7. *In the setting of Theorem 30.3, suppose there is an integer $2 \leq k \leq (p-1)/2$ with $|kb|_p \geq k$. Then evaluation*

$$\pi_1 : F_1 \longrightarrow X$$

is a smooth embedding. Its image is exactly

$$N = \{x \in X : d(x, hx) = \ell\}.$$

The manifold N has codimension two and is a mod-p cohomology sphere. It carries a smooth unit field V whose orbits are ambient geodesics of prime period 2π. The form $\alpha = g(V, \cdot)|_{TN}$ is contact, its Reeb field is V, and its time-ℓ map is $h|_N$.

The squared displacement $f(x) = d(x, hx)^2$ is smooth in a neighborhood of N. It is Morse–Bott there, with critical manifold N and a positive definite Hessian normal to N.

These conclusions hold for every $p \geq 11$. For these primes, evaluation also embeds the circle F_b as a closed geodesic in X.

Proof. For $v \in F_1$ based at x, its geodesic segment of length $k\ell$ joins x to $h^k x$. Since $|k|_p = k$ and $|kb|_p \geq k$, Theorem 30.4 gives

$$d(x, h^k x) = k\ell = \min_{y \in X} d(y, h^k y).$$

This segment, and each of its subsegments, minimizes distance.

Suppose $v, w \in F_1$ have the same base point x. Concatenate the first length-ℓ segment of the geodesic of v with the h-translate of the first length-$(k-1)\ell$ segment of the geodesic of w. This path joins x to $h^k x$ and has length $k\ell$, so it minimizes distance. Its tangent vectors at the junction hx are $dh(v)$ and $dh(w)$. A minimizing path has no corner; hence $v = w$.

For a vertical vector $\xi \in T_v F_1$, the associated Jacobi field satisfies $J(0) = J(\ell) = 0$. Apply Lemma 13.4 to the minimizing segment of length $T = k\ell$, with $t_0 = \ell < T$. It gives $J = 0$, hence $\xi = 0$. Thus π_1 is an injective immersion. Since F_1 is compact, it is an embedding.

Every point of $\pi(F_1)$ has displacement ℓ. Conversely, $\ell = \min_y d(y, hy)$ by Theorem 30.4. Lemma 13.1 therefore gives $G_\ell v = \widehat{hv}$ for the unit initial vector of any minimizing segment from $x \in N$ to hx. Hence $v \in F_1$ and $\pi(F_1) = N$. The dimension and cohomology of N are those of F_1.

Let R_λ be the geodesic vector field on SX. Since F_1 is flow-invariant, $V = (\pi_1)_*(R_\lambda|_{F_1})$ is a smooth unit field on N. Its flow is

$$\Phi_t = \pi_1 \circ G_t \circ \pi_1^{-1}, \qquad \Phi_\ell = h|_N.$$

Thus its orbits are ambient geodesics of prime period 2π. For the canonical contact form λ on SX,

$$\pi_1^*\big(g(V, \cdot)|_{TN}\big) = \lambda|_{F_1}.$$

The latter is contact by Lemma 30.1; its Reeb field is $R_\lambda|_{F_1}$.

For $x \in N$, the segment from x to hx extends to a minimizing segment of length $k\ell > \ell$. Hence hx lies outside the cut locus of x. On some open neighborhood $U \subset X$ of N, the function $L(y) = d(y, hy)$ and the vector field $W_y = \exp_y^{-1}(hy)$ are therefore smooth [52, proof of Lemma 1.1.2, p. 413]. The minimizing segments are

$$\gamma_y(t) = \exp_y(t\, W_y/\ell), \qquad 0 \leq t \leq \ell.$$

Fix $x \in N$, put $v = \pi_1^{-1}(x)$, and write $\gamma = \gamma_x$. For $w \in T_xX$ and small s, let $\beta(s) = \exp_x(sw)$ and

$$J_w(t) = \left.\frac{\partial}{\partial s}\right|_{s=0} \gamma_{\beta(s)}(t).$$

Both β and $h \circ \beta$ are geodesics. The Jacobi field J_w satisfies

$$J_w(0) = w, \qquad J_w(\ell) = dh(w).$$

The segment γ_y has constant speed $L(y)/\ell$, and γ has unit speed. Since $dL_x = 0$,

$$\langle D_t J_w(t), \dot\gamma(t)\rangle = \ell^{-1} dL_x(w) = 0 \qquad (0 \le t \le \ell).$$

With

$$I(Y,Z) = \int_0^\ell \big(\langle D_t Y, D_t Z\rangle - \langle R(Y,\dot\gamma)\dot\gamma, Z\rangle\big)\, dt,$$

the unit-speed specialization of [52, (1.3.1), p. 416] gives

$$\operatorname{Hess}_x L(w,w) = I(J_w, J_w).$$

Since $f = L^2$, $L(x) = \ell$, and $dL_x = 0$, polarization gives

$$\operatorname{Hess}_x f(w,z) = 2\ell I(J_w, J_z).$$

Integration by parts gives

$$I(J_w, J_z) = \langle dh^{-1} D_t J_w(\ell) - D_t J_w(0), z\rangle.$$

In the horizontal–vertical decomposition of T_vTX, the vector $\xi_w = (w, D_tJ_w(0))$ therefore belongs to T_vSX. For $K = \widehat{h}^{-1} G_\ell$, one has

$$dK_v(\xi_w) = \big(w, dh^{-1} D_t J_w(\ell)\big).$$

Consequently

$$w \in \ker \operatorname{Hess}_x f \quad \Longleftrightarrow \quad dK_v(\xi_w) = \xi_w.$$

The maps $\widehat{h}$ and G_ℓ commute, and $K^p = I$. Consequently

$$\ker(dK_v - I) = T_v F_1$$

by the fixed-set theorem for an invariant auxiliary metric [44, Chapter II, Theorem 5.1]. It follows that $\ker \operatorname{Hess}_x f \subset T_xN$. Since $f \ge \ell^2$ and $f|_N = \ell^2$, the Hessian is nonnegative and vanishes on T_xN. Hence

$$\ker \operatorname{Hess}_x f = T_xN,$$

and its restriction to the normal space is positive definite. In coordinates (y,z) with $N = \{z = 0\}$, one has $\partial_z f(y,0) = 0$ and $D_z\partial_z f(y,0) > 0$. The implicit function theorem gives $\{\partial_z f = 0\} = \{z = 0\}$ near each point of N. Thus N is the critical set near N, and f is Morse–Bott there.

For $p \ge 11$, Lemma 30.6 supplies the integer k. Replace h by $h^{b^{-1}}$. The families formerly labeled 1 and b now have phases b^{-1} and 1, respectively. Since $b^{-1} \notin \{1, -1\}$, the same lemma and the marked minimum formula give a longer minimizing segment for the latter family. The injectivity and immersion arguments above apply to it. Thus evaluation embeds F_b; the free geodesic circle action is transitive on F_b, so its image is a closed geodesic. □

Corollary 30.8. *Under the embedding hypothesis of Theorem 30.7,*

$$\widetilde{H}_i(X \setminus N; \mathbb{F}_p) = \begin{cases} \mathbb{F}_p, & i = 1, \\ 0, & i \ne 1. \end{cases}$$

Proof. The manifold X is an oriented integral homology sphere, and N is a compact locally contractible subspace. Integral Alexander duality [31, Theorem 3.44 and the proof of Corollary 3.45], followed by the universal coefficient theorem, gives

$$\widetilde{H}_i(X \setminus N; \mathbb{F}_p) \cong \widetilde{H}^{n-i-1}(N; \mathbb{F}_p).$$

Since N is a mod-p cohomology $(n-2)$-sphere, the latter group is $\mathbb{F}_p$ for $i = 1$ and zero otherwise. □

Lemma 30.9. *Let $p \geq 11$ be prime and $b \in \mathbb{F}_p^* \setminus \{\pm 1\}$. Then the function*

$$\delta(j) = \min\{|j|_p, |bj|_p\}, \qquad j \in \mathbb{F}_p,$$

is not subadditive. More precisely, some r with $\delta(r) = 1$ and some $s \in \{1, -1\}$ satisfy $\delta(r+s) > 2$.

Proof. Changing b to $-b$ does not change δ. Put

$$a = |b|_p, \qquad r = |a^{-1}|_p, \qquad ar \equiv \varepsilon \pmod{p}, \quad \varepsilon \in \{1, -1\}.$$

Then $2 \leq a, r \leq (p-1)/2$ and $\delta(r) = \delta(1) = 1$. If $r \geq 4$, take $s = \varepsilon$. Since $3 \leq r + s \leq (p+1)/2$ and $3 \leq a + 1 \leq (p+1)/2$,

$$|r+s|_p \geq 3, \qquad |a(r+s)|_p = |a+1|_p \geq 3.$$

Thus $\delta(r+s) \geq 3$.

If $r = 3$ and $\varepsilon = 1$, take $s = 1$; then $|4|_p \geq 3$ and $|a(r+s)|_p = |a+1|_p \geq 3$. If $r = 3$ and $\varepsilon = -1$, then $a \geq 4$: the possibilities $a = 2, 3$ would make p divide 7 or 10. Again $s = 1$ gives

$$|r+s|_p = |4|_p \geq 3, \qquad |a(r+s)|_p = |a-1|_p = a - 1 \geq 3.$$

Finally, if $r = 2$, then $4 \leq 2a \leq p - 1$ excludes $2a \equiv 1 \pmod{p}$. Hence $2a = p - 1$ and, for $s = 1$,

$$\delta(r+s) = \delta(3) = \min\{3, (p-3)/2\} = 3.$$

In each case $\delta(r+s) > \delta(r) + \delta(s) = 2$. □

Theorem 30.10. *Under Theorem 30.3, if $p \geq 7$, the images of the large and small families under evaluation are disjoint. For $p \geq 11$, write these embedded images as N and C. With any orientations on N and C, their integer linking number in the oriented homotopy sphere X satisfies*

$$\mathrm{lk}(C, N) \not\equiv 0 \pmod{p}.$$

Proof. Suppose $x \in N_0 \cap C_0$. Arcs on the two geodesics through x give $d(x, h^j x) \leq \ell\delta(j)$, while Theorem 30.4 gives the reverse inequality. Thus

$$d(h^u x, h^v x) = \ell\delta(v - u) \qquad (u, v \in \mathbb{F}_p).$$

For $p \geq 11$, the triangle inequality would make δ subadditive, contrary to Lemma 30.9.

For $p = 7$, choose $b \in \{2, 3\}$ by reversing the small family if necessary. If $b = 2$, put $c = 2$ and $j = 4$; if $b = 3$, put $c = -3 = 4$ and $j = 2$. In both cases

$$cj = 1 \text{ in } \mathbb{F}_7, \qquad \delta(1) = \delta(j) = 1, \qquad \delta(1 + j) = 2.$$

Choose $v \in F_1$ and $w \in F_c$ based at x. Their length-ℓ segments join x to hx and $h^j x$, respectively. Concatenating the first segment with the h-translate of the second gives a path from x to $h^{1+j}x$ of length $2\ell = d(x, h^{1+j}x)$. It has no corner at hx, so $dh_x v = dh_x w$, and hence $v = w$. But $F_1 \cap F_c = \emptyset$: otherwise $G_{(c-1)\ell} v = v$, contrary to the prime period 2π. This proves disjointness.

Suppose now $p \geq 11$. By Theorem 30.7, N and C are embedded, and C is a circle. They are invariant under C_p. Set $Z = X \setminus N$. Choose an invariant closed tubular neighborhood U of N, disjoint from C, and put $K = X \setminus \operatorname{int} U$. Then K is a compact manifold with boundary, and Z retracts equivariantly onto K. Corollary 30.8 shows that K is a mod-p cohomology circle. The actions on C and K are free and induce the identity on their one-dimensional $H^1(-; \mathbb{F}_p)$, since $\mathrm{Hom}(C_p, \mathbb{F}_p^*) = 0$.

Let $e(C), e(K) \in H^2(BC_p;\mathbb{F}_p)$ be their Euler classes. The class $e(K)$ is the transgression of an oriented generator of $H^1(K;\mathbb{F}_p)$ [41, Theorem 2.3, pp. 283–284]. It is nonzero: otherwise the two-row Borel spectral sequence would collapse and $H^*(K/C_p;\mathbb{F}_p)$ would be nonzero in arbitrarily large degrees, whereas K/C_p is a compact manifold with boundary. For the equivariant inclusion $i : C \hookrightarrow K$, the degree formula [41, Theorem 1.9, p. 282] gives

$$\deg_{\mathbb{F}_p}(i)\, e(C) = e(K) \neq 0.$$

Thus $i_* : H_1(C;\mathbb{F}_p) \to H_1(K;\mathbb{F}_p)$ is nonzero.

The contact form orients the connected manifold N. Integral Alexander duality [31, Theorem 3.44, pp. 254–255], together with the exact sequence of (X, Z), gives

$$H_1(Z;\mathbb{Z}) \cong H^{n-2}(N;\mathbb{Z}) \cong \mathbb{Z}.$$

The image of the oriented fundamental class of C is $\mathrm{lk}(C, N)$ times the meridian generator. Its reduction modulo p is nonzero by the preceding degree formula. Reversing either orientation changes only the sign. □

Proposition 30.11. *Under* (30.1)*, let* $F_b \neq \emptyset$ *and* $\dim F_b = 2d_b - 1$*. If the periodic Morse index of a prime geodesic on* X *is* $n-1$*, then, for every* $v \in F_b$*,*

$$\dim\ker(d\pi|_{T_vF_b}) \leq \min\left\{ d_b - 1, \left\lfloor \frac{n-1}{p-1} \right\rfloor \right\}.$$

In particular evaluation is an immersion whenever $p > n$*. Lemma 11.2 gives the prime-index hypothesis under* (30.1)*.*

Proof. Replace h by $h^{b^{-1}}$, so the selected phase is 1, and put $r = \dim\ker(d\pi|_{T_vF_1})$. A vertical vector $\xi \in T_vF_1$ gives a normal Jacobi field with

$$J_\xi(0) = 0, \qquad J_\xi(t + \ell) = dh\, J_\xi(t).$$

The map $\xi \mapsto J_\xi$ is injective by uniqueness for the Jacobi equation. Thus each of the distinct interior instants $j\ell$, $1 \leq j \leq p-1$, has conjugate multiplicity at least r. The Morse index theorem [50, Theorem 15.1, p. 83] gives

$$(p-1)r \leq \mathrm{ind}_D(\gamma|_{[0,2\pi]}) \leq \mathrm{ind}_{\mathrm{per}}(\gamma) = n - 1.$$

The second inequality follows from inclusion of the Dirichlet form domain in the periodic form domain. The endpoint 2π is not included in the conjugate-point count.

The vertical subspace of T_vE lies in $\ker\lambda_v$ and is isotropic for $d\lambda_v$. Its intersection with T_vF_1 is therefore isotropic in the symplectic space $T_vF_1 \cap \ker\lambda_v$, of dimension $2d_b - 2$. Hence $r \leq d_b - 1$ as well. If $p > n$, the integer bound is zero. □

30.4. **Intrinsic index forms.**

Lemma 30.12. *Let* $N \subset X$ *be the embedded minimum submanifold of Theorem 30.7, with induced metric. Along an orbit* γ *of* V*, the intrinsic and ambient index forms agree on* H^1 *vector fields tangent to* N*:*

$$I_N(W, Z) = I_X(W, Z).$$

Consequently the endpoint-fixed intrinsic index on a full period is at most the ambient endpoint-fixed index. If $\gamma|_{[0,T]}$ *minimizes ambient distance, an intrinsically normal Jacobi field vanishing at both ends of a proper subinterval* $[a, b] \subsetneq [0, T]$*, with* $a < b$*, vanishes identically.*

Proof. The orbit is an ambient and intrinsic unit geodesic, so $\mathrm{II}(V, V) = 0$. With the convention that $\langle R(W, V)V, W\rangle$ is positive on the round sphere, Gauss gives

$$\langle R_N(W, V)V, W\rangle = \langle R_X(W, V)V, W\rangle - |\mathrm{II}(V, W)|^2,$$

whereas

$$|D_t^X W|^2 = |D_t^N W|^2 + |\mathrm{II}(V, W)|^2.$$

Subtracting the curvature terms gives

$$|D_t^N W|^2 - \langle R_N(W, V)V, W\rangle = |D_t^X W|^2 - \langle R_X(W, V)V, W\rangle.$$

Integration and polarization prove the equality of forms. The intrinsic Dirichlet form domain is a subspace of the ambient one, which gives the index inequality.

Ambient minimality makes the common Dirichlet form nonnegative on every subinterval of $[0, T]$. If $b < T$, apply Lemma 13.4 to the intrinsic form on $[a, T]$. If $b = T$, then $a > 0$; apply that lemma to the reversed interval $[0, T]$. In either case the Jacobi field is zero. □

31. Spectral comparison at prime return times

Let $m \in \mathbb{Z}$, $m \geq 2$, and let $p \geq 5$ be prime. Put $n = 2m + 1$ and $\ell = 2\pi/p$. Fix

$$Q = L(p; 1^m, a), \qquad 2 \leq a \leq (p-1)/2, \qquad \zeta = e^{2\pi i/p}.$$

Put $P_Q = \sqrt{\Delta_Q + m^2}$. For a closed connected Riemannian manifold M, put $P_M = \sqrt{\Delta_M + m^2}$ and $R_K = m + K + 1/2$.

Proposition 31.1. *Suppose that, for a fixed $0 \leq \eta < 1/4$ and a sequence $K_\nu \to \infty$,*

$$E_\eta(R_{K_\nu}; P_M, P_Q) = o(R_{K_\nu}).$$

Then the universal Riemannian cover $X \to M$ satisfies (30.1)*, and its deck group is cyclic of order p. There is a self-adjoint integer quantizer A strongly commuting with P_M, with $A - P_M \in \Psi_{\mathrm{cl}}^{-1}$, such that, for $R_\nu = R_{K_\nu}$,*

$$\mathrm{Tr}\big(e^{ij\ell A}\chi(A/R_\nu)\big) - \mathrm{Tr}\big(e^{ij\ell P_Q}\chi(P_Q/R_\nu)\big) = o(R_\nu) \tag{31.1}$$

for every $j \in \{0, \dots, p-1\}$ and $\chi \in C_c^\infty((0,1))$.

Proof. Theorem 15.6, with $q = p$, gives $\dim M = n$, a smooth homotopy-sphere cover X, covering degree p, prime period 2π, and volume V_n. A group of prime order is cyclic. The error is also $o(R_\nu^n)$, so Proposition 10.1 applies with lattice spacing one. Its quantizer A satisfies (10.3), which gives

$$\big|\mathrm{Tr}(e^{itA}\chi(A/R_\nu)) - \mathrm{Tr}(e^{itP_Q}\chi(P_Q/R_\nu))\big| \leq \|\chi\|_\infty\big(E_\eta(R_\nu; P_M, P_Q) + O(1)\big) = o(R_\nu).$$

Take $t = j\ell$. □

For the rest of this section, let X, h, $M = X/C_p$, and the families F_c, B_c be as in Section 30. In particular, (30.1) holds. Suppose that a strongly commuting self-adjoint quantizer A satisfies

$$\mathrm{Spec}\, A \subset \mathbb{Z}, \qquad A - P_M \in \Psi_{\mathrm{cl}}^{-1},$$

and that (31.1) holds along $R_\nu \to \infty$.

Corollary 31.2. *The largest component of* $\mathrm{Fix}(G_\ell : SM \to SM)$ *has dimension $n-2$. Consequently, after a change of deck generator, the nonempty families are $F_{\pm 1}$ and $F_{\pm b}$, with dimensions $n-2, n-2, 1, 1$, where $b \not\equiv 0, \pm 1 \pmod p$.*

Proof. Lemma 15.1, with $L = 2\pi$ and $s = p$, shows that every returning geodesic of length ℓ on M is minimizing in its free homotopy class. Its periodic Morse index is zero. The returns are clean by Section 10.1. For $0 \leq \chi \in C_c^\infty((0,1))$, $\chi \neq 0$, the maximal clean coefficient is nonzero by (10.7).

On the round reference the weight-one subspace gives the two Hopf families on S^{2m-1}, and the weight-a subspace gives the two orientations of S^1. Thus the maximal dimension is $2m-1$ and its nonzero annular term has order R^m. The same formula gives identical leading coefficients for A and P_M, since their principal and subprincipal symbols agree. If the comparison maximal dimension were larger or smaller, division of (31.1), at $j = 1$, by the larger leading power would give a nonzero limit and zero simultaneously. Hence that dimension is $2m-1$. Theorem 30.3 gives the families. □

31.1. **The transverse determinant.** Choose the generator h as in Corollary 31.2. Let $F = F_1$, $B = B_1 = F/S^1$, and let $p_F : F \to B$ be the circle projection. Write $\alpha = \lambda|_F$ and $\omega_B = \omega|_B$, so that $p_F^*\omega_B = d\alpha$. The Reeb flow on F has prime period 2π. The integral Euler class $e_B = e|_B$ has real image $[\omega_B/(2\pi)]$. With the symplectic orientation of B, put

$$D = \langle e_B^{m-1}, [B] \rangle \in \mathbb{Z}_{>0}. \tag{31.2}$$

Lemma 31.3. *Let Y be a closed Riemannian manifold whose geodesic flow has a common period, let α be the canonical contact form on SY, and let t be a rational fraction of this period. If a connected component $C \subset \mathrm{Fix}(G_t)$ has dimension $2d+1$, then C is contact and its symplectic normal bundle in the contact distribution has a finite-order return map S without eigenvalue one. Its Duistermaat–Guillemin density is*

$$dv_C = \frac{|\alpha \wedge (d\alpha)^d|/d!}{\sqrt{\det_{\mathbb{R}}(I - S)}}. \tag{31.3}$$

The denominator is constant on C.

Proof. By Section 10.1, the fixed cone $\mathcal{C} = \{r\rho : r > 0,\ \rho \in C\}$ is a clean symplectic submanifold of $T^*Y \setminus 0$, and C is contact. At $\rho \in \mathcal{C}$, put $L = dG_t(\rho)$ and $W = \ker(I - L) = T_\rho \mathcal{C}$. The map L has finite order, so it is semisimple and

$$T_\rho(T^*Y) = W \oplus W^\omega, \qquad (I - L)|_{W^\omega} \text{ is invertible.}$$

The induced map on the quotient by W is therefore $I - S$. The clean density formula [17, Lemma 4.4, p. 60], with the symplectic-volume normalization of [62, Lemma 2.3], gives

$$dv_C = \frac{|\omega_C^{d+1}|/(d+1)!}{\sqrt{\det_{\mathbb{R}}(I - S)}}.$$

Here $\omega_C = d(r\alpha)$, and

$$\frac{\omega_C^{d+1}}{(d+1)!} = r^d\, dr \wedge \frac{\alpha \wedge (d\alpha)^d}{d!}.$$

Since $q = |\xi|_g = r$ on the cone, division by $|dq| = |dr|$ at $r = 1$ gives (31.3), by [17, p. 60]. The eigenvalues of S are roots of unity different from one. Their multiplicities are locally constant, hence constant on C; the determinant is positive and constant. □

Along the image of F in $S(X/C_p)$, the symplectic normal rank is $2n - 2 - (\dim F - 1) = 2(m+1)$. Let

$$\zeta^{w_1}, \zeta^{-w_1}, \dots, \zeta^{w_{m+1}}, \zeta^{-w_{m+1}}$$

be the complexified eigenvalues of its real return map. Each w_i is nonzero modulo p. The list of classes $[w_i] \in \mathbb{F}_p^*/\{\pm 1\}$ is intrinsically defined, including multiplicities. It does not specify the signed symplectic conjugacy class or the path of linearized flows.

Theorem 31.4. *Under* (31.1)*, one has*

$$D = 1, \qquad \{[w_1], \dots, [w_{m+1}]\} = \{[2]^{m-1}, [a-1], [a+1]\}. \tag{31.4}$$

Thus the complexified characteristic polynomial of the symplectic normal return agrees with that of the round reference along its large family. The equality includes all collisions among the displayed residue classes and is valid also for $p = 5$ and $p = 7$.

Proof. Integration along the upstairs circle gives

$$\int_F \frac{\alpha \wedge (d\alpha)^{m-1}}{(m-1)!} = \frac{(2\pi)^m}{(m-1)!} D.$$

The deck group acts along these circles, so passing to F/C_p divides this integral by p. Lemma 31.3 therefore gives

$$\int_{F/C_p} dv = \frac{(2\pi)^m D}{p(m-1)! \prod_{i=1}^{m+1} |1-\zeta^{w_i}|}. \tag{31.5}$$

Reversal identifies the two large components, preserving their positive densities and the absolute normal determinants. Both periodic Morse indices are zero, since their length is ℓ. Their leading clean-trace contributions consequently add with the same phase. The two one-dimensional components have leading order R, below R^m because $m \geq 2$.

Since A and P_M have the same principal and subprincipal symbols, (10.7) gives the same leading coefficient for both operators. Equation (31.1) therefore equates the leading R^m coefficients for P_M and P_Q.

For the round reference, a large family is the Hopf bundle $S^{2m-1} \to \mathbb{CP}^{m-1}$. Its positive Euler class is the integral generator, so $D_Q = 1$. Apply [56, Lemma 6, p. 299] at length ℓ, with weight 1 of multiplicity m and weight a of multiplicity one. The normal eigenvalue pairs have weights

$$[2] \quad (m-1 \text{ times}), \qquad [a-1], \qquad [a+1].$$

The remaining eigenvalue one has multiplicity $2(m-1)$; it is tangent to the fixed family in the contact hyperplane. The pairs with weights ± 2 belong to the tangential summand of [56, Theorem 2, p. 292], but are normal to the fixed family and occur in (31.5).

For the signed deck weights, identify the round geodesic space with the quadric of isotropic lines by $(x, v) \mapsto [x - iv]$. The geodesic flow acts on $x - iv$ by e^{it}, and for $z = (x - iv)/\sqrt{2}$ one has $\operatorname{Im}(\bar{z} \cdot dz) = v \cdot dx = \alpha$. Thus the complex structure on the quadric is compatible with the positive reduced symplectic form. At a line of deck weight 1, the projective tangent weights are

$$0 \quad (m-1 \text{ times}), \qquad -2 \quad (m \text{ times}), \qquad a-1, \qquad -a-1.$$

The differential of the quadric equation removes one direction of weight -2. Hence the signed normal weights of the deck action on the geodesic space are

$$-2 \quad (m-1 \text{ times}), \qquad a-1, \qquad -a-1.$$

Comparing (31.5) on the two sides, all universal factors and the common index phase cancel. Put $u = (2, \dots, 2, a-1, a+1)$, a list of length $m+1$. Then

$$\frac{D}{\prod_i |1-\zeta^{w_i}|} = \frac{1}{\prod_i |1-\zeta^{u_i}|}.$$

Squaring gives an equality in $K = \mathbb{Q}(\zeta)$:

$$D^2 \prod_i (1-\zeta^{u_i})(1-\zeta^{-u_i}) = \prod_i (1-\zeta^{w_i})(1-\zeta^{-w_i}). \tag{31.6}$$

For every nonzero $s \pmod p$, $N_{K/\mathbb{Q}}(1-\zeta^s) = \Phi_p(1) = p$. Taking norms in (31.6) and cancelling the $2(m+1)$ identical factors gives $D^{2(p-1)} = 1$. Since $D > 0$, $D = 1$.

For $s \in \mathbb{F}_p^*$, let c_s be the multiplicity of s in the signed list $(w_1, -w_1, \dots, w_{m+1}, -w_{m+1})$ minus its multiplicity in the analogous list for u. Then

$$c_s = c_{-s}, \qquad \sum_s c_s = 0, \qquad \prod_s (1-\zeta^s)^{c_s} = 1.$$

Every nontrivial pth root is a Galois conjugate of ζ, so the last equality holds at every such root. Franz's independence lemma [21, §§9, 11, pp. 251–252] (see also [16, Chapter I, §5, pp. 11–12]) gives $c_s = 0$ for all s. This is exactly (31.4). The multiplicities in the signed lists include all coincidences among the u_i. □

Corollary 31.5. *For either large family,*

$$\int_F \frac{\alpha \wedge (d\alpha)^{m-1}}{2^{m-1}(m-1)!} = V_{2m-1}.$$

Proof. Integration over the geodesic circles gives $\int_F \alpha(d\alpha)^{m-1} = (2\pi)^m D$. Theorem 31.4 gives $D = 1$, and $V_{2m-1} = (2\pi)^m/(2^{m-1}(m-1)!)$. □

31.2. **Annular coefficient polynomials.** Write $h_a(k)$ for the multiplicity of the frequency $k+m$ on Q and $H_a(z) = \sum_{k\geq 0} h_a(k) z^k$. Lemma 8.1 gives

$$H_a(z) = \frac{1-z^2}{p} \sum_{r=0}^{p-1} \frac{1}{(1-z\zeta^r)^m (1-z\zeta^{-r})^m (1-z\zeta^{ar})(1-z\zeta^{-ar})}. \tag{31.7}$$

For $j = 1, \dots, p-1$, let $C_{j,s}(a)$, $1 \leq s \leq m$, be the partial-fraction coefficients at $z = \zeta^j$:

$$H_a(z) = \sum_{s=1}^{m} \frac{C_{j,s}(a)}{(1-z\zeta^{-j})^s} + \text{a function regular at } \zeta^j.$$

Define the polynomial

$$T_{j,a}(x) = \zeta^{jm} \sum_{s=1}^{m} C_{j,s}(a) \binom{x-m+s-1}{s-1} = \sum_{r=0}^{m-1} t_{j,a,r} x^r. \tag{31.8}$$

Proposition 31.6. *For every $\chi \in C_c^\infty((0,1))$, every $j = 1, \dots, p-1$, and every $N > 0$,*

$$\mathrm{Tr}\big(e^{ij\ell P_Q} \chi(P_Q/R)\big) = \sum_{r=0}^{m-1} t_{j,a,r} R^{r+1} \int_0^1 \chi(u) u^r \, du + O_{\chi,j,N}(R^{-N}).$$

Its leading coefficient is

$$t_{j,a,m-1} = \frac{2\zeta^{jm}}{p(m-1)!(1-\zeta^{2j})^{m-1}(1-\zeta^{(a+1)j})(1-\zeta^{(1-a)j})}. \tag{31.9}$$

Proof. At $z = \zeta^j$, the two terms $r = j, -j$ have a pole of order m. The two terms $r = a^{-1}j, -a^{-1}j$ have simple poles. They are distinct from $j, -j$ because $a \not\equiv \pm 1$. All other terms are regular. Multiplication by $(1-z\zeta^{-j})^m$ and evaluation at $z = \zeta^j$ therefore gives

$$C_{j,m}(a) = \frac{2}{p(1-\zeta^{2j})^{m-1}(1-\zeta^{(a+1)j})(1-\zeta^{(1-a)j})}.$$

This remains valid for $m = 2$, since the other poles have order one. The rational function (31.7) is proper, and every pole is a pth root of unity. Its partial fractions give

$$h_a(k) = \sum_{\xi^p=1} \sum_{s=1}^{s_\xi} C_{\xi,s} \binom{k+s-1}{s-1} \xi^{-k}.$$

At time $j\ell$, the phase $\zeta^{j(k+m)}$ is nonoscillatory precisely when $\xi = \zeta^j$. The proof of Lemma 8.4 applies to each of these finitely many terms: the terms with $\xi \neq \zeta^j$ are $O(R^{-N})$ for every N, and the remaining sum equals

$$\int_{\mathbb{R}} T_{j,a}(x) \chi(x/R) \, dx + O(R^{-N}).$$

Indeed $x = k+m$; since χ is supported in a compact subset of $(0,1)$, the integers $x < m$ make no contribution for large R. Expanding $T_{j,a}$ proves the trace formula. Its leading term is $\frac{\zeta^{jm} C_{j,m}(a)}{(m-1)!}$. □

Theorem 31.7. *Let A be a self-adjoint integer quantizer strongly commuting with P_M, with $A - P_M \in \Psi_{\mathrm{cl}}^{-1}$. Under* (31.1)*, for every $j = 1, \dots, p-1$ the coefficients of $R^m, R^{m-1}, \dots, R$ in the annular clean trace of A equal those in Proposition 31.6.*

Proof. Since $G_\ell^p = I$ and j is a unit modulo p, $\mathrm{Fix}(G_{j\ell}) = \mathrm{Fix}(G_\ell)$. Its components are clean, of dimensions $2m-1$ and one. The operator A is scalar and classical, with principal symbol $|\xi|_g$. A finite-rank smoothing change makes A positive and leaves all annular traces unchanged for large R. The full clean trace expansion [17, Theorem 4.5, equations (4.6)–(4.7)], paired with the annular cutoff as in Section 10.1, gives constants $\tau_{j,r}$ independent of χ such that

$$\mathrm{Tr}(e^{ij\ell A}\chi(A/R)) = \sum_{r=0}^{m-1} \tau_{j,r} R^{r+1} \int_0^1 \chi(u) u^r \, du + O_{\chi,j}(1).$$

The integer exponents follow from the odd dimensions of the fixed components. The first omitted frequency-symbol term has degree -1; its annular integral is bounded because the support of χ is bounded away from zero.

Choose $0 \le \chi \not\equiv 0$. If some $\tau_{j,r} \neq t_{j,a,r}$, let r_0 be the largest such index. Subtract the expansion in Proposition 31.6 and divide by $R_\nu^{r_0+1}$. The limit is

$$(\tau_{j,r_0} - t_{j,a,r_0}) \int_0^1 \chi(u) u^{r_0} \, du \neq 0,$$

contrary to (31.1). Thus $\tau_{j,r} = t_{j,a,r}$ for every r. Since these constants are independent of χ, the equality holds for every cutoff. □

The coefficient of R is the sum of the principal contributions of the one-dimensional families and the degree-zero frequency-symbol terms of the large families.

Corollary 31.8. *Under* (31.1)*, if X is round, then X/C_p is isometric to $L(p; 1^m, a)$.*

Proof. By [39, §1, Example (ii), p. 77], a free order-p orthogonal action is a direct sum of rotation planes with nonzero weights modulo p. Group equal weights modulo sign. A block of multiplicity s gives two fixed components $\mathbb{CP}^{s-1}$ in the geodesic space. Corollary 31.2 therefore gives precisely one block of multiplicity m and one of multiplicity one. After changing the generator and conjugating individual coordinates, the quotient is $L(p; 1^m, b)$ with $2 \le b \le (p-1)/2$.

By (31.9), the absolute leading coefficient at $j = 1$ is

$$|t_{1,b,m-1}| = \frac{2}{p(m-1)![2\sin(2\pi/p)]^{m-1} 4[\sin^2(\pi b/p) - \sin^2(\pi/p)]}.$$

The expression in brackets is strictly increasing for $2 \le b \le (p-1)/2$. Equality of leading coefficients gives $b = a$. The normalized orthogonal representations are then conjugate. Their quotients are isometric by [39, Lemma 1.2, p. 78]. □

32. Signed normal weights and marked phases

An invariant compatible complex structure gives signed weights for a finite-order symplectic map. The principal half-wave determines their product as follows.

Lemma 32.1. *Let P be a scalar, self-adjoint, classical pseudodifferential operator of order one on half-densities over a smooth manifold Y. Assume that its principal symbol q is real and positively homogeneous of degree one, and that its invariant subprincipal symbol vanishes. Put $U(t) = \exp(-itP)$, and let Φ_t be the Hamiltonian flow of q. Suppose that, for an odd positive integer p and $\ell > 0$,*

$$\Phi_{p\ell} = \mathrm{id}, \qquad U(p\ell) = I \pmod{\Psi^{-1}}.$$

For $\rho \in \mathrm{Fix}(\Phi_\ell) \subset T^*Y \setminus 0$, the principal action of $U(\ell)$ on coherent-state profiles at ρ is the unique metaplectic lift of $L = d\Phi_\ell(\rho)$ whose pth power is the identity. It is also the unique such lift in the fourfold extension with central kernel $\mu_4 = \{1, -1, i, -i\}$.

Proof. Work in a compact neighborhood of the trajectory $\{\Phi_t(\rho) : 0 \le t \le p\ell\}$ in $T^*Y \setminus 0$. In cotangent charts covering this neighborhood, the semiclassical Weyl symbol of hP is $q + O(h^2)$. It can be extended away from the compact trajectory to a proper real symbol. The coherent-state propagation theorem and its symbol transport equation [28, Theorems 3.12, 3.15, and 4.2, arXiv version] therefore apply microlocally, successively in these charts. The transport depends only on the symbol near the trajectory. Its two scalar factors are

$$\exp\left(\frac{i}{h}\int_0^t (\xi\cdot\dot{x} - q)\,ds\right), \qquad \exp\left(-i\int_0^t \mathrm{sub}(P)(\Phi_s\rho)\,ds\right).$$

Both equal one: the second by hypothesis, and the first because $\xi\cdot\partial_\xi q = q$. The remaining principal profile map is the quadratic quantization of $d\Phi_t$.

Here the scalar half-density convention is relevant. Under a cotangent coordinate change $x = \psi(y)$, put $A = D\psi(y_0)$ and $B_{ij} = \xi_0\cdot\partial_{ij}\psi(y_0)$. The principal change of a centered profile is

$$f(v) \longmapsto |\det A|^{1/2} e^{iv^{\mathsf{T}}Bv/2} f(Av).$$

The linear change and quadratic multiplication belong to the fourfold extension of [13, §11.2]. Thus a finite sequence of coordinate charts gives a lift $\widetilde{L}$ in this extension; the initial and final profiles are expressed in the same chart at ρ. No metaplectic structure on T^*Y is required for this pointwise construction.

Let $T^h_\rho f$ be a normalized coherent packet with Schwartz profile f, cut off near ρ. With a microlocal cutoff equal to one near ρ understood on the left, the cited propagation and norm estimates give

$$U(\ell)T^h_\rho f = T^h_\rho(\widetilde{L}f) + O_{L^2}(h^{1/2}).$$

The microlocal composition law for principal profiles, with ρ fixed, gives the same formula for $U(p\ell)$ and $\widetilde{L}^p$. On these packets an operator of order -1 is $O_{L^2}(h)$. Hence $\widetilde{L}^p f = f$ for every Schwartz profile, so $\widetilde{L}^p = I$.

Every other lift is $c\widetilde{L}$ with $c \in \mu_4$. The map $c \mapsto c^p$ is a bijection because p is odd. There is also a metaplectic lift with pth power one: for either metaplectic lift T, one has $T^p \in \{I, -I\}$, and exactly one of $T, -T$ has pth power one. This proves both assertions. □

Lemma 32.2. *Let S and S_0 be symplectic maps of real dimension $2d$, with $S^p = S_0^p = I$ for an odd prime p, and with no fixed vectors. Choose invariant compatible complex structures and write their weights as $r_i, r_i^0 \in \{1, \dots, p-1\}$. Suppose that their unsigned weight multisets agree. If the distributional traces of their unique metaplectic lifts whose pth powers are the identity have the same phase, then*

$$\prod_{i=1}^d r_i = \prod_{i=1}^d r_i^0 \quad \text{in } \mathbb{F}_p.$$

Proof. In a compatible unitary basis, a rotation through $\theta = 2\pi r/p$ has continuous metaplectic lift $\exp[-i\theta(N + \frac{1}{2})]$, where N is the oscillator number operator. Its pth power is $(-1)^r$. Thus the lift of S with pth power one is the tensor product of these lifts multiplied by $(-1)^{\sum_i r_i}$. In the quadratic-kernel normalization of [13, §§11.2–11.3], its distributional character is the Abel limit of the oscillator trace. Since $0 < \theta < 2\pi$,

$$\lim_{\epsilon\downarrow 0} \frac{e^{-(\epsilon+i\theta)/2}}{1 - e^{-(\epsilon+i\theta)}} = \frac{1}{2i\sin(\theta/2)}.$$

Consequently its character is

$$\frac{(-1)^{\sum_i r_i} i^{-d}}{\prod_i 2\sin(\pi r_i/p)}. \tag{32.1}$$

Its denominator is positive. Reversing the Hamiltonian or Fourier convention conjugates the characters of both maps.

Match the occurrences of each unsigned weight class in the two multisets. After permutation, each r_i is r_i^0 or $p - r_i^0$; this remains true when a class occurs several times. Each replacement $r_i^0 \mapsto p - r_i^0$ changes the parity of $\sum_i r_i$ and multiplies the product in $\mathbb{F}_p$ by -1. Equality of the phases in (32.1) therefore gives an even number of replacements and equality of the products. For $d = 0$, both products and both characters equal one. □

Corollary 32.3. *Under the hypotheses of Theorem 31.4, the product of the signed symplectic normal weights at $B_1 = F_1/S^1$ agrees with the corresponding product for the round reference. This holds both for the quotient return and for the deck action on the normal bundle in the space of oriented geodesics.*

Proof. On scalar half-densities, $P_M = \sqrt{\Delta_M + m^2}$ has principal symbol $|\xi|_g$ and zero subprincipal symbol [17, p. 58]. The integer spectrum of A, its strong commutation with P_M, and (10.8) give

$$e^{-2\pi i P_M} = I \pmod{\Psi^{-1}}.$$

The homogeneous geodesic flow has period 2π on $T^*M \setminus 0$. Lemma 32.1 applies with $\ell = 2\pi/p$. The same statements hold for P_Q.

Let C be a large fixed cone and $\rho \in C$. Write $T_\rho(T^*M) = W \oplus W^\omega$, where $W = T_\rho C$ has dimension $2m$ and the return is $I_W \oplus S$. By uniqueness, its lift with pth power one is $I \otimes \widetilde{S}$, where $\widetilde{S}$ is the corresponding lift on W^ω. In the clean trace formula [13, Theorem 12 and §11.4], the fixed factor has positive principal density. Indeed, in symplectic coordinates on W, for $0 \le a_W \in C_c^\infty(W)$ with $a_W \not\equiv 0$,

$$\operatorname{Tr}(\operatorname{Op}_h^w(a_W)) = (2\pi h)^{-m} \int_W a_W \, \frac{\omega_W^m}{m!} > 0.$$

The phase of the local leading coefficient is therefore the phase of the distributional character of $\widetilde{S}$.

Every length-ℓ geodesic has periodic Morse index zero, by Lemma 15.1. Equation (10.7) thus gives the same phase on the two large components and on the round large components. The preceding identification and Theorem 31.4 give, respectively, equality of the normal character phases and equality of the unsigned normal weight multisets. Lemma 32.2, with $d = m + 1$, gives equality of the signed products.

On F_1, the quotient return induces $d\bar{h}^{-1}$ on the normal space in $\mathcal{G}$. Replacing it by the deck derivative negates all $m + 1$ weights. Both products acquire the factor $(-1)^{m+1}$, so their equality persists. □

Lemma 32.4. *Under the hypotheses of Theorem 31.4, normalize the generator so that the large family has phase 1, and let $b \in \mathbb{F}_p^* \setminus \{\pm 1\}$ be the phase of the small family. Let $e_H \in H^2_{C_p}(\mathcal{G}; \mathbb{F}_p)$ be the equivariant Euler class of the geodesic circle bundle. Let $v \in H^2(BC_p; \mathbb{F}_p)$ be the first Chern class of the character $h \mapsto \zeta$, and write*

$$e_H|_{B_c} = e_c + \sigma c v, \qquad \sigma \in \{1, -1\},$$

where $e_c = e|_{B_c} \bmod p$ and the sign σ depends on the Euler-class convention. If Π_b is the product of the signed normal deck weights at B_1, relative to v, then

$$\Pi_b = 2^{m-1}\sigma^{m-1}(1 - b^2) \quad \text{in } \mathbb{F}_p. \tag{32.2}$$

In particular,

$$(1 - b^2)^2 = (1 - a^2)^2 \quad \text{in } \mathbb{F}_p, \tag{32.3}$$

and hence

$$b^2 = a^2 \quad \text{or} \quad b^2 = 2 - a^2.$$

If $2 - a^2$ is zero or a quadratic nonresidue modulo p, then $b = \pm a$. This holds for every genuine two-weight case with $p = 5$ or $p = 7$.

Proof. All cohomology groups in this proof have coefficients in $\mathbb{F}_p$. On B_c, the deck generator acts on the geodesic circle fibers by rotation through $c\ell$; hence $e_H|_{B_c} = e_c + \sigma cv$. Give $\mathcal{G}$ and its fixed components their symplectic orientations, and give their normal bundles invariant compatible complex structures. Put

$$K = \mathbb{F}_p(v), \qquad \mathcal{B} = \{B_1, B_{-1}, B_b, B_{-b}\}.$$

Lemma 13.2 applies to the smooth C_p-action on the closed manifold $\mathcal{G}$. For the inclusion $i_B : B \hookrightarrow \mathcal{G}$, the Thom identity on its oriented Borel normal bundle gives

$$i_B^* i_{B!}(x) = e_H(\nu_B)x$$

[31, Corollary 4D.9, Theorem 4D.10, and p. 444]. The degree-zero part on B of $e_H(\nu_B)$ is $v^{\operatorname{rank}_{\mathbb{C}} \nu_B}$ times the product of its nonzero weights. Its remaining part has positive degree on B and is nilpotent. Thus $e_H(\nu_B)$ is invertible over K, and localized restriction and equivariant pushforward give

$$\int_{\mathcal{G}} x = \sum_{B \in \mathcal{B}} \int_B \frac{i_B^* x}{e_H(\nu_B)}.$$

The fixed components have no odd cohomology, by Lemma 30.1. Their even equivariant classes therefore belong to $H^{\text{even}}(B) \otimes \mathbb{F}_p[v]$; the degree-one exterior generator of $H^*(BC_p)$ contributes no term to this formula.

Define a formal Laurent series at infinity by

$$\mathcal{R}(z) = v^{2m+1} \int_{\mathcal{G}} \frac{1}{zv - e_H} = \sum_{k \geq 0} z^{-k-1} v^{2m-k} \int_{\mathcal{G}} e_H^k. \tag{32.4}$$

Each coefficient belongs to $\mathbb{F}_p$: the pushforward has degree $2k - 4m$, and multiplication by v^{2m-k} gives degree zero. For $c = \pm 1$, Lemma 30.1 and Theorem 31.4, with reversal for $c = -1$, give

$$e_c^m = 0, \qquad \int_{B_c} e_c^{m-1} = 1.$$

Consequently

$$\frac{1}{zv - e_H|_{B_c}} = \sum_{j=0}^{m-1} \frac{e_c^j}{v^{j+1}(z - \sigma c)^{j+1}}.$$

The contribution of B_c to $\mathcal{R}$ is rational, with its only possible pole at σc and order at most m. The two point components contribute simple poles at σb and $-\sigma b$. These four points are distinct. Hence the denominator of $\mathcal{R}$ divides

$$Q_b(z) = (z^2 - 1)^m (z^2 - b^2).$$

For $k < 2m$, the pushforward in (32.4) is zero by degree. For $k = 2m$, Lemma 12.2, the prime period 2π, and $\operatorname{Vol}(X) = V_n$ give $\int_{\mathcal{G}} e^{2m} = 2$. Since Q_b is monic of degree $2m + 2$, comparison at infinity gives

$$\mathcal{R}(z) = \frac{2z + C}{Q_b(z)}, \qquad C \in \mathbb{F}_p.$$

Geodesic reversal commutes with C_p and inverts the geodesic circle. It therefore pulls the equivariant circle bundle back to its dual and sends e_H to $-e_H$. It is antisymplectic on the $4m$-dimensional manifold $\mathcal{G}$, so it preserves orientation. Equivariant pushforward then gives

$$\int_{\mathcal{G}} e_H^k = (-1)^k \int_{\mathcal{G}} e_H^k.$$

For odd k this integral is zero, since p is odd. Thus $\mathcal{R}(-z) = -\mathcal{R}(z)$ and $C = 0$:

$$\mathcal{R}(z) = \frac{2z}{(z^2 - 1)^m (z^2 - b^2)}. \tag{32.5}$$

Write $r_1, \dots, r_{m+1}$ for the signed normal weights at B_1. The coefficient of $(z-\sigma)^{-m}$ in the localization formula is

$$v^{2m+1} \int_{B_1} \frac{e_1^{m-1}}{v^m\, v^{m+1} \prod_i r_i} = \frac{1}{\Pi_b}.$$

Indeed, e_1^{m-1} already has top degree on B_1; all positive-degree terms of the reciprocal normal Euler class vanish in this integral. The same coefficient in (32.5) is

$$\frac{2z}{(z+\sigma)^m(z^2-b^2)}\bigg|_{z=\sigma} = \frac{\sigma^{m-1}}{2^{m-1}(1-b^2)}.$$

Taking reciprocals proves (32.2).

At F_1, the quotient return induces $d\bar{h}^{-1}$ on the normal space of B_1 in $\mathcal{G}$. Inversion preserves unsigned weights. Theorem 31.4 therefore gives

$$\Pi_b^2 = 2^{2m-2}(a-1)^2(a+1)^2.$$

Squaring (32.2) and cancelling the nonzero factor 2^{2m-2} gives (32.3). Factoring the difference of squares in $\mathbb{F}_p$ gives the two alternatives. Since $b \neq 0$, the second is impossible when $2 - a^2$ is zero or a quadratic nonresidue. For $p = 5$, $a = 2$ gives the nonresidue 3. For $p = 7$, $a = 2, 3$ give respectively the nonresidue 5 and zero. □

Theorem 32.5. *Under the hypotheses of Theorem 31.4, the phase of the small family is*

$$b \equiv \pm a \pmod{p}.$$

Consequently, after normalizing the generator by a large component, the nonempty phase families are $F_{\pm 1}, F_{\pm a}$, and

$$\min_{x \in X} d(x, h^j x) = \frac{2\pi}{p} \min\{|j|_p, |ja|_p\} \qquad (1 \leq j < p).$$

This holds for every prime $p \geq 5$, without an embedding hypothesis on the evaluated families.

Proof. Choose the same Euler-class and character conventions on M and Q, with a phase-one large component on each side. By Corollary 32.3, the products of their signed normal deck weights are equal. Equation (32.2), applied on both sides, gives

$$2^{m-1}\sigma^{m-1}(1-b^2) = 2^{m-1}\sigma^{m-1}(1-a^2).$$

The common factor is nonzero in $\mathbb{F}_p$. Hence $b^2 = a^2$, and $b = \pm a$. Theorem 30.4 gives the minimum displacement formula. □

33. Integral topology and signed prime return maps

Let X^{2m+1}, $m \geq 2$, be a normalized Zoll homotopy sphere, let h generate a free isometric action of prime order $p \geq 5$, and put $\ell = 2\pi/p$. Write

$$F_b = \{v \in SX : \widehat{h}v = G_{b\ell}v\}.$$

Assume that the nonempty families are F_1, F_{-1}, F_a, F_{-a}, where $2 \leq a \leq (p-1)/2$, with dimensions $2m-1, 2m-1, 1, 1$. Assume that they are connected mod-p homology spheres and that

$$\min_x d(x, h^j x) = \ell \min\{|j|_p, |aj|_p\}. \tag{33.1}$$

At F_1, assume the unsigned transverse weights are

$$[2]^{m-1}, \quad [a-1], \quad [a+1],$$

where $[r] = \{r, -r\} \subset \mathbb{F}_p^\times$ and exponents denote multiplicities. These hypotheses hold in the spectral setting of Proposition 31.1, by Corollary 31.2 and Theorems 30.3, 31.4, and 32.5.

Let λ be the canonical contact form on SX. Return derivatives below are restricted to $\ker \lambda$, with symplectic form $d\lambda$. The Euler class of a geodesic circle bundle has the positive convention of Section 12.

Theorem 33.1. *The families $F_{\pm1}$ are homotopy $(2m-1)$-spheres, and*

$$F_1/S^1 \simeq \mathbb{CP}^{m-1}, \qquad F_1/C_p \simeq L(p;1,\dots,1).$$

The integral cohomology of F_1/S^1 is $\mathbb{Z}[e]/(e^m)$, with $\langle e^{m-1}, [F_1/S^1]\rangle = 1$ in its symplectic orientation. If $m = 2$, then

$$F_1/S^1 \cong S^2, \qquad F_1 \cong S^3, \qquad F_1/C_p \cong L(p;1,1)$$

diffeomorphically.

All four finite symplectic return maps have the conjugacy classes of the corresponding round two-weight maps. With positive compatible complex structures, their nontrivial signed weights are

family	*relative return*	*weights in $\mathbb{F}_p$*
F_1	$d(\widehat{h}^{-1}G_\ell)$	$2^{m-1},\ 1-a,\ 1+a$
F_a	$d(\widehat{h}^{-1}G_{a\ell})$	$(a-1)^m,\ (a+1)^m.$

(33.2)

Reversal gives the other two maps. The deck derivative on the space of oriented geodesics is the inverse of the relative return.

33.1. **Boundary indices and orientation**. A critical path in $\Lambda_{h^j}X$, $j \in \mathbb{F}_p^*$, extends by h^j to a closed geodesic. Its length is $r\ell$ for some $r \geq 1$, with $p \nmid r$, and its initial unit state belongs to $F_{rj^{-1}}$. Conversely, each state in this family determines such a critical path. Thus the critical lengths have residues $j, -j, aj, -aj$ modulo p. The map $\widehat{h}^{-j}G_{r\ell}$ has order dividing p. Its fixed-set tangent space is its fixed derivative space, so the Jacobi-field calculation in the proof of Theorem 29.6 identifies the Hessian kernel with the tangent space of the critical manifold. Every critical manifold is therefore Morse–Bott, and Lemma 29.7 applies. Each h^j has degree one, since its degree has odd order. Consequently

$$\pi_k(\Lambda_{h^j}X) = 0 \qquad (1 \leq k \leq 2m-1).$$

For a family F of dimension $2d-1$, choose the generator g which translates that family by ℓ. Let $j(\theta)$ be the complex normal index with boundary condition $J(\ell) = e^{i\theta}dg\,J(0)$, and let $i(q)$ be the real index of its q-fold arc with endpoint twist g^q. By Lemma 13.1, every nonidentity power of h has minimum displacement at least ℓ. The short arc in F attains this bound, so $j(0) = 0$.

Let $T = d(\widehat{g}^{-1}G_\ell)|_{\ker\lambda}$. Then $T^p = I$, and its fixed space has real dimension $2d-2$. At an eigenphase, the crossing calculation of Lemma 29.8 gives the Hermitian form $Q(u,v) = -i\omega(u,\overline{v})$, where $\omega = d\lambda$ on normal Jacobi data. The map T is semisimple. Its symplectic pairing between reciprocal eigenspaces is nondegenerate, so Q is nondegenerate at every crossing. On the complexification of the real fixed space its inertia is $(d-1, d-1)$. It follows that $j(0+) = j(2\pi-) = d-1$.

At $2\pi s/p$, $1 \leq s < p$, let A_s, B_s count the branches entering and leaving the negative spectrum, and put $\Delta_s = A_s - B_s$. The index at the crossing is its left limit minus B_s, or equivalently its right limit minus A_s. Extend variation fields by $J(t+q\ell) = dg^q J(t)$. The operator

$$(T_q J)(t) = dg^{-1}J(t+\ell), \qquad T_q^q = I,$$

is unitary and preserves the index form. Restriction to $[0,\ell]$ identifies its z-eigenspace with the domain defining $j(\arg z)$; the restricted form is q times that short-arc form. Its character decomposition gives

$$i(q) = \sum_{z^q=1} j(\arg z), \tag{33.3}$$

as in [7, Theorem I and its corollary, pp. 177–178] and [26, Lemma 2.3 and (2.4)]. The tangential index form is nonnegative. If $p \nmid q$, counting the sample points strictly beyond each crossing gives

$$i(q) = (q-1)(d-1) + \sum_{s=1}^{p-1} \Delta_s \left(q - 1 - \left\lfloor \frac{qs}{p} \right\rfloor \right). \tag{33.4}$$

If $p \mid q$, one must subtract $\sum_s B_s$ from the right side, because each nonzero crossing is sampled.

The total crossing nullity is $4m$, the real dimension of the normal Jacobi data. Conjugation gives $B_{p-s} = A_s$, and hence

$$\sum_{s=1}^{p-1} \Delta_s = 0, \qquad \sum_{s=1}^{p-1} B_s = 2m - d + 1.$$

Lemma 11.2 gives $i(p) = 2m$. Substitution in (33.4), including its correction at multiples of p, therefore yields

$$\sum_{s=1}^{p-1} (p - 1 - s)\Delta_s = 4m - p(d - 1).$$

Using

$$\left\lfloor \frac{(q+p)s}{p} \right\rfloor = \left\lfloor \frac{qs}{p} \right\rfloor + s, \qquad \left\lfloor \frac{qs}{p} \right\rfloor + \left\lfloor \frac{(p-q)s}{p} \right\rfloor = s - 1 \quad (1 \le q < p),$$

we obtain

$$\begin{aligned} i(q+p) &= i(q) + 4m \qquad (q \ge 1), \\ i(q) + i(p-q) &= 4m - 2d + 2 \quad (1 \le q < p). \end{aligned} \tag{33.5}$$

All these indices are even:

$$i(q) \equiv 0 \pmod 2. \tag{33.6}$$

Indeed, conjugation pairs the terms in (33.3), except those at 1 and, when q is even, -1. The positive-half crossing nullities sum to $2m - d + 1$, and every jump has the parity of its nullity. Since p is odd, π is not a crossing, and

$$j(\pi) \equiv (d - 1) + (2m - d + 1) = 0 \pmod 2.$$

The negative bundle over each small circle is orientable. On the negative eigenspace of its q-fold arc, nonreal eigenvalues of the real orthogonal operator T_q occur in conjugate pairs. For even q, its -1 eigenspace has dimension $j(\pi)$; for odd q that eigenspace is zero. Thus $\det T_q = 1$. Transport once around the orbit of G_t in SX has transition T_q^p, since $g^p = I$ and $p\ell = 2\pi$. Its determinant is positive, which proves the assertion.

33.2. **The circle indices and the large family.**

Lemma 33.2. *The homotopy types, integral cohomology, Euler number and dimension-three diffeomorphisms are those in Theorem 33.1. In a twisted path space whose minimum is a large family, each of the two small-circle critical manifolds below 2π has index $2m$. If $b = a^{-1} \pmod p$, $g = h^b$, and $d_q = |b\,q|_p$, the q-fold small-family arc has index*

$$i_C(q) = \begin{cases} 0, & q < d_q, \\ 2m, & d_q < q < p - d_q, \\ 4m, & p - d_q < q, \end{cases} \qquad 1 \le q < p. \tag{33.7}$$

Proof. In $\Lambda_h X$, the minimum is F_1, whose mod-p homology is supported in degrees $0, 2m - 1$. The reversed large family has index $2m + 2$, by (33.5). Write the two circle indices as $r, 4m - r$. Their subsequent occurrences add $4m$.

Suppose $q = \min(r, 4m - r) < 2m$. If $q = 0$, a later index-zero circle adds a component. Every subsequent attachment has rank at least $2m + 2$ and cannot join components, contrary to the connectedness of $\Lambda_h X$. Otherwise (33.6) gives $2 \le q \le 2m - 2$. Let A, D be sublevels immediately before and after the index-q circle. All earlier positive-rank attachments have rank at least $2m + 2$, so $H_{q-1}(A; \mathbb{F}_p) = 0$. The oriented Thom isomorphism gives

$$H_q(D, A; \mathbb{F}_p) = H_0(S^1; \mathbb{F}_p) = \mathbb{F}_p.$$

The exact sequence of the pair makes $H_q(D;\mathbb{F}_p)$ nonzero. All later attachments have rank at least $2m+2$ and preserve this group. This contradicts the vanishing of $H_q(\Lambda_h X;\mathbb{F}_p)$. Hence both circle indices are $2m$.

The same argument applies in every marked sector whose minimum is a large family. The opposite large-family index is $2m+2$, the two circle indices sum to $4m$, and later occurrences add $4m$. The preceding relative homology argument therefore gives the same circle indices.

All attachments above F_1 now have rank at least $2m$. Lemma 29.7 and the homotopy sequence of the pair give

$$\pi_j(F_1) = 0 \qquad (1 \le j \le 2m-2).$$

The restriction of λ to F_1 is contact: the fixed contact subspace of a finite-order strict contact map is symplectic, and the Reeb direction is fixed. Thus F_1 is a closed oriented $(2m-1)$-manifold. Hurewicz gives a map $S^{2m-1} \to F_1$ inducing an isomorphism on top homology. It is an integral homology equivalence, hence a homotopy equivalence by [31, Corollary 4.33]. Reversal gives the assertion for F_{-1}.

Put $B = F_1/S^1$. The circle action is free because its prime period is 2π. Its homotopy sequence shows that B is simply connected. The integral Gysin sequence [31, Section 4.D, p. 438] gives

$$H^*(B;\mathbb{Z}) = \mathbb{Z}[e]/(e^m).$$

The class e^{m-1} generates top cohomology. Its symplectic representative is positive, so $\langle e^{m-1}, [B]\rangle = 1$. Give the scalar Hopf bundle the same positive Euler convention, and denote its class by u. Circle bundles are classified by this class, which differs by a fixed sign from the first Chern class of the associated line bundle [32, Proposition 3.10, p. 86]. Choose a classifying map $f : B \to \mathbb{CP}^\infty$, so that $f^*u = e$. Cellular approximation [31, Theorem 4.8, p. 349], using $\dim B = 2m-2$, makes its image lie in $\mathbb{CP}^{m-1}$. Thus f is an integral cohomology equivalence. Both spaces are simply connected; hence f is a homotopy equivalence.

The circle bundle $F_1 \to B$ is the pullback of the Hopf bundle under f. Its bundle map $F_1 \to S^{2m-1}$ is a homotopy equivalence by the homotopy sequences of the two bundles. It is S^1-equivariant and descends to $F_1/C_p \to L(p;1,\dots,1)$. This map induces an isomorphism on fundamental groups and lifts to the preceding equivalence of universal covers, so it is a homotopy equivalence. For $m=2$, B is a simply connected closed surface and hence diffeomorphic to S^2. Its Euler number is ± 1, so circle-bundle classification identifies it with a Hopf bundle. This identification can be chosen smooth by [35, Chapter 4, Section 3, Theorem 3.5, p. 101]; equality of the nonzero Euler classes makes it orientation preserving. Its polar part is an oriented fiberwise isometry of the associated plane bundles. Restriction to the unit circles gives an equivariant diffeomorphism and hence also identifies the quotients by the fiber subgroup C_p.

For $g = h^b$, (33.1) reads

$$\min_x d(x, g^q x) = \ell \min\{|q|_p, d_q\}.$$

If $q \le (p-1)/2$ and $q < d_q$, the small arc is a minimum, so its index is zero. If $d_q < q \le (p-1)/2$, a large family is the minimum, and its circle index is $2m$. Since $d_{p-q} = d_q$, the complementary identity with $d=1$ gives the remaining cases of (33.7). Equality at an endpoint would imply $b = \pm 1 \pmod p$, which is excluded. □

33.3. **The signed return at the large family.** Put $x = p\theta/(2\pi) \in \mathbb{R}/p\mathbb{Z}$. The round boundary index is

$$j_R(x) = (m-1)\sum_{\epsilon=\pm 1} \mathbf{1}_{\{|x-\epsilon|_p<1\}} + \sum_{\epsilon=\pm 1} \mathbf{1}_{\{|x-\epsilon a|_p<1\}}. \tag{33.8}$$

Here $|\cdot|_p$ denotes circular distance on $\mathbb{R}/p\mathbb{Z}$, and the inequalities are strict. On a round normal real two-plane of deck weight w, the Fourier modes of $-D_t^2 - I$ have eigenvalues $(pk + x \pm w)^2 - 1$, $k \in \mathbb{Z}$. Taking $w = 1$ on $m-1$ planes and $w = a$ on the remaining plane gives the formula. In particular, $\sum_{j=0}^{p-1} j_R(j) = 2m$.

Changing one positive-half branch at $r < p/2$ from leaving to entering adds 2 on $(r, p-r)$ and 1 at its endpoints. It changes the integer-grid sum by $2(p-2r)$. Suppose first that $2 \le a \le (p-3)/2$. In the round model, the $m-1$ branches at 2 leave, the branch at $a-1$ enters, and the branch at $a+1$ leaves. Let $0 \le k \le m-1$ count changes among the first branches, and let $\epsilon_b, \epsilon_c \in \{0,1\}$ count changes at the other two. Equality of the full indices gives

$$k(p-4) - \epsilon_b(p-2a+2) + \epsilon_c(p-2a-2) = 0. \tag{33.9}$$

If $\epsilon_b = 0$, all terms are nonnegative, so $k = \epsilon_c = 0$. If $\epsilon_b = \epsilon_c = 1$, then $k(p-4) = 4$, which is impossible for a prime $p \ge 7$. The remaining case gives $k(p-4) = p-2a+2$. Its positive right side is less than $2(p-4)$, so $k = 1$ and $a = 3$. These two changes exchange branches at the same root 2 and leave the crossing inertia unchanged.

If $a = (p-1)/2$, both exceptional positive-half branches enter, at roots $(p-3)/2$ and $(p-1)/2$. The equation is

$$k(p-4) = 3\epsilon_b + \epsilon_c. \tag{33.10}$$

For $p \ge 11$ only the zero solution occurs. For $p = 7$, the only other solution is $(k, \epsilon_b, \epsilon_c) = (1,1,0)$; it exchanges branches at the same root 2. For $p = 5$, $a = 2$, the equation is $k = 3\epsilon_b + \epsilon_c$. The solution $(1,0,1)$ exchanges signs at root 2. The other nonzero solutions are $(3,1,0)$ and $(4,1,1)$, whenever permitted by $k \le m-1$; they differ by that same exchange. Thus a distinct inertia requires $m \ge 4$. It subtracts 2 on $(1,4)$ and adds 6 on $(2,3)$ to the round function, giving

x	$(0,1) \cup (4,5)$	$(1,2) \cup (3,4)$	$(2,3)$
$j(x)$	$m-1$	$m-2$	6.

At the grid points, $j(1) = j(4) = m-2$ and $j(2) = j(3) = 2$. Thus the large-family indices at lengths $2\ell, 3\ell$ are $6, 2m-4$.

In $\Lambda_{h^2}X$, the first family is F_{-a} at length ℓ. Its opposite circle, at 4ℓ, has index $4m$. The four successive indices would therefore be $0, 6, 2m-4, 4m$, with $4m$ added at every subsequent occurrence. Since $m \ge 4$, all attachments after the minimum circle have rank at least four. Lemma 29.7 would give $\pi_1(\Lambda_{h^2}X) \cong \pi_1(S^1) = \mathbb{Z}$, contrary to the evaluation-fibration calculation. Hence the large-family crossing inertias are the round ones for every $p \ge 5$.

33.4. **An invertible finite convolution.**

Lemma 33.3. *Put $H = (p-1)/2$. If real numbers $c_1, \dots, c_H$ satisfy*

$$\sum_{r=1}^{H} (p-2r)c_r = 0, \qquad \sum_{r=1}^{H} c_r \left(q - 1 - 2\left\lfloor \frac{qr}{p} \right\rfloor \right) = 0 \quad (1 \le q < p),$$

then every c_r is zero.

Proof. For $u \in G = (\mathbb{Z}/p\mathbb{Z})^\times$, let $\langle u \rangle$ be its representative in $\{1, \dots, p-1\}$ and put $B(u) = \langle u \rangle / p - 1/2$. Extend c by $c(r) = c_r$ and $c(-r) = -c_r$ for $1 \le r \le H$. Since

$$q - 1 - 2\left\lfloor \frac{qr}{p} \right\rfloor = \frac{q(p-2r)}{p} + 2B(qr),$$

the hypotheses and the oddness of B give $\sum_{u \in G} c(u)B(qu) = 0$ for every $q \in G$. Multiplying by a character $\chi(q)$ and summing over q, with $v = qu$, gives

$$0 = \beta_\chi \sum_{u \in G} c(u)\overline{\chi(u)}, \qquad \beta_\chi = \sum_{v \in G} \chi(v)B(v). \tag{33.11}$$

Suppose $\chi(-1) = -1$, and extend χ by zero on $p\mathbb{Z}$. Since p is prime, χ is nonprincipal and primitive. Put $\tau(\chi) = \sum_{v=1}^{p-1} \chi(v)e^{2\pi i v/p}$. By the Gauss sum formula [16, Exposé I, Proposition (2.1), pp. 4–5],

$$\sum_{v=1}^{p-1} \chi(v)e^{2\pi i n v/p} = \overline{\chi(n)}\tau(\chi), \qquad |\tau(\chi)| = \sqrt{p}.$$

For $0 < t < 1$, the imaginary part of $-\log(1 - \rho e^{2\pi i t})$ as $\rho \uparrow 1$ gives

$$t - \frac{1}{2} = -\frac{1}{\pi} \lim_{\rho \uparrow 1} \sum_{n \geq 1} \frac{\rho^n \sin(2\pi n t)}{n}.$$

Substitute $t = v/p$, multiply by $\chi(v)$, and sum over v. Oddness and the Gauss sum formula yield

$$\beta_\chi = \frac{i\tau(\chi)}{\pi} \lim_{\rho \uparrow 1} \sum_{n \geq 1} \frac{\overline{\chi(n)}\rho^n}{n} = \frac{i\tau(\chi)}{\pi} L(1, \overline{\chi}) \neq 0.$$

The Abel limit equals the convergent Dirichlet series, since the partial sums of $\overline{\chi}$ are bounded. Nonvanishing at one is [16, Exposé I, Proposition (3.2), pp. 8–9]. Equation (33.11) therefore annihilates every odd Fourier coefficient of c. The even coefficients vanish by oddness, so character orthogonality gives $c = 0$. □

33.5. **The signed small-family return.** For $g = h^b$, $b = a^{-1} \pmod p$, write the small-family jumps at positive-half roots as $\Delta_r = A_r - B_r$ and their nullities as $d_r = A_r + B_r$. The fixed normal space is zero, so

$$\sum_{r=1}^{(p-1)/2} d_r = 2m, \qquad d_r \geq |\Delta_r|.$$

Conjugation gives the opposite jump at $p - r$. Equations (33.3)–(33.5) become

$$i_C(q) = \sum_r \Delta_r \left(q - 1 - 2 \left\lfloor \frac{qr}{p} \right\rfloor \right) \quad (1 \leq q < p), \qquad \sum_r (p - 2r)\Delta_r = 4m.$$

The round function is

$$j_R(x) = m \left(\mathbf{1}_{\{|x-b|_p < 1\}} + \mathbf{1}_{\{|x+b|_p < 1\}} \right).$$

Its q-fold index is

$$m \sum_{\epsilon = \pm 1} \#\{k \in \mathbb{Z} : |pk + \epsilon b q| < q\}.$$

For $q < p$, the two possible distances below p from $\epsilon b q$ to a multiple of p are $d_q, p - d_q$. Each count is consequently $0, 1, 2$ as q crosses these two values. This is (33.7). The round full-period index is also $2m$, so its jumps satisfy the same weighted sum $\sum_r (p - 2r)\Delta_r = 4m$. Subtracting the actual and round equations and applying Lemma 33.3 shows that all their jumps agree.

The round roots $[b - 1], [b + 1]$ are nonzero and distinct: equality would give $2 = 0$ or $2b = 0$ in $\mathbb{F}_p$. Each jump has absolute value m, so

$$\sum_r |\Delta_r| = 2m = \sum_r d_r.$$

Since $d_r \geq |\Delta_r|$ term by term, equality holds at every root. There is no opposite-sign pair or additional root, and the complete small-family crossing inertias are the round ones.

Completion of the proof of Theorem 33.1. The topology is Lemma 33.2. The relative returns are real symplectic maps of odd finite order, hence semisimple with no eigenvalue -1. For a nonreal eigenvalue z, put

$$H_z(u, v) = i\omega(u, \overline{v}), \qquad u, v \in E_z,$$

where $\omega = d\lambda$. This nondegenerate form is the negative of the crossing form. Its inertia is therefore determined by the entering and leaving multiplicities. The semisimple case of [30, Theorem 1.2] classifies these maps by their eigenvalue multiplicities and the inertias of H_z: its higher forms vanish, and its form $\widehat{Q}_1^z$ is H_z. The fixed summands are identities of equal dimension. The equality of crossing inertias proved above therefore gives the required real symplectic conjugacies.

On round normal Jacobi data,

$$d\lambda((x,y),(x',y')) = \langle y,x'\rangle - \langle x,y'\rangle, \qquad J(x,y) = (y,-x).$$

Then $d\lambda(v,Jv) > 0$ for $v \neq 0$, and Jacobi evolution is e^{tJ}. On a real normal two-plane, the inverse deck rotation has complex weights $\pm b$ and commutes with J. At the small family, the short return for $g = h^b$ therefore has weights $1-b, 1+b$, each repeated m times. Since $g^a = h$, its a-th power is the return for h at time $a\ell$, with weights $a-1, a+1$. At the large family, the $m-1$ repeated-weight planes give weights $0, 2$, and the exceptional plane gives $1-a, 1+a$. Removing the fixed summand gives (33.2). Reversal conjugates the corresponding real maps and is antisymplectic, so it determines the other two maps with the reversed signed weights. On the geodesic quotient, G_t acts trivially; the relative return induces $d\bar{h}^{-1}$. □

34. Integral linking and displacement extrema

Retain the hypotheses of Theorem 33.1, and suppose $p \geq 11$. Put $n = 2m+1$, $\ell = 2\pi/p$, and $H = (p-1)/2$. Proposition 12.3 gives $\mathrm{Vol}(X) = V_n$, so (30.1) holds. Theorems 30.7 and 30.10 give disjoint embedded submanifolds

$$N = \mathrm{ev}(F_1) = \mathrm{ev}(F_{-1}), \qquad C = \mathrm{ev}(F_a) = \mathrm{ev}(F_{-a}).$$

By Lemma 33.2, N is a homotopy $(n-2)$-sphere; C is a circle. Fix orientations on X, N, C.

Theorem 34.1. *One has* $|\mathrm{lk}(N,C)| = 1$. *The inclusions*

$$N \longrightarrow X \setminus C, \qquad C \longrightarrow X \setminus N$$

are respectively a homotopy equivalence and an integral homology equivalence.

Put $b = a^{-1} \pmod p$. *At least one of the following alternatives holds. If*

$$1 \leq q < r \leq H, \qquad |aq|_p < q < r < |ar|_p, \tag{34.1}$$

then, for $f = h^q$,

$$\min_X d(x,fx) = |aq|_p\,\ell, \quad \operatorname*{argmin}_X d(x,fx) = C,$$

$$\max_X d(x,fx) = q\ell, \quad \operatorname*{argmax}_X d(x,fx) = N.$$

If

$$1 \leq q < r \leq H, \qquad |bq|_p < q < r < |br|_p, \tag{34.2}$$

then the same conclusions hold for $f = (h^b)^q$, *with* N, C *interchanged and* $|aq|_p$ *replaced by* $|bq|_p$. *In either alternative,* $\frac{1}{2}d(x,fx)^2$ *is smooth near its maximum set. Its Hessian has that set's tangent space as kernel and is negative definite on the normal space.*

Lemma 34.2. *Let* $p \geq 11$ *be prime and* $c \not\equiv 0, \pm 1 \pmod p$. *At least one of the permutations*

$$\pi_c(i) = |ci|_p, \qquad \pi_{c^{-1}}(i) = |c^{-1}i|_p, \qquad 1 \leq i \leq H,$$

has integers $q < r$ *with* $\pi(q) < q < r < \pi(r)$.

Proof. Both permutations have no fixed point. If neither has the asserted pair, their sets of strict exceedances are initial intervals, say $\{1,\dots,R\}$ and $\{1,\dots,S\}$. For $j = \pi_c(i)$,

$$i \leq R \iff j > i \iff \pi_c^{-1}(j) < j \iff j > S.$$

Thus $R + S = H$ and $\pi_c(\{1,\dots,R\}) = \{S+1,\dots,H\}$. After replacing c by its inverse, $1 \leq R \leq H/2$. In $\mathbb{F}_p$, put

$$A = \{-R,\dots,R\}, \qquad B = cA = \{0\} \cup J, \qquad J = \{H-R+1,\dots,H+R\}.$$

One has $|A + A| = 4R + 1$ and $J + J = \{-(2R-1), \dots, 2R-1\}$ modulo p. The latter interval and J are disjoint if $3R \le H$; their union fills $\mathbb{F}_p$ if $3R \ge H + 1$. Hence

$$4R + 1 = |B + B| = \min(p, 6R - 1).$$

Either $R = 1$ or $p = 4R + 1$. In the first case $c \equiv \pm H$, and

$$\pi_c(2) = 1 < 2 < 3 < H - 1 = \pi_c(3),$$

a contradiction.

Suppose $p = 4R + 1$. Multiplication by c sends $A \setminus \{0\}$ to its complement in $\mathbb{F}_p^*$. For $j = 2, 4$, the sum of jth powers over $\mathbb{F}_p^*$ is zero: multiplication by an element whose jth power is not one multiplies the sum by a different scalar. Such an element exists because $j < p - 1$. The sums over A are

$$S_2 = \frac{R(R+1)(2R+1)}{3} = -\frac{1}{32}, \qquad S_4 = \frac{R(R+1)(2R+1)(3R^2+3R-1)}{15} = \frac{5}{512}$$

in $\mathbb{F}_p$, where $R = -1/4$. Both are nonzero, since $p \ge 11$. Thus $c^2 S_2 = -S_2$ and $c^4 S_4 = -S_4$, giving $c^2 = c^4 = -1$, a contradiction. □

34.1. **Negative bundles and evaluation.** Let f be an isometry of a closed oriented homotopy n-sphere X. On $\Lambda_f X$, use the energy

$$E(\gamma) = \frac{1}{2} \int_0^1 |\dot\gamma|^2 \, dt.$$

Write $\mathrm{ev}(\gamma) = \gamma(0)$ and let I_γ denote the Hessian at a critical path. Compact Morse–Bott critical manifolds have the negative disk-bundle attachments of Lemma 29.7.

Lemma 34.3. *Let F be a Morse–Bott critical manifold whose evaluation is a diffeomorphism onto $B \subset X$. Suppose every $\gamma \in F$ minimizes between its fixed endpoints. Then evaluation induces an injection*

$$E_\gamma^- \longrightarrow T_{\gamma(0)}X / T_{\gamma(0)}B, \qquad Y \longmapsto Y(0) \bmod T_{\gamma(0)}B,$$

where E_γ^- is the negative space of I_γ. It is an isomorphism if $\mathrm{ind}(F) = \mathrm{codim}\, B$.

Proof. If $Y(0) \in T_{\gamma(0)}B$, choose $J \in T_\gamma F = \ker I_\gamma$ with $J(0) = Y(0)$. The twisted endpoint condition makes $Y - J$ vanish at both endpoints. Fixed-endpoint minimality gives

$$0 \le I_\gamma(Y - J, Y - J) = I_\gamma(Y, Y).$$

This contradicts negative definiteness unless $Y = 0$. The last assertion follows by equality of dimensions. □

Lemma 34.4. *Suppose the only critical manifolds at and below $L^2/2$ are the minimum F_0 and a compact Morse–Bott manifold F, and that F_0 is Morse–Bott. Suppose their evaluations are diffeomorphisms onto disjoint closed connected oriented submanifolds A, B, with*

$$\mathrm{ind}(F) = \mathrm{codim}\, B = r, \qquad \dim A = r - 1, \qquad 2 \le r \le n - 1.$$

Assume that $d(y, fy) = L$ on B and that every path in F minimizes between its fixed endpoints. Then

$$|\,\mathrm{lk}(A, B)| = 1, \qquad d(x, fx) \le L \quad (x \in X), \qquad d(x, fx) < L \quad (x \notin B).$$

Proof. Put $c = L^2/2$. The paths in F have constant speed and energy c. A tubular chart on $\nu^- F$ gives a sufficiently small disk bundle $D(\nu^- F)$ on which $E < c$ off the zero section: compactness of F makes the negative Hessian estimate uniform. Choose a regular value $c_- < c$ above the maximum energy on its sphere bundle and above the minimum. By Lemma 29.7, $E^{c_-} = \{E \le c_-\}$ retracts onto F_0.

No path of energy below c can evaluate to B, since

$$E(\eta) \ge \tfrac{1}{2}\,\mathrm{length}(\eta)^2 \ge \tfrac{1}{2} d(\eta(0), f\eta(0))^2. \tag{34.3}$$

For $\gamma \in F$, the evaluation of the fiber disk D^r_γ therefore meets B only at its centre. By Lemma 34.3, this intersection is transverse with sign ± 1. Deform its boundary in E^{c-} into F_0. The evaluated cylinder avoids B, and gives a map of pairs

$$\Psi : (D^r, S^{r-1}) \longrightarrow (X, A)$$

whose algebraic intersection with B is ± 1. Since $2 \le r \le n-1$ and X is a homology sphere, the boundary map

$$H_r(X, A; \mathbb{Z}) \longrightarrow H_{r-1}(A; \mathbb{Z}) = \mathbb{Z}[A]$$

is an isomorphism. If $\partial[\Psi] = d[A]$, the definition of linking by intersection gives $d \operatorname{lk}(A, B) = \pm 1$. Thus $|\operatorname{lk}(A, B)| = |d| = 1$.

Lemma 34.3 also gives a vector-bundle isomorphism

$$\nu^- F \cong (\operatorname{ev}|_F)^* \nu_X B.$$

The orientations of X and B therefore orient $\nu^- F$ and the n-manifold $D(\nu^- F)$. Its boundary fundamental cycle bounds a finite singular n-chain in E^{c-}, because

$$H_{n-1}(E^{c-}; \mathbb{Z}) = H_{n-1}(A; \mathbb{Z}) = 0.$$

Capping the negative disk bundle by this chain gives an integral n-cycle Γ in $\Lambda_f X$. All its represented paths have energy at most c, with equality only on the zero section.

For $y \in B$, evaluation has just one preimage of y on the negative disk bundle and none on the cap. At that preimage its differential is invertible: the zero-section directions map isomorphically onto $T_y B$, and the negative directions onto the normal quotient. The image of $[\Gamma]$ in $H_n(X, X \setminus \{y\}; \mathbb{Z})$ is consequently a local orientation generator. Excision gives

$$\operatorname{ev}_*[\Gamma] = \pm[X].$$

The evaluated cycle cannot omit a point x, since $H_n(X \setminus \{x\}; \mathbb{Z}) = 0$. Thus every x is the initial point of a represented path of energy at most c. If $x \notin B$, such a path is off the zero section and has energy below c. Equation (34.3) proves the two displacement bounds. □

34.2. **Proof of the linking and extremum theorem.**

Proof of Theorem 34.1. Lemma 34.2 gives (34.1) or (34.2). Assume first (34.1), and put $d = |aq|_p$. The critical lengths below 2π in $\Lambda_{h^q} X$ are

$$d\ell, \quad q\ell, \quad (p-q)\ell, \quad (p-d)\ell;$$

their evaluated images are C, N, N, C, respectively. The first manifold is the full energy minimum. Indeed, (33.1) and (34.3) give its energy as the minimum, and any path attaining it is a minimizing twisted critical path. The classification of critical paths then places it in the first manifold. Corollary 30.5 gives $\operatorname{argmin}_X d(x, h^q x) = C$.

Since $r < |ar|_p$, the marked minimum formula makes the $r\ell$-arc of every large returning geodesic minimizing. Its $q\ell$ prefix is therefore minimizing, and $d(x, h^q x) = q\ell$ on N. Lemma 34.3 bounds the index of the second critical manifold by two. It is even by (33.6), and it is positive. For otherwise this attachment would add a connected component; all subsequent negative ranks are even, so no later attachment could join it to the minimum. Any path joining the two components would have bounded energy and meet only finitely many critical levels, contrary to the connectedness of $\Lambda_{h^q} X$. Here connectedness follows from the evaluation fibration, with connected base X and connected fiber ΩX. Thus the index is two. Lemma 34.4, with $A = C$ and $B = N$, gives the absolute linking number one and $\operatorname{argmax}_X d(x, h^q x) = N$.

Under (34.2), put $g = h^b$. Its small circle has phase one. The first two critical lengths in $\Lambda_{g^q} X$ are $|b\,q|_p \ell$ and $q\ell$, with images N, C. The first is the full minimum, by the same energy argument. Corollary 30.5, with $j = bq$, gives its displacement minimum set N, since $|bq|_p < q = |abq|_p$.

Formula (33.7) gives index $n-1$ for the second. Since $r < |b\,r|_p$, its $r\ell$-arc is minimizing, and so is the $q\ell$ prefix. Lemma 34.4 applies with $A = N$ and $B = C$, and gives the other alternative.

In either case, write B for the maximum set and put $L = q\ell$. The selected arc from $x \in B$ to $f\,x$ is a proper prefix of a minimizing arc. It is the unique minimizing arc to $f\,x$: replacing it by another and retaining the tail would give a minimizing curve, whose velocities must agree at the junction. Uniqueness for the geodesic equation identifies the two prefixes. Lemma 13.4 excludes conjugacy at $f\,x$. Thus the endpoint pair is outside the cut locus, and $u(y) = \frac{1}{2}d(y, f\,y)^2$ is smooth near B [52, proof of Lemma 1.1.2, p. 413].

Parametrize the selected arc by unit speed on $[0, L]$. Let I be its index form and, for $w \in T_xX$, let J_w be the unique Jacobi field with $J_w(0) = w$ and $J_w(L) = d\,f_x\,w$. Second variation gives

$$\operatorname{Hess}_x u(w, z) = LI(J_w, J_z)$$

[52, (1.3.1), p. 416]. Every twisted variation has a unique decomposition $J_w + Z$, where $Z(0) = Z(L) = 0$. Integration by parts gives $I(J_w, Z) = 0$. The Dirichlet form is nonnegative by minimality and has zero kernel by nonconjugacy; hence it is positive definite. Therefore the endpoint form $I(J_w, J_z)$ has the index and nullity of the full twisted form. These are codim B and dim B, respectively, and evaluation identifies its kernel with T_xB. This proves the assertion about the Hessian of u.

Integral duality for a compact submanifold of a closed oriented manifold [31, Theorem 3.44, pp. 254–255], applied to (X, C) and (X, N), and the relative homology sequences give

$$\widetilde{H}_i(X \setminus C; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & i = n-2, \\ 0, & i \neq n-2, \end{cases} \qquad \widetilde{H}_i(X \setminus N; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & i = 1, \\ 0, & i \neq 1. \end{cases}$$

The inclusions send $[N]$ and $[C]$ to their linking numbers times the corresponding meridian generators: this follows by bounding these cycles in X and intersecting with the omitted submanifold. Both coefficients have absolute value one, so both inclusions are integral homology equivalences. The complement $X \setminus C$ is simply connected. Indeed, a disk filling a loop in X can be made transverse to C relative to its boundary, and $2 + 1 - n < 0$ forces the disk to avoid C. Both N and $X \setminus C$ have CW homotopy type and are simply connected. The homological Whitehead theorem [31, Corollary 4.33] makes their inclusion a homotopy equivalence. □

34.3. **The other large-family sectors.**

Theorem 34.5. *If $1 \le q \le H$ and $d = |a\,q|_p < q$, then*

$$d(x, h^q x) \le q\ell \quad (x \in X), \qquad d(x, h^q x) < q\ell \quad (x \notin N).$$

Proof. Equations (33.3), (33.8), and (33.5) give the following critical manifolds below length 2π:

length	$d\ell$	$q\ell$	$(p-q)\ell$	$(p-d)\ell$
image	C	N	N	C
index	0	2	$n-1$	$2n-2$.

For the $q\ell$-arc, each of the two exceptional terms in (33.8) contributes one to the Bott sum, and the repeated terms contribute zero, since $d < q$ and $2q < p$. Complementarity gives the reversed indices. Each added full period increases the index by $2n-2$.

Choose regular sublevels $Z \subset Y$ immediately below and above the second critical value. Then Z retracts onto the minimum circle. The rank-two negative bundle over the simply connected N is oriented, and its Thom isomorphism gives

$$H_n(Y, Z; \mathbb{Z}) \cong H_{n-2}(N; \mathbb{Z}) \cong \mathbb{Z}.$$

Since $H_n(Z; \mathbb{Z}) = H_{n-1}(Z; \mathbb{Z}) = 0$, the relative exact sequence identifies $H_n(Y; \mathbb{Z})$ with this group. The next attachment also has oriented negative bundle over N, of rank $n-1$. Its relative groups in

degrees $n, n+1$ are $H_1(N;\mathbb{Z})$ and $H_2(N;\mathbb{Z})$; both vanish. All later negative ranks exceed $n+1$. The relative exact sequences therefore give

$$H_n(Y;\mathbb{Z}) \xrightarrow{\cong} H_n(\Lambda_{h^q} X;\mathbb{Z}) \xrightarrow[\mathrm{ev}_*]{\cong} H_n(X;\mathbb{Z}).$$

The first isomorphism passes to the full path space because every singular chain has bounded energy. The second is Lemma 29.7: h^q has degree one, since its order is odd, and $n \geq 5$.

Cap the oriented negative disk bundle over N by a finite singular chain in Z, as in the proof of Lemma 34.4. Its relative fundamental class is a Thom generator, so the resulting n-cycle generates $H_n(Y;\mathbb{Z})$. Every represented path has energy at most $(q\ell)^2/2$, with equality only on the zero section. Its evaluation represents $\pm[X]$ and therefore covers X. For $x \notin N$, the represented paths starting at x lie off the zero section and have strictly smaller energy. Equation (34.3) proves the result. □

References

[1] A. Abbondandolo and G. Benedetti, *On the local systolic optimality of Zoll contact forms*, Geom. Funct. Anal. **33** (2023), no. 2, 299–363. doi:10.1007/s00039-023-00624-z. https://arxiv.org/abs/1912.04187.

[2] A. Abbondandolo, G. Benedetti and L. Polterovich, *Lorentz–Finsler metrics on symplectic and contact transformation groups*, Ann. Sc. Norm. Super. Pisa Cl. Sci., published online (2025). DOI: 10.2422/2036-2145.202305_003. Also arXiv:2210.02387v1.

[3] A. A. Agrachev, A. Gentile, and A. Lerario, *Geodesics and horizontal-path spaces in Carnot groups*, Geom. Topol. **19** (2015), no. 3, 1569–1630. doi:10.2140/gt.2015.19.1569.

[4] M. Amann, C. Lange and M. Radeschi, *Odd-dimensional orbifolds with all geodesics closed are covered by manifolds*, Math. Ann. **380** (2021), 1355–1386. DOI: 10.1007/s00208-021-02214-6.

[5] B. Ammann, R. Lauter, V. Nistor, and A. Vasy, *Complex powers and non-compact manifolds*, Comm. Partial Differential Equations **29** (2004), 671–705, doi:10.1081/PDE-120037329. https://arxiv.org/abs/math/0211305v1.

[6] V. N. Berestovskiĭ and Yu. G. Nikonorov, *Clifford–Wolf homogeneous Riemannian manifolds*, J. Differential Geom. **82** (2009), no. 3, 467–500. https://archive.ymsc.tsinghua.edu.cn/pacm_download/21/781-JDG-82-3-A1-berestovskii.pdf.

[7] R. Bott, *On the iteration of closed geodesics and the Sturm intersection theory*, Comm. Pure Appl. Math. **9** (1956), 171–206. doi:10.1002/cpa.3160090204.

[8] K. S. Brown, *Cohomology of Groups*, Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York–Berlin, 1982. doi:10.1007/978-1-4684-9327-6.

[9] A. Cannas da Silva, *Lectures on Symplectic Geometry*, Lecture Notes in Mathematics **1764**, Springer-Verlag, Berlin, 2001. DOI: 10.1007/978-3-540-45330-7. Author's version, revised January 2006.

[10] H. H. Chan, S. H. Chan, and Z.-G. Liu, *Elementary derivations of the Rogers–Fine identity and other q-series identities*, Discrete Math. **348** (2025), 114387. doi:10.1016/j.disc.2024.114387.

[11] D. Cianci, *On the Poisson relation for flat and spherical space forms*, arXiv:1608.02816 (2016). https://arxiv.org/abs/1608.02816.

[12] Y. Colin de Verdière, *Sur le spectre des opérateurs elliptiques à bicaractéristiques toutes périodiques*, Comment. Math. Helv. **54** (1979), 508–522. doi:10.1007/BF02566290.

[13] Y. Colin de Verdière, *Spectrum of the Laplace operator and periodic geodesics: thirty years after*, Ann. Inst. Fourier **57** (2007), 2429–2463, doi:10.5802/aif.2339. https://aif.centre-mersenne.org/item/10.5802/aif.2339.pdf.

[14] M. Colantonio and E. A. Lauret, *The fundamental group of a spherical space form is not audible*, arXiv:2602.12939, version 2 (2026). https://arxiv.org/abs/2602.12939.

[15] F. Dai and Y. Xu, *Spherical harmonics*, arXiv:1304.2585v1 (2013), Chapter 1 of *Approximation Theory and Harmonic Analysis on Spheres and Balls*. https://arxiv.org/abs/1304.2585v1.

[16] G. de Rham, S. Maumary and M. A. Kervaire, *Torsion et type simple d'homotopie*, Lecture Notes in Mathematics, no. 48, Springer-Verlag, Berlin–Heidelberg–New York, 1967. https://webhomes.maths.ed.ac.uk/~v1ranick/papers/derkerma.pdf.

[17] J. J. Duistermaat and V. W. Guillemin, *The spectrum of positive elliptic operators and periodic bicharacteristics*, Invent. Math. **29** (1975), 39–79. doi:10.1007/BF01405172.

[18] J. J. Duistermaat and L. Hörmander, *Fourier integral operators. II*, Acta Math. **128** (1972), 183–269. doi:10.1007/BF02392165.

[19] S. Dwivedi, J. Herman, L. C. Jeffrey, and T. van den Hurk, *Hamiltonian Group Actions and Equivariant Cohomology*, SpringerBriefs in Mathematics, Springer, Cham, 2019. doi:10.1007/978-3-030-27227-2.

[20] P. Etingof, O. Golberg, S. Hensel, T. Liu, A. Schwendner, D. Vaintrob, and E. Yudovina, *Introduction to Representation Theory*, Student Mathematical Library, vol. 59, American Mathematical Society, Providence, RI, 2011. doi:10.1090/stml/059. https://math.mit.edu/~etingof/representationtheorybook.pdf.

[21] W. Franz, *Über die Torsion einer Überdeckung*, J. reine angew. Math. **173** (1935), 245–254. doi:10.1515/crll.1935.173.245.

[22] C. Garetto, C. Jäh, and M. Ruzhansky, *Hyperbolic systems with non-diagonalisable principal part and variable multiplicities, I: well-posedness*, Math. Ann. **372** (2018), 1597–1629. doi:10.1007/s00208-018-1672-1.
[23] P. B. Gilkey, *Invariance Theory, the Heat Equation, and the Atiyah–Singer Index Theorem*, Publish or Perish, 1984; electronic reprint, 1996, https://pages.uoregon.edu/gilkey/dirPDF/InvarianceTheory1Ed.pdf.
[24] C. Gordon, W. Kirwin, D. Schueth and D. Webb, *Quantum equivalent magnetic fields that are not classically equivalent*, Ann. Inst. Fourier **60** (2010), no. 7, 2403–2419. doi:10.5802/aif.2612.
[25] K. Grove, *Condition* (C) *for the energy integral on certain path spaces and applications to the theory of geodesics*, J. Differential Geom. **8** (1973), no. 2, 207–223. doi:10.4310/jdg/1214431639.
[26] K. Grove and M. Tanaka, *On the number of invariant closed geodesics*, Acta Math. **140** (1978), 33–48. doi:10.1007/BF02392302.
[27] V. Guillemin and K. Okikiolu, *Spectral asymptotics of Toeplitz operators on Zoll manifolds*, J. Funct. Anal. **146** (1997), no. 2, 496–516. doi:10.1006/jfan.1996.3044.
[28] V. Guillemin, A. Uribe, and Z. Wang, *Semiclassical states associated with isotropic submanifolds of phase space*, Lett. Math. Phys. **106** (2016), no. 12, 1695–1728. doi:10.1007/s11005-016-0853-7. arXiv:1511.01241v2.
[29] J. Gutt, *Generalized Conley–Zehnder index*, Ann. Fac. Sci. Toulouse Math. (6) **23** (2014), 907–932. arXiv:1307.7239.
[30] J. Gutt, *Normal forms for symplectic matrices*, Portugaliae Math. (N.S.) **71** (2014), no. 2, 109–139. doi:10.4171/PM/1944. https://ems.press/content/serial-article-files/44732.
[31] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.
[32] A. Hatcher, *Vector Bundles and K-Theory*, version 2.2, November 2017. https://pi.math.cornell.edu/~hatcher/VBKT/VB.pdf.
[33] J. Hedicke, *A causal characterization of* $\mathrm{Sp}^{+}_{\mathrm{ell}}(2n)$, Commun. Contemp. Math. **26** (2024), no. 7, 2350040, 13 pp. DOI: 10.1142/S0219199723500402. arXiv:2211.03707v1.
[34] J. W. Helton, *An operator algebra approach to partial differential equations; propagation of singularities and spectral theory*, Indiana Univ. Math. J. **26** (1977), no. 6, 997–1018. doi:10.1512/iumj.1977.26.26081. https://mathweb.ucsd.edu/~helton/BILLSPAPERSscanned/H77a.pdf.
[35] M. W. Hirsch, *Differential Topology*, Graduate Texts in Mathematics, vol. 33, Springer-Verlag, New York–Heidelberg, 1976. https://math.stanford.edu/~ralph/math215b/Hirsch.pdf.
[36] L. Hörmander, *The spectral function of an elliptic operator*, Acta Math. **121** (1968), 193–218. doi:10.1007/BF02391913.
[37] W.-Y. Hsiang, *Cohomology Theory of Topological Transformation Groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 85, Springer-Verlag, Berlin–Heidelberg, 1975. doi:10.1007/978-3-642-66052-8.
[38] E. Humbert, Y. Privat, and E. Trélat, *Geometric and spectral characterization of Zoll manifolds, invariant measures and quantum limits*, arXiv:1811.12717 (2018). https://arxiv.org/abs/1811.12717.
[39] A. Ikeda, *On the spectrum of a Riemannian manifold of positive constant curvature*, Osaka J. Math. **17** (1980), 75–93. DOI: 10.18910/5171.
[40] A. Ikeda and Y. Yamamoto, *On the spectra of 3-dimensional lens spaces*, Osaka J. Math. **16** (1979), 447–469. doi:10.18910/4811.
[41] M. Izydorek and W. Marzantowicz, *Equivariant maps between cohomology spheres*, Topol. Methods Nonlinear Anal. **5** (1995), 279–289. doi:10.12775/TMNA.1995.019.
[42] D. Jakobson and A. Strohmaier, *High energy limits of Laplace-type and Dirac-type eigenfunctions and frame flows*, Communications in Mathematical Physics **270** (2007), no. 3, 813–833. DOI: 10.1007/s00220-006-0176-0. https://arxiv.org/abs/math/0607616.
[43] T. Kato, *Perturbation Theory for Linear Operators*, corrected printing of the second edition, Grundlehren der mathematischen Wissenschaften, vol. 132, Springer-Verlag, Berlin–New York, 1980.
[44] S. Kobayashi, *Transformation Groups in Differential Geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 70, Springer-Verlag, 1972; reprinted in Classics in Mathematics, 1995. doi:10.1007/978-3-642-61981-6.
[45] R. Kuwabara, *Spectrum of the Laplacian on vector bundles over* $C_{2\pi}$*-manifolds*, J. Differential Geom. **27** (1988), no. 2, 241–258. doi:10.4310/jdg/1214441782.
[46] R. Kuwabara, *Correction and addendum to: On the classical and the quantum mechanics in a magnetic field*, J. Math. Tokushima Univ. **30** (1996), 81–87. https://www-math.st.tokushima-u.ac.jp/journal/1996/1996-6.pdf.
[47] E. A. Lauret, *A computational study on lens spaces isospectral on forms*, Experimental Mathematics **30** (2021), no. 2, 268–282. doi:10.1080/10586458.2018.1538908. The theorem and remark numbers refer to arXiv:1703.03077v3 (9 May 2017), https://arxiv.org/abs/1703.03077v3.
[48] E. A. Lauret and R. J. Miatello, *Strong multiplicity one theorems for locally homogeneous spaces of compact-type*, Proc. Amer. Math. Soc. **148** (2020), no. 7, 3163–3173. doi:10.1090/proc/14980. https://arxiv.org/abs/1804.08288v2.
[49] S. Lin and B. Schmidt, *Real projective spaces with all geodesics closed*, Geom. Funct. Anal. **27** (2017), 631–636. DOI: 10.1007/s00039-017-0407-x.
[50] J. Milnor, *Morse Theory*, Annals of Mathematics Studies, no. 51, Princeton University Press, 1963.
[51] E. Meinrenken, *Group actions on manifolds*, lecture notes, University of Toronto, Spring 2003. https://www.math.toronto.edu/mein/teaching/LectureNotes/action.pdf.

[52] V. Ozols, *Critical points of the displacement function of an isometry*, J. Differential Geom. **3** (1969), no. 3–4, 411–432. doi:10.4310/jdg/1214429062.
[53] G. P. Paternain, *Transparent connections over negatively curved surfaces*, J. Mod. Dyn. **3** (2009), 311–333. doi:10.3934/jmd.2009.3.311. arXiv:0809.4360, version 4 (2010).
[54] R. Ponge, *Connes' integration and Weyl's laws*, J. Noncommut. Geom. **17** (2023), 719–767, doi:10.4171/JNCG/509.
[55] F. Prüfer, *On the spectrum and the geometry of the spherical space forms, I*, Ann. Global Anal. Geom. **3** (1985), 129–154. doi:10.1007/BF01000336.
[56] F. Prüfer, *On the spectrum and the geometry of a spherical space form II*, Ann. Global Anal. Geom. **3** (1985), 289–312. doi:10.1007/BF00130482.
[57] M. Radeschi and B. Wilking, *On the Berger conjecture for manifolds all of whose geodesics are closed*, Invent. Math. **210** (2017), 911–962. DOI: 10.1007/s00222-017-0742-4. Also arXiv:1511.07852.
[58] S. Ramanujan, *On certain trigonometrical sums and their applications in the theory of numbers*, Trans. Cambridge Philos. Soc. **22** (1918), no. 13, 259–276. https://ramanujan.sirinudi.org/Volumes/published/ram21.pdf.
[59] S. Smale, *Generalized Poincaré's conjecture in dimensions greater than four*, Ann. of Math. (2) **74** (1961), 391–406.
[60] R. P. Stanley, *Enumerative Combinatorics*, Vol. 1, second edition, Cambridge University Press, 2012. Author's version of 15 July 2011, https://math.mit.edu/~rstan/ec/ec1.pdf.
[61] E. M. Stein and R. Shakarchi, *Fourier Analysis: An Introduction*, Princeton Lectures in Analysis, vol. 1, Princeton University Press, Princeton, NJ, 2003.
[62] C. J. Sutton, *Detecting the moments of inertia of a molecule via its rotational spectrum*, 2013. https://math.dartmouth.edu/~cjsutton/Sutton.2013.Moments1.pdf.
[63] C. J. Sutton, *On the Poisson relation for compact Lie groups*, Ann. Global Anal. Geom. **57** (2020), 537–589. DOI: 10.1007/s10455-020-09712-x. Also arXiv:1606.07426.
[64] S. Tanno, *Eigenvalues of the Laplacian of Riemannian manifolds*, Tôhoku Math. J. (2) **25** (1973), 391–403. doi:10.2748/tmj/1178241341.
[65] M. E. Taylor, *Short Course on Pseudodifferential Operators*, author's version, 2022. https://mtaylor.web.unc.edu/wp-content/uploads/sites/16915/2022/05/psidolect.pdf.
[66] G. Teschl, *Mathematical Methods in Quantum Mechanics: With Applications to Schrödinger Operators*, Graduate Studies in Mathematics, vol. 99, American Mathematical Society, Providence, RI, 2009. https://www.mat.univie.ac.at/~gerald/ftp/book-schroe/schroe.pdf.
[67] C. T. C. Wall, *Free actions of finite groups on spheres*, in *Algebraic and Geometric Topology*, Part 1 (R. J. Milgram, ed.), Proc. Sympos. Pure Math., vol. 32, American Mathematical Society, Providence, RI, 1978, 115–124.
[68] A. Weinstein, *On the volume of manifolds all of whose geodesics are closed*, J. Differential Geom. **9** (1974), 513–517. doi:10.4310/jdg/1214432547.
[69] B. Wilking, *Index parity of closed geodesics and rigidity of Hopf fibrations*, Invent. Math. **144** (2001), 281–295. doi:10.1007/PL00005801.
[70] J. A. Wolf, *Isospectrality for spherical space forms*, Results Math. **40** (2001), 321–338. doi:10.1007/BF03322715.
[71] J.-Y. Wu, *On almost isospectral manifolds*, Indiana Univ. Math. J. **39** (1990), no. 4, 1373–1381. doi:10.1512/iumj.1990.39.39061.
[72] C. T. Yang, *Odd-dimensional Wiedersehen manifolds are spheres*, J. Differential Geom. **15** (1980), no. 1, 91–96. doi:10.4310/jdg/1214435386.
[73] S. Zelditch, *Maximally degenerate Laplacians*, Ann. Inst. Fourier (Grenoble) **46** (1996), 547–587. doi:10.5802/aif.1524.

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