EXPONENTIAL ASYMPTOTIC STABILITY FOR AN ELLIPTIC EQUATION WITH MEMORY ARISING IN ELECTRICAL CONDUCTION

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Abstract. We study an electrical conduction problem in biological tissues in the radiofrequency range, which is governed by an elliptic equation with memory. We prove the time exponential asymptotic stability of the solution, providing in this way a theoretical justification to the complex elliptic problem currently used in electrical impedance tomography.
Our approach relies on the fact that the elliptic equation is the homogenization limit of a sequence of problems for which we are able to prove suitable uniform estimates.

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Contents

1. Introduction

We study the electrical conduction in biological tissues in the radiofrequency range. In this context, a model has been obtained by our group via homogenization theory in [2, 1, 3]. This model is governed by an elliptic equation with memory for the electric potential u_0 (equation (1.1) below).
In this paper we are interested in the behavior of the solution u_0 for large times. In this regard, we prove an asymptotic stability result (Theorem 1.5), which, roughly speaking, states that u_0 exponentially approaches a time-periodic steady state $u_0^{\#}$ as time increases, provided that a time-periodic Dirichlet boundary condition is assigned.
We think that this work is relevant from the point of view of applications, since we give here a theoretical justification to the complex elliptic Problem (1.31)–(1.32) currently used in electrical impedance tomography [6, 8]. Indeed, experimental measurements are performed by assigning time-harmonic boundary data and assuming that the resulting electric potential is time-harmonic, too. In this paper we prove that this assumption is substantially correct for sufficiently large times and that the steady-state electric potential does satisfy the well-known equation (1.31). Moreover, we show how the complex admittivity A^{ω_k} appearing in equation (1.31) depends on the frequency ω_k (equation (1.33) below). Finally, we derive Problem (1.38)–(1.39) which uniquely determines the asymptotic limit $u_0^{\#}$, under time periodic (not necessarily time-harmonic) boundary data. Accordingly, Problem (1.38)–(1.39) can be regarded as a generalization of the standard complex elliptic problem to periodic boundary data. Analogously, the problem for u_0 generalizes the same elliptic problem to nonperiodic (e.g., impulsive) boundary data [2], though here we deal only with the periodic case. We suggest that future inverse-problem research about these problems could bring significant improvements in electrical impedance tomography.
From a mathematical point of view, the asymptotic behavior of evolutive equations with memory is a classical problem [13, 21, 10, 17], currently drawing much interest in the literature [14, 16, 15, 19, 5]. In our context the results of [12] (see also [11, 9]) appear more relevant. There, an elliptic equation with memory, similar to (1.1), is proved to admit a unique solution in a suitable function space. This is done under some assumptions of integrability and coercivity of the integral kernel (see i)-iii) in [12]), which state its compatibility with Thermodynamics. These conditions are far from being obviously satisfied: in fact, the exponential decay of the kernel alone, in general, does not imply the existence of bounded solutions [13, 10].
The results quoted above show the necessity of a detailed study of the structure of the coefficients in (1.1). We recall also that equation (1.1) follows from an homogenization procedure applied to Problem (1.5)–(1.9) below. Hence, we find convenient to obtain the required informations on the structure of the coefficients in (1.1) exploiting this approximation procedure. This approach forces us to obtain estimates for the time

asymptotic convergence rate for Problem (1.5)–(1.9) which are uniform with respect to the homogenization parameter ε.
We note that the coercivity assumptions on the integral kernel, cited above, are a byproduct of this approach (see Proposition 2.2, Remark 4.4 and Remark 5.1).
Moreover, our uniform estimates of the convergence rate could be a useful tool to refine standard error estimates arising in the homogenization procedure.

The paper is organized as follows: in Subsection 1.1 we state our results. In Section 2 we recall the homogenization setting and prove some related decay estimates. Section 3 is devoted to the proof of Theorem 1.1. Section 4, respectively Section 5, contains the proof of Theorem 1.2 in the case $k \neq 0$, respectively in the case $k = 0$. Finally, Theorem 1.3 is proved in Section 6, and Theorem 1.5 is proved in Section 7.

1.1. **Detailed exposition of the results.** It was proved in [2] that the electric potential $u_0(x,t)$ satisfies the equation:

$$-\operatorname{div}\Big(A\nabla u_0 + \int_0^t B(t-\tau)\nabla u_0(x,\tau)\,\mathrm{d}\tau - \mathcal{F}\Big) = 0\,, \quad \text{in } \Omega\times(0,+\infty), \tag{1.1}$$

where Ω is an open connected bounded subset of $\boldsymbol{R}^N$, $N>1$, and the matrices A, $B(t)$, and the vector $\mathcal{F}(x,t)$ are given in equations (2.5) below.
Equation (1.1) is complemented here with a time-periodic Dirichlet boundary condition:

$$u_0(x,t) = \Psi(x)\Phi(t)\,, \quad \text{on } \partial\Omega\times(0,+\infty). \tag{1.2}$$

We assume that

$$\Phi(t)\in H^1_\#(\boldsymbol{R})\,. \tag{1.3}$$

Here and in the following a subscript $\#$ denotes a space of T-periodic functions, for some fixed $T>0$. Moreover, we assume that Ψ is the trace on $\partial\Omega$ of a function, still denoted by Ψ, such that

$$\Psi(x)\in H^1(\boldsymbol{R}^N)\,, \qquad \Delta\Psi = 0 \text{ in } \Omega. \tag{1.4}$$

Problem (1.1)–(1.2) is the homogenization limit as $\varepsilon\searrow 0$ of the problem for $u_\varepsilon(x,t)$ [2]:

$$-\operatorname{div}(\sigma\nabla u_\varepsilon) = 0\,, \qquad \text{in } (\Omega_1^\varepsilon\cup\Omega_2^\varepsilon)\times(0,+\infty); \tag{1.5}$$

$$[\sigma\nabla u_\varepsilon\cdot\nu] = 0\,, \qquad \text{on } \Gamma^\varepsilon\times(0,+\infty); \tag{1.6}$$

$$\frac{\alpha}{\varepsilon}\frac{\partial}{\partial t}[u_\varepsilon] = (\sigma\nabla u_\varepsilon\cdot\nu)^{(\mathrm{out})}\,, \qquad \text{on } \Gamma^\varepsilon\times(0,+\infty); \tag{1.7}$$

$$u_\varepsilon(x,t) = \Psi(x)\Phi(t)\,, \qquad \text{on } \partial\Omega\times(0,+\infty); \tag{1.8}$$

$$[u_\varepsilon](x,0) = S_\varepsilon(x)\,, \qquad \text{on } \Gamma^\varepsilon. \tag{1.9}$$

The operators div and ∇ act with respect to the space variable x; $\Omega = \Omega_1^\varepsilon\cup\Omega_2^\varepsilon\cup\Gamma^\varepsilon$, where Ω_1^ε and Ω_2^ε are two disjoint open subsets of Ω, and $\Gamma^\varepsilon = \partial\Omega_1^\varepsilon\cap\Omega = \partial\Omega_2^\varepsilon\cap\Omega$, with ν as normal unit vector pointing into Ω_2^ε; the typical geometry we have in mind is depicted in Figure 1. We refer to Section 2 for a precise definition of the structure of Ω_1^ε, Ω_2^ε, Γ^ε.

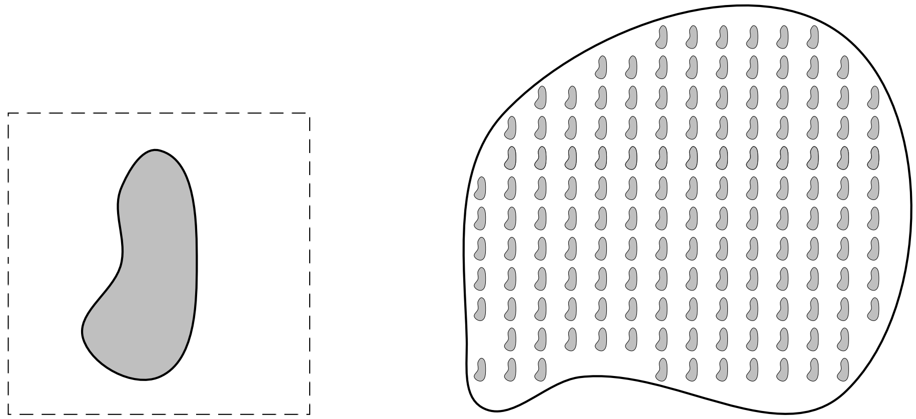

Figure 1. On the left: an example of admissible periodic unit cell $Y = E_1 \cup E_2 \cup \varGamma$ in $\boldsymbol{R}^2$. Here E_1 is the shaded region and $\varGamma$ is its boundary. The remaining part of Y (the white region) is E_2. On the right: the corresponding domain $\varOmega = \varOmega_1^\varepsilon \cup \varOmega_2^\varepsilon \cup \varGamma^\varepsilon$. Here $\varOmega_1^\varepsilon$ is the shaded region and $\varGamma^\varepsilon$ is its boundary. The remaining part of $\varOmega$ (the white region) is $\varOmega_2^\varepsilon$.

Moreover, we assume that:

$$\sigma = \sigma_1 > 0 \quad \text{in } \varOmega_1^\varepsilon, \qquad \sigma = \sigma_2 > 0 \quad \text{in } \varOmega_2^\varepsilon; \qquad \alpha > 0\,, \tag{1.10}$$

where σ_1, σ_2 and α are constant. From a physical point of view, $\varGamma^\varepsilon$ represents the cell membranes, having capacitance α/ε per unit area, whereas $\varOmega_1^\varepsilon$ (resp., $\varOmega_2^\varepsilon$) is the intracellular (resp., extracellular) space, whose conductivity is σ_1 (resp., σ_2).
Since u_ε is not in general continuous across $\varGamma^\varepsilon$ we have set

$$u_\varepsilon^{(\text{out})} := \text{ trace of } u_{\varepsilon|\varOmega_2^\varepsilon} \text{ on } \varGamma^\varepsilon\,, \quad u_\varepsilon^{(\text{int})} := \text{ trace of } u_{\varepsilon|\varOmega_1^\varepsilon} \text{ on } \varGamma^\varepsilon\,,$$
$$\text{and} \quad [u_\varepsilon] := u_\varepsilon^{(\text{out})} - u_\varepsilon^{(\text{int})}\,.$$

A similar convention is employed for the current flux density across the membrane $\sigma\nabla u_\varepsilon \cdot \nu$.
It is known [2] that for every $\overline{T} > 0$, up to a subsequence, u_ε weakly converges in $L^2(\varOmega \times (0, \overline{T}))$ and strongly converges in $L^1_{\text{loc}}(0, \overline{T}; L^1(\varOmega))$ as $\varepsilon \to 0$, provided that the initial datum $S_\varepsilon(x) \in L^2(\varGamma^\varepsilon)$ satisfies:

$$\frac{1}{\varepsilon} \int_{\varGamma^\varepsilon} S_\varepsilon^2(x)\,\mathrm{d}\sigma \le \gamma\,, \tag{1.11}$$

for a constant γ independent of ε. If, moreover, $S_\varepsilon(x)$ satisfies (2.3) and (2.4) below, then any limit $u_0(x,t)$ belongs to $L^2(0, \overline{T}; H^1_0(\varOmega))$ and satisfies Problem (1.1)–(1.2). Therefore, by the uniqueness theorem in [1], the limit is uniquely determined, thus implying the convergence of all the sequence $\{u_\varepsilon\}$.

In this paper we are interested in studying the asymptotic behavior of $u_0(x,t)$ for large times: to this end, we extensively resort to the above approximation procedure of u_0 as homogenization limit of the sequence $\{u_\varepsilon\}$.

In Section 3 we establish the following exponential time-decay for u_0 when homogeneous Dirichlet boundary data prevail on $\partial\Omega \times (0,+\infty)$:

Theorem 1.1. *Let $\Omega_1^\varepsilon, \Omega_2^\varepsilon, \Gamma^\varepsilon$ be as before. Assume that* (1.10) *holds and the initial datum S_ε satisfies* (1.11)*. Let u_ε be the solution of* (1.5)–(1.9)*, with homogeneous Dirichlet boundary data on $\partial\Omega \times (0,+\infty)$, i.e. $\Psi \equiv 0$. Then*

$$\|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le C(\varepsilon + \mathrm{e}^{-\lambda t}) \quad a.e.\ in\ (1,+\infty), \tag{1.12}$$

where C and λ are independent of ε. If, moreover, $u_\varepsilon \to u_0$ weakly in $L^2(\Omega \times (0,\overline{T}))$ for every $\overline{T} > 0$, then

$$\|u_0(\cdot,t)\|_{L^2(\Omega)} \le C\,\mathrm{e}^{-\lambda t} \quad a.e.\ in\ (1,+\infty). \tag{1.13}$$

In order to deal with the nonhomogeneous Dirichlet boudary data (1.2), we construct a function $u_0^\#(x,t)$, as the homogenization limit as $\varepsilon \to 0$ of the sequence $\{u_\varepsilon^\#(x,t)\}$ of the solutions to the following problem:

$$-\operatorname{div}(\sigma\nabla u_\varepsilon^\#) = 0\,, \qquad \text{in } (\Omega_1^\varepsilon \cup \Omega_2^\varepsilon) \times \boldsymbol{R}; \tag{1.14}$$

$$[\sigma\nabla u_\varepsilon^\# \cdot \nu] = 0\,, \qquad \text{on } \Gamma^\varepsilon \times \boldsymbol{R}; \tag{1.15}$$

$$\frac{\alpha}{\varepsilon}\frac{\partial}{\partial t}[u_\varepsilon^\#] = (\sigma\nabla u_\varepsilon^\# \cdot \nu)^{(\mathrm{out})}\,, \qquad \text{on } \Gamma^\varepsilon \times \boldsymbol{R}; \tag{1.16}$$

$$u_\varepsilon^\#(x,t) = \Psi(x)\Phi(t)\,, \qquad \text{on } \partial\Omega \times \boldsymbol{R}; \tag{1.17}$$

$$u_\varepsilon^\#(x,\cdot) \text{ is } T \text{ periodic}, \qquad \forall x \in \Omega\,; \tag{1.18}$$

$$[u_\varepsilon^\#(\cdot,t)] - S_\varepsilon(\cdot) \quad \text{has null average over each connected component of } \Gamma^\varepsilon\,, \tag{1.19}$$

which is derived from Problem (1.5)–(1.9), replacing equation (1.9) with (1.18). Equation (1.19) has been added in order to guarantee uniqueness of the solution, and is suggested by the observation that $[u_\varepsilon(\cdot,t)] - S_\varepsilon(\cdot)$ has null average over each connected component of Γ^ε, as a consequence of (1.5)–(1.7), (1.9).
To solve the above problem, we express the function Φ by means of its Fourier series, i.e.,

$$\Phi(t) = \sum_{k=-\infty}^{+\infty} c_k\,\mathrm{e}^{i\omega_k t} \tag{1.20}$$

where $\omega_k = 2k\pi/T$ is the k-th circular frequency, and represent the solutions $u_\varepsilon^\#(x,t)$ as follows:

$$u_\varepsilon^\#(x,t) = \sum_{k=-\infty}^{+\infty} v_{\varepsilon k}(x)\,\mathrm{e}^{i\omega_k t}\,, \tag{1.21}$$

where the complex-valued functions $v_{\varepsilon k}(x) \in L^2(\Omega)$ are such that $v_{\varepsilon k}|_{\Omega_i^\varepsilon} \in H^1(\Omega_i^\varepsilon)$, $i = 1, 2$, and for $k \neq 0$ satisfy the problem:

$$-\operatorname{div}(\sigma\nabla v_{\varepsilon k}) = 0\,, \qquad \text{in } \Omega_1^\varepsilon \cup \Omega_2^\varepsilon; \tag{1.22}$$

$$[\sigma\nabla v_{\varepsilon k} \cdot \nu] = 0\,, \qquad \text{on } \Gamma^\varepsilon; \tag{1.23}$$

$$\frac{i\omega_k\alpha}{\varepsilon}[v_{\varepsilon k}] = (\sigma\nabla v_{\varepsilon k} \cdot \nu)^{(\mathrm{out})}\,, \qquad \text{on } \Gamma^\varepsilon; \tag{1.24}$$

$$v_{\varepsilon k} = c_k\Psi\,, \qquad \text{on } \partial\Omega, \tag{1.25}$$

whereas for $k=0$ they satisfy the problem:

$$-\operatorname{div}(\sigma\nabla v_{\varepsilon 0}) = 0\,, \qquad \text{in } \Omega_1^\varepsilon \cup \Omega_2^\varepsilon; \tag{1.26}$$

$$[\sigma\nabla v_{\varepsilon 0}\cdot\nu] = 0\,, \qquad \text{on } \Gamma^\varepsilon; \tag{1.27}$$

$$(\sigma\nabla v_{\varepsilon 0}\cdot\nu)^{(\mathrm{out})} = 0\,, \qquad \text{on } \Gamma^\varepsilon; \tag{1.28}$$

$$v_{\varepsilon 0} = c_0\Psi\,, \quad \text{on } \partial\Omega; \tag{1.29}$$

$$[v_{\varepsilon 0}] - S_\varepsilon(\cdot) \quad \text{has null average over each connected component of } \Gamma^\varepsilon\,. \tag{1.30}$$

Note that any solution $v_{\varepsilon k}$ of Problem (1.22)–(1.25) is such that $[v_{\varepsilon k}]$ has null average over each connected component of Γ^ε.
We study the above problems in Sections 4 and 5, and prove the following homogenization result:

Theorem 1.2. *Let $\Omega_1^\varepsilon, \Omega_2^\varepsilon, \Gamma^\varepsilon$ be as before and assume that* (1.4) *and* (1.10) *hold. Then, for $k \in \boldsymbol{Z}\setminus\{0\}$, if $v_{\varepsilon k}$ is the solution of problem* (1.22)–(1.25)*, then $v_{\varepsilon k} \to v_{0k}$, weakly in $L^2(\Omega)$, and strongly in $L^1(\Omega)$, as $\varepsilon \to 0$. The limit $v_{0k} \in H^1(\Omega)$ is the unique solution of the problem*

$$-\operatorname{div}(A^{\omega_k}\,\nabla v_{0k}) = 0\,, \qquad \textit{in } \Omega; \tag{1.31}$$

$$v_{0k} = c_k\Psi\,, \qquad \textit{on } \partial\Omega; \tag{1.32}$$

where

$$A^{\omega_k} = A + \int_0^{+\infty} B(t)\,\mathrm{e}^{-i\omega_k t}\,\mathrm{d}t\,, \tag{1.33}$$

with A and $B(t)$ defined in (2.5)*. An alternative expression for A^{ω_k} is given in equation* (4.31)*.*
For $k=0$, under the further assumption (1.11)*, the solution $v_{\varepsilon 0}$ of problem* (1.26)–(1.30) *strongly converges in $L^2(\Omega)$ to a function $v_{00} \in H^1(\Omega)$ which is the unique solution of the problem*

$$-\operatorname{div}(A^0\,\nabla v_{00}) = 0\,, \qquad \textit{in } \Omega, \tag{1.34}$$

$$v_{00} = c_0\Psi\,, \qquad \textit{on } \partial\Omega, \tag{1.35}$$

where A^0 is defined in equation (5.3)*. Moreover, it turns out that:*

$$A^0 = A + \int_0^{+\infty} B(t)\,\mathrm{d}t\,, \tag{1.36}$$

which formally coincides with (1.33) *after setting $\omega_k = 0$ (see also Remark 5.1).*

Please note that in Section 4, dealing with the case $k \neq 0$, the subscript k is dropped throughout, for the sake of simplicity.
In Section 6 we deal with Problem (1.14)–(1.19), and establish:

Theorem 1.3. *Let $\Omega_1^\varepsilon, \Omega_2^\varepsilon, \Gamma^\varepsilon$ be as before and assume that* (1.3)*,* (1.4)*,* (1.10) *and* (1.11) *hold. Then,*

i) *The series at the right-hand side of equation* (1.21) *strongly converges, uniformly with respect to* ε, *in* $H^1_\#(\boldsymbol{R}; L^2(\Omega))$ *and in* $H^1_\#(\boldsymbol{R}; H^1(\Omega^\varepsilon_i))$, $i = 1, 2$, *to the unique solution* $u^\#_\varepsilon(x,t)$ *of Problem* (1.14)–(1.19).

ii) *The sequence* $\{u^\#_\varepsilon(x,t)\}$ *strongly converges in* $L^\infty_\#(\boldsymbol{R}; L^1(\Omega))$ *and weakly converges in* $L^2_\#(\boldsymbol{R}; L^2(\Omega))$ *as* $\varepsilon \to 0$ *to a function* $u^\#_0(x,t)$, T*-periodic in time, which can be represented by means of the following Fourier series:*

$$u^\#_0(x,t) = \sum_{k=-\infty}^{+\infty} v_{0k}(x)\, \mathrm{e}^{i\omega_k t}\,, \tag{1.37}$$

strongly converging in $H^1_\#(\boldsymbol{R}; H^1(\Omega))$.

iii) *The function* $u^\#_0(x,t)$ *is the unique solution* T*-periodic in time of the problem:*

$$-\operatorname{div}\Big(A\nabla u^\#_0 + \int_0^{+\infty} B(\tau)\nabla u^\#_0(x, t-\tau)\,\mathrm{d}\tau\Big) = 0\,, \qquad \textit{in } \Omega \times \boldsymbol{R}; \tag{1.38}$$

$$u^\#_0 = \Psi(x)\Phi(t)\,, \qquad \textit{on } \partial\Omega \times \boldsymbol{R}. \tag{1.39}$$

Remark 1.4. We note that, with a change of variables, equation (1.38) can be recast as follows:

$$-\operatorname{div}\Big(A\nabla u^\#_0 + \int_{-\infty}^{t} B(t-\tau)\nabla u^\#_0(x, \tau)\,\mathrm{d}\tau\Big) = 0\,, \qquad \text{in } \Omega \times \boldsymbol{R}, \tag{1.40}$$

which closely resembles equation (1.1). In fact, equation (1.1) involves a time integration over $(0,t)$ and contains an exponentially time-decaying source $\mathcal{F}$ accounting for the initial data of the original Problem (1.5)–(1.9) (see Proposition 2.2 below), whereas equation (1.40) involves a time integration over $(-\infty, t)$ and is relevant to periodic functions, i.e., to situations where any transient phenomenon is elapsed. □

Finally, in Section 7 we apply Theorem 1.1 to the function

$$w_\varepsilon = u_\varepsilon - u^\#_\varepsilon\,,$$

which satisfies a homogeneous boundary condition on $\partial\Omega \times (0, +\infty)$, and obtain our main result:

Theorem 1.5. *Let* $\Omega^\varepsilon_1, \Omega^\varepsilon_2, \Gamma^\varepsilon$ *be as before. Assume that* (1.3), (1.4), (1.10) *and* (1.11) *hold. Let* $\{u_\varepsilon\}$ *be the sequence of the solutions of Problem* (1.5)–(1.9) *and* $u^\#_0$ *be defined in Theorem 1.3. If* $u_\varepsilon \to u_0$ *weakly in* $L^2(\Omega \times (0, \overline{T}))$, *for every* $\overline{T} > 0$, *then the following estimate holds:*

$$\|u_0(\cdot,t) - u^\#_0(\cdot,t)\|_{L^2(\Omega)} \le C\,\mathrm{e}^{-\lambda t} \quad \textit{a.e. in } (1, +\infty), \tag{1.41}$$

where C *and* λ *are positive constants.*

2. Notation and preliminary results

Following [2], we introduce a periodic open subset E of $\boldsymbol{R}^N$, so that $E+z=E$ for all $z\in\boldsymbol{Z}^N$. For all $\varepsilon>0$ we define $\Omega_1^\varepsilon=\Omega\cap\varepsilon E$, $\Omega_2^\varepsilon=\Omega\setminus\overline{\varepsilon E}$, $\Gamma^\varepsilon=\Omega\cap\partial(\varepsilon E)$. We assume that Ω, E have regular boundary, say of class C^∞ for the sake of simplicity. We also employ the notation $Y=(0,1)^N$, and $E_1=E\cap Y$, $E_2=Y\setminus\overline{E}$, $\Gamma=\partial E\cap\overline{Y}$. We stipulate that $\overline{E_1}$ is a connected smooth subset of Y such that $\operatorname{dist}(\overline{E_1},\partial Y)>0$. Some generalizations may be possible, but we do not dwell on this point here. Finally, we assume that $\operatorname{dist}(\Gamma^\varepsilon,\partial\Omega)>\gamma\varepsilon$ for some constant $\gamma>0$ independent of ε, by dropping the inclusions contained in the cells $\varepsilon(Y+z)$, $z\in\boldsymbol{Z}^N$ which intersect $\partial\Omega$ (see Figure 1). For later usage, we introduce the set:

$$\boldsymbol{Z}^N_\varepsilon:=\{z\in\boldsymbol{Z}^N:\varepsilon(Y+z)\subseteq\Omega\}\,. \tag{2.1}$$

In [3] we prove existence and uniqueness of a weak solution to (1.5)–(1.9), in the class

$$u_{\varepsilon|\Omega_i^\varepsilon}\in L^2(0,\overline{T};H^1(\Omega_i^\varepsilon))\,,\qquad i=1\,,2\,,\quad \overline{T}>0\,. \tag{2.2}$$

in particular, equation (1.8) is satisfied in the sense of traces.
As it was recalled in the Introduction, if the initial datum $S_\varepsilon(x)$ satisfies (1.11), then for every $\overline{T}>0$, up to a subsequence, as $\varepsilon\to0$, u_ε weakly converges in $L^2(\Omega\times(0,\overline{T}))$ and strongly converges in $L^1_{\mathrm{loc}}(0,\overline{T};L^1(\Omega))$. Under the following more stringent assumption on S_ε:

$$S_\varepsilon(x)=\varepsilon S_1\big(x,\frac{x}{\varepsilon}\big)+\varepsilon R_\varepsilon(x)\,, \tag{2.3}$$

where $S_1:\Omega\times\partial E\to\boldsymbol{R}$, and

$$\begin{aligned}&\|S_1\|_{L^\infty(\Omega\times\partial E)}<\infty\,,\qquad \|R_\varepsilon\|_{L^\infty(\Omega)}\to0\,,\ \text{as } \varepsilon\to0;\\ &S_1(x,y)\text{ is continuous in } x,\text{ uniformly over } y\in\partial E,\\ &\text{and periodic in } y,\text{ for each } x\in\Omega,\end{aligned} \tag{2.4}$$

then all the sequence $\{u_\varepsilon\}$ converges, and the limit $u_0(x,t)$ belongs to $L^2(0,\overline{T};H^1_0(\Omega))$ and satisfies Problem (1.1)–(1.2) [2]. The two matrices A, B and the vector $\mathcal{F}$ appearing there are defined by (see [2], equations (3.31), (4.16) and (4.18)):

$$\begin{aligned}A&=\sigma_0I+\int_\Gamma[\sigma]\nu\otimes\chi^0(y)\,\mathrm{d}\sigma\,,\\ B(t)&=-\alpha\int_\Gamma[\chi^1](y,0)\otimes[\chi^1](y,t)\,\mathrm{d}\sigma=\int_\Gamma\nu\otimes[\sigma\chi^1](y,t)\,\mathrm{d}\sigma\,,\\ \mathcal{F}(x,t)&:=-\alpha\int_\Gamma S_1(x,y)[\chi^1](y,t)\,\mathrm{d}\sigma=\int_\Gamma[\sigma\mathcal{T}(S_1(x,\cdot))](y,t)\nu\,\mathrm{d}\sigma\,,\end{aligned} \tag{2.5}$$

where

$$\sigma_0=\int_Y\sigma\ \mathrm{d}x=\sigma_1|E_1|+\sigma_2|E_2|\,, \tag{2.6}$$

and two cell functions $\chi^0(y)$ and $\chi^1(y)$, and a transform $\mathcal{T}$ appear. They are defined as follows. The components χ^0_h, $h = 1, \ldots, N$, of $\chi^0 : Y \to \boldsymbol{R}^N$ satisfy

$$-\sigma\,\Delta_y\,\chi^0_h = 0\,, \qquad \text{in } E_1,\, E_2; \tag{2.7}$$

$$[\sigma(\nabla_y \chi^0_h - \boldsymbol{e}_h)\cdot\nu] = 0\,, \qquad \text{on } \Gamma; \tag{2.8}$$

$$[\chi^0_h] = 0\,, \qquad \text{on } \Gamma. \tag{2.9}$$

Moreover, χ^0_h is a periodic function with vanishing integral average over Y. The definition of $\chi^1 : Y \times (0,T) \to \boldsymbol{R}^N$ involves the transform $\mathcal{T}$, defined by

$$\mathcal{T}(s)(y,t) = v(y,t)\,, \qquad y \in Y\,,\ t > 0\,, \tag{2.10}$$

where $s : \Gamma \to \boldsymbol{R}$, and v is a periodic null-average function in Y, solving the problem

$$\begin{aligned}
-\sigma\,\Delta_y\,v &= 0\,, && \text{in } (E_1 \cup E_2)\times(0,+\infty);\\
[\sigma\nabla_y v\cdot\nu] &= 0\,, && \text{on } \Gamma\times(0,+\infty);\\
\alpha\frac{\partial}{\partial t}[v] &= (\sigma\nabla_y v\cdot\nu)^{(\mathrm{out})}\,, && \text{on } \Gamma\times(0,+\infty);\\
[v](y,0) &= s(y)\,, && \text{on } \Gamma.
\end{aligned}$$

Finally, χ^1_h is defined by

$$\alpha\chi^1_h = \mathcal{T}\big((\sigma(\nabla_y\chi^0_h - \boldsymbol{e}_h)\cdot\nu)^{(\mathrm{out})}\big)\,. \tag{2.11}$$

Lemma 2.1. *For $s \in L^2(\Gamma)$ such that $\int_\Gamma s\,\mathrm{d}\sigma = 0$, the function $\mathcal{T}(s)(y,t)$ defined in equation* (2.10) *satisfies the following estimate, for some constants $C, \lambda > 0$:*

$$\|[\mathcal{T}(s)](\cdot,t)\|_{L^2(\Gamma)} \le C\,\mathrm{e}^{-\lambda t}\,. \tag{2.12}$$

Proof. The argument is very similar to the one used in Section 3 below, so it is only sketched here. It relies on the application of abstract parabolic theory (e.g., [20], chapter 7) and leads to the explicit solution:

$$[\mathcal{T}(s)](y,t) = \sum_{i=1}^{+\infty} \mathrm{e}^{-\lambda_i t}\,w_i(y)\int_\Gamma s w_i\,\mathrm{d}\sigma\,. \tag{2.13}$$

Here $\{(\lambda_i, w_i)\}_{i\in\boldsymbol{N}}$ are the eigenvalues and eigenvectors of the spectral problem:

$$\text{find } f \in H^{1/2}(\Gamma):\ \ a(f,g) = \lambda\int_\Gamma \alpha\,fg\,\mathrm{d}\sigma\,, \quad \forall g \in H^{1/2}(\Gamma)\,, \tag{2.14}$$

and the bilinear form a is defined as follows:

$$a(f,g) = \int_Y \sigma\nabla z^{(f)}\cdot\nabla z^{(g)}\,\mathrm{d}x\,, \quad f,g \in H^{1/2}(\Gamma)\,, \tag{2.15}$$

where $z^{(s)}$ is the unique Y-periodic solution with vanishing integral average over Y of the problem:

$$-\operatorname{div}(\sigma\nabla z^{(s)}) = 0\,, \qquad \text{in } E_1 \cup E_2; \tag{2.16}$$

$$[\sigma\nabla z^{(s)}\cdot\nu] = 0\,, \qquad \text{on } \Gamma; \tag{2.17}$$

$$[z^{(s)}] = s\,, \qquad \text{on } \Gamma. \tag{2.18}$$

It is easy to show that a is symmetric and continuous, satisfying the coercivity estimate, for every $\beta > 0$:

$$a(f,f) + \beta\int_{\Gamma} \alpha f^2 \,\mathrm{d}\sigma \geq \gamma(\beta)(\|z^{(f)}\|^2_{H^1(E_1)} + \|z^{(f)}\|^2_{H^1(E_2)}) \geq \gamma(\beta)\|f\|^2_{H^{1/2}(\Gamma)}\,.$$

Hence, $\{\lambda_i\}$ is an increasing diverging sequence of nonnegative eigenvalues and $\{w_i^\varepsilon\}$ constitutes a Hilbert orthonormal basis of $L^2(\Gamma)$. In particular, it is easy to show that $\lambda_1 = 0$ and the corresponding eigenspace is generated by the constant function w_1 on Γ, so that the first term of the sum in (2.13) disappears, since s has null average over Γ. Moreover, $\lambda_2 > 0$ and the assert follows from (2.13), with $C := \|s\|_{L^2(\Gamma)}$ and $\lambda := \lambda_2$. □

Proposition 2.2. *The constant matrix A is positive definite and symmetric. The function χ^1 satisfies the estimate:*

$$\|[\chi^1_h(\cdot,t)]\|_{L^2(\Gamma)} \leq C\,\mathrm{e}^{-\lambda t}\,, \quad h = 1\dots N\,; \tag{2.19}$$

the matrix $B(t)$ belongs to $L^\infty(0,+\infty)$, is symmetric and satisfies the estimate:

$$|B_{hj}(t)| \leq C\,\mathrm{e}^{-\lambda t}\,, \quad h,j = 1\dots N\,; \tag{2.20}$$

the vector $\mathcal{F}(x,t)$, under the further assumption (2.4)*, belongs to $L^\infty(\Omega\times(0,+\infty))$ and satisfies the estimate:*

$$\|\mathcal{F}_h(\cdot,t)\|_{L^\infty(\Omega)} \leq C\,\mathrm{e}^{-\lambda t}\,, \quad h = 1\dots N\,. \tag{2.21}$$

In equations (2.19)–(2.21) *C and λ are positive constants.*

Proof. The positive definiteness of A is proved in Proposition 4.1 of [2]. Equation (2.19) follows from (2.11), Lemma 2.1 and Lemma 7.3 of [2]. Equations (2.20) and (2.21) follows from (2.19) and (2.5), using the Cauchy-Schwarz inequality and, in the proof of (2.21), also the regularity stipulated in (2.4). □

3. Proof of Theorem 1.1

We introduce the space $\widetilde{H}^{1/2}(\Gamma^\varepsilon) \subset H^{1/2}(\Gamma^\varepsilon)$ of the functions which have null average over each connected component of Γ^ε, i.e. on $\varepsilon(\Gamma + z)$, for each z belonging to the set $\boldsymbol{Z}^N_\varepsilon$ defined in (2.1).
We decompose the initial datum $S_\varepsilon(x)$ in (1.9) as $S_\varepsilon(x) = \overline{S}_\varepsilon(x) + \widetilde{S}_\varepsilon(x)$, where

$$\begin{aligned} &\overline{S}_\varepsilon(x) = \fint_{\varepsilon(\Gamma+z)} S_\varepsilon \,\mathrm{d}\sigma =: C_{\varepsilon z} \qquad \text{on each } \varepsilon(\Gamma+z),\ z\in\boldsymbol{Z}^N_\varepsilon\,; \\ &\widetilde{S}_\varepsilon(x) \in \widetilde{H}^{1/2}(\Gamma^\varepsilon)\,. \end{aligned} \tag{3.1}$$

Accordingly, the solution u_ε of problem (1.5)–(1.9) with $\Psi \equiv 0$ is decomposed as $\overline{u}_\varepsilon + \widetilde{u}_\varepsilon$.
Clearly,

$$\overline{u}_\varepsilon(x,t) = \begin{cases} 0 & \text{for } (x,t) \in \Omega_2^\varepsilon \times (0,+\infty)\,, \\ \\ -C_{\varepsilon z} & \text{for } (x,t) \in (\varepsilon(E_1+z)) \times (0,+\infty),\ z \in \boldsymbol{Z}_\varepsilon^N\,. \end{cases} \tag{3.2}$$

Using the previous equation, we compute:

$$\int_\Omega |\overline{u}_\varepsilon|^2 \,\mathrm{d}x = \sum_{z\in\boldsymbol{Z}_\varepsilon^N} \int_{\varepsilon(E_1+z)} |\overline{u}_\varepsilon|^2 \,\mathrm{d}x = \varepsilon^N |E_1| \sum_{z\in\boldsymbol{Z}_\varepsilon^N} \left| \fint_{\varepsilon(\Gamma+z)} S_\varepsilon \,\mathrm{d}\sigma \right|^2 .$$

On the other hand, by Hölder's inequality, we estimate:

$$\sum_{z\in\boldsymbol{Z}_\varepsilon^N} \left| \fint_{\varepsilon(\Gamma+z)} S_\varepsilon \,\mathrm{d}\sigma \right|^2 \le \frac{\gamma}{\varepsilon^{N-1}} \int_{\Gamma^\varepsilon} S_\varepsilon^2 \,\mathrm{d}\sigma\,.$$

Hence, as a consequence of (1.11), it follows that

$$\|\overline{u}_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le C\varepsilon\,, \tag{3.3}$$

where C is a constant independent of ε.
An estimate for $\widetilde{u}_\varepsilon$, follows from an application of abstract parabolic theory, as summarized for example in [20], chapter 7. We consider the two Hilbert spaces $H^{1/2}(\Gamma^\varepsilon) \subset L^2(\Gamma^\varepsilon)$ and the bilinear form on $H^{1/2}(\Gamma^\varepsilon)$:

$$a_\varepsilon(f,g) = \int_\Omega \sigma\nabla z_\varepsilon^{(f)} \cdot \nabla z_\varepsilon^{(g)} \,\mathrm{d}x\,, \quad f,g \in H^{1/2}(\Gamma^\varepsilon)\,, \tag{3.4}$$

where $z_\varepsilon^{(s)}$ is the unique solution of the problem:

$$-\operatorname{div}(\sigma\nabla z_\varepsilon^{(s)}) = 0\,, \qquad \text{in } \Omega_1^\varepsilon \cup \Omega_2^\varepsilon; \tag{3.5}$$

$$[\sigma\nabla z_\varepsilon^{(s)} \cdot \nu] = 0\,, \qquad \text{on } \Gamma^\varepsilon; \tag{3.6}$$

$$[z_\varepsilon^{(s)}] = s\,, \qquad \text{on } \Gamma^\varepsilon; \tag{3.7}$$

$$z_\varepsilon^{(s)} = 0\,, \qquad \text{on } \partial\Omega. \tag{3.8}$$

It is easy to show (e.g., [3, Th. 6]) that a_ε is a symmetric and continuous bilinear form. Moreover, we have the coercivity estimate, for every $\beta > 0$:

$$a_\varepsilon(f,f) + \beta \int_{\Gamma^\varepsilon} \frac{\alpha}{\varepsilon} f^2 \,\mathrm{d}\sigma \ge \gamma(\beta)(\|z_\varepsilon^{(f)}\|^2_{H^1(\Omega_1^\varepsilon)} + \|z_\varepsilon^{(f)}\|^2_{H^1(\Omega_2^\varepsilon)}) \ge \gamma(\beta,\varepsilon)\|f\|^2_{H^{1/2}(\Gamma^\varepsilon)}\,,$$

where we have used the Poincaré's inequality in [2, Lemma 7.1], and classical trace inequalities.

Then we consider the spectral problem:

$$\text{find } f \in H^{1/2}(\Gamma^\varepsilon): \; a_\varepsilon(f,g) = \lambda^\varepsilon \int_{\Gamma^\varepsilon} \frac{\alpha}{\varepsilon} fg \, \mathrm{d}\sigma \,, \quad \forall g \in H^{1/2}(\Gamma^\varepsilon) \,, \tag{3.9}$$

and the associate evolution problem, for an arbitrary $\overline{T} > 0$:

$$\begin{aligned} &\text{given } f_0 \in L^2(\Gamma^\varepsilon), \text{ find } F \in L^2(0,\overline{T}; H^{1/2}(\Gamma^\varepsilon)) \cap C([0,\overline{T}]; L^2(\Gamma^\varepsilon)): \\ &F(0) = f_0 \,, \quad \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma^\varepsilon} \frac{\alpha}{\varepsilon} F(t) g \, \mathrm{d}\sigma + a_\varepsilon(F(t), g) = 0 \,, \quad \forall g \in H^{1/2}(\Gamma^\varepsilon) \,. \end{aligned} \tag{3.10}$$

Problem (3.9) admits an increasing diverging sequence $\{\lambda_i^\varepsilon\}$ of nonnegative eigenvalues and there exists a Hilbert orthonormal basis of $L^2(\Gamma^\varepsilon)$ composed by eigenvectors w_i^ε such that [20, Th. 6.2-1]:

$$a_\varepsilon(w_i^\varepsilon, g) = \lambda_i^\varepsilon \int_{\Gamma^\varepsilon} \frac{\alpha}{\varepsilon} w_i^\varepsilon g \, \mathrm{d}\sigma \,, \quad \forall g \in H^{1/2}(\Gamma^\varepsilon) \,, \; i \in \boldsymbol{N} \,.$$

Moreover, for every $f_0 \in L^2(\Gamma^\varepsilon)$, Problem (3.10) admits a unique solution [20, Th. 7.2-1], which can be represented as follows [20, Lemma 7.2-1]:

$$F(x,t) = \sum_{i=1}^{+\infty} \mathrm{e}^{-\lambda_i^\varepsilon t} \, w_i^\varepsilon(x) \int_{\Gamma^\varepsilon} f_0 w_i^\varepsilon \, \mathrm{d}\sigma \,. \tag{3.11}$$

Since problem (3.10) is a weak formulation of Problem (1.5)–(1.9) with homogeneous Dirichlet boundary conditions, i.e. $\Psi \equiv 0$, and initial data f_0, we conclude that:

$$[\widetilde{u}_\varepsilon(x,t)] = \sum_{i=1}^{+\infty} \mathrm{e}^{-\lambda_i^\varepsilon t} \, w_i^\varepsilon(x) \int_{\Gamma^\varepsilon} \widetilde{S}_\varepsilon w_i^\varepsilon \, \mathrm{d}\sigma \,. \tag{3.12}$$

Let N_ε be the number of connected components of Γ^ε. It is easy to show that

$$\lambda_i^\varepsilon = 0, \qquad i \in \{1, \dots, N_\varepsilon\} \,,$$

and the corresponding eigenspace is generated by the characteristic functions of $\varepsilon(\Gamma + z)$, $z \in \boldsymbol{Z}_\varepsilon^N$: indeed, by (3.4)–(3.8), $a_\varepsilon(f,g) = 0$ for all $g \in H^{1/2}(\Gamma^\varepsilon)$ when f is piecewise constant on Γ^ε. However we can neglect those eigenvalues, since $\widetilde{S}_\varepsilon \in \widetilde{H}^{1/2}(\Gamma^\varepsilon)$ and hence they disappear from equation (3.12).
Our aim is to prove that the next eigenvalue, i.e. $\lambda_{N_\varepsilon+1}^\varepsilon$, here denoted by $\widetilde{\lambda}^\varepsilon$, is bounded below by a positive constant independent of ε. To this purpose, we introduce the space

$$\widetilde{H}^1(\Omega) := \{ v \in L^2(\Omega) \;:\; v_{|\Omega_i^\varepsilon} \in H^1(\Omega_i^\varepsilon), \; i = 1,2, \; [v] \in \widetilde{H}^{1/2}(\Gamma^\varepsilon) \} \,, \tag{3.13}$$

and, using Lemma 3.1 and Remark 3.2 below, we estimate, for any $v \in \widetilde{H}^1(\Omega)$:

$$\int_{\Omega} \sigma |\nabla v|^2 \,\mathrm{d}x \geq \sum_{z \in \boldsymbol{Z}^N_\varepsilon} \int_{\varepsilon(Y+z)} \sigma |\nabla v|^2 \,\mathrm{d}x$$
$$\geq \frac{\alpha \widetilde{\lambda}}{\varepsilon} \sum_{z \in \boldsymbol{Z}^N_\varepsilon} \int_{\varepsilon(\Gamma+z)} [v]^2 \,\mathrm{d}\sigma = \frac{\alpha \widetilde{\lambda}}{\varepsilon} \int_{\Gamma^\varepsilon} [v]^2 \,\mathrm{d}\sigma \,, \quad (3.14)$$

where $\widetilde{\lambda}$ is defined in (3.18). Hence (cfr. [20], eq. (6.2-20)),

$$\widetilde{\lambda}^\varepsilon := \min_{s \in \widetilde{H}^{1/2}(\Gamma^\varepsilon) \setminus \{0\}} \frac{\displaystyle\int_{\Omega} \sigma |\nabla z^{(s)}_\varepsilon|^2 \,\mathrm{d}x}{\displaystyle\frac{\alpha}{\varepsilon} \int_{\Gamma^\varepsilon} s^2 \,\mathrm{d}\sigma} \geq \widetilde{\lambda} \,, \quad (3.15)$$

for $\widetilde{\lambda} > 0$ and independent of ε.
Estimate (3.15), together with (3.12), gives

$$\|\widetilde{u}_\varepsilon(\cdot, t)\|_{L^2(\Omega)} \leq C \,\mathrm{e}^{-\widetilde{\lambda} t/2} \quad \text{a.e. in } (1, +\infty). \quad (3.16)$$

In order to prove (3.16), we reason as follows. For every $t > 0$ fixed, using the Poincaré's inequality [2, Lemma 7.1], Lemma 3.3 and equation (3.29) below, equations (3.12) and (3.15), the Parseval identity and equation (1.11), we have

$$\int_t^{t+h} \int_{\Omega} |\widetilde{u}_\varepsilon|^2 \,\mathrm{d}x \,\mathrm{d}\tau \leq C \left(\int_t^{t+h} \int_{\Omega} |\nabla u_\varepsilon|^2 \,\mathrm{d}x \,\mathrm{d}\tau + \frac{1}{\varepsilon} \int_t^{t+h} \int_{\Gamma^\varepsilon} [\widetilde{u}_\varepsilon]^2 \,\mathrm{d}\sigma \,\mathrm{d}\tau \right)$$
$$\leq \frac{C f(h,t)}{\varepsilon} \int_{\Gamma^\varepsilon} [\widetilde{u}_\varepsilon(x, t/2)]^2 \,\mathrm{d}\sigma \leq \frac{C f(h,t)}{\varepsilon} \sum_{i=1}^{+\infty} \mathrm{e}^{-2\lambda^\varepsilon_i t/2} \left(\int_{\Gamma^\varepsilon} \widetilde{S}_\varepsilon w^\varepsilon_i \,\mathrm{d}\sigma \right)^2$$
$$\leq \frac{C f(h,t)}{\varepsilon} \,\mathrm{e}^{-\widetilde{\lambda} t} \|S_\varepsilon\|^2_{L^2(\Gamma^\varepsilon)} \leq C f(h,t) \,\mathrm{e}^{-\widetilde{\lambda} t} \,, \quad (3.17)$$

where $f(h,t) = \log(1 + h/t) + h$. Dividing by h and letting $h \to 0$, equation (3.16) follows. This equation together with (3.3) gives (1.12), with $\lambda = \widetilde{\lambda}/2$.
In order to derive equation (1.13), we use the L^2-weak convergence of u_ε to u_0 in $\Omega \times (t, t+h)$, for every fixed $t > 1$ and $h > 0$, and estimate (1.12) as follows:

$$\int_t^{t+h} \int_{\Omega} u_0^2 \,\mathrm{d}x \,\mathrm{d}\tau \leq \liminf_{\varepsilon \to 0} \int_t^{t+h} \int_{\Omega} u_\varepsilon^2 \,\mathrm{d}x \,\mathrm{d}\tau \leq h (C \,\mathrm{e}^{-\lambda t})^2 \,.$$

Dividing by h and letting $h \to 0$, equation (1.13) follows.

Lemma 3.1. *Set* $\widetilde{H}^1(Y) := \{ v \in L^2(Y) \; : \; v_{|E_i} \in H^1(E_i), \; i = 1, 2, \; [v] \in \widetilde{H}^{1/2}(\Gamma) \}$, *where* $\widetilde{H}^{1/2}(\Gamma)$ *is comprised by the functions of* $H^{1/2}(\Gamma)$ *with null integral average.*

Then, it results that

$$\widetilde{\lambda} := \min_{v\in\widetilde{H}^1(Y),\ [v]\not\equiv 0} \frac{\int\limits_Y \sigma|\nabla v|^2\,\mathrm{d}y}{\alpha\int\limits_\Gamma [v]^2\,\mathrm{d}\sigma} > 0\,. \tag{3.18}$$

Proof. We introduce the bilinear form:

$$a(f,g) = \int\limits_Y \sigma\nabla z^{(f)}\cdot\nabla z^{(g)}\,\mathrm{d}x\,,\quad f,g\in H^{1/2}(\Gamma)\,, \tag{3.19}$$

where $z^{(s)}$ is the unique solution with vanishing integral average over Y of the problem:

$$-\operatorname{div}(\sigma\nabla z^{(s)}) = 0\,,\qquad \text{in } E_1\cup E_2; \tag{3.20}$$

$$[\sigma\nabla z^{(s)}\cdot\nu] = 0\,,\qquad \text{on } \Gamma; \tag{3.21}$$

$$[z^{(s)}] = s\,,\qquad \text{on } \Gamma; \tag{3.22}$$

$$\sigma_2\nabla z^{(s)}\cdot n = 0\,;\qquad \text{on } \partial Y. \tag{3.23}$$

where n is the outward unit normal to ∂Y.
Reasoning as before, it can be shown that the spectral problem:

$$\text{find } f\in H^{1/2}(\Gamma):\ \ a(f,g) = \lambda\int\limits_\Gamma \alpha\, fg\,\mathrm{d}\sigma\,,\quad \forall g\in H^{1/2}(\Gamma)\,, \tag{3.24}$$

admits an increasing diverging sequence of nonnegative eigenvalues $\{\lambda_i\}$. It is easy to show that the first one is zero and the corresponding eigenspace is composed by the constant functions on Γ. The space orthogonal to the first eigenspace is $\widetilde{H}^{1/2}(\Gamma)$ and hence the second eigenvalue, denoted by $\overline{\lambda}$, satisfies (cfr. [20], eq. (6.2-20)):

$$\overline{\lambda} = \min_{s\in\widetilde{H}^{1/2}(\Gamma)\setminus\{0\}} \frac{\int\limits_Y \sigma|\nabla z^{(s)}|^2\,\mathrm{d}y}{\alpha\int\limits_\Gamma s^2\,\mathrm{d}\sigma}\,, \tag{3.25}$$

thus we have that $\overline{\lambda} > 0$, since otherwise the corresponding eigenvector would be constant, and hence zero.
Clearly, the infimum at the right-hand side of (3.18) is less than or equal to $\overline{\lambda}$, since for $s\in\widetilde{H}^{1/2}(\Gamma)\setminus\{0\}$, it results that $z^{(s)}\in\widetilde{H}^1(Y)$, and $[z^{(s)}] = s$.
On the other hand, for every $v\in\widetilde{H}^1(Y)\setminus\{0\}$, the function $z^{([v])}\in\widetilde{H}^1(Y)$ is such that:

$$\int\limits_Y \sigma|\nabla v|^2\,\mathrm{d}y = \int\limits_Y \sigma|\nabla z^{([v])} + \nabla(v - z^{([v])})|^2\,\mathrm{d}y \ge \int\limits_Y \sigma|\nabla z^{([v])}|^2\,\mathrm{d}y\,, \tag{3.26}$$

since by (3.20)–(3.23) we have:

$$\int_Y \sigma \nabla z^{([v])} \cdot \nabla (v - z^{([v])}) \, \mathrm{d}y = - \int_Y (v - z^{([v])}) \operatorname{div}(\sigma \nabla z^{([v])}) \, \mathrm{d}y$$
$$- \int_\Gamma [v - z^{([v])}] (\sigma \nabla z^{([v])} \cdot \nu)^{(\mathrm{out})} \, \mathrm{d}\sigma + \int_{\partial Y} (v - z^{([v])}) (\sigma_2 \nabla z^{([v])} \cdot n) \, \mathrm{d}\sigma = 0.$$

As a consequence of (3.26), we conclude that the infimum at the right-hand side of (3.18) is attained and is equal to $\overline{\lambda}$.

□

Remark 3.2. The change of variables $y = x/\varepsilon$ applied to equation (3.18) yields:

$$\min_{v \in \widetilde{H}^1(\varepsilon Y),\ [v] \not\equiv 0} \frac{\displaystyle\int_{\varepsilon Y} \sigma |\nabla v|^2 \, \mathrm{d}x}{\displaystyle\frac{\alpha}{\varepsilon} \int_{\varepsilon \Gamma} [v]^2 \, \mathrm{d}\sigma} = \widetilde{\lambda} > 0 \, , \tag{3.27}$$

where $\widetilde{H}^1(\varepsilon Y) := \{ v \in L^2(\varepsilon Y) \ : \ v_{|\varepsilon E_i} \in H^1(\varepsilon E_i), \ i = 1, 2, \ [v] \in \widetilde{H}^{1/2}(\varepsilon \Gamma) \}$, and $\widetilde{H}^{1/2}(\varepsilon \Gamma)$ is comprised by the functions of $H^{1/2}(\varepsilon \Gamma)$ with null integral average. In particular, we emphasize that $\widetilde{\lambda}$ is a positive constant independent of ε. □

Lemma 3.3. *Under the assumptions of Theorem 1.1, there exists a constant $\gamma > 0$ independent of ε, such that the following estimate holds for $t > 0$:*

$$\sup_{\tau \geq t} \int_\Omega \sigma |\nabla u_\varepsilon(x, \tau)|^2 \, \mathrm{d}x \leq \frac{\gamma}{\varepsilon t} \int_{\Gamma^\varepsilon} [u_\varepsilon(x, t/2)]^2 \, \mathrm{d}\sigma \, . \tag{3.28}$$

Proof. For $0 \leq t_1 \leq t_2$, we multiply equation (1.5) by u_ε, integrate by parts over $(\Omega_1^\varepsilon \cup \Omega_2^\varepsilon) \times (t_1, t_2)$, use equations (1.6), (1.7) and the homogeneous Dirichlet boundary data on $\partial \Omega$, and obtain:

$$\int_{t_1}^{t_2} \int_\Omega \sigma |\nabla u_\varepsilon|^2 \, \mathrm{d}x \, \mathrm{d}\tau + \frac{\alpha}{2\varepsilon} \int_{\Gamma^\varepsilon} [u_\varepsilon(x, t_2)]^2 \, \mathrm{d}\sigma = \frac{\alpha}{2\varepsilon} \int_{\Gamma^\varepsilon} [u_\varepsilon(x, t_1)]^2 \, \mathrm{d}\sigma \, . \tag{3.29}$$

Then we fix $t > 0$ and choose a cutoff function $\zeta(\tau) \in C^1(0, +\infty)$ such that

$$\zeta(\tau) = \begin{cases} 0 \, , & \tau \leq t/2 \, ; \\ 1 \, , & \tau \geq t \, ; \end{cases} \qquad 0 \leq \zeta' \leq \frac{\widetilde{\gamma}}{t} \, . \tag{3.30}$$

We multiply equation (1.5) by $u_{\varepsilon t} \zeta$, and integrate by parts over $(\Omega_1^\varepsilon \cup \Omega_2^\varepsilon) \times (t/2, t)$. These computations can be made rigorous using a Steklov averaging procedure. Using equations (1.6), (1.7), (3.30) and the homogeneous Dirichlet boundary data on $\partial \Omega$, we obtain:

$$\int_\Omega \frac{\sigma}{2} |\nabla u_\varepsilon(x, t)|^2 \, \mathrm{d}x + \frac{\alpha}{\varepsilon} \int_{t/2}^{t} \int_{\Gamma^\varepsilon} \zeta [u_{\varepsilon t}]^2 \, \mathrm{d}\sigma \, \mathrm{d}\tau = \int_{t/2}^{t} \int_\Omega \frac{\sigma}{2} |\nabla u_\varepsilon|^2 \zeta' \, \mathrm{d}x \, \mathrm{d}\tau \, ,$$

Hence,

$$\sup_{\tau \geq t} \int_{\Omega} \sigma |\nabla u_\varepsilon(x,\tau)|^2 \,\mathrm{d}x \leq \int_{t/2}^{+\infty} \int_{\Omega} \sigma |\nabla u_\varepsilon|^2 \zeta' \,\mathrm{d}x \,\mathrm{d}\tau \,, \tag{3.31}$$

and the assert follows from equations (3.29), with $t_1 = t/2$ and $t_2 = t$, and (3.30). □

4. Homogenization limit of time-harmonic solutions: case $k \neq 0$

In this Section we prove Theorem 1.2 in the case $k \neq 0$. For the sake of simplicity, we omit here the subscript k and set

$$\psi(x) := c_k \Psi(x) \,. \tag{4.1}$$

4.1. **Energy estimate.** We establish the following energy estimate:

$$\int_{\Omega} \sigma |\nabla v_\varepsilon|^2 \,\mathrm{d}x + \frac{\omega}{\varepsilon} \int_{\Gamma^\varepsilon} |[v_\varepsilon]|^2 \,\mathrm{d}\sigma \leq \gamma \int_{\Omega} \sigma |\nabla \psi|^2 \,\mathrm{d}x \,, \tag{4.2}$$

where γ is independent of ε and ω. This estimate, together with Poincaré's inequality [2, Lemma 7.1] imply the following L^2 estimate:

$$\int_{\Omega} v_\varepsilon^2 \,\mathrm{d}x \leq \gamma (1 + \omega^{-1}) \int_{\Omega} \sigma |\nabla \psi|^2 \,\mathrm{d}x \,. \tag{4.3}$$

In order to carry out the proof, we set:

$$z_\varepsilon = v_\varepsilon - \psi \,. \tag{4.4}$$

The complex-valued function $z_\varepsilon(x,t)$ satisfies the equations:

$$-\operatorname{div}(\sigma \nabla z_\varepsilon) = 0 \,, \qquad \text{in } \Omega_1^\varepsilon \cup \Omega_2^\varepsilon; \tag{4.5}$$

$$[\sigma \nabla z_\varepsilon \cdot \nu] = -[\sigma] \nabla \psi \cdot \nu \,, \qquad \text{on } \Gamma^\varepsilon; \tag{4.6}$$

$$\frac{i\omega\alpha}{\varepsilon} [z_\varepsilon] = (\sigma \nabla z_\varepsilon \cdot \nu)^{(\mathrm{out})} + \sigma_2 \nabla \psi \cdot \nu \,, \qquad \text{on } \Gamma^\varepsilon; \tag{4.7}$$

$$z_\varepsilon = 0 \,, \qquad \text{on } \partial\Omega. \tag{4.8}$$

We multiply (4.5) by $\overline{z}_\varepsilon$, integrate over $\Omega_1^\varepsilon \cup \Omega_2^\varepsilon$, use the Gauss-Green identity and equation (4.8), and arrive to:

$$\int_{\Omega} \sigma |\nabla z_\varepsilon|^2 \,\mathrm{d}x + \int_{\Gamma^\varepsilon} [\overline{z}_\varepsilon \, \sigma \nabla z_\varepsilon \cdot \nu] \,\mathrm{d}\sigma = 0 \,. \tag{4.9}$$

Using equations (4.6)–(4.7), and then the Gauss-Green identity and equations (1.4) and (4.8), we obtain

$$\int_{\Omega} \sigma |\nabla z_\varepsilon|^2 \,\mathrm{d}x + \frac{i\omega\alpha}{\varepsilon} \int_{\Gamma^\varepsilon} |[z_\varepsilon]|^2 \,\mathrm{d}\sigma = \int_{\Gamma^\varepsilon} [\overline{z}_\varepsilon \, \sigma \nabla \psi \cdot \nu] \,\mathrm{d}\sigma = -\int_{\Omega} \sigma \nabla \overline{z}_\varepsilon \cdot \nabla \psi \,\mathrm{d}x \,. \tag{4.10}$$

Taking the real and imaginary parts of equation (4.10) and adding them, we get

$$\int_{\Omega} \sigma |\nabla z_\varepsilon|^2 \,\mathrm{d}x + \frac{\omega\alpha}{\varepsilon} \int_{\Gamma^\varepsilon} |[z_\varepsilon]|^2 \,\mathrm{d}\sigma = -\int_{\Omega} \sigma (\Re \nabla z_\varepsilon - \Im \nabla z_\varepsilon) \cdot \nabla \psi \,\mathrm{d}x \,. \tag{4.11}$$

Then, we estimate, using Young's inequality:

$$\int_{\Omega} \sigma|\nabla z_\varepsilon|^2 \,\mathrm{d}x + \frac{\omega\alpha}{\varepsilon} \int_{\Gamma^\varepsilon} |[z_\varepsilon]|^2 \,\mathrm{d}\sigma \le \frac{1}{2} \int_{\Omega} \sigma|\nabla z_\varepsilon|^2 \,\mathrm{d}x + 2 \int_{\Omega} \sigma|\nabla\psi|^2 \,\mathrm{d}x \,, \qquad (4.12)$$

whence equation (4.2) follows.

4.2. **Existence.** We prove existence of solution of Problem (4.5)–(4.8), for the unknown z_ε defined in equation (4.4), in the class

$$\mathcal{H} = \{z_\varepsilon \in L^2(\Omega)\,; \quad z_{\varepsilon|\Omega_i^\varepsilon} \in H^1(\Omega_i^\varepsilon)\,,\ i = 1\,,2\,; \quad z_{\varepsilon|\partial\Omega} = 0\}\,, \qquad (4.13)$$

which is identified with the Hilbert space $H^1(\Omega_1^\varepsilon) \times H_0^1(\Omega_2^\varepsilon)$. The weak formulation of Problem (4.5)–(4.8) is

$$a(z_\varepsilon, \phi) := \int_{\Omega} \sigma \nabla z_\varepsilon \cdot \nabla\overline{\phi} \,\mathrm{d}x + \frac{i\alpha\omega}{\varepsilon} \int_{\Gamma^\varepsilon} [z_\varepsilon][\overline{\phi}] \,\mathrm{d}\sigma = \int_{\Gamma^\varepsilon} [\overline{\phi}\, \sigma\nabla\psi\cdot\nu] \,\mathrm{d}\sigma\,, \quad \forall \phi \in \mathcal{H}\,. \qquad (4.14)$$

Existence of $z_\varepsilon \in \mathcal{H}$ satisfying (4.14) follows from the Lax-Milgram Theorem [22, Chp. 6, Th. 1.4]: indeed, the continuity of the bilinear form $a(\cdot,\cdot)$ and of the linear functional at the right-hand side of (4.14) follows from standard trace inequalities, and the coercivity estimate $|a(\phi,\phi)| \ge m\|\phi\|_{\mathcal{H}}$, for some $m > 0$, follows from the Poincaré's inequality in [2, Lemma 7.1].

4.3. **Formal homogenization asymptotics.** In this Section we aim at identifying the homogenized equation of Problem (1.22)–(1.25), via the two-scale method. The argument is standard, so we only sketch it.
Introduce the microscopic variables $y \in Y$, $y = x/\varepsilon$, assuming

$$v_\varepsilon = v_\varepsilon(x,y) = v_0(x,y) + \varepsilon v_1(x,y) + \varepsilon^2 v_2(x,y) + \dots\,. \qquad (4.15)$$

Note that v_0, v_1, v_2 are periodic in y, and v_1, v_2 are assumed to have zero integral average over Y.
Applying (4.15) to (1.22)–(1.24) we find, at the leading-order term:

$$-\sigma\,\Delta_y v_0 = 0\,, \qquad \text{in } E_1,\ E_2; \qquad (4.16)$$

$$[\sigma\nabla_y v_0\cdot\nu] = 0\,, \qquad \text{on } \Gamma; \qquad (4.17)$$

$$i\omega\alpha[v_0] = (\sigma\nabla_y v_0\cdot\nu)^{(\mathrm{out})}\,, \qquad \text{on } \Gamma. \qquad (4.18)$$

Multiplying (4.16) by $\overline{v}_0$, integrating by parts over $E_1 \cup E_2$ and taking into account (4.17)–(4.18), it easily follows that:

$$v_0 = v_0(x)\,. \qquad (4.19)$$

Proceeding as above, but taking into consideration the next-order terms in the ε–expansion, we obtain

$$-\sigma\,\Delta_y v_1 = 0\,, \qquad \text{in } E_1,\ E_2; \qquad (4.20)$$

$$[\sigma\nabla_y v_1\cdot\nu] = -[\sigma\nabla_x v_0\cdot\nu]\,, \qquad \text{on } \Gamma; \qquad (4.21)$$

$$i\omega\alpha[v_1] = (\sigma\nabla_y v_1\cdot\nu)^{(\mathrm{out})} + \sigma_2\nabla_x v_0\cdot\nu\,, \qquad \text{on } \Gamma. \qquad (4.22)$$

In (4.20) and in (4.22) we have made use of (4.19), and of its consequence $[v_0] = 0$.

We represent v_1 in the form

$$v_1(x,y) = -\chi^\omega(y) \cdot \nabla_x v_0(x)\,, \tag{4.23}$$

where the cell function $\chi^\omega : Y \to \boldsymbol{C}^N$, is such that its components χ^ω_h, $h = 1, \dots, N$, satisfy

$$-\sigma\,\Delta_y\,\chi^\omega_h = 0\,, \qquad \text{in } E_1,\ E_2; \tag{4.24}$$

$$[\sigma(\nabla_y\chi^\omega_h - \boldsymbol{e}_h)\cdot\nu] = 0\,, \qquad \text{on } \Gamma; \tag{4.25}$$

$$i\omega\alpha[\chi^\omega_h] = (\sigma(\nabla_y\chi^\omega_h - \boldsymbol{e}_h)\cdot\nu)^{(\mathrm{out})}\,, \qquad \text{on } \Gamma. \tag{4.26}$$

and are periodic functions with vanishing integral average over Y. Existence and uniqueness of the solution of Problem (4.24)–(4.26) is proved in Lemma 4.1 below.
Finally, the next-order terms in the ε–expansion give:

$$-\sigma\,\Delta_y\,v_2 = \sigma\,\Delta_x\,v_0 + 2\sigma\frac{\partial^2 v_1}{\partial x_j \partial y_j}\,, \qquad \text{in } E_1, E_2 \tag{4.27}$$

$$[\sigma\nabla_y v_2\cdot\nu] = -[\sigma\nabla_x v_1\cdot\nu]\,, \qquad \text{on } \Gamma; \tag{4.28}$$

$$i\omega\alpha[v_2] = (\sigma\nabla_y v_2\cdot\nu)^{(\mathrm{out})} + (\sigma\nabla_x v_1\cdot\nu)^{(\mathrm{out})}\,, \qquad \text{on } \Gamma. \tag{4.29}$$

Integrating by parts equation (4.27) both in E_1 and in E_2, using equation (4.28) and adding the two contributions, we get

$$-\sigma_0\,\Delta_x\,v_0 = -2\int_\Gamma [\sigma\nabla_x v_1\cdot\nu]\,\mathrm{d}\sigma + \int_\Gamma [\sigma\nabla_x v_1\cdot\nu]\,\mathrm{d}\sigma = -\int_\Gamma [\sigma\nabla_x v_1\cdot\nu]\,\mathrm{d}\sigma\,,$$

where σ_0 is defined in equation (2.6).
Then, we use the representation (4.23) and infer from the equality above the PDE for v_0 as

$$-\operatorname{div}(A^\omega\,\nabla v_0) = 0\,, \qquad \text{in } \Omega\,, \tag{4.30}$$

where the matrix A^ω is given by (here the superscript t denotes transposition)

$$A^\omega = \sigma_0 I + \int_\Gamma \nu\otimes[\sigma\chi^\omega]\,\mathrm{d}\sigma = \sigma_0 I - \int_Y \sigma\nabla^t\chi^\omega\ \mathrm{d}y\,. \tag{4.31}$$

Lemma 4.1. *Under the assumptions on E_1, E_2, Γ reported in Section 2, Problem* (4.24)–(4.26) *admits a unique solution in the class*

$$\widehat{H}^1(Y) := \{f \in L^2(\boldsymbol{R}^N)\colon f|_{E_i} \in H^1(E_i),\ i = 1,2,\ f \textit{ is } Y\textit{-periodic with vanishing integral average over } Y\}\,. \tag{4.32}$$

Proof. First, we prove the uniqueness result. Assuming, by contradiction, that two different solutions $\chi^\omega_{h,1}$ and $\chi^\omega_{h,2}$ to Problem (4.24)–(4.26) exist, the function $z^\omega_h := \chi^\omega_{h,2} - \chi^\omega_{h,1}$ satisfies:

$$-\sigma\,\Delta_y\,z^\omega_h = 0\,, \qquad \text{in } E_1,\ E_2; \tag{4.33}$$

$$[\sigma\nabla_y z^\omega_h\cdot\nu] = 0\,, \qquad \text{on } \Gamma; \tag{4.34}$$

$$i\omega\alpha[z^\omega_h] = (\sigma\nabla_y z^\omega_h\cdot\nu)^{(\mathrm{out})}\,, \qquad \text{on } \Gamma. \tag{4.35}$$

Multiplying (4.33) by $\overline{z}_h^\omega$, integrating by parts and using (4.34) and (4.35), we obtain:

$$\int_Y \sigma|\nabla z_h^\omega|^2 \,\mathrm{d}y + i\omega\alpha \int_\Gamma |[z_h^\omega]|^2 \,\mathrm{d}\sigma = 0\,. \tag{4.36}$$

This estimate, recalling that $z_h^\omega \in \widehat{H}^1(Y)$, implies that $z_h^\omega \equiv 0$.
As far as existence is concerned, we refer to equation (4.56) below, where a solution to Problem (4.24)–(4.26) is explicitly exhibited. Alternatively, one could appeal to the Lax-Milgram theorem as in Subsection 4.2. □

4.4. **Homogenization limit.** Introduce for $i = 1, \dots, N$, the functions

$$q_i^\varepsilon(x,t) = x_i - \varepsilon\chi_i^\omega\big(\frac{x}{\varepsilon}\big)\,, \tag{4.37}$$

so that explicit calculations reveal

$$-\sigma\,\Delta\, q_i^\varepsilon = 0\,, \qquad \text{in } \Omega_1^\varepsilon,\ \Omega_2^\varepsilon; \tag{4.38}$$

$$[\sigma\nabla q_i^\varepsilon\cdot\nu] = 0\,, \qquad \text{on } \Gamma^\varepsilon; \tag{4.39}$$

$$\frac{i\omega\alpha}{\varepsilon}[q_i^\varepsilon] = (\sigma\nabla q_i^\varepsilon\cdot\nu)^{(\mathrm{out})}\,, \qquad \text{on } \Gamma^\varepsilon. \tag{4.40}$$

Let $\varphi \in C_o^\infty(\Omega)$, and select $q_i^\varepsilon\varphi$ as a testing function in the weak formulation of (1.22)–(1.25) and use equations (1.23)–(1.24). We obtain

$$\int_\Omega \sigma\nabla v_\varepsilon\cdot\nabla q_i^\varepsilon\,\varphi\,\mathrm{d}x + \int_\Omega \sigma\nabla v_\varepsilon\cdot\nabla\varphi\, q_i^\varepsilon\,\mathrm{d}x + \frac{i\omega\alpha}{\varepsilon}\int_{\Gamma^\varepsilon}[v_\varepsilon][q_i^\varepsilon]\varphi\,\mathrm{d}\sigma = 0\,. \tag{4.41}$$

Next select $v_\varepsilon\varphi$ as a testing function in the weak formulation of (4.38)–(4.40). We get

$$\int_\Omega \sigma\nabla q_i^\varepsilon\cdot\nabla v_\varepsilon\,\varphi\,\mathrm{d}x + \int_\Omega \sigma\nabla q_i^\varepsilon\cdot\nabla\varphi\, v_\varepsilon\,\mathrm{d}x + \frac{i\omega\alpha}{\varepsilon}\int_{\Gamma^\varepsilon}[q_i^\varepsilon][v_\varepsilon]\varphi\,\mathrm{d}\sigma = 0\,. \tag{4.42}$$

Subtract (4.42) from (4.41) and find,

$$\int_\Omega \sigma\nabla v_\varepsilon\cdot\nabla\varphi\, q_i^\varepsilon\,\mathrm{d}x = \int_\Omega \sigma\nabla q_i^\varepsilon\cdot\nabla\varphi\, v_\varepsilon\,\mathrm{d}x\,. \tag{4.43}$$

The energy inequality (4.2) and the L^2 estimate (4.3) imply that, extracting subsequences if needed, we may assume

$$-\sigma\nabla v_\varepsilon \to \xi^\omega\,, \quad v_\varepsilon \to v_0\,, \qquad \text{weakly in } L^2(\Omega), \tag{4.44}$$

$$v_\varepsilon \to v_0\,, \qquad \text{strongly in } L^1(\Omega), \tag{4.45}$$

for some $\xi^\omega \in L^2(\Omega)^N$, $v_0 \in L^2(\Omega)$. On the other hand, recalling (4.37) and (4.31), it is easy to show, that:

$$q_i^\varepsilon \to x_i\,, \qquad \text{strongly in } L^2(\Omega), \tag{4.46}$$

$$\sigma\nabla q_i^\varepsilon \to A^\omega \boldsymbol{e}_i\,, \qquad \text{weakly in } L^2(\Omega). \tag{4.47}$$

Thus, using [2, Lemma 7.5], it follows

$$-\int_{\Omega} \xi^{\omega} \cdot \nabla\varphi\, x_i \,\mathrm{d}x = \int_{\Omega} A^{\omega} \boldsymbol{e}_i \cdot \nabla\varphi\, v_0 \,\mathrm{d}x \,. \tag{4.48}$$

As usual, next we take φx_i as a testing function in the weak formulation of (1.22)–(1.25). On letting $\varepsilon \to 0$, we get

$$-\int_{\Omega} \xi^{\omega} \cdot \nabla\varphi\, x_i \,\mathrm{d}x - \int_{\Omega} \xi^{\omega} \cdot \boldsymbol{e}_i\, \varphi \,\mathrm{d}x = 0 \,. \tag{4.49}$$

We substitute (4.49) in (4.48), and, recalling that A^{ω} is symmetric (see Subsection 4.6), we obtain

$$\int_{\Omega} v_0 A^{\omega} \nabla\varphi \,\mathrm{d}x = \int_{\Omega} \xi^{\omega}\, \varphi \,\mathrm{d}x \,.$$

By the arbitrariness of $\varphi \in C^{\infty}_{o}(\Omega)$, recalling also equation (4.49) above, it follows that

$$\xi^{\omega} = -A^{\omega}\nabla v_0 \quad \text{and} \quad \operatorname{div}\xi^{\omega} = 0 \text{ in the sense of distributions,}$$

and hence equation (1.31) is in force.
For future usage, we note that equations (4.3) and (4.44) imply:

$$\int_{\Omega} v_0^2 \,\mathrm{d}x \le \gamma(1+\omega^{-1}) \int_{\Omega} \sigma |\nabla\psi|^2 \,\mathrm{d}x \,. \tag{4.50}$$

4.5. **Dirichlet boundary condition for v_0.** In this section we prove equation (1.32) using an argument similar to [2], § 5.1. We define:

$$V_{\varepsilon}(x) = \begin{cases} v_{\varepsilon}(x) & \text{in } \Omega, \\ \psi & \text{in } \boldsymbol{R}^N \setminus \overline{\Omega}. \end{cases}$$

Since the jump of V_{ε} across $\partial\Omega$ is zero, we infer that for each bounded open set $G \subset \boldsymbol{R}^N$, the variation $|DV_{\varepsilon}|(G)$ is given by

$$|DV_{\varepsilon}|(G) = \int_{G} |\nabla V_{\varepsilon}| \,\mathrm{d}x + \int_{\Gamma^{\varepsilon}\cap G} |[V_{\varepsilon}]| \,\mathrm{d}\sigma \le \gamma\big(|G|^{1/2} + (\varepsilon |\Gamma^{\varepsilon} \cap G|_{N-1})^{1/2}\big) \,, \tag{4.51}$$

where we have made use of Hölder's inequality and of equations (1.4), (4.1), (4.2). As a first consequence of this estimate, we may invoke classical compactness and semicontinuity results to show that (extracting subsequences if needed)

$$V_{\varepsilon} \to V_0 \,, \qquad \text{in } L^1(\boldsymbol{R}^N), \qquad |DV_0|(G) \le \liminf_{\varepsilon\to 0} |DV_{\varepsilon}|(G) \,, \tag{4.52}$$

for every set $G \subset \boldsymbol{R}^N$ as above. On the other hand, according to [4, Th. 3.77],

$$|DV_0|(\partial\Omega) = \int_{\partial\Omega} |V_0^+ - V_0^-| \,\mathrm{d}\sigma = \int_{\partial\Omega} |V_0^+ - \psi| \,\mathrm{d}\sigma \,, \tag{4.53}$$

where the symbol V_0^+ (respectively, V_0^-) denotes the trace on $\partial\Omega$ of $V_{0|\Omega}$ (respectively, of $V_{0|\boldsymbol{R}^N\setminus\overline{\Omega}} \equiv \psi$).

Define for $0 < h < 1$ the open set

$$G_h = \{x \in \boldsymbol{R}^N \mid \operatorname{dist}(x, \partial\Omega) < h\} .$$

Combining (4.51)–(4.53), we obtain, as $\partial\Omega \subset G_h$ for all h,

$$\int_{\partial\Omega} |V_0^+ - \psi| \,\mathrm{d}\sigma \leq |DV_0(G_h)| \leq \gamma \liminf_{\varepsilon\to 0} \left(|G_h|^{1/2} + (\varepsilon|\Gamma^\varepsilon \cap G_h|_{N-1})^{1/2}\right) \leq \gamma h^{1/2} .$$

Indeed, it is readily seen that $|G_h| \leq \gamma h$, and that $|\Gamma^\varepsilon \cap G_h|_{N-1} \leq \gamma h/\varepsilon$ for all sufficiently small h. Therefore, letting $h \to 0$ above we obtain that $V_0^+ = \psi$ a.e. on $\partial\Omega$. As a consequence, $v_0 = \psi$ a.e. on $\partial\Omega$.

Remark 4.2. Due to Proposition 4.3 below and the Lax-Milgram Theorem, the problem

$$-\operatorname{div}(A^\omega \nabla v) = 0 , \qquad \text{in } \Omega; \tag{4.54}$$

$$v = \psi , \qquad \text{on } \partial\Omega, \tag{4.55}$$

admits a unique solution $v \in H^1(\Omega)$. As a consequence, the function $v_0 = \lim_{\varepsilon\to 0} v_\varepsilon$, which was proved to satisfy the problem above, coincides with v. Hence, $v_0 \in H^1(\Omega)$. In passing, we note that the uniqueness of v_0 also implies that actually the whole sequence $\{v_\varepsilon\}$ converges to v_0. □

4.6. **Structure of the limit equation.** First, we show that equations (1.33) and (4.31) yield the same matrix A^ω. To this end, we set

$$\theta^\omega = \chi^0 + \int_0^{+\infty} \chi^1(\cdot, t)\ \mathrm{e}^{-i\omega t}\,\mathrm{d}t . \tag{4.56}$$

Recalling (2.7)–(2.8) and (2.11), it follows that θ^ω satisfies equations (4.24)–(4.25). Indeed, it satisfies also equation (4.26):

$$\begin{aligned}(\sigma(\nabla_y\theta_h^\omega - \boldsymbol{e}_h)\cdot\nu)^{(\mathrm{out})} &= (\sigma(\nabla_y\chi_h^0 - \boldsymbol{e}_h)\cdot\nu)^{(\mathrm{out})} + \int_0^{+\infty} (\sigma(\nabla_y\chi^1(\cdot,t))\cdot\nu)^{(\mathrm{out})}\ \mathrm{e}^{-i\omega t}\,\mathrm{d}t \\ &= \alpha[\chi_h^1(\cdot,0)] + \int_0^{+\infty} \alpha\frac{\partial}{\partial t}[\chi_h^1(\cdot,t)]\ \mathrm{e}^{-i\omega t}\,\mathrm{d}t = i\omega\alpha[\theta_h^\omega],\end{aligned} \tag{4.57}$$

where we used (2.11), (2.9), and Proposition 2.2. Thus $\theta_h^\omega = \chi_h^\omega$, since both of them satisfy Problem (4.24)–(4.26), which admits a unique solution in the class $\widehat{H}^1(Y)$ (see Lemma 4.1 above). In turn, recalling (2.5), this implies the equivalence between equations (1.33) and (4.31).
Then we prove the following result, which, in particular, implies the well-posedness of Problem (4.54)–(4.55), used in Remark 4.2.

Proposition 4.3. A^ω *is symmetric; its real part and its imaginary part are positive definite;* $|A_{hj}^\omega|$, $h, j = 1, \dots, N$, *is uniformly bounded with respect to* ω. *Moreover,*

$\Re(A^\omega\zeta,\zeta)\geq\gamma|\zeta|^2$, *for all* $\zeta\in\boldsymbol{C}^N$, *where* $(\cdot,\cdot)$ *is the scalar product in* $\boldsymbol{C}^N$ *and* γ *is a positive constant.*

Proof. The symmetry of A^ω follows from equation (1.33) and the fact that the matrices A and $B(t)$ therein are symmetric (see Proposition 2.2). The uniform upper bound on $\|A^\omega\|$ follows from equation (1.33) and Proposition 2.2.
In order to prove the strict positivity of $\Re(A^\omega)$ and $\Im(A^\omega)$, we compute:

$$\int_Y \sigma(\nabla\chi_j^\omega-\boldsymbol{e}_j)\cdot(\nabla\overline{\chi}_h^\omega-\boldsymbol{e}_h)\,\mathrm{d}y=-\int_\Gamma(\sigma(\nabla\chi_j^\omega-\boldsymbol{e}_j)\cdot\nu)^{(\mathrm{out})}[\overline{\chi}_h^\omega]\,\mathrm{d}\sigma$$
$$-\int_Y\sigma(\nabla\chi_j^\omega-\boldsymbol{e}_j)\cdot\boldsymbol{e}_h\,\mathrm{d}y=-i\omega\alpha\int_\Gamma[\chi_j^\omega][\overline{\chi}_h^\omega]\,\mathrm{d}\sigma+A_{hj}^\omega\,. \quad (4.58)$$

where we used the Gauss-Green theorem, equations (4.24)–(4.26) and (4.31), and the fact that χ^ω is Y-periodic. As a consequence,

$$\Re(A^\omega)=S^\omega+W^\omega\,,\quad \text{and}\quad \Im(A^\omega)=T^\omega+Z^\omega\,, \quad (4.59)$$

where, setting $\alpha^\omega=\Re(\chi^\omega)$ and $\beta^\omega=\Im(\chi^\omega)$,

$$S_{hj}^\omega=\int_Y\sigma(\nabla\alpha_j^\omega-\boldsymbol{e}_j)\cdot(\nabla\alpha_h^\omega-\boldsymbol{e}_h)\,\mathrm{d}y+\int_Y\sigma\nabla\beta_j^\omega\cdot\nabla\beta_h^\omega\,\mathrm{d}y\,,$$
$$W_{hj}^\omega=-\omega\alpha\int_\Gamma\Big([\alpha_j^\omega][-\beta_h^\omega]+[\beta_j^\omega][\alpha_h^\omega]\Big)\,\mathrm{d}\sigma\,,$$
$$T_{hj}^\omega=\omega\alpha\int_\Gamma\Big([\alpha_j^\omega][\alpha_h^\omega]+[\beta_j^\omega][\beta_h^\omega]\Big)\,\mathrm{d}\sigma\,,\quad\text{and}$$
$$Z_{hj}^\omega=\int_Y\sigma\nabla\beta_j^\omega\cdot(\nabla\alpha_h^\omega-\boldsymbol{e}_h)\,\mathrm{d}y-\int_Y\sigma(\nabla\alpha_j^\omega-\boldsymbol{e}_j)\cdot\nabla\beta_h^\omega\,\mathrm{d}y\,. \quad (4.60)$$

Clearly, the matrices S^ω and T^ω are symmetric, whereas the matrices W^ω and Z^ω are skew-symmetric: hence, $W^\omega=Z^\omega=0$, due to the symmetry of A^ω. Exploiting the Y-periodicity of α^ω, we have for all $\eta\in\boldsymbol{R}^N$

$$(\Re(A^\omega)\eta,\eta)=\textstyle\sum_{j,h}S_{jh}^\omega\eta_j\eta_h\geq$$
$$\sigma_m\int_Y|\nabla\textstyle\sum_j(\alpha_j^\omega\eta_j-y_j\eta_j)|^2\,\mathrm{d}y+\sigma_m\int_Y|\nabla\textstyle\sum_j(\beta_j^\omega\eta_j)|^2\,\mathrm{d}y\geq\gamma|\eta|^2\,, \quad (4.61)$$

where $\sigma_m=\min(\sigma_1,\sigma_2)$ and γ is a positive constant. In order to prove the last inequality, first we fix η such that $|\eta|=1$, and observe that

$$\sigma_m\int_Y|\nabla\textstyle\sum_j(\alpha_j^\omega\eta_j-y_j\eta_j)|^2\,\mathrm{d}y=0$$

implies that $\sum_j(\alpha_j^\omega \eta_j - y_j\eta_j)$ is constant in E_2, which is a contradiction, since the functions α_j^ω are Y-periodic, while y_j are not. Then, the result follows by compactness and homogeneity with respect to η.
Analogously, we compute for all $\eta \in \boldsymbol{R}^N$

$$(\Im(A^\omega)\eta,\eta) = \sum_{j,h} T^\omega_{jh}\eta_j\eta_h = \omega\alpha \int_\Gamma \Big[\Big(\sum_j [\alpha_j^\omega \eta_j]\Big)^2 + \Big(\sum_j [\beta_j^\omega \eta_j]\Big)^2\Big]\,\mathrm{d}\sigma \geq \gamma|\eta|^2\,.$$

Indeed, reasoning as above, if $\eta \in \boldsymbol{R}^N$, $|\eta| = 1$ exists such that $\sum_j[\chi_j^\omega \eta_j] = 0$, by (4.24)–(4.26) it results that $\sum_j(\chi_j^\omega - y_j)\eta_j$ is constant, and this contradicts the Y-periodicity of χ^ω.
Finally, for $\zeta \in \boldsymbol{C}^N$ we set $\eta = \Re(\zeta)$, $\upsilon = \Im(\zeta)$ and compute, by exploiting (4.61) and the symmetry of $\Im(A^\omega)$:

$$\Re(A^\omega\zeta,\zeta) = (\Re(A^\omega)\eta,\eta) + (\Re(A^\omega)\upsilon,\upsilon) \geq \gamma|\zeta|^2\,.$$

□

Remark 4.4. We emphasize that the condition of strict positivity of $\Im(A^\omega)$ implies assumption iii) in [12]. This assumption was stipulated there as a consequence of the Second Law of Thermodynamics. In this paper, the same condition is proved to be a direct consequence of the homogenization of equations (1.5)–(1.9), which are derived from Maxwell equations. □

5. Homogenization limit of time-harmonic solutions: case $k = 0$

In this Section we prove Theorem 1.2 in the case $k = 0$, so that we study problem (1.26)–(1.30). It amounts to solving independent Neumann problems on Ω_2^ε and on each connected component $\varepsilon(E_1 + z)$, $z \in \boldsymbol{Z}^N_\varepsilon$, of Ω_1^ε. The first one was considered in [7, Chp. 1] in the context of homogenization in perforated media, where the authors obtained that there exists a positive constant γ, independent of ε such that

$$\int_{\Omega_2^\varepsilon} |\nabla v_{\varepsilon 0}|^2\,\mathrm{d}x \leq \gamma\,. \tag{5.1}$$

Moreover, they proved that:

$$P_\varepsilon v_{\varepsilon 0} \to v_{00} \qquad \text{weakly in } H^1(\Omega), \text{ as } \varepsilon \to 0\,, \tag{5.2}$$

where we use the following notation. Setting $V_\varepsilon = \{v \in H^1(\Omega_2^\varepsilon) \colon v = c_0\Psi \text{ on } \partial\Omega\}$, P_ε is any extension operator from $L^2(\Omega_2^\varepsilon)$ to $L^2(\Omega)$ and from V_ε to $H^1(\Omega)$ such that, for any $v \in V_\varepsilon$, $\|P_\varepsilon v\|_{L^2(\Omega)} \leq C\|v\|_{L^2(\Omega_2^\varepsilon)}$ and $\|\nabla P_\varepsilon v\|_{[L^2(\Omega)]^N} \leq C\|\nabla v\|_{[L^2(\Omega_2^\varepsilon)]^N}$ for a constant C independent of ε. Moreover, v_{00} is the solution of (1.34)–(1.35) and

$$A^0 = \sigma_2|E_2|I + \int_\Gamma \sigma_2\nu \otimes \chi^{00}(y)\,\mathrm{d}\sigma\,. \tag{5.3}$$

The components χ_h^{00}, $h = 1, \dots, N$, of $\chi^{00} : E_2 \to \boldsymbol{R}^N$ satisfy

$$-\sigma_2\,\Delta_y\,\chi_h^{00} = 0\,, \qquad \text{in } E_2; \tag{5.4}$$

$$\sigma_2(\nabla_y\chi_h^{00} - \boldsymbol{e}_h)\cdot\nu = 0\,, \qquad \text{on } \Gamma. \tag{5.5}$$

In addition, χ_h^{00} is a Y-periodic function with vanishing integral average over E_2. For every $z \in \boldsymbol{Z}_\varepsilon^N$, the Neumann problem in $\varepsilon(E_1+z)$ can be explicitly solved, giving

$$v_{\varepsilon 0}(x) = \fint_{\varepsilon(\Gamma+z)} v_{\varepsilon 0}^{(\mathrm{out})} \, \mathrm{d}\sigma - \fint_{\varepsilon(\Gamma+z)} S_\varepsilon(x) \, \mathrm{d}\sigma =: v_{\varepsilon 0}^{(a)}(x) + v_{\varepsilon 0}^{(b)}(x) \,, \quad x \in \varepsilon(E_1+z) \,. \tag{5.6}$$

By (5.2), it follows that $v_{\varepsilon 0} \to v_{00}$ strongly in $L^2(\Omega)$, since

$$\|v_{\varepsilon 0} - v_{00}\|_{L^2(\Omega)} \leq \|v_{\varepsilon 0} - P_\varepsilon v_{\varepsilon 0}\|_{L^2(\Omega)} + \|P_\varepsilon v_{\varepsilon 0} - v_{00}\|_{L^2(\Omega)} \,, \tag{5.7}$$

and the first term at the right-hand side of the previous inequality is estimated as follows:

$$\|v_{\varepsilon 0} - P_\varepsilon v_{\varepsilon 0}\|^2_{L^2(\Omega)} \leq \gamma\varepsilon \int_{\Gamma^\varepsilon} [v_{\varepsilon 0}]^2 \, \mathrm{d}\sigma + \gamma\varepsilon^2 \int_{\Omega_2^\varepsilon} |\nabla v_{\varepsilon 0}|^2 \, \mathrm{d}x \leq \gamma\varepsilon^2 \tag{5.8}$$

where we used [18, Lemma 6], the fact that $P_\varepsilon v_{\varepsilon 0} = v_{\varepsilon 0}$ on Ω_2^ε, estimate (5.1) and the estimate:

$$\int_{\Gamma^\varepsilon} [v_{\varepsilon 0}]^2 \, \mathrm{d}\sigma \leq 2 \int_{\Gamma^\varepsilon} (v_{\varepsilon 0}^{(\mathrm{out})} - v_{\varepsilon 0}^{(a)})^2 \, \mathrm{d}\sigma + 2 \int_{\Gamma^\varepsilon} (v_{\varepsilon 0}^{(b)})^2 \, \mathrm{d}\sigma =: I_1 + I_2 \,. \tag{5.9}$$

Here I_1 is estimated as follows:

$$I_1 \leq \gamma\varepsilon \int_{\Omega_2^\varepsilon} |\nabla v_{\varepsilon 0}|^2 \, \mathrm{d}x \,,$$

obtained reasoning as in (3.14), and using Lemma 3.1 and Remark 3.2 above applied to the function

$$w_\varepsilon(x) = \begin{cases} v_{\varepsilon 0}^{(a)}(x) & \text{for } x \in \Omega_1^\varepsilon \,, \\ v_{\varepsilon 0}^{(\mathrm{out})}(x) & \text{for } x \in \Omega_2^\varepsilon \,, \end{cases}$$

whose jump across Γ^ε, $[w_\varepsilon] = v_{\varepsilon 0}^{(\mathrm{out})} - v_{\varepsilon 0}^{(a)}$, has null average over each connected components of Γ^ε by (5.6). On the other hand, using (1.11), we compute:

$$I_2 \leq 2\varepsilon^{N-1}|\Gamma| \sum_{z \in \boldsymbol{Z}_\varepsilon^N} \left| \fint_{\varepsilon(\Gamma+z)} S_\varepsilon(x) \, \mathrm{d}\sigma \right|^2 \leq 2 \sum_{z \in \boldsymbol{Z}_\varepsilon^N} \int_{\varepsilon(\Gamma+z)} S_\varepsilon^2(x) \, \mathrm{d}\sigma$$

$$= 2 \int_{\Gamma^\varepsilon} S_\varepsilon^2(x) \, \mathrm{d}\sigma \leq \gamma\varepsilon \,.$$

It remains to prove equation (1.36). To this end, we set:

$$\theta^0 = \chi^0 + \int_0^{+\infty} \chi^1(\cdot, t) \, \mathrm{d}t \,. \tag{5.10}$$

We remark that θ^0 coincides with θ^ω defined in (4.56) after setting $\omega = 0$. Using equations (2.7), (2.8), (2.11), and Proposition 2.2, we note that the components θ_h^0,

$h = 1, \ldots, N$, of $\theta^0 : Y \to \boldsymbol{R}^N$ satisfy

$$-\sigma \, \Delta_y \, \theta^0_h = 0 \,, \qquad \text{in } E_1, \, E_2; \tag{5.11}$$

$$[\sigma(\nabla_y \theta^0_h - \boldsymbol{e}_h) \cdot \nu] = 0 \,, \qquad \text{on } \mathit{\Gamma}; \tag{5.12}$$

$$(\sigma_2(\nabla_y \theta^0_h - \boldsymbol{e}_h) \cdot \nu)^{(\mathrm{out})} = 0 \,, \qquad \text{on } \mathit{\Gamma}. \tag{5.13}$$

In addition, θ^0_h is a Y-periodic function with vanishing integral average over Y. The above problem is comprised by two independent Neumann problems in E_1 and E_2. Comparing with Problem (5.4)–(5.5), we obtain that

$$\theta^0_h(y) = \begin{cases} y_h + d_1 & \text{for } y \in E_1 \,, \\ \chi^{00}_h(y) + d_2 & \text{for } y \in E_2 \,, \end{cases}$$

for some constants d_1, d_2. Hence, recalling (2.5) and (2.6), we get:

$$A + \int_0^{+\infty} B(t) \, \mathrm{d}t = \sigma_0 I + \int_{\mathit{\Gamma}} \nu \otimes [\sigma \theta^0](y) \, \mathrm{d}\sigma = \sigma_0 I - \sigma_1 |E_1| I + \int_{\mathit{\Gamma}} \nu \otimes \sigma_2 \chi^{00}(y) \, \mathrm{d}\sigma = A^0$$

Remark 5.1. In passing, we note that our hypotheses on the geometry of $\mathit{\Omega}^\varepsilon_2$ imply that A^0 is a positive definite real symmetric matrix [7, Chp. 1]. □

6. Time-periodic solutions: proof of Theorem 1.3

6.1. **Fourier representation of the time-periodic solution $\{u^{\#}_\varepsilon\}$.** Here we prove Theorem 1.3, Part i). In order to show the convergence in $H^1_{\#}(\boldsymbol{R}; L^2(\mathit{\Omega}))$ of the series at the right-hand side of equation (1.21), we use the Parseval identity and equations (4.3), (4.1), (1.3), and we get:

$$\int_0^T \int_{\mathit{\Omega}} \left| \sum_{k=-\infty}^{+\infty} i\omega_k v_{\varepsilon k}(x) \, \mathrm{e}^{i\omega_k t} \right|^2 \mathrm{d}x \, \mathrm{d}t = T \int_{\mathit{\Omega}} \sum_{k=-\infty}^{+\infty} \omega_k^2 |v_{\varepsilon k}(x)|^2 \, \mathrm{d}x$$

$$\leq \gamma \sum_{k=-\infty}^{+\infty} \omega_k^2 |c_k|^2 < +\infty \,.$$

The convergence in $H^1_{\#}(\boldsymbol{R}; H^1(\mathit{\Omega}^\varepsilon_i))$, $i = 1, 2$ can be shown analogously.
It remains to show that the function $u^{\#}_\varepsilon(x, t)$ defined in (1.21) solves Problem (1.14)–(1.19). Weak solutions to this problem are defined to be in the class

$$u_\varepsilon(x, \cdot) \text{ is } T\text{-periodic in time;} \qquad u_{\varepsilon | \mathit{\Omega}^\varepsilon_i} \in L^2_{\#}(\boldsymbol{R}; H^1(\mathit{\Omega}^\varepsilon_i)) \,, \quad i = 1 \,, 2 \,, \tag{6.1}$$

and $u_{\varepsilon | \partial \mathit{\Omega}} = \Psi \Phi$ in the sense of traces. The weak formulation is

$$\int_0^T \int_{\mathit{\Omega}} \sigma \nabla u_\varepsilon \cdot \nabla \overline{\psi} \, \mathrm{d}x \, \mathrm{d}t - \frac{\alpha}{\varepsilon} \int_0^T \int_{\mathit{\Gamma}^\varepsilon} [u_\varepsilon] \frac{\partial}{\partial t} [\overline{\psi}] \, \mathrm{d}\sigma \, \mathrm{d}t = 0 \,, \tag{6.2}$$

for each $\psi \in L^2_{\#}(\boldsymbol{R}; L^2(\mathit{\Omega}))$ such that ψ is in the class (6.1), $[\psi] \in H^1_{\#}(\boldsymbol{R}; L^2(\mathit{\Gamma}^\varepsilon))$, and ψ vanishes on $\partial \mathit{\Omega} \times (0, T)$.

The left-hand side in equation (6.2), after substituting u_ε from the series at the right-hand side of (1.21), becomes:

$$\sum_{k=-\infty}^{+\infty} \int_0^T \left[\int_\Omega \sigma \nabla v_{\varepsilon k} \cdot \nabla \overline{\psi} \, \mathrm{d}x + \frac{i\alpha\omega_k}{\varepsilon} \int_{\Gamma^\varepsilon} [v_{\varepsilon k}][\overline{\psi}] \, \mathrm{d}\sigma \right] \mathrm{e}^{i\omega_k t} \, \mathrm{d}t \,,$$

which vanishes, since $v_{\varepsilon k}$ satisfies Problem (1.22)–(1.24) for $k \neq 0$ and Problem (1.26)–(1.28) for $k = 0$. The series over k can be exchanged with the integrals, since using Hölder's inequality and equations (4.2), (4.1), (5.1), (1.3) we obtain:

$$\sum_{k=-\infty}^{+\infty} \int_0^T \left[\int_\Omega |\sigma \nabla v_{\varepsilon k} \cdot \nabla \overline{\psi}| \, \mathrm{d}x + \frac{\alpha\omega_k}{\varepsilon} \int_{\Gamma^\varepsilon} |[v_{\varepsilon k}][\overline{\psi}]| \, \mathrm{d}\sigma \right] \mathrm{d}t \leq \gamma(\varepsilon) \sum_{k=-\infty}^{+\infty} |c_k|^2 < +\infty \,.$$

On the other hand, the boundary condition (1.8) is satisfied, as it is easily verified by exchanging the trace operator on $\partial\Omega$ with the series and recalling equations (1.25), (1.29) and (1.20), taking into account the linearity and continuity of the trace operator and Theorem 1.3, Part i).

Uniqueness of T-periodic solutions to Problem (1.14)–(1.19) is easily proved. Indeed, by linearity, the difference $w_\varepsilon^\#(x,t)$ of two such solutions satisfies

$$\int_0^T \int_\Omega \sigma |\nabla w_\varepsilon^\#|^2 \, \mathrm{d}x = 0 \,,$$

hence it is piece-wise constant. This relation follows integrating (1.14) over $\Omega \times (0,T)$, using the Gauss-Green identity, the homogeneous Dirichlet boundary data for $w_\varepsilon^\#$, and equations (1.15), (1.16), (1.18). By equation (1.19), it follows that $w_\varepsilon^\#$ has null average over each connected component of Γ^ε, hence it is constant over $\Omega \times \boldsymbol{R}$, and so it vanishes, due to the homogeneous Dirichlet boundary data.

6.2. **Convergence of $\{u_\varepsilon^\#\}$ to $u_0^\#$ as $\varepsilon \to 0$.** Here we prove Theorem 1.3, Part ii). We estimate, using the monotone convergence theorem, for $k_0 \in \boldsymbol{N}$ fixed:

$$\begin{aligned}
\int_\Omega |u_\varepsilon^\#(x,t) - u_0^\#(x,t)| \, \mathrm{d}x &= \int_\Omega \left| \sum_{k=-\infty}^{+\infty} (v_{\varepsilon k}(x) - v_{0k}(x)) \, \mathrm{e}^{i\omega_k t} \right| \mathrm{d}x \\
&\leq \int_\Omega \sum_{k=-\infty}^{+\infty} |v_{\varepsilon k}(x) - v_{0k}(x)| \, \mathrm{d}x = \sum_{k=-\infty}^{+\infty} \int_\Omega |v_{\varepsilon k}(x) - v_{0k}(x)| \, \mathrm{d}x \\
&= \sum_{|k| \leq k_0} \int_\Omega |v_{\varepsilon k}(x) - v_{0k}(x)| \, \mathrm{d}x + \sum_{|k| > k_0} \int_\Omega |v_{\varepsilon k}(x) - v_{0k}(x)| \, \mathrm{d}x =: I_1 + I_2
\end{aligned}$$

Using Hölder's inequality and equations (4.3), (4.50), (4.1), we compute:

$$I_2 \leq \gamma \sum_{|k| > k_0} \left(\int_\Omega (|v_{\varepsilon k}(x)|^2 + |v_{0k}(x)|^2) \, \mathrm{d}x \right)^{1/2} \leq \gamma \sum_{|k| > k_0} |c_k|$$

By hypothesis (1.3), the right-hand term of the above inequality can be made arbitrarily small by choosing k_0 sufficiently large. For such k_0 fixed, I_1 can be made arbitrarily small letting $\varepsilon \to 0$, by virtue of the strong L^1 convergence of $v_{\varepsilon k}$ to v_{0k} as $\varepsilon \to 0$. We conclude that $u_\varepsilon^\#(x,t) \to u_0^\#(x,t)$ in $L^\infty_\#(\boldsymbol{R}; L^1(\Omega))$ as $\varepsilon \to 0$.
In order to prove that $u_\varepsilon^\#(x,t) \to u_0^\#(x,t)$ weakly in $L^2_\#(\boldsymbol{R}; L^2(\Omega))$ as $\varepsilon \to 0$, we represent test functions $\phi \in L^2_\#(\boldsymbol{R}; L^2(\Omega))$ by means of their Fourier series, as follows:

$$\phi(x,t) = \sum_{k=-\infty}^{+\infty} \phi_k(x)\,\mathrm{e}^{i\omega_k t}\,. \tag{6.3}$$

We compute, for $k_0 \in \boldsymbol{N}$ fixed:

$$\int_0^T \int_\Omega (u_\varepsilon^\#(x,t) - u_0^\#(x,t))\overline{\phi}(x,t)\,\mathrm{d}x\,\mathrm{d}t = T \int_\Omega \sum_{k=-\infty}^{+\infty} (v_{\varepsilon k}(x) - v_{0k}(x))\overline{\phi}_k(x)\,\mathrm{d}x$$
$$= T \int_\Omega \sum_{|k|\le k_0} (v_{\varepsilon k}(x) - v_{0k}(x))\overline{\phi}_k(x)\,\mathrm{d}x + T \int_\Omega \sum_{|k|>k_0} (v_{\varepsilon k}(x) - v_{0k}(x))\overline{\phi}_k(x)\,\mathrm{d}x =: I_1 + I_2\,.$$

Using the monotone convergence theorem and Hölder's inequality, we compute:

$$|I_2| \le T \sum_{|k|>k_0} \int_\Omega |v_{\varepsilon k}(x) - v_{0k}(x)||\overline{\phi}_k(x)|\,\mathrm{d}x$$
$$\le \gamma \sum_{|k|>k_0} \left(\int_\Omega (|v_{\varepsilon k}(x)|^2 + |v_{0k}(x)|^2)\,\mathrm{d}x \right)^{1/2} \left(\int_\Omega |\overline{\phi}_k(x)|^2\,\mathrm{d}x \right)^{1/2}\,.$$

By equations (1.3), (4.1), (4.3), (4.50) and (6.3), the right-hand term of the above inequality can be made arbitrarily small by choosing k_0 sufficiently large. For such fixed k_0, I_1 can be made arbitrarily small letting $\varepsilon \to 0$, by virtue of the weak L^2 convergence of $v_{\varepsilon k}$ to v_{0k} as $\varepsilon \to 0$, and the assert follows.
It remains to prove that the series (1.37) strongly converges in $H^1_\#(\boldsymbol{R}; H^1(\Omega))$. To this end, we set:

$$z_{0k} = v_{0k} - c_k \Psi\,, \tag{6.4}$$

and compute, for $k \neq 0$, from equations (1.31)–(1.32):

$$\int_\Omega A^{\omega_k}\,\nabla z_{0k} \cdot \nabla \overline{z_{0k}}\,\mathrm{d}x = -\int_\Omega c_k A^{\omega_k}\,\nabla \Psi \cdot \nabla \overline{z_{0k}}\,\mathrm{d}x\,.$$

Taking the real part of the previous equation, using Proposition 4.3, Young's inequality and assumption (1.4), we obtain:

$$\int_\Omega |\nabla z_{0k}|^2\,\mathrm{d}x \le \gamma |c_k|^2\,, \tag{6.5}$$

for a constant γ independent of k. Recalling that z_{0k} vanishes on $\partial\Omega$, the assert follows from Parseval's identity and assumption (1.3).

6.3. **Equation for the time-periodic asymptotic solution.** Here we prove Theorem 1.3, Part iii). Equation (1.39) follows from equations (1.32), (1.35), (1.20) and the $H^1_\#(\boldsymbol{R}; H^1(\Omega))$-convergence of the series (1.37). In order to prove equation (1.38), we consider its weak formulation:

$$\int_0^T \int_\Omega \nabla\phi \cdot A\nabla u_0^\# \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega \nabla\phi \cdot \int_0^{+\infty} B(\tau)\nabla u_0^\#(x, t-\tau)\,\mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}t = 0\,, \qquad \forall \phi \in L^2_\#(\boldsymbol{R}; H^1(\Omega))\,. \tag{6.6}$$

A direct computation shows that any partial sum of the series (1.37), i.e.,

$$\hat{u}_0^N(x,t) = \sum_{k=-N}^{N} v_{0k}(x)\,\mathrm{e}^{i\omega_k t}\,, \qquad N \in \boldsymbol{N}\,, \tag{6.7}$$

satisfies equation (6.6), by virtue of equations (1.31), (1.33), (1.34), (1.36). Then we let $N \to +\infty$: to this regard, as far as the second integral in equation (6.6) is concerned, we proceed as follows. We exchange the integration order, use Hölder's inequality, Parseval identity, Beppo-Levi theorem, Proposition 2.2, and equations (6.4) and (6.5), thus obtaining the following estimate:

$$\left| \int_0^{+\infty} B(\tau)\,\mathrm{e}^{-i\omega_k\tau}\,\mathrm{d}\tau \int_0^T \int_\Omega \nabla\phi \cdot \sum_{|k|>N} \nabla v_{0k}(x)\,\mathrm{e}^{i\omega_k t}\,\mathrm{d}x\,\mathrm{d}t \right| \le \gamma \|\phi\|_{L^2_\#(\boldsymbol{R};H^1(\Omega))} \sum_{|k|>N} |c_k|^2\,, \tag{6.8}$$

which tends to zero by (1.3) and (1.20).

Remark 6.1. Theorem 1.3, Part iii) is related to the results in [12], where, however, the setting is slightly different. □

7. Stability result: proof of Theorem 1.5

In this Section we prove Theorem 1.5. Let u_ε and $u_\varepsilon^\#$ be the solutions of Problem (1.5)–(1.9) and Problem (1.14)–(1.19), respectively. We set:

$$w_\varepsilon = u_\varepsilon - u_\varepsilon^\#\,. \tag{7.1}$$

Since w_ε satisfies Problem (1.5)–(1.9) with homogeneous Dirichlet boundary data on $\partial\Omega \times (0, +\infty)$, i.e. $\Psi \equiv 0$, and with S_ε replaced by $S_\varepsilon - u_\varepsilon^\#(\cdot, 0)$, the assert follows from Theorem 1.1, after proving that $u_\varepsilon^\#(\cdot, 0)$ satisfies (1.11).
To this end, we first observe that a classical trace inequality implies that

$$\frac{1}{\varepsilon}\int_{\Gamma^\varepsilon} [u_\varepsilon^\#(x,0)]^2\,\mathrm{d}\sigma \le \frac{\gamma}{\varepsilon}\int_0^T \int_{\Gamma^\varepsilon} \left(|[u_\varepsilon^\#]|^2 + |[u_{\varepsilon t}^\#]|^2 \right)\mathrm{d}\sigma\,\mathrm{d}t\,. \tag{7.2}$$

Then we use equation (1.21), the Parseval identity, (4.2), (4.1), (5.9) and following, (5.1), and estimate:

$$\frac{\gamma}{\varepsilon}\int_0^T\int_{\Gamma^\varepsilon}\Big(|[u^{\#}_\varepsilon]|^2+|[u^{\#}_{\varepsilon t}]|^2\Big)\,\mathrm{d}\sigma\,\mathrm{d}t=\frac{\gamma}{\varepsilon}\sum_{k=-\infty}^{+\infty}\int_{\Gamma^\varepsilon}|[v_{\varepsilon k}]|^2(1+\omega_k^2)\,\mathrm{d}\sigma$$

$$\leq\gamma\sum_{k=-\infty}^{+\infty}|c_k|^2(1+k^2)\,. \quad (7.3)$$

The assert follows since the right-hand term of (7.3) is estimated by a constant independent of ε, by (1.20) and (1.3).

References

[1] M. Amar, D. Andreucci, P. Bisegna, and R. Gianni. An elliptic equation with history. *C. R. Acad. Sci. Paris, Ser. I*, 338:595–598, 2004. Elsevier.

[2] M. Amar, D. Andreucci, P. Bisegna, and R. Gianni. Evolution and memory effects in the homogenization limit for electrical conduction in biological tissues. *Mathematical Models and Methods in Applied Sciences*, 14:1261–1295, 2004. World Scientific.

[3] M. Amar, D. Andreucci, P. Bisegna, and R. Gianni. Existence and uniqueness for an elliptic problem with evolution arising in electrodynamics. *Nonlinear Analysis Real World Applications*, 6:367–380, 2005. Elsevier.

[4] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of Bounded Variation and Free Discontinuity Problems*. Oxford Mathematical Monographs. Oxford University Press, Oxford, United Kingdom, 2000.

[5] J. Appleby, M. Fabrizio, B. Lazzari, and D. Reynolds. On exponential asymptotic stability in linear viscoelasticity. *Mathematical Models and Methods in Applied Sciences*, 16:1677–1694, 2006.

[6] L. Borcea. Electrical impedance tomography. *Inverse Problems*, 18:R99–R136, 2002.

[7] D. Cioranescu and J. Saint Jean Paulin. *Homogenization of reticulated structures*, volume 136 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1999.

[8] H. Dehghani and N. K. Soni. Electrical impedance spectroscopy: theory. In K. D. Paulsen, P. M. Meaney, and L. C. Gilman, editors, *Alternative breast imaging: Four model-based approaches*, pages 85–105. Springer, 2005.

[9] M. Fabrizio and B. Lazzari. Sulla stabilità di un sistema viscoelastico lineare. In *Acc. Naz. Lincei, Tavola rotonda sul tema: Continui con Memoria*. Roma, 1992.

[10] M. Fabrizio and A. Morro. Viscoelastic relaxation functions compatible with thermodynamics. *Journal of Elasticity*, 19:63–75, 1988.

[11] M. Fabrizio. Sul problema di Fichera in viscoelasticità lineare. In *Atti del III Meeting: Waves and Stability*. Bari, 1989.

[12] M. Fabrizio. Sulla correttezza di un problema integrodifferenziale della viscoelasticità. In *Seminari di Analisi*. Dipartimento di Matematica, Università di Bologna, 1992.

[13] G. Fichera. Avere una memoria tenace crea gravi problemi. *Arch. Ration. Mech. Anal.*, 70:101–112, 1979.

[14] C. Giorgi, M. Naso, and V. Pata. Exponential stability in linear heat conduction with memory: a semigroup approach. *Commun. Appl. Anal.*, 5:121–133, 2001.

[15] C. Giorgi, M. Naso, and V. Pata. Energy decay of electromagnetic systems with memory. *Mathematical Models and Methods in Applied Sciences*, 15:1489–1502, 2005.

[16] B. Lazzari and R. Nibbi. Sufficient conditions for the exponential stability in linear conductors with memory. *Int. J. Eng. Sci.*, 30:533–544, 1992.

[17] B. Lazzari and E. Vuk. Constitutive equations and quasi-static problem in linear thermoviscoelaticity. *Int. J. Eng. Sci.*, 40:1131–1149, 2002.

[18] F. Lene and D. Leguillon. Étude de l'influence d'un glissement entre les constituants d'un matériau composite sur ses coefficients de comportement effectifs. *Journal de Mécanique*, 20:509–536, 1981.

[19] M. Medjden and N.-E. Tatar. Asymptotic behavior for a viscoelastic problem with not necessarily decreasing kernel. *Appl. Math. Comput.*, 167:1221–1235, 2005.

[20] P. A. Raviart and J.-M. Thomas. *Introduction á l'analyse numérique des équations aux dérivées partielles.* Masson, Paris, France, 1983.

[21] M. Slemrod. Global existence, uniqueness, and asymptotic stability of classical smooth solutions in one-dimensional nonlinear thermoelasticity. *Arch. Ration. Mech. Anal.*, 76:97–133, 1981.

[22] A. E. Taylor and D. C. Lay. *Introduction to functional analysis.* John Wiley & Sons, New York-Chichester-Brisbane, second edition, 1980.