

SPECTRAL IDENTITIES AND CLOSED-GEODESIC PERIODS

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ABSTRACT. We construct closed real-analytic locally symmetric nine-manifolds with equal scalar Laplace spectra and explicitly determine the infinite difference of their closed-geodesic period sets. Their common half-wave trace is real analytic near every period in this difference, although both geodesic flows are clean. In dimension eight, all finite-rank unitary flat connection spectra agree under a fixed $\mathbb{Z}/7\mathbb{Z}$ -marking, while the period sets differ. On a fixed closed simply connected eighteen-manifold, one of two real-analytic scalar-isospectral metrics has infinitely many primitive periods absent from the period set of the other. These spectral equalities follow from intrinsic equivariant identities for $U(2n+1)$ and $Sp(n) \times SO(2n+2)$, $n \geq 1$. For every central extension of a finite group with finitely generated kernel, prescribed finite families of central quotients occur as fundamental groups of closed homogeneous manifolds. Their canonical rough connection spectra agree for every finite-dimensional unitary representation of a common compact isotropy group. Simple connectivity varies in dimension eight, and rational formality in dimension seven. For a closed connected Riemannian manifold, marked finite-holonomy spectra determine the profinite conjugacy envelope of least lengths. Among compact connected Lie groups with bi-invariant metrics and a $U(n) \times U(1)$ marking, prescribed finite jets of the rate and logarithmic amplitude of the normalized equivariant heat trace, dimension and a volume lower bound characterize exactly these two models. The ambient weighted identity gives scalar-isospectral left-invariant normal homogeneous metrics on every $SU(m)$, $m \geq 6$, which are not locally isometric, answering Gordon–Sutton’s uniqueness question negatively.

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1. INTRODUCTION

Zelditch [103, Problem 4.1], Colin de Verdière [25, §13], and Sutton [89, arXiv v1, Problem 1] ask whether the scalar Laplace spectrum determines the set of lengths of smoothly closed geodesics. The answer is negative, even for closed real-analytic locally symmetric manifolds of nonnegative sectional curvature.

All operator spectra include multiplicities. The representations indexing flat spectra are finite dimensional. A closed manifold is compact and has no boundary. Write $\text{Lsp}(X)$ for the set of lengths of nonconstant closed geodesics, including iterates and without length multiplicities. For a closed connected Riemannian manifold X , let $\ell_X(c)$ be the minimum length of a loop in the free homotopy class of $c \in \Pi = \pi_1(X)$, with $\ell_X(1) = 0$. The minimum is attained (Lemma 5.3); conjugation and inversion preserve it.

1.1. Closed-geodesic periods and the Poisson relation. Put $h = 2\pi/7$ and $S_3 = \{a^2 + b^2 + c^2 : a, b, c \in \mathbb{Z}\}$.

Theorem 1.1 (Closed-geodesic periods). *There are closed connected real-analytic nine-manifolds M_9, N_9 , locally symmetric and of nonnegative sectional curvature, such that*

$$\begin{aligned} \text{Spec}(\Delta_{M_9}) &= \text{Spec}(\Delta_{N_9}), & \text{Lsp}(M_9) &\subsetneq \text{Lsp}(N_9), \\ \text{Lsp}(N_9) \setminus \text{Lsp}(M_9) &= \{h\sqrt{n} : n \in S_3, 7 \mid n, 49 \nmid n\}. \end{aligned} \quad (1.1)$$

Both geodesic flows are clean. The common half-wave trace is real analytic near every length in the difference, which contains infinitely many lengths of embedded primitive closed geodesics.

The manifolds are the order-seven quotients (6.4) (Theorem 7.1, Corollary 7.4). By (7.3), as $T \rightarrow \infty$,

$$\#((\text{Lsp}(N_9) \setminus \text{Lsp}(M_9)) \cap (0, T]) = \frac{5}{4\pi^2} T^2 + O(\log(2 + T)).$$

For a closed Riemannian manifold X , write $w_X(t) = \text{Tr}(e^{it\sqrt{\Delta_X}}) \in \mathcal{S}'(\mathbb{R})$. The Poisson relation [89, arXiv v1, §1.2, equation (1.1)] is

$$\text{singsupp } w_X \subseteq \{0\} \cup \text{Lsp}(X) \cup (-\text{Lsp}(X)).$$

Zelditch [103, Problem 4.1] asks whether w_X can be smooth at an actual closed-geodesic length; Sutton [89, arXiv v1, Problem 2] asks whether this inclusion is always an equality. Theorem 1.1 gives a negative answer to the latter question. For $\ell = h\sqrt{14}$, the fixed set of the time- ℓ geodesic flow on the unit tangent bundle of N_9 consists of twelve clean three-dimensional tori. Their vectors are the initial velocities of embedded primitive closed geodesics, and exactly two components correspond to each nonidentity deck transformation (Proposition 7.5). In each pair, the complete singular expansions cancel coefficient by coefficient (Proposition 7.6).

The period sets can differ even when every unitary flat spectrum is fixed under one fundamental-group marking. In dimension eight, Theorem 7.7 gives

$$\pi_1(M_8) \simeq \pi_1(N_8) \simeq C_7, \quad \text{Spec}(\Delta_{M_8, \rho}) = \text{Spec}(\Delta_{N_8, \rho}) \quad (\rho : C_7 \rightarrow \text{U}(V)),$$

where $C_7 = \mathbb{Z}/7\mathbb{Z}$, each marking sends 1 to the corresponding deck generator, and ρ ranges over all finite-dimensional unitary representations. Theorem 7.7 and Corollary 7.9 give

$$3h \in \text{Lsp}(N_8) \setminus \text{Lsp}(M_8), \quad \ell_{M_8}(3) = \ell_{N_8}(3) = h\sqrt{2}. \quad (1.2)$$

In fact $\ell_{M_8}(c) = \ell_{N_8}(c)$ for every $c \in C_7$, by Theorem 5.4 with the identity homomorphism of C_7 .

On one simply connected eighteen-manifold, scalar-isospectral analytic metrics have infinitely many exclusive lengths of embedded primitive geodesics. Such pairs with positive Ricci curvature occur in every C^∞ neighborhood of a fixed product metric (Theorem 8.8). Theorem 8.2 gives seventeen-dimensional pairs on one manifold with every marked unitary flat spectrum equal on every connected finite cover and an explicit exclusive primitive period. Theorem 9.2 gives an exact period-set difference

and counting asymptotics for two locally symmetric metrics on one nineteen-manifold. Their flat spectra agree for representations pulled back from the base circle (Proposition 9.1).

Miatello–Rossetti [74, arXiv v1, Example 2.3 and Theorem 4.1(ii)] construct compact flat manifolds with different full length sets and equal spectra on certain p -forms, $p > 0$, whereas the scalar spectrum determines their full length sets. Their scalar and Sunada examples with different length spectra differ in multiplicities [74, arXiv v1, Examples 3.4 and 3.6].

Gordon–Rossetti [49, Corollary 1.6] show that, on a closed nonorientable $2m$ -manifold, an isolated nondegenerate simple closed geodesic with orientation-reversing holonomy has zero contribution to the complexified heat trace on m -forms. For scalar-isospectral Heisenberg manifolds, Gornet [50, Example 2.15 and §5] computes the leading wave coefficients and explains their equality despite different length multiplicities.

Colin de Verdière [25, §13] describes cancellation of complete semiclassical expansions as a possible obstruction to length recovery. Koval–Vig [61, Theorem 1.1] obtain finite-order cancellation for planar billiards: for each finite N there are smooth strictly convex domains, arbitrarily C^∞ -close to an ellipse, whose Dirichlet cosine trace is C^N near a billiard length. The analytic regularity in Theorem 1.1 follows from the analytic Poisson inclusion of Proposition A.3.

1.2. Equivariant identities and descent. For $n \geq 1$, put

$$G_n = \mathrm{U}(2n+1), \quad H_n = \mathrm{Sp}(n) \times \mathrm{SO}(2n+2), \quad P_n = \mathrm{U}(n) \times \mathrm{U}(1).$$

The embeddings are $i_n(A, z) = \mathrm{diag}(A, \bar{A}, z)$ and $j_n(A, z) = (\mathrm{diag}(A, \bar{A}), r(A \oplus z))$, where r is realification. Give the groups the bi-invariant metrics

$$Q_G = -\mathrm{Re} \, \mathrm{Tr}_\mathbb{C}, \quad Q_H = -\frac{1}{2} \mathrm{Re} \, \mathrm{Tr}_\mathbb{C} \oplus -\frac{1}{2} \mathrm{Tr}_\mathbb{R}.$$

Their restrictions to $\mathfrak{p}_n$ agree. There is a unitary equivalence for the right- P_n actions,

$$(L^2(G_n), \Delta_{G_n}, P_n) \simeq (L^2(H_n), \Delta_{H_n}, P_n), \quad (1.3)$$

which preserves smooth functions and the self-adjoint domains (Theorem 3.2). It also intertwines the left actions of $L_n = \mathrm{U}(n)$ through $i_n(A, 1)$ and $j_n(A, 1)$. For $n = 1$, restriction to the invariant subspaces of the order-seven actions in (6.4) gives the scalar spectral equality for M_9, N_9 . On the subspaces invariant under the common right circle, the same unitary intertwines the residual C_7 -actions; their character subspaces give the marked unitary flat spectral equalities for M_8, N_8 (Corollary 4.14, Theorem 7.7).

On a normal homogeneous space J/K , the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{j}$ defines the canonical connection. For a finite-dimensional unitary K -module (V, ρ) , let $\Delta_{J/K, V}$ be its rough Laplacian on $J \times_K V$. The trivial module gives the scalar Laplacian. On functions satisfying $f(gk) = \rho(k)^{-1} f(g)$, one has

$$\Delta_{J/K, V} = C_J - C_{K, V}, \quad C_{K, V} = - \sum_a d\rho(Y_a)^2,$$

where $C_J = \Delta_J$ acts componentwise and (Y_a) is orthonormal in $\mathfrak{k}$. For a closed subgroup $K \leq \prod_j P_{n_j}$, the metric on $\mathfrak{k}$ is the same for every choice of ambient factors G_{n_j} or H_{n_j} . Tensor products of (1.3) therefore give equal canonical rough spectra for every common K -module (Theorem 4.3). For the central extensions in Theorem 1.2, coordinate deletion between lifted stabilizers induces the homomorphisms $E \rightarrow E/A_I$.

For the bilateral $P_n \times P_n$ action, the difference of the eigenspaces at $2n+1$, the first positive eigenvalue, generates the ideal of all eigenspace differences in $R(P_n \times P_n)$. After restriction to any closed subgroup, all these differences vanish if and only if the first one vanishes (Corollary 3.10). The $L_1 \times P_1$ equivalence also gives the scalar spectral equalities for the connection metrics of Theorem 8.8. A common principal connection gives the associated-bundle spectral identity of Proposition 4.7.

Sutton [88, Theorems 2.3 and 3.6] obtained simply connected, locally non-isometric normal homogeneous spaces from representation-equivalent connected subgroups. The groups G_n and H_n

occur in the dimension-data identity of An–Yu–Yu [5, Theorem 1.5(1)]; their construction gives simply connected isospectral homogeneous spaces of different homotopy types [5, Theorem 1.7]. Yu obtained nontrivial τ -dimension-data identities and vector bundles with isospectral ambient Casimir operators [101, Theorem 2.1 and Corollary 2.6] [102, arXiv version 2, §2.3]. Yu’s weighted identities also imply the ambient Casimir identity of Theorem 20.1, by finite support separation and the leading Weyl-denominator term. The intrinsic equivalence (1.3) identifies the right- P_n representations in each scalar eigenspace of G_n and H_n , with the metrics above.

1.3. Realization and topology. With K fixed, the boundary maps $\pi_1(J/K) \twoheadrightarrow \pi_0(K)$ for the ambient products J have the same finite target. Their kernels are the finitely generated central groups $\text{coker}(\pi_1(K^0) \rightarrow \pi_1(J))$ (Lemma 10.2). By Wilking’s theorem [99, Theorem 2.2(a),(d)], a group is the fundamental group of a compact connected Riemannian homogeneous manifold if and only if it contains a finitely generated central subgroup of finite index.

Let

$$1 \longrightarrow A \longrightarrow E \xrightarrow{p} \Gamma \longrightarrow 1$$

be central, with Γ finite and A finitely generated. Choose generators $c_1, \dots, c_r$ of A , where $r \geq 1$; identity entries are allowed. Put $A_I = \langle c_j : j \in I \rangle$ for $I \subseteq [r]$.

Theorem 1.2 (Realization). *There are compact connected Lie groups J_I , a compact Lie group K with embeddings into every J_I , and normal homogeneous metrics on $M_I = J_I/K$, $I \subseteq [r]$, such that*

$$\pi_1(M_I) \simeq E/A_I, \quad \pi_0(K) \simeq \Gamma, \quad \dim \ker \text{Ric}_{M_I} = \text{rank}(A/A_I). \quad (1.4)$$

The boundary homomorphism $\pi_1(M_I) \rightarrow \pi_0(K)$ is the prescribed map $E/A_I \rightarrow \Gamma$. For every finite-dimensional unitary K -module V , the spectrum of $\Delta_{M_I, V}$ is independent of I . The tangent spaces at the identity cosets are isomorphic as metric K -modules. The metrics are real analytic and have nonnegative sectional curvature; their Ricci tensors are positive definite exactly when E/A_I is finite.

The common tangent module gives equal canonical rough spectra on all its tensor powers and invariant tensor summands. The manifolds admit compatible homogeneous spin structures for which the cubic spin squares have equal spectra with every common homogeneous unitary twist. On the graded exterior Clifford module, the signed cubic spectra agree (Theorem 4.9 and Proposition 4.11). For $L \leq \Gamma$, these equalities hold on the degree- $[\Gamma : L]$ covers with fundamental groups $p^{-1}(L)/A_I$, including all flat twists pulled back from finite-dimensional unitary representations of L .

A nonempty finite list of groups $E_1, \dots, E_m$ admits a realization $E_i \simeq \pi_1(J_i/K)$, with compact connected Lie groups J_i and one compact isotropy group K , if and only if there are surjections

$$p_i : E_i \twoheadrightarrow \Gamma \quad (1.5)$$

onto one finite group, with finitely generated central kernels. For prescribed maps p_i , the realization may be chosen with these boundary homomorphisms and the canonical spectral equalities above (Theorem 11.1).

Let A be a nontrivial finitely generated abelian group, and let $d(A)$ be its least number of generators. Among pairs $(\prod_{j=1}^r J_j)/K$ and $(\prod_{j=1}^r J'_j)/K$, where $r, n_j \geq 1$, $J_j, J'_j \in \{G_{n_j}, H_{n_j}\}$ and $K \leq \prod_{j=1}^r P_{n_j}$ is one common closed subgroup, the least dimension for fundamental groups A and 1 is

$$8d(A) \quad (1.6)$$

(Theorem 11.5). Bérard asks whether a closed manifold scalar-isospectral to a simply connected manifold must itself be simply connected [12, VIII.4, p. 141]. For every $q \geq 2$, Corollary 11.12 gives eight-dimensional manifolds with fundamental groups C_q and 1, equal scalar and canonical connection spectra, positive Ricci curvature and nonnegative sectional curvature, answering this question negatively.

The seven-dimensional quotients

$$\begin{aligned} A_7 &= G_1/P_1 \cong \mathrm{SU}(3)/S_{(1,-1,0)}^1, \\ B_7 &= H_1/P_1 \cong \mathrm{SU}(2)^3/\ker(\alpha - \beta - \gamma), \end{aligned}$$

are simply connected and carry the normal metrics induced by Q_G and Q_H . Here α, β, γ are the 2π -periodic angles of the three maximal circles. Their canonical rough spectra agree, their Ricci tensors are positive, and their second Betti numbers are 1 and 2 (Theorem 12.1). The Aloff–Wallach space $A_7 = N_{1,-1}$ is rationally formal [40, Theorem 5.3]; the space $B_7 = Q^{1,1,1}$ has a triple Massey product which does not contain zero [17, proof of Theorem 3.2, case $n = 3$]. Their Hodge two-form spectra differ in the multiplicity of zero. Since every simply connected closed manifold of dimension at most six is formal [80, Proposition 4.6, p. 574], seven is the least dimension for this formality contrast.

Cornelissen–Peyerimhoff ask whether closed connected oriented Riemannian manifolds with large fundamental groups must be isometric if a bijection between all finite-dimensional unitary representations of their fundamental groups preserves the corresponding flat spectra, without assuming a common finite orbifold quotient [27, Chapter 1, Open Problem, p. 6]. A group is large if a finite-index subgroup surjects onto a nonabelian free group. For a closed oriented surface Σ_g , $g \geq 2$, products with the simply connected pair of [5, Theorem 1.7] already give a negative answer: every marked unitary flat spectrum agrees, while the second homotopy groups differ. For a hyperbolic metric on Σ_g , a common positive analytic warping function for the fibers A_7, B_7 gives nine-dimensional metrics with restricted holonomy $\mathrm{SO}(9)$ and equal marked unitary flat spectra on every corresponding finite cover (Corollary 13.3). Their second Betti numbers are 2 and 3, and their rational formality is opposite. Their universal covers have second Betti numbers 1 and 2, so they have no common connected Riemannian cover.

1.4. Finite holonomy and least lengths. For the manifolds of Theorem 1.2,

$$\ell_{M_{[r]}}(\gamma) = \min_{p_I(c)=\gamma} \ell_{M_I}(c), \quad p_I : E/A_I \twoheadrightarrow \Gamma, \quad \gamma \in \Gamma.$$

The minimum is attained for every $I \subseteq [r]$ (Corollary 14.10). The source group need not be conjugacy separable.

Let X be an arbitrary closed connected smooth Riemannian manifold of positive dimension, with marked fundamental group Π . For a finite quotient $q : \Pi \twoheadrightarrow Q$, let $Y_q \rightarrow X$ be its regular cover. Put $R_g f(y) = f(g^{-1}y)$ and $H_{q,g}(t) = \mathrm{Tr}(R_g e^{-t\Delta_{Y_q}})$. These traces are positive, and character inversion recovers them from the flat spectra indexed by the representations of Q . Their small-time asymptotics give

$$\lim_{t \downarrow 0} -4t \log H_{q,g}(t) = \ell_q(g)^2, \quad \ell_q(g) = \min_{q(c) \sim_Q g} \ell_X(c). \quad (1.7)$$

Every minimum is attained. Let $\hat{\Pi}$ be the profinite completion of Π , with canonical map $\iota : \Pi \rightarrow \hat{\Pi}$, and let $\hat{q}$ be the continuous extension of q . The function $\hat{\ell}_X(z) = \sup_q \ell_q(\hat{q}(z))$ satisfies

$$\{z : \hat{\ell}_X(z) \leq R\} = \bigcup_{\ell_X(c) \leq R} \mathrm{Conj}_{\hat{\Pi}}(\iota(c)), \quad 0 \leq R < \infty. \quad (1.8)$$

Only finitely many distinct orbits occur. For each $0 \leq R < \infty$, one finite quotient attains the supremum at every $\iota(c)$ with $\ell_X(c) \leq R$ (Theorem 14.2). If Π is conjugacy separable, the marked finite-holonomy spectra determine every $\ell_X(c)$.

For every smooth metric and every $g \neq 1$, the equivariant cosine trace $\mathrm{Tr}(R_g \cos(t\sqrt{\Delta_{Y_q}}))$ is singular at $t = \ell_q(g)$ (Theorem 15.3). The probabilities on $\hat{\Pi}$ whose pushforwards to Q assign mass $H_{q,g}(t)/(|Q| \mathrm{Tr}(e^{-t\Delta_X}))$ to g satisfy a large-deviation principle with speed $1/t$ and rate $\hat{\ell}_X^2/4$ (Theorem 14.8).

For flat connections on trivial bundles with compact Lie structure group and positive constant killing, Le Jan [65, §3, proof of Proposition 5] identifies Brownian-loop integrals of normalized holonomy characters minus 1 with differences of normalized twisted and scalar zeta derivatives at 0. His Remark 2 relates conjugacy separability to the separation of free homotopy classes by holonomy and notes that every finite-dimensional unitary representation of Higman's group is trivial. Wang–Xie–Yu prove finite-quotient approximation of delocalized traces for equivariant bounded smooth ℓ^1 -summable kernels, along sequences of finite-index normal subgroups distinguishing the chosen nontrivial conjugacy class [98, Lemmas 2.4 and 2.6]. For a smooth closed connected three-manifold, Li–Wu–Zhou prove that the distributional trace of $\sin(t\sqrt{\Delta})/\sqrt{\Delta}$ is not C^1 on any neighborhood of the shortest positive closed-geodesic length when that length is below the conjugate radius [66, Theorem 4.1].

1.5. Marked recognition. Fix $n \geq 1$. Let J be a compact connected Lie group with a positive bi-invariant metric and a homomorphism $\iota : P_n \rightarrow J$. For $X \in \mathfrak{p}_n$ near zero, let $I_J(X)$ and $e^{\Phi_J(X)}$ be the exponential rate and leading amplitude of the normalized heat trace for right translation by $\iota(\exp_{P_n} X)$, as in (16.1). Then

$$I_J(X) = \|d\iota X\|^2, \quad \Phi_J(X) = -\frac{1}{2} \log j_J(d\iota X),$$

where j_J is the Jacobian of the Lie exponential. Together with $\dim J$, these germs determine the metric adjoint P_n -module $\mathfrak{j}$ and $d\iota$, up to an orthogonal P_n -intertwiner (proof of Theorem 16.2). If

$$\begin{aligned} \dim J &= (2n+1)^2, & \text{jet}_0^2 I_J &= \text{jet}_0^2 I_{G_n}, & \text{Vol}(J) &\geq \text{Vol}(G_n), \\ \text{jet}_0^6(\Phi_J|_{\mathfrak{j}(\mathfrak{p}_n)}) &= \text{jet}_0^6(\Phi_{G_n}|_{\mathfrak{j}(\mathfrak{p}_n)}), \end{aligned} \tag{1.9}$$

then (J, ι) is isometrically isomorphic, with its marking, to (G_n, i_n) or (H_n, j_n) (Theorem 19.1). Besides dimension and the volume bound, nine coefficients of the central restriction of Φ_J and three central rate coefficients suffice, together with the rate coefficient along one fixed nonzero direction in $\mathfrak{su}(n)$ when $n \geq 2$: thirteen real local quantities, or twelve when $n = 1$. Three identities with nonnegative summands restrict the support of the central adjoint weights; the moment equations and the dimension then determine their multiplicities. The resulting marked metric Lie algebras and the periods of the marking reduce J to finite central quotients; the volume bound leaves precisely the two groups above.

Fix $P = \prod_{j=1}^r P_{n_j}$ and positive scales c_j . The scalar eigenspaces of $\prod_{j=1}^r (G_{n_j}, c_j Q_G)$, with their eigenvalues and right- P actions, have exactly the 2^r marked realizations

$$\prod_{j=1}^r (U_{j_j}, c_j Q_{J_j}), \quad J_j \in \{G_{n_j}, H_{n_j}\}, \tag{1.10}$$

among compact connected Lie groups with positive bi-invariant metrics (Theorem 19.6). Here Q_{J_j} is the corresponding reference form. These marked classes remain distinct when ranks or scales repeat.

1.6. Metrics on compact simple groups. For $m \geq 6$, equip $\text{SU}(m)$ with its trace metric. The nonisomorphic reference groups $G_1 = \text{U}(3)$ and $H_1 = \text{Sp}(1) \times \text{SO}(4)$ embed in a six-dimensional complex block by (22.1)–(22.2). Their Casimirs for the restricted trace form have identical spectra on every finite-dimensional complex $\text{SU}(m)$ -module (Theorem 20.1 and Lemma 22.2). Contracting the metric along either subgroup by the same factor $s \in (0, 1)$ gives the scalar Laplacian $C_{\text{SU}(m)} + (s^{-1} - 1)D_K$, where D_K is the subgroup Casimir.

Gordon–Sutton [44, Introduction] ask whether spectral isolation among naturally reductive left-invariant metrics on a fixed compact simple group can be strengthened to uniqueness or finiteness. Theorem 22.1 gives a negative answer to uniqueness on every $\text{SU}(m)$, $m \geq 6$. For each $0 < s < 1$, the two metrics above are scalar-isospectral, left-invariant, and normal homogeneous. Both are locally irreducible, with positive Ricci curvature and nonnegative sectional curvature. Their least Ricci eigenvalues differ, so they are not locally isometric.

For fixed integers $r \geq 1$ and $m \geq 6r$, the block metrics of Theorem 22.5 form a family $(g_{\omega, \varepsilon})$, with $\omega \in \{A, B\}^r$ and $0 < \varepsilon < 1$. Within this family, two metrics are scalar-isospectral if and only if their

parameters ε agree. Each spectral fiber therefore consists of exactly 2^r isometry classes, represented by metrics that are pairwise not locally isometric. Every C^∞ neighborhood of the bi-invariant trace metric on $SU(m)$ contains one of these 2^r -member sets.

Part I. Spectral identities and descent

With the reference metrics, the scalar eigenspaces of G_n and H_n are equivalent $L_n \times P_n$ -modules (Theorem 3.2). For a closed subgroup of $P_n \times P_n$, equivalence of the first positive eigenspaces implies equivalence at every eigenvalue (Theorem 3.9). Tensor products of the right- P_n equivalences restrict to every closed $K \leq \prod_j P_{n_j}$, giving equal canonical rough spectra on the corresponding normal homogeneous quotients for every finite-dimensional unitary K -module (Theorem 4.3).

2. REFERENCE GROUPS, METRICS, AND WEYL LATTICES

All Lie groups have the indicated compact global forms, and all Laplacians are nonnegative. Haar measure has mass one when it defines a Hilbert space; volumes refer to Riemannian measure. For $n \geq 1$, set

$$N = 2n + 1, \quad G_n = U(N), \quad H_n = \mathrm{Sp}(n) \times \mathrm{SO}(2n + 2), \quad P_n = U(n) \times U(1).$$

If $r(A)$ is the realification of a unitary matrix and $c(A) = \mathrm{diag}(A, \bar{A})$, the two embeddings are

$$i_n(A, z) = \mathrm{diag}(A, \bar{A}, z), \quad j_n(A, z) = (c(A), r(A \oplus z)). \quad (2.1)$$

Indeed $c(A)$ preserves the alternating form with matrix $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, and $\det_{\mathbb{R}} r(A) = |\det_{\mathbb{C}} A|^2 = 1$. The defining blocks recover A, z , so both maps are injective.

The invariant inner products are

$$Q_G = -\mathrm{Re} \mathrm{Tr}_{\mathbb{C}}, \quad Q_C = -\frac{1}{2} \mathrm{Re} \mathrm{Tr}_{\mathbb{C}}, \quad Q_D = -\frac{1}{2} \mathrm{Tr}_{\mathbb{R}}, \quad (2.2)$$

where the traces are evaluated on products of Lie algebra elements. The forms Q_C on $\mathfrak{sp}(m)$ and Q_D on $\mathfrak{so}(2m)$ refer to the defining representations of complex and real dimension $2m$, respectively. These positive adjoint-invariant forms define bi-invariant metrics. On H_n put $Q_H = Q_C \oplus Q_D$.

Lemma 2.1. *The two induced metrics on $\mathfrak{p}_n$ are equal:*

$$Q_P((X, it), (Y, is)) = -2 \mathrm{Re} \mathrm{Tr}_{\mathbb{C}}(XY) + ts. \quad (2.3)$$

Proof. For $u = (X, it)$ and $v = (Y, is)$ in $\mathfrak{p}_n$,

$$\begin{aligned} Q_G(di_n u, di_n v) &= -\mathrm{Re} \mathrm{Tr}_{\mathbb{C}}(XY + \bar{X}\bar{Y}) + ts = -2 \mathrm{Re} \mathrm{Tr}_{\mathbb{C}}(XY) + ts, \\ Q_H(dj_n u, dj_n v) &= -\frac{1}{2} \mathrm{Re} \mathrm{Tr}_{\mathbb{C}}(XY + \bar{X}\bar{Y}) - \frac{1}{2} \mathrm{Tr}_{\mathbb{R}} r(XY) + ts \\ &= -2 \mathrm{Re} \mathrm{Tr}_{\mathbb{C}}(XY) + ts. \end{aligned}$$

Here $\mathrm{Tr}_{\mathbb{R}} r(XY) = 2 \mathrm{Re} \mathrm{Tr}_{\mathbb{C}}(XY)$. □

Write $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. The lattice-coordinate identification $\mathbb{R}/\mathbb{Z} \simeq \mathbb{T}$ is $[x] \mapsto [2\pi x]$. Put

$$c_m(x) = \mathrm{diag}(e^{ix_1}, \dots, e^{ix_m}, e^{-ix_1}, \dots, e^{-ix_m}), \quad d_m(x) = \mathrm{diag}(R(x_1), \dots, R(x_m)).$$

Here $R(x)$ is rotation through x in an oriented real plane. The differentials of these torus parametrizations have squared norm $\sum x_j^2$ for both Q_C and Q_D . For $U(N)$, $Q_G(i \mathrm{diag} x, i \mathrm{diag} x) = \sum x_j^2$. Each displayed maximal torus has period lattice $(2\pi\mathbb{Z})^m$ and character lattice $\mathbb{Z}^m$, with $m = N$ for $U(N)$.

2.1. Highest weights and the scalar Casimir. In orthonormal dual coordinates e_j , the positive roots and dominant integral weights are

J	positive roots	dominant weights
$U(N)$	$e_i - e_j \ (i < j)$	$\lambda_1 \geq \dots \geq \lambda_N, \ \lambda_i \in \mathbb{Z}$
$Sp(m)$	$e_i \pm e_j \ (i < j), \ 2e_i$	$\lambda_1 \geq \dots \geq \lambda_m \geq 0, \ \lambda_i \in \mathbb{Z}$
$SO(2m)$	$e_i \pm e_j \ (i < j)$	$\lambda_1 \geq \dots \geq \lambda_{m-1} \geq \lambda_m , \ \lambda_i \in \mathbb{Z}$

(2.4)

The last row has $m \geq 2$; the characters of $SO(2)$ are e^{ikx} , $k \in \mathbb{Z}$. Half-integral spin weights do not belong to the character lattice of the displayed orthogonal group. The half-sums of positive roots are

$$\rho_A = (n, n-1, \dots, -n), \quad \rho_{C,m} = (m, m-1, \dots, 1), \quad \rho_{D,m} = (m-1, m-2, \dots, 0).$$

For $m = 1$, put $\rho_{D,1} = 0$. The weight parametrizations and Weyl character formula are those of [20, Chapter V, §6, pp. 218–222; Chapter VI, Theorem 1.7, p. 242, and Proposition 2.6, p. 252]. The Peter–Weyl decomposition is [37, Theorem 35.9]. For products, weights and ρ -vectors are concatenated, and the squared norms are added.

Lemma 2.2. *Let $J = U(N)$, $Sp(m)$, or $SO(2m)$, with the metric in (2.2), or a product of these groups with the product metric. On the matrix-coefficient space of an irreducible representation of highest weight λ , the scalar Laplacian acts by*

$$c_J(\lambda) = |\lambda + \rho_J|^2 - |\rho_J|^2. \quad (2.5)$$

Proof. The nonnegative scalar Laplacian is the Casimir operator $-\sum_a Z_a^2$ for an orthonormal basis of invariant fields. Apply [90, Propositions 9.A.1–9.A.2 and equation (9.A.21)], with the opposite sign convention for the Laplacian. The torus norm computations above identify the dual metric with the displayed Euclidean norm. This norm includes the central direction of $U(N)$; for $SO(2)$ the root system is empty. $\square$

For $x, y \in J$, $t > 0$, and $F_{x,y}(g) = xgy^{-1}$, let

$$B_J(t; x, y) = \text{Tr}(F_{x,y}^* e^{-t\Delta_J}).$$

The pullback operator on the coefficient space of π_λ has trace $\chi_\lambda(x)\overline{\chi_\lambda(y)}$: the left and right matrix indices transform by $\pi_\lambda(x)$ and $\pi_\lambda(y)^{-1}$. Hence

$$B_J(t; x, y) = \sum_{\lambda \text{ dominant}} \chi_\lambda(x)\overline{\chi_\lambda(y)} e^{-tc_J(\lambda)}. \quad (2.6)$$

At $x = 1$ the first character is $\dim V_\lambda$; at $x = y = 1$ the multiplicity is $(\dim V_\lambda)^2$. The Weyl dimension formula [37, Proposition 28.1] gives $\dim V_\lambda \leq C(1 + |\lambda + \rho_J|)^a$ for a fixed integer a , and unitarity gives $|\chi_\lambda(x)| \leq \dim V_\lambda$. On $0 < \varepsilon \leq t \leq T$, the summands in (2.6) are bounded by $C_{\varepsilon,T} e^{-\varepsilon|\lambda + \rho_J|^2/2}$. The integer Gaussian sum is finite. Thus the series converges uniformly in x, y on compact positive time intervals, including singular torus arguments. Bounded eigenvalue intervals contain only finitely many highest weights.

Lemma 2.3. *Let $\mathcal{E} \subset [0, \infty)$ meet every bounded interval in a finite set. Let $a_E \in \mathbb{C}$ and $t_0 \in \mathbb{R}$, and suppose $\sum_{E \in \mathcal{E}} |a_E| e^{-t_0 E} < \infty$. If $\sum_E a_E e^{-tE} = 0$ for every $t \geq t_0$, then every a_E is zero.*

Proof. Apply the uniqueness theorem for general Dirichlet series [46, Chapter II, §5, Theorem 6] to $\sum_E (a_E e^{-t_0 E}) e^{-sE}$, where $s = t - t_0$. The exponents are distinct and locally finite, the series converges absolutely at $s = 0$, and its sum vanishes for every $s \geq 0$. Finite sums are included by adjoining larger exponents with zero coefficients. $\square$

Lemma 2.4. *Let D be a Hausdorff topological group. Suppose $\mathcal{H}_i = \widehat{\bigoplus}_{\eta} E_{i,\eta}$ are orthogonal sums of finite-dimensional continuous unitary D modules and $S_{\eta} : E_{1,\eta} \rightarrow E_{2,\eta}$ are equivariant unitaries. Their direct sum is a unitary $S : \mathcal{H}_1 \rightarrow \mathcal{H}_2$. For every complex-valued function f on the common index set, it intertwines the diagonal operators $f_i u = \sum_{\eta} f(\eta) u_{\eta}$, with their maximal domains*

$$\mathcal{D}_{i,f} = \left\{ u = \sum_{\eta} u_{\eta} \in \mathcal{H}_i : \sum_{\eta} |f(\eta)|^2 \|u_{\eta}\|^2 < \infty \right\}.$$

It intertwines averaging over each compact subgroup of D and the generators of continuous one-parameter subgroups, with their domains. If $\mathcal{H}_i = L^2(M_i)$ for closed Riemannian manifolds M_i and a common nonnegative diagonal function gives their scalar Laplacians, S induces isomorphisms $H^s(M_1) \rightarrow H^s(M_2)$ for every $s \in \mathbb{R}$ and restricts to an isomorphism $C^\infty(M_1) \rightarrow C^\infty(M_2)$. The same conclusions hold for finite tensor products, with diagonal functions of the tuples of factor labels.

Proof. Let $P_{i,\eta}$ be the orthogonal projection onto $E_{i,\eta}$. The equalities $SP_{1,\eta} = P_{2,\eta}S$ and

$$\sum_{\eta} |f(\eta)|^2 \|S_{\eta} u_{\eta}\|^2 = \sum_{\eta} |f(\eta)|^2 \|u_{\eta}\|^2$$

give $SD_{1,f} = \mathcal{D}_{2,f}$ and $Sf_1 = f_2S$; the inverse is $\widehat{\bigoplus}_{\eta} S_{\eta}^{-1}$. The displayed domain is the spectral-calculus domain [92, §3.1, equation (3.25)]. Equivariance gives $S \int_K U_1(k) dk = (\int_K U_2(k) dk)S$ for every compact subgroup $K \leq D$, with Haar probability measure. For a continuous one-parameter subgroup, $SU_1(t)S^{-1} = U_2(t)$. Stone's theorem [92, Theorems 5.1–5.2] therefore gives the same conjugacy for the generators, including their domains. For each vector, only countably many summands occur, so these applications may be restricted to invariant separable Hilbert sums.

For the Laplacians, S preserves $\sum_{\lambda} (1 + \lambda)^s \|u_{\lambda}\|^2$. The elliptic estimate [35, Theorem 15.7], applied to $(1 + \Delta_{M_i})^k$, identifies these completions with $H^{2k}(M_i)$ for integers $k \geq 0$; interpolation and duality give every real s . The extensions agree on intersections, and $\bigcap_s H^s(M_i) = C^\infty(M_i)$. For finite tensor products, the coordinate projections and the unitaries are tensor products of the factor projections and unitaries, so the same domain formula applies to the tuples of labels. $\square$

2.2. Global periods of the markings. The fundamental groups and the maps from the maximal tori are given by [20, Chapter V, Theorem 7.1, p. 223]. With determinant winding coordinates on $\pi_1(P_n) = \mathbb{Z}^2$, the two maps induced by (2.1) are

$$(m, l) \mapsto l \quad \text{in } \pi_1(G_n) = \mathbb{Z}, \quad (m, l) \mapsto m + l \pmod{2} \quad \text{in } \pi_1(H_n) = \mathbb{Z}/2. \quad (2.7)$$

The determinant of $i_n(A, z)$ is z . The symplectic factor is simply connected. A determinant generator of $U(n)$ realifies to a full rotation in one real coordinate plane, and the z generator rotates the last plane. Each represents the nonzero element of $\pi_1(\mathrm{SO}(2n+2))$, including $\mathrm{SO}(4)$. In particular both maps in (2.7) are onto.

For finitely many ranks n_j and scales $s_j > 0$, write

$$P = \prod_{j=1}^r P_{n_j}, \quad J_I = \prod_{j \notin I} G_{n_j} \times \prod_{j \in I} H_{n_j}, \quad I \subseteq \{1, \dots, r\}.$$

The metric on the j th factor is s_j times its reference metric. The induced product metric on P is independent of I .

3. INTRINSIC SPECTRAL IDENTITY, BILATERAL DEFECT, AND MAXIMAL EQUIVARIANCE

Fix $n \geq 1$ and put $N = 2n + 1$, $G = G_n$, $H = H_n$, and $P = P_n$. The action on functions associated with two markings is

$$T_{p,q}f(g) = f(\iota_-(p)^{-1}g\iota_+(q)), \quad (p, q) \in P \times P.$$

On the coefficient space $V_\lambda \otimes V_\lambda^*$ its character is

$$\overline{\chi_\lambda(\iota_-(p))} \chi_\lambda(\iota_+(q)).$$

On G both markings are i_n ; on H both are j_n . Put

$$J'_n = \mathrm{SO}(2n) \times \mathrm{Sp}(n+1),$$

with its reference product metric. Its dimension is N^2+2 , and $\pi_1(J'_n) = \pi_1(\mathrm{SO}(2n))$, equal to $\mathbb{Z}$ for $n = 1$ and C_2 for $n \geq 2$. In the real coordinates of the rotation two-planes, put $s = \mathrm{diag}(1, -1, 1, \dots, 1) \in \mathrm{O}(2n)$ and $\sigma(g) = sgs^{-1}$. Its two markings are

$$k_-(A, z) = (\sigma(r(A)), c(A \oplus z)), \quad k_+(A, z) = (r(A), c(A \oplus z)).$$

For $u = (X, it)$ and $v = (Y, it')$, the trace forms give

$$Q_{J'_n}(dk_\pm u, dk_\pm v) = -\mathrm{Re} \mathrm{Tr}_{\mathbb{C}}(XY) - \mathrm{Re} \mathrm{Tr}_{\mathbb{C}}(XY) + tt' = Q_P(u, v).$$

For $n = 1$, σ is inversion on $\mathrm{SO}(2)$. The map $(g, h) \mapsto (\sigma(g), h)$ exchanges these two markings and is an isometry.

For $R = G, H, J'_n$, let $M_R(\lambda)$ be the multiset of the absolute coordinates of $\lambda + \rho_R$, with both factors concatenated on a product. For a multiset M of N nonnegative integers, set

$$\mathcal{V}_R(M) = \bigoplus_{M_R(\lambda)=M} V_\lambda \otimes V_\lambda^* \subset L^2(R). \quad (3.1)$$

This is a finite-dimensional $P \times P$ module: a fixed absolute multiset admits only finitely many signed ordered tuples. An empty sum is zero. The multiplicities in (3.1) are the full matrix-coefficient multiplicities.

3.1. Finite Weyl determinants. Choose $D \in \mathbb{Z}_{\geq 0}$ and independent variables $a_0, \dots, a_D$. Put

$$\theta(t) = \sum_{k=-D}^D a_{|k|} e^{ikt}, \quad E(t, s) = \theta(t-s) + \theta(t+s), \quad O(t, s) = \theta(t-s) - \theta(t+s).$$

For a group R , let $F_R(x, y)$ be the finite sum

$$\sum_{\substack{\lambda \text{ dominant} \\ |\lambda_j + \rho_{R,j}| \leq D}} \chi_\lambda(x) \overline{\chi_\lambda(y)} \prod_j a_{|\lambda_j + \rho_{R,j}|}.$$

The bar conjugates the character value and leaves the formal variables fixed. Define

$$\delta_{A,N}(x) = \prod_{i < j} 2i \sin((x_i - x_j)/2), \quad (3.2)$$

$$\delta_{D,m}(x) = \prod_{i < j} 2(\cos x_i - \cos x_j), \quad (3.3)$$

$$\delta_{C,m}(x) = (2i)^m \prod_j \sin x_j \delta_{D,m}(x). \quad (3.4)$$

Here $\delta_{D,1} = 1$. These are the Weyl denominators $\prod_{\alpha > 0} 2i \sin(\alpha(x)/2)$ for the positive roots in (2.4) [37, Corollary 25.5, p. 112]. The Weyl character formula [20, Chapter VI, Theorem 1.7, p. 242] gives the finite identities

$$\begin{aligned} F_{\mathrm{U}(N)}(x, y) &= \frac{\det[\theta(x_i - y_j)]}{\delta_{A,N}(x) \delta_{A,N}(y)}, \\ F_{\mathrm{Sp}(m)}(x, y) &= \frac{\det O[x, y]}{\delta_{C,m}(x) \delta_{C,m}(y)}, \\ F_{\mathrm{SO}(2m)}(x, y) &= \frac{\det E[x, y] + \det O[x, y]}{2\delta_{D,m}(x) \delta_{D,m}(y)}. \end{aligned} \quad (3.5)$$

These equalities hold at regular torus arguments.

For the unitary formula, finite Cauchy–Binet gives a sum over $k_1 > \dots > k_N$ in $[-D, D] \cap \mathbb{Z}$, with terms

$$\left(\prod_j a_{|k_j|} \right) \det[e^{ik_i x_j}] \overline{\det[e^{ik_i y_j}]}$$

Reversing the selected columns changes both determinants by the same sign. Thus each decreasing tuple occurs once, and division by the Weyl denominators gives the first formula. For the symplectic formula,

$$O(t, s) = 4 \sum_{k=1}^D a_k \sin(kt) \sin(ks).$$

Cauchy–Binet sums over $k_1 > \dots > k_m > 0$. The conjugate alternants $\det[2i \sin(k_i x_j)]$ contribute 4^m , giving the second formula.

For $m \geq 2$, the orthogonal Weyl group consists of signed permutations with an even number of sign changes. Its alternant vanishes when two coordinates have the same absolute value. A regular orbit has a unique representative $k_1 > \dots > k_{m-1} > |k_m|$ and trivial stabilizer, also when $k_m = 0$: changing the sign of the zero coordinate alone is not in the Weyl group.

For $k_1 > \dots > k_m > 0$, put

$$C_k(x) = \det[2 \cos(k_i x_j)], \quad S_k(x) = \det[2i \sin(k_i x_j)].$$

The alternants for the two signs of the last coordinate are $(C_k + S_k)/2$ and $(C_k - S_k)/2$. The sum of their bilateral products is

$$\frac{1}{2} (C_k(x) C_k(y) + S_k(x) \overline{S_k(y)}).$$

If $k_m = 0$, there is one alternant, $C_k/2$, and its product is $C_k(x) C_k(y)/4$. On the other hand,

$$E(t, s) = 2a_0 + \sum_{k=1}^D a_k (2 \cos kt) (2 \cos ks), \quad O(t, s) = \sum_{k=1}^D a_k (2i \sin kt) \overline{(2i \sin ks)}.$$

In Cauchy–Binet for E , the constant row 2 has weight $a_0/2$. Thus $(\det E + \det O)/2$ has precisely the preceding coefficients, including those with a zero coordinate. Division by the two Weyl denominators proves the orthogonal formula.

For $m = 1$, the last line of (3.5) is $\frac{1}{2}(E(x, y) + O(x, y)) = \theta(x - y)$, the full character sum of $\mathrm{SO}(2)$.

Lemma 3.1. *For square matrices A, B of order n , column vectors c, d, e, f , and scalars g, h , put*

$$M = \begin{pmatrix} A & B & c \\ B & A & d \\ e^T & f^T & g \end{pmatrix}, \quad E = \begin{pmatrix} A+B & c+d \\ (e+f)^T & g+h \end{pmatrix}, \quad O = \begin{pmatrix} A-B & c-d \\ (e-f)^T & g-h \end{pmatrix}.$$

Then

$$2 \det M = \det(A-B) \det E + \det(A+B) \det O. \quad (3.6)$$

Proof. The simultaneous orthogonal change $(x, y) \mapsto (x+y, x-y)/\sqrt{2}$ in the first $2n$ rows and columns gives

$$\begin{pmatrix} A+B & 0 & (c+d)/\sqrt{2} \\ 0 & A-B & (c-d)/\sqrt{2} \\ (e+f)^T/\sqrt{2} & (e-f)^T/\sqrt{2} & g \end{pmatrix}.$$

The bordered determinant formula [95, §3.7.1, Theorem 3.9, p. 47] gives its determinant as

$$g \det(A+B) \det(A-B) - \frac{1}{2} \det(A-B) (e+f)^T \operatorname{adj}(A+B) (c+d) - \frac{1}{2} \det(A+B) (e-f)^T \operatorname{adj}(A-B) (c-d).$$

Expanding the two bordered determinants on the right of (3.6) gives twice this expression; the two terms containing h cancel. The adjugate formulas are polynomial identities and include singular matrices. $\square$

The odd symmetric Toeplitz factorization in [5, Lemma 5.2, equation (5.1)] is given in [95, §4.5.2, p. 88, equation (4.5.9)]; it yields the dimension-data identity of [5, Theorem 1.5(1)]. Highest-weight versions of these factorizations, with opposite paired entries, are given in [101, Proposition 2.5] and [102, arXiv version 2, Proposition 2.5]. Classical-group heat determinants are treated in [28, §7.2].

3.2. The intrinsic identity. Let $L_n = U(n)$ act on the left through $A \mapsto i_n(A, 1)$ or $A \mapsto j_n(A, 1)$.

Theorem 3.2 (The intrinsic identity). *For every multiset M , the spaces $\mathcal{V}_G(M)$ and $\mathcal{V}_H(M)$ are equivalent unitary $L_n \times P_n$ modules. In particular, corresponding scalar eigenspaces are equivalent as right- P_n modules. One unitary $S : L^2(G_n) \rightarrow L^2(H_n)$ fixes the normalized constants and intertwines these actions and the Laplacians, including their self-adjoint domains. A common positive scaling of the metrics preserves these intertwining.*

Proof. Put $x = (u, -u, 0)$ and $y = (v, -v, b)$. The last row of $O[(u, 0), (v, b)]$ is zero. Lemma 3.1, with $A_{ij} = \theta(u_i - v_j)$, $B_{ij} = \theta(u_i + v_j)$, $c_i = \theta(u_i - b)$, $d_i = \theta(u_i + b)$, $e_j = f_j = \theta(v_j)$, and $g = h = \theta(b)$, gives

$$\det[\theta(x_i - y_j)] = \frac{1}{2} \det O[u, v] \det E[(u, 0), (v, b)]. \quad (3.7)$$

The denominators satisfy

$$\delta_{A,N}(u, -u, a) = (-1)^{n(n+1)/2} \delta_{C,n}(u) \delta_{D,n+1}(u, a). \quad (3.8)$$

For $\tilde{x} = (u_1, -u_1, \dots, u_n, -u_n, a)$,

$$\begin{aligned} \delta_{A,N}(\tilde{x}) &= (2i)^n \prod_i \sin u_i \prod_{i < j} 4(\cos u_i - \cos u_j)^2 \prod_i 2(\cos a - \cos u_i) \\ &= (-1)^n \delta_{C,n}(u) \delta_{D,n+1}(u, a). \end{aligned}$$

The permutation from $\tilde{x}$ to $(u, -u, a)$ has sign $(-1)^{n(n-1)/2}$, proving (3.8). The two signs cancel in the bilateral quotient.

Equations (3.5)–(3.8) equate the finite bilateral character polynomials of G, H at the indicated arguments. The regular parameters are dense: the excluded equations are finitely many proper angle hyperplanes, also with the left final angle zero. Each coefficient is a finite character sum, so the equality extends by continuity to all parameters. Independent unitary diagonalization extends it to every left L_n and right P_n argument. For a given M , choose $D \geq \max M$ and extract the coefficient of $\prod_{m \in M} a_m$, with repetitions. Simultaneous inversion gives equal characters of $\mathcal{V}_G(M)$ and $\mathcal{V}_H(M)$ for the action T . Character orthogonality on the compact group [37, Corollary 35.12] gives equal irreducible multiplicities, hence a unitary equivalence.

The two root norms are

$$|\rho_A|^2 = |\rho_{C,n}|^2 + |\rho_{D,n+1}|^2 = 2 \sum_{j=1}^n j^2. \quad (3.9)$$

Thus each multiset M gives the same Laplace eigenvalue on both sides. Choose the finite-dimensional unitaries and apply Lemma 2.4. The zero eigenspaces are the constants, whose normalized generators can be matched. Scaling by $s > 0$ replaces both Laplacians by $s^{-1}\Delta$. $\square$

3.3. The bilateral correction. Write $\alpha(A, z; B, w) = z$, $\beta(A, z; B, w) = w$ and set

$$\omega = \alpha\beta^{-1} + \alpha^{-1}\beta - \alpha\beta - \alpha^{-1}\beta^{-1}. \quad (3.10)$$

At $z = e^{ia}$, $w = e^{ib}$, its value is $4 \sin a \sin b$.

Theorem 3.3. *For every multiset M ,*

$$[\mathcal{V}_G(M)] - [\mathcal{V}_H(M)] = \omega[\mathcal{V}_{J'_n}(M)] \quad \text{in } R(P \times P). \quad (3.11)$$

Proof. For $x = (u, -u, a)$, $y = (v, -v, b)$, Lemma 3.1 gives

$$\det[\theta(x_i - y_j)] = \frac{1}{2}(\det O_n \det E_{n+1} + \det E_n \det O_{n+1}), \quad (3.12)$$

where $E_n = E[u, v]$, $O_n = O[u, v]$, and the order $n + 1$ matrices have arguments $(u, a), (v, b)$. Put

$$\mathcal{D} = \delta_{C,n}(u) \overline{\delta_{C,n}(v)} \delta_{D,n+1}(u, a) \delta_{D,n+1}(v, b).$$

By (3.5) and (3.8), subtraction gives

$$F_G - F_H = \frac{(\det E_n - \det O_n) \det O_{n+1}}{2\mathcal{D}}. \quad (3.13)$$

Reflection in one orthogonal angle leaves $E_n, \delta_{D,n}$ unchanged and negates one row of O_n . Hence the numerator in (3.13), divided by

$$2\delta_{D,n}(u) \delta_{D,n}(v) \delta_{C,n+1}(u, a) \overline{\delta_{C,n+1}(v, b)},$$

is $F_{J'_n}$. This holds for $n = 1$ as well, since the reflected circle character sum is $\theta(u + v)$. Finally

$$\frac{\delta_{C,n+1}(u, a) \delta_{D,n}(u)}{\delta_{C,n}(u) \delta_{D,n+1}(u, a)} = 2i \sin a. \quad (3.14)$$

The quotient of the two bilateral denominators is therefore $(2i \sin a) \overline{(2i \sin b)} = 4 \sin a \sin b$. The finite polynomial identity holds on a dense regular set. Continuity and independent diagonalization extend it to $P \times P$. Since $\omega(p^{-1}, q^{-1}) = \omega(p, q)$, simultaneous inversion gives the identity for the action T . For each M , choose $D \geq \max M$ and extract the coefficient of $\prod_{m \in M} a_m$. This proves (3.11). $\square$

Put $E^+ = \alpha\beta^{-1} \oplus \alpha^{-1}\beta$ and $E^- = \alpha\beta \oplus \alpha^{-1}\beta^{-1}$. Equation (3.11) and complete reducibility give

$$\mathcal{V}_G(M) \oplus E^- \otimes \mathcal{V}_{J'_n}(M) \simeq \mathcal{V}_H(M) \oplus E^+ \otimes \mathcal{V}_{J'_n}(M) \quad (3.15)$$

as unitary $P_n \times P_n$ modules.

3.4. The common central algebra. For $1 \leq r \leq N$, set

$$p_r(x) = \sum_{a=1}^N x_a^{2r}, \quad \kappa_r = 2 \sum_{a=1}^n a^{2r}, \quad d_r = (n+1)^{2r} - n^{2r}.$$

The Harish–Chandra isomorphism defines a central differential operator $Z_{R,r}$ on $C^\infty(R)$, of order at most $2r$, with highest-weight eigenvalue

$$p_r(\lambda + \rho_R) - p_r(\rho_R), \quad R = G, H, J'_n. \quad (3.16)$$

This convention includes reductive algebras [38, Theorem 14.1(ii)–(v) and Remark 14.2]. In particular $Z_{R,1} = \Delta_R$, $p_r(\rho_G) = p_r(\rho_H) = \kappa_r$, and $p_r(\rho_{J'_n}) = \kappa_r + d_r$. Put $d = (d_1, \dots, d_N)$, and let $E_R(\eta)$ be the joint eigenspace with eigenvalue vector $\eta \in \mathbb{R}^N$, with $E_R(\eta) = 0$ when η is not a joint eigenvalue.

Theorem 3.4. *The operators $Z_{R,r}$ are essentially self-adjoint, and their closures strongly commute. With the same notation for these closures, the joint eigenspaces of $(Z_{G,r})_{r=1}^N$ and $(Z_{H,r})_{r=1}^N$ are equivalent $L_n \times P_n$ modules. The unitary in Theorem 3.2 can intertwine all their joint Borel functional calculi, with their domains, and preserve smooth functions. For the full bilateral actions the joint eigenspaces satisfy*

$$E_G(\eta) \oplus E^- \otimes E_{J'_n}(\eta - d) \simeq E_H(\eta) \oplus E^+ \otimes E_{J'_n}(\eta - d). \quad (3.17)$$

Proof. Write $z_{R,r}(\lambda)$ for (3.16). Then

$$|z_{R,r}(\lambda)| \leq (c_R(\lambda) + |\rho_R|^2)^r + p_r(\rho_R) \leq C_r(1 + c_R(\lambda))^r.$$

For $f \in C^\infty(R)$, its Laplace spectral truncations converge to f in H^{2r} . Since $Z_{R,r}$ has order at most $2r$, their images converge to $Z_{R,r}f$ in L^2 . The displayed bound also gives convergence in the maximal diagonal graph norm. Thus the differential and diagonal expressions agree on $C^\infty(R)$. Finite spectral

sums form a graph core for the diagonal operator and belong to $C^\infty(R)$; hence the closures agree. The diagonal eigenvalues are real and the common spectral projections commute, proving essential self-adjointness and strong commutation.

Newton's identities

$$e_0 = 1, \quad r e_r = \sum_{h=1}^r (-1)^{h-1} e_{r-h} p_h$$

determine the elementary symmetric functions of $x_1^2, \dots, x_N^2$ from the first N power sums. Thus the joint labels determine exactly the absolute-coordinate multiset. Theorem 3.2 and (3.15) give the joint-space isomorphisms, with the shift $\eta - d$ on J'_n . Lemma 2.4 gives the joint functional-calculus domains and preservation of smooth functions. $\square$

Corollary 3.5. *Let K be a closed subgroup of a finite product $P = \prod_i P_{n_i}$, and let τ be a finite-dimensional unitary K module. Let K act on $L^2(J_I)$ through the right markings and on $L^2(J_I) \otimes \tau$ diagonally. On the spaces $(L^2(J_I) \otimes \tau)^K$, all central operators $Z_{J_i, i, r}$, indexed by the factors and $1 \leq r \leq 2n_i + 1$, have the same joint spectral multiplicities for every binary choice I . The intertwiners preserve every joint functional-calculus domain. Write J_i for the i th factor of J_I . For fixed positive scales s_i , common to all binary choices, the same assertions hold for the metrics $s_i Q_{J_i}$ and the operators $Z_{i, r}^{(s_i)} = s_i^{-r} Z_{i, r}$.*

Proof. Tensor the joint unitaries of Theorem 3.4 with 1_τ and restrict to the K invariants by Lemma 2.4. In orthonormal dual Cartan coordinates, the change $Q_{J_i} \mapsto s_i Q_{J_i}$ multiplies each coordinate by $s_i^{-1/2}$. Both p_r and its value at ρ_{J_i} are multiplied by s_i^{-r} , giving the stated normalization and the same joint labels on both sides. $\square$

Under the displayed common Cartan coordinates, the intersection of the two Harish–Chandra invariant polynomial algebras is exactly

$$\mathbb{C}[x]^{S_N} \cap \mathbb{C}[x]^{W(C_n) \times W(D_{n+1})} = \mathbb{C}[p_1, \dots, p_N]. \quad (3.18)$$

Indeed $W(C_n)$ contains a single-coordinate sign change; conjugation by S_N gives every such sign change. The two groups therefore generate $W_{BC, N} = \{\pm 1\}^N \rtimes S_N$. Its invariants are the symmetric polynomials in the squares, which Newton's identities express in the p_r .

Corollary 3.6. *The inclusions in (3.18) into the two full invariant algebras are finite, of generic degrees*

$$2^N \quad \text{and} \quad 2 \binom{N}{n},$$

respectively. These count the full central characters above a common orbit represented by x with $x_a \neq 0$ and $x_a^2 \neq x_b^2$ for $a \neq b$.

Proof. The signed-permutation actions are faithful and finite. The polynomial ring $\mathbb{C}[x]$ is finite over the Noetherian ring $\mathbb{C}[x]^{W_{BC, N}}$, and its intermediate invariant subrings are finite over that ring [11, Corollary 1.2.2 and Theorem 1.3.1]. Their fraction-field degrees are the subgroup indices [11, Proposition 1.1.1]. Here the orders are $2^N N!$, $N!$, and $(2^n n!)(2^n(n+1)!)$. Division gives the stated degrees. The stipulated orbit has trivial stabilizer, so its subgroup orbits are counted by the same indices. $\square$

3.5. Heat traces and maximal subgroups. For $p = (A, z)$ and $q = (B, w)$ in P , the substitution $a_k = e^{-tk^2}$ in the finite identities, followed by $D \rightarrow \infty$, is justified by the Gaussian bound following (2.6). It gives

$$B_G(t; i(p), i(q)) - B_H(t; j(p), j(q)) = 4 \operatorname{Im} z \operatorname{Im} w e^{-Nt} B_{J'_n}(t; k_-(p), k_+(q)). \quad (3.19)$$

The shift is $d_1 = N$. All $Z_{R, r}$ are nonnegative. Among $2n + 1$ distinct integers, $\{-n, \dots, n\}$ minimizes every even power sum. Among m distinct positive integers the minimizing set is $\{1, \dots, m\}$, and among m distinct nonnegative integers it is $\{0, \dots, m - 1\}$. These are exactly the shifted tuples at zero highest

weight for the unitary, symplectic and orthogonal factors. For $\mathbf{t} = (t_1, \dots, t_N)$ with $t_1 > 0$ and $t_r \geq 0$ when $r > 1$,

$$0 \leq \exp\left(-\sum_r t_r z_{R,r}(\lambda)\right) \leq e^{-t_1 c_R(\lambda)}.$$

Thus the joint heat traces

$$B_R(\mathbf{t}; a, b) = \text{Tr}\left(F_{a,b}^* \exp\left(-\sum_r t_r Z_{R,r}\right)\right)$$

converge absolutely. Evaluating (3.17) at (p^{-1}, q^{-1}) gives

$$\begin{aligned} B_G(\mathbf{t}; i(p), i(q)) - B_H(\mathbf{t}; j(p), j(q)) \\ = \omega(p, q) e^{-\sum_r t_r d_r} B_{J'_n}(\mathbf{t}; k_-(p), k_+(q)). \end{aligned}$$

Lemma 3.7. *For an isometry F of a closed connected Riemannian manifold X and $t > 0$, $\text{Tr}(F^* e^{-t\Delta}) > 0$.*

Proof. The scalar heat kernel satisfies $K_t(x, y) > 0$ for $t > 0$ [45, Corollary 8.12, p. 229]. Hence

$$\text{Tr}(F^* e^{-t\Delta}) = \int_X K_t(Fx, x) dx > 0. \quad \square$$

Proposition 3.8. *The first positive eigenvalue of G_n and H_n is $N = 2n + 1$, with multiplicity $2N^2 + 2$. Its bilateral character difference is*

$$\chi_{E_N(G)} - \chi_{E_N(H)} = \omega. \quad (3.20)$$

The defining representations are the unique irreducible representations of $\text{Sp}(n)$ and $\text{SO}(2n + 2)$ with Casimir N ; their dimensions are $2n$ and $2n + 2$.

Proof. For G , the shifted tuple is a set S of N distinct integers. If k entries of $S_0 = \{-n, \dots, n\}$ are replaced, then

$$\sum_{s \in S} s^2 - \sum_{s \in S_0} s^2 \geq k((n+1)^2 - n^2) = kN.$$

Equality at N permits one removal from $\{-n, n\}$ and one insertion from $\{-n-1, n+1\}$. The four resulting representations are the defining representation, its dual, determinant and inverse determinant; their dimensions are $N, N, 1, 1$. For $\text{Sp}(n)$, the shifted coordinates are distinct positive integers. The same inequality compares them with $\{1, \dots, n\}$ and permits only $n \mapsto n+1$ at energy N . For $\text{SO}(2n+2)$, their absolute values are distinct nonnegative integers; comparison with $\{0, \dots, n\}$ again permits only that replacement. The zero last coordinate has a single sign. In both cases subtraction of ρ gives highest weight $(1, 0, \dots, 0)$. The first product eigenspace is the direct sum of these two defining coefficient spaces. This gives $(2n)^2 + (2n+2)^2 = 2N^2 + 2$. At energy N in (3.19), the auxiliary space is its one-dimensional constant space. Lemma 2.3 gives (3.20). $\square$

Theorem 3.9. *For a closed subgroup $K \leq P \times P$, the following conditions are equivalent:*

- (1) *all corresponding multiset components $\mathcal{V}_G(M), \mathcal{V}_H(M)$ are equivalent as K modules;*
- (2) *all corresponding scalar eigenspaces are equivalent as K modules;*
- (3) *their eigenspaces at N are equivalent as K modules;*
- (4) *their bilateral heat characters agree on K at one positive time;*
- (5) *$K \subset \ker \alpha^2$ or $K \subset \ker \beta^2$.*

The two kernels in the last condition are the maximal such closed subgroups. The maximal connected subgroups are $\ker \alpha$ and $\ker \beta$, of dimension $2n^2 + 1$. On the maximal torus $(\mathbb{T}^n \times \mathbb{T})^2$, their dimensions are $2n + 1$.

Proof. The first condition implies the second, and the second implies both the third and fourth. By (3.20), the third makes ω vanish on K . By positivity and (3.19), the fourth has the same consequence. Thus $K \subset \ker \alpha^2 \cup \ker \beta^2$. A group contained in a union of two subgroups is contained in one:

otherwise choose x in the second but outside the first, and y in the first but outside the second; xy belongs to neither. The last condition follows. It makes (3.11) zero after restriction, proving the first condition. The two kernels are incomparable. Every admissible subgroup lies in one of them, so they are maximal. A connected subgroup has connected image in $\{1, -1\}$ and hence trivial image. The dimensions are those of the indicated circle kernels. $\square$

Corollary 3.10. *For every closed $K \leq P \times P$,*

$$[E_\lambda(G)|_K] - [E_\lambda(H)|_K] = \omega|_K[E_{\lambda-N}(J'_n)|_K] \quad (3.21)$$

for all real λ , with missing spaces zero. In $R(K)$, the ideal generated by these differences is $(\omega|_K)$, generated already at the first positive eigenvalue.

Proof. Sum (3.11) over multisets of the prescribed scalar energy. The auxiliary energy is lower by N . Every difference is a multiple of $\omega|_K$, and the energy- N difference equals $\omega|_K$ by (3.20). $\square$

3.6. Finite quotients and their flat bundles. Let $q \in \mathbb{Z}_{\geq 1}$ and $u_i, v_i, A, B \in \mathbb{Z}$. For the cyclic torus subgroup generated by

$$\frac{2\pi}{q}(u_1, \dots, u_n, A; v_1, \dots, v_n, B),$$

the condition of Theorem 3.9 is

$$2A \equiv 0 \pmod{q} \quad \text{or} \quad 2B \equiv 0 \pmod{q}. \quad (3.22)$$

For odd q this is $A \equiv 0$ or $B \equiv 0$.

Let $K \leq P \times P$ be finite and act freely on both G and H by

$$k \cdot g = \iota_-(p)g\iota_+(q)^{-1}, \quad k = (p, q).$$

For $R = G, H$ and a finite-dimensional unitary representation τ of K , let $\mathcal{E}_{R,\tau} = (R \times V_\tau)/K$ over $K \backslash R$, where

$$k(g, v) = (k \cdot g, \tau(k)v).$$

The trivial connection on $R \times V_\tau$ descends to a flat connection on $\mathcal{E}_{R,\tau}$. Its smooth sections are the functions $f \in C^\infty(R, V_\tau)$ satisfying $f(k \cdot g) = \tau(k)f(g)$. The lifted connection Laplacian is $\Delta_R \otimes 1$, so

$$E_\lambda(\mathcal{E}_{R,\tau}) \simeq (E_\lambda(R) \otimes V_\tau)^K.$$

For irreducible τ ,

$$\dim(E_\lambda(R) \otimes V_\tau^*)^K = \dim \text{Hom}_K(V_\tau, E_\lambda(R)).$$

By Theorem 3.9, equality of the spectra of $\mathcal{E}_{G,\tau}$ and $\mathcal{E}_{H,\tau}$ for every irreducible τ , equality of their multiplicities at N for every irreducible τ , and condition (5) of that theorem are equivalent. For the cyclic groups above, the last condition is (3.22).

4. DESCENT, CANONICAL TENSORS, AND CUBIC CLIFFORD OPERATORS

Let J be a compact connected Lie group with a positive bi-invariant metric Q , let $K \leq J$ be closed, and put $\mathfrak{m} = \mathfrak{k}^\perp$. The metric on J/K is the normal quotient metric. The principal connection is $\text{pr}_\mathfrak{k}(g^{-1}dg)$.

Lemma 4.1. *For Riemannian probability measures, pullback identifies $L^2(J/K)$ with $L^2(J)^K$, and identifies their closed Dirichlet forms and Laplacians, including their domains. In particular,*

$$\Delta_J(f \circ p) = (\Delta_{J/K}f) \circ p. \quad (4.1)$$

Proof. The quotient map has smooth local sections, without a connectedness assumption on K [37, Theorem 3.1(i)]. Every component of a fibre is totally geodesic: for left-invariant fields, $\nabla_X Y = [X, Y]/2$, and $\mathfrak{k}$ is a Lie subalgebra. The submersion identity for minimal fibres [42, Lemma 1.3] gives (4.1).

Fibre integration and the horizontal isometry give

$$\|f \circ p\|_{L^2(J)} = \|f\|_{L^2(J/K)}, \quad \|d(f \circ p)\|_{L^2(J)} = \|df\|_{L^2(J/K)}$$

with probability measures. Haar averaging $P_K u(g) = \int_K u(gk) dk$ is an orthogonal projection and preserves the Dirichlet form decomposition into invariant functions and their orthogonal complement. It is a contraction for the form norm. Averaging smooth approximations therefore makes smooth invariant functions dense in the invariant form domain; these functions descend through the local sections. Thus pullback identifies the closed forms and their domains. Equality of the associated self-adjoint operators then gives equality of their operator domains. $\square$

Proposition 4.2. *Let a compact Lie group D act freely and isometrically on a closed Riemannian manifold J . Let $Q = J/D$ have the quotient metric, and let $v(q)$ be the Riemannian volume of the orbit above q . The restriction of Δ_J to invariant functions is unitarily equivalent to*

$$L_v = \Delta_Q - \langle \text{grad} \log v, \text{grad}(\cdot) \rangle \quad \text{on } L^2(Q, v dv_Q), \quad (4.2)$$

including the self-adjoint domains. Consequently, for every common free action of $D \leq L_n \times P_n$ on G_n, H_n , these weighted quotient operators have equal spectra. If both orbit-volume functions are constant, the ordinary quotient Laplacians are isospectral.

Proof. With Riemannian measure on J , a constant normalization only rescales the unitary identification. The free compact action gives a smooth principal bundle. In local bundle coordinates the vertical metric varies smoothly, so v is smooth and positive. Fibre integration and the quotient metric give, for smooth f, g on Q ,

$$\begin{aligned} \int_J (f \circ p) \overline{g \circ p} dv_J &= \int_Q f \bar{g} v dv_Q, \\ \int_J \langle d(f \circ p), d(\bar{g} \circ p) \rangle dv_J &= \int_Q \langle df, d\bar{g} \rangle v dv_Q. \end{aligned}$$

Integration by parts identifies the latter form with $-v^{-1} \text{div}(v \text{grad} f)$, which is (4.2). Haar averaging of smooth approximations gives a core for the invariant closed form. The displayed identities therefore identify the closed forms and their associated self-adjoint operators, with their domains. Restrict the $L_n \times P_n$ -equivariant unitary of Theorem 3.2 to the D invariants, adjusting the constant measure normalizations if necessary. This gives the weighted spectral equivalence. For constant v , the first-order term vanishes. $\square$

Theorem 4.3. *For every closed $K \leq \prod_j P_{n_j}$, the manifolds $M_I = J_I/K$ are scalar-isospectral. For every fixed finite-dimensional unitary K module W , their canonical rough connection Laplacians on $J_I \times_K W$ are isospectral. The unitaries preserve operator domains and commute with fibre intertwiners and finite direct sums. If K^0 acts trivially on W , these connections are flat and their holonomy is trivial on loops that lift to loops in $J_I/K^0 \rightarrow J_I/K$.*

Proof. Tensoring the intertwiners of Theorem 3.2 gives a unitary $S_{IL} : L^2(J_I) \rightarrow L^2(J_L)$ commuting with the whole right action of P and intertwining the scalar Laplacians with their domains. Each product eigenspace is the finite orthogonal sum of the tensor products with the prescribed sum of eigenvalues. Haar averaging over K commutes with S_{IL} . Lemma 4.1 gives scalar isospectrality.

Write the action on W as τ . A section is represented by $f(gk) = \tau(k)^{-1} f(g)$. For $R_X f(g) = \frac{d}{dt}|_0 f(g \exp(tX))$ and an orthonormal basis (u_a) of $\mathfrak{k}$, $R_{u_a} f = -d\tau(u_a)f$. Splitting the energy into its horizontal and vertical parts gives, on equivariant functions,

$$\Delta_J = \nabla_\tau^{0*} \nabla_\tau^0 + C_\tau, \quad C_\tau = - \sum_a d\tau(u_a)^2. \quad (4.3)$$

Here Δ_J acts componentwise. The horizontal energy is the canonical connection energy; fibre integration introduces only the constant fibre volume. The invariant form on $\mathfrak{k}$ is preserved by every component of K , so conjugating the orthonormal basis sum shows that C_τ commutes with $\tau(K)$. It is a bounded self-adjoint fibre endomorphism. Haar averaging gives a smooth equivariant form core. Thus the energy identity proves (4.3) for the associated self-adjoint operators; subtracting C_τ leaves the operator domain unchanged.

The induced forms on $\mathfrak{k}$ agree for all I by Lemma 2.1. Hence $S_{IL} \otimes 1_W$ preserves equivariance and commutes with the same endomorphism C_τ . It gives the required rough intertwiners. Tensoring with the fibre identity commutes with fibre intertwiners and finite direct sums. If $\tau|_{K^0} = 1$, the associated bundle is obtained from the discrete principal K/K^0 bundle $J_I/K^0 \rightarrow J_I/K$. Its induced connection pulls back to the trivial connection. $\square$

Theorem 4.4. *Let J_1, J_2 be compact connected metric Lie groups with bi-invariant metrics and embeddings of the same compact group K . Suppose the induced forms on $\mathfrak{k}$ agree. The following are equivalent:*

- (1) *their scalar eigenspaces, with eigenvalues, are equivalent right- K modules;*
- (2) *their canonical rough spectra agree for every irreducible unitary representation of the full group K ;*
- (3) *their canonical rough spectra agree for every finite-dimensional unitary representation of K .*

Proof. For irreducible τ , full K invariance of C_τ and Schur's lemma give $C_\tau = c_\tau 1$, even if $\tau|_{K^0}$ is reducible. The number c_τ is the same for J_1 and J_2 . By (4.3), the rough multiplicity at $\lambda - c_\tau$ is

$$\dim(E_\lambda(J_i) \otimes V_\tau)^K = \dim \operatorname{Hom}_K(V_\tau^*, E_\lambda(J_i)).$$

For $\lambda < c_\tau$ this invariant space is zero, since the rough Laplacian is nonnegative. For $\lambda \geq c_\tau$, its dimension is recovered from the rough spectrum; an absent eigenvalue has multiplicity zero. As τ^* runs through all irreducible K modules, (2) determines every irreducible multiplicity in every ambient eigenspace, and proves (1).

Under (1), choose equivariant unitaries on the scalar eigenspaces and take their Hilbert direct sum. Tensoring with 1_W preserves the invariant spaces and commutes with C_τ for arbitrary finite-dimensional τ . Lemma 2.4 and (4.3) give (3), including operator domains. Condition (3) includes (2). $\square$

4.1. The tangent representation.

Proposition 4.5. *There is a real P_n -equivariant orthogonal isomorphism $F_n : \mathfrak{g}_n \rightarrow \mathfrak{h}_n$ satisfying $F_n \circ di_n = dj_n$ on $\mathfrak{p}_n$.*

Proof. Let V be the defining $U(n)$ module and z the final circle character. Since $\bar{V} \simeq V^*$ as unitary modules,

$$\begin{aligned} \mathfrak{g}_{n,\mathbb{C}} &= 2 \operatorname{End} V \oplus V^{\otimes 2} \oplus (V^*)^{\otimes 2} \oplus \mathbb{C} \\ &\quad \oplus Vz \oplus Vz^{-1} \oplus V^*z \oplus V^*z^{-1}. \end{aligned} \tag{4.4}$$

On the other side,

$$\begin{aligned} \mathfrak{sp}(n)_\mathbb{C} &= \operatorname{Sym}^2 V \oplus \operatorname{End} V \oplus \operatorname{Sym}^2 V^*, \\ \mathfrak{so}(2n+2)_\mathbb{C} &= \Lambda^2 V \oplus \operatorname{End} V \oplus \Lambda^2 V^* \oplus \mathbb{C} \\ &\quad \oplus Vz \oplus Vz^{-1} \oplus V^*z \oplus V^*z^{-1}. \end{aligned}$$

The identity $\operatorname{Sym}^2 V \oplus \Lambda^2 V = V \otimes V$ equates the ambient complex modules, including $n = 1$, where $\Lambda^2 V = 0$. The embedded real $\mathfrak{p}_n$ has the same invariant metric on both sides. Cancelling its character equates the complexified orthogonal complements. These real complements are therefore isomorphic [96, §3.6, Theorem 5, p. 39]. For a real equivariant isomorphism C between them, C^*C commutes with P_n , so $C(C^*C)^{-1/2}$ is an orthogonal intertwiner. Combine it with the prescribed isometry $di_n u \mapsto dj_n u$. $\square$

Corollary 4.6. *For every closed $K \leq P$, the tangent isotropy modules $\mathfrak{m}_I = \mathfrak{k}^\perp \subset \mathfrak{j}_I$ are orthogonally equivalent as representations of the full group K . All corresponding canonical rough tensor spectra agree, including exterior, symmetric and arbitrary tensor powers, invariant tensor summands, and common finite-dimensional unitary twists.*

Proof. The direct sum of the maps F_n is an isometry for the prescribed factor scales, is P -equivariant, and identifies the embeddings of $\mathfrak{p}$. It therefore maps $\mathfrak{k}$ and its orthogonal complement onto their counterparts, equivariantly under every component of K . Apply each tensor construction to this identification, then apply Theorem 4.3 to the resulting common fibre module. $\square$

At the base point, the canonical tangent connection has torsion $T^0(X, Y) = -[X, Y]_{\mathfrak{m}}$ for $X, Y \in \mathfrak{m}$. For the Levi-Civita connection [2, §2],

$$\nabla_X^{\text{LC}} Y = \nabla_X^0 Y + \frac{1}{2}[X, Y]_{\mathfrak{m}}.$$

The tensor spectra above refer to ∇^0 .

4.2. Equivariant fibres and connection metrics. Related associated-bundle constructions were considered by Sutton [88, Proposition 4.1, pp. 712–713] and Bordoni–Marino [19, Theorem 2.1]. Principal connection metrics also occur in Schueth [84, §1, Theorem 2].

Proposition 4.7. *Let a compact Lie group P act isometrically on the right on two closed Riemannian manifolds (F_i, h_i) , $i = 1, 2$. Suppose there is a unitary*

$$S : L^2(F_1) \longrightarrow L^2(F_2), \quad S\Delta_{F_1} = \Delta_{F_2}S, \quad SR_p^{(1)} = R_p^{(2)}S,$$

where $R_p f(x) = f(xp)$ and the Laplacians are intertwined as self-adjoint operators. Let $\mathcal{Q} \rightarrow B$ be a smooth principal right P bundle over a closed Riemannian base (B, g_B) , and choose a smooth principal connection.

On

$$E_i = (\mathcal{Q} \times F_i)/P, \quad (q, x) \cdot p = (qp, xp),$$

give E_i the connection metric: the chosen connection induces the horizontal distribution, the horizontal metric is the pullback of g_B , and every vertical fibre carries h_i , orthogonal to the horizontal directions. Then (E_1, g_1) and (E_2, g_2) are scalar isospectral. For every finite-rank Hermitian bundle $W \rightarrow B$ with a unitary connection, the degree-zero connection Laplacians on $\pi_i^* W \rightarrow E_i$ are also unitarily equivalent.

Suppose in addition that self-adjoint fibre operators $Z_{i,r}$ commute strongly with one another and with Δ_{F_i} , their spectral projections commute with the P action, and S intertwines $Z_{1,r}$ and $Z_{2,r}$ as self-adjoint operators. Then their fibrewise realizations on E_i are intertwined, including their self-adjoint domains. Their spectral projections commute with those of the total connection Laplacians.

Proof. The left fibre actions are $\mu_i(p)x = xp^{-1}$. Apply [8, Theorem 3.4, pp. 9–10] with the same principal connection on both sides and the identity bundle automorphism for every irreducible representation of P . Both compatibility conditions are identities. Applying the theorem on each connected component of B gives scalar isospectrality. The theorem allows disconnected P and F_i ; finite fibres occur in [8, Remark 3.5(2)].

The connection metric has totally geodesic fibres [13, §2.1, pp. 185–186]. Put $V_{i,\lambda} = \ker(\Delta_{F_i} - \lambda)$ and $\mathcal{V}_{i,\lambda} = \mathcal{Q} \times_P V_{i,\lambda}$, with the representation $R^{(i)}$ on the fibre. These are finite-rank Hermitian bundles. The diagonal quotient convention gives $u(qp) = R_{p^{-1}}^{(i)} u(q)$. Fibre integration and the orthogonal fibre spectral decomposition give

$$L^2(E_i, \pi_i^* W) \simeq \widehat{\bigoplus}_{\lambda} L^2(B, \mathcal{V}_{i,\lambda} \otimes W).$$

Let $\nabla_{i,\lambda}$ be the induced tensor connection. For a vector field X on B , its horizontal lift X^h , a vertical field V , a scalar function f on E_i and a local section s of W ,

$$\begin{aligned} \nabla_{X^h}^{\pi_i^* W} (f \pi_i^* s) &= X^h(f) \pi_i^* s + f \pi_i^* (\nabla_X^W s), \\ \nabla_V^{\pi_i^* W} (f \pi_i^* s) &= V(f) \pi_i^* s. \end{aligned}$$

Apply [8, Lemma 3.2, pp. 7–8] to the irreducible P summands of $V_{i,\lambda}$. Together with these identities and the vertical eigenvalue λ , its horizontal Laplacian formula gives, on smooth sections of $\mathcal{V}_{i,\lambda} \otimes W$,

$$\Delta_{E_i, \pi_i^* W} = \nabla_{i,\lambda}^* \nabla_{i,\lambda} + \lambda.$$

The operator on the right has a smooth orthonormal eigenbasis on the closed base. Its sections lift to smooth total eigensections, and the union of these bases over λ is complete in the displayed Hilbert sum. Applying Lemma 2.4 to the finite-dimensional eigenspaces of the operators on the right, then grouping by λ , gives

$$\Delta_{E_i, \pi_i^* W} \simeq \bigoplus_{\lambda} A_{i,\lambda}, \quad A_{i,\lambda} = \nabla_{i,\lambda}^* \nabla_{i,\lambda} + \lambda. \quad (4.5)$$

The operator domain consists of the tuples $u = (u_{\lambda})$ in the displayed Hilbert sum such that

$$u_{\lambda} \in \mathcal{D}(A_{i,\lambda}), \quad \sum_{\lambda} \|A_{i,\lambda} u_{\lambda}\|^2 < \infty.$$

Each restriction $S_{\lambda} : V_{1,\lambda} \rightarrow V_{2,\lambda}$ is a P -equivariant unitary. It induces a parallel unitary on the associated bundles, also after tensoring with 1_W . The Hilbert sum of these maps intertwines the operators in (4.5) and preserves the displayed domains.

For the operators $Z_{i,r}$, strong commutation with Δ_{F_i} makes every $V_{i,\lambda}$ a reducing subspace. The restriction of each $Z_{i,r}$ is self-adjoint on this finite-dimensional space, hence is defined everywhere. The commuting restrictions therefore have common orthogonal eigenspaces $V_{i,\lambda,\zeta}$, where $\zeta = (\zeta_r)_r$. Their projections commute with P , so these spaces define parallel associated subbundles, and the decomposition (4.5) refines accordingly. On the corresponding Hilbert sum, the fibrewise realization of $Z_{i,r}$ is multiplication by ζ_r , with domain

$$\left\{ u = (u_{\lambda,\zeta}) \in L^2(E_i, \pi_i^* W) : \sum_{\lambda,\zeta} |\zeta_r|^2 \|u_{\lambda,\zeta}\|^2 < \infty \right\}.$$

The maps induced by S preserve the joint labels and these domains. Each joint subbundle is parallel, so its projection reduces the total connection Laplacian. The spectral projections of each fibrewise $Z_{i,r}$ are sums of these reducing projections and therefore commute with the spectral projections of that Laplacian. $\square$

4.3. Equivariant Clifford modules. Take the Clifford convention $c(X)c(Y) + c(Y)c(X) = -2\langle X, Y \rangle$. Let (W, c, τ) be a finite-dimensional Hermitian Clifford module with a unitary representation of the full group K , satisfying

$$\tau(k)c(X)\tau(k)^{-1} = c(\text{Ad}(k)X), \quad (4.6)$$

and with $c(X)$ skew-Hermitian for real X . For orthonormal bases (e_i) of $\mathfrak{m}$ and (u_a) of $\mathfrak{k}$, put

$$\sigma_c(u) = \frac{1}{4} \sum_{i,j} \langle [u, e_i], e_j \rangle c(e_i)c(e_j), \quad \kappa(u) = d\tau(u) - \sigma_c(u), \quad C_{\kappa} = - \sum_a \kappa(u_a)^2. \quad (4.7)$$

Lemma 4.8. *The map κ is a skew-Hermitian Lie algebra representation commuting with the Clifford action. Its Casimir C_{κ} is nonnegative and commutes with $\tau(K)$.*

Proof. The infinitesimal spin representation satisfies

$$[\sigma_c(u), c(X)] = c([u, X]), \quad [\sigma_c(u), \sigma_c(v)] = \sigma_c([u, v])$$

[59, (1.10)–(1.12)]. Differentiating (4.6) gives the first commutator also for $d\tau(u)$. Hence $\kappa(u)$ commutes with every Clifford product, in particular every $\sigma_c(v)$. Since $d\tau = \sigma_c + \kappa$, subtracting the Lie bracket identities gives $[\kappa(u), \kappa(v)] = \kappa([u, v])$. Both $d\tau(u)$ and $\sigma_c(u)$ are skew-Hermitian. Conjugation by $\tau(k)$ carries $\kappa(u)$ to $\kappa(\text{Ad}(k)u)$ for every $k \in K$. The invariant orthonormal sum is therefore K -invariant, and $\langle C_{\kappa} w, w \rangle = \sum_a \|\kappa(u_a)w\|^2$. $\square$

Define on equivariant functions

$$D_{J,W} = \sum_i c(e_i)R_{e_i} + v_J, \quad v_J = \frac{1}{2} \sum_{i < j < k} \langle [e_i, e_j]_{\mathfrak{m}}, e_k \rangle c(e_i)c(e_j)c(e_k). \quad (4.8)$$

Write $\sigma(u)$ and v_J for the elements of $\text{Cl}(\mathfrak{m}_{\mathbb{C}}, -Q)$ obtained from (4.7) and (4.8) by replacing each $c(e_i)$ by e_i . Thus $c(\sigma(u)) = \sigma_c(u)$ and $c(v_J) = v_J$. The two contractions in (4.8) are invariant under K , hence descend to $J \times_K W$. On $C^\infty(J, W)$, each R_X is formally skew-adjoint and commutes with every constant fibre endomorphism. Since $c(e_i)$ is skew-Hermitian and a product of three distinct Clifford generators is Hermitian, the descended operator is formally self-adjoint. Its principal symbol is Clifford multiplication, so it is of Dirac type. Since J/K is closed, [9, Theorems 4.4 and 4.11, with empty boundary] give a self-adjoint closure with domain $H^1(J \times_K W)$. We denote this closure again by $D_{J,W}$.

Theorem 4.9. *For the above operator,*

$$D_{J,W}^2 = \Delta_J - C_\kappa + b_J, \quad b_J = \|\rho_j\|^2 - \|\rho_{\mathfrak{k}}\|^2. \quad (4.9)$$

The root norms are measured in the dual invariant metrics; the roots vanish on central directions. In the product construction the constants b_{J_i} are equal. Therefore all corresponding cubic squares have equal spectra. If W has a K -invariant grading for which $c(X)$ is odd, the signed spectra of the cubic operators also agree.

Proof. Apply Kostant's identities to $\mathfrak{k}_{\mathbb{C}} \subset \mathfrak{j}_{\mathbb{C}}$ with bilinear form $-Q$. Both algebras are reductive, the restricted form is nondegenerate, and the orthogonal complement is $\mathfrak{m}_{\mathbb{C}}$. Kostant's spin map is σ , and his cubic element is $-v_J$ [59, (1.20)–(1.23)]. Since the dual basis to (u_a) is $(-u_a)$, his Theorem 1.91 gives

$$v_J^2 - \sum_a \sigma(u_a)^2 = b_J 1 \quad \text{in } \text{Cl}(\mathfrak{m}_{\mathbb{C}}, -Q). \quad (4.10)$$

Under $X \mapsto R_X$ and the Clifford action c , Kostant's cubic Dirac element acts as $-D_{J,W}$: its linear contraction is $-\sum_i R_{e_i}c(e_i)$. The ambient Casimir acts as Δ_J , and the diagonal isotropy Casimir acts as $-\sum_a (R_{u_a} + \sigma_c(u_a))^2$. Thus [59, Theorem 2.16] gives, on smooth sections,

$$D_{J,W}^2 = \Delta_J + \sum_a (R_{u_a} + \sigma_c(u_a))^2 + b_J.$$

On equivariant functions $R_u = -d\tau(u)$, so the sum is $-C_\kappa$. The identities hold for arbitrary ranks. The endomorphism C_κ is bounded and self-adjoint. By [9, Theorem 4.4, with empty boundary],

$$\mathcal{D}(D_{J,W}^2) = \{f \in H^1 : D_{J,W}f \in H^1\} = H^2,$$

where the Sobolev spaces are those of $J \times_K W$. By (4.3), the restriction of Δ_J is the canonical rough Laplacian plus the bounded endomorphism C_τ . Its domain is H^2 by elliptic regularity [9, §2.3]; adding $-C_\kappa + b_J$ preserves this domain. Thus (4.9) holds as an identity of self-adjoint operators on H^2 .

By (3.9), the ambient root norms agree in each reference block. Products add their squared norms, and scaling a block by s_j divides its dual squared norm by s_j . The common embedded $\mathfrak{k}$ has its common metric, so its root norm agrees as well. The tangent identification makes κ common. Consequently $S_{IL} \otimes 1_W$ commutes with C_κ and intertwines the squares, including their domains.

If ϵ is the grading involution, then $\epsilon D = -D\epsilon$, since both terms in (4.8) have odd Clifford degree. Each positive square eigenvalue λ therefore gives $\sqrt{\lambda}$ and $-\sqrt{\lambda}$ with equal multiplicities. The square also determines the zero multiplicity. This proves equality of the signed spectra. $\square$

4.4. Exterior modules and spinors.

Corollary 4.10. *On $W = \Lambda^* \mathfrak{m}_{\mathbb{C}}^*$, with $c(X) = X^\flat \wedge -\iota_X$ and the natural K action, the cubic operators have equal signed spectra on all M_I , also after any common finite-dimensional unitary K twist.*

Proof. The exterior action satisfies (4.6) for every orthogonal transformation. Exterior parity is invariant and Clifford multiplication is odd. Placing a twist in even degree preserves these properties. Apply Theorem 4.9. $\square$

The exterior operator is defined by (4.8). Its cubic term is left Clifford multiplication by the specified three-form. The Hodge operator $d + d^*$ is defined by the Levi-Civita connection. The zero multiplicities of the Hodge Laplacians on two-forms differ in Theorem 12.1.

Suppose the oriented tangent isotropy representation has a lift $\tilde{\lambda} : K \rightarrow \text{Spin}(\mathfrak{m})$. For a finite-dimensional unitary K representation ρ , the spin module $\Delta \otimes V_\rho$ has $\kappa = 1 \otimes d\rho$. Therefore

$$(D_{J,\rho}^{1/3})^2 = \Delta_J - C_\rho + b_J. \quad (4.11)$$

Here the spin connection belongs to the family $\nabla_X^t Y = \nabla_X^0 Y + t[X, Y]_{\mathfrak{m}}$, and the twist has its canonical connection. The spin Dirac correction is

$$\frac{3t}{2} \sum_{i < j < k} \langle [e_i, e_j], e_k \rangle c_i c_j c_k$$

[2, §3.1, equation (5)]. Thus $t = 1/3$ gives (4.8). The Levi-Civita value is $t = 1/2$ [2, §2].

Proposition 4.11. *If K^0 is central in K , every M_I admits a homogeneous spin structure. The structures can be chosen from one common isotropy lift. For these structures and every common finite-dimensional unitary K twist, the cubic spin squares have equal spectra. In even dimension their full spinor signed spectra also agree.*

Proof. For the connected reductive complexifications of $U(2n+1)$, $\text{Sp}(n)$ and $\text{SO}(2n+2)$, and their products, the half-sums of positive roots belong to the torus character lattices in (2.4). The adjoint-lifting criterion [56, arXiv version 3, Corollary 4, p. 11] therefore gives algebraic spin lifts. The inverse image of $\text{SO}(j_I, Q)$ under the complex spin covering is $\text{Spin}(j_I, Q)$. Restriction gives homomorphisms $J_I \rightarrow \text{Spin}(j_I)$ lifting the ambient adjoint actions. The criterion includes $n = 1$, hence the factor $\text{SO}(4)$.

Centrality of K^0 makes $\text{Ad}(K)$ fix $\mathfrak{k}$ pointwise. The ambient adjoint action is oriented, so its action on $\mathfrak{m}$ is oriented. The inverse image of $1_{\mathfrak{k}} \oplus \text{SO}(\mathfrak{m})$ in $\text{Spin}(\mathfrak{j})$ is the embedded $\text{Spin}(\mathfrak{m})$. Restricting an ambient lift therefore gives an isotropy lift on the full group K . Transport one such lift by the tangent isometries and apply (4.11). Even-dimensional full spin modules have the invariant chirality grading, so their cubic operators have equal signed spectra. $\square$

Two lifts differ by a character $K \rightarrow \{1, -1\}$; the preceding comparison fixes one common lift. In odd dimension it gives equality of the cubic spin-square spectra.

Proposition 4.12. *The universal Clifford elements have equal squares under the tangent identifications. For one block with $K = P_n$, no P_n -equivariant orthogonal isomorphism $F : \mathfrak{g}_n \rightarrow \mathfrak{h}_n$ satisfying $F \circ di_n = dj_n$ carries $\mathfrak{v}_{G_n}$ to $\mathfrak{v}_{H_n}$ under the induced Clifford isomorphism.*

Proof. Put $C_\sigma = -\sum_a \sigma(u_a)^2$. Equation (4.10) gives $\mathfrak{v}_J^2 = b_J - C_\sigma$ in the universal Clifford algebra; both terms on the right are common. For $K = P_n$, the bracket on $\mathfrak{p}_n$, its action on $\mathfrak{m}$, the invariant metric, and the cubic three-form determine the full Lie bracket [59, (1.16) and (1.23)]. A map as in the statement identifying the cubic elements would therefore be a Lie algebra isomorphism. This is impossible: $\mathfrak{u}(2n+1)$ has a one-dimensional center, whereas $\mathfrak{sp}(n) \oplus \mathfrak{so}(2n+2)$ has zero center. $\square$

4.5. Residual flows in rank one.

Corollary 4.13. *Let K satisfy Theorem 3.9. The spectral unitary intertwines the generators of every one-parameter subgroup of K , with their domains, and the operators $(1 + \Delta)^z$ for every $z \in \mathbb{C}$, with their maximal spectral domains.*

Proof. Apply Lemma 2.4 to the K -equivariant scalar eigenspaces in Theorem 3.9. $\square$

Write $G_A = G_1$, $G_B = H_1$, $\iota_A = i_1$, $\iota_B = j_1$ and $\iota_i(u, v) = \iota_i(e^{iu}, e^{iv})$ for angular arguments. At $n = 1$ the bilateral identity gives, for $t > 0$ and $u, s, v \in \mathbb{T}$,

$$B_{G_A}(t; \iota_A(u, 0), \iota_A(s, v)) = B_{G_B}(t; \iota_B(u, 0), \iota_B(s, v)). \quad (4.12)$$

Let $h = 2\pi/7$ and put

$$H_i = \{\iota_i(0, v) : v \in \mathbb{T}\}, \quad A = G_A/H_A, \quad B = G_B/H_B.$$

The common subgroup

$$S_1 = \{(u, 0; 3u, b) : u, b \in \mathbb{T}\} \subset \ker \alpha \quad (4.13)$$

contains the right-circle subgroup and the commuting flow subgroup. Since $\iota_i(3hs, 0)$ commutes with H_i , the latter descends to

$$F_i(s)(gH_i) = \iota_i(hs, 0)g\iota_i(3hs, 0)^{-1}H_i, \quad s \in \mathbb{R}.$$

Corollary 4.14. *There is a unitary $S : L^2(A) \rightarrow L^2(B)$ fixing normalized constants, intertwining the Laplacians and the seven-periodic flows $F_i(s)$, and intertwining their generators with domains. In particular, for $s \in \mathbb{R}$ and $t > 0$,*

$$\mathrm{Tr}(F_A(s)^* e^{-t\Delta_A}) = \mathrm{Tr}(F_B(s)^* e^{-t\Delta_B}). \quad (4.14)$$

Proof. Let $U : L^2(G_A) \rightarrow L^2(G_B)$ be an S_1 -equivariant spectral unitary fixing constants, and let P_i be right-circle averaging. Then $UP_A = P_B U$, and each P_i commutes with the flow and the Laplacian. Restrict U to $\mathrm{ran} P_A$ and apply Lemma 4.1. Corollary 4.13 gives the intertwining of the generators with their domains. Since $7h = 2\pi$, one has $F_i(s+7) = F_i(s)$. The products in (4.14) are trace class and are conjugate by S . $\square$

4.6. Curvature of normal homogeneous quotients.

Proposition 4.15. *Let $M = J/K$, where J is a compact connected Lie group and K is a closed subgroup. Give M the metric induced by a positive $\mathrm{Ad}(J)$ -invariant form Q , and write $\mathfrak{m} = \mathfrak{k}^\perp$. Then*

$$\ker \mathrm{Ric}_o = \mathfrak{z}(\mathfrak{j}) \cap \mathfrak{m}, \quad \dim \ker \mathrm{Ric} = b_1(M).$$

The universal Riemannian cover is

$$\tilde{M} \cong \mathbb{R}^{b_1(M)} \times N,$$

where N is compact and simply connected, and has positive Ricci curvature if it has positive dimension. Moreover,

$$\mathrm{Ric}_M > 0 \iff \pi_1(M) \text{ is finite.}$$

In dimension zero, the Ricci inequality is interpreted as vacuous. If $J = \mathrm{U}(N)$, $Q = -\mathrm{Re} \mathrm{Tr}$, and $0 \neq T \in \mathfrak{k}$, then

$$\mathrm{Ric}_{J/K} \geq \frac{|\mathrm{Tr} T|^2}{2Q(T, T)} g_{J/K}. \quad (4.15)$$

Proof. For an orthonormal pair $X, Y \in \mathfrak{m}$, the normal homogeneous curvature formula [77, (3.6), p. 250] is

$$\sec(X, Y) = \frac{1}{4} \|[X, Y]_{\mathfrak{m}}\|^2 + \|[X, Y]_{\mathfrak{k}}\|^2. \quad (4.16)$$

For an orthonormal basis (e_a) of $\mathfrak{m}$, put

$$A_X = \sum_a \|[X, e_a]_{\mathfrak{m}}\|^2, \quad C_X = \sum_a \|[X, e_a]_{\mathfrak{k}}\|^2.$$

Then $\mathrm{Ric}(X, X) = A_X/4 + C_X$. If (u_α) is an orthonormal basis of $\mathfrak{k}$, invariance of Q gives

$$C_X = \sum_\alpha \|[u_\alpha, X]\|^2, \quad -B_j(X, X) = A_X + 2C_X.$$

Thus

$$\text{Ric}(X, X) = -\frac{1}{4}B_j(X, X) + \frac{1}{2} \sum_{\alpha} \| [u_{\alpha}, X] \|^2. \quad (4.17)$$

Since $-B_j(X, X) = \|\text{ad } X\|_{\text{HS}}^2$, the Ricci tensor is nonnegative and its kernel is $\mathfrak{z}(\mathfrak{j}) \cap \mathfrak{m}$.

For $J = \text{U}(N)$, put $Z = iI_N/\sqrt{N}$. Since $X \in \mathfrak{m}$ is orthogonal to T , projection onto $T^{\perp}$ gives

$$|Q(X, Z)|^2 \leq \left(1 - \frac{|\text{Tr } T|^2}{NQ(T, T)}\right) \|X\|^2, \quad \|X_{\text{su}(N)}\|^2 \geq \frac{|\text{Tr } T|^2}{NQ(T, T)} \|X\|^2.$$

Now $-B_{\text{u}(N)}(X, X) = 2N\|X_{\text{su}(N)}\|^2$, so (4.17) implies (4.15). If $N = 1$, the hypothesis $T \neq 0$ forces $\mathfrak{m} = 0$.

Since J is connected, every central $Z \in \mathfrak{m}$ is fixed by $\text{Ad}(J)$, and hence by all of $\text{Ad}(K)$. It therefore defines a global J -invariant field on M , even when K is disconnected. This field is Killing, has constant length and lies in the Ricci kernel. The Killing-field Bochner identity [18, Lemmas 2–3, pp. 779–781] makes it parallel; its dual is a harmonic one-form. Conversely, by Bochner's theorem [18, Theorem 1, p. 780], every harmonic one-form on M is parallel and takes values in the Ricci kernel. Evaluation of their metric-dual fields at o therefore identifies the harmonic one-forms with $\mathfrak{z}(\mathfrak{j}) \cap \mathfrak{m}$. Hodge theory gives $\dim \ker \text{Ric} = b_1(M)$.

Put $r = b_1(M)$ and $\mathcal{D} = \ker \text{Ric}_M$. A basis of $\mathfrak{z}(\mathfrak{j}) \cap \mathfrak{m}$ gives a global parallel frame of $\mathcal{D}$, so $\mathcal{D}$ and $\mathcal{D}^{\perp}$ are parallel. The universal Riemannian cover is complete and simply connected. The de Rham theorem [32, §7, Theorem II, pp. 339–340] gives $\tilde{M} = \mathbb{R}^r \times N$, where N is complete and simply connected and its tangent spaces are the lifts of $\mathcal{D}^{\perp}$; zero-dimensional factors are interpreted as points. If $\dim N > 0$, let $c > 0$ be the least positive eigenvalue of the Ricci endomorphism at o . Homogeneity gives the same lower bound on $\mathcal{D}^{\perp}$ at every point. The product Ricci formula therefore gives $\text{Ric}_N \geq c g_N$. Dimension one is impossible, since its Ricci tensor would vanish. For $\dim N \geq 2$, Myers's theorem [76, Theorem I, p. 404] makes N compact; for $\dim N = 0$, take N to be a point.

Nash's criterion [77, Proposition 3.4, pp. 249–250] gives $\text{Ric}_M > 0$ if and only if $\pi_1(M)$ is finite. In dimension zero, M is a point. $\square$

5. FINITE-QUOTIENT HEAT TRACES AND MINIMUM LENGTHS

Let X be a closed connected smooth Riemannian manifold of positive dimension, let $\Pi = \pi_1(X)$ with a fixed marking, and let Δ_X be the nonnegative scalar Laplacian. Write $\ell_X(c)$ for the infimum of lengths of piecewise smooth loops in the free homotopy class of $c \in \Pi$, with $\ell_X(1) = 0$.

Fix a left deck action on $\tilde{X}$. For $\rho : \Pi \rightarrow \text{U}(d)$, set $E_{\rho} = (\tilde{X} \times \mathbb{C}^d)/\Pi$, where $\gamma(x, v) = (\gamma x, \rho(\gamma)v)$, and let $\Delta_{X, \rho} = \nabla^* \nabla$ for the descended flat connection. Sections satisfy

$$f(\gamma x) = \rho(\gamma)f(x). \quad (5.1)$$

A path lifting from x to γx has parallel transport $\rho(\gamma)^{-1}$ in the initial trivialization. Put $\Theta_{X, \rho}(s) = \text{Tr}(e^{-s\Delta_{X, \rho}})$ for $s > 0$. For the trivial one-dimensional representation, $\Delta_{X, 1} = \Delta_X$. Write $\Theta_X := \Theta_{X, 1}$. Representations of a deck group are pulled back to $\pi_1(X)$.

5.1. The finite character transform. For a finite quotient $q : \Pi \twoheadrightarrow Q$, let $Y_q \rightarrow X$ be its connected regular cover. Put

$$R_g f(x) = f(g^{-1}x), \quad H_{q, g}(s) = \text{Tr}(R_g e^{-s\Delta_{Y_q}}), \quad \ell_q(g) = \min_{y \in Y_q} d(y, gy).$$

More generally, let a finite group Γ act freely by isometries on a closed connected Riemannian manifold Z , with quotient X . Put $E_{\lambda} = \ker(\Delta_Z - \lambda)$. For $\gamma \in \Gamma$ and $\rho \in \text{Irr}(\Gamma)$, put

$$a_{\lambda}(\gamma) = \text{Tr}(R_{\gamma}|_{E_{\lambda}}), \quad m_{\rho}(\lambda) = \dim \text{Hom}_{\Gamma}(V_{\rho}, E_{\lambda}).$$

Equivariant eigenfunctions identify $\ker(\Delta_{X, \rho^*} - \lambda)$ with $(E_{\lambda} \otimes V_{\rho}^*)^{\Gamma}$. Scalar complex conjugation commutes with the deck action and the Laplacian and exchanges ρ with ρ^* . Thus

$$m_{\rho}(\lambda) = \dim \ker(\Delta_{X, \rho^*} - \lambda) = \dim \ker(\Delta_{X, \rho} - \lambda).$$

Define

$$D_\gamma(t) = \sum_{\lambda} a_\lambda(\gamma) e^{it\sqrt{\lambda}}, \quad W_\rho(t) = \sum_{\lambda} m_\rho(\lambda) e^{it\sqrt{\lambda}}.$$

For $N > \dim Z$, Weyl's law [35, Theorem 16.7] gives

$$\sum_{\lambda} \dim E_\lambda (1 + \sqrt{\lambda})^{-N} < \infty.$$

Since $|a_\lambda(\gamma)| \leq \dim E_\lambda$ and $m_\rho(\lambda) \leq \dim E_\lambda$, both series converge in $S'(\mathbb{R})$. The real form of E_λ gives $a_\lambda(\gamma) \in \mathbb{R}$ and $a_\lambda(\gamma^{-1}) = a_\lambda(\gamma)$. Each a_λ is a class function.

Lemma 5.1. *For $s > 0$, every irreducible representation and every deck element,*

$$\Theta_{X,\rho}(s) = \frac{1}{|Q|} \sum_{g \in Q} \overline{\chi_\rho(g)} H_{q,g}(s), \quad H_{q,g}(s) = \sum_{\rho \in \text{Irr}(Q)} \chi_\rho(g) \Theta_{X,\rho}(s), \quad (5.2)$$

$$W_\rho = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \overline{\chi_\rho(\gamma)} D_\gamma, \quad D_\gamma = \sum_{\rho \in \text{Irr}(\Gamma)} \chi_\rho(\gamma) W_\rho. \quad (5.3)$$

The second pair holds in $S'(\mathbb{R})$. All W_ρ are smooth on an open interval if and only if all D_γ are. The same equivalence holds for real analyticity and for germs at any $t \in \mathbb{R}$. Moreover,

$$\bigcup_{\rho} \text{WF}(W_\rho) = \bigcup_{\gamma} \text{WF}(D_\gamma), \quad \bigcup_{\rho} \text{WF}_A(W_\rho) = \bigcup_{\gamma} \text{WF}_A(D_\gamma).$$

Proof. Character orthogonality [27, §3.3], applied to the finite-dimensional Γ -module E_λ , gives both transforms at each eigenvalue. Summation against $e^{-s\lambda}$ is absolute for $s > 0$; summation against $e^{it\sqrt{\lambda}}$ converges in $S'(\mathbb{R})$. This proves the displayed identities.

Each family is a finite constant linear combination of the other. Hence smoothness and real analyticity are equivalent on every open interval and at every germ. For either $\text{WF}_* = \text{WF}$ or $\text{WF}_* = \text{WF}_A$,

$$\text{WF}_*(W_\rho) \subseteq \bigcup_{\gamma} \text{WF}_*(D_\gamma), \quad \text{WF}_*(D_\gamma) \subseteq \bigcup_{\rho} \text{WF}_*(W_\rho).$$

Taking unions proves the two equalities. $\square$

For a flat connection and positive constant killing, Le Jan expresses the Brownian-loop integral of a normalized holonomy character minus 1 as the difference between the normalized twisted and scalar zeta derivatives at 0 [65, §3, proof of Proposition 5].

In (5.2) and (5.3), the characters are ordinary traces. The scalar trace on the ρ -isotypic subspace is $\dim \rho W_\rho$. In particular,

$$W_1 = \frac{1}{|\Gamma|} \sum_{\gamma} D_\gamma, \quad D_1 = \sum_{\rho} \dim \rho W_\rho.$$

The flat bundles here correspond to representations of $\pi_1(X)$ that factor through the fixed deck quotient $\pi_1(X) \twoheadrightarrow \Gamma$.

5.2. Heat exponents and attained minima.

Lemma 5.2. *Let F be an isometry of a closed connected Riemannian n -manifold Z , $n > 0$. Put $R_F f = f \circ F^{-1}$ and $a = \min_{x \in Z} d(x, Fx)$, and let K_s be the heat kernel of Δ_Z . For $s > 0$,*

$$H_F(s) := \text{Tr}(R_F e^{-s\Delta_Z}) = \int_Z K_s(x, Fx) dx > 0.$$

There are $c, C, s_0 > 0$ such that, for $0 < s < s_0$,

$$cs^{n/2} e^{-a^2/(4s)} \leq H_F(s) \leq Cs^{-n-1} e^{-a^2/(4s)}. \quad (5.4)$$

Hence $-4s \log H_F(s) \rightarrow a^2$. In particular,

$$\lim_{s \downarrow 0} -4s \log H_{q,g}(s) = \ell_q(g)^2. \quad (5.5)$$

The $H_{q,g}$ are real, positive, and invariant under conjugacy and inversion. For $g \neq 1$, $\ell_q(g) > 0$. There are $a_q, N_q > 0$ such that

$$H_{q,g}(s) = O(s^{-N_q} e^{-a_q/s})$$

uniformly for $g \neq 1$.

Proof. The kernel of $R_F e^{-s\Delta_Z}$ is $K_s(F^{-1}x, y)$. Since F preserves Riemannian volume,

$$\text{Tr}(R_F e^{-s\Delta_Z}) = \int_Z K_s(F^{-1}x, x) dx = \int_Z K_s(x, Fx) dx.$$

The scalar heat kernel is positive on the connected manifold Z . Applied to Δ_Z , [68, Theorems 3.5 and 3.8] give

$$c_0 s^{-n/2} e^{-d(x,y)^2/(4s)} \leq K_s(x, y) \leq C_0 s^{-n-1} e^{-d(x,y)^2/(4s)}$$

for all $x, y \in Z$ and $0 < s < s_0$. The upper bound integrates to the upper bound in (5.4).

Choose x_0 with $d(x_0, Fx_0) = a$. For $x \in B(x_0, s)$,

$$d(x, Fx) \leq d(x, x_0) + a + d(Fx_0, Fx) \leq a + 2s.$$

After decreasing s_0 , one has $\text{vol } B(x_0, s) \geq b s^n$ with $b > 0$. Thus

$$H_F(s) \geq c_0 b s^{n/2} e^{-(a+2s)^2/(4s)} = c_0 b s^{n/2} e^{-a^2/(4s)} e^{-a-s},$$

which proves the lower bound. The limit follows from (5.4) and $s \log s \rightarrow 0$, also when $a = 0$.

For deck transformations, the identities $K_s(x, y) = K_s(y, x) = K_s(hx, hy)$ give $H_{q,g^{-1}} = H_{q,g}$ and $H_{q,hgh^{-1}} = H_{q,g}$. If $g \neq 1$, the continuous function $y \mapsto d(y, gy)$ is positive on the compact manifold Y_q , since the deck action is free. For $Q \neq \{1\}$, put

$$\delta_q = \min_{g \neq 1} \ell_q(g) > 0.$$

For $g \neq 1$, the upper bound gives

$$H_{q,g}(s) \leq C_q s^{-\dim X - 1} e^{-\delta_q^2/(4s)}.$$

□

Lemma 5.3. *Every nontrivial free homotopy class has a positive minimum represented by a smoothly closed geodesic. Only finitely many classes have minimum at most a given $R < \infty$. Moreover,*

$$\ell_q(g) = \min\{\ell_X(c) : q(c) \text{ is conjugate to } g\}, \quad (5.6)$$

with attainment, including $g = 1$.

Proof. Every nontrivial free homotopy class contains a nonconstant closed geodesic of least length [63, Proposition 6.28, p. 174]. Closed geodesics of bounded length on a compact Riemannian manifold represent only finitely many free homotopy classes [31, §2, p. 3647].

If a loop α represents c and $q(c)$ is conjugate to g , its lift to Y_q joins a point y to hy for some h conjugate to g . Since the deck transformations are isometries,

$$L(\alpha) \geq d(y, hy) \geq \ell_q(h) = \ell_q(g).$$

Thus $\ell_X(c) \geq \ell_q(g)$. Conversely, compactness of Y_q gives y with $d(y, gy) = \ell_q(g)$. A minimizing geodesic segment from y to gy projects to a loop in a class c with $q(c)$ conjugate to g . For this class,

$$\ell_q(g) \leq \ell_X(c) \leq d(y, gy) = \ell_q(g).$$

This proves (5.6) and attainment. For $g = 1$, the constant loop gives $\ell_q(1) = \ell_X(1) = 0$. □

5.3. The minimum formula for a finite target.

Theorem 5.4. *Let M, N be closed connected Riemannian manifolds with marked fundamental groups E, Γ , where Γ is finite. Let $p : E \twoheadrightarrow \Gamma$ have kernel C . Suppose*

$$\text{Spec } \Delta_{M, p^* \rho} = \text{Spec } \Delta_{N, \rho} \quad (\rho \in \text{Irr}(\Gamma)).$$

Then $Y_p = C \backslash \tilde{M}$ and $\tilde{N}$ have equal Γ -equivariant heat and half-wave traces, and

$$\ell_N(\gamma) = \min_{p(e)=\gamma} \ell_M(e) \quad (\gamma \in \Gamma). \quad (5.7)$$

The minimum is attained for arbitrary E . If E is finite, let $H_{M,e}$ and $D_{M,e}$ denote the equivariant heat and half-wave traces on $\tilde{M}$, and define $H_{N,\gamma}$ and $D_{N,\gamma}$ on $\tilde{N}$ in the same way. Then

$$H_{N,\gamma}(s) = \frac{1}{|C|} \sum_{p(e)=\gamma} H_{M,e}(s), \quad D_{N,\gamma} = \frac{1}{|C|} \sum_{p(e)=\gamma} D_{M,e}. \quad (5.8)$$

Equality for irreducibles is equivalent to equality for all finite-dimensional unitary representations of Γ .

Proof. The trivial representation gives equal scalar spectra. If either manifold is a point, both are points: a positive-dimensional closed manifold has unbounded Laplace spectrum. In this case $E = \Gamma = 1$ and the formulas hold. Suppose henceforth that both dimensions are positive.

The covers Y_p and $\tilde{N}$ are compact, with deck group Γ . Lemma 5.1 gives equality of their equivariant heat and half-wave traces. Equations (5.5) and (5.6) give

$$\ell_N(\gamma) = \min_{p(e)=\gamma} \ell_M(e),$$

with attainment. If a minimizing e satisfies $p(e) = h\gamma h^{-1}$, choose $a \in E$ with $p(a) = h$. Then

$$p(a^{-1}ea) = \gamma, \quad \ell_M(a^{-1}ea) = \ell_M(e),$$

which proves (5.7).

Suppose E is finite. Pullback under $\tilde{M} \rightarrow Y_p$, multiplied by $|C|^{-1/2}$, identifies $L^2(Y_p)$ unitarily with $L^2(\tilde{M})^C$. The orthogonal projector is

$$P_C = \frac{1}{|C|} \sum_{c \in C} R_c.$$

It commutes with the Laplacian and, since $C \trianglelefteq E$, with every R_e . For $p(e) = \gamma$, the restriction of R_e to the C -invariant subspace corresponds to R_γ . On each Laplace eigenspace, therefore,

$$\text{Tr}(R_\gamma | \ker(\Delta_{Y_p} - \lambda)) = \frac{1}{|C|} \sum_{p(h)=\gamma} \text{Tr}(R_h | \ker(\Delta_{\tilde{M}} - \lambda)).$$

Summation with weights $e^{-s\lambda}$ gives the heat identity in (5.8). Summation with weights $e^{it\sqrt{\lambda}}$ in $S'(\mathbb{R})$ gives its half-wave identity.

Every finite-dimensional unitary representation of Γ is a finite direct sum of irreducibles, and the corresponding flat Laplacian is the orthogonal direct sum of the irreducible flat Laplacians. This proves the final equivalence. $\square$

When E is finite and C is central, an irreducible representation of E is inflated from Γ if and only if its central character is trivial on C .

Corollary 5.5. *Under the hypotheses of Theorem 5.4, let $H \leq E$ contain C . The connected covers for H and H/C have equal flat spectra for every finite-dimensional unitary representation of H/C , pulled back to H . If $H \leq E$ has finite index and $C \not\subseteq H$, the covers for H and $p(H)$ have different degrees and cannot have equal scalar spectra.*

Proof. Put $\tilde{H} = H/C$, and let $M_H \rightarrow M$ and $N_{\tilde{H}} \rightarrow N$ be the corresponding covers. For a finite-dimensional unitary representation σ of $\tilde{H}$, put $\tau = \text{Ind}_{\tilde{H}}^{\Gamma} \sigma$. Apply the induction identity [27, Lemma 3.8.1] to the finite towers

$$Y_p \longrightarrow M_H \longrightarrow M, \quad \tilde{N} \longrightarrow N_{\tilde{H}} \longrightarrow N.$$

The composite covering has group Γ and the first covering has group $\tilde{H}$ in each tower. The coset intertwiner in that identity is unitary for Riemannian densities and commutes with the componentwise scalar Laplacian, so orientability is unnecessary. Theorem 5.4 gives

$$\begin{aligned} \text{Spec } \Delta_{M_H, (p|_H)^* \sigma} &= \text{Spec } \Delta_{M, p^* \tau} \\ &= \text{Spec } \Delta_{N, \tau} = \text{Spec } \Delta_{N_{\tilde{H}}, \sigma}. \end{aligned}$$

For any finite-index subgroup $H \leq E$, normality of C gives

$$[E : H] = [E : HC][HC : H] = [\Gamma : p(H)][C : C \cap H].$$

Suppose $C \not\subseteq H$. Since $C \neq 1$, M has positive dimension. The trivial representation gives equal scalar spectra of M and N ; N therefore has positive dimension as well. Weyl's law gives equal dimensions and volumes. Since $[C : C \cap H] > 1$,

$$\text{vol}(M_H) = [C : C \cap H] \text{vol}(N_{p(H)}) > \text{vol}(N_{p(H)}).$$

Weyl's law excludes equality of their scalar spectra. $\square$

Part II. Closed geodesics and wave traces

The rank-one $L_1 \times P_1$ -equivalence intertwines the cyclic actions and residual flows of Corollary 4.14. The geodesic closing equations also involve the exponential periods of G_1 and H_1 . Suitable cyclic quotients have different full period sets: in dimension eight all unitary flat spectra agree under one C_7 -marking, while in dimension nine the common half-wave trace is analytic near every period in the difference (Theorems 7.7 and 7.1). Equivariance under the residual flows also gives scalar-isospectral connection metrics on fixed manifolds with exclusive primitive periods, including infinitely many on a simply connected eighteen-manifold (Theorem 8.8).

6. CLOSING EQUATIONS AND CONNECTION METRICS

Let $R(s)$ denote rotation through angle s in an oriented real two-plane. Put

$$G_A = \text{U}(3), \quad G_B = \text{Sp}(1) \times \text{SO}(4).$$

Their bi-invariant metrics are specified by

$$Q_A(X, Y) = -\text{Re Tr}(XY), \quad Q_C(X, Y) = -\frac{1}{2} \text{Re Tr}(XY), \quad Q_D(X, Y) = -\frac{1}{2} \text{Tr}(XY). \quad (6.1)$$

The first two traces are taken in the defining complex representations; the third is taken in the defining real representation. On G_B the metric is $Q_C \oplus Q_D$. Define

$$\iota_A(s, v) = \text{diag}(e^{is}, e^{-is}, e^{iv}), \quad (6.2)$$

$$\iota_B(s, v) = (e^{is}, \text{diag}(R(s), R(v))). \quad (6.3)$$

The first component on the second line is a unit quaternion. Throughout,

$$h = \frac{2\pi}{7}, \quad \zeta = e^{ih}.$$

For $i = A, B$ set

$$\ell_i = \iota_i(h, 0), \quad r_i = \iota_i(3h, h), \quad \Phi_i(g) = \ell_i g r_i^{-1},$$

and define orbit quotients

$$M_9 = G_A / \langle \Phi_A \rangle, \quad N_9 = G_B / \langle \Phi_B \rangle. \quad (6.4)$$

6.1. Quaternionic coordinates and metric scales. Write $\mathbb{H} = \mathbb{C}\mathbf{j} \oplus \mathbb{C}$, where the ordered bases of the two real planes are $(\mathbf{j}, \mathbf{k})$ and $(1, \mathbf{i})$. The homomorphism

$$\mathrm{Sp}(1) \times \mathrm{Sp}(1) \longrightarrow \mathrm{SO}(4), \quad (a, b) : z \longmapsto azb^{-1}, \quad (6.5)$$

is the double covering with kernel $\{(1, 1), (-1, -1)\}$ [90, §4.1, equations (4.1.40)–(4.1.42)]. Its differential is $(u, v) \mapsto L_u - R_v$.

For $a = e^{i\alpha}$ and $b = e^{i\beta}$, the two plane angles are

$$(s, v) = (\alpha + \beta, \alpha - \beta). \quad (6.6)$$

Indeed, for $w \in \mathbb{C}$, the identity $\mathbf{j}e^{-i\beta} = e^{i\beta}\mathbf{j}$ gives $e^{i\alpha}(\mathbf{w}\mathbf{j})e^{-i\beta} = e^{i(\alpha+\beta)}\mathbf{w}\mathbf{j}$, whereas $e^{i\alpha}\mathbf{w}e^{-i\beta} = e^{i(\alpha-\beta)}\mathbf{w}$. For imaginary quaternions u, v , let L_u, R_v denote left and right multiplication. One has

$$-\frac{1}{2} \mathrm{Tr}_{\mathbb{R}}(L_u^2) = 2|u|^2, \quad -\frac{1}{2} \mathrm{Tr}_{\mathbb{R}}(R_v^2) = 2|v|^2, \quad \mathrm{Tr}_{\mathbb{R}}(L_u R_v) = 0.$$

The first two identities follow from $u^2 = -|u|^2$, $v^2 = -|v|^2$. For the third identity, bilinearity reduces to $u, v \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$. If $u = v$, the diagonal entries of $L_u R_u$ in the quaternionic basis consist of $-1, -1, 1, 1$. If $u \neq v$, quaternion multiplication sends every basis vector to a signed different basis vector, so all four diagonal entries are zero. Since $L_u R_v = R_v L_u$, expansion of $-\frac{1}{2} \mathrm{Tr}_{\mathbb{R}}(L_u - R_v)^2$ gives the pullback of Q_D :

$$2(|u|^2 + |v|^2). \quad (6.7)$$

The defining complex matrix of an imaginary quaternion u has eigenvalues $i|u|$ and $-i|u|$. Hence $Q_C(u, u) = -\frac{1}{2} \mathrm{Re} \mathrm{Tr}_{\mathbb{C}}(u^2) = |u|^2$. Consequently G_B has the Riemannian double cover

$$S^3(1) \times S^3(\sqrt{2}) \times S^3(\sqrt{2}).$$

These conventions also give Euclidean norm in the torus coordinates (x_0, x_1, x_2) consisting of one quaternionic angle and the two real plane angles.

6.2. The finite actions and their torus coordinates. Both ℓ_i and r_i have order seven, so $\Phi_i^7 = I$. For $1 \leq j \leq 6$, a fixed point g would satisfy $g^{-1}\ell_i^j g = r_i^j$, so the two factors would be conjugate in G_i .

On the unitary side,

$$\det \ell_A^j = 1, \quad \det r_A^j = e^{ijh} \neq 1.$$

Thus they are not conjugate. On the other side, their first components in $\mathrm{Sp}(1)$ have angles jh and $3jh$. Conjugacy of unit quaternions identifies the angles up to sign modulo 2π . It would require $j \equiv \pm 3j \pmod{7}$, impossible for $j \neq 0$. Thus both cyclic actions are free and have exactly seven elements.

With their bi-invariant metrics, G_A, G_B are globally symmetric and have nonnegative sectional curvature; their geodesics are $g \exp(tV)$ [90, Lemma E.4.1, Proposition E.4.2 and equation (E.4.13)]. The finite free quotients are therefore closed real-analytic locally symmetric nine-manifolds of nonnegative sectional curvature.

The torus coordinates are $i \operatorname{diag}(x_1, x_2, x_3)$ on $\mathfrak{u}(3)$, $i x_0$ on $\mathfrak{sp}(1)$, and the two plane-rotation generators of angular speeds x_1, x_2 on $\mathfrak{so}(4)$. Equations (6.1) give Euclidean squared norm in these coordinates. The full torus period lattices are

$$2\pi\mathbb{Z}^3 \quad \text{on both groups.} \quad (6.8)$$

For $\mathrm{SO}(4)$, each real two-plane has independent period 2π . The spin covering group instead has plane-angle period lattice $\{2\pi(a, b) : a, b \in \mathbb{Z}, a \equiv b \pmod{2}\}$, since both $(s+v)/2$ and $(s-v)/2$ must be in $2\pi\mathbb{Z}$ there.

6.3. Scalar spectra. Let $B_G(t; L, R)$ denote the bilateral heat trace in (2.6).

Corollary 6.1. *The manifolds M_9, N_9 have equal scalar Laplace spectra, including multiplicities.*

Proof. The invariant projection is $\frac{1}{7} \sum_{j=0}^6 \Phi_i^{j*}$. Finite-quotient descent in Section 3.6 gives

$$Z_{M_9}(t) = \frac{1}{7} \sum_{j=0}^6 B_{G_A}(t; \ell_A^j, r_A^j), \quad Z_{N_9}(t) = \frac{1}{7} \sum_{j=0}^6 B_{G_B}(t; \ell_B^j, r_B^j).$$

Equation (4.12), at $(u, s, v) = (jh, 3jh, jh)$, equates each summand. Lemma 2.3 then equates the spectral multiplicities; its absolute-convergence hypothesis follows by bounding their differences by the two scalar heat traces. $\square$

6.4. Circle quotients and marked connection spectra. The induced metric on either parameter torus is $2 ds^2 + dv^2$. Let

$$H_i = \{u_i(0, v) : v \in \mathbb{R}/(2\pi\mathbb{Z})\}, \quad X_i = G_i/H_i \quad (i = A, B), \quad X_A = A, \quad X_B = B,$$

with the normal homogeneous submersion metrics. Set

$$r_{i,0} = u_i(3h, 0), \quad \phi_i(gH_i) = \ell_i g r_{i,0}^{-1} H_i. \quad (6.9)$$

The elements $r_{i,0}$ centralize H_i , so these are well-defined isometries. Put

$$M_8 = A/\langle \phi_A \rangle, \quad N_8 = B/\langle \phi_B \rangle. \quad (6.10)$$

The period-seven flows extending these deck transformations are

$$F_i(s)(gH_i) = u_i(hs, 0) g u_i(3hs, 0)^{-1} H_i, \quad \phi_i = F_i(1). \quad (6.11)$$

Lemma 6.2. *The spaces A, B are simply connected. The actions in (6.10) are free of order seven.*

Proof. The map

$$U(3)/H_A \longrightarrow SU(3), \quad gH_A \longmapsto g \operatorname{diag}(1, 1, (\det g)^{-1})$$

is well defined: right multiplication by $\operatorname{diag}(1, 1, z)$ multiplies $\det g$ by z and cancels in the displayed expression. Its inverse is $g \mapsto gH_A$. Both maps are smooth. Thus $A \simeq SU(3)$. The fibration $SU(2) \rightarrow SU(3) \rightarrow S^5$, given by the first column, and its homotopy sequence give $\pi_1(A) = \pi_2(A) = 0$ [47, Corollary 4.9 and Theorem 4.41].

The group H_B fixes the ordered pair (j, k) , so $SO(4)/H_B \simeq V_2(\mathbb{R}^4)$. The maps

$$(x, y) \longmapsto (x, x^{-1}y), \quad (x, z) \longmapsto (x, xz)$$

identify the Stiefel manifold with $S^3 \times S^2$: $\operatorname{Re}(x^{-1}y) = \langle x, y \rangle = 0$ and $|x^{-1}y| = 1$. Consequently $B \simeq S^3 \times S^3 \times S^2$, with $\pi_1(B) = 0$ and $\pi_2(B) = \mathbb{Z}$. A full rotation in H_B lifts in the spin group from $(1, 1)$ to $(-1, -1)$, so $\pi_1(H_B) \rightarrow \pi_1(G_B) = C_2$ is reduction modulo two.

If ϕ_A^j fixed a coset for $1 \leq j \leq 6$, then ℓ_A^j would be conjugate to $r_{A,0}^j b$ for some $b = \operatorname{diag}(1, 1, e^{iv})$. Determinants force $e^{iv} = 1$. The remaining eigenvalue sets would require

$$\{j, -j, 0\} = \{3j, -3j, 0\} \quad \text{in } \mathbb{Z}/7\mathbb{Z},$$

which is impossible. On B , a fixed coset would require the first quaternionic components e^{ijh} and e^{3ijh} to be conjugate. This requires $j \equiv \pm 3j \pmod{7}$, also impossible. Both actions have seventh power the identity, hence have order seven and are free. The markings of the quotient fundamental groups send $1 \in C_7$ to the displayed deck generator. $\square$

The simply connected product presentation is

$$B = \frac{S^3(1) \times S^3(\sqrt{2}) \times S^3(\sqrt{2})}{\tilde{H}}, \quad \tilde{H} = \{(1, e^{iv}, e^{-iv}) : v \in \mathbb{R}/(2\pi\mathbb{Z})\}. \quad (6.12)$$

Equation (6.6) identifies $\tilde{H}$ with the full preimage of H_B : the first plane angle is zero precisely when $\alpha + \beta \in 2\pi\mathbb{Z}$. This preimage is connected and contains the spin kernel $(1, -1, -1)$ at $v = \pi$. The lifts of the two deck factors are

$$\tilde{\ell} = (e^{ih}, e^{ih/2}, e^{ih/2}), \quad \tilde{r} = (e^{3ih}, e^{3ih/2}, e^{3ih/2}). \quad (6.13)$$

Their seventh powers are the same central element $(1, -1, -1)$, so their two-sided action has order seven even before quotienting by $\tilde{H}$.

6.4.1. The quotient metric and its curvature. Lemma 4.1 identifies the quotient Laplacians with the restrictions of the group Laplacians to right- H_i -invariant functions, including their form and operator domains. The curvature identities of Proposition 4.15 apply to both quotients. The commutator bounds are $\|[X, Y]\| \leq 2\|X\|\|Y\|$ on the unitary and unit-quaternion factors, and $\sqrt{2}\|X\|\|Y\|$ on each spin factor with its metric multiplied by two. The first follows from the Frobenius inequality and the second from $[u, v] = 2u \times v$. For the product, squaring and summing gives the bound with constant two. Equation (4.16) therefore gives $0 \leq K \leq 4$ on both quotients.

On a round three-sphere of radius R , the Ricci endomorphism is $2R^{-2}I$. The three covering spheres in Section 6 therefore give $\text{Ric}_{G_B} \geq g_{G_B}$. Equation (4.17) implies $\text{Ric}_B \geq g_B$. For $A = \text{U}(3)/H_A$, the circle generator $T = iE_{33}$ satisfies $\text{Tr } T = i$ and $Q_A(T, T) = 1$. Equation (4.15) therefore gives

$$\text{Ric}_A \geq \frac{1}{2}g_A, \quad \text{Ric}_B \geq g_B.$$

The group G_B has volume

$$\frac{1}{2} \text{Vol}(S^3(1)) \text{Vol}(S^3(\sqrt{2}))^2 = 32\pi^6.$$

Equation (4.12) at the identity gives the same group heat trace on G_A . The leading term $\text{Vol}(G)(4\pi t)^{-\dim G/2}$ of the scalar heat trace [27, Section 3.10] consequently gives $\text{Vol}(G_A) = 32\pi^6$. The circles H_i have length 2π , so

$$\text{Vol}(A) = \text{Vol}(B) = 16\pi^5. \quad (6.14)$$

The finite quotients have volume $16\pi^5/7$. These quotient metrics are analytic: in a slice given by $\exp(\mathfrak{h}^\perp)$ the metric coefficients are obtained from analytic group operations and inversion of the nondegenerate horizontal metric. The finite free quotient preserves this regularity.

6.4.2. The seven flat line bundles. Since $\phi_i = F_i(1)$, Corollary 4.14 gives

$$\text{Tr}(\phi_A^{j*} e^{-t\Delta_A}) = \text{Tr}(\phi_B^{j*} e^{-t\Delta_B}) \quad (j \in C_7, t > 0). \quad (6.15)$$

For $\chi_r(j) = e^{2\pi i r j / 7}$, character projection gives

$$\Pi_{i,r} = \frac{1}{7} \sum_{j=0}^6 \overline{\chi_r(j)} \phi_i^{j*}, \quad \text{ran } \Pi_{i,r} = \{f : \phi_i^* f = e^{2\pi i r / 7} f\}.$$

Since A, B are the universal covers, this range represents the flat line bundle labelled by χ_r in (5.1). For a smooth section s , let $\tilde{s}$ be its equivariant lift and put $J_i s = 7^{-1/2} \tilde{s}$. Fibre integration gives

$$\begin{aligned} \|J_i s\|_{L^2(X_i)}^2 &= \|s\|_{L^2(X_i/\langle \phi_i \rangle)}^2, \\ \|dJ_i s\|_{L^2(X_i)}^2 &= \|\nabla s\|_{L^2(X_i/\langle \phi_i \rangle)}^2. \end{aligned}$$

Thus J_i extends to a unitary onto $\text{ran } \Pi_{i,r}$ and identifies the closed forms and their associated self-adjoint operators, including their domains. Equation (6.15) equates the projected heat traces and hence their multiplicities. Every finite-dimensional unitary representation of C_7 is an orthogonal sum of the χ_r ; therefore (7.13) holds with the same marking.

6.5. Closing equations for normal homogeneous quotients. Let G have a bi-invariant metric, let $H \leq G$ be closed, and give G/H its normal homogeneous metric. Its geodesics are $\gamma(t) = g \exp(tV)H$, where $V \perp \mathfrak{h}$ [90, Proposition E.4.2], [63, Problem 5-6(b), equation (5.27)]. The lift $g \exp(tV)$ is horizontal, with constant speed $|V|$. The segment $0 \leq t \leq 1$ therefore has length $|V|$.

Lemma 6.3. *Suppose r centralizes H and set $\Phi(gH) = \ell g r^{-1} H$. For $j \in \mathbb{Z}$, the endpoints and velocities of $\gamma|_{[0,1]}$ match through Φ^j if and only if, for some $b \in H$,*

$$V \perp \mathfrak{h}, \quad e^V = g^{-1} \ell^j g r^{-j} b, \quad \text{Ad}_{r^j b^{-1}} V = V. \quad (6.16)$$

For $H = \{1\}$ these equations reduce to

$$e^V = g^{-1} \ell^j g r^{-j}, \quad \text{Ad}_{r^j} V = V. \quad (6.17)$$

For $V = 0$ they mean that gH is fixed by Φ^j . If $V \neq 0$ and Φ generates a finite free action, they characterize smoothly closed quotient geodesics of length $|V|$.

Proof. Put $a = \ell^j g r^{-j}$. Endpoint equality is $g e^V = ab$ for some $b \in H$. At the representative a , the terminal velocity is represented by $\text{Ad}_b V$, since $ab \exp(tV)H = a \exp(t \text{Ad}_b V)H$. The transformed initial velocity is represented by $\text{Ad}_{r^j} V$. Both vectors lie in $\mathfrak{h}^\perp$, on which the quotient differential is injective. Their equality is precisely the last equation in (6.16). Conversely the equations give both endpoint and velocity equality. Uniqueness of the geodesic equation then extends the segment smoothly across every translated copy. Every quotient geodesic lifts horizontally, so all initial data are covered. Setting $H = \{1\}$ gives (6.17). $\square$

Coordinates for the closing equations. For G_A and G_B , the torus exponential has kernel $2\pi\mathbb{Z}^3$ in the coordinates of (6.8). On the covering group $\tilde{G}_B = \text{Sp}(1)^3$, quaternionic angular coordinates (α, β, γ) project to

$$(x_0, x_1, x_2) = (\alpha, \beta + \gamma, \beta - \gamma), \quad |V|^2 = \alpha^2 + 2\beta^2 + 2\gamma^2.$$

The regular cyclic quotients. For $1 \leq j \leq 6$, the three unitary eigenvalues of r_A^j have distinct residues $3j, -3j, j$ modulo seven. If $R = \text{diag}(z_1, z_2, z_3)$, then

$$[V, R]_{ab} = (z_b - z_a) V_{ab}.$$

Thus $[V, r_A^j] = 0$ forces every off-diagonal entry of V to vanish. For the first factor of G_B , write $r = e^{i\alpha}$ with $\sin \alpha \neq 0$. An imaginary quaternion u commutes with r precisely when $[u, i] = 0$, or equivalently $u \in \mathbb{R}i$. In the orthogonal factor, the four complex eigenvalues of r_B^j have residues $3j, -3j, j, -j$, all distinct. A commuting real operator preserves each complex eigenline and therefore each of the two real rotation planes. Its skew-symmetric restriction to either plane is a real multiple of that plane's rotation generator. Hence the Lie-algebra centralizer is precisely the specified two-plane torus. Equation (6.17) therefore makes V toral on both groups for every nonidentity deck power.

In the Euclidean torus coordinates of Section 6.2, put

$$a_A = (1, -1, 0), \quad b_A = (3, -3, 1), \quad a_B = (1, 1, 0), \quad b_B = (3, 3, 1).$$

The orbit of a_A under the unitary Weyl group is its six coordinate permutations. The orbit of a_B under the product Weyl group is

$$\mathcal{W}_{BaB} = \{(\epsilon, \eta, 0), (\epsilon, 0, \eta) : \epsilon, \eta \in \{1, -1\}\}. \quad (6.18)$$

Indeed the first coordinate may change sign, and even signed permutations in the last two coordinates take $(1, 0)$ to $(\pm 1, 0)$ or $(0, \pm 1)$.

Proposition 6.4. *For $i = A, B$, let $P_A = M_9$, $P_B = N_9$, let T_i be the specified maximal torus of G_i , and let $\mathcal{W}_i$ be its Weyl group. Then*

$$\text{Lsp}(P_i) = \left\{ h|m| : \begin{array}{l} m \in \mathbb{Z}^3 \setminus \{0\}, \text{ and either } m \in 7\mathbb{Z}^3, \\ \text{or } m \equiv j(wa_i - b_i) \pmod{7} \text{ for some } 1 \leq j \leq 6, w \in \mathcal{W}_i \end{array} \right\}. \quad (6.19)$$

In particular both length sets are locally discrete in $(0, \infty)$.

Proof. For $1 \leq j \leq 6$, the centralizer calculations give $V \in \mathfrak{t}_i$. Equation (6.17) makes $\exp V r_i^j \in T_i$ conjugate to ℓ_i^j . The groups G_i are compact and connected, so conjugacy in T_i is equivalence under $\mathcal{W}_i$ [26, Proposition 0.1]. Thus $\exp V r_i^j = w(\ell_i^j)$ for some $w \in \mathcal{W}_i$. By (6.8),

$$V = hm, \quad m \equiv j(wa_i - b_i) \pmod{7}.$$

Conversely, choose $g \in N_{G_i}(T_i)$ with $g^{-1}\ell_i g = w(\ell_i)$ and any integer lift m of the congruence. The vector $V = hm$ commutes with r_i^j , and both equations in (6.17) hold.

For $j = 0$, one has $\exp V = 1$. The closure of $\exp(RV)$ is a torus, and [90, Proposition 6.A.3 and equations (6.A.12), (6.A.14), (6.A.16)] places it in a conjugate of T_i . Hence V is conjugate to a vector of $\mathfrak{t}_i$ with coordinates $2\pi k$, $k \in \mathbb{Z}^3$. Conjugation preserves the norm, so these lengths are exactly $h|m|$ with $m \in 7\mathbb{Z}^3 \setminus \{0\}$. Conversely, every such torus vector has exponential one. A length at most T has squared length $h^2 n$ with $1 \leq n \leq T^2/h^2$; only finitely many lengths occur. $\square$

Lemma 6.5. *Let T be a maximal torus of a compact Lie group with a bi-invariant metric, let $n \in N_G(T)$, and let a finite free cyclic action be generated by $x \mapsto \ell x r^{-1}$ with $\ell, r \in T$. The coset Tn descends to an embedded totally geodesic flat torus in the quotient. Every primitive closed geodesic contained in this torus is embedded.*

Proof. The map $t \mapsto tn$ identifies the restricted deck action with translation by $\delta = \ell n r^{-1} n^{-1} \in T$, since $\ell(tn)r^{-1} = t\delta n$. Therefore the image is the quotient torus $T/\langle\delta\rangle$. Its map into the ambient quotient is injective: two representatives in Tn with the same image differ by one of these restricted deck translations. It is an immersion because both quotient maps are local isometries, and it is an embedding because its domain is compact and the ambient quotient is Hausdorff. The bi-invariant connection restricts to zero on the abelian Lie algebra $\mathfrak{t}$; hence Tn is flat and totally geodesic, and these local properties descend. A straight line on a flat torus is a translate of a homomorphism $\mathbb{R} \rightarrow \mathfrak{t}/\Lambda$. If its least positive period is p , its kernel is $p\mathbb{Z}$. It therefore induces an injective immersion of $\mathbb{R}/p\mathbb{Z}$, which is an embedding by compactness. $\square$

6.6. Twisted geodesics at rational flow times. Let F be an isometry of a Riemannian manifold X . A geodesic $\gamma : [0, 1] \rightarrow X$ is F -twisted if

$$\gamma(1) = F(\gamma(0)), \quad \dot{\gamma}(1) = dF_{\gamma(0)}\dot{\gamma}(0).$$

A constant geodesic is F -twisted precisely when its value is a fixed point of F .

Lemma 6.6. *Let $q \geq 1$, $\gcd(q, 6) = 1$, $N = 7q$, and $h_q = 2\pi/N = h/q$. For every integer j , every $F_A(j/q)$ - or $F_B(j/q)$ -twisted geodesic has squared length in $h_q^2 \mathbb{Z}_{\geq 0}$. On either A or B , every nonconstant ordinary closed geodesic has length at least 2π and squared length in $4\pi^2 \mathbb{Z}_{>0}$.*

Proof. Since $F_i(s+7) = F_i(s)$, the twisting map depends only on j modulo N . Assume $j \in \{0, \dots, N-1\}$. By Lemma 6.3, a twisted segment has a horizontal velocity V and, for some $g \in G_i$ and $b \in H_i$,

$$R_i = \iota_i(3jh_q, 0)b^{-1}, \quad e^V R_i = g^{-1}\iota_i(jh_q, 0)g, \quad [V, R_i] = 0.$$

Since $\gcd(N, 6) = 1$, the relation $6j \equiv 0 \pmod{N}$ forces $j = 0$. For $j \neq 0$, the angle $3jh_q$ is therefore neither zero nor π modulo 2π , and $e^{3ijh_q} \neq e^{-3ijh_q}$.

On A , write $V = iB$ with B Hermitian. Horizontality is $B_{33} = 0$. The effective right element in (6.16) is

$$R = R_A = \text{diag}(e^{3ijh_q}, e^{-3ijh_q}, e^{iv}).$$

The first two entries are distinct. If all three entries are distinct, $[B, R] = 0$ makes B diagonal and horizontality makes its third entry zero. For each of its first two eigenvalues θ , the corresponding eigenvalue of $e^{iB}R$ is $e^{i(\theta+3\epsilon jh_q)}$ for an $\epsilon \in \{1, -1\}$. If the last entry of R equals one of the first two, its eigenspaces have dimensions two and one, and B preserves them. On each of these two blocks,

$R = e^{3i\epsilon jh_q} I$ for one of the same two signs. In either case each eigenangle on a block with right phase $e^{3i\epsilon jh_q}$ satisfies

$$e^{i(\theta+3\epsilon jh_q)} \in \{e^{ijh_q}, e^{-ijh_q}, 1\}, \quad \theta = h_q(t - 3\epsilon j + Nn)$$

for some $t \in \{j, -j, 0\}$ and $n \in \mathbb{Z}$. Thus all three eigenvalues of B belong to $h_q\mathbb{Z}$, and $|V|^2 = \text{Tr}(B^2) \in h_q^2\mathbb{Z}_{\geq 0}$.

On B , the symplectic component centralizes the nonreal quaternion e^{3ijh_q} , so it is $\theta_C i$. Its exponential equation gives $\theta_C + 3jh_q \equiv \pm jh_q \pmod{2\pi}$ and hence $\theta_C \in h_q\mathbb{Z}$. In $\text{SO}(4)$ the effective right element is

$$R_D = \text{diag}(R(\alpha), R(v)), \quad \alpha = 3jh_q.$$

If $v \neq \pm\alpha$, the two real planes have disjoint complex spectra. A commuting real matrix preserves both planes; its skew-symmetric restriction to each plane is a multiple of the plane-rotation generator. This remains true when $v = 0$ or π , since the second eigenspace is the second real plane and $\sin \alpha \neq 0$. Write the two angular speeds as θ, η . Horizontality is $\eta = 0$. The complex eigenvalues of $e^{V_D} R_D$ are those of a rotation through jh_q and a fixed plane. Hence $\theta + \alpha$ is congruent to 0, jh_q , or $-jh_q$ modulo 2π . Since $\alpha = 3jh_q$ and $2\pi = Nh_q$, one has $\theta \in h_q\mathbb{Z}$, and $|V_D|^2 = \theta^2$.

If $v \equiv \pm\alpha$, put $J = (R_D - \cos \alpha I) / \sin \alpha$. Then $J^2 = -I$ and $J^T J = I$. The equations $[V_D, R_D] = 0$ and $[V_D, J] = 0$ are equivalent. A real skew-symmetric operator commuting with J is skew-Hermitian on the complex two-dimensional space $(\mathbb{R}^4, J)$. The unitary spectral theorem therefore gives its complex eigenvalues as $i\theta_1, i\theta_2$, with $\theta_1, \theta_2 \in \mathbb{R}$. On this complex space $R_D = e^{i\alpha} I$, so the complex eigenvalues of $e^{V_D} R_D$ are $e^{i(\theta_1+\alpha)}$ and $e^{i(\theta_2+\alpha)}$. Their complex conjugates complete the real spectrum, which by the closing equation is $\{e^{ijh_q}, e^{-ijh_q}, 1, 1\}$. Thus each $\theta_a + \alpha$ is congruent to 0, jh_q , or $-jh_q$ modulo 2π , and both θ_a are multiples of h_q . Each eigenvalue $-\theta_a^2$ of V_D^2 occurs twice over $\mathbb{R}$, whence

$$|V_D|^2 = -\frac{1}{2} \text{Tr}_{\mathbb{R}}(V_D^2) = \theta_1^2 + \theta_2^2.$$

For $j = 0$ on A , $R = \text{diag}(1, 1, e^{iv})$. If $e^{iv} \neq 1$, commutation splits off the third coordinate and $B_{33} = 0$ leaves the third eigenvalue of $e^{iB} R$ unequal to one, a contradiction. Thus $R = I$ and the eigenvalues of B are multiples of 2π .

For $j = 0$ on B , the symplectic exponential must be one. The orthogonal effective element is $\text{diag}(I_2, R(v))$. When $v \neq 0$ modulo 2π , the spectra of its two plane restrictions are disjoint, even at $v = \pi$, so its Lie-algebra centralizer is block diagonal. The last horizontal angular speed is zero, so its exponential cannot cancel $R(v)$. Therefore $R_D = I$ and $e^{V_D} = I$. Its two toral eigenangles and the magnitude of the symplectic vector are multiples of 2π . On either space the resulting squared norm is

$$|V|^2 = 4\pi^2(n_1^2 + n_2^2 + n_3^2), \quad n_1, n_2, n_3 \in \mathbb{Z}.$$

For $V \neq 0$ this is at least $4\pi^2$, and for $V = 0$ it is zero. $\square$

6.7. Connection metrics and flat connection spectra.

Lemma 6.7. *Let (W_i, g_i) , $i = 0, 1$, be closed Riemannian manifolds with Killing fields K_i . Suppose that a unitary map $T : L^2(W_0, dv_{g_0}) \rightarrow L^2(W_1, dv_{g_1})$ intertwines their scalar Laplacians and the one-parameter groups generated by K_0, K_1 . Let (Z, b) be a closed Riemannian manifold, let α be a smooth real one-form on Z , and let $E \rightarrow Z$ be a finite-rank Hermitian bundle with a metric connection ∇^E . On $W_i \times Z$, set*

$$G_i(v + X, w + Y) = g_i(v + \alpha(X)K_i, w + \alpha(Y)K_i) + b(X, Y). \quad (6.20)$$

Then $T \otimes I$ intertwines the connection Laplacians on the pullbacks of E to $(W_i \times Z, G_i)$, including their self-adjoint domains. In particular, the scalar Laplacians are unitarily equivalent.

Proof. Let Ψ_i be the flow of K_i , and let P be the closure of $\{(\Psi_0(t), \Psi_1(t)) : t \in \mathbb{R}\}$ in $\text{Isom}(W_0, g_0) \times \text{Isom}(W_1, g_1)$. The isometry groups are compact Lie groups [39, Theorem 2.1]; their connected abelian subgroup P is therefore a compact torus, possibly trivial. Write $\exp(t\xi) = (\Psi_0(t), \Psi_1(t))$, with $\xi \in \mathfrak{p}$, and regard the two actions as right actions. Strong continuity extends the equivariance of T to P .

On the trivial principal bundle $Z \times P$, take the connection $\omega = \theta_P - \alpha\xi$, where θ_P is the Maurer–Cartan form. Under the identification $[(z, p), x] \mapsto (xp^{-1}, z)$, its horizontal vectors are $(-\alpha(X)K_i, X)$. Its connection metric is therefore (6.20). Proposition 4.7, with coefficient bundle E , gives the unitary $T \otimes I$ and the stated equivalence, including the operator domains.

The shear $A_i(v, X) = (v + \alpha(X)K_i, X)$ has determinant one and $G_i = A_i^*(g_i \oplus b)$. Hence $dv_{G_i} = dv_{g_i} dv_b$ and, for every smooth section f of the pullback of E ,

$$\mathcal{E}_i(f) = \int_{W_i \times Z} (|d_{W_i} f|_{g_i}^2 + |\nabla_Z^E f - \alpha K_i f|_b^2) dv_{g_i} dv_b. \quad (6.21)$$

□

Proposition 6.8. *Let (W, g_W) be closed and connected, let Ψ be an isometric flow of period seven with generator K , and let $L > 0$. On $P = W \times (\mathbb{R}/\mathbb{Z})$, for $\tau \in \mathbb{R}$ define*

$$g_\tau(v + a\partial_s, w + b\partial_s) = g_W(v + \tau aK, w + \tau bK) + L^2 ab.$$

This metric is isometric to the quotient of $(W \times \mathbb{R}, g_W + L^2 ds^2)$ by $T_\tau(x, s) = (\Psi(\tau)x, s + 1)$, and its volume form is $L dv_W ds$. Its Riemannian universal cover is $(\widetilde{W} \times \mathbb{R}, \widetilde{g}_W + L^2 ds^2)$.

For $d \in \mathbb{Z}_{>0}$, on $P_d = W \times (\mathbb{R}/d\mathbb{Z})$ let E_ϑ be the flat line bundle pulled back from the character $d \mapsto e^{2\pi i \vartheta}$ of $d\mathbb{Z}$. If $(\lambda_a, i h r_a)$, $h = 2\pi/7$, are the joint eigenvalues of (Δ_W, K) , repeated with their multiplicities, then $r_a \in \mathbb{Z}$ and the complete connection spectrum is

$$E_{a,m} = \lambda_a + \frac{4\pi^2}{L^2} \left(\frac{m + \vartheta}{d} - \frac{\tau r_a}{7} \right)^2, \quad m \in \mathbb{Z}. \quad (6.22)$$

Thus equality of joint eigenspace multiplicities for two flows on (W, g_W) gives equality of all these spectra, with every fixed finite-dimensional unitary representation of $d\mathbb{Z}$ and every $d \geq 1$.

For $\tau = 1/q$, $q \in \mathbb{Z}_{>0}$, the product $(W \times \mathbb{R}/(7q\mathbb{Z}), g_W + L^2 ds^2)$ is a Riemannian cover of degree $7q$. For every τ ,

$$\text{inj}(P, g_\tau) \geq \min\{\text{inj}(W, g_W), L/2\} > 0. \quad (6.23)$$

All covariant derivatives of curvature have bounds independent of τ . If W, g_W, K are real analytic, the metrics are jointly analytic in space and τ and converge to $g_W + L^2 ds^2$ in C^∞ as $\tau \rightarrow 0$.

Proof. Translation in s makes $\langle T_\tau \rangle$ free and properly discontinuous. The maps

$$(x, [s]) \mapsto [\Psi(\tau s)x, s], \quad [y, s] \mapsto (\Psi(-\tau s)y, [s]) \quad (6.24)$$

are inverse diffeomorphisms. Their differentials give the displayed metric and its volume form. Composing the universal covering map of W with the quotient map gives the simply connected Riemannian cover.

For the line bundle on P_d , take $Z = \mathbb{R}/d\mathbb{Z}$, $b = L^2 ds^2$, and $\alpha = \tau ds$ in (6.21). In the parallel frame on $\mathbb{R}$, sections satisfy $f(x, s + d) = e^{2\pi i \vartheta} f(x, s)$, and the closed form is the closure of

$$\mathcal{E}_\tau(f, f) = \int_0^d \int_W (|d_W f|^2 + L^{-2} |\partial_s f - \tau K f|^2) L dv_W ds. \quad (6.25)$$

The initial domain consists of smooth equivariant functions on $W \times \mathbb{R}$, so every derivative has the same boundary multiplier. Unitarity of that multiplier cancels the boundary terms in integration by parts. Since the flow preserves volume, $K^* = -K$; moreover K commutes with ∂_s and Δ_W . The associated differential operator is

$$\Delta_\tau = \Delta_W - L^{-2} (\partial_s - \tau K)^2. \quad (6.26)$$

Diagonalize the circle action on each finite-dimensional Laplace eigenspace. Its period seven gives a complete orthonormal joint basis v_a with $Kv_a = i h r_a v_a$, $r_a \in \mathbb{Z}$. The sections

$$e_{a,m}(x, s) = (Ld)^{-1/2} v_a(x) e^{2\pi i(m+\vartheta)s/d}$$

form a complete orthonormal basis of smooth sections, and (6.26) gives their eigenvalues (6.22). Lemma 2.4, applied to this basis, gives

$$\begin{aligned} D(\mathcal{E}_\tau) &= \left\{ \sum c_{a,m} e_{a,m} : \sum (1 + E_{a,m}) |c_{a,m}|^2 < \infty \right\}, \\ D(\Delta_\tau) &= \left\{ \sum c_{a,m} e_{a,m} : \sum (1 + E_{a,m}^2) |c_{a,m}|^2 < \infty \right\}. \end{aligned}$$

For $E_{a,m} \leq R$, $\lambda_a \leq R$ and $|m + \vartheta - d\tau r_a/7| \leq Ld\sqrt{R}/(2\pi)$; hence only finitely many pairs occur. A finite-dimensional unitary representation of $d\mathbb{Z}$ splits into the eigenspaces of its generator, and its connection spectrum is the sum of the corresponding character spectra.

At $\tau = 1/q$, $T_\tau^{7q}(x, s) = (x, s + 7q)$. The covering map in the fixed coordinates of P is

$$(x, [s]_{7q}) \mapsto (\Psi(-s/q)x, [s]_1).$$

It pulls back $g_{1/q}$ to the product metric and has exactly $7q$ preimages over each point. Every nontrivial deck transformation of $W \times \mathbb{R} \rightarrow P$ has displacement at least L . If $a < \min\{\text{inj}(W, g_W), L/2\}$, the product exponential is injective and nonsingular on each tangent ball of radius a . Two endpoints in such a ball cannot differ by a nontrivial deck transformation, since their distance is less than $2a < L$. The quotient exponential is therefore injective and nonsingular there, proving (6.23).

Local isometry to the fixed product gives all curvature bounds. With $\eta = g_W(K, \cdot)$, the exact tensor difference is

$$g_\tau - (g_W + L^2 ds^2) = \tau(ds \otimes \eta + \eta \otimes ds) + \tau^2 |K|^2 ds \otimes ds. \quad (6.27)$$

Its coefficients are analytic under the stated hypotheses, and on the compact product its C^m norm is bounded by $C_m(|\tau| + \tau^2)$ for each $m \geq 0$. $\square$

Equip $W = A \times B$ with $g_W = g_A \oplus g_B$, and extend the flows by $\Psi_0(s) = (F_A(s), I)$ and $\Psi_1(s) = (I, F_B(s))$. Let S be the unitary of Corollary 4.14. On $L^2(W)$, set

$$U(f \otimes g) = S^{-1}g \otimes Sf.$$

Then U is unitary, $U^2 = I$, and

$$U\Delta_W = \Delta_W U, \quad UK_0 = K_1 U. \quad (6.28)$$

The identities hold on the joint eigenbasis and hence on the respective spectral domains. Since S fixes normalized constants, $U1 = 1$. Proposition 6.8 proves (8.6) for every $d\mathbb{Z}$ and its fixed representation. Since W is simply connected, these are all connected finite covers and all finite-dimensional unitary representations of their full fundamental groups. Their last coordinates induce the inclusions $de\mathbb{Z} \hookrightarrow d\mathbb{Z}$; pullback restricts the representation, as in Theorem 8.1.

7. LENGTH SETS IN DIMENSIONS EIGHT AND NINE

Let $S_3 = \{a^2 + b^2 + c^2 : a, b, c \in \mathbb{Z}\}$.

Theorem 7.1 (Different period sets in dimension nine). *The two actions in (6.4) are free of order seven. The quotients are smooth closed connected real-analytic nine-manifolds, locally symmetric and of nonnegative sectional curvature. They satisfy*

$$\text{Spec}(\Delta_{M_9}) = \text{Spec}(\Delta_{N_9}), \quad h\sqrt{14} \in \text{Lsp}(N_9) \setminus \text{Lsp}(M_9). \quad (7.1)$$

The length $h\sqrt{14}$ on N_9 is realized by an embedded primitive geodesic. Moreover, $\text{Lsp}(M_9) \subsetneq \text{Lsp}(N_9)$ and

$$\text{Lsp}(N_9) \setminus \text{Lsp}(M_9) = \{h\sqrt{n} : n \in S_3, 7 \mid n, 49 \nmid n\}. \quad (7.2)$$

As $T \rightarrow +\infty$,

$$\#((\text{Lsp}(N_9) \setminus \text{Lsp}(M_9)) \cap (0, T]) = \frac{5}{4\pi^2} T^2 + O(\log(2 + T)). \quad (7.3)$$

The common scalar half-wave trace is real analytic near every length in the difference. In particular, the difference contains the embedded primitive lengths

$$h\sqrt{(49k - 2)^2 + 10}, \quad k \in \mathbb{Z}_{\geq 0}. \quad (7.4)$$

7.1. A primitive length absent from the unitary quotient.

Lemma 7.2. *Let $q \in \mathbb{Z}_{>0}$. If $h\sqrt{q} \in \text{Lsp}(M_9)$ and $7 \mid q$, then $49 \mid q$.*

Proof. For a nonidentity deck element, Proposition 6.4 gives

$$q = |m|^2 \equiv j^2 |wa_A - b_A|^2 = j^2 (21 - 2\langle wa_A, b_A \rangle) \pmod{7}.$$

The scalar product is a difference of two distinct entries of $(3, -3, 1)$, so it belongs to $\{\pm 2, \pm 4, \pm 6\}$. Since 2 and j are invertible in $\mathbb{Z}/7\mathbb{Z}$, the residue $-2j^2\langle wa_A, b_A \rangle$ is nonzero. Thus $7 \mid q$ is incompatible with any nonidentity deck element. The remaining case has $m \in 7\mathbb{Z}^3$, and hence $49 \mid q$. $\square$

Corollary 7.3. *The lengths in (7.4) are pairwise distinct and belong to $\text{Lsp}(N_9) \setminus \text{Lsp}(M_9)$. Each has an embedded primitive representative on N_9 .*

Proof. Put $J = \text{diag}(1, -1, 1, -1)$, $g = (1, J)$, and

$$V_k = h(49k - 2, 3, -1) \quad (k \in \mathbb{Z}_{\geq 0}). \quad (7.5)$$

Conjugation by J reverses the two rotation angles, so $g^{-1}\ell_B g$ has torus angles $(h, -h, 0)$. Since

$$(49k - 2, 3, -1) \equiv (-2, -4, -1) \pmod{7},$$

one has

$$\exp V_k = g^{-1}\ell_B g r_B^{-1}, \quad \text{Ad}_{r_B} V_k = V_k.$$

Lemma 6.3 gives a closed geodesic of period one. A smaller period $0 < t < 1$ would give $\exp(tV_k) = g^{-1}\ell_B^j g r_B^{-j}$ for some $j \in \mathbb{Z}$. The last coordinate would then satisfy $t - j \in 7\mathbb{Z}$, a contradiction. Thus the least period is one. Moreover, $g \in N_{G_B}(T_B)$, so Lemma 6.5 makes the projected geodesic embedded.

Its squared length is

$$|V_k|^2 = h^2((49k - 2)^2 + 10), \quad (49k - 2)^2 + 10 \equiv 14 \pmod{49}.$$

Lemma 7.2 excludes this length from M_9 . Finally,

$$(49(k + 1) - 2)^2 - (49k - 2)^2 = 49(98k + 45) > 0,$$

so the lengths are distinct. $\square$

7.2. The length-set difference in dimension nine. Let $\mathfrak{W} = \{\pm 1\}^3 \rtimes \mathfrak{S}_3$ act on $(\mathbb{Z}/7\mathbb{Z})^3$ by signed coordinate permutations. Each orbit has a unique representative $v(m) = (a, b, c)$ with $0 \leq a \leq b \leq c \leq 3$: replace each coordinate by its least absolute residue and order the three values. If $v(m) = v(m')$, there is $w \in \mathfrak{W}$ such that $wm \equiv m' \pmod{7}$. The integral map w preserves squared norm. Consequently, for any set $C \subset (\mathbb{Z}/7\mathbb{Z})^3$, the two sets

$$\{|m|^2 : m \bmod 7 \in C\}, \quad \{|m|^2 : m \bmod 7 \in \mathfrak{W}C\}$$

are equal: if $m \bmod 7 = wc$ with $c \in C$, then $w^{-1}m$ has residue c and squared norm $|m|^2$. Thus the squared norms in (6.19) are determined by the orbit representatives $v(j(wa_i - b_i))$.

Write abc for the ordered triple (a, b, c) . Multiplication by 2 modulo seven induces the permutation

$$\kappa(0) = 0, \quad \kappa(1) = 2, \quad \kappa(2) = 3, \quad \kappa(3) = 1$$

on least absolute residues. Let $C(a, b, c)$ be the increasing rearrangement of $(\kappa(a), \kappa(b), \kappa(c))$. Thus

$$Cv(m) = v(2m).$$

Since $\mathbb{F}_7^\times = \{\pm 1, \pm 2, \pm 4\}$ and $v(-m) = v(m)$, the representatives obtained by multiplying a fixed nonzero deck residue by all $j \in \mathbb{F}_7^\times$ are exactly its C -orbit. These orbits are

$$\begin{aligned} \mathcal{O}_1 &= \{033, 011, 022\}, & \mathcal{O}_2 &= \{133, 112, 223\}, \\ \mathcal{O}_3 &= \{023, 013, 012\}, & \mathcal{O}_4 &= \{233, 113, 122\}. \end{aligned}$$

Indeed, applying $1 \mapsto 2 \mapsto 3 \mapsto 1$ to each displayed triple, then arranging its entries in increasing order, gives its successor. In the following table, the two entries in the last column and last row are distinct Weyl images; every other entry occurs once.

ν	$ m ^2 \pmod{7}$	$wa_A - b_A$	$wa_B - b_B$
033	4	$(-4, 3, 0)$	$(-4, -3, 0)$
133	5	$(-4, 4, -1)$	$(-4, -4, -1)$
023	6	$(-3, 2, 0)$	$(-2, -3, 0)$
233	1	$(-3, 4, -2)$	$(-4, -3, -2)$
122	2	$(-2, 2, -1)$	$(-2, -2, -1)$
223	3	$(-2, 3, -2)$	$(-2, -3, -2)$
123	0	none	$(-4, -2, -1), (-2, -4, -1)$

(7.6)

Subtracting b_A from the six permutations of a_A , and b_B from the eight vectors in (6.18), gives all entries. The first six types generate $\mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_4$, whereas $C(123) = 123$. The identity deck transformation contributes 000 on both sides. Therefore

$$\begin{aligned}\mathcal{R}_A &= \{000\} \cup \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_4, \\ \mathcal{R}_B &= \mathcal{R}_A \cup \{123\},\end{aligned}\tag{7.7}$$

where $\mathcal{R}_i$ is the set of representatives of the residue classes in (6.19). The norm-preserving action of $\mathcal{W}$ therefore gives $\text{Lsp}(M_9) \subset \text{Lsp}(N_9)$.

The only square residues modulo seven are 0, 1, 2, 4. The set $\{1, 2, 4\}$ is disjoint from its negative; hence a zero coordinate in a triple of square residues summing to zero forces all three to be zero. Two equal nonzero entries a would require the nonsquare $-2a$ as the third. Thus

$$7 \mid |m|^2 \iff \nu(m) = 000 \text{ or } \nu(m) = 123.\tag{7.8}$$

If $7 \mid q$ and $49 \nmid q$, every three-square representation therefore has representative 123: the alternative $m \in 7\mathbb{Z}^3$ would force $49 \mid q$. For $q \in \mathcal{S}_3$ such a representation exists, and (7.7) gives a length on N_9 but not on M_9 . Conversely, a squared norm occurring on N_9 and not on M_9 must arise from the additional representative 123, so it is divisible by seven. The Gauss–Legendre criterion [87, Introduction, equation (1.2), arXiv version] gives, for $n \in \mathbb{Z}_{\geq 0}$,

$$n \notin \mathcal{S}_3 \iff n = 4^a(8b + 7) \text{ for some } a, b \in \mathbb{Z}_{\geq 0}.$$

If $q = 49u > 0$, the maximal power 4^a dividing u also is the maximal power dividing $49u$, and $49 \equiv 1 \pmod{8}$. Hence $49u \in \mathcal{S}_3$ if and only if $u \in \mathcal{S}_3$. A representation $u = x^2 + y^2 + z^2$ gives $q = (7x)^2 + (7y)^2 + (7z)^2$, an identity-deck representation on M_9 . Consequently a squared norm occurring only on N_9 is divisible by seven and not by 49, and (7.2) follows.

For $R \geq 0$, let $A(R)$ count the positive integer parameters in (7.2) through R . The number with seven-adic valuation one is

$$\lfloor R/7 \rfloor - \lfloor R/49 \rfloor = \frac{6R}{49} + O(1).$$

The three-square exceptions with that valuation are precisely $4^a(8b + 7)$, where $a, b \geq 0$ and $v_7(8b + 7) = 1$. Their exponents a are unique, because $8b + 7$ is odd. For fixed a , put $X = R/4^a$. The allowed values of b satisfy $0 \leq b \leq (X - 7)/8$ and belong to the six residue classes 0, 7, 14, 21, 28, 35 modulo 49: divisibility by 49 removes the seventh class 42. Their number is $6X/392 + O(1)$, with an absolute error uniform in a, R . There are $O(\log(2 + R))$ contributing values of a , and

$$\sum_{a \geq 0} \frac{6}{392 \cdot 4^a} = \frac{1}{49}.$$

The omitted geometric tail contributes $O(1)$, since its first term has $R/4^a < 7$. Subtracting these disjoint exceptions yields

$$A(R) = \frac{5R}{49} + O(\log(2 + R)).$$

Since $h\sqrt{n} \leq T$ is equivalent to $n \leq 49T^2/(4\pi^2)$, this gives (7.3).

7.3. Regularity of the common wave trace. For a closed Riemannian manifold P , put

$$W_P(t) = \text{Tr}(e^{it\sqrt{\Delta_P}}) \in \mathcal{S}'(\mathbb{R}).$$

The polynomial spectral bound in Section 5, applied to the trivial deck group on each connected component, gives convergence in $\mathcal{S}'(\mathbb{R})$. Equal spectral multiplicities give equal wave distributions.

The Chazarain–Duistermaat–Guillemin Poisson relation [89, §1.2, equation (1.1), arXiv version] gives

$$\text{sing supp } W_P \subset \{0\} \cup \overline{\text{Lsp}(P) \cup (-\text{Lsp}(P))}. \quad (7.9)$$

For the real-analytic metrics and flat unitary bundles considered here, Proposition A.3 gives the corresponding inclusion for the analytic singular support.

The metrics on M_9, N_9 are real analytic and scalar-isospectral. Corollary A.5 makes their common half-wave trace real analytic near every length in $\text{Lsp}(N_9) \setminus \text{Lsp}(M_9)$. Proposition 6.4 gives $\text{Lsp}(M_9) \subset h\sqrt{\mathbb{Z}_{>0}}$. Proposition A.3 therefore gives

$$W_{M_9} = W_{N_9} \in C^\omega((h\sqrt{13}, h\sqrt{15})), \quad (7.10)$$

since the only integer strictly between 13 and 15 is 14, excluded by Lemma 7.2.

Corollary 7.4. *The metrics on M_9 and N_9 are clean. At every $\ell \in \text{Lsp}(N_9) \setminus \text{Lsp}(M_9)$, every coefficient in the clean wave-trace expansion vanishes after summing the component contributions with the same asymptotic degree.*

Proof. The groups G_A, G_B are globally symmetric by Section 6.2, hence clean by [89, Theorem 1.8, arXiv version]. Their finite actions are free, so Lemma A.1 makes both quotients clean. For a fixed component Θ of dimension d_Θ , the clean trace formula [89, §1.2, equations (1.5)–(1.6), arXiv version] has Fourier coefficients $c_{\Theta,r}$ at degrees $(d_\Theta - 1)/2 - r$, $r \in \mathbb{Z}_{\geq 0}$. The common trace is real analytic near ℓ by Corollary A.5. Its product with a smooth cutoff supported in that neighborhood has a rapidly decreasing Fourier transform. There are finitely many fixed components; the asymptotic degrees form a discrete set bounded above. Uniqueness of the asymptotic expansion therefore gives

$$\sum_{\substack{\Theta, r \geq 0 \\ (d_\Theta - 1)/2 - r = p}} c_{\Theta,r} = 0$$

for every occurring degree p . □

7.3.1. The fixed components at $h\sqrt{14}$. Put $G = G_B$ with the bi-invariant metric (6.1). Its specified torus T has Euclidean coordinates in $\mathbb{R}^3$ and period lattice $2\pi\mathbb{Z}^3$. Write $\exp_T$ for its exponential map, and set

$$h = \frac{2\pi}{7}, \quad a = (1, 1, 0), \quad b = (3, 3, 1).$$

$$l = \exp_T(ha), \quad r = \exp_T(hb), \quad \Psi(g) = lgr^{-1} = \Phi_B(g).$$

The cyclic group $\Gamma = \langle \Psi \rangle$ has order seven and acts freely. Let $N = N_9 = G/\Gamma$, and let $p : SG \rightarrow SN$ denote the induced unit-tangent covering. Let $\bar{\Phi}_t$ be the geodesic flow on SN , and put $\ell = h\sqrt{14}$.

Proposition 7.5. *The fixed set of the time- ℓ geodesic flow on SN consists of exactly twelve connected components. Each is a three-dimensional torus and consists of initial velocities of embedded primitive closed geodesics. There are exactly two components associated with each nonidentity deck transformation Ψ^j , $1 \leq j \leq 6$. Their free-loop Morse indices are*

$$(2, 0), \quad (2, 4), \quad (4, 2), \quad (4, 2), \quad (2, 4), \quad (2, 0),$$

in the order $j = 1, \dots, 6$. The two components associated with each j therefore have opposite Duistermaat–Guillemin phases.

Proof. The component count. For $1 \leq j \leq 6$, both l^j and r^j are regular torus elements. For the symplectic factor, their adjoint root angles are respectively $2jh$ and $6jh$. For the orthogonal factor, their two positive-root angles are respectively (jh, jh) and $(2jh, 4jh)$. None is a multiple of 2π . Their group centralizers are the connected torus T . For the orthogonal factor of l^j , one real plane rotates through $jh \not\equiv 0, \pi$, while the second plane is fixed. A commuting orthogonal map preserves these planes. Its restriction to the rotating plane is a rotation and has determinant one; its restriction to the fixed plane must consequently have determinant one as well. The centralizer in $\mathrm{SO}(4)$ is therefore $\mathrm{SO}(2) \times \mathrm{SO}(2)$, which is connected. For r^j its four distinct complex eigenlines give the same centralizer, and each nonreal quaternionic component has centralizer $\{e^{it} : t \in \mathbb{R}\}$.

A geodesic $\gamma(t) = g \exp_T(thm)$, $0 \leq t \leq 1$, closes through Ψ^j precisely when

$$\exp_T(hm) = g^{-1} l^j g r^{-j}, \quad \mathrm{Ad}_{r^j}(hm) = hm.$$

Regularity of r^j forces every such velocity to be toral. Thus, for some Weyl image $a' \in \mathcal{W}a$,

$$m \in \mathbb{Z}^3, \quad m \equiv j(a' - b) \pmod{7}, \quad g^{-1} l^j g = \exp_T(hja'),$$

where

$$\mathcal{W}a = \{(\epsilon, \eta, 0), (\epsilon, 0, \eta) : \epsilon, \eta \in \{1, -1\}\}.$$

If $|m|^2 = 14$, the absolute coordinates of m are 1, 2, 3. Indeed each coordinate has absolute value at most three, and $14 = 9 + 4 + 1$ is the only decomposition into three such squares. Consequently each coordinate is its unique representative modulo seven in $\{-3, \dots, 3\}$.

Table (7.6) shows that exactly two Weyl images have residue type 123:

$$a^{(1)} = (-1, 1, 0), \quad a^{(2)} = (1, -1, 0).$$

Multiplication by $j \neq 0$ permutes the three nonzero absolute residues, so these are the only two images contributing to $|m|^2 = 14$ for every j . Define $m_{j,v}$ by reducing $j(a^{(v)} - b)$ coordinatewise to $\{-3, \dots, 3\}$.

Choose a Weyl representative $g_v \in N_G(T)$ with $g_v^{-1} l g_v = \exp_T(ha^{(v)})$. For every $j \in \{1, \dots, 6\}$, the set of allowed base points for (j, v) is exactly the torus coset Tg_v : the equality $g^{-1} l^j g = g_v^{-1} l^j g_v$ is equivalent to $g g_v^{-1} \in C_G(l^j) = T$. The corresponding lifted component is

$$\tilde{\Theta}_{j,v} = \left\{ \left(g, \frac{hm_{j,v}}{\ell} \right) : g \in Tg_v \right\} \subset SG,$$

where tangent vectors are expressed in left trivialization. The deck action preserves each component: conjugating the closing equation by r preserves its toral right-hand side, and $\mathrm{Ad}_r m_{j,v} = m_{j,v}$. On Tg_v , the deck action is a translation after identification with T , since g_v normalizes T . It acts freely, so $\Theta_{j,v} = p(\tilde{\Theta}_{j,v})$ is again a torus. Different labels cannot be identified: the deck group preserves them, and freeness makes the closing deck element unique at a given lifted initial velocity. No component is associated with the identity deck transformation, because the shortest nonconstant closed geodesic on G has length $2\pi > \ell$. Consequently

$$\mathrm{Fix}(\bar{\Phi}_\ell) = \prod_{j=1}^6 \prod_{v=1}^2 \Theta_{j,v}, \quad \Theta_{j,v} \simeq (\mathbb{R}/\mathbb{Z})^3.$$

The twisted Hessian. Consider the energy

$$E(\gamma) = \frac{1}{2} \int_0^1 |\dot{\gamma}(t)|^2 dt$$

on the Sobolev path space $\{\gamma \in H^1([0, 1], G) : \gamma(1) = \Psi^j \gamma(0)\}$. A sufficiently small neighborhood of a loop in N has unique nearby lifts after choosing its initial point in one evenly covered neighborhood. The closing deck element is locally constant, since it takes values in the finite discrete group. Projection

consequently identifies this local path space with a neighborhood in the free loop space of N and preserves energy. Its Hessian index is therefore the ordinary free-loop index of the projected closed geodesic. Along $\gamma(t) = g \exp(tV)$ with $V = hm$, left-trivialize a variation field as $Y(t) = dL_{\gamma(t)}y(t)$ and put $A = \text{ad}_V$. Write $B_j = \text{Ad}_{r^j}$. The tangent space to this path space consists of H^1 fields with boundary condition

$$y(1) = \text{Ad}_{r^j} y(0).$$

The bi-invariant connection and curvature give

$$\nabla_t Y \longleftrightarrow y' + \frac{1}{2}Ay, \quad R(Y, \dot{\gamma})\dot{\gamma} \longleftrightarrow -\frac{1}{4}A^2y.$$

The endpoint terms in the second variation cancel, since Ψ^j is an isometry and $\dot{\gamma}(1) = d\Psi^j \dot{\gamma}(0)$. Hence

$$\begin{aligned} I_j(y, y) &= \int_0^1 (|y' + \frac{1}{2}Ay|^2 + \frac{1}{4}\langle A^2y, y \rangle) dt \\ &= \int_0^1 (|y'|^2 + \langle y', Ay \rangle) dt, \end{aligned}$$

because $A^* = -A$. The Jacobi equation and its endpoint conditions are

$$y'' + Ay' = 0, \quad y(1) = B_j y(0), \quad y'(1) = B_j y'(0).$$

Indeed, $AB_j = B_jA$, so matching the covariant derivatives $y' + \frac{1}{2}Ay$ is equivalent to the last condition.

The toral directions give the periodic form $\int_0^1 |y'|^2 dt$. Its index is zero and its kernel is the three-dimensional space of constant toral fields. The three positive roots are

$$\alpha_1 = 2e_1, \quad \alpha_2 = e_2 - e_3, \quad \alpha_3 = e_2 + e_3.$$

Each root plane has real dimension two. Identify the plane for α isometrically with $\mathbb{C}$ so that A acts by $i h \alpha(m)$ and B_j by $e^{i h j \alpha(b)}$. Put $\omega_n = h(7n + j \alpha(b))$. Every twisted H^1 field on this plane has the expansion

$$z(t) = \sum_{n \in \mathbb{Z}} c_n e^{i \omega_n t}, \quad \sum_n (1 + \omega_n^2) |c_n|^2 < \infty.$$

Multiplication by $e^{-i h j \alpha(b) t}$ identifies this expansion with the periodic Fourier expansion. Parseval's identity gives

$$I_j(z, z) = \sum_{n \in \mathbb{Z}} \omega_n (\omega_n + h \alpha(m)) |c_n|^2.$$

The series converges absolutely under the displayed H^1 condition. The coefficient of $|c_n|^2$ is

$$\lambda_{j,m,\alpha,n} = h^2 (7n + j \alpha(b)) (7n + j \alpha(b) + \alpha(m)). \quad (7.11)$$

Each mode has real dimension two. The index is therefore twice the number of negative coefficients. Replacing α by $-\alpha$ and n by $-n$ leaves this list unchanged.

Here $(\alpha_1(b), \alpha_2(b), \alpha_3(b)) = (6, 2, 4)$. Neither factor in (7.11) can vanish: the first is nonzero modulo seven, and the second is congruent to $j \alpha(a^{(v)})$ modulo seven. All the roots of $a^{(1)}$ and $a^{(2)}$ are nonzero modulo seven. Thus every kernel field has $y(t) = y_0 \in \mathfrak{t}$. It is the variation field of $\gamma_\epsilon(t) = g \exp(\epsilon y_0) \exp(tV)$. These variations move the initial point in Tg_v , leave the toral velocity and its norm fixed, and exhaust its three tangent directions. Conversely, a vector in $\ker(D_v \bar{\Phi}_\ell - I)$, with $v \in \Theta_{j,v}$, lifts to a Jacobi field satisfying the two twisted boundary conditions after rescaling $[0, \ell]$ to $[0, 1]$. Uniqueness for the Jacobi initial-value problem and the preceding constant toral variations give

$$\ker(D_v \bar{\Phi}_\ell - I) = T_v \Theta_{j,v}.$$

Let

$$q_\alpha(j, m) = \#\{n \in \mathbb{Z} : (7n + j \alpha(b))(7n + j \alpha(b) + \alpha(m)) < 0\}.$$

Equivalently, $q_\alpha(j, m)$ counts the integers satisfying

$$\min\{0, -\alpha(m)\} < 7n + j\alpha(b) < \max\{0, -\alpha(m)\}.$$

The Morse index is $\mu(j, m) = 2 \sum_{\alpha \in \{\alpha_1, \alpha_2, \alpha_3\}} q_\alpha(j, m)$. Writing $x = j\alpha(b)$, $u = \min(0, -\alpha(m))$, and $v = \max(0, -\alpha(m))$, the number of integers in the strict interval $(u - x)/7 < n < (v - x)/7$ is

$$q_\alpha(j, m) = \max \left\{ 0, \left\lceil \frac{v - x}{7} \right\rceil - \left\lfloor \frac{u - x}{7} \right\rfloor - 1 \right\}.$$

The resulting integers are

j	$m_{j,1}$	$q(j, m_{j,1})$	$\mu_{j,1}$	$m_{j,2}$	$q(j, m_{j,2})$	$\mu_{j,2}$
1	3, -2, -1	(1, 0, 0)	2	-2, 3, -1	(0, 0, 0)	0
2	-1, 3, -2	(0, 1, 0)	2	3, -1, -2	(1, 0, 1)	4
3	2, 1, -3	(1, 1, 0)	4	1, 2, -3	(0, 1, 0)	2
4	-2, -1, 3	(1, 1, 0)	4	-1, -2, 3	(0, 1, 0)	2
5	1, -3, 2	(0, 1, 0)	2	-3, 1, 2	(1, 0, 1)	4
6	-3, 2, 1	(1, 0, 0)	2	2, -3, 1	(0, 0, 0)	0

Each triple in a q column is ordered by $(\alpha_1, \alpha_2, \alpha_3)$. In particular $i^{-\mu_{j,1}} = -i^{-\mu_{j,2}}$ for every j .

Primitivity and embeddedness. Every positive closed-geodesic length on N is $h\sqrt{n}$ for an integer $n \geq 1$, by the closing equations. If a geodesic of length $h\sqrt{14}$ were a k -fold iterate with $k \geq 2$, its primitive length would give $14 = k^2 n$ for a positive integer n . The squarefreeness of 14 excludes this. Moreover, the image of every displayed geodesic lies in Tg_ν , because g_ν normalizes T . Lemma 6.5 applies to this invariant coset. The primitive projected straight lines in its quotient torus are embedded, so all geodesics in the twelve components are embedded. $\square$

7.3.2. Cancellation for each deck transformation. Write $\Psi_A = \Phi_A$, $\Psi_B = \Phi_B$ for the cyclic isometries of G_A, G_B with their metrics (6.1). For the direct pullback $F^*f = f \circ F$, put

$$W_{i,j}(t) = \text{Tr} \left((\Psi_i^j)^* e^{-it\sqrt{\Delta_{G_i}}} \right), \quad i \in \{A, B\}, \quad j \in \{0, \dots, 6\}.$$

Time reversal gives

$$W_{i,j}(-t) = \text{Tr} \left((\Psi_i^j)^* e^{it\sqrt{\Delta_{G_i}}} \right).$$

The diagonal kernel in this convention is $K_i(t, \Psi_i^j x, x)$. Its positive-time stationary points are precisely geodesics satisfying the endpoint and tangent conditions through Ψ_i^j . The index j is the deck index in the preceding table.

Proposition 7.6 (Complete cancellation in each deck class). *For every $j \in \{1, \dots, 6\}$, the complete singular expansions of the two clean components $\Theta_{j,1}$ and $\Theta_{j,2}$ cancel coefficient by coefficient at $\ell = h\sqrt{14}$. Each component has a strictly positive Duistermaat–Guillemin mass, and the two masses associated with the same j are equal. These statements hold separately for each j , before the cyclic average that produces the scalar wave trace on N .*

Proof. The maximal-equivariance identity, specialized as in Equation (4.12) to $(u, s, v) = (jh, 3jh, jh)$, gives

$$\text{Tr} \left((\Psi_A^j)^* e^{-t_0 \Delta_{G_A}} \right) = \text{Tr} \left((\Psi_B^j)^* e^{-t_0 \Delta_{G_B}} \right) \quad (t_0 > 0)$$

for each individual j . Put $E_{i,\lambda} = \ker(\Delta_{G_i} - \lambda)$, with $E_{i,\lambda} = \{0\}$ if λ is absent. For every $\lambda \geq 0$,

$$|\text{Tr}((\Psi_i^j)^*|_{E_{i,\lambda}})| \leq \dim E_{i,\lambda}.$$

The finite heat traces give absolute convergence. Lemma 2.3 therefore gives

$$\text{Tr}((\Psi_A^j)^*|_{E_{A,\lambda}}) = \text{Tr}((\Psi_B^j)^*|_{E_{B,\lambda}}) \quad (\lambda \geq 0).$$

The distributional convergence in Section 5 then gives $W_{A,j} = W_{B,j}$ in $S'(\mathbb{R})$.

Put $I = (h\sqrt{13}, h\sqrt{15})$. By Section 7.3, $I \cap \text{Lsp}(M_9) = \emptyset$. Proposition A.3 makes every flat half-wave trace over M_9 real analytic on $I \cup (-I)$. For $\chi_r(j) = e^{2\pi i r j/7}$, let W_{M_9, χ_r} denote the half-wave trace of the flat line bundle obtained by composing χ_r with the epimorphism $\pi_1(M_9) \rightarrow C_7$ defined by $G_A \rightarrow M_9$. The convention $R_\gamma f = f \circ \gamma^{-1}$ in Lemma 5.1 gives

$$W_{A,j}(t) = D_{\Psi_A^{-j}}(-t) = \sum_{r=0}^6 \chi_r(-j) W_{M_9, \chi_r}(-t).$$

Thus $W_{A,j} = W_{B,j} \in C^\omega(I)$.

The finite-cover trace identity is

$$W_N^-(t) := \text{Tr}(e^{-it\sqrt{\Delta_N}}) = \frac{1}{7} \sum_{j=0}^6 W_{B,j}(t).$$

Microlocally near ℓ , the term $\frac{1}{7} W_{B,j}$ receives exactly the two contributions from $\Theta_{j,1}$ and $\Theta_{j,2}$. Choose zeroth-order pseudodifferential cutoffs $A_{j,v}$ on N which are microlocally the identity in a conic neighborhood of $\Theta_{j,v}$ and zero near all other fixed components. Pulling their kernels back in evenly covered charts gives deck-invariant operators $\tilde{A}_{j,v}$, modulo smoothing operators. For $W_{j,v}^{\text{loc}} = \text{Tr}(A_{j,v} e^{-it\sqrt{\Delta_N}})$, finite-cover averaging gives

$$W_{j,v}^{\text{loc}} = \frac{1}{7} \sum_{k=0}^6 \text{Tr}(\tilde{A}_{j,v}(\Psi^k)^* e^{-it\sqrt{\Delta_G}}) \pmod{C^\infty}.$$

Smoothing errors have rapidly decreasing spectral matrix coefficients and hence smooth traces. At the selected component only $k = j$ has stationary points. The sum of the two cutoffs is the identity near all stationary points of that sector. The localized clean trace formula [89, §1.2, equations (1.5)–(1.6), arXiv version] gives

$$W_{j,1}^{\text{loc}} + W_{j,2}^{\text{loc}} = \frac{1}{7} W_{B,j} \pmod{C^\infty \text{ near } \ell}. \quad (7.12)$$

The common factor $1/7$ is positive. Changing the cutoffs away from these components changes only a smooth remainder.

By Proposition 6.4, $I \cap \text{Lsp}(N) = \{\ell\}$. Choose $\chi \in C_c^\infty(I)$ equal to one near ℓ and supported where (7.12) holds. Define the shifted Fourier transform by

$$\alpha_{j,v}(s) = \langle W_{j,v}^{\text{loc}}(t), \chi(t) e^{is(t-\ell)} \rangle.$$

All components have dimension three. Their shifted, localized Fourier amplitudes therefore have expansions

$$\alpha_{j,v}(s) \sim \sum_{r=0}^{\infty} C_{j,v,r} s^{1-r} \quad (s \rightarrow +\infty).$$

There is a constant $\kappa_3 \in \mathbb{C} \setminus \{0\}$, independent of j, v , such that

$$C_{j,v,0} = \kappa_3 i^{-\mu_{j,v}} D_{j,v}, \quad D_{j,v} = \int_{\Theta_{j,v}} d\mu_\ell > 0,$$

where $d\mu_\ell$ is the positive Duistermaat–Guillemin measure [89, §1.2, equations (1.5)–(1.6), arXiv version]. In this Riemannian formula, $\mu_{j,v}$ is the free-loop Morse index computed in Proposition 7.5. The constant κ_3 depends only on the Fourier and density conventions and the component dimension.

By (7.12) and smoothness of $W_{B,j}$, $\alpha_{j,1}(s) + \alpha_{j,2}(s) = O(s^{-M})$ for every $M > 0$. Uniqueness of the asymptotic expansion gives

$$C_{j,1,r} + C_{j,2,r} = 0 \quad (r \geq 0).$$

For $r = 0$, the identity $i^{-\mu_{j,1}} = -i^{-\mu_{j,2}}$ implies $D_{j,1} = D_{j,2} > 0$. □

7.4. The eight-dimensional quotients.

Theorem 7.7 (Equal flat spectra and different period sets). *The spaces A, B are simply connected, and the finite actions just specified are free of order seven. Mark each quotient fundamental group by sending $1 \in C_7$ to its displayed deck generator. Then*

$$\text{Spec}(\Delta_{M_8, \rho}) = \text{Spec}(\Delta_{N_8, \rho}) \quad \text{for every finite-dimensional unitary representation } \rho \text{ of } C_7, \quad (7.13)$$

whereas

$$3h \in \text{Lsp}(N_8) \setminus \text{Lsp}(M_8). \quad (7.14)$$

The geodesic of length $3h$ constructed on N_8 is embedded and primitive. The metrics are real analytic and satisfy

$$0 \leq \sec \leq 4, \quad \text{Ric} \geq \frac{1}{2}g, \quad \text{Vol}(M_8) = \text{Vol}(N_8) = \frac{16\pi^5}{7}.$$

Moreover, $\pi_2(M_8) = 0$, $\pi_2(N_8) = \mathbb{Z}$, $b_2(M_8) = 0$, and $b_2(N_8) = 1$.

7.4.1. Exclusion of the length $3h$ on M_8 . On G_A write $V = iB$, with B Hermitian. Horizontality is exactly

$$B_{33} = 0. \quad (7.15)$$

The squared norm is $|V|^2 = \text{Tr}(B^2)$, the sum of the squares of the three real eigenvalues of B . With $b \in H_A$, the effective right element $R = r_{A,0}^j b^{-1}$ is

$$R = \text{diag}(\zeta^{3j}, \zeta^{-3j}, e^{iv}), \quad \zeta = e^{2\pi i/7}.$$

The closing conditions say $[B, R] = 0$ and that $e^{iB}R$ is conjugate to ℓ_A^j .

For $j = 0$, Lemma 6.6 gives length at least $2\pi > 3h$ for every nonconstant closed geodesic.

Let $1 \leq j \leq 6$. The first two diagonal entries of R are distinct. There are exactly the following possibilities.

Three distinct eigenvalues. If the third eigenvalue differs from both, B is diagonal. Equation (7.15) makes its third entry zero, and the first two entries have the form hm, hn with $m, n \in \mathbb{Z}$. Neither can be zero: otherwise ζ^{3j} or ζ^{-3j} would belong to $\{\zeta^j, \zeta^{-j}, 1\}$, contrary to $3 \not\equiv 0, \pm 1 \pmod{7}$. If $m, n \neq 0$ and $m^2 + n^2 = 9$, each square is at most eight, so $m^2, n^2 \in \{1, 4\}$. Their sum is then 2, 5, or 8, a contradiction. Thus this case does not give length $3h$.

A repeated eigenvalue. Suppose the third eigenvalue equals $\zeta^{3\sigma j}$, with $\sigma \in \{1, -1\}$. Its eigenspace is spanned by the third coordinate and the first coordinate when $\sigma = 1$, or the second when $\sigma = -1$. Let the eigenvalues of B on this two-dimensional space be hm, hn , and its eigenvalue on the remaining line be hr . The scalars m, n, r are integers because R is scalar on each block and $e^{iB}R$ has eigenvalues $\zeta^j, \zeta^{-j}, 1$.

None of m, n, r is zero: the repeated and the remaining eigenvalues of R are both excluded from that set. The equation $|V| = 3h$ would give

$$m^2 + n^2 + r^2 = 9. \quad (7.16)$$

Every nonzero square is at least one, so each of the three squares is at most seven and belongs to $\{1, 4\}$. If k of them equal four, their sum is $3 + 3k = 9$, whence $k = 2$. Thus their absolute values are 1, 2, 2. In the coordinate order placing the third coordinate last, the repeated Hermitian block is $\begin{pmatrix} a & z \\ \bar{z} & 0 \end{pmatrix}$. Hence

$$h^2 mn = \det \begin{pmatrix} a & z \\ \bar{z} & 0 \end{pmatrix} = -|z|^2 < 0, \quad (7.17)$$

since $m, n \neq 0$.

Put $S = m + n + r$. Comparing the sums and sums of squares of the three exponent residues gives

$$\begin{aligned} S + 3\sigma j &\equiv 0 \pmod{7}, \\ (m + 3\sigma j)^2 + (n + 3\sigma j)^2 + (r - 3\sigma j)^2 &\equiv 2j^2 \pmod{7}. \end{aligned}$$

Equation (7.16) and the first congruence give

$$0 \equiv 9 + 6\sigma j(S - 2r) + 25j^2 \equiv 9 + 7j^2 - 12\sigma jr \equiv 2 + 2\sigma jr \pmod{7}.$$

Thus $\sigma jr \equiv -1 \pmod{7}$, and hence

$$Sr \equiv 3 \pmod{7}. \quad (7.18)$$

If $|r| = 1$, the opposite signs of m, n , both of magnitude two, give $Sr = 1$. If $|r| = 2$, then $|m|, |n|$ are one and two with opposite signs, giving $Sr = 4 \pm 2 \in \{2, 6\}$. Neither agrees with (7.18). Both resonant cases are excluded.

Thus $3h \notin \text{Lsp}(M_8)$. For an arbitrary closed length, Lemma 6.6 at $q = 1$ gives

$$\text{Lsp}(M_8) \subset \{h\sqrt{q} : q \in \mathbb{Z}_{>0}\}. \quad (7.19)$$

7.4.2. *An embedded primitive geodesic on N_8 .* In the product presentation (6.12), the metric norm on imaginary quaternion triples is

$$|(v_0, v_1, v_2)|^2 = |v_0|^2 + 2|v_1|^2 + 2|v_2|^2.$$

Put

$$g = \left(1, \mathbf{j}, \frac{1 - \mathbf{k}}{\sqrt{2}}\right), \quad V = h(\mathbf{i}, 0, -2\mathbf{j}), \quad b = (1, e^{-ih}, e^{ih}) \in \tilde{H}. \quad (7.20)$$

The generator of $\tilde{H}$ is $(0, \mathbf{i}, -\mathbf{i})$, so V is horizontal. Quaternion multiplication gives

$$\mathbf{j}^{-1}\mathbf{i}\mathbf{j} = -\mathbf{i}, \quad \left(\frac{1 - \mathbf{k}}{\sqrt{2}}\right)^{-1} \mathbf{i} \left(\frac{1 - \mathbf{k}}{\sqrt{2}}\right) = \mathbf{j}.$$

Consequently

$$g^{-1}\tilde{\ell}^3 g = (e^{3ih}, e^{-3ih/2}, e^{3jh/2}),$$

while

$$R := \tilde{r}^3 b^{-1} = (e^{9ih}, e^{11ih/2}, e^{7ih/2}) = (e^{9ih}, e^{11ih/2}, -1).$$

The first component of $e^V R$ is $e^{10ih} = e^{3ih}$; the second is $e^{11ih/2} = e^{-3ih/2}$. The third is

$$e^{-2jh}(-1) = e^{j(\pi-2h)} = e^{3jh/2}.$$

The first right component commutes with its velocity, the second velocity is zero, and the third right component is central. Consequently

$$e^V R = g^{-1}\tilde{\ell}^3 g, \quad \text{Ad}_R V = V.$$

Lemma 6.3 makes $g \exp(tV)\tilde{H}$ a geodesic closing on N_8 through deck power three. Its length is

$$|V| = h\sqrt{1 + 2 \cdot 4} = 3h.$$

Suppose the points at $0 \leq t_1 < t_2 < 1$ have the same image in N_8 . The first quaternionic component, on which $\tilde{H}$ acts trivially, would satisfy

$$e^{iht_2} = e^{ijh} e^{iht_1} e^{-3ijh}$$

for some $j \in \{0, \dots, 6\}$. Thus $t_2 - t_1 + 2j \in 7\mathbb{Z}$, forcing $t_2 - t_1$ to be an integer, contrary to $0 < t_2 - t_1 < 1$. The induced map from $\mathbb{R}/\mathbb{Z}$ is an injective immersion; compactness of $\mathbb{R}/\mathbb{Z}$ makes it an embedding. A multiple traversal would identify two times in $[0, 1)$, so the geodesic is primitive. Its lift to the simply connected space B ends at ϕ_B^3 of its initial point, so its marked homotopy class is $3 \in C_7$.

7.4.3. *Topology and marked wave traces.* By [47, Proposition 4.1], the covering maps give

$$\pi_2(M_8) = \pi_2(A) = 0, \quad \pi_2(N_8) = \pi_2(B) = \mathbb{Z}.$$

Each connected component of a common finite cover would have second homotopy group both zero and $\mathbb{Z}$, which is impossible. The isotopies $F_i(t)$, $0 \leq t \leq 1$, give $\phi_i^* = I$ on $H^*(X_i; \mathbb{Q})$. Since $X_i \rightarrow X_i/C_7$ is a regular covering of degree seven, [47, Proposition 3G.1] gives

$$H^*(X_i/C_7; \mathbb{Q}) \simeq H^*(X_i; \mathbb{Q})^{C_7} = H^*(X_i; \mathbb{Q}).$$

The spaces A, B are simply connected, so [47, Theorem 4.32] gives $H_2(A; \mathbb{Z}) = 0$ and $H_2(B; \mathbb{Z}) = \mathbb{Z}$. Therefore

$$b_2(M_8) = 0, \quad b_2(N_8) = 1.$$

The Hodge theorem identifies these numbers with the zero-eigenvalue multiplicities of the Hodge Laplacians on two-forms [27, Example 3.1.2].

Both metrics are real analytic, and the bundles in (5.1) have constant transition matrices in parallel frames. Proposition A.3 therefore gives real analyticity of each trace on every open interval disjoint from zero and the signed closed-geodesic lengths.

By (7.19) and the exclusion of $3h$, no length of M_8 lies in

$$(h\sqrt{8}, h\sqrt{10}). \tag{7.21}$$

Consequently, for every $\rho : C_7 \rightarrow \mathcal{U}(r)$,

$$\mathrm{Tr}(e^{it\sqrt{\Delta_{M_8, \rho}}}) = \mathrm{Tr}(e^{it\sqrt{\Delta_{N_8, \rho}}}) \in C^\omega((h\sqrt{8}, h\sqrt{10})).$$

Since $\phi_i^{j*} = R_{\phi_i^{-j}}$, Lemma 5.1 gives

$$\mathrm{Tr}(\phi_i^{j*} e^{it\sqrt{\Delta_{X_i}}}) \in C^\omega((h\sqrt{8}, h\sqrt{10})) \quad (j \in C_7).$$

The geodesic in (7.20) closes through $j = 3$. Lemma 7.8 gives a shorter geodesic in that class, of length $h\sqrt{2}$.

Lemma 7.8. *No $F_A(j)$ -twisted geodesic has length $3h$ for any integer j . The $F_B(3)$ -twisted geodesic in (7.20) has length $3h$ and projects to an embedded curve in N_8 . Each of A and B has an $F_i(3)$ -twisted geodesic of squared length $2h^2$.*

Proof. Every $F_A(j)$ -twisted geodesic projects to a closed geodesic on M_8 , so Section 7.4.1 excludes length $3h$. The $F_B(3)$ -twisted geodesic of length $3h$ in (7.20) has an embedded projection by Section 7.4.2. For $i = A, B$, put

$$V_i = \left. \frac{d}{dt} \right|_{t=0} \iota_i(ht, 0).$$

The metric $2ds^2 + dv^2$ on the parameter torus gives $V_i \perp \mathfrak{h}_i$ and $|V_i|^2 = 2h^2$. In G_i ,

$$e^{V_i} = \iota_i(h, 0) = \iota_i(-6h, 0) = \ell_i^3 r_{i,0}^{-3}, \quad \mathrm{Ad}_{r_{i,0}^3} V_i = V_i.$$

Lemma 6.3, with $g = b = 1$, gives an $F_i(3)$ -twisted geodesic of length $h\sqrt{2}$. □

Corollary 7.9. *For the marked class $3 \in C_7$,*

$$\ell_{M_8}(3) = \ell_{N_8}(3) = h\sqrt{2}, \quad 3h \in \mathrm{Lsp}(N_8) \setminus \mathrm{Lsp}(M_8).$$

The geodesic of length $3h$ in (7.20) is embedded and primitive.

Proof. Consider a shortest loop on M_8 in class 3. By Lemma 5.3, it is a closed geodesic. In the three-distinct-eigenvalue case of Section 7.4.1, its two nonzero horizontal eigenvalues belong to $h\mathbb{Z}$, so its squared length is at least $2h^2$. In a repeated-eigenvalue case, all three eigenvalues are nonzero elements of $h\mathbb{Z}$, so the squared length is at least $3h^2$. Lemma 7.8 supplies a class-three geodesic of length $h\sqrt{2}$ on M_8 . Theorem 5.4, with the identity epimorphism of C_7 and the equal flat spectra of Theorem 7.7, gives the same minimum on N_8 . Sections 7.4.1 and 7.4.2 exclude $3h$ on M_8 and give the primitive embedded geodesic of length $3h$ on N_8 . □

8. FIXED MANIFOLDS AND SIMULTANEOUS TOPOLOGY

8.1. Products and marked finite covers.

Theorem 8.1. *Let F be any smooth closed connected Riemannian manifold, and put*

$$P = M_8 \times F, \quad Q = N_8 \times F, \quad \Gamma = C_7 \times \pi_1(F).$$

Fix the left deck action of $\pi_1(F)$ on its universal cover $\tilde{F}$. For each finite-index subgroup $\Lambda \leq \Gamma$, put

$$P_\Lambda = \Lambda \backslash (A \times \tilde{F}), \quad Q_\Lambda = \Lambda \backslash (B \times \tilde{F}),$$

where (j, γ) acts as (ϕ_A^j, γ) or (ϕ_B^j, γ) , respectively. Chosen points in the two universal covers determine the basepoints and the identifications $\pi_1(P_\Lambda) = \Lambda = \pi_1(Q_\Lambda)$. For every finite-dimensional unitary representation $\rho : \Lambda \rightarrow \text{U}(d)$,

$$\text{Spec}(\Delta_{P_\Lambda, \rho}) = \text{Spec}(\Delta_{Q_\Lambda, \rho}). \quad (8.1)$$

For finite-index $\Lambda' \leq \Lambda \leq \Gamma$, the natural covering maps on both sides induce the inclusion $\Lambda' \hookrightarrow \Lambda$; the pullback of the bundle labelled by ρ is labelled by $\rho|_{\Lambda'}$. If every nonconstant smoothly closed geodesic on F has length greater than $3h$, then $3h \in \text{Lsp}(Q) \setminus \text{Lsp}(P)$.

Proof. Write $\Pi = \pi_1(F)$. A finite-dimensional unitary representation of $C_7 \times \Pi$ splits under its central C_7 action as

$$V = \bigoplus_{a=0}^6 V_a, \quad \rho|_{V_a} = \chi_a \otimes \sigma_a, \quad \chi_a(j) = e^{2\pi i a j / 7}.$$

The subspaces are invariant under Π , and each σ_a is unitary. Each σ_a may have arbitrary unitary image. The associated bundle is a direct sum of exterior products. On a tensor $u \otimes v$ the connection Laplacian is

$$\Delta(u \otimes v) = (\Delta u) \otimes v + u \otimes (\Delta v).$$

Tensor products of complete factor eigenbases form a complete eigenbasis for the product connection Laplacian, with eigenvalues $\lambda_j + \mu_k$ and their full multiplicities. Lemma 2.4, applied to the function $(\lambda_j, \mu_k) \mapsto \lambda_j + \mu_k$, gives its spectral domain. Tonelli's theorem gives

$$Z_{M_8 \times F, \rho}(t) = \sum_{a=0}^6 Z_{M_8, \chi_a}(t) Z_{F, \sigma_a}(t). \quad (8.2)$$

Theorem 7.7 gives $Z_{M_8, \chi_a}(t) = Z_{N_8, \chi_a}(t)$ for every a and $t > 0$; therefore $Z_{P, \rho}(t) = Z_{Q, \rho}(t)$ for every unitary representation ρ of Γ .

Let $\Lambda \leq \Gamma$ have finite index and let $\rho : \Lambda \rightarrow \text{U}(V)$. Realize $\tau = \text{Ind}_\Lambda^\Gamma \rho$ as the maps $v : \Gamma \rightarrow V$ satisfying $v(\lambda\gamma) = \rho(\lambda)v(\gamma)$, with squared norm $\sum_{\Lambda\gamma} |v(\gamma)|^2$ and action $(\tau(\delta)v)(\gamma) = v(\gamma\delta)$. Right multiplication permutes the finitely many left cosets, so τ is a unitary representation of dimension $[\Gamma : \Lambda] \dim V$.

For a ρ -equivariant function f on the universal cover, put

$$\hat{f}(x)(\gamma) = f(\gamma x).$$

Then $\hat{f}(\delta x) = \tau(\delta)\hat{f}(x)$, and the inverse is $f(x) = \hat{f}(x)(1)$. If D is a measurable fundamental domain for Γ , the union of γD over $\Lambda \backslash \Gamma$ is a fundamental domain for Λ . Consequently

$$\int_D \|\hat{f}(x)\|^2 dx = \sum_{\Lambda\gamma} \int_D |f(\gamma x)|^2 dx = \int_{\Lambda \backslash \tilde{P}} |f|^2.$$

The same identity holds for energies, because the deck maps are local isometries and the lifted flat connections are trivial. The inverse maps therefore identify the closed forms and their self-adjoint operators, proving

$$\text{Spec}(\Delta_{P_\Lambda, \rho}) = \text{Spec}(\Delta_{P, \text{Ind}_\Lambda^\Gamma \rho}). \quad (8.3)$$

The norm and energy identities hold for every finite-index subgroup, including nonnormal subgroups of infinite Γ , because $\Lambda \backslash \Gamma$ is finite. For a tower dominated by a finite Galois cover, the corresponding spectral identity is [27, Lemma 3.8.1]. Equation (8.2) for $\tau = \text{Ind}_\Lambda^\Gamma \rho$ gives equal heat traces on P_Λ and Q_Λ . Lemma 2.3 then gives (8.1).

For $\Lambda' \leq \Lambda$, a based loop has the same lift and endpoint in both quotients of the universal cover, so the induced map on fundamental groups is inclusion. The map

$$[(x, v)]_{\Lambda'} \mapsto ([x]_{\Lambda'}, [(x, v)]_\Lambda)$$

identifies the bundle for $\rho|_{\Lambda'}$ with the pullback bundle. In local lifted coordinates it is the identity on V ; therefore it is well defined, invertible, and preserves the flat connection.

By [63, Problem 5-7(a), p. 146], a product curve is a geodesic if and only if both components are geodesics. For a smoothly closed product geodesic, both components are smoothly closed and have constant speeds; its squared length is the sum of their squared lengths. Under the stated hypothesis on F , a closed product geodesic of length $3h$ has constant F component. It would therefore give a geodesic of length $3h$ on M_8 , contrary to Theorem 7.7. On N_8 , the geodesic of length $3h$ paired with a constant point of F gives one on Q . $\square$

For $F = \mathbb{R}/(2\pi\mathbb{Z})$, the boundary condition $f(x + 2\pi) = e^{2\pi i \alpha} f(x)$ has the complete eigenbasis $e^{i(m+\alpha)x}$, $m \in \mathbb{Z}$, with eigenvalues $(m + \alpha)^2$. For $k \geq 1$ and $\Lambda = C_7 \times k\mathbb{Z}$, the marked covers are

$$P_\Lambda = M_8 \times \mathbb{R}/(2\pi k\mathbb{Z}), \quad Q_\Lambda = N_8 \times \mathbb{R}/(2\pi k\mathbb{Z}).$$

The character sending the generator $k \in k\mathbb{Z}$ to $e^{2\pi i \alpha}$ has circle eigenvalues $(m + \alpha)^2/k^2$, for every $\alpha \in \mathbb{R}/\mathbb{Z}$ and $m \in \mathbb{Z}$. A nonconstant closed geodesic in this circle has length at least $2\pi k$. Since $3h < 2\pi k$, every closed product geodesic of length $3h$ has constant circle component. Consequently $3h \in \text{Lsp}(Q_\Lambda) \setminus \text{Lsp}(P_\Lambda)$ for each $k \geq 1$. The circle direction has zero Ricci curvature.

The smooth manifolds underlying A and B are $A \simeq \text{SU}(3)$ and $B \simeq S^3 \times S^3 \times S^2$. Put

$$W = A \times B, \quad g_W = g_A + g_B, \quad L = h 2^{1/4}, \quad P = W \times (\mathbb{R}/\mathbb{Z}).$$

Let K_A, K_B generate the flows, and set $K_0 = (K_A, 0)$, $K_1 = (0, K_B)$. For $\sigma = 0, 1$ and $\tau \in \mathbb{R}$ define

$$g_{\sigma, \tau}(v + a\partial_s, w + b\partial_s) = g_W(v + \tau a K_\sigma, w + \tau b K_\sigma) + L^2 ab. \quad (8.4)$$

These are positive metrics on the fixed smooth manifold

$$P \simeq \text{SU}(3) \times S^3 \times S^3 \times S^2 \times S^1, \quad \pi_1(P) = \mathbb{Z}, \quad (8.5)$$

marked by the last coordinate.

Theorem 8.2. *For every real τ , every integer $d \geq 1$, and every finite-dimensional unitary representation $\rho : d\mathbb{Z} \rightarrow \text{U}(a)$, the pullback metrics on the connected cover $P_d = W \times (\mathbb{R}/d\mathbb{Z})$ satisfy*

$$\text{Spec}(\Delta_{g_{0, \tau, d, \rho}}) = \text{Spec}(\Delta_{g_{1, \tau, d, \rho}}). \quad (8.6)$$

These equalities hold with the marking (8.5) and the subgroup inclusions $d\mathbb{Z} \subset d\mathbb{Z}$ for integers $d, e \geq 1$. Every metric has Riemannian universal cover $(W \times \mathbb{R}, g_W + L^2 ds^2)$.

If $q \geq 1$ is an integer and $\gcd(q, 6) = 1$, the pair at $\tau = 1/q$ has a common connected finite Riemannian cover of degree $7q$ over each, but

$$\ell_q = 3h\sqrt{1 + q^2 \sqrt{2}} \in \text{Lsp}(P, g_{1, 1/q}) \setminus \text{Lsp}(P, g_{0, 1/q}). \quad (8.7)$$

The length ℓ_q is represented by an embedded primitive geodesic of marked winding $3q$. The families (8.4) are real analytic in space and parameter, agree at $\tau = 0$, and have volume $L(16\pi^5)^2$ for every parameter. On $|\tau| \leq 1$ they have uniform positive injectivity radius and bounds on every covariant derivative of curvature. Their sectional curvatures belong to $[0, 4]$; their Ricci tensors have a one-dimensional kernel and positive eigenvalues at least $1/2$.

For each τ , the minimum length of a loop in every nonzero marked free homotopy class agrees for the two metrics.

The common spectrum may vary with τ . At each parameter the two spectra agree. As $q \rightarrow \infty$, the two metrics approach the same metric in C^∞ , while the cover degrees $7q$ and the lengths ℓ_q tend to infinity.

8.2. Common finite covers and unequal length sets. By Proposition 6.8, for $q \in \mathbb{Z}_{>0}$ and $\tau = 1/q$ the product

$$C_q = (W \times \mathbb{R} / (7q\mathbb{Z}), g_W + L^2 ds^2) \quad (8.8)$$

is a common Riemannian cover of degree $7q$ of the two metrics. Write $T_{\sigma, \tau}(x, s) = (\Psi_\sigma(\tau)x, s + 1)$.

Lemma 8.3. *Let q be a positive integer coprime to six and let $C > 0$. The product curve $t \mapsto (\gamma_B(t), 3qt)$, where γ_B is the geodesic of (7.20), induces an embedded primitive geodesic in*

$$B(q, C) = (B \times \mathbb{R}, g_B + C^2 du^2) / \langle (x, u) \mapsto (F_B(1/q)x, u + 1) \rangle.$$

Its length is $3\sqrt{h^2 + q^2 C^2}$ and its winding is $3q$. For $C = L$, denote this manifold and curve by B_q and Γ_q ; the length is $\ell_q = 3h\sqrt{1 + q^2}\sqrt{2}$.

Proof. The endpoints and tangents of the product curve match through the $3q$ th deck transformation, and its squared speed is $9h^2 + 9q^2 C^2$. If two points at parameter difference $0 < \delta < 1$ were identified, their real-coordinate difference would give $k = 3q\delta \in \mathbb{Z}$. The first quaternionic coordinate in (7.20), unaffected by H_B , then gives

$$\delta + 2k/q \in 7\mathbb{Z}.$$

Substitution gives $\delta \in \mathbb{Z}$, a contradiction. The curve has nonzero speed and is injective on the parameter circle. Compactness makes it an embedding, and injectivity excludes a smaller period. $\square$

Fix henceforth $q \in \mathbb{Z}_{>0}$ with $\gcd(q, 6) = 1$.

Every closed geodesic on the suspension lifts to a product geodesic on $A \times B \times \mathbb{R}$. If its winding is $k \in \mathbb{Z}$, its real-coordinate increment is k and its squared length is the sum of its two fibre squared lengths and $k^2 L^2$. For $g_{0,1/q}$ the A curve is twisted by $F_A(k/q)$ and the B curve is ordinarily closed. Lemma 6.6 implies that its squared length has the form

$$h^2 \left(\frac{R}{q^2} + k^2 \sqrt{2} \right), \quad R \in \mathbb{Z}_{\geq 0}. \quad (8.9)$$

The squared lengths for $g_{1,1/q}$ have the same form (8.9).

The metric $g_{1,1/q}$ has the geodesic given by Γ_q and a constant A -component, of squared length ℓ_q^2 . Suppose the same length occurred for $g_{0,1/q}$. Equality with (8.9) gives

$$\left(\frac{R}{q^2} - 9 \right) + (k^2 - 9q^2) \sqrt{2} = 0.$$

Both coefficients are rational. Since $\sqrt{2} \notin \mathbb{Q}$, they vanish:

$$k^2 = 9q^2, \quad R = 9q^2.$$

The total fibre squared length is then $9h^2 < 4\pi^2$. By Lemma 6.6, its ordinarily closed B component must be constant. Its A component would be an $F_A(3)$ - or $F_A(-3)$ -twisted geodesic of length $3h$, contradicting Lemma 7.8. Thus $\ell_q \notin \text{Lsp}(P, g_{0,1/q})$, and (8.7) follows for every $q \geq 1$ with $\gcd(q, 6) = 1$.

Lemma 8.3 gives its embeddedness and primitivity. Its marked homotopy class is $3q \in \mathbb{Z}$.

The set in (8.9) is locally finite: a length at most T satisfies $|k| \leq T/L$ and $0 \leq R \leq q^2 T^2 / h^2$, permitting only finitely many integer pairs. There is therefore an open interval $I_q \subset (0, \infty)$ containing ℓ_q and disjoint from $\text{Lsp}(P, g_{0,1/q})$. Every closed geodesic on a finite Riemannian cover projects to a closed geodesic of the same length, so I_q is disjoint from the length set of every such cover of $(P, g_{0,1/q})$. The metrics are real analytic. Corollary A.5, applied to the isospectral flat Laplacians in (8.6), gives

$$\text{Tr}(e^{it\sqrt{\Delta_{g_{0,1/q}, d, \rho}}}) = \text{Tr}(e^{it\sqrt{\Delta_{g_{1,1/q}, d, \rho}}}) \in C^\omega(I_q)$$

for every $d \geq 1$ and every finite-dimensional unitary representation $\rho : d\mathbb{Z} \rightarrow \mathcal{U}(a)$. The interval I_q is independent of d and ρ .

8.2.1. *A common quotient.* Write $N = 7q$ and let T_0, T_1 denote $T_{0,1/q}, T_{1,1/q}$. They commute, and for $a, b \in \mathbb{Z}$,

$$T_0^a T_1^b(x_A, x_B, s) = (F_A(a/q)x_A, F_B(b/q)x_B, s + a + b).$$

For $0 < j < N$, the action $F_A(j/q)$ has no fixed coset. The determinant condition would first remove its right H_A factor; conjugacy would then require

$$\{j, -j, 0\} = \{3j, -3j, 0\} \quad \text{in } \mathbb{Z}/N\mathbb{Z}.$$

Because N is odd and 3 is invertible modulo N , this requires $2j = 0$ or $4j = 0$, contrary to $0 < j < N$. For $F_B(j/q)$, its first quaternionic component gives the same two congruences. Thus $F_i(a/q) = I$ exactly when $N \mid a$.

On $C_q = W \times \mathbb{R}/(N\mathbb{Z})$, the homomorphism $(a, b) \mapsto T_0^a T_1^b$ has kernel $N\mathbb{Z} \oplus N\mathbb{Z}$. Its image is therefore $C_N \times C_N$. If an element of this image has a fixed point, its translation coordinate gives $a + b \equiv 0 \pmod{N}$. If $a \not\equiv 0$, its A component is fixed-point-free; if $a \equiv 0$, then also $b \equiv 0$. Hence the finite action is free. The quotient $C_q/(C_N \times C_N)$ is a smooth common base. Each intermediate quotient by $\langle T_0 \rangle$ or $\langle T_1 \rangle$ has degree N over that base.

Put $D = C_N \times C_N$. Since W is simply connected, $H_1(C_q; \mathbb{F}_\ell) \cong \mathbb{F}_\ell$ for every prime ℓ . Each element $T_0^a T_1^b$ is isotopic to the identity through

$$(x_A, x_B, [s]_N) \mapsto (F_A(ta/q)x_A, F_B(tb/q)x_B, [s + t(a + b)]_N), \quad 0 \leq t \leq 1.$$

The D -action on this homology group is therefore trivial. In particular, it contains neither the regular module $\mathbb{F}_\ell[D]$, of dimension N^2 , nor either permutation module $\mathbb{F}_\ell[D/\langle T_\sigma \rangle]$, of dimension N . Neither permutation module is a quotient of this homology group either: every linear map from a one-dimensional space has rank at most one, whereas $N \geq 7$. Thus this covering diagram fails the regular-module condition for homological wideness and the weaker permutation-module condition, in both its quotient and submodule forms [27, Theorem 5.4.1, §6.1, and Definition 7.1.1, arXiv version 1].

8.3. **Minimum lengths in free homotopy classes and curvature bounds.** Let

$$d_i(s) = \min_{x \in X_i} d_{X_i}(x, F_i(s)x), \quad X_A = A, X_B = B.$$

The minimum exists by compactness, and $d_i(-s) = d_i(s)$ by the change of variable $x \mapsto F_i(s)x$. Lemma 5.2 applied to $F_i(-s)$ gives the positive traces

$$\Theta_i(t, s) = \text{Tr}(F_i(s)^* e^{-t\Delta_i}) > 0$$

and

$$-4t \log \Theta_i(t, s) \longrightarrow d_i(s)^2. \quad (8.10)$$

The flow identity (4.14) therefore gives $d_A(s) = d_B(s)$ for every real s .

For $k \in \mathbb{Z} \setminus \{0\}$, define $\ell_{\min, \sigma, \tau}(k)$ as the infimum of lengths of piecewise smooth loops in the marked free homotopy class k . The universal cover $W \times \mathbb{R}$ is simply connected and its deck group is the marked abelian group $\mathbb{Z}$, so this is exactly the class of paths with endpoint $T_{\sigma, \tau}^k$ times their initial point. For $z = (x_A, x_B, s)$, product distance gives

$$d(z, T_{\sigma, \tau}^k z)^2 = k^2 L^2 + \begin{cases} d_A(x_A, F_A(\tau k)x_A)^2, & \sigma = 0, \\ d_B(x_B, F_B(\tau k)x_B)^2, & \sigma = 1, \end{cases}$$

where $d_A(x, y)$ and $d_B(x, y)$ denote the respective Riemannian distances. Consequently minimizing displacement gives

$$\ell_{\min, \sigma, \tau}(k)^2 = k^2 L^2 + d_A(\tau k)^2 = k^2 L^2 + d_B(\tau k)^2 \quad (k \neq 0). \quad (8.11)$$

Indeed, every loop of winding k lifts to a path joining a point to its k th deck translate, so its length is at least the minimum displacement in (8.11). Compactness of the fiber attains that displacement, and a minimizing product segment projects to a loop of winding k with that length. Hence it is the class minimum. Lemma 5.3 supplies a smoothly closed representative; its length is positive because it is at least $|k|L$. This proves equality of the two marked minimum-length functions for every τ .

By Proposition 6.8, both metrics are locally isometric to the fixed product $W \times \mathbb{R}$. Thus their curvature and all its covariant derivatives have the product bounds. In the coordinates (6.24), the real direction is $\partial_s - \tau K_\sigma$, so

$$\ker \text{Ric}_{g_{\sigma,\tau}} = \mathbb{R}(\partial_s - \tau K_\sigma), \quad 0 \leq \sec(g_{\sigma,\tau}) \leq 4.$$

The other Ricci eigenvalues are those of W , hence at least $1/2$. The same proposition gives the positive uniform injectivity bound and C^m convergence estimate, and (6.14) gives volume $L(16\pi^5)^2$. This proves Theorem 8.2.

8.4. Infinite length-set differences on a simply connected manifold. Retain the eight-dimensional normal homogeneous manifolds (A, g_A) and (B, g_B) , the period-seven flows F_A, F_B , and their infinitesimal generators K_A, K_B from Section 6.4. By Corollary 4.14, there is a unitary operator

$$U : L^2(A, dv_A) \longrightarrow L^2(B, dv_B), \quad U\Delta_A = \Delta_B U, \quad UK_A = K_B U.$$

The domains satisfy

$$UD(\Delta_A) = D(\Delta_B), \quad UD(K_A) = D(K_B).$$

By Theorem 7.7, the fixed manifold

$$M = A \times B \times S^2$$

is simply connected and has dimension eighteen.

8.4.1. A prescribed length in an analytic family.

Lemma 8.4. *Let P be a closed real-analytic manifold, let $I \subset \mathbb{R}$ be an open interval, and let $(g_s)_{s \in I}$ be a jointly real-analytic family of Riemannian metrics. Fix a compact interval $J \subset I$ and $\ell > 0$. Suppose that every closed g_s -geodesic $\gamma : \mathbb{R}/\mathbb{Z} \rightarrow P$ of constant speed ℓ , for $s \in J$, satisfies*

$$\int_0^1 (\partial_s g_s)_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t)) dt > 0. \quad (8.12)$$

Then $\{s \in J : \ell \in \text{Lsp}(P, g_s)\}$ is finite.

Proof. If no such geodesic occurs on J , the exceptional set is empty. Otherwise, write $J = [s_-, s_+]$ and choose an auxiliary Riemannian metric g_{ref} on P . Compactness of $J \times P$ gives a number $c > 0$ such that $g_s \geq c g_{\text{ref}}$ for $s \in J$. Let Φ_s^t be the geodesic flow on TP . The Christoffel symbols depend analytically on (s, x) , so the geodesic flow is analytic in (s, t, v) ; completeness holds because P is closed. The set

$$Z_\ell = \{(s, v) \in J \times TP : \Phi_s^1(v) = v, g_s(v, v) = \ell^2\}$$

is closed and semianalytic in $I \times TP$. The fixed-point condition is the inverse image of the analytic diagonal in $TP \times TP$ under $(s, v) \mapsto (\Phi_s^1(v), v)$, the speed condition is one analytic equation, and $s_- \leq s \leq s_+$ gives two analytic inequalities. Its vectors satisfy $|v|_{g_{\text{ref}}} \leq \ell/\sqrt{c}$, so Z_ℓ is compact. The uniformization theorem [15, Theorem 0.1] gives a real-analytic manifold N and a proper analytic map $\psi : N \rightarrow I \times TP$ with image Z_ℓ . Since $N = \psi^{-1}(Z_\ell)$, properness makes N compact; hence N has finitely many connected components.

Let $p = \text{pr}_I \circ \psi$. For a smooth curve $a \mapsto n(a)$ in N , write $\psi(n(a)) = (s(a), v(a))$ and set

$$\gamma_a(t) = \pi_{TP}(\Phi_{s(a)}^t(v(a))), \quad V_a(t) = \partial_a \gamma_a(t).$$

The equations defining Z_ℓ give $\gamma_a(0) = \gamma_a(1)$, $\dot{\gamma}_a(0) = \dot{\gamma}_a(1)$, and $\int_0^1 g_{s(a)}(\dot{\gamma}_a, \dot{\gamma}_a) dt = \ell^2$. Differentiating the first equality gives $V_a(0) = V_a(1)$. The first variation of energy is therefore

$$\begin{aligned} 0 &= \frac{s'(a)}{2} \int_0^1 (\partial_s g_{s(a)})(\dot{\gamma}_a, \dot{\gamma}_a) dt \\ &\quad + [g_{s(a)}(V_a, \dot{\gamma}_a)]_0^1 - \int_0^1 g_{s(a)}(V_a, \nabla_t \dot{\gamma}_a) dt \\ &= \frac{s'(a)}{2} \int_0^1 (\partial_s g_{s(a)})(\dot{\gamma}_a, \dot{\gamma}_a) dt. \end{aligned}$$

The endpoint term vanishes by periodicity, and $\nabla_t \dot{\gamma}_a = 0$ because γ_a is a geodesic. Hypothesis (8.12) gives $s'(a) = 0$. Every tangent vector to N is the velocity of such a curve, so $dp = 0$. Thus p is constant on each connected component of N , and $p(N) = \{s \in J : \ell \in \text{Lsp}(P, g_s)\}$ is finite. $\square$

8.4.2. *Norms and periods of twisted geodesics.* Lemma 6.6 gives

$$\text{Lsp}(A, g_A) \cup \text{Lsp}(B, g_B) \subseteq \{2\pi\sqrt{N} : N \in \mathbb{Z}_{>0}\}. \quad (8.13)$$

In particular, a closed geodesic in $(A \times B, g_A \oplus \beta g_B)$ has squared length $4\pi^2(N + \beta D)$ for some $N, D \in \mathbb{Z}_{\geq 0}$.

Put $\zeta = e^{ih}$, where $h = 2\pi/7$, and write

$$L_j = \text{diag}(\zeta^j, \zeta^{-j}, 1), \quad R_j = \text{diag}(\zeta^{3j}, \zeta^{-3j}, 1).$$

Thus $F_A(j)(gH_A) = L_j g R_j^{-1} H_A$. By Lemma 7.8, no $F_A(j)$ -twisted geodesic has length $3h$ for an integer j .

Lemma 8.5. *For $r \in \mathbb{Z}_{\geq 0}$ and $k = 2^r$, there is no $F_A(3k)$ - or $F_A(-3k)$ -twisted geodesic of length $3kh$.*

Proof. Suppose such a geodesic exists, and let $j = 3k$ or $j = -3k$. Its horizontal lift has the form $g \exp(itC)$, where

$$C = C^*, \quad C_{33} = 0, \quad \text{Tr}(C^2) = 9k^2 h^2.$$

By the horizontal closing criterion (6.16), for some $b \in H_A$,

$$[C, S] = 0, \quad e^{iC} S = g^{-1} L_j g, \quad S = R_j b^{-1}. \quad (8.14)$$

The eigenvalue argument in the proof of Lemma 6.6, with $q = 1$, puts all three eigenvalues of C in $h\mathbb{Z}$: $j = \pm 3 \cdot 2^r$ is nonzero modulo seven, so the first two diagonal entries of S are distinct.

Write these eigenvalues as hm_1, hm_2, hm_3 . They satisfy

$$m_1^2 + m_2^2 + m_3^2 = 9 \cdot 4^r.$$

An integral square is zero or one modulo four. Since at most three such residues occur, divisibility of their sum by four forces every residue to be zero. Induction on r gives $k \mid m_a$ for each a . Choose $a \in \{1, \dots, 6\}$ with $ak \equiv 1 \pmod{7}$. On an eigenvector whose C -eigenvalue is $h k z$, one has $e^{ihz} = e^{iahkz}$, hence

$$e^{iC/k} = (e^{iC})^a.$$

The two factors on the left in (8.14) commute. Raising that equation to the power a gives

$$e^{iC/k} S^a = g^{-1} L_{j/k} g, \quad S^a = R_{j/k} b^{-a}, \quad [C/k, S^a] = 0.$$

Here $j/k = 3$ or -3 , and the identities follow from $aj \equiv j/k \pmod{7}$. Together with $(C/k)_{33} = 0$, these are the closing and horizontality equations for an $F_A(j/k)$ -twisted geodesic of length $3h$, contrary to Lemma 7.8. $\square$

Let

$$\mathcal{N} = \{n \in \mathbb{Z}_{>0} : n \equiv 1, 2, 3, 5, 6 \pmod{8}\}. \quad (8.15)$$

Lemma 8.6. *For each $n \in \mathcal{N}$ there are an odd positive integer d and a closed geodesic $c_n : \mathbb{R} \rightarrow A$ of speed $2\pi\sqrt{n}$ such that*

$$c_n(t + \delta) = c_n(t) \iff d\delta \in \mathbb{Z}. \quad (8.16)$$

Proof. The residue classes defining $\mathcal{N}$ contain no integer of the form $4^a(8b + 7)$. The three-square theorem [87, Introduction, equation (1.2), arXiv version] therefore gives $n = m_1^2 + m_2^2 + m_3^2$ with $m_a \in \mathbb{Z}$. Then $d = \gcd(|m_1|, |m_2|, |m_3|)$ is odd, since $4 \nmid n$. By changing signs, arrange $\min_a m_a \leq 0 \leq \max_a m_a$. Choose nonnegative t_1, t_2, t_3 with $t_1 + t_2 + t_3 = 1$ and $\sum_a t_a m_a = 0$. For an explicit choice, if some $m_a = 0$, put all weight at that index. Otherwise, for $m_p < 0 < m_q$, set $t_p = m_q/(m_q - m_p)$, $t_q = -m_p/(m_q - m_p)$, and the remaining weight to zero. Take a unitary matrix V whose third column is $(\sqrt{t_1}, \sqrt{t_2}, \sqrt{t_3})^\top$. Then

$$C_n = 2\pi V^* \text{diag}(m_1, m_2, m_3) V, \quad (C_n)_{33} = 0.$$

The curve $c_n(t) = \exp(itC_n)H_A$ is horizontal and is a geodesic of speed $2\pi\sqrt{n}$. Since $\exp(iC_n) = I$, it is closed with period one.

Equality of its values at t and $t + \delta$ is equivalent to $\exp(i\delta C_n) \in H_A$. Write this exponential as $\text{diag}(1, 1, e^{i\delta})$. If $e^{i\delta} \neq 1$, commutation with C_n forces C_n to preserve $\mathbb{C}e_3$, on which its eigenvalue is $(C_n)_{33} = 0$. Its exponential would then act as one on that line, a contradiction. Thus $\exp(i\delta C_n) = I$. Equivalently, $\delta m_a \in \mathbb{Z}$ for each a , which is equivalent to $d\delta \in \mathbb{Z}$ by Bézout's identity. $\square$

Let B_q and Γ_q be the suspension and embedded primitive geodesic of Lemma 8.3.

8.4.3. *The metrics and their spectra.* On $W = A \times B$, put

$$g_{W,0,\beta} = g_A \oplus \beta g_B, \quad g_{W,1,\beta} = \beta g_A \oplus g_B, \quad K_0 = (K_A, 0), \quad K_1 = (0, K_B), \quad \beta > 0.$$

Let (x, y, z) be the coordinates on the unit sphere $S^2 \subset \mathbb{R}^3$, and let b_* be its round metric of curvature one. Define

$$R = \frac{L}{2\pi}, \quad \alpha_\tau = \frac{\tau}{2\pi}(x dy - y dx), \quad b_s = R^2(1 + sz^2)b_*, \quad \tau \in \mathbb{R}, \quad s > -1. \quad (8.17)$$

For $\sigma \in \{0, 1\}$, $v, w \in TW$, and $X, Y \in TS^2$, set

$$G_{\sigma,\beta,\tau,s}(v + X, w + Y) = g_{W,\sigma,\beta}(v + \alpha_\tau(X)K_\sigma, w + \alpha_\tau(Y)K_\sigma) + b_s(X, Y). \quad (8.18)$$

These are jointly real-analytic metrics in space and the parameters. Their positive definiteness follows directly from the invertible shear in Lemma 6.7 and $1 + sz^2 \geq \min\{1, 1 + s\} > 0$ on S^2 .

Lemma 8.7. *For every $\beta > 0$, $\tau \in \mathbb{R}$, and $s > -1$, the metrics $G_{0,\beta,\tau,s}$ and $G_{1,\beta,\tau,s}$ are scalar-isospectral.*

Proof. The unitary map on $L^2(A \times B, dv_A dv_B)$ defined by

$$T(f \otimes g) = U^{-1}g \otimes Uf$$

satisfies

$$T(\Delta_A + \beta^{-1}\Delta_B) = (\beta^{-1}\Delta_A + \Delta_B)T, \quad TK_0 = K_1T.$$

Since $\dim A = \dim B = 8$, both vertical volume forms equal $\beta^4 dv_A dv_B$. Thus T is also unitary for the two scaled vertical metrics. Apply Lemma 6.7. $\square$

The reflection $(x, y, z) \mapsto (x, y, -z)$ preserves b_s and α_τ . Its product with the identity of W is an isometry of (8.18). Its connected fixed set $\Sigma = W \times \{z = 0\}$ is totally geodesic by [63, Problem 8-32, p. 262]. In the equatorial coordinate $(x, y, z) = (\cos 2\pi u, \sin 2\pi u, 0)$,

$$b_s = L^2 du^2, \quad \alpha_\tau = \tau du.$$

Consequently the induced metric is the suspension metric

$$\widehat{G}_{\sigma,\beta,\tau}(v + a\partial_u, w + b\partial_u) = g_{W,\sigma,\beta}(v + \tau a K_\sigma, w + \tau b K_\sigma) + L^2 ab. \quad (8.19)$$

It is independent of s . For $\sigma = 1$ and $\tau = 1/q$, the map

$$(x_A, x_B, [u]) \mapsto (x_A, [F_B(u/q)x_B, u])$$

is a well-defined isometry from this equatorial submanifold to $(A, \beta g_A) \times \mathcal{B}_q$. Increasing u by one applies the defining deck transformation of $\mathcal{B}_q$; differentiating the map gives the shear in (8.19).

8.4.4. Infinitely many embedded primitive geodesics. Choose $\beta > 0$ with $1, \beta, \sqrt{2}$ linearly independent over $\mathbb{Q}$. For $q \in \mathbb{Z}_{>0}$ coprime to six, $r \in \mathbb{Z}_{\geq 0}$, and $n \in \mathcal{N}$, define

$$k_r = 2^r, \quad \Lambda_{r,n,q,\beta}^2 = h^2(9k_r^2 + 49\beta n + 9q^2 k_r^2 \sqrt{2}). \quad (8.20)$$

Theorem 8.8 (Infinitely many exclusive primitive periods). *For each such (q, β) , there is a countable set $E_{q,\beta} \subset (-1, \infty)$ such that, for $s \in (-1, \infty) \setminus E_{q,\beta}$, the two real-analytic scalar-isospectral metrics on the fixed simply connected eighteen-manifold M satisfy*

$$\Lambda_{r,n,q,\beta} \in \text{Lsp}(M, G_{1,\beta,1/q,s}) \setminus \text{Lsp}(M, G_{0,\beta,1/q,s}) \quad (r \in \mathbb{Z}_{\geq 0}, n \in \mathcal{N}). \quad (8.21)$$

Each length $\Lambda_{r,n,q,\beta}$ is represented by an embedded primitive geodesic on $(M, G_{1,\beta,1/q,s})$, and the common half-wave trace is real analytic on an open interval about that length. The lengths are pairwise distinct and satisfy

$$\#\{(r, n) \in \mathbb{Z}_{\geq 0} \times \mathcal{N} : \Lambda_{r,n,q,\beta} \leq T\} = \frac{5}{32\beta\pi^2 \log 2} T^2 \log T + O_{q,\beta}(T^2). \quad (8.22)$$

Moreover, there is a product metric g_ on M with positive Ricci curvature such that every C^∞ neighborhood of g_* contains a pair $(G_{0,\beta,1/q,s}, G_{1,\beta,1/q,s})$ with positive Ricci curvature for some q, β as above and some $s \in (-1, \infty) \setminus E_{q,\beta}$.*

Proof. Abbreviate $G_{\sigma,s} = G_{\sigma,\beta,1/q,s}$. On $(\Sigma, G_{1,s}|\Sigma)$, the curve

$$t \mapsto (c_n(t), \Gamma_q(k_r t)) \in (A, \beta g_A) \times \mathcal{B}_q$$

has squared length (8.20). By Lemma 8.6, equality at parameter difference δ in its first component is equivalent to $d\delta \in \mathbb{Z}$, for some odd positive integer d . Embeddedness of Γ_q gives equality in the second component precisely when $k_r \delta \in \mathbb{Z}$. Since $\gcd(d, k_r) = 1$, Bézout's identity implies $\delta \in \mathbb{Z}$. The product has nonzero constant speed, so it is an injective immersion of the compact parameter circle and hence an embedding. Injectivity also excludes every smaller positive period, so the geodesic is primitive. Total geodesy of the equator makes it a geodesic of $G_{1,\beta,1/q,s}$ for every $s > -1$.

Suppose that the same length belongs to $\text{Lsp}(\Sigma, G_{0,s}|\Sigma)$. A lifted closed geodesic has real-coordinate displacement $m \in \mathbb{Z}$; its A -component is $F_A(m/q)$ -twisted, and its B -component is closed or constant. Indeed, the untwisting isometry identifies the universal cover of this equatorial metric with the product $(A \times B \times \mathbb{R}, g_A \oplus \beta g_B \oplus L^2 du^2)$; the m th deck map is $(F_A(m/q), I, u \mapsto u + m)$. Lifting a smooth closed geodesic gives a product geodesic whose endpoints and tangents match through precisely this deck map. For every $m \in \mathbb{Z}$, including zero, Lemma 6.6 therefore gives the squared length

$$h^2(q^{-2}N + 49\beta D + m^2 \sqrt{2}), \quad N, D \in \mathbb{Z}_{\geq 0}. \quad (8.23)$$

Rational independence of $1, \beta, \sqrt{2}$, together with (8.20), gives

$$N = 9q^2 k_r^2, \quad D = n, \quad m = \pm 3q k_r.$$

The A -component would be $F_A(\pm 3k_r)$ -twisted with length $3k_r h$, contrary to Lemma 8.5.

Fix (r, n) and a compact interval $J \subset (-1, \infty)$. The derivative of $G_{0,s}$ with respect to s is the nonnegative tensor $R^2 z^2 b_*$. If a closed geodesic $\gamma = (w, b)$ of length $\Lambda_{r,n,q,\beta}$ had zero integrated variation, continuity would give

$$z(b(t))^2 |\dot{b}(t)|_{b_*}^2 = 0 \quad (t \in \mathbb{R}/\mathbb{Z}).$$

The base curve b then lies in the equator or is constant. Indeed, on each connected component of the inverse image of either open hemisphere, $\dot{b} = 0$. Its value is constant there, and any such nonempty component is also closed in the parameter circle, since its constant value is strictly inside that hemisphere. It must therefore be the entire circle.

In the equatorial case the curve, already an ambient geodesic, is also an intrinsic geodesic of Σ , since its tangential covariant acceleration is zero. It is excluded by (8.23). If b is constant, variations of w with the same base point have energy $\frac{1}{2} \int_0^1 (g_A \oplus \beta g_B)(\dot{w}, \dot{w}) dt$. Since γ is a critical point of the ambient energy, w is an intrinsic closed geodesic of $(W, g_A \oplus \beta g_B)$. By (8.13), its squared length belongs to

$$49h^2\{N' + \beta D' : N', D' \in \mathbb{Z}_{\geq 0}\}.$$

This cannot equal (8.20), whose coefficient of $\sqrt{2}$ is nonzero. Thus (8.12) holds, and Lemma 8.4 gives finitely many exceptional parameters in J for this (r, n) . For example, the compact intervals $[-1 + 1/a, a]$, $a \in \mathbb{Z}_{>0}$, exhaust $(-1, \infty)$. The union of the finite exceptional sets over these intervals and all (r, n) is countable; denote it by $E_{q,\beta}$. For $s \notin E_{q,\beta}$, (8.21) holds for every (r, n) .

If two quantities in (8.20) are equal, the coefficients of $\sqrt{2}$ give $4^r = 4^{r'}$, and the coefficients of β then give $n = n'$. Hence the lengths $\Lambda_{r,n,q,\beta}$ are distinct. Corollary A.5 makes the common half-wave trace of the scalar-isospectral analytic metrics $G_{0,\beta,1/q,s}$ and $G_{1,\beta,1/q,s}$ real analytic near each excluded positive length.

Put

$$H(T) = \#\{(r, n) \in \mathbb{Z}_{\geq 0} \times \mathcal{N} : \Lambda_{r,n,q,\beta} \leq T\},$$

and set

$$X = (T/h)^2, \quad A_q = 9(1 + q^2 \sqrt{2}), \quad B_\beta = 49\beta.$$

The five residue classes defining $\mathcal{N}$ give

$$\#(\mathcal{N} \cap [1, Y]) = \frac{5}{8}Y + O(1) \quad (Y \geq 0),$$

with absolute error at most five. Consequently, for $X \geq A_q$ and $K = \lfloor \log_4(X/A_q) \rfloor$,

$$\begin{aligned} H(T) &= \sum_{r=0}^K \left(\frac{5}{8B_\beta} (X - A_q 4^r) + O(1) \right) \\ &= \frac{5}{8B_\beta} \left((K+1)X - \frac{A_q(4^{K+1} - 1)}{3} \right) + O(K+1) \\ &= \frac{5}{8B_\beta \log 4} X \log X + O_{q,\beta}(X). \end{aligned}$$

Here $K+1 = \log X / \log 4 + O_q(1)$, $A_q 4^{K+1} \leq 4X$, and the sum of the residue-class errors has absolute value at most $5(K+1)$. Substituting $h^2 = 4\pi^2/49$ gives (8.22).

Take $q_j = 6j + 1$ and put $\beta_j = 1 + \sqrt{3}/q_j$. The numbers $1, \beta_j, \sqrt{2}$ are linearly independent over $\mathbb{Q}$, since $1, \sqrt{3}, \sqrt{2}$ are. Choose

$$s_j \in (0, 1/q_j) \setminus E_{q_j, \beta_j}.$$

Such a choice is possible because the exceptional set is countable. Put

$$g_* = g_A \oplus g_B \oplus R^2 b_*.$$

Put $\alpha_* = (x dy - y dx)/(2\pi)$, and let η_σ be the metric dual of K_σ in its unscaled active factor. Write $g_{\text{other},0} = g_B$ and $g_{\text{other},1} = g_A$, each pulled back to M . Since the active factor is unscaled for both values of σ , the exact expansion is

$$\begin{aligned} G_{\sigma,\beta,\tau,s} - g_* &= (\beta - 1)g_{\text{other},\sigma} + \tau(\eta_\sigma \otimes \alpha_* + \alpha_* \otimes \eta_\sigma) \\ &\quad + \tau^2 |K_\sigma|^2 \alpha_* \otimes \alpha_* + sR^2 z^2 b_*. \end{aligned}$$

All tensors on the right are fixed and smooth on the compact manifold. Taking their $C^m(g_*)$ norms gives, for every $m \geq 0$,

$$\|G_{\sigma,\beta,\tau,s} - g_*\|_{C^m(g_*)} \leq C_m(|\beta - 1| + |\tau| + \tau^2 + |s|)$$

when $1/2 \leq \beta \leq 2$ and $|\tau|, |s| \leq 1$. Thus $G_{\sigma,\beta_j,1/q_j,s_j} \rightarrow g_*$ in C^∞ for both values of σ . By Theorem 7.7,

$$\text{Ric}_{g_*} \geq \rho g_*, \quad \rho = \min\{1/2, R^{-2}\} > 0.$$

By continuity of the Ricci tensor in the C^2 topology,

$$\text{Ric}_{G_{\sigma,\beta_j,1/q_j,s_j}} \geq (\rho/2)g_* > 0$$

for both values of σ and all sufficiently large j . $\square$

Remark 8.9. The counting function in (8.22) is that of the subset $\{\Lambda_{r,n,q,\beta} : r \geq 0, n \in \mathcal{N}\}$ of the ordinary length-set difference. The spectral equality is

$$\text{Spec}(\Delta_{G_{0,\beta,\tau,s}}) = \text{Spec}(\Delta_{G_{1,\beta,\tau,s}}) \quad (\beta > 0, \tau \in \mathbb{R}, s > -1),$$

with the same parameters on both sides.

8.5. Simple connectivity and length sets.

Lemma 8.10. *Let $(L_0, g_0), (L_1, g_1)$ and $(X, h_X), (Y, h_Y)$ be two pairs of scalar-isospectral closed connected Riemannian manifolds of positive dimension. Fix $T > 0$ and choose $c > 0$ such that*

$$2c \min\{\text{inj}(X, h_X), \text{inj}(Y, h_Y)\} > T. \quad (8.24)$$

Set $P_0 = (L_0 \times X, g_0 + c^2 h_X)$ and $P_1 = (L_1 \times Y, g_1 + c^2 h_Y)$. Then P_0, P_1 are scalar-isospectral and

$$\text{Lsp}(P_i) \cap (0, T] = \text{Lsp}(L_i, g_i) \cap (0, T] \quad (i = 0, 1).$$

A closed geodesic of L_i , regarded as a curve in P_i with constant second component, remains embedded and primitive whenever it has these properties in L_i .

Proof. Common scaling sends each factor eigenvalue μ to $c^{-2}\mu$. The product spectral decomposition in Theorem 8.1 therefore gives equal spectra on P_0, P_1 .

By [63, Problem 5-7(a), p. 146], the nonconstant closed geodesics of P_i are precisely the products of factor geodesics parametrized affinely on $\mathbb{R}/\mathbb{Z}$, with at most one constant component. Their squared lengths are the sums of the factor squared lengths. Thus, for $(Z_0, h_0) = (X, h_X)$ and $(Z_1, h_1) = (Y, h_Y)$,

$$\text{Lsp}(P_i) = \left\{ \sqrt{a^2 + c^2 b^2} : \begin{array}{l} a \in \text{Lsp}(L_i, g_i) \cup \{0\}, \quad b \in \text{Lsp}(Z_i, h_i) \cup \{0\}, \\ (a, b) \neq (0, 0) \end{array} \right\}. \quad (8.25)$$

Every nonconstant closed geodesic on Z_i has length at least $2 \text{inj}(Z_i)$: otherwise its two half-arcs, with opposite initial velocities, would be distinct minimizing geodesics from one endpoint to the other in a normal ball, contrary to [63, Proposition 6.11, p. 161]. Thus (8.24) forces the second component of every closed product geodesic of length at most T to be constant. Its first component is a closed geodesic of the same positive length. Conversely, every closed geodesic of the first factor gives a product geodesic of the same length. This proves the set identities. Equality of two points, or occurrence of a smaller period, for a curve with constant second component is equivalent to the same property for its first component; its velocity is $(\dot{y}_1, 0)$. $\square$

9. EXACT LENGTH DIFFERENCES AND THEIR ASYMPTOTICS

9.1. Length sets and their asymptotics on a locally symmetric manifold. Let g_{G_A} and g_{G_B} be the bi-invariant Riemannian metrics induced by (6.1) on $G_A = \text{U}(3)$ and $G_B = \text{Sp}(1) \times \text{SO}(4)$. Set

$$W_G = G_A \times G_B, \quad g_{W_G} = g_{G_A} \oplus g_{G_B}, \quad P_G = W_G \times (\mathbb{R}/\mathbb{Z}), \quad h = \frac{2\pi}{7}, \quad L = h^{1/4}.$$

The symbols g_{G_A}, g_{G_B} denote the group metrics, while g_A, g_B denote the quotient metrics of Section 6.4. Define the period-seven flows

$$\mathcal{F}_i(u)(x) = \iota_i(hu, 0)x\iota_i(3hu, hu)^{-1}, \quad i = A, B.$$

Let Z_0 and Z_1 be the generators of $(\mathcal{F}_A(u), I)$ and $(I, \mathcal{F}_B(u))$ on W_G . For $\sigma \in \{0, 1\}$ and $\tau \in \mathbb{R}$, put

$$g_{\sigma, \tau}^G(v + a\partial_s, w + b\partial_s) = g_{W_G}(v + \tau a Z_\sigma, w + \tau b Z_\sigma) + L^2 ab. \quad (9.1)$$

Proposition 9.1. *For every $\tau \in \mathbb{R}$, the metrics $g_{0, \tau}^G$ and $g_{1, \tau}^G$ are scalar-isospectral, real analytic, locally symmetric, and have nonnegative sectional curvature and the same Riemannian universal cover. For every $d \in \mathbb{Z}_{>0}$, their pullbacks to $P_{G, d} = W_G \times (\mathbb{R}/d\mathbb{Z})$ are isospectral for each flat unitary bundle pulled back from the same representation of the base group $d\mathbb{Z}$. For $\tau = 1/q$, $q \in \mathbb{Z}_{>0}$, they have a common connected finite Riemannian cover of degree $7q$ and their geodesic flows are clean.*

Proof. The closed flow subgroup

$$\mathcal{K} = \{(ht, 0; 3ht, ht) : t \in \mathbb{R}\} \subset \mathcal{T}_1$$

is contained in $\ker \alpha$. Corollary 4.13 gives a unitary $S : L^2(G_A) \rightarrow L^2(G_B)$ intertwining the scalar Laplacians and the pullbacks by $\mathcal{F}_A(t), \mathcal{F}_B(t)$ for every $t \in \mathbb{R}$. On $L^2(W_G)$ the tensor map $T(f \otimes g) = S^{-1}g \otimes Sf$ therefore satisfies

$$T\Delta_{W_G} = \Delta_{W_G}T, \quad TZ_0 = Z_1T.$$

Proposition 6.8 gives the stated scalar and connection spectra, local isometry to $(W_G \times \mathbb{R}, g_{W_G} + L^2 ds^2)$, and the common finite cover

$$(W_G \times \mathbb{R}/(7q\mathbb{Z}), g_{W_G} + L^2 ds^2)$$

of degree $7q$ when $\tau = 1/q$. The product metric is analytic, locally symmetric, and of nonnegative sectional curvature, as are its locally isometric quotients. The compact product is a Lie group with a bi-invariant metric; it is globally symmetric and clean [89, arXiv v1, §1.3 and Theorem 1.8]. Lemma A.1 gives cleanliness of both quotients. $\square$

Write $g_\sigma = g_{\sigma, 1}^G$ and let $N_\sigma(T) = \#(\text{Lsp}(P_G, g_\sigma) \cap (0, T])$.

Theorem 9.2 (Exact period differences). *The length sets satisfy $\text{Lsp}(P_G, g_0) \subsetneq \text{Lsp}(P_G, g_1)$, with*

$$\text{Lsp}(P_G, g_1) \setminus \text{Lsp}(P_G, g_0) = \left\{ h\sqrt{7m + \sqrt{2}k^2} : m, k \in \mathbb{Z}_{>0}, \quad 7 \nmid k, \quad m \notin \{1, 4\} \right\}. \quad (9.2)$$

Every length in this difference is a clean period of (P_G, g_1) at which the common scalar half-wave trace is real analytic. As $T \rightarrow \infty$,

$$N_0(T) = \frac{253}{12 \cdot 2^{1/4} \pi^3} T^3 + \frac{1}{8\pi^2} T^2 + O(T), \quad (9.3)$$

$$N_1(T) = \frac{295}{12 \cdot 2^{1/4} \pi^3} T^3 + \frac{1}{8\pi^2} T^2 + O(T). \quad (9.4)$$

Consequently,

$$\#((\text{Lsp}(P_G, g_1) \setminus \text{Lsp}(P_G, g_0)) \cap (0, T]) = \frac{7}{2 \cdot 2^{1/4} \pi^3} T^3 + O(T), \quad (9.5)$$

and

$$\lim_{T \rightarrow \infty} \frac{\#((\text{Lsp}(P_G, g_1) \setminus \text{Lsp}(P_G, g_0)) \cap (0, T])}{N_1(T)} = \frac{42}{295}.$$

9.1.1. *The norm sets for a fixed deck element.* Let S_3 and ν be the three-square set and residue type defined in Section 7.2. For $j \in \mathbb{Z}/7\mathbb{Z}$, let $Q_{A,j}$ and $Q_{B,j}$ denote the squared norm parameters of $F_A(j)$ -twisted and $F_B(j)$ -twisted geodesic segments, respectively: a parameter n means that the segment has squared length $h^2 n$. Constant segments are included whenever they satisfy the closing equations.

Signed coordinate permutations preserve squared norms, so the norm sets are unchanged by saturating the allowed residue classes under these permutations, as in Section 7.2. Proposition 6.4 gives

$$Q_{A,0} = Q_{B,0} = 49S_3.$$

For $j = 1$, Table (7.6) gives the six A types, whose squared norms have the six nonzero residues modulo seven. The B types are exactly these six together with 123; the multiplicity two of the latter does not affect a norm set. For general $j \neq 0$, multiply each residue vector by j before taking its type. Signed coordinate permutations commute with scalar multiplication. Moreover, multiplication by any $j \neq 0$ permutes the three nonzero least absolute residues $\{1, 2, 3\}$. Therefore, with

$$\mathcal{H} = \{|v|^2 : v \in \mathbb{Z}^3, \nu(v) = 123\},$$

one has

$$Q_{B,j} = Q_{A,j} \dot{\cup} \mathcal{H} \quad (j \neq 0). \quad (9.6)$$

The union is disjoint: every parameter in $Q_{A,j}$ is nonzero modulo seven, while every parameter in $\mathcal{H}$ is divisible by seven. The six A types, after multiplication by j , have all six nonzero squared norm residues exactly once.

Define the total fibre parameter sets

$$F_{A,j} = Q_{A,j} + 49S_3, \quad F_{B,j} = Q_{B,j} + 49S_3.$$

The second summand is the squared-length parameter of a closed geodesic in the group factor on which the monodromy is trivial. The period lattices in (6.8) have the same Euclidean norm set, so these are the fibre norm sets of both metrics. Lagrange's four-square theorem [97, Theorem 11.4.3] gives $S_3 + S_3 = \mathbb{Z}_{\geq 0}$, by grouping three squares and one square. Hence

$$F_{A,0} = F_{B,0} = 49\mathbb{Z}_{\geq 0}. \quad (9.7)$$

9.1.2. *Representations by vectors outside $7\mathbb{Z}^3$.*

Lemma 9.3. *Every positive member of S_3 has a representation $x^2 + y^2 + z^2$ in which x, y, z are not all divisible by seven. Consequently*

$$\mathcal{H} = \{n \in S_3 : n > 0, 7 \nmid n\}. \quad (9.8)$$

Proof. Write $n = 4^a u$, with $a \geq 0$, $u > 0$ and $4 \nmid u$. Since $n \in S_3$, one has $u \not\equiv 0, 4, 7 \pmod{8}$. Gauss's primitive three-square theorem [97, Theorem 30.1.3] gives $u = x^2 + y^2 + z^2$ with $\gcd(x, y, z) = 1$. The vector $2^a(x, y, z)$ has squared norm n and coordinate 2^a , so it lies outside $7\mathbb{Z}^3$.

By (7.8), a vector of squared norm divisible by seven has residue type 000 or 123. Thus a vector outside $7\mathbb{Z}^3$ whose squared norm is a multiple of seven has least absolute residue type 123. This proves (9.8). $\square$

Lemma 9.4. *One has the exact sum-set identity*

$$\mathcal{H} + 49S_3 = 7\mathbb{Z}_{>0} \setminus \{7, 28\}. \quad (9.9)$$

Proof. Every member of the left-hand side is a positive multiple of seven. Neither 7 nor 28 belongs to it: both are smaller than 49 and neither is a sum of three squares.

Let $n > 0$ be divisible by seven, with $n \notin \{7, 28\}$. If $n \in S_3$, then $n = n + 49 \cdot 0 \in \mathcal{H} + 49S_3$ by (9.8). Otherwise write

$$n = 4^a(8b + 7) = 7 \cdot 4^a(8d + 1),$$

where divisibility by seven gives $b = 7d$. If $a = 0$, the exclusion $n \neq 7$ gives $n \geq 63$, and $n - 49 \equiv 6 \pmod{8}$. If $a = 1$, the exclusion $n \neq 28$ gives $n \geq 252$, and $n - 49 \equiv 3 \pmod{8}$. In both cases $n - 49$ is a positive three-square integer divisible by seven, hence belongs to $\mathcal{H}$. If $a \geq 2$, then $n \geq 112$ and $n - 98 \equiv 6 \pmod{8}$. Thus $n - 98$ is again a positive member of $\mathcal{H}$. Since $1, 2 \in S_3$, these three cases prove the reverse inclusion. $\square$

Since $F_{A,j} \cap 7\mathbb{Z} = \emptyset$ for $j \neq 0$, (9.6) and (9.9) give

$$F_{B,j} = F_{A,j} \dot{\cup} (7\mathbb{Z}_{>0} \setminus \{7, 28\}) \quad (j \neq 0). \quad (9.10)$$

9.1.3. Eventual completeness of the nonzero residue classes.

Lemma 9.5. *For every $j \neq 0$ and every integer $n \geq 7203$,*

$$n \in F_{A,j} \iff 7 \nmid n.$$

In particular $F_{B,j}$ contains every integer at least 7203. Uniformly in $j \neq 0$ and $X \geq 0$,

$$\#(F_{A,j} \cap [0, X]) = \frac{6}{7}X + O(1), \quad \#(F_{B,j} \cap [0, X]) = X + O(1). \quad (9.11)$$

Proof. Let $n \geq 7203$ with $7 \nmid n$. Choose one of the allowed A residue vectors r whose squared norm is n modulo seven. Such a vector exists because the six types have the six nonzero squared norm residues. Choose a coordinate i with $r_i \not\equiv 0 \pmod{7}$. Replacing r by $r + 7te_i$ changes its squared norm modulo 49 by $14r_it$. Since $2r_i$ is invertible modulo seven, choose t so that the new vector r' satisfies

$$|r'|^2 \equiv n \pmod{49}.$$

Choose a parity vector with exactly b odd coordinates, where $b \in \{0, 1, 2, 3\}$ is congruent to $n - 1$ modulo four. The Chinese remainder theorem supplies an integer vector x having this parity vector and congruent to r' modulo 49. Choose each coordinate in a centered residue interval modulo 98; then $|x_i| \leq 49$ and $|x|^2 \leq 7203$. Its residue modulo seven is allowed for $Q_{A,j}$, and

$$u := \frac{n - |x|^2}{49} \in \mathbb{Z}_{\geq 0}, \quad u \equiv n - b \equiv 1 \pmod{4}.$$

The last congruence excludes the three-square obstruction, so $u \in S_3$. Hence $n = |x|^2 + 49u \in F_{A,j}$. Conversely, $F_{A,j} \cap 7\mathbb{Z} = \emptyset$. Equation (9.10) gives $[7203, \infty) \cap \mathbb{Z} \subset F_{B,j}$. Both sets differ from the indicated unions of residue classes by a subset of $[0, 7203) \cap \mathbb{Z}$. Counting complete blocks of seven and the remaining block therefore gives (9.11), with an absolute error bound independent of the six nonzero values of j . $\square$

9.1.4. Suspension lengths and their counting functions.

Proof of Theorem 9.2. Put $c = \sqrt{2}$, $F_{0,j} = F_{A,j}$, and $F_{1,j} = F_{B,j}$. A closed suspension geodesic with base winding k lifts to a product geodesic whose real-coordinate displacement is k and whose fibre endpoints and velocities match through the k th power of the monodromy. Conversely, any pair of group segments satisfying these conditions, together with the linear real-coordinate segment, projects to a closed geodesic. Reversing the segment replaces the monodromy by its inverse and preserves its squared norm. In the residue description this is $v \mapsto -v$, so $F_{\sigma,-j} = F_{\sigma,j}$. The squared lengths for $g_\sigma = g_{\sigma,1}^G$ are therefore precisely

$$h^2(n + ck^2), \quad k \in \mathbb{Z}_{\geq 0}, \quad n \in F_{\sigma,k \bmod 7}, \quad (n, k) \neq (0, 0). \quad (9.12)$$

The parametrization by (n, k) is injective: rational independence of 1 and $\sqrt{2}$ gives $n = n'$ and $k^2 = (k')^2$, and nonnegative k, k' are then equal. Equations (9.7) and (9.10) prove both length-set inclusion and (9.2).

Put $R = (T/h)^2$ and $K = \sqrt{R/c}$. For each fixed positive integer a , the identities for $\sum_{l=1}^m 1$ and $\sum_{l=1}^m l^2$ give

$$\begin{aligned} S_a(R) &:= \sum_{1 \leq l \leq K/a} (R - ca^2 l^2) \\ &= \frac{2}{3a\sqrt{c}} R^{3/2} - \frac{R}{2} + O_a(\sqrt{R}). \end{aligned} \quad (9.13)$$

Indeed, put $x = \sqrt{R}/(a\sqrt{c})$, $m = \lfloor x \rfloor$, and $\theta = x - m \in [0, 1)$. In $S_a(R) = Rm - ca^2 m(m+1)(2m+1)/6$, substitution gives

$$S_a(R) = \frac{2}{3a\sqrt{c}} R^{3/2} - \frac{R}{2} + a\sqrt{c} \left(-\theta^2 + \theta - \frac{1}{6} \right) \sqrt{R} + O_{a,c}(1).$$

The bounded range of θ proves (9.13). It follows that

$$\begin{aligned} \sum_{\substack{1 \leq k \leq K \\ 7 \nmid k}} (R - ck^2) &= S_1(R) - S_7(R) = \frac{4}{7\sqrt{c}} R^{3/2} + O(\sqrt{R}), \\ \sum_{\substack{1 \leq k \leq K \\ 7 \mid k}} (R - ck^2) &= S_7(R) = \frac{2}{21\sqrt{c}} R^{3/2} - \frac{R}{2} + O(\sqrt{R}). \end{aligned}$$

Write $d_0 = 6/7$ and $d_1 = 1$. The nonzero winding classes not divisible by seven contribute $d_\sigma(S_1 - S_7) + O(\sqrt{R})$, by (9.11). The positive winding classes divisible by seven contribute $S_7/49 + O(\sqrt{R})$, by (9.7); their allowed zero fibre parameter adds at most $\lfloor K/7 \rfloor$ terms. The zero winding class contributes $\lfloor R/49 \rfloor$, since the zero total length is excluded. Therefore

$$N_\sigma(T) = \frac{2}{3\sqrt{c}} \left(\frac{6}{7} d_\sigma + \frac{1}{343} \right) R^{3/2} + \frac{R}{98} + O(\sqrt{R}).$$

The parentheses equal $253/343$ for $\sigma = 0$ and $295/343$ for $\sigma = 1$. Substitution of $h = 2\pi/7$ proves (9.3)–(9.5).

The difference also has the exact count

$$\sum_{\substack{1 \leq k \leq \sqrt{R/c} \\ 7 \nmid k}} \left(\left\lfloor \frac{R - ck^2}{7} \right\rfloor - \mathbf{1}_{\{R - ck^2 \geq 7\}} - \mathbf{1}_{\{R - ck^2 \geq 28\}} \right).$$

The two removed fibre parameters are precisely those excluded in (9.2).

There are only finitely many pairs (n, k) under any length bound, so the length sets in (9.12) are locally discrete. Thus every member of the difference has an open interval disjoint from $\text{Lsp}(P_G, g_0)$. Proposition A.3 makes the common half-wave trace real analytic there. Cleanliness follows from Proposition 9.1. $\square$

9.1.5. Base covers and nearby metrics. For $p \in \mathbb{Z}_{>0}$, let $g_{\sigma,p}^{\text{cov}}$ be the pullback of $g_\sigma = g_{\sigma,1}^G$ to $P_{G,p} = W_G \times (\mathbb{R}/p\mathbb{Z})$.

Corollary 9.6. *If $7 \nmid p$, the two lifted length sets are nested and their difference is*

$$\left\{ h\sqrt{7m + \sqrt{2}p^2k^2} : m, k \in \mathbb{Z}_{>0}, \quad 7 \nmid k, \quad m \notin \{1, 4\} \right\}.$$

Its counting function is

$$\frac{7}{2p^{1/4}\pi^3} T^3 + O_p(T),$$

and its proportion among all positive distinct lengths of $g_{1,p}^{\text{cov}}$ tends to $42/295$. If $7 \mid p$, the two lifted metrics are isometric to $(W_G \times \mathbb{R}/p\mathbb{Z}, g_{W_G} + L^2 ds^2)$.

Proof. A geodesic of base winding k in this cover has real displacement pk and monodromy $\mathcal{F}_i(pk)$. If $7 \nmid p$, multiplication by p permutes the nonzero residues modulo seven. The difference for a fixed deck transformation (9.10) is independent of the nonzero residue, and the fibre densities in (9.11) are uniform in it. The proof of Theorem 9.2 therefore applies with $c = \sqrt{2} p^2$. The leading coefficients of both length counts, and of their difference, are divided by p , proving the stated count and ratio. If $7 \mid p$, both monodromies are the identity, so the two mapping tori are the same product metric. $\square$

Under the product marking,

$$\pi_1(P_G) = (\mathbb{Z} \times C_2) \times \mathbb{Z}, \quad \pi_1(P_{G,p}) = (\mathbb{Z} \times C_2) \times p\mathbb{Z}.$$

The representations in Proposition 9.1 are the unitary representations of the latter group that are trivial on $\mathbb{Z} \times C_2 = \pi_1(W_G)$.

Lemma 9.7. *Let G be a compact Lie group with a bi-invariant metric, $\ell, r, g \in G$, $N \in \mathbb{Z}_{>0}$, and $V \in \mathfrak{g}$. Suppose $\ell^N = r^N = 1$ and the geodesic $g \exp(tV)$, $0 \leq t \leq 1$, satisfies the position and tangent conditions*

$$\exp V = g^{-1} \ell g r^{-1}, \quad \text{Ad}_r V = V.$$

Then $\exp(NV) = 1$. In particular, for either group G_A, G_B with the metrics (6.1), every $\mathcal{F}_i(j/q)$ -twisted geodesic has squared length in $(h/q)^2 \mathbb{Z}_{\geq 0}$ for integers j and $q \geq 1$.

Proof. The tangent condition implies that r commutes with $\exp V$. Hence

$$\exp(NV) = (\exp V r)^N r^{-N} = g^{-1} \ell^N g = 1.$$

For the displayed flows, both translating elements have $(7q)$ th power one. Conjugate $7qV$ into the torus period lattice $2\pi\mathbb{Z}^3$ of (6.8). Its squared norm is $4\pi^2 N'$ for some $N' \in \mathbb{Z}_{\geq 0}$, so $|V|^2 = (h/q)^2 N'$. $\square$

Corollary 9.8. *For each $q \in \mathbb{Z}_{>0}$, the metrics $g_{0,1/q}^G$ and $g_{1,1/q}^G$ on the fixed manifold P_G have a length-set difference containing*

$$\begin{aligned} \mathcal{D}_q &= \left\{ h \sqrt{7m + \sqrt{2} q^2 k^2} : m, k \in \mathbb{Z}_{>0}, \quad 7 \nmid k, \quad m \notin \{1, 4\} \right\} \\ &\subseteq \text{Lsp}(P_G, g_{1,1/q}^G) \setminus \text{Lsp}(P_G, g_{0,1/q}^G). \end{aligned} \tag{9.14}$$

Its counting function is

$$\#(\mathcal{D}_q \cap (0, T]) = \frac{7}{2q 2^{1/4} \pi^3} T^3 + O_q(T). \tag{9.15}$$

Each member is a clean period of the second metric at which the common half-wave trace is real analytic. Both metric sequences converge in C^∞ to $g_{W_G} \oplus L^2 ds^2$ as $q \rightarrow \infty$.

Proof. At winding qk , the monodromy is $\mathcal{F}_i(k)$ and the circle energy is $\sqrt{2} q^2 k^2$. Equation (9.10) therefore gives $\mathcal{D}_q$ as the difference between the length sets of geodesics whose winding is divisible by q . Every squared length of either full suspension is, by Lemma 9.7, of the form

$$h^2 (q^{-2} N + \sqrt{2} l^2), \quad N \in \mathbb{Z}_{\geq 0}, \quad l \in \mathbb{Z}.$$

Equality with a member of (9.14) forces $l^2 = q^2 k^2$, by irrationality of $\sqrt{2}$. Hence its winding is $\pm qk$, and (9.10) excludes it from $\text{Lsp}(P_G, g_{0,1/q}^G)$. Equations (9.13) for $a = 1, 7$ and $c = \sqrt{2} q^2$ give (9.15) for every integer $q \geq 1$. Cleanliness follows from Proposition 9.1. The squared-length set above is locally finite, so every member of $\mathcal{D}_q$ has an open neighbourhood disjoint from $\text{Lsp}(P_G, g_{0,1/q}^G)$. Proposition A.3 gives real analyticity of the common trace there. The convergence follows directly from (9.1). $\square$

At $q = 1$, the set $\mathcal{D}_1$ is the entire length-set difference by Theorem 9.2.

All metrics in the following table are real analytic. In dimension eighteen, positive Ricci curvature holds for the nearby pairs in Theorem 8.8.

Theorem	Manifolds	Equal spectra	Periods on only one side
7.7	Dimension 8; $\pi_1 = C_7$	Every marked unitary flat spectrum	$3h$; the minima in class 3 agree
7.1	Dimension 9; locally symmetric	Scalar spectrum	Exact difference (1.1)
8.2	Dimension 17; one manifold, $\pi_1 = \mathbb{Z}$	Every marked unitary flat spectrum on every finite cover	An embedded primitive period
8.8	Dimension 18; one simply connected manifold	Scalar spectrum	Infinitely many embedded primitive periods for the stated parameters
9.2	Dimension 19; one locally symmetric manifold	Flat spectra pulled back from the base circle	Exact difference at $\tau = 1$; both total counting asymptotics

For the nearby nineteen-dimensional pairs, the length-set difference contains $\mathcal{D}_q$ of Corollary 9.8.

Part III. Central quotients and topology

For the binary products J_I and a closed $K \leq \prod_j P_{n_j}$, the groups $\pi_1(J_I/K)$ are central extensions of the finite group $\pi_0(K)$ by finitely generated abelian groups (Lemma 10.2). Conversely, every nonempty finite list of such extensions over one finite group has a common-isotropy realization whose boundary homomorphisms are the prescribed quotient maps. The realizations can be chosen to have equal canonical rough spectra for every common finite-dimensional unitary isotropy representation and pairwise different higher rational homotopy groups (Theorem 11.1).

10. PRESCRIBED CENTRAL QUOTIENTS

10.1. Central kernels and covering groups.

Lemma 10.1. *For $m \geq 1$, the determinant identifies $\pi_1(U(m))$ with $\mathbb{Z}$, whereas $SU(m)$ and $Sp(m)$ are simply connected. These groups have zero π_2 . For $m \geq 3$, $\pi_1(SO(m)) = C_2$ and $\pi_2(SO(m)) = 0$; a full rotation in one coordinate plane generates $\pi_1(SO(m))$. For every compact connected Lie group L and maximal torus $T \leq L$, the homomorphism $\pi_1(T) \rightarrow \pi_1(L)$ is onto.*

Proof. For every compact connected Lie group, the second homotopy group vanishes, and maximal-torus inclusion identifies the fundamental group with the integral lattice modulo the coroot lattice [20, Chapter V, Theorem 7.1]. The classical lattices are given in [20, Chapter V, §6]. For $SU(m)$ and $Sp(m)$ they coincide; for $U(m)$ their quotient is $\mathbb{Z}$, with quotient homomorphism the coordinate sum, induced by the determinant. For $SO(m)$ the coroot lattice consists of the integer vectors with even coordinate sum, so a full coordinate-plane rotation represents the nonzero class. The remaining low-dimensional cases are $U(1) = S^1$, $SU(1) = \{1\}$, $Sp(1) = SU(2) = S^3$, and $SO(3) = \mathbb{RP}^3$; see [20, Chapter I, (6.18)–(6.19)] for the last two identifications. $\square$

Lemma 10.2. *Let J be a compact connected Lie group and $K \leq J$ closed. Let $\tilde{J} \rightarrow J$ be the universal covering homomorphism and $\tilde{K}$ the inverse image of K . Then*

$$\pi_1(J/K) \cong \pi_0(\tilde{K}), \quad 1 \longrightarrow C \longrightarrow \pi_1(J/K) \longrightarrow \pi_0(K) \longrightarrow 1,$$

where

$$C = \text{coker}(\pi_1(K^0) \longrightarrow \pi_1(J))$$

is finitely generated and central. For right quotients, the boundary homomorphisms are defined by inverse endpoints of lifted loops.

Proof. The homotopy sequence of $\tilde{K} \rightarrow \tilde{J} \rightarrow J/K$ gives the first isomorphism. An inverse endpoint multiplies in the order of concatenation of loops, so the isomorphism is an isomorphism of groups. The homotopy sequence of $K \rightarrow J \rightarrow J/K$ gives the second sequence and the formula for C . The kernel of the covering homomorphism $\tilde{J} \rightarrow J$ is central [37, Proposition 2.11(ii)]; its image in $\pi_0(\tilde{K})$ is C . Finally $\pi_1(J)$ is a finitely generated abelian group and K has finitely many components. Thus C is finitely generated and has finite index. $\square$

Lemma 10.3. *Let $1 \rightarrow A \rightarrow E \rightarrow \Gamma \rightarrow 1$ be central, with A finitely generated and Γ finite. For every real vector space V , restriction $\text{Hom}(E, V) \rightarrow \text{Hom}(A, V)$ is an isomorphism. In particular $\dim_{\mathbb{R}} \text{Hom}(E, \mathbb{R}) = \text{rank } A$.*

Proof. Choose a classifying space $X = K(E, 1)$ and its regular covering $\pi : Y \rightarrow X$ corresponding to A . Then $Y = K(A, 1)$ and the deck group is Γ . Under $H^1(X; V) = \text{Hom}(E, V)$ and $H^1(Y; V) = \text{Hom}(A, V)$, the map π^* is restriction. The deck action on $H^1(Y; V)$ is induced by conjugation on A , hence is trivial. The transfer identities [47, §3.G, pp. 321–322] give

$$\text{tr } \pi^* = |\Gamma| \text{ id}, \quad \pi^* \text{ tr} = \sum_{\gamma \in \Gamma} \gamma^* = |\Gamma| \text{ id}.$$

Since multiplication by $|\Gamma|$ is invertible on the real vector space V , restriction is an isomorphism with inverse $|\Gamma|^{-1} \text{ tr}$. Finally $\dim_{\mathbb{R}} \text{Hom}(A, \mathbb{R}) = \text{rank } A$. $\square$

10.2. The central extension. Fix the groups, embeddings, and metrics of Section 2.

Theorem 10.4 (Prescribed central quotients). *Let $1 \rightarrow A \rightarrow E \xrightarrow{p} \Gamma \rightarrow 1$ be a central extension, where Γ is finite and A is finitely generated. Fix a generating tuple $c_1, \dots, c_r$ of A , with $r \geq 1$; identity entries and repetitions are allowed. For $I \subseteq [r]$, put $A_I = \langle c_j : j \in I \rangle$. There are odd integers $n_1 < \dots < n_r$ and one closed subgroup $K \leq \prod_{j=1}^r P_{n_j}$, with K^0 central in K , whose 2^r ambient quotients M_I have the following properties.*

(1) *There are isomorphisms*

$$\pi_1(M_I) \simeq E/A_I, \quad \pi_0(K) \simeq \Gamma,$$

under which the boundary homomorphism is the quotient $E/A_I \rightarrow \Gamma$. For $I \subseteq I'$, the quotient homomorphism $E/A_I \rightarrow E/A_{I'}$ is induced by a homomorphism of lifted stabilizers. Distinct subsets give different higher rational homotopy groups, even when the fundamental groups are isomorphic.

(2) *The quotients are closed connected real analytic normal homogeneous manifolds with nonnegative sectional curvature and*

$$\dim \ker \text{Ric}_{M_I} = \text{rank}(A/A_I).$$

(3) *Their scalar spectra agree, as do their canonical rough connection spectra for every common finite-dimensional unitary K -representation. Their metric tangent isotropy modules are isomorphic, giving the same spectra for all canonical natural tensor connections. They admit compatible homogeneous spin structures for which all homogeneously twisted cubic spin squares are isospectral. The graded cubic exterior operators have equal signed spectra with all common homogeneous unitary twists, in the conventions of Section 4.*

(4) *For every $L \leq \Gamma$, the corresponding connected covers have degree $[\Gamma : L]$, fundamental groups $p^{-1}(L)/A_I$, and equal flat spectra for every finite-dimensional unitary L -representation.*

Proof. The compact group.

Write A additively and define

$$\phi : \mathbb{Z}^r \rightarrow A, \quad \phi(e_j) = c_j, \quad \Lambda = \ker \phi, \quad W = \text{span}_{\mathbb{R}} \Lambda, \quad V = \mathbb{R}^r / W, \quad T = \mathbb{R}^r / \Lambda.$$

Set $a = \text{rank } A$ and $h = r - a$. The lattice Λ has rank h , so W/Λ is a compact torus. The integer classes embed $A = \mathbb{Z}^r / \Lambda$ as a closed discrete subgroup $\iota(A) \leq T$. Indeed, its inverse image under the

quotient map $\mathbb{R}^r \rightarrow T$ is $\mathbb{Z}^r$, which is closed. The image of a ball of radius less than $1/2$ meets this subgroup only at zero, because $\Lambda \subset \mathbb{Z}^r$.

Give E the discrete topology and form the Lie group

$$\bar{K} = (E \times T) / \{(c, \iota(c)^{-1}) : c \in A\}.$$

The denominator is closed, discrete and central, so the quotient map is a covering map. Both E and T embed. The full inverse image of E is $E \times \iota(A)$, so E is closed and discrete. Their intersection is A , with $[c, 1] = [1, \iota(c)]$. The identity component is the central subgroup T , and $\pi_0(\bar{K}) = \Gamma$.

Let $L : \mathbb{R}^r \rightarrow V$ be the quotient map. It descends to T and induces $\ell : A \rightarrow V$. By Lemma 10.3, the homomorphism ℓ extends uniquely to a homomorphism $B_E : E \rightarrow V$. The formula

$$\beta([e, t]) = B_E(e) + L(t)$$

is well defined on $\bar{K}$: its value on $(c, \iota(c)^{-1})$ is zero. Choose a linear section $V \rightarrow \mathbb{R}^r$ of L . Its image in T is central and splits β . Consequently

$$\bar{K} \cong V \times K_c, \quad K_c = \ker \beta.$$

The identity component of K_c is W/Λ . Every Γ -component meets K_c , since multiplication by the central section cancels its β -value. Thus

$$K_c^0 = W/\Lambda, \quad \pi_0(K_c) = \Gamma.$$

There are finitely many components and each is a translate of a compact torus; hence K_c is compact.

The embedding.

The map

$$z : \bar{K} \longrightarrow \mathbb{R}^r / \mathbb{Z}^r \simeq \mathbb{T}^r, \quad z([e, [x]_\Lambda]) = [x]_{\mathbb{Z}^r}$$

is a well-defined surjective homomorphism with kernel the embedded E . Let $z_1, \dots, z_r$ be its coordinate characters, $z_j([e, [x]_\Lambda]) = e^{2\pi i x_j}$, restricted to K_c .

There is an explicit faithful unitary representation of K_c of dimension

$$d = |\Gamma|(h + 1).$$

Indeed, take the regular representation of Γ , pulled back to K_c . If $\chi_1, \dots, \chi_h$ is a basis of the character lattice of the central torus K_c^0 , add $\text{Ind}_{K_c^0}^{K_c} \chi_j$ for each j . Induction here is finite-dimensional because the index is $|\Gamma|$: its representation space consists of equivariant functions on the finitely many cosets. The sum over cosets defines an invariant Hermitian form. On K_c^0 , each induced representation restricts to $|\Gamma|$ copies of χ_j , with the conventions $f(tk) = \chi_j(t)^{-1}f(k)$ and $(g \cdot f)(k) = f(g^{-1}k)$. An element in the kernel of the regular summand lies in K_c^0 . If all χ_j are trivial on it, it is the identity. Their direct sum ρ is therefore faithful. For $h = 0$, the pulled-back regular representation of Γ is faithful.

Let $K = K_c \times \mathbb{T}^r$, with auxiliary circle coordinates w_j . Set

$$A_j = \rho \oplus \rho \oplus w_j, \quad k \longmapsto ((A_j(k), z_j(k)))_{j=1}^r \in \prod_{j=1}^r P_{n_j}.$$

Trivial summands may be appended to any A_j . The resulting homomorphism is injective: an element of its kernel has trivial K_c -coordinate by faithfulness of ρ , and each w_j -coordinate is one. Its image is compact and hence closed.

Fundamental groups.

Consider the full pullback of the determinant covering:

$$\tilde{K}_c = \{(k, x) \in K_c \times \mathbb{R}^r : z(k) = [x]_{\mathbb{Z}^r}\}.$$

The formula

$$\Psi(k, x) = k[1, -x] \in \bar{K}$$

takes values in $E = \ker z$. It is a homomorphism because T is central. It is onto: given $e \in E$, choose x with $L(x) = -B_E(e)$ and set $k = e[1, x]$. Then $k \in K_c$ and $\Psi(k, x) = e$. Its kernel is

$$\{([1, x], x) : x \in W\} \cong W,$$

which is connected. Since E is discrete, this kernel is exactly $\tilde{K}_c^0$, and $\pi_0(\tilde{K}_c) = E$.

For $I \subseteq [r]$, keep only the real coordinates in I^c :

$$\tilde{K}_{c,I} = \{(k, x_{I^c}) : z_{I^c}(k) = [x_{I^c}]_{\mathbb{Z}^{I^c}}\}.$$

Forgetting x_I gives an onto covering homomorphism $F_I : \tilde{K}_c \rightarrow \tilde{K}_{c,I}$, with kernel $\{(1, x) : x \in \mathbb{Z}^I\}$. For such an integer vector,

$$\Psi(1, x) = [1, -x] = \phi(x)^{-1}.$$

Thus $\Psi(\ker F_I) = A_I$, and Ψ induces an onto homomorphism $\Psi_I : \tilde{K}_{c,I} \rightarrow E/A_I$. Put

$$W_I = F_I(\tilde{K}_c^0) \simeq W/(\Lambda \cap \mathbb{Z}^I).$$

Indeed, on the identity component parametrized by $x \mapsto ([1, x], x)$, the kernel of F_I consists exactly of $x \in \Lambda \cap \mathbb{Z}^I$. If $\Psi(k, x) \in A_I$, choose $y \in \mathbb{Z}^I$ with $\Psi(1, y) = \Psi(k, x)$. Then $(k, x)(1, y)^{-1} \in \ker \Psi$, so $\ker \Psi_I = W_I$. In particular this kernel is connected. The homomorphisms fit into the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & W & \longrightarrow & \tilde{K}_c & \xrightarrow{\Psi} & E & \longrightarrow & 1 \\ & & \downarrow & & \downarrow F_I & & \downarrow & & \\ 1 & \longrightarrow & W_I & \longrightarrow & \tilde{K}_{c,I} & \xrightarrow{\Psi_I} & E/A_I & \longrightarrow & 1. \end{array}$$

Since E/A_I is discrete and W_I is connected,

$$\pi_0(\tilde{K}_{c,I}) \simeq E/A_I. \quad (10.1)$$

For $I \subseteq I'$, coordinate deletion gives $F_{I,I'} : \tilde{K}_{c,I} \rightarrow \tilde{K}_{c,I'}$, and $\Psi_{I'} F_{I,I'}$ is Ψ_I followed by $E/A_I \rightarrow E/A_{I'}$. Coordinate deletions compose, so the component homomorphisms have the prescribed compositions.

Let J_I have G_{n_j} when $j \notin I$ and H_{n_j} when $j \in I$. Pull back the universal covering of the determinant torus of the remaining G -factors. Denote this connected covering group by $\hat{J}_I$. Its fundamental group is $(C_2)^I$: each covered unitary factor is $SU(2n_j + 1) \times \mathbb{R}$, while only the special orthogonal factors contribute a nontrivial fundamental group. The inverse image of K is $\hat{K}_I = \tilde{K}_{c,I} \times \mathbb{T}^r$. For $I \subseteq I'$, the homomorphism between these lifted stabilizers is $F_{I,I'} \times \text{id}_{\mathbb{T}^r}$.

By (2.1) and Lemma 10.1, a loop in P_n induces the following ambient fundamental-group elements:

$$(m, l) \mapsto l \quad \text{on } G_n, \quad (m, l) \mapsto m + l \pmod{2} \quad \text{on } H_n.$$

For $j \in I$, define a loop in $\hat{K}_I^0$ by keeping the K_c -coordinate equal to 1, all real coordinates equal to 0, all w_k with $k \neq j$ equal to 1, and putting $w_j(t) = e^{2\pi i t}$, $0 \leq t \leq 1$. Along this loop every z_k is 1, every A_k with $k \neq j$ is the identity, and A_j has determinant $e^{2\pi i t}$. Its image in $\pi_1(\hat{J}_I)$ is therefore the j -th coordinate generator. Hence the first arrow in

$$\pi_1(\hat{K}_I^0) \longrightarrow (C_2)^I \longrightarrow \pi_1(M_I) \longrightarrow \pi_0(\hat{K}_I) \longrightarrow 1$$

is onto. Exactness and (10.1) give $\pi_1(M_I) \simeq E/A_I$. The projection $\hat{K}_I \rightarrow K$ induces $E/A_I \rightarrow \Gamma$, since the central torus has trivial component. With the inverse-endpoint boundary convention of Lemma 10.2, this is the prescribed homomorphism $\pi_1(M_I) \rightarrow \pi_0(K)$.

Geometry and spectra.

Theorem 4.3, applied to the common closed subgroup K , gives equal scalar and canonical connection spectra. For $L \leq \Gamma$, let K_L be its inverse image in K . Then $K_L^0 = K^0$, and $J_I/K_L \rightarrow J_I/K$ is a connected cover of degree $[\Gamma : L]$. Let $\hat{K}_{I,L}$ be the inverse image of K_L in $\hat{J}_I$. The homomorphism

$$(k, x_{I^c}, w) \mapsto \Psi_I(k, x_{I^c})$$

sends $\widehat{K}_{I,L}$ onto $p^{-1}(L)/A_I$ and has connected kernel $W_I \times \mathbb{T}^r$. The auxiliary circle loops again map onto $(C_2)^I$, so

$$\pi_1(J_I/K_L) = p^{-1}(L)/A_I.$$

A unitary representation of $L = \pi_0(K_L)$, inflated to K_L , is trivial on K_L^0 . Theorem 4.3, applied to K_L and this representation, gives equal flat spectra.

The quotients are smooth, compact, connected and real analytic. Normal homogeneity gives $\sec \geq 0$. Proposition 4.15 and Lemma 10.3 give

$$\dim \ker \text{Ric}_{M_I} = b_1(M_I) = \dim_{\mathbb{R}} \text{Hom}(E/A_I, \mathbb{R}) = \text{rank}(A/A_I),$$

since $1 \rightarrow A/A_I \rightarrow E/A_I \rightarrow \Gamma \rightarrow 1$ is central and Γ is finite.

Take $n_j = 2d + 2j - 1$. Then $\dim K = h + r$ and

$$\dim M_I = \sum_{j=1}^r (2n_j + 1)^2 - (h + r).$$

This bound depends only on $|\Gamma|$, r , and $\text{rank } A$, not on the orders of the torsion elements or on the extension cocycle. The common spectrum may depend on the extension and its chosen generators.

The chosen ranks $n_j = 2d + 2j - 1$ are odd and strictly increasing. Since K^0 is toral, Theorem 12.8 gives

$$\dim_{\mathbb{Q}}(\pi_{4n_j+1}(M_I) \otimes \mathbb{Q}) = \#\{k \geq j : k \notin I\}.$$

Successive differences distinguish all switch sets I . Here K^0 is central in K . Thus Corollary 4.6, Theorem 4.9, and Proposition 4.11 give the common metric tangent module, compatible spin structures, equal cubic spin squares, and equal signed cubic exterior spectra. $\square$

10.3. Fundamental groups of compact homogeneous spaces.

Proposition 10.5. *The fundamental group of a closed connected Riemannian homogeneous manifold is finitely generated and central-by-finite. Conversely, each finitely generated central-by-finite group occurs as the fundamental group of a compact normal homogeneous member of the product construction. For every central extension $1 \rightarrow A \rightarrow E \rightarrow \Gamma \rightarrow 1$ with A finitely generated and Γ finite, the members with fundamental groups E and Γ have equal scalar and canonical homogeneous connection spectra, and equal flat spectra for the bundles pulled back from the same unitary representation of Γ .*

Proof. Wilking's theorem [99, Theorem 2.2(a),(d)] gives the first assertion. Conversely, let E be finitely generated and central-by-finite, and let $A \leq Z(E)$ have finite index. Then A is finitely generated. Apply Theorem 10.4 to a generating tuple of A , taking an identity entry if $A = 1$. The members $M_{\emptyset}$ and $M_{[r]}$ have fundamental groups E and $E/A = \Gamma$, respectively, and the required spectra. $\square$

The vector-group pushout occurs in Wilking's proof [99, §2.2]. For each central extension and generating tuple, Theorem 10.4 gives a compact isotropy group realizing all the quotients E/A_I , with the canonical spectral equalities of Section 4. The homomorphisms $E/A_I \rightarrow E/A_{I'}$ arise from coordinate deletions between lifted stabilizers.

11. FUNDAMENTAL GROUPS AND DIMENSION BOUNDS

11.1. The common finite quotient.

Theorem 11.1 (Fundamental-group lists with common isotropy). *Let $E_1, \dots, E_m$ be a nonempty finite list of groups. The following conditions are equivalent.*

- (1) *There are compact connected Lie groups J_i and embeddings of one compact Lie group K in all the J_i such that $\pi_1(J_i/K) \simeq E_i$.*
- (2) *There are a finite group Γ and surjections $p_i : E_i \rightarrow \Gamma$ whose kernels A_i are finitely generated central subgroups of E_i .*

- (3) *Such a common-isotropy realization exists in the finite-product family of Theorem 10.4, with its scalar, canonical bundle, natural tensor, and cubic operator spectra. The chosen manifolds may moreover be required to have pairwise different higher rational homotopy groups.*

The manifolds in (3) have real analytic normal homogeneous metrics and nonnegative sectional curvature. Their Ricci nullities are $\text{rank } A_i = b_1(J_i/K)$, so their Ricci curvatures are positive exactly when E_i is finite. The canonical rough spectra agree for every common finite-dimensional unitary isotropy representation and for all the associated natural tensor modules. They admit compatible homogeneous spin structures for which all twisted cubic spin squares are isospectral; their graded cubic exterior operators, with unitary twists, have equal signed spectra.

For each $L \leq \Gamma$, the corresponding degree- $[\Gamma : L]$ covers have fundamental groups $p_i^{-1}(L)$. The full canonical bundle, tensor, and cubic spectral equalities also hold on these covers. In particular, all their flat spectra pulled back from representations of L agree.

Proof. If (1) holds, Lemma 10.2, with $\Gamma = \pi_0(K)$, gives the central epimorphisms in (2).

Conversely, form the fibre product

$$E = E_1 \times_{\Gamma} \cdots \times_{\Gamma} E_m.$$

It is a central extension of Γ by $A = \prod_i A_i$. Its projection to E_i is onto and has kernel $\prod_{j \neq i} A_j$. For each i , choose a finite generating tuple of A_i and denote its index set by B_i , with the B_i pairwise disjoint. Append identity generators at m further indices $d_1, \dots, d_m$, and put

$$I_i = \left(\bigcup_{j \neq i} B_j \right) \cup \{d_i\}.$$

The resulting tuple generates A , and its entries indexed by I_i generate $\prod_{j \neq i} A_j$. Thus Theorem 10.4 gives a member with fundamental group

$$E / \prod_{j \neq i} A_j \simeq E_i.$$

The sets I_i are distinct, including when some A_i are trivial. Choose the odd strictly increasing ranks and trivial summands of Theorem 10.4. By Theorem 12.8, the selected members have pairwise different higher rational homotopy groups.

Theorem 10.4 gives the scalar and canonical connection spectra. Corollary 4.6, Theorem 4.9, and Proposition 4.11 give the tensor spectra, the cubic spectra, and compatible spin structures.

For $L \leq \Gamma$, let K_L be its full inverse image in K . Then K_L is closed, $K_L^0 = K^0$, and $[K : K_L] = [\Gamma : L]$. Writing $p : E \rightarrow \Gamma$ for the common projection, we obtain

$$\pi_1(J_{I_i}/K_L) \simeq p^{-1}(L) / \prod_{j \neq i} A_j \simeq p_i^{-1}(L)$$

from Theorem 10.4. The ambient right-module equivalence restricts to K_L , and Theorem 4.3 applies to every finite-dimensional unitary K_L -module. The isotropy identifications and the common spin lift restrict to K_L as well, so the tensor and cubic spectral equalities follow from the same results. A representation of L has zero differential and gives the indicated flat bundle. Finally, Proposition 4.15 and Lemma 10.3 give Ricci nullity $\text{rank } A_i$, and (3) implies (1). $\square$

Corollary 11.2. *Every finite list of finitely generated abelian groups occurs as the fundamental groups of one such isospectral family together with a simply connected member. Conversely, every group in a common-subgroup family containing a simply connected member is abelian. The first Betti numbers, and hence the dimensions of the Euclidean factors of the universal covers, can be prescribed by any finite list of nonnegative integers.*

Proof. For existence, apply Theorem 11.1 with $\Gamma = 1$ to the given list augmented by 1. Conversely, a simply connected member forces $\pi_0(K) = 1$, so all the other fundamental groups are central boundary kernels. Apply Proposition 4.15 to obtain the Euclidean factors of the universal covers. $\square$

Proposition 11.3. *Let J be a compact connected Lie group, and let $H_0, H_1 \leq J$ be closed subgroups with identical dimension data:*

$$\dim V^{H_0} = \dim V^{H_1} \quad (V \in \text{Irr}(J)).$$

Then J/H_0 is simply connected if and only if J/H_1 is simply connected. Their fundamental groups are simultaneously finite or infinite; in the finite case they have equal orders.

Proof. By [5, Theorem 1.3], maximal tori $T_i \leq H_i^0$ are conjugate in J , and $|\pi_0(H_0)| = |\pi_0(H_1)|$. Lemma 10.1 gives surjectivity of $\pi_1(T_i) \rightarrow \pi_1(H_i^0)$.

Put $I_i = \text{im}(\pi_1(H_i^0) \rightarrow \pi_1(J))$. These are the images of $\pi_1(T_i)$. The tori are conjugate, and inner automorphisms of connected J are homotopic to the identity through maps fixing the identity. Hence $I_0 = I_1 = I$. The connected-isotropy fibration and the covering $J/H_i^0 \rightarrow J/H_i$ give

$$1 \longrightarrow \pi_1(J)/I \longrightarrow \pi_1(J/H_i) \longrightarrow \pi_0(H_i) \longrightarrow 1.$$

The right groups are finite of equal orders. Thus the middle groups are finite exactly when their common kernel is finite, and their orders in that case are the kernel order multiplied by that common component count. Each middle group is trivial exactly when both its kernel and quotient are trivial. $\square$

11.2. Dimension bounds. In this subsection, $J = J_I$ and $K \leq P = \prod_{j=1}^r P_{n_j}$ are as in Section 2, with $r \geq 1$, $n_j \geq 1$, and K closed. The subgroup K is fixed when I varies. For a finitely generated abelian group C , let $d(C)$ denote its minimal number of generators; set $d(1) = 0$.

Theorem 11.4. *Put*

$$C = \ker(\pi_1(J/K) \longrightarrow \pi_0(K))$$

and let $d = d(C)$, with $d(1) = 0$. Then

$$r \geq d, \quad \dim(J/K) \geq \sum_j (3n_j^2 + 4n_j) + d \geq 7r + d \geq 8d. \quad (11.1)$$

If $\pi_1(J/K)$ is nonabelian, then some $n_j \geq 2$, and the last bound improves to $\dim(J/K) \geq 8d + 13$.

Proof. Choose a maximal torus T_K of K^0 , contained in a maximal torus T_P of P , and write

$$L_K = \pi_1(T_K), \quad L_P = \pi_1(T_P).$$

The inclusion $L_K \subset L_P$ is primitive: its quotient is the lattice of the torus T_P/T_K and is free. The map $\pi_1(T_K) \rightarrow \pi_1(K^0)$ is onto by Lemma 10.1.

If J has u unitary and h symplectic–orthogonal coordinates, $u + h = r$, then

$$B = \pi_1(J) = \mathbb{Z}^u \oplus (C_2)^h.$$

The cocharacter map $f : L_P \rightarrow B$ is onto: it is the distinguished z -coordinate in a G -factor and the sum of the $U(n)$ determinant and z windings modulo two in an H -factor, by (2.7). The homotopy exact sequence therefore gives

$$C = B/f(L_K).$$

Thus C is a quotient both of the r -generated group B and of the free abelian group L_P/L_K . Consequently

$$d \leq r, \quad d \leq \text{rank } P - \text{rank } K.$$

By [20, Chapter IV, Theorem 2.3(i)], the centralizer of T_K in the compact connected Lie group K^0 is T_K . Since $\mathfrak{t}_P$ centralizes T_K , this gives $\mathfrak{t}_P \cap \mathfrak{k} = \mathfrak{k}$. It follows that

$$\dim P - \dim K \geq \text{rank } P - \text{rank } K \geq d.$$

Since $\dim G_n = \dim H_n = (2n + 1)^2$ and $\dim P_n = n^2 + 1$, this proves (11.1).

If all $n_j = 1$, then P is a torus contained in a maximal torus T of J . By Lemma 10.1, the map $\pi_1(T) \rightarrow \pi_1(J)$ is onto. Hence the full inverse image $\tilde{T}$ in the universal covering group is connected. Its Lie algebra is abelian, so $\tilde{T}$ is abelian. The full inverse image $\tilde{K}$ lies in $\tilde{T}$; thus $\pi_1(J/K) \simeq \pi_0(\tilde{K})$ is abelian by Lemma 10.2. Thus a nonabelian fundamental group requires some $n_j \geq 2$. Replacing one minimal contribution 7 by $3(2)^2 + 4(2) = 20$ adds 13. $\square$

Theorem 11.5 (Sharp dimension for prescribed abelian groups). *Let A be a nontrivial finitely generated abelian group. Among pairs $J_I/K, J_{I'}/K$ with the same subgroup K , the least dimension of an isospectral pair with fundamental groups A and 1 is $8d(A)$.*

Proof. By Theorem 11.1, the simply connected member forces K to be connected. Theorem 11.4 gives the lower bound $\dim M \geq 8d(A)$.

Write $A = \mathbb{Z}^a \oplus \bigoplus_{j=1}^t C_{q_j}$ in nontrivial invariant factors, so $d(A) = a + t$. Corollary 11.12, with $n = 1$, gives an eight-dimensional pair with fundamental groups $C_{q_j}, 1$ for each finite factor, and one with groups $\mathbb{Z}, 1$ for each free factor. Their products attain dimension $8(a + t)$. $\square$

Corollary 11.6. *Let E be finite and $1 \neq C \leq Z(E)$. Among the product realizations for which the all-unitary boundary is the specified map $E \rightarrow E/C$, the least number of factors is $d(C)$.*

Proof. The all-unitary member has central boundary kernel C , so Theorem 11.4 gives $r \geq d(C)$. A generating tuple of length $d(C)$ in Theorem 10.4 attains this number. $\square$

Example 11.7. Let D_8 denote the dihedral group of order eight. The groups Q_8, D_8 , and C_2^2 occur in one isospectral family with common isotropy: their maps to C_2^2 are the two quotients by the centers and the identity map. Their kernels are central, so Theorem 11.1 applies.

11.3. A primitive torus and all its coordinate quotients. For a free central kernel, the groups $P_1 = U(1) \times U(1)$ suffice.

Theorem 11.8. *Let $\Phi : \mathbb{Z}^r \rightarrow \mathbb{Z}^a$, where $r \geq 1$ and $0 \leq a \leq r$, and write $c_j = \Phi(e_j)$. There is a common connected torus $K \leq (P_1)^r$ whose ambient quotients have dimension $7r + a$ and satisfy*

$$\pi_1(M_I) = \mathbb{Z}^a / \langle c_j : j \in I \rangle \quad (I \subseteq [r]).$$

All 2^r quotients have one scalar spectrum and the same canonical rough connection spectrum for every common finite-dimensional unitary K -representation. Their normal homogeneous metrics are real analytic and have nonnegative sectional curvature. The Ricci nullity at each member is the rank of its displayed fundamental group.

Proof. Put $\Lambda = \ker \Phi$ and $W = \text{span}_{\mathbb{R}} \Lambda$. Since $\mathbb{Z}^r / \Lambda \simeq \mathbb{Z}^a$ is torsion-free, Λ is primitive and $W \cap \mathbb{Z}^r = \Lambda$. Consequently the torus

$$T_\Lambda = W / \Lambda \longrightarrow \mathbb{R}^r / \mathbb{Z}^r \simeq \mathbb{T}^r$$

embeds as a closed connected subtorus of dimension $r - a$. Write its coordinate characters as $z_1, \dots, z_r$. Take

$$K = T_\Lambda \times \mathbb{T}_w^r, \quad A_j = w_j, \quad k \mapsto ((w_j, z_j))_{j=1}^r \in (P_1)^r.$$

The joint z -map is injective on T_Λ , and the w -entries are the coordinates of the other factor. The displayed map is therefore injective; its compact image is closed. Equivalently K is the kernel of the torus homomorphism $(P_1)^r \rightarrow \mathbb{T}^a$ defined by applying the integer matrix Φ to the z -coordinates.

Because K is connected, $\pi_1(M_I)$ is the cokernel of the map from $\pi_1(K)$ to the ambient fundamental group. In the coordinates $\pi_1(K) = \Lambda \oplus \mathbb{Z}^r$, that map is

$$(\lambda, m) \mapsto ((\lambda_j)_{j \notin I}, (\lambda_j + m_j \bmod 2)_{j \in I}) \in \mathbb{Z}^{I^c} \oplus (\mathbb{Z}/2)^I.$$

The images of the m_j loops contain $\{0\} \oplus (\mathbb{Z}/2)^I$. The remaining cokernel is

$$\mathbb{Z}^{I^c} / \text{pr}_{I^c} \Lambda \simeq \mathbb{Z}^r / (\Lambda + \mathbb{Z}^I) \simeq \mathbb{Z}^a / \langle c_j : j \in I \rangle.$$

These are marked identifications: deleting additional coordinates induces the corresponding quotient of $\mathbb{Z}^a$.

Each ambient factor has dimension nine, whereas $\dim K = (r - a) + r$. The quotient dimension is therefore $9r - (2r - a) = 7r + a$. Theorem 4.3 gives the scalar and canonical connection spectra. The normal quotient is real analytic, and (4.16) gives nonnegative sectional curvature. Proposition 4.15 identifies its Ricci nullity with the rank of the displayed abelian fundamental group. $\square$

Corollary 11.9. *Let $p_1, \dots, p_r$ be distinct primes, with $r \geq 1$, and put $P = \prod_j p_j$. There are 2^r pairwise nonhomeomorphic scalar-isospectral normal homogeneous manifolds of dimension $7r + 1$, with fundamental groups*

$$\pi_1(M_\emptyset) = \mathbb{Z}, \quad \pi_1(M_I) = C_{P/\prod_{i \in I} p_i} \quad (\emptyset \neq I \subseteq [r]).$$

Here C_1 denotes the trivial group. Every member has nonnegative sectional curvature. If $I \neq \emptyset$, then M_I has positive Ricci curvature, whereas $M_\emptyset$ has Ricci nullity one. Among families J_I/K with one closed subgroup $K \leq \prod_j P_{n_j}$, the least dimension for this prescribed list is $7r + 1$.

Proof. Take $a = 1$ and $c_j = P/p_j$. Their greatest common divisor is one: a prime dividing all of them would divide P , but the entry indexed by that prime omits it. This also holds for $r = 1$, when the single entry is one. Thus they define the surjection required in Theorem 11.8. For nonempty I ,

$$\gcd(c_i : i \in I) = P / \prod_{i \in I} p_i.$$

A prime p_k divides every selected entry exactly when $k \notin I$, proving the equality. Different nonempty subsets give different finite group orders, and the empty subset gives the infinite group $\mathbb{Z}$. All 2^r groups are therefore different, which proves nonhomeomorphism. The member with fundamental group $\mathbb{Z}$ has Ricci nullity one; the others have positive Ricci curvature.

For optimality, a product with r' factors has at most $2^{r'}$ ambient choices, so $r' \geq r$. The simply connected member makes the common isotropy connected. Thus the member with fundamental group $\mathbb{Z}$ has central boundary kernel $\mathbb{Z}$, and Theorem 11.4 gives $\dim M \geq 7r' + 1 \geq 7r + 1$. The quotients above attain this value. $\square$

Example 11.10 (Four members in dimension fifteen). Take $\Phi = (2, 3) : \mathbb{Z}^2 \rightarrow \mathbb{Z}$. An explicit common subgroup is $K = \mathbb{T}^3$, embedded by

$$(u, w_1, w_2) \mapsto ((w_1, u^3), (w_2, u^{-2})) \in P_1 \times P_1.$$

The joint powers of u are faithful because $\gcd(3, 2) = 1$. The four ambient quotients have fundamental groups $\mathbb{Z}, C_2, C_3, 1$, respectively, and dimension $18 - 3 = 15$. By Corollary 11.9, this dimension is least among the families J_I/K for that four-group list. For three primes, the same corollary gives eight members in dimension twenty-two; primes 2, 3, 5 give $\mathbb{Z}, C_{15}, C_{10}, C_6, C_5, C_3, C_2, 1$.

11.4. Primitive circles and their cyclic quotients.

Theorem 11.11. *Let $n \geq 1$ and let $(a_1, \dots, a_n, b) \in \mathbb{Z}^{n+1}$ be primitive. In P_n set*

$$k(\theta) = (\text{diag}(e^{ia_1\theta}, \dots, e^{ia_n\theta}), e^{ib\theta}), \quad K = \{k(\theta) : \theta \in \mathbb{R}/2\pi\mathbb{Z}\}.$$

Write $s = \sum_j a_j + b$, $D = 2 \sum_j a_j^2 + b^2$, and

$$X_{n;a,b} = G_n/K, \quad Y_{n;a,b} = H_n/K.$$

These are real analytic normal homogeneous manifolds of dimension $4n(n+1)$ with nonnegative sectional curvature and the common canonical spectra of Theorem 4.3. Their homotopy groups are

$$\begin{aligned} \pi_1(X_{n;a,b}) &= \mathbb{Z}/b\mathbb{Z}, & \pi_2(X_{n;a,b}) &= \begin{cases} 0, & b \neq 0, \\ \mathbb{Z}, & b = 0, \end{cases} \\ \pi_1(Y_{n;a,b}) &= \begin{cases} \mathbb{Z}/2, & s \text{ even}, \\ 1, & s \text{ odd}, \end{cases} & \pi_2(Y_{n;a,b}) &= \mathbb{Z}. \end{aligned}$$

Here $\mathbb{Z}/0\mathbb{Z} = \mathbb{Z}$. The Ricci tensor of $Y_{n;a,b}$ is positive. On the other member,

$$\text{Ric}_{X_{n;a,b}} \geq \frac{b^2}{2D} g_{X_{n;a,b}},$$

and its Ricci tensor is positive exactly when $b \neq 0$. The circle generators satisfy

$$Q_G(\dot{k}(0), \dot{k}(0)) = Q_H(\dot{k}(0), \dot{k}(0)) = D. \quad (11.2)$$

If $b \neq 0$, then

$$X_{n;a,b} \cong SU(2n+1)/F, \quad F = K \cap SU(2n+1) \cong \mathbb{Z}/|b|\mathbb{Z}, \quad b_2(X_{n;a,b}) = 0.$$

If s is odd, then $b_2(Y_{n;a,b}) = 1$. In particular $X_{n;a,1}$ is diffeomorphic to $SU(2n+1)$.

Proof. Primitivity gives integers r_j, r with $\sum r_j a_j + rb = 1$. If every $a_j \theta, b \theta$ is in $2\pi\mathbb{Z}$, so is θ ; thus K is an embedded circle. Its squared generator norm is D by (2.3), and the quotient dimension is $(2n+1)^2 - 1 = 4n(n+1)$. Theorem 4.3 and (4.16) give the common canonical spectra and nonnegative sectional curvature. Lemma 10.1 gives the exact sequence

$$0 \rightarrow \pi_2(J/K) \rightarrow \mathbb{Z} \xrightarrow{k_*} \pi_1(J) \rightarrow \pi_1(J/K) \rightarrow 1. \quad (11.3)$$

By (2.7), k_* is multiplication by b into $\mathbb{Z}$ or by s into C_2 . Their kernels and cokernels give the table; the latter kernel is either $\mathbb{Z}$ or $2\mathbb{Z}$, both infinite cyclic.

The algebra of H_n is semisimple, so its Ricci kernel is zero. For the generator $T = \dot{k}(0)$ in $\mathfrak{u}(2n+1)$, $\text{Tr } T = ib$ and $Q_G(T, T) = D$. Equation (4.15) gives the Ricci bound. If $b = 0$, the nonzero central vector iI_{2n+1} is horizontal and belongs to the Ricci kernel. When $b \neq 0$, the determinant $e^{ib\theta}$ maps K onto $U(1)$. Thus $SU(2n+1)$ acts transitively, with stabilizer F of order $|b|$, and $SU(2n+1)/F \cong X$ by [37, Corollary 3.13]. Since $\pi_1(SU(2n+1)) = \pi_2(SU(2n+1)) = 0$, the Hurewicz theorem gives $H^2(SU(2n+1); \mathbb{R}) = 0$ [47, Theorem 4.32]. The finite covering $SU(2n+1) \rightarrow X$ induces an injection on real cohomology [47, Proposition 3G.1], so $b_2(X) = 0$. If s is odd, then Y is simply connected and $\pi_2(Y) = \mathbb{Z}$; the same Hurewicz theorem gives $b_2(Y) = 1$. $\square$

Corollary 11.12. *For every $n \geq 1$ and $q \geq 1$, the choice $\mathbf{a} = (q-1, 0, \dots, 0)$, $b = q$ gives a pair of dimension $4n(n+1)$ with*

$$\pi_1(X) = \mathbb{Z}/q\mathbb{Z}, \quad \pi_1(Y) = 1, \quad \pi_2(X) = 0, \quad \pi_2(Y) = \mathbb{Z}, \quad b_2(X) = 0, \quad b_2(Y) = 1.$$

Both normal metrics have positive Ricci curvature. The choice $\mathbf{a} = (1, 0, \dots, 0)$, $b = 0$ gives fundamental groups $\mathbb{Z}$ and 1, with positive Ricci curvature on the latter. For fixed n , the scalar spectra of the finite cyclic pairs are distinct as q varies. For $n = 1$, write $K_q = \{(e^{i(q-1)\theta}, e^{iq\theta})\} \leq P_1$, $X_q = G_1/K_q$ and $Y_q = H_1/K_q$. These eight-dimensional manifolds satisfy

$$X_q \cong SU(3)/\mu_q, \quad Y_q \cong S^2 \times S^3 \times S^3, \quad \mu_q = \{\text{diag}(z, z^{-1}, 1) : z^q = 1\}.$$

Here $C_1 = 1$.

Proof. The two primitive vectors have $s = 2q-1$ and $s = 1$, respectively. Theorem 11.11 gives the displayed homotopy and Betti groups and positive Ricci curvature. Scalar isospectrality and fibre integration give

$$\text{Vol } X = \text{Vol } Y = \frac{\text{Vol}(G_n, Q_G)}{2\pi\sqrt{2(q-1)^2 + q^2}}.$$

Put $D_q = 2(q-1)^2 + q^2$. Since $D_{q+1} - D_q = 6q - 1 > 0$, the volumes are distinct for distinct q . Weyl's law [35, Theorem 16.7] therefore distinguishes their scalar spectra. For $n = 2$ the unitary circle is

$$\{\text{diag}(e^{i(q-1)\theta}, 1, e^{-i(q-1)\theta}, 1, e^{iq\theta}) : \theta \in \mathbb{R}/2\pi\mathbb{Z}\}; \quad (11.4)$$

its orthogonal rotation weights are $(q-1, 0, q)$ and its quotient has dimension 24. For $n = 1$, the finite unitary stabilizer is μ_q because $a_1 = q-1$.

For the topology of Y_q , lift to

$$\mathrm{SU}(2)_0 \times \mathrm{SU}(2)_+ \times \mathrm{SU}(2)_- \longrightarrow \mathrm{Sp}(1) \times \mathrm{SO}(4).$$

Choose torus angles so that (β, γ) maps to two-plane rotations $(\beta + \gamma, \beta - \gamma)$. The kernel is $\{(1, 1, 1), (1, -1, -1)\}$. The full preimage of K_q is the connected primitive circle

$$\{(e^{i(2q-2)t}, e^{i(2q-1)t}, e^{-it}) : t \in \mathbb{R}/2\pi\mathbb{Z}\}. \quad (11.5)$$

It contains the covering kernel at $t = \pi$ and maps onto K_q twice, with $\theta = 2t$, so the displayed circle is the full inverse image of K_q .

Its last projection is an isomorphism onto the maximal torus in $\mathrm{SU}(2)$. Thus it is the graph of a homomorphism from that torus to $F = \mathrm{SU}(2)^2$, after reversing its parameter. The quotient is a principal F -bundle over $\mathrm{SU}(2)/S^1 = S^2$: left multiplication by F is free and transitive on each fibre, and local Hopf sections give local trivializations. Since $\pi_1(F) = 0$, its smooth clutching loop extends smoothly over a disk. Hence the bundle is smoothly trivial, and $Y_q \cong S^2 \times S^3 \times S^3$. $\square$

Proposition 11.13. *Take $\mathbf{a} = (1, 0, \dots, 0)$ and $b = 1$. Then X is diffeomorphic to $\mathrm{SU}(2n+1)$ and $\pi_1(Y) = \mathbb{Z}/2$. Both metrics have positive Ricci curvature. Under that diffeomorphism, the metric on $\mathrm{SU}(2n+1)$ is left invariant. At the identity, for $U, V \in \mathfrak{su}(2n+1)$,*

$$g(U, V) = Q_G(U, V) - \frac{1}{3}Q_G(U, T)Q_G(V, T), \quad T = \dot{k}(0), \quad Q_G(T, T) = 3.$$

It is not bi-invariant.

Proof. For $b = 1$ and $s = 2$, Theorem 11.11 gives $X \cong \mathrm{SU}(2n+1)$, $\pi_1(Y) = C_2$, and positive Ricci curvature. The metric formula is the orthogonal projection away from the vertical line $\mathbb{R}T$. Write $T_0 = T - (\mathrm{Tr} T)/(2n+1)I_{2n+1} \neq 0$. The form $Q_G - g$ has rank one on $\mathfrak{su}(2n+1)$. If g were bi-invariant, the kernel of $Q_G - g$ would be a nonzero proper ideal by [37, Proposition 14.12], contrary to the simplicity of $\mathfrak{su}(2n+1)$. $\square$

Products of the $n = 2$ pairs give $\mathbb{Z}^r \times \prod_{j=1}^s C_{q_j}$ and 1 in dimension $24(r+s)$ for $r+s > 0$, by the product formulas for spectra and fundamental groups. The rank-one choices attain the sharper dimension in Theorem 11.5. The vector $\mathbf{a} = 0, b = 1$ gives two simply connected members with second Betti numbers zero and one.

Proposition 11.14. *Every quotient J_1/K with at least one factor has dimension at least seven. In dimension seven the only possible parameters are $r = 1, n_1 = 1, K = T^2$, so both ambient choices are simply connected. In dimension eight, if either choice is simply connected, K is a primitive circle with weights (a, b) , and*

$$\pi_1(G_1/K) = \mathbb{Z}/b\mathbb{Z}, \quad \pi_1(H_1/K) = \begin{cases} 1, & a+b \text{ odd}, \\ C_2, & a+b \text{ even}. \end{cases}$$

Here $\mathbb{Z}/0\mathbb{Z} = \mathbb{Z}$. Exactly one member is simply connected precisely when either $a+b$ is odd and $|b| \neq 1$, or $|b| = 1$ and a is odd. Both quotient Ricci tensors are positive precisely when $b \neq 0$.

Proof. Theorem 11.4 gives $\dim(J/K) \geq \sum_j (3n_j^2 + 4n_j) \geq 7r$. Dimension at most eight forces $r = n_1 = 1$. In dimension seven a closed subgroup of T^2 of dimension two is all of T^2 . By (2.7), both maps from $\pi_1(T^2)$ to the ambient fundamental groups are onto; hence both quotients are simply connected. In dimension eight a simply connected quotient forces K connected, hence a primitive circle. Theorem 11.11, with $n = 1$, gives the fundamental groups and the Ricci criterion. The unitary quotient is simply connected exactly when $|b| = 1$, and the other quotient exactly when $a+b$ is odd. These conditions give the stated alternatives. $\square$

11.5. Free torus actions.

Proposition 11.15. *For $r \geq 1$ and any closed $K \leq \prod_{j=1}^r P_{n_j}$, every quotient J_I/K admits a free isometric action of a torus of dimension r . Its orbits are flat and totally geodesic. For fixed K and matched block scales the acting torus and its orbit metric are the same at all choices of I . Every such quotient has Euler characteristic zero and bounds unorientedly; if it is orientable, it bounds orientedly.*

Proof. In one block define

$$\ell_G(t) = \text{diag}(e^{it}I_n, e^{it}I_n, 1), \quad \ell_H(t) = (\text{diag}(e^{it}I_n, e^{-it}I_n), \text{real}(e^{-it}I_n) \oplus I_2).$$

These circles centralize the respective copies of P_n . Their generators are orthogonal to $\mathfrak{p}_n$ and have squared norm $2n$ in the specified metrics. In the H case the two cross inner products with $u(n)$ cancel; in the G case the X and $\bar{X}$ traces cancel. In each case the intersection with P_n consists of $t = 0, \pi$, corresponding to the abstract elements $(I_n, 1), (-I_n, 1)$.

The product torus L centralizes K , and $D = L \cap K$ is the same subgroup of $(\mathbb{Z}/2)^r$ for every I . Right multiplication defines an isometric action on J/K . At every gK its stabilizer is exactly D , because $g\ell K = gK$ if and only if $\ell \in K$. Thus L/D acts freely. For block scales s_j , its metric on the covering space $\mathbb{R}^r$ is $\sum_j 2s_j n_j dt_j^2$ and its period lattice is $(2\pi\mathbb{Z})^r$ enlarged by the half-periods representing D . Both the lattice and the inner product are independent of I .

The subgroup KL is closed, and $J/K \rightarrow J/(KL)$ is a principal L/D bundle. Every geodesic of an orbit is the projection of $g \exp(tX)$ for $X \in \text{Lie } L$. Since $X \perp \mathfrak{k}$, this is a geodesic of the normal homogeneous quotient by Section 6.5. The orbits are therefore totally geodesic and, being flat tori with the displayed invariant metric, flat.

A primitive circle in L/D acts freely. Its fundamental vector field has no zero, giving Euler characteristic zero. The associated complex line bundle over the circle quotient has disk bundle with boundary J/K . If J/K is oriented, the connected circle action preserves its orientation, the base obtains the induced orientation, and the disk bundle is oriented with this boundary orientation. $\square$

11.6. Prescribed finite abelian groups and finitely many periods.

Corollary 11.16. *Let D be a nontrivial finite abelian group, and let $d(D)$ be its least number of generators. Fix a positive-Ricci pair $(L_0, g_0), (L_1, g_1)$ from Theorem 8.8, and a finite nonempty subset F of the lengths in (8.21). There are closed connected real-analytic scalar-isospectral manifolds P_0, P_1 of dimension $18 + 8d(D)$ such that*

$$\text{Ric}_{P_i} > 0, \quad \pi_1(P_0) \simeq D, \quad \pi_1(P_1) = 1, \quad F \subset \text{Lsp}(P_1) \setminus \text{Lsp}(P_0).$$

Every member of F has an embedded primitive representative on P_1 , and the common half-wave trace is real analytic near each member of F . For a cyclic group, the dimension is 26.

Proof. Theorem 11.5 gives real analytic normal homogeneous manifolds $(X, h_X), (Y, h_Y)$ of dimension $8d(D)$ with equal scalar spectra and fundamental groups D and 1. Their Ricci tensors are positive by Proposition 4.15, because their fundamental groups are finite. Choose $c > 0$ with

$$2c \min\{\text{inj}(X, h_X), \text{inj}(Y, h_Y)\} > \max F.$$

Put $P_0 = (L_0 \times X, g_0 + c^2 h_X)$ and $P_1 = (L_1 \times Y, g_1 + c^2 h_Y)$. Lemma 8.10 gives the spectral identity, the length exclusions, and the embedded primitive representatives. Since the first factors are simply connected, the product fundamental groups are D and 1. The Ricci tensor of a product is the sum of the Ricci tensors of its factors; constant scaling does not change the latter as covariant tensors. It is therefore positive on every nonzero product tangent vector. The product metric coefficients are analytic. Corollary A.5 makes the common half-wave trace analytic near each $\ell \in F$. $\square$

The case $D = 1$ gives the eighteen-dimensional pair. The scale c depends on the finite set F .

12. SEVEN-DIMENSIONAL TOPOLOGY AND PRESCRIBED FUNDAMENTAL GROUPS

The $n = 1$, $K = P_1 = \mathbb{T}^2$ case of Theorem 4.3 gives a simply connected pair

$$\begin{aligned} A = A_7 &= \mathrm{U}(3)/\mathbb{T}^2 \cong \mathrm{SU}(3)/S_{(1,-1,0)}^1, \\ B = B_7 &= (\mathrm{Sp}(1) \times \mathrm{SO}(4))/\mathbb{T}^2 \cong \mathrm{SU}(2)^3 / \ker(\alpha - \beta - \gamma). \end{aligned}$$

Here each angle in the last expression is a 2π -periodic maximal-torus angle of $\mathrm{SU}(2)$. Both metrics are the normal quotient metrics above. The second manifold is usually denoted $Q^{1,1,1}$, up to Weyl changes of signs. The first is the Aloff–Wallach manifold $N_{1,-1}$. Both winding maps in (2.7) are onto, so both quotients are simply connected. The determinant kernel $\mathrm{SU}(3)$ is transitive on the first quotient with stabilizer $\mathrm{diag}(e^{it}, e^{-it}, 1)$. On the finite cover $\mathrm{SU}(2)_0 \times \mathrm{SU}(2)_+ \times \mathrm{SU}(2)_- \rightarrow H_1$, choose angles so the last two factors give real rotations $\beta + \gamma$ and $\beta - \gamma$. The preimage of P_1 is then exactly $\alpha = \beta + \gamma \pmod{2\pi}$. This is a connected torus, as the defining integer character is primitive, and it contains the covering kernel.

For a closed connected manifold, formality over $\mathbb{Q}$ is equivalent to formality over $\mathbb{R}$ by [86, Corollary 4.17], since its cohomology is finite-dimensional.

Theorem 12.1 (Seven-dimensional topology). *The scalar-isospectral manifolds A and B have dimension seven, nonnegative sectional curvature, and positive Ricci curvature. The manifold A is rationally formal. The manifold B has a nonzero triple Massey product and is not rationally formal. Their second Betti numbers are respectively 1 and 2, so the zero eigenvalue of the Hodge Laplacian on two-forms has different multiplicities. Moreover*

$$H^4(A; \mathbb{Z}) = 0, \quad H^4(B; \mathbb{Z}) = \mathbb{Z}/2.$$

Dimension seven is the smallest dimension in which a pair of simply connected closed manifolds can have opposite rational formality.

Proof. The manifold $A = N_{1,-1}$ is formal by [40, Theorem 5.3]. Proposition C.24 gives the two second Betti numbers and the integral fourth cohomology groups. It also identifies B with the circle bundle over $(S^2)^3$ whose first Chern class is the sum of the three positive generators. This is the seven-dimensional bundle in [17, proof of Theorem 3.2].

Let $u, v \in H^2(B; \mathbb{Q})$ be the pullbacks of the generators of the first two sphere factors. Since $H^4(B; \mathbb{Q}) = 0$, the product $\langle u, u, v \rangle$ is defined. Poincaré duality gives $H^3(B; \mathbb{Q}) = 0$, so its indeterminacy is zero. After extension of scalars to $\mathbb{R}$, it is the nonzero product computed in [17, proof of Theorem 3.2, case $n = 3$]. It is therefore nonzero over $\mathbb{Q}$, and B is not formal. Hodge theory gives the different zero multiplicities on two-forms.

Scalar isospectrality follows from Theorem 4.3 with the trivial representation. The normal homogeneous metrics have nonnegative sectional curvature. Their Ricci curvatures are positive by Proposition 4.15, since A and B are simply connected. Every simply connected compact manifold of dimension at most six is formal by [80, Proposition 4.6, p. 574]. $\square$

Corollary 12.2. *The simply connected seven-dimensional pair in Theorem 12.1 has equal canonical rough tensor spectra and equal signed exterior cubic spectra, with all common homogeneous unitary twists. Its Hodge two-form spectra differ. For its unique spin structures the homogeneous twisted cubic spin squares agree. The least untwisted square eigenvalue is 2, with multiplicity 2.*

Proof. Apply Theorems 4.3 and 4.9 with $n = 1$, $K = P_1$. The Hodge two-form spectra differ by Theorem 12.1. The tangent module consists of a fixed line and three rotation planes of torus weights $2a, a + b, a - b$. Their sum is $4a$, even on the integral torus lattice, and the spin weights are the eight half-sums with independent signs. Their character is

$$2 + e^{2ia} + e^{-2ia} + e^{i(a+b)} + e^{-i(a+b)} + e^{i(a-b)} + e^{-i(a-b)}.$$

Thus the invariant spinor dimension is two. The isotropy is abelian, the ambient Weyl vector satisfies $\|\rho_j\|^2 = 2$, and the untwisted formula is $(D^{1/3})^2 = \Delta_j + 2$. Only constant equivariant spinors attain

the value two. Simple connectivity makes the spin structure unique. For the torus metric $2a^2 + b^2$, the spin Casimir eigenvalues are 0, 2, 3/2 with multiplicities 2, 2, 4; hence the cubic endomorphism square has eigenvalues 2, 0, 1/2 with those same multiplicities. $\square$

12.1. Formality of the dimension-datum quotients.

Proposition 12.3. *Let $n \geq 1$ and $N = 2n + 1$. The quotients $SU(2N)/U(N)$ and $SU(2N)/(\mathrm{Sp}(n) \times \mathrm{SO}(2n + 2))$ in An–Yu–Yu’s construction [5, Theorem 1.5] are both rationally formal, for the defining embeddings in that theorem.*

Proof. Write H for either isotropy group. It is connected and has rank N . The defining representation of $SU(2N)$ has torus weights $\pm x_1, \dots, \pm x_N$ on H , so every odd Chern class restricts to zero in $H^*(BH; \mathbb{R})$. Order the polynomial generators of $H^*(BSU(2N); \mathbb{R})$ as

$$c_2, c_4, \dots, c_{2N}, \quad c_3, c_5, \dots, c_{2N-1}.$$

The restrictions of the last $N - 1$ generators belong to the ideal generated by those of the first $N = \mathrm{rank} H$ generators, since they vanish. Thus the homogeneous space is of Cartan type in the sense of [60, arXiv version 2, §2.3]. Its formality follows from [60, arXiv version 2, §3, Theorem 6 and Remark 2]. $\square$

12.2. Products and all unitary flat spectra.

Theorem 12.4. *Let Z be any closed connected Riemannian manifold, and give $Z \times A$ and $Z \times B$ their product metrics. Under the canonical identification of both fundamental groups with $\pi_1(Z)$, one has:*

- (1) *Every finite-dimensional unitary flat-bundle spectrum agrees, including multiplicities.*
- (2) *Every corresponding finite cover has equal flat-bundle spectra for every finite-dimensional unitary representation of its fundamental group.*
- (3) *The minimum length in every free homotopy class is exactly the minimum length of the corresponding class in Z .*
- (4) *The two products are not homeomorphic: their second Betti numbers differ by one. The product $Z \times B$ is nonformal. If Z is formal, then $Z \times A$ is formal.*

Proof. Since A, B are simply connected, every flat bundle is pulled back from the representation $\rho : \pi_1(Z) \rightarrow U(V)$. Its Laplacian is

$$\Delta_{Z,\rho} \otimes 1 + 1 \otimes \Delta_A \quad \text{or} \quad \Delta_{Z,\rho} \otimes 1 + 1 \otimes \Delta_B.$$

Tensor eigenbases give the same sums with the same multiplicities. For $L \leq \pi_1(Z)$ of finite index, the corresponding cover is $Z_L \times A$ or $Z_L \times B$. Replacing Z, ρ by Z_L and any unitary representation of L gives the equal flat spectra on these covers.

Projection to Z does not increase length, while a loop γ in Z lifts as (γ, a_0) or (γ, b_0) without changing its length. These operations preserve the fundamental-group marking, so the three length infima are equal. Lemma 5.3 gives an attained minimum in every nontrivial free homotopy class of Z ; a constant loop minimizes the trivial class. A constant-fiber lift of a minimizing loop attains the same minimum in either product.

The Künneth formula and $H^1(A; \mathbb{Q}) = H^1(B; \mathbb{Q}) = 0$ give

$$b_2(Z \times A) = b_2(Z) + 1, \quad b_2(Z \times B) = b_2(Z) + 2.$$

The product of formal commutative differential graded algebras is formal. Pull a defining system for the nonzero triple Massey product on B in Theorem 12.1 back along $Z \times B \rightarrow B$. If the resulting Massey set contained zero, its restriction to $\{z_0\} \times B$ would put zero in the original Massey set, a contradiction. $\square$

Corollary 12.5. *For every finite group E , there are closed normal homogeneous manifolds with fundamental group E , nonnegative sectional curvature, positive Ricci curvature, and opposite rational formality which have all corresponding unitary flat-bundle spectra equal. The equality holds on all corresponding finite covers and for all unitary twists on those covers; all marked minimum lengths also agree. For $E = 1$, dimension seven is possible. If E embeds in $SU(N)$, dimension $N^2 + 6$ is possible.*

Proof. A faithful representation $\sigma : E \rightarrow U(d)$ gives a faithful map $\sigma \oplus \det(\sigma)^{-1} : E \rightarrow SU(d+1)$. Set $Z = SU(N)/E$, with its normal metric. Every deck right translation on $SU(N)$ is homotopic to the identity, through right translations along a path in the connected group. By the transfer theorem [47, Proposition 3G.1], the finite covering therefore induces an isomorphism

$$H^*(Z; \mathbb{Q}) \longrightarrow H^*(SU(N); \mathbb{Q}).$$

The induced map of rational polynomial form algebras is a quasi-isomorphism. The compact Lie group $SU(N)$ is formal by [91, §3.1, Theorem 5], applied with $H = 1$; hence Z is formal.

Apply Theorem 12.4. Products of the normal metrics remain normal homogeneous, and their Ricci curvatures are positive. For $E = 1$, take Z to be a point. $\square$

Corollary 12.6. *For every closed oriented hyperbolic surface Σ_g of genus $g \geq 2$, the nine-dimensional manifolds $\Sigma_g \times A$ and $\Sigma_g \times B$ have identical marked unitary flat spectra on all corresponding finite covers and identical complete minimum marked length spectra. They have opposite rational formality.*

Proof. A closed oriented surface admits a Kähler structure and is therefore formal by [30, §6]. Apply Theorem 12.4. $\square$

12.3. Higher homotopy groups.

Lemma 12.7. *If J is a connected Lie group and $K \leq J$ is closed with toral identity component, then*

$$\pi_d(J) \longrightarrow \pi_d(J/K)$$

is an isomorphism for every $d \geq 3$.

Proof. For $q \geq 2$, one has $\pi_q(K) = \pi_q(K^0) = 0$. The homotopy exact sequence of $K \rightarrow J \rightarrow J/K$ [47, Theorem 4.41] gives the result at degrees $d, d-1$. $\square$

Theorem 12.8. *Let $n_1 < \dots < n_r$ be odd positive integers, and let the common closed subgroup $K \leq \prod_j P_{n_j}$ have toral identity component. For the manifolds M_I in Theorem 4.3, put*

$$s_j(I) = \dim_{\mathbb{Q}}(\pi_{4n_j+1}(M_I) \otimes \mathbb{Q}), \quad s_{r+1}(I) = 0.$$

Then

$$s_j(I) = \#\{k \geq j : k \notin I\}, \quad s_j(I) - s_{j+1}(I) = \begin{cases} 1, & j \notin I, \\ 0, & j \in I. \end{cases}$$

Consequently all 2^r manifolds have distinct rational higher homotopy groups, even if some or all fundamental groups agree.

Proof. The rational primitive degrees of $U(2n+1)$ are $1, 3, \dots, 4n+1$, with multiplicity one. For H_n the multiset consists of two copies of

$$3, 7, \dots, 4n-1$$

and one further degree $2n+1$. The repeated occurrence of $2n+1$ when it is already on the list is retained. These are the A, C, D invariant-degree formulas [91, §3.1 and Appendix B, (28)–(30)]. For $n = 1$, the formula follows from the finite cover $SU(2)^3 \rightarrow H_1$, so it does not require a table convention for the low-rank root system D_2 .

For odd n , every rational primitive degree of H_n is congruent to 3 modulo 4. It therefore contributes nothing to any $\pi_{4k+1} \otimes \mathbb{Q}$, $k \geq 1$. A G_n factor contributes one precisely when $k \leq n$. Lemma 12.7 and the product formula for higher homotopy groups give the displayed tail sums, whose successive differences recover I . $\square$

Theorem 12.9. *Fix a finite group E and a faithful representation $\sigma : E \rightarrow \mathrm{U}(d)$. Let m be the least odd positive integer at least d , and set $n_j = m + 2(j - 1)$. For every $r \geq 1$, there exist 2^r mutually isospectral closed normal homogeneous manifolds of dimension*

$$D_r = 4 \sum_{j=1}^r n_j(n_j + 1) = \frac{16}{3}r^3 + (8m - 4)r^2 + \left(4m^2 - 4m - \frac{4}{3}\right)r$$

with marked fundamental group E , pairwise distinct rational higher homotopy groups, nonnegative sectional curvature and positive Ricci curvature. All common canonical associated-bundle spectra agree, including every marked unitary flat spectrum. The flat equalities also hold on all corresponding finite covers for every unitary twist there. As $r \rightarrow \infty$, the number of distinct homotopy types in this family is

$$2^r = 2^{(3/16)^{1/3}D_r^{1/3} + O_m(1)}.$$

Proof. Take $K = E \times \mathbb{T}^r$ and embed it in $\prod_j P_{n_j}$ by

$$A_j(e, t) = \sigma(e) \oplus I_{n_j-d}, \quad z_j(e, t) = t_j.$$

The map is injective: the A_j recover e by faithfulness, and the z_j recover t . Its image is compact and hence closed. Its identity component is the central torus $\mathbb{T}^r$.

The map on fundamental groups from K^0 to any ambient product is the product of the maps

$$\mathbb{Z} \longrightarrow \mathbb{Z}, \quad l \longmapsto l \quad \text{on a } G \text{ factor}, \quad \mathbb{Z} \longrightarrow \mathbb{Z}/2, \quad l \longmapsto l \bmod 2 \quad \text{on an } H \text{ factor}.$$

It is surjective. The exact sequence therefore identifies $\pi_1(M_I)$ with $\pi_0(K) = E$. Fix the same boundary convention for every I .

Theorem 4.3 gives the canonical bundle statements. A representation of E , inflated to K , is trivial on K^0 ; its canonical connection is flat. Moreover J_I/K^0 is simply connected by the same surjectivity, so these are exactly the flat bundles with the specified E -marking. For $L \leq E$, replace K by $L \times \mathbb{T}^r$, which gives the corresponding connected cover and all its flat twists. Theorem 12.8 separates all the resulting base manifolds. The normal metrics have nonnegative sectional curvature; their finite fundamental groups give positive Ricci curvature by Proposition 4.15.

Finally $\dim G_n = \dim H_n = (2n + 1)^2$ and $\dim K = r$. This gives $D_r = 4 \sum n_j(n_j + 1)$. The stated polynomial follows from the sums of $j - 1$ and $(j - 1)^2$. Inverting its leading term gives $r = (3D_r/16)^{1/3} + O_m(1)$ and the stated count. $\square$

For each fixed r , these 2^r manifolds have one spectrum in dimension D_r . Both the spectrum and the dimension may depend on r .

13. UNITARY FLAT SPECTRA AND RIEMANNIAN HOLONOMY

Ejiri [36] proved the scalar spectral comparison for warped products. Here all flat Laplacians act on sections of finite-rank unitary flat bundles.

Theorem 13.1 (Equality of all unitary flat spectra). *Let (F_0, h_0) and (F_1, h_1) be closed connected simply connected scalar-isospectral Riemannian manifolds of dimension $d \geq 1$. Let (B, g_B) be a closed connected Riemannian manifold of positive dimension b , and let $f : B \rightarrow (0, \infty)$ be smooth. Put*

$$X_i = B \times F_i, \quad g_i = g_B + f^2 h_i, \quad i = 0, 1.$$

The projections identify both fundamental groups with $\pi_1(B)$. For every finite-dimensional unitary representation $\rho : \pi_1(B) \rightarrow \mathrm{U}(V_\rho)$, the corresponding flat connection Laplacians on (X_0, g_0) and (X_1, g_1) have identical spectra, including multiplicities. These equalities hold on every corresponding connected finite cover and for every unitary flat twist on that cover.

If $\mathrm{Hess}_{g_B} f$ is nondegenerate at one point of B , then

$$\mathrm{Hol}^0(X_i, g_i) = \mathrm{SO}(b + d), \quad i = 0, 1.$$

In particular their universal Riemannian covers have no nontrivial de Rham product decomposition. If B and the fibers are oriented, their full holonomy groups are also $\mathrm{SO}(b + d)$. If the Hessian is nondegenerate on a dense subset of B , no nonempty open subset of either total manifold is a nontrivial Riemannian product.

Proof. Let $\pi_i : X_i \rightarrow B$ be the projection and let $E_\rho \rightarrow B$ be the Hermitian flat bundle of ρ . Since F_i is simply connected, every unitary flat bundle on X_i is the pullback of one on B , with the marking induced by the projection $X_i \rightarrow B$. All L^2 spaces carry the Riemannian measures. Choose a unitary scalar spectral intertwiner $S : L^2(F_0, h_0) \rightarrow L^2(F_1, h_1)$ once, by matching each fiber eigenspace. It preserves smooth functions, since it preserves all Sobolev scales defined by $1 + \Delta_{F_i}$.

The volume density of g_i is $f^d dv_B dv_{h_i}$. Thus the section Hilbert spaces are

$$L^2(B, E_\rho; f^d dv_B) \hat{\otimes} L^2(F_i, dv_{h_i}),$$

and $U_\rho = I \otimes S$ is unitary between them. The flat connection Laplacian is

$$\Delta_{X_i, \rho} = \Delta_{B, \rho} - d\nabla_{\mathrm{grad} \log f}^\rho + f^{-2} \Delta_{F_i}. \quad (13.1)$$

Integration by parts in the quadratic form

$$\int_{B \times F_i} (|\nabla_B u|^2 + f^{-2} |d_{F_i} u|^2) f^d dv_B dv_{h_i}$$

gives (13.1). The first two terms of (13.1) act on the base factor and the multiplier f^{-2} is a base function. Hence U_ρ intertwines the displayed operators on smooth sections. It also intertwines the unwarped product connection Laplacians, so

$$U_\rho H^2(X_0, \pi_0^* E_\rho) = H^2(X_1, \pi_1^* E_\rho).$$

These Sobolev spaces are independent of the choice of smooth metric on a compact manifold. They are the self-adjoint domains of the warped elliptic operators [35, Theorem 16.1 and §16.3]. Taking closures gives $U_\rho \Delta_{X_0, \rho} U_\rho^{-1} = \Delta_{X_1, \rho}$, with equality of domains.

Every connected finite cover of X_i corresponds to a finite-index subgroup of $\pi_1(B)$ and is the product of the corresponding connected cover $B' \rightarrow B$ with F_i . Its lifted metric has the same warped form, with the lifted function f . Applying the preceding argument on B' gives equal spectra for every unitary twist on these covers.

Let $p \in B$ be a point where the Hessian is nondegenerate, and let $q \in F_i$ be arbitrary. At (p, q) split the tangent space as $H \perp V$ into horizontal and vertical subspaces, with their warped inner products. Let $A : H \rightarrow H$ be the self-adjoint invertible map defined by

$$g_B(AX, Y) = f(p)^{-1} \mathrm{Hess}_{g_B} f(X, Y).$$

Set $(X \wedge U)Z = g(U, Z)X - g(X, Z)U$ and take the convention

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]}.$$

The warped curvature formulas [16, Lemma 7.4(2),(4)], with this convention, give, for $X, Y \in H$ and $U, W \in V$,

$$R(X, U)Y = g_B(AX, Y)U, \quad R(X, U)W = -g(U, W)AX.$$

Hence

$$R(X, U) = -(AX) \wedge U. \quad (13.2)$$

By the Ambrose–Singer theorem [4, §2, Theorem 2], every curvature endomorphism at (p, q) belongs to the restricted holonomy Lie algebra. Since A is onto, the holonomy Lie algebra contains every mixed skew map $X \wedge U$ by (13.2). For orthonormal bases (e_a) of H and (u_j) of V , the brackets

$$[e_a \wedge u_j, e_c \wedge u_j] = -e_a \wedge e_c, \quad [e_a \wedge u_j, e_a \wedge u_k] = -u_j \wedge u_k$$

show that all horizontal and vertical skew maps belong to the holonomy Lie algebra. Thus the holonomy Lie algebra is $\mathfrak{so}(b+d)$. Restricted Riemannian holonomy is connected and lies in $\mathrm{SO}(b+d)$, proving equality. If the total manifold is oriented, the full group is contained in the same special orthogonal group and hence also equals it. Lifting to a finite cover preserves this argument at a lift of p . If the Hessian is nondegenerate on a dense set, each nonempty open subset contains a point where the curvature endomorphisms generate the full orthogonal Lie algebra. The parallel factor subspaces of a nontrivial product would be invariant under these endomorphisms, which is impossible. This excludes a nontrivial Riemannian product on that open subset. $\square$

Lemma 13.2. *Every closed positive-dimensional Riemannian manifold admits positive smooth functions f with a nondegenerate Hessian at some point. Such f may be chosen arbitrarily close to 1 in C^∞ . If the metric and manifold are real analytic, one may choose f real analytic with its Hessian nondegenerate on a dense open subset.*

Proof. First suppose that the manifold is connected. Fix normal coordinates centered at p . Choose a smooth function with compact support ψ in the coordinate ball which equals $1 - \sum_a (x^a)^2$ near p . Its Hessian at p is $-2g_B$. For sufficiently small $\varepsilon > 0$, the function $f = 1 + \varepsilon\psi$ is positive and its Hessian at p is $-2\varepsilon g_B$. Taking $\varepsilon \downarrow 0$ proves the convergence $f \rightarrow 1$ in C^∞ .

Suppose that B and g_B are real analytic. Expand the smooth function ψ in real Laplace eigenfunctions. Its spectral partial sums converge in every Sobolev space, since ψ is smooth; Sobolev embedding therefore gives convergence in C^2 . For an eigenfunction u of eigenvalue λ , the operator $\Delta_B - \lambda$ has analytic coefficients and principal symbol $|\xi|_{g_B}^2 > 0$ for $\xi \neq 0$. Analytic elliptic regularity [21, Theorem 1.1] applied in analytic coordinate balls makes u real analytic. Every finite sum is therefore real analytic. Choose one such sum ϕ sufficiently close to ψ in C^2 that $\mathrm{Hess} \phi(p)$ remains nondegenerate. For sufficiently small $\varepsilon > 0$, $f = 1 + \varepsilon\phi$ is positive and real analytic. The determinant of the endomorphism $g_B^{-1} \mathrm{Hess} f$ is a globally defined analytic function which is nonzero at p . On connected B , its nonzero set is open and dense. With this fixed ϕ , taking $\varepsilon \downarrow 0$ also gives C^∞ convergence to 1. On a disconnected manifold, perform this construction on each of its finitely many connected components and take a common sufficiently small $\varepsilon > 0$. $\square$

Corollary 13.3. *Let A_7 and B_7 be the simply connected scalar-isospectral seven-manifolds from Theorem 12.1, with $b_2(A_7) = 1$, $b_2(B_7) = 2$, with A_7 rationally formal and B_7 having a triple Massey product which does not contain zero. For every closed connected positive-dimensional Riemannian base B , there are warped metrics on $B \times A_7$ and $B \times B_7$ with full restricted holonomy $\mathrm{SO}(\dim B + 7)$ and equal spectra for all corresponding unitary flat bundles and all corresponding finite covers. If $\mathrm{Ric}_{g_B} > 0$, the common warping function may be chosen so that both total Ricci tensors are positive. Their second Betti numbers are*

$$b_2(B \times A_7) = b_2(B) + 1, \quad b_2(B \times B_7) = b_2(B) + 2.$$

They are therefore not homotopy equivalent. If B is rationally formal, these products have opposite rational formality. If B is aspherical, their universal covers have different second Betti numbers, so the two spaces have no common connected Riemannian cover. Their scalar curvature functions are equal after projection to the common base.

In particular, $B = S^1$ gives dimension eight and fundamental group $\mathbb{Z}$. For an oriented closed surface $B = \Sigma_g$, $g \geq 2$, the examples have dimension nine, full holonomy $\mathrm{SO}(9)$, fundamental group $\pi_1(\Sigma_g)$, second Betti numbers 2 and 3, and opposite rational formality. Give this surface a hyperbolic metric and choose the warping function real analytically as in the lemma. Then neither total metric admits a nontrivial Riemannian product on any nonempty open subset.

Proof. Apply Theorem 13.1 with a function from Lemma 13.2. If $\mathrm{Ric}_{g_B} > 0$, write h_0, h_1 for the given fiber metrics. Both product metrics $g_B + h_i$ have positive Ricci curvature. Their compact unit tangent bundles give a common positive lower bound. The Ricci tensor depends continuously on the metric in the C^2 topology, so both warped metrics remain Ricci-positive when f is sufficiently close to 1 in C^2 . The functions in the lemma approach 1 in C^∞ while retaining a nondegenerate Hessian somewhere;

if B and g_B are real analytic, the analytic choices retain the dense-set Hessian condition as well. The Künneth formula gives the two Betti numbers because $H^1(A_7; \mathbb{Q}) = H^1(B_7; \mathbb{Q}) = 0$. The formality assertions follow from Theorem 12.4, since the underlying manifolds are unchanged by the warping function. If B is aspherical, its universal cover is contractible. The universal covers $\tilde{B} \times A_7$ and $\tilde{B} \times B_7$ consequently have second Betti numbers 1 and 2. A common connected cover would have a universal cover homeomorphic to both, which is impossible.

Both homogeneous fibers have constant scalar curvature. The heat coefficients a_0, a_2 give their volume and integrated scalar curvature [42, Theorem A.1(a),(c), p. 239]; their scalar curvature is therefore the same constant σ . Tracing the warped curvature formulas [16, Lemma 7.4] gives, with the positive base Laplacian,

$$\text{Scal}_{g_i} = \text{Scal}_{g_B} + f^{-2}\sigma + 14f^{-1}\Delta_B f - 42f^{-2}|df|^2.$$

Indeed the mixed trace is $-14f^{-1} \text{tr Hess } f$, and the vertical trace is $f^{-2}\sigma - 42f^{-2}|df|^2$. The displayed expression is independent of the fiber. The circle is formal, and the formality of closed oriented surfaces was established in Corollary 12.6. The circle and surfaces of genus at least two are aspherical, giving the stated specializations. $\square$

Remark 13.4. The one-point Hessian hypothesis gives full restricted holonomy on the connected manifold. It alone does not exclude a product patch elsewhere: a smooth warping function might be constant on an open set. The dense-set hypothesis gives the stronger exclusion stated above. On a circle of length L , let $s \in \mathbb{R}/L\mathbb{Z}$ be the arclength coordinate. Then

$$f(s) = 1 + \varepsilon \cos(2\pi s/L), \quad 0 < \varepsilon < 1,$$

has nondegenerate Hessian away from the two zeros of $\cos(2\pi s/L)$. For an analytic metric and an analytic function on a connected base, nondegeneracy at one point implies nondegeneracy on a dense open set, as in Lemma 13.2.

Part IV. Finite holonomy and least lengths

For a closed connected Riemannian manifold X of positive dimension, marked finite-holonomy spectra determine the deck heat traces on every finite regular cover. After division by the deck-group orders, these traces define central heat measures on the profinite completion; the finite-quotient minima have supremum equal to the minimum of ℓ_X on each nonempty profinite conjugacy fiber (Theorems 14.4 and 14.2). If the fundamental group is conjugacy separable, every nontrivial class minimum is recovered and is a singularity of some finite-holonomy flat half-wave trace (Proposition 15.5). For the C_7 -pair of Theorem 7.7, these minima agree while the full period sets differ.

14. PROFINITE HEAT MEASURES AND THE CONJUGACY ENVELOPE

Let X be a closed connected Riemannian manifold of positive dimension with a fixed marking, and put $\Pi = \pi_1(X)$. For each finite quotient $q : \Pi \twoheadrightarrow Q$, retain the functions $H_{q,g}$ and ℓ_q of Section 5. Write $K = \hat{\Pi}$ for the profinite completion and $\iota : \Pi \rightarrow K$ for its canonical map. Since Π is finitely generated, K is compact and metrizable. The map ι need not be injective.

14.1. The exact conjugacy envelope.

Definition 14.1. For $z \in K$, define

$$\hat{\ell}_X(z) = \sup_q \ell_q(\hat{q}(z)) \in [0, \infty],$$

where q ranges over finite quotients of Π and $\hat{q} : K \rightarrow Q$ is its continuous extension.

Theorem 14.2 (The profinite conjugacy envelope). *The function $\widehat{\ell}_X$ is lower semicontinuous, invariant under conjugacy and inversion, and zero precisely at 1. For every finite $R \geq 0$,*

$$\{z : \widehat{\ell}_X(z) \leq R\} = \bigcup_{[c] : \ell_X(c) \leq R} \text{Conj}_K(\iota(c)), \quad (14.1)$$

a finite union of compact conjugacy orbits. Consequently

$$\widehat{\ell}_X(z) = \min\{\ell_X(c) : z \in \text{Conj}_K(\iota(c))\}, \quad (14.2)$$

$$\sup_q \ell_q(q(c)) = \min\{\ell_X(d) : q(d) \sim_Q q(c) \text{ for every finite quotient } q\}. \quad (14.3)$$

The minimum of an empty set is ∞ . For every $c \in \Pi$, a finite quotient attains the supremum in (14.3). For every finite $R \geq 0$, a single quotient attains it for all classes of minimum length at most R .

Proof. Every continuous finite quotient of K restricts to a quotient of Π , since $\iota(\Pi)$ is dense. By [100, Lemma 2.1, arXiv version 2], two elements of K are conjugate if and only if their images are conjugate in every finite quotient. Each orbit is compact as the image of K under conjugation.

Every $\ell_q \circ \widehat{q}$ is locally constant and invariant under conjugacy and inversion. Their supremum is lower semicontinuous. A nonidentity z survives in a finite quotient, whose nonidentity deck minimum is positive. Thus $\widehat{\ell}_X$ vanishes precisely at 1. By (5.6), every orbit on the right of (14.1) lies in its left side. Conversely, the set S_R of actual classes of minimum at most R is finite by Lemma 5.3. If z lies in none of their profinite orbits, choose one quotient separating it from each class in S_R and take a common refinement. Its deck minimum at $\widehat{q}(z)$ exceeds R , a contradiction. This proves (14.1). Taking $R = \widehat{\ell}_X(z)$ when finite proves (14.2); if infinite, no actual class lies in that orbit. Profinite conjugacy gives (14.3).

For $c \in \Pi$, put $a = \widehat{\ell}_X(\iota(c)) \leq \ell_X(c)$. If $a > 0$, separate c simultaneously from the finitely many actual classes of minimum less than a . The resulting quotient has deck minimum at least a and, by definition of the supremum, at most a . If $a = 0$, every quotient already attains it. Refinement cannot decrease a deck minimum, by (5.6); a common refinement therefore works for every class in the finite set S_R . $\square$

An element $c \in \Pi$ is *conjugacy distinguished* if every element not conjugate to c is separated from it by conjugacy in some finite quotient; Π is *conjugacy separable* if every element has this property.

Corollary 14.3. *For a fixed metric, $\widehat{\ell}_X(\iota(c)) = \ell_X(c)$ exactly when c has smallest minimum length in its profinite conjugacy fiber. Equality holds for every c exactly when ℓ_X is constant on each such fiber; in particular it holds if Π is conjugacy separable.*

Proof. Apply (14.3) to one fiber or to every element of each fiber. Conjugacy separability makes each fiber a single ordinary class. $\square$

14.2. Central heat measures and their class masses.

Theorem 14.4. *For every $t > 0$ there is a unique positive finite central Borel measure $\lambda_{X,t}$ on K with*

$$(\widehat{q})_* \lambda_{X,t}(\{g\}) = \frac{H_{q,g}(t)}{|Q|}. \quad (14.4)$$

Its mass is $\Theta_X(t)$. The probability $\nu_{X,t} = \lambda_{X,t}/\Theta_X(t)$ is invariant under conjugacy and inversion, and for every finite-image unitary representation ρ ,

$$\int_K \overline{\chi_\rho(z)} d\nu_{X,t}(z) = \frac{\Theta_{X,\rho}(t)}{\Theta_X(t)}. \quad (14.5)$$

The complete marked finite-image flat spectra are equivalent to $(\lambda_{X,t})_{t>0}$, or to $(\Theta_X, (\nu_{X,t})_{t>0})$.

Proof. Suppose $q = pq'$ and put $C = \ker p$. Normalized pullback identifies $L^2(Y_q)$ with $L^2(Y_{q'})^C$. The finite-cover trace identity in the proof of Theorem 5.4, applied to $Y_{q'} \rightarrow Y_q$, gives

$$H_{q,g}(t) = \frac{1}{|C|} \sum_{p(h)=g} H_{q',h}(t). \quad (14.6)$$

The trivial-character case of (5.2) gives $|Q|^{-1} \sum_g H_{q,g}(t) = \Theta_X(t)$. Thus

$$\nu_{q,t}(\{g\}) = \frac{H_{q,g}(t)}{|Q|\Theta_X(t)}$$

defines compatible probabilities on the finite quotients. Apply the projective measure theorem [41, 418M]: the projections $\hat{q} : K \rightarrow Q$ are continuous, onto, compatible, and separate points; the compact set K has full image measure in every quotient. The resulting Radon probability, restricted to the Borel sets, is $\nu_{X,t}$. Every finite Borel measure on the compact metrizable space K is regular, so uniqueness holds among Borel measures. Set $\lambda_{X,t} = \Theta_X(t)\nu_{X,t}$. Its invariance under conjugacy and inversion follows from the corresponding invariances of $H_{q,g}$.

Equation (14.5) is (5.2). The spectrum of the trivial representation gives $\Theta_X(t)$. Finite-group character inversion determines every central pushforward of $\nu_{X,t}$ from these moments, hence determines $\lambda_{X,t} = \Theta_X(t)\nu_{X,t}$. Conversely the measures determine every twisted heat trace. Lemma 2.3 determines its spectrum: the union of two candidate spectra is locally finite in $[0, \infty)$, and convergence of their heat traces at any $t_0 > 0$ gives the required absolute summability of the multiplicity difference. $\square$

Remark 14.5. For an irreducible finite-image unitary representation ρ , Schur's lemma and (14.5) give

$$\int_K \rho(z^{-1}) d\nu_{X,t}(z) = \frac{\Theta_{X,\rho}(t)}{\dim \rho \Theta_X(t)} \text{Id}.$$

If $K = \{1\}$, then $\nu_{X,t} = \delta_1$ for every metric. Thus the family $(\nu_{X,t})_{t>0}$ alone need not determine Θ_X .

14.2.1. Holonomy and exact aggregation. The flat holonomy heat identity follows from vector-bundle Feynman–Kac [82, Section 8, equation (34)], with zero potential and source time $2t$ for the heat operator $e^{-t\Delta}$. Le Jan's integrated character identity with nonzero killing is [65, §3, proof of Proposition 5 and final display]; both his characters and his twisted zeta functions are divided by the representation dimension. His generator is $-\Delta/2 - k$, so at $k = 0$ his time $2t$ is the time t for $-\Delta$. Here $\chi_\rho = \text{Tr } \rho$, and $\Theta_{X,\rho}$ is the heat trace of the associated flat bundle of rank $\dim \rho$, without a topological triviality assumption. For graph covers and finite-quotient approximation of delocalized traces, see [64, Sections 4.1 and 4.3] and [98, Lemmas 2.4 and 2.6].

Let $C(\Pi)$ be the ordinary conjugacy classes and set

$$b_c(t) = \int_X Q_t^{x,x} \{\omega : [\omega] = c\} dx,$$

where $Q_t^{x,x}$ is the unnormalized bridge for generator $-\Delta_X$, of mass $K_X(t, x, x)$, with paths parametrized on $[0, 1]$. Two loops at uniform distance less than the injectivity radius are freely homotopic by pointwise minimizing geodesics. Thus the class sets are open and Borel in the continuous loop space. Since Π is countable, they form a countable partition, and

$$b_c(t) \geq 0, \quad \sum_c b_c(t) = \Theta_X(t) < \infty. \quad (14.7)$$

Lifting bridges gives the equivalent delocalized heat trace $b_c(t) = \int_F \sum_{g \in c} \tilde{K}(t, x, gx) dx$ on a measurable fundamental domain F ; positivity justifies the sum and integral. Let m_c be the orbital probability obtained by conjugating $\iota(c)$ by Haar measure on K , and put $c \approx d$ when their profinite orbits coincide.

Proposition 14.6. *One has the total-variation convergent positive decomposition*

$$\lambda_{X,t} = \sum_c b_c(t) m_c = \sum_{B \in C(\Pi)/\approx} B_X(B, t) m_B, \quad B_X(B, t) = \sum_{c \in B} b_c(t). \quad (14.8)$$

The measure $\lambda_{X,t}$ gives full measure to $\bigcup_{c \in \Pi} \text{Conj}_K(\iota(c))$. The complete marked finite-image flat spectra are equivalent to all $B_X(B, t)$; if c is conjugacy distinguished, they recover $b_c(t)$.

Proof. Feynman–Kac and flatness give $\Theta_{X,\rho}(t) = \sum_c b_c(t) \overline{\chi_\rho(c)}$. The sum is absolute by $|\chi_\rho| \leq \dim \rho$ and (14.7). The inverse holonomy agrees with (5.1); replacing all classes by inverses makes no difference, since time reversal gives $b_c = b_{c^{-1}}$. The sum $\sum_c b_c(t) m_c$ converges in total variation by (14.7) and has these character moments, so uniqueness proves (14.8). Distinct compact conjugacy orbits are disjoint; the measure of the orbit of c is exactly $B_X([c]_\approx, t)$. Conversely the marked orbits and these aggregates reconstruct the measure, including its scalar mass. A distinguished class has a singleton fiber. $\square$

Proposition 14.7. *For $c \in C(\Pi)$, put $A_{q,c}(t) = |Q|^{-1} \sum_{g \in \text{Conj}_Q(q(c))} H_{q,g}(t)$. Along every nested cofinal sequence of finite quotients it decreases to $B_X([c]_\approx, t)$. For every finite $R \geq 0$, there is a finite quotient q such that, for every c with $\ell_X(c) \leq R$,*

$$0 \leq A_{q,c}(t) - B_X([c]_\approx, t) \leq \sum_{\ell_X(d) > R} b_d(t) \quad (t > 0). \quad (14.9)$$

Proof. An orbital probability pushes forward uniformly to its finite conjugacy orbit. Hence $A_{q,c}(t) = \sum_{q(d) \sim q(c)} b_d(t)$. Refinement decreases these sets to $[c]_\approx$, by profinite conjugacy; continuity from above for the finite measure (14.7) proves the limit. A cofinal sequence exists because a finitely generated group has finitely many subgroups of each bounded index. To prove the tail bound, separate every pair of classes of minimum at most R lying in different profinite fibers in one common finite quotient. Finiteness of these classes leaves only terms of length greater than R in the difference. $\square$

14.3. The geodesic rate and its functoriality.

Theorem 14.8 (The geodesic rate). *As $t \downarrow 0$, the probabilities $\nu_{X,t}$ satisfy a large-deviation principle on K with speed $1/t$ and good rate $I_X(z) = \widehat{\ell}_X(z)^2/4$. Thus, for open O and closed F ,*

$$\liminf_{t \downarrow 0} t \log \nu_{X,t}(O) \geq -\inf_O I_X, \quad \limsup_{t \downarrow 0} t \log \nu_{X,t}(F) \leq -\inf_F I_X.$$

The finite locus of I_X is exactly the union of the actual profinite conjugacy orbits.

Proof. Put $\nu_{q,t} = (\widehat{q})_* \nu_{X,t}$ and $I_q(g) = \ell_q(g)^2/4$. Since $t \log \Theta_X(t) \rightarrow 0$, (14.4) and (5.5) give

$$\lim_{t \downarrow 0} t \log \nu_{q,t}(\{g\}) = -I_q(g).$$

For a nonempty subset $A \subset Q$, the largest atom in A is at most $\nu_{q,t}(A)$ and at least $\nu_{q,t}(A)/|A|$. Hence $\nu_{q,t}$ satisfies the large-deviation principle on the finite discrete space Q with good rate I_q . The Dawson–Gärtner projective-limit theorem [29, Theorem 3.3 and §3.1], applied to the compact Hausdorff inverse limit K , gives the good rate

$$\sup_q I_q(\widehat{q}(z)) = \frac{1}{4} \left(\sup_q \ell_q(\widehat{q}(z)) \right)^2 = I_X(z).$$

Its finite locus is given by Theorem 14.2. $\square$

14.3.1. *Epimorphisms.* For a one-point manifold, set $K = \{1\}$, $\Theta(t) = 1$, $\lambda_t = \nu_t = \delta_1$, and $\widehat{\ell} = I = 0$.

Theorem 14.9. *Let $p : \pi_1(M) \twoheadrightarrow \pi_1(N)$ be an epimorphism for closed connected smooth Riemannian manifolds. Suppose*

$$\text{Spec } \Delta_{M,p^*\rho} = \text{Spec } \Delta_{N,\rho} \quad (14.10)$$

for every finite-image unitary representation ρ , with multiplicities. Then $\widehat{p}$ is onto and

$$(\widehat{p})_* \nu_{M,t} = \nu_{N,t}, \quad I_N(z) = \min_{\widehat{p}(w)=z} I_M(w), \quad \widehat{\ell}_N(z) = \min_{\widehat{p}(w)=z} \widehat{\ell}_M(w). \quad (14.11)$$

If the target group is conjugacy separable, then

$$\ell_N(\gamma) = \min_{p(e)=\gamma} \ell_M(e) \quad (14.12)$$

for every $\gamma \in \pi_1(N)$, and the minimum is attained.

Proof. If either manifold is a point, equality of the scalar heat traces and the small-time heat asymptotic imply that both are points; the conclusions follow from the stated conventions. Suppose henceforth that both have positive dimension. The image of $\widehat{p}$ is compact and contains the dense image of $\pi_1(N)$, so $\widehat{p}$ is onto. The trivial representation in (14.10) gives $\Theta_M = \Theta_N$. Since $\widehat{p}$ is onto, $(\widehat{p})_* \nu_{M,t}$ is central. Equation (14.5) and finite-group character inversion therefore give $(\widehat{p})_* \nu_{M,t} = \nu_{N,t}$. For every finite quotient $q : \pi_1(N) \twoheadrightarrow Q$, the same inversion and (5.5) give

$$H_{q \circ p, g}^M(t) = H_{q, g}^N(t), \quad \ell_{q \circ p}^M(g) = \ell_q^N(g). \quad (14.13)$$

By the contraction principle [29, Theorem 3.2], the image probabilities have good rate $J(z) = \min_{\widehat{p}(w)=z} I_M(w)$. The map $\widehat{p}$ is continuous between compact Hausdorff spaces, and I_M is good by Theorem 14.8. Uniqueness of the rate function [29, §3.1] gives $J = I_N$. Every fiber is nonempty and compact; lower semicontinuity gives attainment. Taking nonnegative square roots gives the length identity in (14.11).

Suppose $\pi_1(N)$ is conjugacy separable, and let $\gamma \in \pi_1(N)$. Corollary 14.3 gives $\widehat{\ell}_N(\iota(\gamma)) = \ell_N(\gamma) < \infty$. Choose a minimizing w above $\iota(\gamma)$. By (14.2), there is $e \in \pi_1(M)$ such that w is conjugate to $\iota(e)$ and $\ell_M(e) = \widehat{\ell}_M(w) = \ell_N(\gamma)$. Thus $p(e)$ is profinitely conjugate to γ , hence conjugate to it in $\pi_1(N)$. Choose $v \in \pi_1(N)$ with $vp(e)v^{-1} = \gamma$, and choose $u \in \pi_1(M)$ with $p(u) = v$. Then $p(ueu^{-1}) = \gamma$ and $\ell_M(ueu^{-1}) = \ell_N(\gamma)$. For every e' with $p(e') = \gamma$, (14.11) gives

$$\ell_N(\gamma) \leq \widehat{\ell}_M(\iota(e')) \leq \ell_M(e').$$

This proves (14.12). □

Corollary 14.10. *In Theorem 10.4, put $M_* = M_{[r]}$. For each prescribed $p_I : E/A_I \twoheadrightarrow \Gamma$,*

$$\ell_{M_*}(\gamma) = \min_{p_I(u)=\gamma} \ell_{M_I}(u).$$

The minimum is attained, even when E/A_I is infinite. The same holds on the corresponding covers for each $L \leq \Gamma$, with source group $p^{-1}(L)/A_I$ and target L .

Proof. The realization gives the pulled-back flat spectral identities for the prescribed epimorphisms and covers. Their targets are finite, so Theorem 5.4 applies directly. □

14.4. Limits and a circle example. As $t \downarrow 0$, $\nu_{X,t} \Rightarrow \delta_1$; as $t \rightarrow \infty$, it converges weakly to Haar probability on K . Indeed, in a fixed finite quotient all nonidentity masses vanish exponentially at zero. At infinity, the constant eigenfunction on connected Y_q gives $H_{q,g}(t) \rightarrow 1$ and $\Theta_X(t) \rightarrow 1$, hence uniform probability. Density of cylinder functions gives both weak limits on K .

Proposition 14.11. *For a circle of circumference L , viewed with $\Pi = \mathbb{Z} \subset \widehat{\mathbb{Z}}$,*

$$\nu_{X,t} = \frac{\sum_{k \in \mathbb{Z}} e^{-L^2 k^2 / (4t)} \delta_k}{\sum_{k \in \mathbb{Z}} e^{-L^2 k^2 / (4t)}}. \quad (14.14)$$

Its rate is $L^2 k^2 / 4$ on integers and infinite elsewhere. Its total-variation distance from Haar probability is one for each $t > 0$, although it converges weakly to Haar. For $\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$, $\ell_m(k) = L \min_{r \in \mathbb{Z}} |k + rm|$. The least degree recovering $L|k|$ simultaneously for $|k| \leq N$, $N \in \mathbb{Z}_{\geq 1}$, is $2N$; no finite quotient recovers all these deck minima at once.

Proof. Periodization on the covering circle of length mL gives

$$H_{m,j}(t) = \frac{mL}{\sqrt{4\pi t}} \sum_{k \equiv j(m)} e^{-L^2 k^2 / (4t)}.$$

Divide by $m\Theta_X(t)$ to obtain every finite pushforward of (14.14). The envelope theorem gives its rate. The countable support $\mathbb{Z}$ is Haar-null in the infinite compact group, proving total-variation distance one in the convention $\sup_A |\mu(A) - \eta(A)|$.

The deck formula is the least distance between translates on the covering circle. It equals $L|k|$ for every $|k| \leq N$ when $m \geq 2N$. If $m < 2N$, a shortest representative of N modulo m has absolute value at most $m/2 < N$. Since every finite quotient of $\mathbb{Z}$ is cyclic, degree $2N$ is necessary and sufficient for simultaneous attainment when $|k| \leq N$. $\square$

Remark 14.12. The first positive scalar eigenvalue is $(2\pi/L)^2$, so the scalar spectrum determines L and all marked lengths $L|k|$. For $N \in \mathbb{Z}_{\geq 1}$, equality $\ell_m(k) = L|k|$ for every $|k| \leq N$ holds precisely when $m \geq 2N$.

15. CLASS MINIMA AND EQUIVARIANT WAVE SINGULARITIES

For a marked manifold X , write $\Pi = \pi_1(X)$ and $\iota : \Pi \rightarrow \widehat{\Pi}$ as above.

15.1. A criterion valid for every metric.

Theorem 15.1 (Separation of unoriented classes). *For a fixed closed connected smooth manifold X of dimension at least three, the following are equivalent:*

- (1) *for all $c, d \in \Pi$, if $\iota(c)$ and $\iota(d)$ are conjugate in $\widehat{\Pi}$, then d is conjugate in Π to c or c^{-1} ;*
- (2) *for every smooth metric g and every $c \in \Pi$, $\widehat{\ell}_{(X,g)}(\iota(c)) = \ell_{(X,g)}(c)$.*

Proof. Under (1), the profinite conjugacy fiber of c is contained in $[c]_{\Pi} \cup [c^{-1}]_{\Pi}$. Conjugation and reversal preserve length, and the fiber contains c . Hence (14.3) gives (2). If (1) fails and a nontrivial element maps to $1 \in \widehat{\Pi}$, its envelope is zero whereas its minimum is positive for every metric. Otherwise there are nontrivial c, d in one profinite fiber but in distinct unoriented classes. The prescription theorem [33, Theorem 5.1, arXiv version 2] realizes a positive assignment m on finitely many elements whenever $m(x) \leq |k|m(y)$ for every relation $x \sim y^k$. Assign $m(c) = 1$, $m(d) = 3/2$. Cross relations have $|k| \geq 2$, so these inequalities hold; diagonal relations also satisfy them. The resulting metric has $\ell(c) = 1$, $\ell(d) = 3/2$, but $\widehat{\ell}(\iota(d)) \leq 1$, contradicting (2). $\square$

Remark 15.2 (Failure of a triangle inequality). In general $\widehat{\ell}(ab) \leq \widehat{\ell}(a) + \widehat{\ell}(b)$ fails. Take $X = \mathbb{RP}^2 \times \mathbb{RP}^3$ and write $\pi_1(X) = \{1, a, b, ab\} \simeq (\mathbb{Z}/2)^2$. Distinct nonidentity elements are not powers of one another. The prescription theorem [33, Theorem 5.1] therefore gives a smooth metric with $\ell_X(a) = \ell_X(b) = 1$ and $\ell_X(ab) = 3$. Since Π is finite, (14.3) gives $\widehat{\ell}_X(a) = \widehat{\ell}_X(b) = 1$ and $\widehat{\ell}_X(ab) = 3$.

15.2. The first singularity of a deck cosine trace.

Theorem 15.3 (Nonidentity deck minima). *Let a finite group Γ act freely by isometries on closed connected Z , and put $X = \Gamma \backslash Z$. For $\gamma \neq 1$, set $\ell = \ell_Z(\gamma) = \min_x d_Z(x, \gamma x) > 0$. It is the attained minimum of the loop lengths in that deck class, and has a smoothly closed geodesic representative on X . The deck cosine trace*

$$C_\gamma(t) = \text{Tr}(R_\gamma \cos(t\sqrt{\Delta_Z}))$$

is singular at both ℓ and $-\ell$. The deck half-wave trace and some irreducible flat half-wave trace are singular at ℓ . The heat trace $H_\gamma(s) = \text{Tr}(R_\gamma e^{-s\Delta_Z})$ has exponential rate (5.5), also for $\gamma = 1$ with $\ell_Z(1) = 0$.

Proof. Lemmas 5.3 and 5.2 identify ℓ with the minimum of the corresponding free homotopy lengths and give the sharp bounds (5.4). Cosine finite propagation gives

$$C_\gamma = 0 \quad \text{on } (-\ell, \ell). \quad (15.1)$$

For $\phi \in C_c^\infty((-\ell, \ell))$, the operator $\int \phi(t) \cos(t\sqrt{\Delta_Z}) dt$ is smoothing. Finite propagation makes its kernel zero on the deck graph, since $d(x, \gamma x) \geq \ell$. Hence $\langle C_\gamma, \phi \rangle = 0$. The spectral theorem and the Fourier transform of the Gaussian give

$$H_\gamma(s) = (4\pi s)^{-1/2} \langle C_\gamma(t), e^{-t^2/(4s)} \rangle; \quad (15.2)$$

compare [68, Theorem 2.1, equation (2.5)]. Polynomial spectral growth justifies pairing the tempered spectral series with this Schwartz function.

Suppose C_γ were smooth near ℓ . It is real and even, hence smooth near $-\ell$, and (15.1) makes both endpoint germs flat. Choose an even smooth cutoff η , equal to one for $|t| \leq \ell + \epsilon$ and zero for $|t| \geq \ell + 2\epsilon$, with transition inside these smooth neighborhoods. Then ηC_γ is smooth and compactly supported, and for each integer $m \geq 0$, $|\eta C_\gamma(t)| \leq A_m(|t| - \ell)^m$ outside $[-\ell, \ell]$. Its contribution to (15.2) is bounded by

$$A'_m s^{-1/2} e^{-\ell^2/(4s)} \int_0^{2\epsilon} u^m e^{-\ell u/(2s)} du \leq A''_m s^{m+1/2} e^{-\ell^2/(4s)}.$$

Here $\ell > 0$ is essential.

For $T = (1 - \eta)C_\gamma$, choose $\beta \in C^\infty(\mathbb{R})$ with $\beta = 0$ for $|t| \leq \ell + \epsilon/2$ and $\beta = 1$ for $|t| \geq \ell + 3\epsilon/4$. Then $\beta = 1$ on a neighborhood of $\text{supp } T$, so

$$\langle T, e^{-t^2/(4s)} \rangle = \langle T, \beta e^{-t^2/(4s)} \rangle.$$

Temperedness gives fixed nonnegative integers a, b and $B > 0$ with

$$|\langle T, \phi \rangle| \leq B \sum_{j \leq b} \sup_t (1 + |t|)^a |\phi^{(j)}(t)|.$$

For $\phi = \beta e^{-t^2/(4s)}$ and $0 < s \leq 1$, differentiation gives

$$\sum_{j \leq b} \sup_t (1 + |t|)^a |\phi^{(j)}(t)| \leq C s^{-b'} \sup_{|t| \geq \ell + \epsilon/2} (1 + |t|)^{a'} e^{-t^2/(4s)} \leq C' s^{-b'} e^{-c'/(4s)}$$

for fixed a', b' and any $\ell^2 < c' < (\ell + \epsilon/2)^2$. Thus $|\langle T, e^{-t^2/(4s)} \rangle| \leq B' s^{-b'} e^{-c'/(4s)}$. Even after the factor $s^{-1/2}$, this is $O(s^N e^{-\ell^2/(4s)})$ for every N . Put $n = \dim Z$ and choose m, N with $m + 1/2 > n/2$ and $N > n/2$. The two estimates give $H_\gamma(s) = o(s^{n/2} e^{-\ell^2/(4s)})$, contrary to the lower bound $cs^{n/2} e^{-\ell^2/(4s)}$ in (5.4). Thus C_γ is singular at both endpoints. Finally $C_\gamma = (D_\gamma + \overline{D_\gamma})/2$, so smoothness of D_γ would imply smoothness of C_γ ; character inversion then forces some W_ρ to be singular. $\square$

Remark 15.4. The cosine propagator has finite propagation speed. On a finite universal cover, $\ell_Z(\gamma)$ is the minimum of the marked class; in another finite cover it is the minimum over all free classes mapping to that deck conjugacy class. The identity class has minimum zero and gives no recovery of positive contractible periods.

15.3. Class recovery and later clean periods.

Proposition 15.5. *If X is closed and connected and $\pi_1(X)$ is conjugacy separable, its marked finite-image flat spectra determine the minimum of every nontrivial free homotopy class. Each such minimum is a singularity of some irreducible finite-image flat half-wave trace.*

Proof. Let c be a nontrivial class and put $L = \ell_X(c) > 0$. Corollary 14.3 and Theorem 14.2 give

$$L = \sup_q \ell_q(q(c)),$$

with attainment in some finite quotient q . Since $\ell_q(1) = 0$, one has $q(c) \neq 1$. Theorem 15.3 gives an irreducible flat half-wave trace singular at L . Each term of the supremum is determined by (5.2) and (5.5). $\square$

Proposition 15.6. *Let $Z \rightarrow X$ be a finite regular Riemannian cover of closed connected manifolds, with deck group Γ . Let Φ_t be the geodesic flow on SX . Suppose $L > 0$ is an isolated period and $\text{Fix}(\Phi_L)$ is clean. Fix a deck conjugacy class a occurring at L . Let $\mathcal{F}_a$ be the connected components of $\text{Fix}(\Phi_L)$ whose projected closed geodesics have deck class a , and put $d_a = \max_{F \in \mathcal{F}_a} \dim F$. For $F \in \mathcal{F}_a$, let $m_F > 0$ be its Duistermaat–Guillemin mass and ζ_F its Maslov phase, in the half-wave convention $e^{it\sqrt{\Delta}}$. If*

$$\sum_{\substack{F \in \mathcal{F}_a \\ \dim F = d_a}} \zeta_F m_F \neq 0,$$

then an irreducible flat half-wave trace associated with Γ is singular at L . In particular, this holds when the phases of the components of dimension d_a are equal.

Proof. The deck conjugacy class is locally constant on $\text{Fix}(\Phi_L)$, hence constant on each of its connected components. Choose a scalar pseudodifferential operator A_a of order zero which is microlocally the identity near the components in $\mathcal{F}_a$ and zero near the other fixed components. For $\gamma \in a$, the finite-cover localization in the proof of Proposition 7.6 gives

$$\text{Tr}(A_a e^{it\sqrt{\Delta_X}}) = \frac{|a|}{|\Gamma|} D_\gamma(t) \pmod{C^\infty \text{ near } L}.$$

The localized clean trace formula [34, Theorem 4.5], in the Riemannian form [89, §1.2, equations (1.5)–(1.6), arXiv version], has leading Fourier order $(d_a - 1)/2$ and coefficient

$$\kappa_{d_a} \sum_{\substack{F \in \mathcal{F}_a \\ \dim F = d_a}} \zeta_F m_F, \quad \kappa_{d_a} \neq 0.$$

Components of smaller dimension have smaller leading order. The displayed coefficient is nonzero, so D_γ is singular at L . Equation (5.3) gives a singular irreducible twist. $\square$

15.4. Equal flat spectra and unequal marked minima. Every finite-dimensional unitary representation of Higman’s group is trivial [65, Remark 2].

Lemma 15.7. *Every finite-dimensional complex representation of a finitely generated group with no nontrivial finite quotient is trivial.*

Proof. The image of such a representation is a finitely generated linear group, hence residually finite by Malcev’s theorem [72]; see [81, p. 1]. If the image were nontrivial, a nontrivial finite quotient of it would give a nontrivial finite quotient of the original group. $\square$

Lemma 15.8. *There are a closed connected oriented smooth five-manifold X , a diffeomorphism $F : X \rightarrow X$, a smooth metric g , and $c \in \pi_1(X)$ such that $\pi_1(X)$ is infinite and finitely presented with no nontrivial finite-dimensional unitary representation, while*

$$\ell_{(X,g)}(c) = 1, \quad \ell_{(X,F^*g)}(c) = 3/2.$$

The two metrics are isometric.

Proof. Let

$$H = \langle a, b, c, d \mid a^{-1}ba = b^2, b^{-1}cb = c^2, c^{-1}dc = d^2, d^{-1}ad = a^2 \rangle.$$

Higman's group H is infinite and has no nontrivial finite quotient [51]; see [7, arXiv version 1, p. 1]. By [73, Theorem 6], there is a closed connected smooth stably parallelizable five-manifold Y with $\pi_1(Y) = H$. Choose an orientation and set $X = Y \# Y$. Van Kampen gives $\pi_1(X) = H * H$. This group is infinite and finitely presented. Every homomorphism from $H * H$ to a finite group is trivial on both factors, hence trivial. Lemma 15.7 makes every finite-dimensional unitary representation trivial.

Remove corresponding balls from the two copies of Y . Identify the ends of $S^4 \times [-1, 1]$ with the boundary spheres by $(x, -1) \mapsto x$ and $(x, 1) \mapsto r(x)$, where $r : S^4 \rightarrow S^4$ is an involutive reflection. The exchange of the punctured copies extends over the neck by $(x, t) \mapsto (r(x), -t)$. The maps agree on the attaching spheres and on their product collars, and define a diffeomorphism F inducing the factor swap up to an inner automorphism.

Choose corresponding nonidentity elements c_1, c_2 in the two factors. Under the retraction onto the first factor, c_1 remains nonidentity whereas every conjugate of every power of c_2 maps to 1. The other retraction gives the reverse assertion. Thus neither c_i is conjugate to a power of the other. The assignments $m(c_1) = 1, m(c_2) = 3/2$ satisfy the hypotheses of [33, Theorem 5.1], which gives a smooth metric g with these minima. The isometry $F : (X, F^*g) \rightarrow (X, g)$ gives

$$\ell_{(X, F^*g)}(c_1) = \ell_{(X, g)}(F_*c_1) = \ell_{(X, g)}(c_2) = 3/2.$$

Take $c = c_1$ and compare the two metrics under the identity marking. □

Theorem 15.9. *Suppose A, B are closed connected simply connected scalar-isospectral Riemannian d -manifolds with $b_2(A) \neq b_2(B)$. There are closed connected Riemannian $(d + 5)$ -manifolds M, N with the same marked fundamental group, equal flat spectra for every finite-dimensional unitary representation of that group, distinct minimum marked length functions, and different homeomorphism types. The seven-dimensional pair $A = N_{1,-1}, B = Q^{1,1,1}$ in Theorem 12.1, with second Betti numbers 1, 2, gives dimension 12.*

Proof. Take X, F, g from Lemma 15.8 and put $M = (X, g) \times A, N = (X, F^*g) \times B$, with the identity marking on $G = \pi_1(X)$. Simple connectivity of A and B identifies both fundamental groups with G . For $s > 0$, the product heat formula gives

$$\Theta_M(s) = \Theta_{(X, g)}(s)\Theta_A(s) = \Theta_{(X, F^*g)}(s)\Theta_B(s) = \Theta_N(s).$$

Every rank- r unitary representation of G is trivial, so its flat Laplacian is the direct sum of r scalar Laplacians. The heat identity and Lemma 2.3 give equality of all these flat spectra.

Projection to X does not increase loop length. A minimizing loop in X paired with a constant point of A or B has the same length. Hence, for every $e \in G$,

$$\ell_M(e) = \ell_{(X, g)}(e), \quad \ell_N(e) = \ell_{(X, F^*g)}(e).$$

For the element c of Lemma 15.8, these values are 1 and $3/2$. Since $H^1(A; \mathbb{Q}) = H^1(B; \mathbb{Q}) = 0$, the rational Künneth formula gives

$$b_2(M) = b_2(X) + b_2(A), \quad b_2(N) = b_2(X) + b_2(B).$$

These Betti numbers differ, so M and N are not homeomorphic. □

Remark 15.10. The example has trivial profinite completion: every finite heat-trace probability is δ_1 and the sole rate is zero. Its different minimum functions are compared under the identity marking on the X factor.

Part V. Inverse theorems for marked groups

Fix $P = \prod_{j=1}^r P_{n_j}$ and positive factor scales. Among compact connected Lie groups with bi-invariant metrics, the graded right- P scalar data of $\prod_j G_{n_j}$, at these scales, have exactly the 2^r binary products as marked realizations (Theorem 19.6). The equivariant heat traces recover the pullback metric and the adjoint module with its differential marking (Theorem 16.2). The relative bracket classification gives the binary Lie algebra alternatives; the periods of the marking and the volume determine their global forms.

16. METRIC AND ADJOINT RECOVERY FROM EQUIVARIANT HEAT DATA

Let P be a compact connected Lie group, let (J, Q) be a compact connected Lie group with a positive bi-invariant metric, and let $\iota : P \rightarrow J$ be a homomorphism. Its graded right- P scalar data are the scalar eigenspaces $E_\lambda(J)$, with their eigenvalues and actions $R_p f(g) = f(g\iota(p))$. For $X \in \mathfrak{p}$ near zero, put

$$H_J(t, X) = \frac{\text{Tr}(R_{\exp_P X} e^{-t\Delta_J})}{\text{Tr}(e^{-t\Delta_J})},$$

and define

$$I_J(X) = \lim_{t \downarrow 0} -4t \log H_J(t, X), \quad A_J(X) = \lim_{t \downarrow 0} e^{I_J(X)/(4t)} H_J(t, X), \quad \Phi_J = \log A_J. \quad (16.1)$$

The scalar heat kernel is positive, so these logarithms are real.

Lemma 16.1. *The limits in (16.1) exist near zero, and*

$$I_J(X) = Q(d\iota X, d\iota X), \quad A_J(X) = j_J(d\iota X)^{-1/2}, \quad (16.2)$$

where

$$j_J(Y) = \det \left(\int_0^1 e^{-u \text{ad } Y} du \right). \quad (16.3)$$

The positive square root is taken near zero. Thus I_J is quadratic and Φ_J is an even analytic germ with zero constant term.

Proof. Let $k_t(g) = K_t(e, g)$ be the heat kernel for Riemannian measure, and put $v = \text{Vol}(J)$. The integral kernel of $R_p e^{-t\Delta_J}$ on the diagonal is $K_t(g\iota(p), g)$. Left invariance and symmetry give

$$\text{Tr}(R_p e^{-t\Delta_J}) = \int_J K_t(g\iota(p), g) dv_g = v k_t(\iota(p)). \quad (16.4)$$

In a sufficiently small normal neighborhood, the Lie exponential is the Riemannian exponential. The relative heat asymptotic is

$$k_t(\exp Y) = (4\pi t)^{-d/2} e^{-|Y|^2/(4t)} (j_J(Y)^{-1/2} + O(t)), \quad (16.5)$$

uniformly on compact subsets of that neighborhood, where $d = \dim J$. This relative estimate follows from [68, Theorem 1.1]. The leading coefficient is the inverse square root of the exponential Jacobian [67, equations (1.13)–(1.14)]. The scalar bundle has trivial parallel transport, and the diagonal leading coefficient is one. Consequently, for $Y = d\iota X$,

$$H_J(t, X) = e^{-|Y|^2/(4t)} \frac{j_J(Y)^{-1/2} + O(t)}{1 + O(t)}.$$

Since $j_J(Y)^{-1/2} > 0$, this gives

$$-4t \log H_J(t, X) = |Y|^2 + O(t), \quad e^{|Y|^2/(4t)} H_J(t, X) = j_J(Y)^{-1/2} + O(t),$$

and hence (16.2).

The left-translated differential of the exponential is $\int_0^1 e^{-u \operatorname{ad} Y} du$ [90, equations (3.C.15)–(3.C.16)]. Left translation preserves volume, giving (16.3). Since $\operatorname{ad} Y$ is skew-adjoint, its nonzero eigenvalues occur in pairs $\pm i\omega_j$. Each pair contributes $(\sin(\omega_j/2)/(\omega_j/2))^2$ to the determinant, and each zero eigenvalue contributes one. The determinant is analytic, even and equal to one at zero. Thus $\Phi_J(X) = -\frac{1}{2} \log j_J(d\iota X)$ is even and analytic, with $\Phi_J(0) = 0$. $\square$

For $X \in \mathfrak{p}$, let $\omega_1(X), \dots, \omega_r(X) > 0$ be the rotation frequencies of $\operatorname{ad}(d\iota X)$, with multiplicities. Then

$$j_J(s d\iota X) = \prod_{j=1}^r \left(\frac{\sin(s\omega_j(X)/2)}{s\omega_j(X)/2} \right)^2. \quad (16.6)$$

The logarithmic sine expansion [79, equation (25.8.8)] is

$$-\log \frac{\sin z}{z} = \sum_{k \geq 1} \frac{\zeta(2k)}{k\pi^{2k}} z^{2k} \quad (|z| < \pi).$$

Thus

$$\Phi_J(sX) = \sum_{k \geq 1} \frac{\zeta(2k)}{k(2\pi)^{2k}} s^{2k} \sum_{j=1}^r \omega_j(X)^{2k} \quad (16.7)$$

for sufficiently small s . An empty rotation list gives $\Phi_J(sX) = 0$.

Theorem 16.2. *Suppose (J_1, Q_1, ι_1) and (J_2, Q_2, ι_2) have equal graded right- P scalar data. Then*

$$\ker \iota_1 = \ker \iota_2, \quad \iota_1^* Q_1 = \iota_2^* Q_2,$$

and there is a real orthogonal P -intertwiner

$$F : \mathfrak{j}_1 \longrightarrow \mathfrak{j}_2, \quad F \circ d\iota_1 = d\iota_2.$$

Proof. An element p belongs to $\ker \iota_i$ exactly when R_p is the identity on every scalar eigenspace. If the latter holds, density of the eigenspaces gives $R_p = 1$ on $L^2(J_i)$. If $\iota_i(p) \neq e$, a continuous function separating e and $\iota_i(p)$ has a nonzero translation difference on an open set of positive volume, a contradiction. This proves equality of the kernels.

The character series converge absolutely, bounded termwise by the ordinary heat trace. Equality of the graded data therefore gives equality of all equivariant heat traces. At the identity, the leading heat coefficient gives equal dimensions d and volumes v . Equation (16.4) yields

$$k_{1,t}(\iota_1(p)) = k_{2,t}(\iota_2(p)). \quad (16.8)$$

For $d = 0$ both connected groups are points. Otherwise Lemma 16.1 gives equality of the pullback quadratic forms near zero, hence everywhere by homogeneity. Polarization gives $\iota_1^* Q_1 = \iota_2^* Q_2$. The leading coefficients in (16.5) then give

$$j_1(s d\iota_1 X) = j_2(s d\iota_2 X) \quad (16.9)$$

for every $X \in \mathfrak{p}$ and all sufficiently small real s .

Put $m = \lfloor d/2 \rfloor$. For each i , pad the list of squared positive rotation frequencies of $\operatorname{ad}(d\iota_i X)$ with zeros to obtain m numbers. By (16.7), the two lists have equal power sums of orders $1, \dots, m$. Newton's identities, in the proof of Theorem 3.4, give equal elementary symmetric functions, hence equal monic polynomials with these roots. The multisets coincide. Removing the padding and taking positive square roots gives the same frequencies with multiplicities; the common dimension gives the same zero-eigenvalue multiplicity. If $m = 0$, both frequency lists are empty.

The two adjoint characters therefore agree at $\exp_p X$ for every X . Every element of a compact connected group is an exponential [37, Theorem 43.11], so the complexified adjoint P modules are equivalent by character orthogonality [37, Corollary 35.12]. The kernel of either differential is the radical of the common pullback form. Hence $d\iota_1 X \mapsto d\iota_2 X$ is a well-defined isometry on the

differential images; it is equivariant since the ι_i are homomorphisms. Subtracting the character of this submodule gives equal complex characters on the orthogonal complements. The real-form and polar-decomposition argument in the proof of Proposition 4.5 gives an orthogonal real P -intertwiner on these complements. Its direct sum with the prescribed image isometry is F . $\square$

Corollary 16.3. *Under the equivalent conditions of Theorem 4.4, orthogonally equivalent tangent K modules give equal cubic-square spectra for every common equivariant Hermitian Clifford module. A common K -invariant grading for which Clifford multiplication is odd gives equal signed spectra. If K is connected, equality of graded right- K scalar data already implies equality of the induced metric and tangent module, hence of all common canonical tensor and cubic-square spectra and of the graded signed cubic spectra. In the product construction these spectral equalities hold for every closed $K \leq P$, including disconnected K .*

Proof. For an orthonormal invariant frame, Kostant's identity [59, Proposition 1.84], with invariant form $-Q$, and the bi-invariant curvature formula give

$$\|\rho_j\|^2 = \frac{1}{24} \sum_{a,b,c} \langle [e_a, e_b], e_c \rangle^2 = \frac{\text{Scal}_J}{6}.$$

The scalar heat coefficients determine $\text{Vol}(J)$ and $\text{Vol}(J) \text{Scal}_J / 6$ [42, Theorem A.1(a),(c)]; scalar curvature is constant. The ambient root norms therefore agree. The common induced metric on $\mathfrak{k}$ gives equal isotropy root norms. Thus the constant and the common fiber endomorphism in the cubic-square identity of Theorem 4.9 agree. The equivariant scalar unitary in the proof of Theorem 4.4, tensored with the fiber identity, intertwines the cubic squares. The grading argument in the proof of Theorem 4.9 gives equal signed spectra.

For connected K , Theorem 16.2 supplies the metric and tangent identification. Theorem 4.4 then gives all common canonical tensor spectra. In the product construction, Proposition 4.5 supplies the identification on every component of any closed $K \leq P$; Theorem 4.3 gives the tensor spectra. $\square$

17. FINITE MOMENTS, COMPACT RELATIVE EMBEDDINGS, AND GENERATED IDEALS

Fix $n \geq 1$. The central parameters of $\mathfrak{p}_n$ are denoted by a, b : an element is $(X + iaI_n, ib)$ with $X \in \mathfrak{su}(n)$. The reference central adjoint weights are

$$(\pm 2, 0) \text{ with multiplicity } n^2 \text{ each,} \quad (1, 1), (1, -1), (-1, 1), (-1, -1) \text{ with multiplicity } n \text{ each.} \quad (17.1)$$

The zero-weight multiplicity is $2n^2 + 1$.

For an orthogonal representation $\rho : T^2 \rightarrow \text{O}(W)$, put

$$\Phi_\rho(X) = -\frac{1}{2} \log \det \left(\int_0^1 e^{-u d\rho(X)} du \right)$$

near zero, where the determinant is positive.

Lemma 17.1. *Let $\dim W = (2n + 1)^2$. If the sixth jet of Φ_ρ agrees with the reference jet, then the T^2 module W has the weights (17.1), including the zero weights. Nine coefficients suffice: those determining*

$$M_{02}, M_{04}, M_{06}, M_{20}, M_{40}, M_{60}, M_{22}, M_{42}, M_{11}, \quad M_{uv} = \sum_{(\alpha, \beta)} \alpha^u \beta^v. \quad (17.2)$$

The sum runs over all complex weights with multiplicity.

Proof. Write $\Phi = \Phi_\rho$. The logarithmic sine expansion in (16.7) gives

$$\Phi(a, b) = \frac{1}{2} \sum_{k \geq 1} \frac{\zeta(2k)}{k(2\pi)^{2k}} \sum_{(\alpha, \beta)} (\alpha a + \beta b)^{2k}.$$

The factor $1/2$ counts the two conjugate complex weights of each real rotation plane. Thus the coefficient of $a^u b^v$, $u + v = 2k$, is the nonzero constant $\frac{1}{2} \frac{\zeta(2k)}{k(2\pi)^{2k}} \binom{2k}{u}$ times M_{uv} . The reference moments give

$$\sum \beta^2(\beta^2 - 1)^2 = M_{06} - 2M_{04} + M_{02} = 0.$$

Each summand is nonnegative, so $\beta \in \{0, 1, -1\}$. Next

$$\sum \beta^2(\alpha^2 - 1)^2 = M_{42} - 2M_{22} + M_{02} = 0$$

forces $\alpha = \pm 1$ when $\beta \neq 0$. If the multiplicities of $(1, 1), (1, -1)$ are u, v , conjugation gives the same multiplicities at their negatives. The equations $2(u + v) = M_{02} = 4n$ and $2(u - v) = M_{11} = 0$ give $u = v = n$.

Remove these $4n$ weights. The remaining weights have $\beta = 0$, and their pure second, fourth and sixth moments are $8n^2, 32n^2, 128n^2$. Therefore

$$\sum_{\beta=0} \alpha^2(\alpha^2 - 4)^2 = 128n^2 - 8(32n^2) + 16(8n^2) = 0.$$

Thus $\alpha \in \{0, 2, -2\}$ there. The second moment and conjugation give multiplicity n^2 at each nonzero weight. Subtraction from the ambient dimension gives $2n^2 + 1$ zero weights. $\square$

Lemma 17.2. *For $n \geq 2$, an n -dimensional unitary $U(n)$ module on which $e^{ia}I_n$ acts by e^{ia} is the defining module V . With central weight -1 it is V^* .*

Proof. If the $SU(n)$ action is trivial, the representation is a sum of characters $\det^k$, whose central weights are multiples of n . Hence it has a nontrivial $SU(n)$ constituent. Lemma B.1 makes the module irreducible, with $SU(n)$ restriction V or V^* . By [90, Proposition 4.5.2], it is $V \otimes \det^k$ or $V^* \otimes \det^k$ for some $k \in \mathbb{Z}$. Central weight 1 gives $nk = 0$ in the first case and $nk = 2$ in the second. The latter is possible only for $n = 2$, when $V^* \otimes \det \simeq V$. Duality gives V^* at central weight -1 . $\square$

17.1. The semisimple ideal generated by the nontrivial module. Let V_n denote the sum of the nontrivial real P_n -isotypic components of (4.4). Its orthogonal fixed complement has dimension three.

Lemma 17.3. *Let $\mathfrak{s}$ be compact semisimple and let $\phi : \mathfrak{p}_n \rightarrow \mathfrak{s}$ be a homomorphism whose adjoint action integrates to P_n . Suppose its nontrivial P_n module is V_n and generates $\mathfrak{s}$ as an ideal. Then its marked Lie algebra is exactly one of*

$$\mathfrak{su}(2n + 1), \quad \mathfrak{sp}(n) \oplus \mathfrak{so}(2n + 2).$$

In the first case the marking is the traceless projection of d_{I_n} ; in the second it is d_{J_n} . Their fixed subalgebras have dimensions two and three, respectively.

Proof. The infinitesimal action on V_n is faithful: its central weights include $2a$ and $a \pm b$, and its $\mathfrak{su}(n)$ action is faithful for $n \geq 2$. Hence ϕ is injective. A simple ideal with trivial P_n action has zero projection of V_n , contrary to generation.

Assume $n \geq 2$ and put $\mathfrak{h} = \mathfrak{su}(n)$. The positive- b space is $(V \oplus V^*)z$, with two inequivalent summands distinguished by their a weights. A simple ideal in which b has nonzero projection has the three-grading of (B.1), and its positive space is an irreducible zero-grade module. Its contribution is a P_n submodule, so there are at most two such ideals.

If there are two, each has positive tangent V or V^* . Lemma B.2 makes both projective $\mathfrak{su}(n + 1)$ ideals. They contain the two $\mathfrak{h}_{\mathbb{C}}$ summands of (4.4) with (a, b) -weight $(0, 0)$, and have no (a, b) weight $(2, 0)$. Every remaining simple ideal has zero $\mathfrak{h}$ projection: a nonzero projection would contribute a third adjoint summand at (a, b) -weight $(0, 0)$. Thus $\mathfrak{h}$ would act trivially on the remaining module, which contains $\text{Sym}^2 V$. This is impossible. Thus exactly one simple ideal has nonzero b projection. By Lemma B.2, it is the projective $\mathfrak{su}(2n + 1)$ or the quadric $\mathfrak{so}(2n + 2)$, with its marked projection fixed by the tangent action.

The projective alternative contains all of V_n , so generation makes it all of $\mathfrak{s}$. In the quadric alternative, the remaining nontrivial module is

$$\mathfrak{h}_{\mathbb{C}} \oplus \text{Sym}^2 V \oplus \text{Sym}^2 V^*.$$

Its positive- a part is irreducible and belongs to one simple ideal. Divide the projection of a by two to normalize its grading. The $\mathfrak{h}$ action on $\text{Sym}^2 V$ is nontrivial, so this ideal contains the unique remaining adjoint copy of $\mathfrak{h}$. Any other remaining ideal has both zero a weights and zero $\mathfrak{h}$ projection, hence is P_n -trivial and excluded.

The zero-grade algebra of this last ideal has, under $\mathfrak{h}$, one adjoint copy and a trivial complement. The adjoint copy is its embedded $\mathfrak{h}$, and the complement centralizes it. By Schur's lemma and compactness, the complement acts by purely imaginary scalars on $\text{Sym}^2 V$. Faithfulness of the zero-grade tangent action bounds its real dimension by one; the nonzero grading element belongs to this complement. Thus the zero-grade algebra is $\mathfrak{u}(n)$. In the projective row of (B.1), its simple derived algebra forces one block to have dimension one and the other dimension n ; the tangent dimension is then n , not $n(n+1)/2$. In the alternating row, the derived algebra forces $m = n$, giving tangent dimension $n(n-1)/2$. In the vector row, $\mathfrak{so}(d) \simeq \mathfrak{su}(n)$ gives $(d, n) = (3, 2)$ or $(6, 4)$ by the root-system classification. The first is $\mathfrak{so}(5) \simeq \mathfrak{sp}(2)$; the second has tangent dimension $6 \neq 10$. The exceptional zero-grade derived algebras have dimensions 45 and 78, neither of the form $n^2 - 1$. The symplectic row is therefore the only remaining pair. The zero-grade embedding is fixed by the positive tangent $\text{Sym}^2 V$ and its central weight two. By [37, Proposition 39.3], an outer automorphism of $\mathfrak{su}(n)$ would replace it by its dual; for $n = 2$ all automorphisms are inner. This determines the marked symplectic projection.

For $n = 1$, put $\mathfrak{w} = \mathfrak{c}_{\mathfrak{s}}(\phi(\mathfrak{p}_1))$. Its orthogonal complement consists of three distinct real rotation planes of weights $2a, a+b, a-b$. The action of $\mathfrak{w}$ on them factors through $\mathfrak{so}(2)^3$. Consequently $[\mathfrak{w}, \mathfrak{w}]$ annihilates the planes. Its centralizer in $\mathfrak{s}$ contains their sum V_1 and is stable under $\mathfrak{w}$: this follows from the Jacobi identity and the fact that $[\mathfrak{w}, \mathfrak{w}]$ is an ideal of $\mathfrak{w}$. Since $\mathfrak{s} = \mathfrak{w} \oplus V_1$, this centralizer is an ideal containing V_1 , hence equals $\mathfrak{s}$. Semisimplicity gives $[\mathfrak{w}, \mathfrak{w}] = 0$. Since $\phi(\mathfrak{p}_1) \subset \mathfrak{w}$, any element centralizing $\mathfrak{w}$ lies in $\mathfrak{c}_{\mathfrak{s}}(\phi(\mathfrak{p}_1)) = \mathfrak{w}$. Thus $\mathfrak{w}$ is maximal abelian and is a Cartan algebra. Its rank k satisfies $k \geq 2$ by injectivity and $k + 6 = \dim \mathfrak{s} \geq 3k$, so $k \leq 3$. The root-system classification gives $\mathfrak{su}(3)$ for $(k, \dim \mathfrak{s}) = (2, 8)$ and $\mathfrak{su}(2)^3$ for $(3, 9)$ [37, Theorems 19.10 and 22.1]. In type A_2 , $a+b, a-b$ are simple roots with sum $2a$; in type A_1^3 the three displayed weights are the restrictions of the three positive roots. Root-labelled compact isomorphisms therefore align the markings. The explicit modules in both ranks give the stated fixed dimensions. $\square$

Theorem 17.4. *On the reference Euclidean P_n module $\mathfrak{g}_n$, with its distinguished embedded $\mathfrak{p}_n$, there are exactly two classes of Lie brackets, up to orthogonal Lie isomorphisms fixing $\mathfrak{p}_n$ pointwise, for which the metric is invariant, the distinguished subalgebra has its prescribed bracket, and its adjoint action is the prescribed P_n action. They are*

$$(\mathfrak{u}(2n+1), Q_G, di_n), \quad (\mathfrak{sp}(n) \oplus \mathfrak{so}(2n+2), Q_C \oplus Q_D, dj_n).$$

Proof. The positive invariant form makes the algebra the orthogonal sum of its center and compact simple ideals [104, Exercise 3.30(c), Proposition 3.25(b), and Theorem 3.21(a)]. The ideal generated by its nontrivial P_n module lies in its semisimple part and satisfies Lemma 17.3. In the unitary alternative it has codimension one, so the remaining ideal is central; in the symplectic-orthogonal alternative it already has the full dimension.

For $n \geq 2$ in the unitary case, write the simple metric as sQ_G . Its restriction to $\mathfrak{su}(n)$ is $2sQ_{\mathfrak{U}(n)}$, so $s = 1$. For $Y \in \mathfrak{u}(n)$, the traceless projection of $di_n(Y, ib)$ has squared norm

$$2Q_{\mathfrak{U}(n)}(Y, Y) + \frac{2n}{2n+1}b^2.$$

The central pullback is therefore $b^2/(2n+1)$. Reflection of the central line aligns its possible sign with the model. For $n = 1$, the simple pullback is $2sa^2 + (2s/3)b^2$. The residual form $2(1-s)a^2 + (1-2s/3)b^2$

is nonnegative of rank at most one. Its determinant vanishes only at $s = 1$ or $s = 3/2$, and the latter has a negative a^2 coefficient. Thus $s = 1$ again.

For $n \geq 2$ in the other case, write the metric as $sQ_C \oplus tQ_D$. Its restriction is $(s + t)Q_{U(n)} + tb^2$, forcing $s = t = 1$. For $n = 1$, the three rank-one ideals have positive coefficients q_1, q_2, q_3 with pullback $q_1(2a)^2 + q_2(a + b)^2 + q_3(a - b)^2 = 2a^2 + b^2$. Coefficient comparison gives $q_1 = 1/4$, $q_2 = q_3 = 1/2$. These are precisely $Q_C \oplus Q_D$. Existence on the fixed Euclidean module follows from Proposition 4.5. The center dimensions distinguish the two types. Fixing $\mathfrak{p}_n$ intertwines its adjoint action, and connectedness gives full P_n equivariance. $\square$

17.2. Relative embeddings from central moments.

Proposition 17.5. *Let J be a Lie group of dimension $(2n + 1)^2$ with a positive bi-invariant metric and a homomorphism $P_n \rightarrow J$ inducing Q_P . Suppose the nine central moments (17.2) are the reference moments. Its marked metric Lie algebra is a reference one, except that for $n = 4$ the symplectic projection can be replaced by*

$$(X + iaI_4, ib) \mapsto -X^T + iaI_4 \quad (X \in \mathfrak{su}(4))$$

in the zero-grade $\mathfrak{u}(4) \subset \mathfrak{sp}(4)$. The orthogonal projection remains dj_4 .

Proof. Lemma 17.1 recovers the full central torus module. For $n = 1$ this is the whole parameter module, and Theorem 17.4 applies after real polar identification fixing its differential image. Assume $n \geq 2$. The positive- b spaces have a weights $1, -1$ and dimension n each. Lemma 17.2 identifies them with V, V^* . If two simple ideals have nonzero b projection, each is projective by Lemma B.2. Their joint (a, b) -zero dimensions are n^2, n^2 . Their orthogonal complement therefore has zero- a dimension one and positive- a dimension n^2 , with weight two. After dividing a by two, every simple ideal with nonzero a projection contributes a central grading direction to its zero grade. Hence there is exactly one, with zero grade of dimension one. Table (B.1) makes it $\mathfrak{su}(2)$ with positive dimension one, contradicting $n^2 > 1$.

There is one simple ideal with nonzero b projection, projective or quadric by Lemma B.2; the same lemma fixes its marked projection. In the projective case it has codimension one, and the metric calculation of Theorem 17.4 gives the reference $\mathfrak{u}(2n + 1)$ metric and marking.

In the quadric case its orthogonal complementary ideal $\mathfrak{l}$ has

$$\dim \mathfrak{l} = 2n^2 + n, \quad \dim \mathfrak{l}_0 = n^2, \quad \dim_{\mathbb{C}} \mathfrak{l}_+ = n(n + 1)/2, \quad (17.3)$$

where the subscripts are the zero and positive a grades, the latter of weight two. The b action there is zero. The projection of $\mathfrak{h} = \mathfrak{su}(n)$ into $\mathfrak{l}$ is nonzero. Otherwise the metric tQ_D on the $\mathfrak{so}(2n + 2)$ factor would restrict to $2Q_{U(n)}$ on $\mathfrak{h}$, forcing $t = 2$; its projection of b would then have squared norm two, exceeding the prescribed total norm one.

The injective $\mathfrak{h}$ projection and the nonzero grading element A are independent in $\mathfrak{l}_0$, and their dimensions sum to n^2 . Thus $\mathfrak{l}_0 = \mathfrak{h} \oplus \mathbb{R}A$. Its center has dimension one, so exactly one simple ideal has nonzero a projection. No extra central direction remains. A zero-graded simple ideal would equal the whole $\mathfrak{h}$ summand; the simple ideal with nonzero a projection would then have zero grade only $\mathbb{R}A$, hence positive dimension one, contrary to (17.3). Therefore $\mathfrak{l}$ is simple with zero grade $\mathfrak{u}(n)$ and positive dimension $n(n + 1)/2$. The row comparison in Lemma 17.3 gives $\mathfrak{l} = \mathfrak{sp}(n)$.

Its zero-grade $\mathfrak{h}$ embedding is either inner or the composition of complex conjugation with an inner automorphism [37, Proposition 39.3]. If $X \mapsto CXC^{-1}$ preserves $\mathfrak{su}(n)$, then C^*C commutes with that algebra and is scalar; rescaling C makes it unitary. Thus the inner conjugations may be taken in the compact group. The central weight two fixes the a projection. Its b projection commutes with every a -graded summand of $\mathfrak{l}$, since all their b weights are zero. It is therefore central in the simple algebra $\mathfrak{l}$ and vanishes. Thus the prescribed b norm makes the orthogonal metric scale one, and the $\mathfrak{h}$ norm makes the symplectic metric scale one as well. In the inner case the marking is dj_n . In the outer case the positive- a module restricts to $\text{Sym}^2 V^*$ on $\text{SU}(n)$, while its central weight is two. Put $\xi = e^{2\pi i/n}$. The element ξI_n acts by ξ^{-2} through $\text{SU}(n)$ and by ξ^2 through the central circle of

$U(n)$. Hence $\xi^4 = 1$, so $n \mid 4$. For $n \geq 3$ this leaves $n = 4$; for $n = 2$ the automorphism is inner. The resulting differential at $n = 4$ is the one displayed. $\square$

18. GLOBAL PERIODS AND EXCEPTIONAL FORMS

Lemma 18.1. *Let P be a connected Lie group, and let J_0, J be compact connected Lie groups with positive bi-invariant metrics. Let $\iota_0 : P \rightarrow J_0$ and $\iota : P \rightarrow J$ be homomorphisms. Suppose $(\iota_0)_* : \pi_1(P) \rightarrow \pi_1(J_0)$ is surjective and $\psi : \mathfrak{j}_0 \rightarrow \mathfrak{j}$ is an isometric Lie algebra isomorphism satisfying $\psi \circ d\iota_0 = d\iota$. Then ψ integrates to a surjective local isometry $q : J_0 \rightarrow J$ with $q \circ \iota_0 = \iota$ and finite central kernel.*

Proof. Identify the simply connected covers of J_0 and J by [37, Theorem 8.2 and Corollary 8.4]. Let Ω_0, Ω be their central deck subgroups, and let $\Lambda = \ker(\tilde{P} \rightarrow P)$ [37, Proposition 2.11]. The two lifted markings agree by uniqueness in Theorem 8.2 of that reference. Denote the common lift by $\tilde{\iota}$. Surjectivity on fundamental groups gives

$$\Omega_0 = \tilde{\iota}(\Lambda) \subseteq \Omega.$$

The identity of the common cover descends to q , with $\ker q = \Omega/\Omega_0 \subset Z(J_0)$ and $q \circ \iota_0 = \iota$. This kernel is closed and discrete in compact J_0 , hence finite. Since $dq = \psi$, the map q is a local isometry. $\square$

Let $C_m \leq Z(U(2n+1))$ be the scalar subgroup of order m , and put

$$G_{n,m} = U(2n+1)/C_m, \quad H_n^S = (\mathrm{Sp}(n)/\{\pm 1\}) \times \mathrm{SO}(2n+2),$$

$$H_n^O = \mathrm{Sp}(n) \times (\mathrm{SO}(2n+2)/\{\pm I\}).$$

All quotient metrics and markings are induced from the reference ones. Write $v_n = \mathrm{Vol}(G_n) = \mathrm{Vol}(H_n)$; the equality follows from Theorem 3.2 and the leading heat coefficient.

Theorem 18.2. *The compact connected Lie groups J with a positive bi-invariant metric Q , a faithful marking $\iota : P_n \rightarrow J$, and an orthogonal P_n -intertwiner $F : \mathfrak{j} \rightarrow \mathfrak{g}_n$ satisfying $F \circ d\iota = d\iota_n$ are exactly*

$$G_{n,m} \quad (m \geq 1 \text{ odd}), \quad H_n, \quad H_n^S, \quad H_n^O,$$

up to an isometric Lie group isomorphism intertwining the marking.

Proof. Theorem 17.4 gives a reference marked metric Lie algebra. By (2.7) and Lemma 18.1, $J = J_0/\Gamma$ for a finite central subgroup Γ . The composed marking is faithful exactly when $\Gamma \cap \iota_0(P_n) = \{1\}$. The central intersections are (B.2). Every finite subgroup of the scalar circle is C_m ; it avoids $-I$ exactly when m is odd. In $Z(H_n) = \{\pm 1\} \times \{\pm I\}$, the subgroups avoiding the diagonal nontrivial element are the trivial subgroup and the two coordinate subgroups. They give the three listed semisimple forms. Conversely these intersections are trivial and their local metric adjoint modules are unchanged. The unitary and semisimple forms have different Lie algebra center dimensions. Within each type, the volume and the multiplicities computed below distinguish the listed forms. $\square$

Theorem 18.3. *For a fixed n , the unique pair of distinct marked groups in Theorem 18.2 with equal scalar spectra is G_n, H_n . Different ranks yield different dimensions.*

Proof. The quotient volumes are

$$\mathrm{Vol}(G_{n,m}) = v_n/m, \quad \mathrm{Vol}(H_n^S) = \mathrm{Vol}(H_n^O) = v_n/2.$$

The scalar spectrum determines volume. Different m cannot give equal spectra, and a cross-type equality would require $m = 1$ or 2 . Only the odd value 1 is permitted. By Proposition 3.8, the eigenspace of H_n at $N = 2n+1$ is the direct sum of the defining coefficient spaces of $\mathrm{Sp}(n)$ and $\mathrm{SO}(2n+2)$. The center -1 of $\mathrm{Sp}(n)$ and the center $-I$ of $\mathrm{SO}(2n+2)$ act by minus the identity on their respective defining representations. Thus

$$\mathrm{mult}_{H_n^S}(N) = (2n+2)^2, \quad \mathrm{mult}_{H_n^O}(N) = (2n)^2.$$

These differ. The pair G_n, H_n has equal spectra by Theorem 3.2 and different Lie algebra center dimensions. Finally their common dimension $(2n+1)^2$ determines n . $\square$

If faithfulness is not imposed, every finite central quotient of G_n or H_n , with its induced metric and marking, occurs.

18.1. The exceptional relative form. At $n = 4$, define

$$J_* = (\mathrm{Sp}(4)/\{\pm 1\}) \times \mathrm{SO}(10),$$

with the quotient reference metric, and

$$\iota_*(A, z) = \left(\left[c((\det A)^{1/2} \bar{A}) \right], r(A \oplus z) \right). \quad (18.1)$$

A different square root changes the first matrix by its central minus identity. Products of square roots differ from the square root of a product by a sign, so the map is well defined and multiplicative. Local smooth choices of the square root agree in the quotient on overlaps, so ι_* is smooth. For $X \in \mathfrak{su}(4)$,

$$d\iota_*(X + iaI_4, ib) = (dc(-X^T + iaI_4), dr((X + iaI_4) \oplus ib)).$$

Theorem 18.4. *Let J be compact and connected with a positive bi-invariant metric and a homomorphism $\iota : P_n \rightarrow J$. Suppose $\dim J = (2n + 1)^2$, $\iota^*Q = Q_P$, and its nine central moments (17.2) are the reference ones. Then (J, ι) is a finite central quotient, with induced metric and marking, of G_n or H_n , or, when $n = 4$, of (J_*, ι_*) . Every listed quotient has the prescribed local data. The exceptional forms have volume at most $v_4/2$, attained by J_* .*

Proof. Proposition 17.5 gives the local alternatives. For the two reference differentials, (2.7) and Lemma 18.1 give the stated quotients. For the relative-dual alternative, the common universal cover is $\mathrm{Sp}(4) \times \mathrm{Spin}(10)$. Along $A(\theta) = \mathrm{diag}(e^{i\theta}, 1, 1, 1)$, $z = 1$, the symplectic lift is $c(e^{i\theta/2} \bar{A}(\theta))$ and ends at -1 . The orthogonal lift of the coordinate-plane rotation ends at -1 in $\mathrm{Spin}(10)$. This period therefore has endpoint $(-1, -1)$. The independent z loop has endpoint $(1, -1)$. Their generated subgroup is $\{\pm 1\} \times \{\pm 1\}$, and its quotient is J_* with (18.1). Thus $(\iota_*)_* : \pi_1(P_4) \rightarrow \pi_1(J_*)$ is surjective. Lemma 18.1 gives a finite central quotient of (J_*, ι_*) .

A finite central quotient leaves the metric Lie algebra and local heat germs unchanged. For $X \in \mathfrak{su}(4)$,

$$\|d\iota_*(X + iaI_4, ib)\|^2 = 2Q_{U(4)}(X, X) + 8a^2 + b^2.$$

On the central two-torus its adjoint weights remain (17.1). Equations (16.2) and (16.7) therefore give the reference central germs. Finally $\mathrm{Vol}(J_*) = v_4/2$, and a quotient by a finite group F divides this volume by $|F|$. $\square$

Corollary 18.5. *Let J be compact and connected with a positive bi-invariant metric and a homomorphism $\iota : P_n \rightarrow J$. If $\dim J = (2n + 1)^2$ and the rate 2-jet and amplitude-log 6-jet agree with the reference on a full maximal torus of P_n , then (J, ι) is a finite central quotient of G_n or H_n . Every such quotient has those jets. If moreover $\mathrm{Vol}(J) \geq v_n$, then J is one of the two reference marked groups.*

Proof. The rate determines the full invariant pullback form by conjugation. The central restriction gives Theorem 18.4. For the exceptional local marking, take $X = i \mathrm{diag}(x_1, \dots, x_n)$ with $\sum x_i = 0$ and set $b = 0$. The symplectic positive- a weights are respectively $2a + x_i + x_j$ or $2a - x_i - x_j$, $i \leq j$. The opposite weights give the conjugate half, and the zero- a weights contribute no term linear in a . The fourth-power sum over all complex weights of the symplectic factor therefore has coefficient of a equal to

$$16(n + 4) \sum_i x_i^3 \quad \text{and} \quad -16(n + 4) \sum_i x_i^3, \quad (18.2)$$

because

$$\sum_{i \leq j} (x_i + x_j)^3 = \frac{1}{2} \left(\sum_{i, j} (x_i + x_j)^3 + \sum_i (2x_i)^3 \right) = (n + 4) \sum_i x_i^3.$$

The orthogonal factor is the same. At $n = 4$, choose $(x_i) = (1, 1, -2, 0)$, whose cubic sum is nonzero. By (16.7), the homogeneous degree-four term of Φ is $1/5760$ times the sum of fourth powers over

all complex weights. The two symplectic contributions to its coefficient of a are therefore $-2/15$ and $2/15$. Equality of the full jet excludes the dual sign. Central quotients preserve the local marked metric algebra, proving the converse. Their volumes are $v_n/|F|$, so the stated lower bound forces $F = 1$. $\square$

19. MARKED RECOGNITION AND PRODUCT CLASSIFICATION

Thirteen local coefficients suffice for the following recognition theorem, and twelve suffice when $n = 1$.

Theorem 19.1 (Marked recognition). *Fix $n \geq 1$ and put $v_n = \text{Vol}(G_n)$ for the reference metric. Let J be any compact connected Lie group with a positive bi-invariant metric and a homomorphism $\iota : P_n \rightarrow J$. Suppose*

- (1) $\dim J = (2n + 1)^2$;
- (2) the three coefficients of $I_J(a, b)$ on the central two-torus agree with those of $2na^2 + b^2$;
- (3) for $n \geq 2$, the coefficient of t^2 in $I_J(tH_0, 0)$ is $2Q_{U(n)}(H_0, H_0)$ for one fixed nonzero $H_0 \in \mathfrak{su}(n)$;
- (4) the nine coefficients of $\Phi_J(a, b)$ determining (17.2) equal the reference coefficients;
- (5) $\text{Vol}(J) \geq v_n$.

Then (J, ι) is isometrically isomorphic, with its marking, to (G_n, i_n) or (H_n, j_n) . Conversely both groups satisfy these conditions. The marking is therefore faithful and their complete graded right- P_n scalar data are determined.

Proof. For $n \geq 2$, apply Schur's lemma [37, Lemma 9.9 and Corollary 9.12] to $\mathfrak{p}_{n, \mathbb{C}} = \mathfrak{sl}_n(\mathbb{C}) \oplus \mathbb{C}^2$. The invariant form ι^*Q has no mixed term between $\mathfrak{su}(n)$ and the center, and its restriction to $\mathfrak{su}(n)$ is a scalar multiple of $Q_{U(n)}$. Conditions (2)–(3) therefore give $\iota^*Q = Q_P$. For $n = 1$, condition (2) determines the whole form. Condition (4) gives Theorem 18.4. Every finite central quotient of G_n or H_n has volume $v_n/|F|$, and every exceptional form has volume at most $v_4/2$. Condition (5) excludes the latter and makes F trivial for the reference quotients. Theorem 3.2 equates their graded data. Both reference groups satisfy the local conditions by their common metric adjoint module and have volume v_n . $\square$

The nine moment values are explicitly

$$\begin{aligned} M_{0,2k} &= 4n, & M_{2k,0} &= 2n^2 2^{2k} + 4n \quad (k = 1, 2, 3), \\ M_{22} &= M_{42} = 4n, & M_{11} &= 0. \end{aligned} \tag{19.1}$$

For $u + v = 2k$, the corresponding coefficient of $a^u b^v$ is $\frac{1}{2} \frac{\zeta(2k)}{k(2\pi)^{2k}} \binom{2k}{u} M_{uv}$. Together with the three central rate coefficients and, for $n \geq 2$, the coefficient along tH_0 , these are thirteen real local quantities, or twelve at $n = 1$. They are derivatives of order at most six of the analytic germs in (16.1); dimension and volume are separate conditions.

Corollary 19.2. *The graded right- P_n scalar data of (G_n, i_n) have exactly the two compact connected bi-invariant marked realizations (G_n, i_n) and (H_n, j_n) .*

Proof. The full graded data give the rate and amplitude germs, dimension and volume. Theorem 19.1 leaves the two reference groups, whose graded data agree by Theorem 3.2. $\square$

The groups H_n^S and H_n^O have the same marked metric adjoint module and volume, but their scalar multiplicities at $2n + 1$ differ by Theorem 18.3.

19.1. Equal local germs with different kernels.

Example 19.3. Let $J = \mathbb{R}^2/(2\pi\mathbb{Z})^2$ have its Euclidean metric and let $P = S^1$. The markings

$$\iota_1(e^{is}) = (e^{5is}, 1), \quad \iota_2(e^{is}) = (e^{3is}, e^{4is})$$

have the same ambient spectrum, volume and dimension. Both give $I(s) = 25s^2$ and $A(s) = 1$, with equal complete formal local heat expansions. Their kernels are C_5 and 1, respectively. Indeed $3s, 4s \in 2\pi\mathbb{Z}$ implies $s \in 2\pi\mathbb{Z}$. For sufficiently small s , the zero term in the Gaussian lattice sum is $e^{-25s^2/(4t)}(4\pi t)^{-1}$ in both cases. Every other term has displacement $Y + 2\pi k$ with $k \in \mathbb{Z}^2 \setminus \{0\}$. For $|Y| \leq \pi/2$,

$$|Y + 2\pi k|^2 - |Y|^2 \geq 4\pi^2|k|^2 - 2\pi^2|k| \geq 2\pi^2|k|^2.$$

After division by the zero lattice term, the remaining sum is $O(e^{-\pi^2/(2t)})$, uniformly on this neighborhood. It has no nonzero coefficient in an expansion in powers of t .

For every integer $m \geq 1$, the same torus with metric $Q_m = \text{diag}(m^{-2}, m^2)$ and marking $\iota_m(e^{is}) = (e^{ims}, 1)$ has fixed dimension and volume, $I(s) = s^2$, $A(s) = 1$, and kernel C_m . This gives unbounded kernel orders at fixed local data and volume.

19.2. Equal complete local heat expansions on a two-torus.

Proposition 19.4. *There exists a faithfully T^2 -marked compact connected Lie group J with a positive bi-invariant metric, having the same dimension 25, volume, scalar curvature 60, and complete formal local equivariant heat expansions as $(G_2, i_2|_{Z(P_2)})$, but a different scalar spectrum.*

Proof. Take $G = \text{U}(5)$ with Q_G and marking

$$(e^{ia}, e^{ib}) \mapsto \text{diag}(e^{ia}, e^{ia}, e^{-ia}, e^{-ia}, e^{ib}).$$

Take

$$J = (\text{U}(3), Q_G/2)^2 \times (\text{Sp}(1), Q_C)^2 \times S_L^1,$$

where S_L^1 has length L . The marking is $\text{diag}(e^{ia}, e^{-ia}, e^{ib})$ in each unitary factor, e^{ia} in each quaternionic factor, and 1 on the circle. The diagonal entries make both markings faithful. Their induced quadratic forms are

$$I_G = 4a^2 + b^2, \quad I_J = 2(a^2 + b^2/2) + 2a^2 = 4a^2 + b^2.$$

Both adjoint modules have four real rotation planes of weight $2a$, two of weight $a + b$, two of weight $a - b$, and a fixed space of dimension nine. Their common exponential Jacobian is

$$j(a, b) = \left(\frac{\sin a}{a} \right)^8 \left(\frac{\sin((a+b)/2)}{(a+b)/2} \right)^4 \left(\frac{\sin((a-b)/2)}{(a-b)/2} \right)^4,$$

with each quotient equal to 1 at zero. For $\text{U}(N)$ with Q_G , the Killing form is $-2NQ_G$ on $\mathfrak{su}(N)$ and zero on the center. Hence $\text{Scal} = N(N^2 - 1)/2$. Each scaled unitary factor has scalar curvature 24; $(\text{Sp}(1), Q_C)$ is the unit round three-sphere and has scalar curvature 6. Thus $\text{Scal}(J) = 48 + 12 = 60 = \text{Scal}(G)$. Choose $L > 0$ so that their volumes agree; the other four factor volumes are positive and fixed.

For a compact bi-invariant metric group, the complete local heat coefficient formula is

$$(4\pi t)^{d/2} e^{|Y|^2/(4t)} k_t(e, \exp Y) \sim j(Y)^{-1/2} e^{t \text{Scal}/6}. \quad (19.2)$$

This follows from [53, Theorem 3.3 and Proposition 3.5] for a compact Lie group with an arbitrary positive invariant form. In that reference, $\Omega = -\Delta$ and $j_j = j^{1/2}$. The kernel q_t of $e^{t(\Omega - \|\rho\|^2)/2}$ satisfies

$$q_{2t}(\exp Y) \sim (4\pi t)^{-d/2} e^{-|Y|^2/(4t)} j(Y)^{-1/2}, \quad k_t = e^{t\|\rho\|^2} q_{2t}.$$

The identity $\|\rho\|^2 = \text{Scal}/6$ is the one in the proof of Corollary 16.3. For the negative Killing metric, the coefficient formula is already in [94, proof of Theorem 3, pp. 290–291]. The relative remainder at every order follows from [68, Theorem 1.1]. Thus both marked traces, after multiplication by $(4\pi t)^{25/2} e^{(4a^2+b^2)/(4t)}/v$, have expansion $j(a, b)^{-1/2} e^{10t}$. Dividing by the common diagonal expansion gives equality for normalized traces as well.

An eigenfunction of a quaternionic factor with eigenvalue three pulls back to J with that eigenvalue. Proposition 3.8 at $n = 2$ shows that the first positive eigenvalue of $\text{U}(5)$ is five. Thus the scalar spectra differ. $\square$

The centralizer of the displayed T^2 in J is a torus: each unitary factor has three distinct one-dimensional character spaces, and each quaternionic centralizer is a circle. Identify T^2 with the central torus of $P_2 = \mathrm{U}(2) \times \mathrm{U}(1)$ by $(e^{ia}, e^{ib}) \mapsto (e^{ia}I_2, e^{ib})$. The displayed marking cannot be the restriction of an infinitesimally injective P_2 marking: the image of $\mathfrak{su}(2)$ would lie in that abelian centralizer and would therefore vanish.

19.3. Metric splitting without an assumed ambient product. Fix $r \geq 1$, ranks $n_i \geq 1$ and scales $c_i > 0$. Let $P = \prod_i P_{n_i}$ and let $J_0 = \prod_i G_{n_i}$ have the product marking and metrics $c_i Q_G$. For $\varepsilon \in \{G, H\}^r$, write J_ε for the corresponding marked product, with the same scales.

Lemma 19.5. *A compact metric Lie algebra with the prescribed metric adjoint P module of J_0 , including its distinguished embedded $\mathfrak{p}$, is a marked orthogonal direct sum of the metric Lie algebras of one J_ε .*

Proof. Let $W = \mathfrak{g}^P$ and let V_i be the nontrivial isotypic part supported on P_{n_i} . Then

$$\mathfrak{g} = W \perp \bigoplus_i V_i, \quad \dim W = 3r.$$

For $i \neq j$, every constituent of $V_i \otimes V_j$ is nontrivial on both factors, while no constituent of $\mathfrak{g}$ is. Equivariance of the bracket therefore gives $[V_i, V_j] = 0$. Also $[W, V_i] \subset V_i$. The subalgebra generated by V_i is preserved by W and commutes with each V_j , $j \neq i$, by the derivation identity. It is consequently an ideal $\mathfrak{a}_i$. The center of $\mathfrak{g}$ is P -fixed, so V_i has zero central projection. The compact reductive decomposition [104, Proposition 3.25(b) and Theorem 3.21(a),(b)] places each $\mathfrak{a}_i$ in $[\mathfrak{g}, \mathfrak{g}]$ as a sum of simple ideals. Since the $\mathfrak{a}_i$ commute, these sums have disjoint simple factors and are orthogonal. Hence

$$\mathfrak{g} = \mathfrak{a}_0 \perp \bigoplus_i \mathfrak{a}_i, \tag{19.3}$$

with $\mathfrak{a}_0$ P -trivial.

For $j \neq i$, the marked $\mathfrak{p}_{n_j}$ projection into $\mathfrak{a}_i$ centralizes a semisimple ideal and is zero. The marked $\mathfrak{p}_{n_i}$ projection into $\mathfrak{a}_i$ has the prescribed action on V_i , so Lemma 17.3 applies. Let u be the number of its unitary alternatives. Counting fixed dimensions in (19.3) gives

$$\dim \mathfrak{a}_0 = 3r - 2u - 3(r - u) = u. \tag{19.4}$$

Let $L_i : \mathfrak{p}_{n_i} \rightarrow \mathfrak{a}_0$ be the residual projection. Since the P action on $\mathfrak{a}_0$ is trivial, $[L_i X, Y] = 0$ for $X \in \mathfrak{p}_{n_i}$ and $Y \in \mathfrak{a}_0$. Its image is therefore central, and it vanishes on $\mathfrak{su}(n_i)$. The images of different L_i are orthogonal: the marked factors were orthogonal, and their semisimple projections have disjoint support. Their residual quadratic forms are nonnegative with ranks $\dim \operatorname{im} L_i$. Every unitary alternative has nonzero residual form. For $n_i \geq 2$, its simple scale is c_i , fixed by the $\mathfrak{su}(n_i)$ norm, and its residual form is $c_i b^2 / (2n_i + 1)$. For $n_i = 1$, write the simple scale as $s > 0$; its residual form is

$$2(c_i - s)a^2 + (c_i - 2s/3)b^2. \tag{19.5}$$

Nonnegativity gives $s \leq c_i$, so the b^2 coefficient is at least $c_i/3 > 0$.

The u unitary alternatives therefore give u nonzero orthogonal residual images in the u -dimensional space of (19.4). Each has dimension one, their sum exhausts $\mathfrak{a}_0$, and every other residual image is zero. Since all images are central, $\mathfrak{a}_0$ is abelian. For each unitary alternative, adjoin its one-dimensional residual image to $\mathfrak{a}_i$; for the other alternatives retain $\mathfrak{a}_i$. These orthogonal blocks have precisely the reference P_{n_i} module, distinguished $\mathfrak{p}_{n_i}$ and induced form $c_i Q_P$. Their marked subspaces are isometric. By the real representation argument and polar decomposition in the proof of Proposition 4.5, their orthogonal complements are equivariantly isometric. Thus each block is orthogonally equivalent to the reference Euclidean module by a map fixing its marked subspace. After division of the metric by c_i , Theorem 17.4 identifies each block with its reference marked metric Lie algebra. Taking their orthogonal direct sum gives the decomposition. $\square$

Theorem 19.6 (The marked product classification). *Let J be any compact connected Lie group with a positive bi-invariant metric and a homomorphism $\iota : P \rightarrow J$. Its graded right- P scalar data agree with those of J_0 if and only if it is isometrically isomorphic, with marking, to one of the 2^r groups J_ε . These marked classes are distinct, including repeated ranks or scales.*

Proof. Theorem 16.2 recovers the metric adjoint module and the differential marking, and makes ι faithful. Lemma 19.5 gives the marked metric Lie algebra of one J_ε . Its marking induces a surjection $\pi_1(P) \rightarrow \pi_1(J_\varepsilon)$ by the factorwise maps (2.7). Lemma 18.1 therefore gives a marked surjective local isometry $J_\varepsilon \rightarrow J$ with finite central kernel F .

The product intrinsic identity gives $\text{Vol}(J_\varepsilon) = \text{Vol}(J_0)$. The assumed scalar data give $\text{Vol}(J) = \text{Vol}(J_0)$. Since the quotient volume is $\text{Vol}(J_\varepsilon)/|F|$, one has $|F| = 1$.

Conversely, apply the finite tensor-product assertion of Lemma 2.4 to the intertwiners of Theorem 3.2, with diagonal function

$$f(\lambda_1, \dots, \lambda_r) = \sum_{i=1}^r c_i^{-1} \lambda_i,$$

where λ_i is an eigenvalue for the unscaled reference metric. This gives the graded right- P equivalence. Finally a marked isomorphism preserves each V_i and its generated semisimple ideal. The two alternatives in Lemma 17.3 have dimensions $(2n_i + 1)^2 - 1$ and $(2n_i + 1)^2$, respectively. Thus it preserves every ε_i . $\square$

For $r = 0$, equality with the scalar data of the point forces $\dim J = 0$. A connected realization is therefore the point.

Part VI. Metrics on compact simple groups

For $n \geq 1$, the groups G_n and H_n embed in $L = \text{SU}(4n + 2)$, equipped with its trace metric. Yu's weighted dimension-data identities [101], combined with finite support separation and the leading Weyl-denominator term, give equal spectra for their induced Casimirs in every finite-dimensional L -module (Theorem 20.1). The scalar Laplacian formula for subgroup scaling then gives scalar-isospectral left-invariant metrics on L for every $s > 0$; for $0 < s < 1$, they are normal homogeneous and not locally isometric (Corollary 22.4). For $m \geq 6r \geq 6$, commuting block embeddings give 2^r scalar-isospectral metrics on $\text{SU}(m)$ that are pairwise not locally isometric (Theorem 22.5).

20. CASIMIR OPERATORS IN AN AMBIENT REPRESENTATION

Fix an integer $n \geq 1$, and put $N = 2n + 1$ and $J = \text{SU}(2N)$. Equip $\mathfrak{j}$ with $Q(X, Y) = -\text{Re Tr}_{\mathbb{C}}(XY)$. Consider the injectively embedded connected compact subgroups

$$\begin{aligned} K_A &= \{\text{diag}(A, \overline{A}) : A \in \text{U}(N)\}, \\ K_B &= \{\text{diag}(S, R) : S \in \text{Sp}(n), R \in \text{SO}(2n + 2)\}. \end{aligned}$$

Their dimensions are both N^2 . On the abstract groups take the reference forms

$$Q_A = -\text{Re Tr}_{\mathbb{C}}, \quad Q_B = -\frac{1}{2} \text{Re Tr}_{\mathbb{C}} \oplus -\frac{1}{2} \text{Tr}_{\mathbb{R}}.$$

The first summand in Q_B is on $\mathfrak{sp}(n)$, and the second is on $\mathfrak{so}(2n + 2)$. Direct trace evaluation gives

$$Q|_{\mathfrak{k}_A} = 2Q_A, \quad Q|_{\mathfrak{k}_B} = 2Q_B. \quad (20.1)$$

Write $C_A^{\text{ref}}, C_B^{\text{ref}}$ for the positive Casimir elements associated with the reference forms. Every finite-dimensional J -module admits an invariant Hermitian form by [37, Proposition 35.5]. The two Casimir endomorphisms are then nonnegative and self-adjoint.

Define

$$D_R = \frac{1}{2} C_R^{\text{ref}}, \quad R \in \{A, B\}. \quad (20.2)$$

By (20.1), D_R is the positive Casimir for the restriction of the ambient form Q to $\mathfrak{k}_R$.

Yu's weighted dimension-data identities [101, Theorem 2.1 and Proposition 2.2] imply the following equality for the reference Casimirs.

Theorem 20.1 (The ambient Casimir identity). *For every finite-dimensional complex representation V of $SU(4n+2)$, the endomorphisms induced by C_A^{ref} and C_B^{ref} on V have the same eigenvalues with the same multiplicities. Equivalently,*

$$\text{Tr}_V(e^{-tC_A^{\text{ref}}}) = \text{Tr}_V(e^{-tC_B^{\text{ref}}}) \quad (t \geq 0).$$

Write $X = (x, y, z)$, with $x, y \in \mathbb{R}^n$, and put

$$p_C(x) = \prod_{i=1}^n x_i \prod_{1 \leq i < j \leq n} (x_i^2 - x_j^2), \quad p_D(y, z) = \prod_{1 \leq i < j \leq n+1} (w_i^2 - w_j^2), \quad w = (y_1, \dots, y_n, z).$$

Let S be the group generated by the independent swaps $x_i \leftrightarrow y_i$ and by $z \mapsto -z$. Denote averaging over this group by Av_S . For $v(t) = (1, t, \dots, t^{2n})^T$, set

$$D(X) = \det[v(x_1), v(-y_1), \dots, v(x_n), v(-y_n), v(z)], \quad \rho = (n, n-1, \dots, 1, n, n-1, \dots, 1, 0).$$

Lemma 20.2. *The denominators in the following formula are nonzero, and*

$$\text{Av}_S \frac{D(X)}{D(\rho)} = \text{Av}_S \frac{p_C(x)p_D(y, z)}{p_C(\rho_C)p_D(\rho_D)}, \quad (20.3)$$

where $\rho_C = (n, n-1, \dots, 1)$ and $\rho_D = (n, n-1, \dots, 1, 0)$.

Proof. Put $W = \{\pm 1\}^N \rtimes S_N$ and

$$\alpha = (n, n-1, \dots, 1, 0, -1, \dots, -n), \quad LX = (x_1, \dots, x_n, z, -y_n, \dots, -y_1). \quad (20.4)$$

Then $L \in W$, $L\rho = \alpha$, and $\kappa = |\rho|^2 = |\alpha|^2 = 2 \sum_{j=1}^n j^2$. For a Weyl element v , write $\epsilon(v) = \det(v)$. In $\mathbb{Q}[\mathbb{Z}^N]$, set

$$b_A = \sum_{u \in S_N} \epsilon(u) e^{-L^{-1}u\alpha}, \quad b_B = \sum_{v \in W(C_n) \times W(D_{n+1})} \epsilon(v) e^{-v\rho}.$$

The dominant integral weights $(m-1)\alpha$ and $(m-1)\rho$ satisfy the hypotheses of [101, Theorem 2.1] for every integer $m \geq 1$. The unitary weight has opposite tails and its middle coordinate is zero. The orthogonal weight has last coordinate zero and belongs to the character lattice of a maximal torus of $SO(2n+2)$. The group-algebra identity of [101, Proposition 2.2], after the change of variables L , is

$$\text{Av}_W(e^{m\rho} b_A) = \text{Av}_W(e^{m\rho} b_B).$$

Every exponent of b_A and b_B has norm $\sqrt{\kappa}$. Distinct points of $W\rho$ are integral and have distance at least one. For $m = 2\kappa + 1 > 2\sqrt{\kappa}$, the closed balls of radius $\sqrt{\kappa}$ centered at the distinct points of $mW\rho$ are disjoint. The terms in the ball centered at $m\rho$ come exactly from $S = \text{Stab}_W(\rho)$. Restriction to this ball and cancellation of $\frac{|S|}{|W|} e^{m\rho}$ give

$$\text{Av}_S b_A = \text{Av}_S b_B. \quad (20.5)$$

The stabilizer consists of the independent exchanges $x_i \leftrightarrow y_i$ and the sign change of z .

Write $p_A(U) = \prod_{i < j} (U_i - U_j)$ and $q = n(2n+1)$. Under $e^\mu \mapsto \exp(t\langle \mu, X \rangle)$, the Weyl denominator formula [37, Corollary 25.5] gives

$$b_A(tX) = (-t)^q p_A(LX) + O(t^{q+2}), \quad b_B(tX) = (-t)^q 2^n p_C(x)p_D(y, z) + O(t^{q+2}).$$

The factor 2^n is the product of the coefficients of the roots $2e_i$ of C_n . The column permutation from LX to the order in D has n^2 inversions. Since $q \equiv n^2 \pmod{2}$,

$$p_A(LX) = D(X). \quad (20.6)$$

Comparing degree q in (20.5) gives

$$\text{Av}_S D = 2^n \text{Av}_S(p_C p_D). \quad (20.7)$$

Evaluation at the S -fixed point ρ yields

$$D(\rho) = p_A(\alpha) = 2^n p_C(\rho_C) p_D(\rho_D) > 0.$$

Division proves (20.3), including $n = 1$. □

Proof of Theorem 20.1. Conjugating maximal tori inside J , their elements have eigenvalues

$$e^{i\theta_1}, \dots, e^{i\theta_N}, e^{-i\theta_1}, \dots, e^{-i\theta_N}.$$

Consequently the restrictions of the ambient character agree and define a Laurent polynomial $f = \sum_{v \in \mathbb{Z}^N} f_v e^v$ invariant under the full signed permutation group $W = \{\pm 1\}^N \rtimes S_N$. The sum is finite. A permutation matrix giving any permutation or sign change of the paired eigenvalues can be multiplied by a scalar of modulus one to have determinant one; its conjugation action is unchanged. Thus the invariance holds under conjugation in J .

The character lattices of these maximal tori are $\mathbb{Z}^N$: for K_A these are the torus characters of $U(N)$; for K_B they are the integer symplectic weights and integer orthogonal weights of $SO(2n+2)$. No half-integral spin weights occur. The reference forms induce the Euclidean dual form in these coordinates. Choose positive roots with

$$\rho_A = (n, n-1, \dots, 0, -1, \dots, -n), \quad \rho_B = \rho = (\rho_C, \rho_D).$$

Their squared norms are equal:

$$|\rho_A|^2 = |\rho_B|^2 = 2 \sum_{j=1}^n j^2 =: \kappa_n.$$

Let

$$p_A(X) = \prod_{1 \leq i < j \leq N} (X_i - X_j), \quad p_B(X) = p_C(x) p_D(y, z).$$

The Weyl dimension formula [37, Proposition 28.1] gives

$$\dim V_\lambda = \frac{p_R(\lambda + \rho_R)}{p_R(\rho_R)}.$$

By Lemma 2.2, the reference Casimir eigenvalue is $|\lambda + \rho_R|^2 - |\rho_R|^2$. The reference forms here are those of (2.2), including the central direction of $U(N)$.

If $f = \sum_\lambda m_\lambda \chi_\lambda$ as a character of K_R , then

$$\mathcal{H}_R(f; t) := \sum_\lambda m_\lambda \dim V_\lambda e^{-t(|\lambda + \rho_R|^2 - |\rho_R|^2)} = \frac{e^{\kappa_n t}}{p_R(\rho_R)} \sum_v f_v p_R(v + \rho_R) e^{-t|v + \rho_R|^2}. \quad (20.8)$$

Put $W_A = S_N$ and $W_B = W(C_n) \times W(D_{n+1})$, and set $\delta_R = \sum_{w \in W_R} \epsilon(w) e^{w\rho_R}$. Weyl's character formula [37, Theorem 25.4] makes $f\delta_R$ a sum of alternating orbit sums at $\lambda + \rho_R$. Summing its coefficients at μ against $p_R(\mu) e^{-t|\mu|^2}$ gives

$$|W_R| \sum_\lambda m_\lambda p_R(\lambda + \rho_R) e^{-t|\lambda + \rho_R|^2}.$$

The weights on root hyperplanes contribute zero. Expand δ_R . The identities $f_{wv} = f_v$ and $p_R(wX) = \epsilon(w)p_R(X)$, together with orthogonality of w , give

$$|W_R| \sum_v f_v p_R(v + \rho_R) e^{-t|v + \rho_R|^2}.$$

Dividing proves (20.8); all sums here are finite.

For the signed permutation L of (20.4), $L\rho = \rho_A$ and (20.6) gives

$$\frac{p_A(LX)}{p_A(\rho_A)} = \frac{D(X)}{D(\rho)}.$$

Changing variables by L in (20.8), and using $f_{L\nu} = f_\nu$, gives

$$\mathcal{H}_A(f; t) = e^{\kappa_n t} \sum_\nu f_\nu \frac{D(\nu + \rho)}{D(\rho)} e^{-t|\nu + \rho|^2}.$$

The same expression for $\mathcal{H}_B$ has $p_B(\nu + \rho)/p_B(\rho)$ in place of the determinant ratio. Since S fixes ρ , the factors f_ν and $e^{-t|\nu + \rho|^2}$ are invariant under S . Averaging the polynomial factor and applying Lemma 20.2 proves equality of the two traces. Lemma 2.3 equates all Casimir multiplicities. $\square$

21. SUBGROUP CONTRACTIONS ON A COMPACT SIMPLE GROUP

Let L be a compact connected Lie group with simple Lie algebra $\mathfrak{l}$. Let Q be a positive $\text{Ad}(L)$ -invariant inner product, and write

$$B_{\mathfrak{l}} = -4cQ, \quad c > 0.$$

For a closed connected subgroup $K \leq L$, put $\mathfrak{m} = \mathfrak{k}^\perp_Q$. For $0 < s < 1$, define the left-invariant metric

$$g_{K,s}|_{\mathfrak{m}} = Q|_{\mathfrak{m}}, \quad g_{K,s}|_{\mathfrak{k}} = sQ|_{\mathfrak{k}}, \quad \mathfrak{m} \perp_{g_{K,s}} \mathfrak{k}. \quad (21.1)$$

Lemma 21.1. *The metric $g_{K,s}$ is normal homogeneous for $L \times K$ and has nonnegative sectional curvature. Its normal presentation is*

$$(L, g_{K,s}) \cong \left((L, Q) \times \left(K, \frac{s}{1-s} Q|_{\mathfrak{k}} \right) \right) / \Delta K.$$

The diffeomorphism is induced by $(x, k) \mapsto xk^{-1}$.

Proof. Apply [3, Theorem 2.21] with trivial initial isotropy, one deformation subgroup K , and

$$t_0 = r_0 = 1, \quad t_1 = s, \quad r_1 = \frac{s}{1-s}.$$

The parameter relation is $r_1^{-1} = t_1^{-1} - t_0^{-1}$. Since $0 < s < 1$, both factors have positive bi-invariant metrics. The horizontal lifts of $X \in \mathfrak{m}$ and $Y \in \mathfrak{k}$ are, respectively,

$$(X, 0), \quad (sY, -(1-s)Y).$$

Nonnegative sectional curvature follows from (4.16). $\square$

For a normal homogeneous space J/D , write $\mathfrak{j} = \mathfrak{d} \oplus \mathfrak{h}$ orthogonally for its positive invariant form q , and let (V_i) be an orthonormal basis of $\mathfrak{d}$. Equation (4.17) gives

$$\text{Ric}(X, X) = -\frac{1}{4}B_{\mathfrak{j}}(X, X) + \frac{1}{2} \sum_i \|[V_i, X]\|_q^2, \quad X \in \mathfrak{h}. \quad (21.2)$$

Lemma 21.2. *The splitting $\mathfrak{l} = \mathfrak{k} \oplus \mathfrak{m}$ is orthogonal for the Ricci tensor. Its restrictions are*

$$\text{Ric}_{K,s}(X, X) = cs^2Q(X, X) - \frac{1-s^2}{4}B_{\mathfrak{k}}(X, X), \quad X \in \mathfrak{k}, \quad (21.3)$$

$$\text{Ric}_{K,s}(X, X) = cQ(X, X) + \frac{1-s}{2} \sum_{i=1}^{\dim K} Q([E_i, X], [E_i, X]), \quad X \in \mathfrak{m}, \quad (21.4)$$

where (E_i) is a Q -orthonormal basis of $\mathfrak{k}$. Every $g_{K,s}$ has positive Ricci curvature. If $\mathfrak{z}(\mathfrak{k}) \neq 0$, the smallest eigenvalue of the Ricci endomorphism is cs . If $\mathfrak{k}$ is semisimple, its smallest Ricci eigenvalue is strictly larger than cs .

Proof. Suppose first that $0 < \dim K < \dim L$. Decompose $\mathfrak{k}$ into its simple ideals and one-dimensional central summands, orthogonally for Q . Apply [6, Proposition 2.2] with background form $Q_0 = -B_{\mathfrak{l}} = 4cQ$ and parameters

$$\alpha = \frac{1}{4c}, \quad \alpha_i = \frac{s}{4c}.$$

In that notation, $\alpha_i/\alpha = s$ and $\kappa_i Q_0 = -B_{\mathfrak{k}_i}$ on each ideal. If $(E_{i,j})$ is a Q -orthonormal basis of $\mathfrak{k}_i$, then its form A_i is

$$A_i(X, Y) = - \sum_j Q([X, E_{i,j}], [Y, E_{i,j}]), \quad X, Y \in \mathfrak{m}.$$

The three formulas of that proposition give the vanishing mixed terms and (21.3)–(21.4). For $K = \{1\}$ and $K = L$, the metrics are Q and sQ ; their Ricci tensors are both cQ , as follows from (4.17) with trivial isotropy. The same formulas therefore hold in these cases.

Let $T_{\mathfrak{k}}$ be the nonnegative Q -self-adjoint operator defined by $Q(T_{\mathfrak{k}}X, Y) = -\frac{1}{4}B_{\mathfrak{k}}(X, Y)$. On $\mathfrak{k}$ the Ricci endomorphism is

$$csI + (s^{-1} - s)T_{\mathfrak{k}}.$$

The identity $-B_{\mathfrak{k}}(X, X) = \sum_i \|[X, E_i]\|_Q^2$ identifies the kernel of $T_{\mathfrak{k}}$ with $\mathfrak{z}(\mathfrak{k})$. On $\mathfrak{m}$, every Ricci eigenvalue is at least $c > cs > 0$. Thus Ricci curvature is positive, with least eigenvalue cs exactly when $\mathfrak{z}(\mathfrak{k}) \neq 0$. For nonzero semisimple $\mathfrak{k}$, the operator $T_{\mathfrak{k}}$ is positive definite, so the least Ricci eigenvalue exceeds cs . If $\mathfrak{k} = 0$, then $\text{Ric} = cQ$ and the same strict inequality holds. $\square$

Corollary 21.3. *Let K_1, K_2 be closed connected subgroups of the same compact simple group (L, Q) . If $\mathfrak{z}(\mathfrak{k}_1) \neq 0$ and $\mathfrak{k}_2$ is semisimple, then for every $0 < s < 1$ the metrics $g_{K_1, s}$ and $g_{K_2, s}$ are not locally isometric.*

Proof. By Lemma 21.2, the two least Ricci eigenvalues are respectively cs and a number strictly greater than cs . A local isometry intertwines the Ricci endomorphisms. $\square$

Lemma 21.4. *Let J be a compact connected Lie group, K a closed subgroup, and Q a positive $\text{Ad}(J)$ -invariant form on $\mathfrak{j}$. Give J/K the normal homogeneous metric induced by Q . Suppose that $\mathfrak{k}$ contains no nonzero ideal of $\mathfrak{j}$. Every local Riemannian product decomposition of J/K induces a Q -orthogonal decomposition of $\mathfrak{j}$ into commuting ideals, and $\mathfrak{k}$ decomposes as its intersections with those ideals. In particular, if $\mathfrak{j}$ is simple, the normal homogeneous space is locally irreducible.*

Proof. Pass to the complete simply connected Riemannian cover and the effective lifted action; the Lie algebra pair $(\mathfrak{j}, \mathfrak{k})$ is unchanged. Homogeneity makes the dimension of local holonomy constant. By [58, Vol. I, Chapter II, Theorem 10.3], local and restricted holonomy agree. Thus a local parallel splitting extends to the cover by parallel transport. The de Rham theorem [58, Vol. I, Chapter IV, Theorem 6.2] reduces the assertion to the de Rham factors and orthogonal splittings of the Euclidean factor.

Put $\mathfrak{m} = \mathfrak{k}^{\perp_Q}$, and let $\mathfrak{m} = \bigoplus_i \mathfrak{m}_i$ be the de Rham tangent decomposition. Invariance of Q makes this presentation naturally reductive. The decomposition theorem for naturally reductive spaces [58, Vol. II, Chapter X, Theorem 5.2] gives ideals

$$\mathfrak{f} = \mathfrak{m} + [\mathfrak{m}, \mathfrak{m}], \quad \mathfrak{j}_i = \mathfrak{m}_i + [\mathfrak{m}_i, \mathfrak{m}_i], \quad \mathfrak{f} = \bigoplus_i \mathfrak{j}_i.$$

The ideal $\mathfrak{f}^{\perp_Q}$ lies in $\mathfrak{k}$, so it vanishes and $\mathfrak{f} = \mathfrak{j}$. For $i \neq j$, the ideals commute, and invariance gives

$$Q([\mathfrak{j}_i, \mathfrak{j}_i], \mathfrak{j}_j) = Q(\mathfrak{j}_i, [\mathfrak{j}_j, \mathfrak{j}_j]) = 0.$$

Together with $Q(\mathfrak{m}_i, \mathfrak{m}_j) = 0$, this proves $\mathfrak{j}_i \perp_Q \mathfrak{j}_j$. Hence $\mathfrak{k} = \bigoplus_i (\mathfrak{k} \cap \mathfrak{j}_i)$. For the Euclidean factor, (4.16) gives $[\mathfrak{m}_0, \mathfrak{m}] = 0$. Since $\mathfrak{j} = \mathfrak{m} + [\mathfrak{m}, \mathfrak{m}]$, the subspace $\mathfrak{m}_0$ is central. Its orthogonal subspaces are therefore commuting ideals as well. $\square$

Lemma 21.5. *For every positive $\text{Ad}(K)$ -invariant inner product q_K on $\mathfrak{k}$, the normal homogeneous metric on*

$$((L, Q) \times (K, q_K)) / \Delta K \cong L$$

is locally irreducible. In particular, this holds for every metric $g_{K,s}$ in (21.1).

Proof. Write $\mathfrak{j} = \mathfrak{l} \oplus \mathfrak{k}$, $\mathfrak{d} = \Delta \mathfrak{k}$ and $\mathfrak{h} = \mathfrak{d}^{\perp_q}$, where $q = Q \oplus q_K$. If an ideal contained in $\mathfrak{d}$ contains (X, X) , then $([Y, X], 0) \in \mathfrak{d}$ for all $Y \in \mathfrak{l}$. Hence $X \in \mathfrak{z}(\mathfrak{l}) = 0$.

By Lemma 21.4, a nontrivial local product would give $\mathfrak{j} = \mathfrak{j}_1 \perp \mathfrak{j}_2$ with $\mathfrak{d} = \bigoplus_i (\mathfrak{d} \cap \mathfrak{j}_i)$. The projections of the $\mathfrak{j}_i$ to $\mathfrak{l}$ are commuting ideals with sum $\mathfrak{l}$. Since $\mathfrak{l}$ is simple, one projection vanishes; say $\mathfrak{j}_2 \subset 0 \oplus \mathfrak{k}$. Then $\mathfrak{d} \cap \mathfrak{j}_2 = 0$, so $\mathfrak{j}_2 \subset \mathfrak{h}$. But $(0, X) \in \mathfrak{h}$ implies $q_K(X, Y) = 0$ for every $Y \in \mathfrak{k}$, and hence $X = 0$. This contradicts $\mathfrak{j}_2 \neq 0$. $\square$

21.1. The two subgroups of $\text{SU}(4n+2)$. Take $L = \text{SU}(4n+2)$, Q , K_A , and K_B as in Section 20, and put $N = 2n+1$. The Lie algebra of K_A has one-dimensional center; that of K_B is semisimple, including $n = 1$. The ambient Killing form is $B_{\mathfrak{l}} = -4NQ$.

Proposition 21.6. *For every $n \geq 1$ and $0 < s < 1$, the two metrics $g_{A,s} = g_{K_A,s}$ and $g_{B,s} = g_{K_B,s}$ are locally irreducible, left invariant and normal homogeneous on the fixed simply connected simple group $\text{SU}(4n+2)$. They have positive Ricci curvature and nonnegative sectional curvature, but they are not locally isometric. Their smallest Ricci eigenvalues are*

$$\rho_{\min}(g_{A,s}) = Ns, \quad \rho_{\min}(g_{B,s}) = \min \left\{ N, \frac{(3n+2)s + ns^{-1}}{2} \right\} > Ns. \quad (21.5)$$

Their volumes and scalar curvatures agree and are

$$\text{Vol}(g_{A,s}) = \text{Vol}(g_{B,s}) = s^{N^2/2} \text{Vol}(Q), \quad (21.6)$$

$$\begin{aligned} \text{Scal}(g_{A,s}) &= \text{Scal}(g_{B,s}) \\ &= N(4N^2 - 1 + (1-s)N^2) + \frac{N(N^2 - 1)(1-s)^2}{4s}. \end{aligned} \quad (21.7)$$

Neither metric is Einstein for $0 < s < 1$. For $n = 1$, the common manifold is $\text{SU}(6)$ of dimension 35, and the scalar curvature is $120 - 21s + 6/s$.

Proof. Normal homogeneity, local irreducibility, both curvature inequalities, and the first Ricci minimum follow from Lemmas 21.1, 21.2, and 21.5, with $c = N$. By (20.1), the induced forms are $2Q_A$ and $2Q_B$. The classical Killing forms [14, Remark 6.15, p. 88], restricted to the compact real forms, are

$$\begin{aligned} B_{\mathfrak{su}(N)}(X, Y) &= 2N \text{Re Tr}_{\mathbb{C}}(XY), \\ B_{\mathfrak{sp}(n)}(X, Y) &= 2(n+1) \text{Re Tr}_{\mathbb{C}}(XY), \\ B_{\mathfrak{so}(2n+2)}(X, Y) &= 2n \text{Tr}_{\mathbb{R}}(XY), \end{aligned}$$

for X, Y in the respective Lie algebras. Thus the ambient restrictions in (20.1) give

$$-\frac{B_{\mathfrak{su}(N)}}{4Q|_{\mathfrak{su}(N)}} = \frac{N}{4}, \quad -\frac{B_{\mathfrak{sp}(n)}}{4Q|_{\mathfrak{sp}(n)}} = \frac{n+1}{2}, \quad -\frac{B_{\mathfrak{so}(2n+2)}}{4Q|_{\mathfrak{so}(2n+2)}} = \frac{n}{2}.$$

Each quotient here denotes the scalar relating the two bilinear forms on the indicated summand. The orthogonal trace formula holds in dimension 4, so these ratios include $n = 1$. Thus the smallest vertical Ricci eigenvalue for B is

$$Ns + \frac{n}{2}(s^{-1} - s) = \frac{(3n+2)s + ns^{-1}}{2}.$$

The element $\text{diag}((n+1)iI_{2n}, -niI_{2n+2})$ lies in $\mathfrak{m}_B$, commutes with $\mathfrak{k}_B$, and is nonzero. Equation (21.4) therefore attains the horizontal minimum N , proving (21.5).

The volume formula follows from the N^2 contracted directions. For the scalar formula, put $b_K = -\text{tr}_{Q|_{\mathfrak{k}}} B_{\mathfrak{k}}$. Taking traces in (21.3)–(21.4) yields

$$\text{Scal}(g_{K,s}) = c(\dim L + (1-s)\dim K) + \frac{(1-s)^2}{4s} b_K.$$

Here $\text{tr}_{\mathfrak{m}} D_K^{\text{ad}} = 4c \dim K - b_K$, where $D_K^{\text{ad}} = -\sum_i \text{ad}(E_i)^2$. Indeed the trace of this operator on $\mathfrak{l}$ is $4c \dim K$ and its trace on $\mathfrak{k}$ is b_K . For K_A , the semisimple summand has dimension $N^2 - 1$ and $B_{\text{su}(N)} = -NQ|_{\text{su}(N)}$, so $b_{K_A} = N(N^2 - 1)$. For K_B , the Killing ratios are $2(n+1)$ on its nN -dimensional symplectic summand and $2n$ on its $(n+1)N$ -dimensional orthogonal summand. Hence

$$b_{K_B} = 4n(n+1)N = N(N^2 - 1) = b_{K_A}.$$

Substitute $c = N$, $\dim L = 4N^2 - 1$, and $\dim K = N^2$.

The A metric has eigenvalue $Ns < N$ while its horizontal Ricci eigenvalues are at least N . The B metric has different Ricci eigenvalues on its symplectic and orthogonal summands, their difference being $(s^{-1} - s)/2 > 0$. Thus neither metric is Einstein. $\square$

22. ISOSPECTRAL METRICS ON COMPACT SIMPLE GROUPS

Schueth constructed nontrivial continuous isospectral families of left-invariant metrics on $\text{SU}(m)$, $m \geq 6$, arbitrarily close to a bi-invariant metric.¹ Those families have sectional curvatures of both signs. The metrics below with contraction parameters in $(0, 1)$ are normal homogeneous and have nonnegative sectional curvature.

Put $L = \text{SU}(m)$, $m \geq 6$, and let $Q(X, Y) = -\text{Re Tr}_{\mathbb{C}}(XY)$. In a fixed six-dimensional block define

$$K_A^{(m)} = \{\text{diag}(A, \overline{A}, I_{m-6}) : A \in \text{U}(3)\}, \quad (22.1)$$

$$K_B^{(m)} = \{\text{diag}(S, R, I_{m-6}) : S \in \text{Sp}(1), R \in \text{SO}(4)\}. \quad (22.2)$$

For either subgroup K , and $s > 0$, define a left-invariant metric by

$$g_{K,s} = Q|_{\mathfrak{k}^{\perp Q}} \oplus sQ|_{\mathfrak{k}}. \quad (22.3)$$

Write $g_{A,s}$ and $g_{B,s}$ for these two metrics.

Theorem 22.1 (Isospectral metrics on $\text{SU}(m)$). *Let $m \geq 6$. The metrics $g_{A,s}$ and $g_{B,s}$ on $\text{SU}(m)$ are scalar-isospectral for every $s > 0$. Their spectral equivalence is induced by a single unitary operator, independent of s , which commutes with the left $\text{SU}(m)$ -action. For $0 < s < 1$, both metrics are normal homogeneous and locally irreducible, with positive Ricci curvature and nonnegative sectional curvature. They are not locally isometric for $0 < s < 1$. In this range, their least Ricci endomorphism eigenvalues are*

$$\rho_{\min}(g_{A,s}) = \frac{ms}{2}, \quad \rho_{\min}(g_{B,s}) = \min \left\{ \frac{m}{2}, \frac{(m-1)s + s^{-1}}{2} \right\} > \frac{ms}{2}. \quad (22.4)$$

Within the union of the two curves with $0 < s < 1$ and fixed Q , each spectral fiber has exactly two isometry classes.

22.1. Subgroup Casimirs and the Laplacian. For a compact connected Lie group L with a positive $\text{Ad}(L)$ -invariant inner product Q on $\mathfrak{l}$, write C_L for its positive Casimir and D_K for the positive Casimir defined by $Q|_{\mathfrak{k}}$, where $K \leq L$ is a connected closed subgroup. For $s > 0$, define $g_{K,s}$ by (22.3).

Lemma 22.2. *Let L be a compact connected Lie group containing an injective copy of $\text{SU}(6)$, and let Q_L be a positive invariant form on $\mathfrak{l}$. Take inside that copy the subgroups K_A, K_B of Section 20 with $n = 1$. Then their Q_L -induced Casimirs have the same eigenvalue distribution on every finite-dimensional complex representation of L .*

¹D. Schueth, *Isospectral manifolds with different local geometries*, arXiv:math/0102123v1, Example 3.9 and Remarks 3.12(i)–(ii).

Proof. The restriction of Q_L to the simple algebra $\mathfrak{su}(6)$ is aQ for some $a > 0$, where $Q = -\text{Re Tr}$. Both subgroup Casimirs are therefore scaled by a^{-1} . Restrict an L -representation to $\text{SU}(6)$ and apply Theorem 20.1, with (20.2), to that finite-dimensional representation. $\square$

Proposition 22.3. *Suppose two connected closed subgroups K_1, K_2 of (L, Q) have identical induced-Casimir eigenvalue distributions on every finite-dimensional complex L -representation. Then the metrics $g_{K_1,s}$ and $g_{K_2,s}$ in (22.3) are scalar-isospectral for every $s > 0$. There is one unitary U on $L^2(L)$, for normalized Haar measure, which commutes with the left L -action and Δ_Q , preserves smooth functions, and satisfies*

$$U\Delta_{K_1,s} = \Delta_{K_2,s}U \quad (s > 0)$$

with $UD(\Delta_{K_1,s}) = D(\Delta_{K_2,s})$. For $0 < s < 1$, both metrics are normal homogeneous.

Proof. The normalized volume measure of each $g_{K,s}$ is Haar probability measure. The scalar Laplacian is the negative sum of squares in a left-invariant orthonormal frame [43, proof of Lemma 2.2, p. 1621]. Splitting a Q -orthonormal basis into subgroup and complementary directions gives

$$\Delta_{K,s} = C_L + (s^{-1} - 1)D_K. \quad (22.5)$$

For simple L and semisimple K , compare [44, Section 5]; the branching formula for commuting subgroups is [3, Theorem 2.23]. For the subgroups of Section 20, (20.2) gives the coefficient $(s^{-1} - 1)/2$ for the reference Casimir. The difference $C_L - D_K$ is the negative sum of squares in the complementary directions and is nonnegative. Hence, as quadratic forms on every finite-dimensional unitary representation,

$$\min\{1, s^{-1}\}C_L \leq \Delta_{K,s} \leq \max\{1, s^{-1}\}C_L.$$

In the Peter–Weyl decomposition [37, Theorem 35.9], choose conventions so that the left L -action is on V_π in $V_\pi \otimes V_\pi^*$. The operator in (22.5) is

$$I \otimes (c_L(\pi)I + (s^{-1} - 1)D_K|_{V_\pi^*}).$$

By hypothesis, choose a unitary u_π on V_π^* intertwining the two subgroup Casimirs themselves. These choices are independent of s . The Hilbert direct sum of $I \otimes u_\pi$ is the required unitary U . It commutes with Δ_Q , which is scalar on each ambient type, and it intertwines the displayed Laplacians. For $i \in \{1, 2\}$ and each subgroup-Casimir eigenvalue μ , put

$$E_{i,\pi,\mu} = V_\pi \otimes \ker(D_{K_i}|_{V_\pi^*} - \mu I).$$

The maps $I \otimes u_\pi$ identify these finite-dimensional L -modules. Apply Lemma 2.4 to the labels (π, μ) and the diagonal functions $c_L(\pi) + (s^{-1} - 1)\mu$ and $c_L(\pi)$. It gives the stated equality of operator domains and preservation of the Sobolev spaces defined by Δ_Q , hence of smooth functions and all eigenvalue multiplicities.

For $0 < s < 1$, [3, Theorem 2.21], with trivial initial isotropy, deformation subgroup K , and $t_0 = r_0 = 1$, $t_1 = s$, $r_1 = s/(1 - s)$, gives the normal presentation of Lemma 21.1. $\square$

Proof of Theorem 22.1. The six-dimensional block embedding of $\text{SU}(6)$ in $\text{SU}(m)$ is injective, and the ambient trace form restricts to the same trace form. Lemma 22.2 and Proposition 22.3 give the common spectra and a single left-equivariant unitary for all $s > 0$. Lemmas 21.1, 21.2, and 21.5 give normal homogeneity, nonnegative sectional curvature, positive Ricci curvature, and local irreducibility.

For $L = \text{SU}(m)$, $B_l = -2mQ$, so $c = m/2$. The subgroup $K_A^{(m)}$ has a one-dimensional center and $K_B^{(m)}$ is semisimple. Lemma 21.2 gives $\rho_{\min}(g_{A,s}) = ms/2$ and a strictly larger minimum for $g_{B,s}$. The eigenvalues of $T_{\mathfrak{t}_B}$ are 1 on $\mathfrak{sp}(1)$ and $1/2$ on $\mathfrak{so}(4)$, as computed in Proposition 21.6. Thus the least vertical eigenvalue for $g_{B,s}$ is $((m - 1)s + s^{-1})/2$. The nonzero vector

$$Z = \text{diag}(2iI_2, -iI_4, 0_{m-6})$$

lies in $\mathfrak{k}_B^{\perp Q}$ and commutes with $\mathfrak{k}_B$. It attains the horizontal lower bound $m/2$. This proves (22.4). The strict inequality excludes every local isometry, since the Ricci eigenvalues are constant on each homogeneous manifold and are preserved by local isometries.

Both subgroups have dimension nine. Hence

$$\text{Vol}(g_{A,s}) = \text{Vol}(g_{B,s}) = s^{9/2} \text{Vol}(Q). \quad (22.6)$$

Weyl's law [35, Theorem 16.7] makes volume a scalar spectral invariant. The expression in (22.6) is strictly increasing in s . Within the two stated curves, isospectrality therefore forces the same parameter, and exactly the two already distinguished isometry classes occur at each parameter. $\square$

Corollary 22.4. *The subgroups of Section 20 give two isospectral metrics $g_{A,s}, g_{B,s}$ on $\text{SU}(4n+2)$ for every $n \geq 1$ and $s > 0$. For $0 < s < 1$ their least Ricci eigenvalues, volumes, and scalar curvatures are those of Proposition 21.6; in particular, they are not locally isometric. With Q fixed, the spectral fibers in these two curves have exactly two isometry classes for $0 < s < 1$.*

Proof. Theorem 20.1, with $D_R = C_R^{\text{ref}}/2$, and Proposition 22.3 give scalar isospectrality. The remaining statements follow from Proposition 21.6 and the strict monotonicity of $s^{(2n+1)^2/2} \text{Vol}(Q)$. $\square$

22.2. Binary families on a fixed simple group. Retain the trace form Q and the two nine-dimensional subgroups $K_A, K_B \subset \text{SU}(6)$ in (22.1)–(22.2).

Fix integers m, r with $m \geq 6r \geq 6$ and embed $\text{SU}(6)^r$ block diagonally in $L = \text{SU}(m)$, with the identity on the remaining $m - 6r$ coordinates. For $\omega = (\omega_1, \dots, \omega_r) \in \{A, B\}^r$, define $K_\omega = \prod_{j=1}^r K_{\omega_j}$ in these blocks, and put $\mathfrak{m}_\omega = \mathfrak{k}_\omega^{\perp Q}$. For $0 < \varepsilon < 1$, set

$$s_j(\varepsilon) = 1 - \varepsilon + \frac{\varepsilon j}{100mr}, \quad j = 1, \dots, r. \quad (22.7)$$

These parameters are distinct and belong to $(0, 1)$. Let $g_{\omega, \varepsilon}$ be the left-invariant metric defined by

$$g_{\omega, \varepsilon}|_{\mathfrak{m}_\omega} = Q, \quad g_{\omega, \varepsilon}|_{\mathfrak{k}_{\omega_j}} = s_j Q, \quad (22.8)$$

with all the displayed summands orthogonal.

Theorem 22.5 (Exact binary families). *For fixed $m \geq 6r \geq 6$ and $\varepsilon \in (0, 1)$, the 2^r metrics in (22.8) have the same scalar Laplace spectrum, including multiplicities, and are pairwise not locally isometric. Every metric is normal homogeneous and locally irreducible, with positive Ricci curvature and nonnegative sectional curvature.*

Within the displayed family, the spectral equivalence is exactly

$$\text{Spec}(g_{\omega, \varepsilon}) = \text{Spec}(g_{\omega', \varepsilon'}) \iff \varepsilon = \varepsilon'. \quad (22.9)$$

Thus each spectral fiber within this family contains exactly 2^r isometry classes. Every C^∞ neighborhood of Q contains one of these 2^r -member isospectral sets. In particular, every fixed $\text{SU}(m)$ with $m \geq 6$ admits such a set of at least $2^{\lfloor m/6 \rfloor}$ isometry classes.

Proof. The connected block subgroups K_{ω_j} commute and have Q -orthogonal Lie algebras. Their product map is injective, since the blocks are disjoint. Apply [3, Theorem 2.21] with trivial initial isotropy and

$$t_0 = r_0 = 1, \quad t_j = s_j, \quad r_j = \frac{s_j}{1 - s_j}.$$

The parameter relations are $r_j^{-1} = t_j^{-1} - t_0^{-1}$. The theorem identifies (22.8) with the metric on $(L \times K_\omega)/\Delta K_\omega$ induced by

$$Q \oplus \bigoplus_{j=1}^r \frac{s_j}{1 - s_j} Q|_{\mathfrak{k}_{\omega_j}}. \quad (22.10)$$

This form is positive and invariant. Thus the metric is normal homogeneous and has nonnegative sectional curvature by (4.16). Lemma 21.5, applied to the second-factor form in (22.10), gives local irreducibility.

Let C_L denote the positive ambient Casimir for Q , and let $D_{K_{\omega_j}}$ denote the subgroup Casimir for the restriction of Q . In the notation of Theorem 20.1,

$$D_{K_{\omega_j}} = \frac{1}{2} C_{K_{\omega_j}}^{\text{ref}}.$$

The factor $1/2$ is unchanged by the larger ambient group, since its trace metric restricts to the same trace metric on each six-dimensional block. The scalar Laplacian is

$$\Delta_{\omega, \varepsilon} = C_L + \sum_{j=1}^r (s_j^{-1} - 1) D_{K_{\omega_j}} \quad (22.11)$$

under the right regular representation.

Fix two words ω, ω' . For each irreducible L -module V_π , choose an orthogonal decomposition

$$V_\pi^*|_{\text{SU}(6)^r} \cong \bigoplus_{\tau} M_{\pi, \tau} \otimes V_{\tau_1} \otimes \cdots \otimes V_{\tau_r},$$

where the $M_{\pi, \tau}$ are multiplicity spaces; see [44, Section 4.6(ii)] for the irreducible representations of a product. Theorem 20.1, with $n = 1$ and the common factor $1/2$, gives, for every j , a unitary $u_{\tau, j}$ on V_{τ_j} satisfying

$$u_{\tau, j} D_{K_{\omega_j}} = D_{K_{\omega'_j}} u_{\tau, j}.$$

Consequently

$$u_\pi = \bigoplus_{\tau} I_{M_{\pi, \tau}} \otimes u_{\tau, 1} \otimes \cdots \otimes u_{\tau, r}$$

intertwines all r subgroup Casimirs simultaneously. In the Peter–Weyl decomposition, apply Lemma 2.4 to the corresponding joint eigenspaces, with labels $(\pi, \mu_1, \dots, \mu_r)$ and diagonal value

$$c_L(\pi) + \sum_{j=1}^r (s_j^{-1} - 1) \mu_j.$$

This gives unitary equivalence of $\Delta_{\omega, \varepsilon}$ and $\Delta_{\omega', \varepsilon}$, including their self-adjoint domains and all eigenvalue multiplicities.

Put $c = m/2$. The subgroup K_ω is nontrivial and proper, since $0 < 9r < m^2 - 1$. Decompose each $\mathfrak{k}_{\omega_j}$ into simple ideals and Q -orthogonal central lines; in particular, $\mathfrak{so}(4)$ has two simple ideals. Apply the Ricci formula of D'Atri and Ziller, as stated in [6, Proposition 2.2], with

$$Q_0 = -B_{\text{su}(m)} = 4cQ, \quad \alpha = \frac{1}{4c}, \quad \alpha_i = \frac{s_j}{4c}$$

on every ideal or central line in the j th block. The normalization is that in the proof of Lemma 21.2. Define T_j by

$$Q(T_j X, Y) = -\frac{1}{4} B_{\mathfrak{k}_{\omega_j}}(X, Y).$$

The Ricci endomorphism on $\mathfrak{k}_{\omega_j}$ is

$$c s_j I + (s_j^{-1} - s_j) T_j. \quad (22.12)$$

On $\mathfrak{m}_\omega$, its quadratic form is

$$cQ(X, X) + \frac{1}{2} \sum_{j=1}^r (1 - s_j) \sum_i Q([E_{j,i}, X], [E_{j,i}, X]), \quad (22.13)$$

where $(E_{j,i})_i$ is a Q -orthonormal basis of $\mathfrak{k}_{\omega_j}$. The same proposition gives zero mixed Ricci terms between distinct vertical blocks and between the vertical and horizontal spaces. Since $T_j \geq 0$ and $0 < s_j < 1$, both displayed restrictions are positive definite.

By Proposition 21.6 with $n = 1$, the subgroup Killing ratios are unchanged under the block embeddings. For type A , T_j has a one-dimensional kernel and eigenvalue $3/4$ on the eight-dimensional $\mathfrak{su}(3)$ summand. For type B , its eigenvalues are 1 on $\mathfrak{sp}(1)$ and $1/2$ on $\mathfrak{so}(4)$. Set

$$R_\varepsilon = c(1 - \varepsilon) + \frac{\varepsilon}{4}.$$

Each type- A center gives the Ricci eigenvalue cs_j , and

$$cs_j = c(1 - \varepsilon) + \frac{\varepsilon j}{200r} \leq c(1 - \varepsilon) + \frac{\varepsilon}{200} < R_\varepsilon. \quad (22.14)$$

Every other vertical eigenvalue satisfies

$$\begin{aligned} cs_j + \frac{1}{2}(s_j^{-1} - s_j) &\geq c(1 - \varepsilon) + (1 - s_j) \\ &\geq c(1 - \varepsilon) + \left(1 - \frac{1}{100m}\right) \varepsilon \\ &\geq c(1 - \varepsilon) + \frac{599}{600} \varepsilon > R_\varepsilon. \end{aligned} \quad (22.15)$$

Here $(s^{-1} - s)/2 \geq 1 - s$ for $0 < s < 1$ and $m \geq 6$. Every horizontal Ricci eigenvalue is at least $c > R_\varepsilon$. Consequently the Ricci multiset below R_ε is exactly

$$\{cs_j : \omega_j = A\}, \quad (22.16)$$

with multiplicity one at each value. Since the s_j are distinct, this multiset recovers the binary word. The 2^r metrics are therefore pairwise not locally isometric.

Finally, their common volume is

$$V(\varepsilon) = \text{Vol}(Q) \prod_{j=1}^r s_j(\varepsilon)^{9/2}. \quad (22.17)$$

Its logarithmic derivative is

$$\frac{V'(\varepsilon)}{V(\varepsilon)} = \frac{9}{2} \sum_{j=1}^r \frac{-1 + j/(100mr)}{s_j(\varepsilon)} < 0.$$

Equality of spectra implies equality of volume, and hence $\varepsilon = \varepsilon'$. The converse was proved above. This proves (22.9) and the exact fiber count. For each ω , the tensor $g_{\omega,\varepsilon} - Q$ is ε times a fixed smooth left-invariant symmetric tensor. Hence $g_{\omega,\varepsilon} \rightarrow Q$ in C^∞ as $\varepsilon \downarrow 0$. The set of words is finite, so every prescribed neighborhood of Q contains all 2^r metrics for sufficiently small positive ε . Taking $r = \lfloor m/6 \rfloor$ gives $2^{\lfloor m/6 \rfloor}$ classes in each of these spectral fibers. □

23. FURTHER DETERMINATION PROBLEMS

Question 23.1. What is the least dimension of a closed simply connected manifold admitting two real-analytic metrics with positive Ricci curvature, equal scalar spectra, and different sets of closed-geodesic periods? What is the least dimension if both metrics are also required to have nonnegative sectional curvature?

Theorem 8.8 gives the upper bound eighteen for the first question.

Theorem 19.6 concerns bi-invariant metrics and graded right- P scalar data. For arbitrary metrics, the spectral-isolation question for a bi-invariant metric is [43, p. 1620]:

Question 23.2. Let g_0 be a bi-invariant metric on a compact connected Lie group J . Is there a C^∞ neighborhood $\mathcal{U}$ of g_0 such that every $g \in \mathcal{U}$ with $\text{Spec}(\Delta_g) = \text{Spec}(\Delta_{g_0})$ is isometric to g_0 ?

For the metrics of Theorem 22.5, (22.17) gives

$$\text{Vol}(g_{\omega,\varepsilon}) < \text{Vol}(Q) \quad (0 < \varepsilon < 1).$$

Thus $\text{Spec}(\Delta_{g_{\omega,\varepsilon}}) \neq \text{Spec}(\Delta_Q)$. The spectra for distinct values of ε are different by (22.9).

APPENDIX A. CLEAN QUOTIENTS AND ANALYTIC WAVE TRACES

A.1. Clean geodesic flows.

A.1.1. *Finite Riemannian quotients.* Let (M, g) be a closed Riemannian manifold, let SM be its unit tangent bundle, and let Φ_t be its geodesic flow. Say that Φ_T is clean if its fixed set is a finite disjoint union of closed embedded submanifolds and

$$T_u \text{Fix}(\Phi_T) = \ker(D_u \Phi_T - I) \quad (u \in \text{Fix}(\Phi_T)).$$

The empty fixed set is allowed. The metric is clean when this condition holds for every $T > 0$. All tangent kernels in this definition are taken in the unit tangent bundle.

Lemma A.1. *If a finite group Γ acts freely by isometries on a closed clean Riemannian manifold (M, g) , the quotient $N = M/\Gamma$, with its quotient metric, is clean.*

Proof. Let $p : SM \rightarrow SN$ be the unit-tangent covering. Fix $T > 0$ and choose q divisible by the orders of all elements of Γ . Put

$$S = \text{Fix}(\Phi_{qT}), \quad F_Y = \gamma^{-1} \Phi_T, \quad B_Y = \text{Fix}(F_Y).$$

Deck transformations commute with the flow, so $F_Y^q = \Phi_{qT}$, $B_Y \subset S$, and $F_Y|_S$ has finite order. If S is empty there is nothing to prove. Otherwise cleanliness makes it a finite union of compact embedded manifolds. After averaging a Riemannian metric on S over $\langle F_Y|_S \rangle$, the map $F_Y|_S$ is an isometry. Its fixed set is locally an embedded submanifold, and

$$T_u B_Y = \ker(D_u F_Y - I) \cap T_u S,$$

by the fixed-locus theorem and equivariance of the exponential map [63, Problem 8-32, p. 262, and Proposition 5.20, pp. 130–131]. Every point of B_Y has a connected relative neighborhood. Compactness gives a finite cover by such neighborhoods, hence finitely many components. If $D_u F_Y v = v$, then $D_u \Phi_{qT} v = v$, so cleanliness puts v in $T_u S$. Thus $T_u B_Y = \ker(D_u F_Y - I)$.

The covering identity gives

$$p^{-1} \text{Fix}(\bar{\Phi}_T) = \coprod_{\gamma \in \Gamma} B_Y.$$

The union is disjoint by freeness. Its connected components are finitely many closed embedded submanifolds. The identity $\delta B_Y = B_{\delta_Y \delta^{-1}}$ shows that Γ permutes these components. Distinct orbits of components have disjoint images under p . These images are closed by compactness, and covering charts identify them locally with the components upstairs. They are therefore embedded submanifolds. At $u \in B_Y$,

$$(D_{p(u)} \bar{\Phi}_T - I) D_u p = D_u p (D_u F_Y - I).$$

Since $D_u p$ is an isomorphism, the tangent-kernel equality upstairs descends to the required equality downstairs. $\square$

Corollary A.2. *Every round spherical space form is clean.*

Proof. On the unit sphere,

$$\Phi_t(x, v) = (\cos(t)x + \sin(t)v, -\sin(t)x + \cos(t)v), \quad |x| = |v| = 1, \quad \langle x, v \rangle = 0.$$

Its fixed set is empty for $t \notin 2\pi\mathbb{Z}$ and is the whole unit tangent bundle for $t \in 2\pi\mathbb{Z}$. In the latter case $D\Phi_t = I$. Thus the round sphere is clean. Rescaling preserves this property, and Lemma A.1 applies to every finite free isometric quotient. $\square$

A.2. The analytic Poisson inclusion. Let (X, g) be a closed real-analytic Riemannian manifold of dimension d , and write $\mathcal{L}(X, g) = \text{Lsp}(X, g)$. Let $0 \leq \lambda_0 \leq \lambda_1 \leq \dots$ be the square roots of the scalar Laplace eigenvalues, repeated according to multiplicity. Put

$$w_g(t) = \sum_j e^{it\lambda_j}, \quad c_g(t) = \sum_j \cos(t\lambda_j).$$

Weyl's law [35, Theorem 16.7, p. 224] gives $\#\{j : \lambda_j \leq R\} = O((1+R)^d)$, hence $\sum_j (1+\lambda_j)^{-m} < \infty$ for every $m > d$. Thus both series converge in $S'(\mathbb{R})$. The Fourier convention is $\hat{u}(\tau) = \int e^{-it\tau} u(t) dt$. The projection of the analytic wavefront set $\text{WF}_A(u)$ to the base is denoted by $\text{singsupp}_A(u)$; its complement is the largest open set on which u is real analytic [54, Theorem 8.4.5, p. 284].

Proposition A.3 (The analytic Poisson inclusion). *One has*

$$\text{singsupp}_A w_g \subseteq \{0\} \cup \mathcal{L}(X, g) \cup (-\mathcal{L}(X, g)). \quad (\text{A.1})$$

The same inclusion holds for the cosine trace and for both traces of the connection Laplacian on every finite-rank unitary flat bundle over X .

Proof. For $\lambda \geq 0$, set

$$s_\lambda(t) = \begin{cases} \sin(t\lambda)/\lambda, & \lambda > 0, \\ t, & \lambda = 0. \end{cases}$$

Let H be the Heaviside function, and choose an orthonormal eigenbasis (e_j) . Relative to $dV_g(y)$, put

$$E_+(t, x, y) = \sum_j H(t) s_{\lambda_j}(t) e_j(x) \overline{e_j(y)}.$$

The series converges as a distribution. For a test function ψ supported in $[-T, T] \times X \times X$, integration by parts in x bounds its j th pairing by

$$T^2(1 + \lambda_j^2)^{-m} \sup_{|t| \leq T} \|(1 + \Delta_x)^m \psi(t, \cdot, \cdot)\|_{L^2}.$$

Here $|s_\lambda(t)| \leq |t|$ and $\|e_j \otimes \bar{e}_j\|_2 = 1$. The bound is summable when $2m > d$. The identities $s_\lambda(0) = 0$ and $s'_\lambda(0) = 1$ give

$$P_x E_+ = P_y E_+ = \delta_0(t) \delta_{\text{diag}}(x, y), \quad E_+|_{t < 0} = 0, \quad P_x = \partial_t^2 + \Delta_x, \quad P_y = \partial_t^2 + \Delta_y. \quad (\text{A.2})$$

The second equation follows from $\Delta_y \overline{e_j(y)} = \lambda_j^2 \overline{e_j(y)}$. The source is the analytic density on the analytic submanifold $\{(0, x, x) : x \in X\}$, regarded as a distribution. Its analytic wavefront set is contained in its conormal bundle [54, Theorem 8.5.2 and the following paragraph, p. 297],

$$\{(0, x, x; \tau, \xi, -\xi) : (\tau, \xi) \neq 0\}. \quad (\text{A.3})$$

Over $t > 0$, analytic microlocal elliptic regularity [54, Theorem 8.6.1, p. 306], applied separately to both equations in (A.2), gives

$$(t, x, y; \tau, \xi, \eta) \in \text{WF}_A(E_+) \implies \tau^2 = |\xi|_x^2 = |\eta|_y^2. \quad (\text{A.4})$$

Such a covector is nonzero, hence $\tau \neq 0$. In particular $p_x = |\xi|_x^2 - \tau^2$ satisfies $\partial_\tau p_x = -2\tau \neq 0$ there. On every conic neighborhood with $\tau \neq 0$, the characteristic roots are simple and the slices $t = \text{constant}$ are noncharacteristic, since $p_x(dt) = -1$. Analytic singularities therefore propagate along the p_x -bicharacteristics in these neighborhoods [57, Theorem 3.3', p. 387].

Consider the p_x -bicharacteristic through a covector in (A.4). Its variables (y, η) and its temporal covector τ are constant. Since $dt/ds = -2\tau$, it crosses $t = 0$ once. Completeness of the geodesic flow extends it across this crossing. Its temporal covector remains nonzero. Each compact segment is covered by finitely many of the preceding coordinate and conic neighborhoods, so the local propagation statements apply successively along it. If the covector at $t = 0$ were absent from (A.3),

the entire segment extending to a negative time would avoid $\text{WF}_A(P_x E_+)$. Propagation would then carry membership in $\text{WF}_A(E_+)$ to that negative time, contradicting $E_+|_{t<0} = 0$. Thus the covector at $t = 0$ has spatial components $(y, -\eta)$.

Let Φ^s denote the Hamiltonian flow of $q(x, \xi) = |\xi|_x$ on $T^*X \setminus 0$. On $p_x = 0$ one has $q = |\tau|$, and Hamilton's equations, with t as parameter, give

$$\frac{d(x, \xi)}{dt} = -\frac{1}{2\tau} H_{q^2} = -\text{sgn}(\tau) H_q.$$

Consequently

$$(t, x, y; \tau, \xi, \eta) \in \text{WF}_A(E_+), \quad t > 0 \implies (x, \xi) = \Phi^{-\text{sgn}(\tau)t}(y, -\eta). \quad (\text{A.5})$$

Define $\iota(t, x) = (t, x, x)$ and $\pi(t, x) = t$ for $t > 0$. The conormal bundle of ι consists of $(t, x, x; 0, \xi, -\xi)$ with $\xi \neq 0$, and is disjoint from $\text{WF}_A(E_+)$ by (A.4). Since $\text{WF}(E_+) \subseteq \text{WF}_A(E_+)$, the pullback $\iota^* E_+$ exists by [54, Theorem 8.2.4, p. 263]. The analytic pullback estimate [54, Theorem 8.5.1, p. 296] gives

$$\text{WF}_A(\iota^* E_+) \subseteq \{(t, x; \tau, \xi + \eta) : (t, x, x; \tau, \xi, \eta) \in \text{WF}_A(E_+)\}.$$

The map π is a proper analytic submersion, and dV_g is an analytic density. Analytic integration along the fiber [57, Lemma 2.2, pp. 375–376] implies that

$$(t; \tau) \in \text{WF}_A(\pi_* \iota^* E_+)$$

requires a covector in (A.5) with $y = x$ and $\eta = -\xi$. It follows that

$$(x, \xi) = \Phi^{-\text{sgn}(\tau)t}(x, \xi), \quad \xi \neq 0.$$

The flow of q commutes with positive dilation in the cotangent fibers. A fixed point of Φ^{-t} is also a fixed point of Φ^t . Replacing ξ by $\xi/|\xi|_x$ therefore gives a closed unit-speed geodesic of period t . Hence $\pi_* \iota^* E_+$ is real analytic on $(0, \infty) \setminus \mathcal{L}(X, g)$.

For $\varphi \in C_c^\infty((0, \infty))$, set

$$a_\varphi(\lambda) = \int \varphi(t) s_\lambda(t) dt.$$

Repeated integration by parts gives, for every $m \geq 0$, $|a_\varphi(\lambda)| \leq C_{\varphi, m}(1 + \lambda)^{-m}$. The elliptic estimate and Sobolev embedding [35, Theorems 15.7 and 12.16, pp. 204 and 147] bound every C^k norm of e_j by a fixed power of $1 + \lambda_j$. Thus the series

$$A_\varphi(x, y) = \sum_j a_\varphi(\lambda_j) e_j(x) \overline{e_j(y)}$$

converges in $C^\infty(X \times X)$, and $\text{Tr } A_\varphi = \sum_j a_\varphi(\lambda_j)$ converges absolutely. Take an analytic coordinate chart $U \subset \mathbb{R}^d$, $\chi \in C_c^\infty(U)$, and $\rho \in C_c^\infty(\mathbb{R}^d)$ with $\int \rho = 1$. Put $\rho_\varepsilon(z) = \varepsilon^{-d} \rho(z/\varepsilon)$ and $\kappa(t, x, z) = (t, x, x + z)$. A nonzero conormal to $z = 0$ has $(\tau, \xi, \eta) = (0, -\xi, \xi)$ and is absent from $\text{WF}(E_+)$ by (A.4). Multiply $\kappa^* E_+$ by a compactly supported cutoff equal to one near $\text{supp}(\varphi \otimes \chi) \times \{0\}$. The restriction formula [54, Theorem 8.2.4 and (8.2.5), pp. 263–264] is an absolutely convergent Fourier integral. Since $|\hat{\rho}| \leq \|\rho\|_1$ and $\hat{\rho}(-\varepsilon\xi) \rightarrow 1$, dominated convergence gives

$$\begin{aligned} \langle \iota^* E_+, \varphi \otimes \chi \rangle &= \lim_{\varepsilon \downarrow 0} \int_U \chi(x) \int \rho_\varepsilon(z) A_\varphi(x, x + z) dz dV_g(x) \\ &= \int_U \chi(x) A_\varphi(x, x) dV_g(x). \end{aligned}$$

The coordinate pairing before restriction is with respect to $dt dV_g(x) dz$; pairing in t gives A_φ , and its smoothness gives the last equality. A finite partition of unity therefore yields

$$\langle \pi_* \iota^* E_+, \varphi \rangle = \int_X A_\varphi(x, x) dV_g(x) = \sum_j a_\varphi(\lambda_j).$$

Since $s'_\lambda = \cos(t\lambda)$, differentiation in positive time gives $c_g = \partial_t(\pi_* l^* E_+)$. Therefore c_g is real analytic off $\mathcal{L}(X, g)$ in positive time. Since $c_g(-t) = c_g(t)$, it is real analytic off $-\mathcal{L}(X, g)$ in negative time.

The series

$$W_g(z) = \sum_j e^{iz\lambda_j}, \quad \text{Im } z > 0,$$

is holomorphic, with all derivatives locally uniformly convergent, because $\sum_j (1 + \lambda_j)^m e^{-\varepsilon\lambda_j} < \infty$ for every m and $\varepsilon > 0$. For $0 < \varepsilon \leq 1$, polynomial counting gives

$$|W_g(t + i\varepsilon)| \leq \varepsilon \int_0^\infty e^{-\varepsilon R} \#\{j : \lambda_j \leq R\} dR \leq C\varepsilon^{-d}.$$

This identity includes zero eigenvalues. For a compactly supported smooth φ , rapid decay of $\widehat{\varphi}(-\lambda_j)$ permits termwise passage to the boundary and gives w_g in distributions. The analytic boundary-value theorem [85, Corollary 2.9], applied to the upper half-plane, yields

$$\text{WF}_A(w_g) \subset \{(t; \tau) : \tau > 0\}, \quad \text{WF}_A(w_g(-)) \subset \{(t; \tau) : \tau < 0\}. \quad (\text{A.6})$$

If c_g is analytic on an interval I , the identity $w_g = 2c_g - w_g(-)$ implies

$$\text{WF}_A(w_g)|_I \subseteq \text{WF}_A(w_g(-))|_I.$$

Since a nonzero real number cannot be both positive and negative, (A.6) gives $\text{WF}_A(w_g)|_I = \emptyset$. This proves (A.1).

Let $E \rightarrow X$ be a finite-rank unitary flat bundle. Local parallel unitary frames have constant transition matrices and define an analytic bundle structure. In these frames the connection Laplacian is the scalar analytic Laplacian on each component. Its retarded kernel is a section of $E \boxtimes E^*$ and satisfies (A.2), with the y operator acting on E^* . Equations (A.4)–(A.5) hold componentwise. Diagonal restriction followed by the contraction $E_x \otimes E_x^* \rightarrow \mathbb{C}$ is analytic, so the same proper pushforward gives (A.1) for the cosine trace. The nonnegative discrete connection spectrum satisfies a polynomial counting bound; applying (A.6) to this spectrum gives the half-wave inclusion. $\square$

Remark A.4. For every closed Riemannian manifold, $\mathcal{L}(X, g)$ is closed in $(0, \infty)$. If $t_j \rightarrow t > 0$ and $\Phi^{t_j}(v_j) = v_j$ in S^*X , compactness gives a subsequence with $v_j \rightarrow v$. Continuity of the flow then gives $\Phi^t(v) = v$. Thus every positive number outside $\mathcal{L}(X, g)$ has an open interval disjoint from that set.

Corollary A.5. *Let $E_X \rightarrow (X, g)$ and $E_Y \rightarrow (Y, h)$ be finite-rank unitary flat bundles over closed real-analytic Riemannian manifolds. Suppose their connection Laplacians are isospectral, and write*

$$w(t) = \text{Tr}(e^{it\sqrt{\Delta_{E_X}}}) = \text{Tr}(e^{it\sqrt{\Delta_{E_Y}}}).$$

The distribution w is real analytic on every open interval $I \subset (0, \infty)$ disjoint from $\mathcal{L}(X, g)$. If $\ell \in \mathcal{L}(Y, h) \setminus \mathcal{L}(X, g)$, then w is real analytic on an open interval about ℓ , and

$$\ell \notin \text{singsupp}_A w, \quad \text{singsupp}_A w \cap (0, \infty) \subseteq \mathcal{L}(Y, h).$$

The trivial line bundles give the scalar case.

Proof. Proposition A.3, applied to E_X , makes w analytic on I . If $\ell \in \mathcal{L}(Y, h) \setminus \mathcal{L}(X, g)$, Remark A.4 gives such an interval containing ℓ . The same proposition, applied to E_Y , gives the displayed inclusion, which is strict because it omits ℓ . $\square$

A.3. Spherical space forms. Prüfer's shifted Poisson formula [83, Corollary 4] gives the ordinary scalar formula of Corollary A.8 by Lemma A.7. Spectral determination of the spherical period set is already contained in [83, Corollary 5]; for odd-dimensional space forms, it also follows from [55, Proposition 2.7 and Corollary 2.8, pp. 83–84].

Let $\Gamma \leq \text{O}(n+1)$ be finite and act freely on the unit sphere, where $n \geq 2$. Endow $M = \mathbb{S}^n/\Gamma$ with its quotient metric, and put

$$\rho = \frac{n-1}{2}, \quad \mathcal{O}(\Gamma) = \{\text{ord}(\gamma) : \gamma \in \Gamma\}, \quad C_\rho(t) = \text{Tr} \cos(t\sqrt{\Delta_M + \rho^2}).$$

Theorem A.6 (Prüfer). *One has*

$$\text{sing supp } C_\rho = \bigcup_{q \in \mathcal{O}(\Gamma)} \frac{2\pi}{q} \mathbb{Z} = \{0\} \cup \text{Lsp}(M) \cup (-\text{Lsp}(M)). \quad (\text{A.7})$$

Proof. Put $L_\pm(M) = \text{Lsp}(M) \cup (-\text{Lsp}(M))$. By [83, Corollary 1, p. 135],

$$L_\pm(M) = \left\{ \frac{2\pi k}{q} : q \in \mathcal{O}(\Gamma), k \in \mathbb{Z} \setminus \{0\}, \gcd(k, q) = 1 \right\}.$$

If $d \mid q$, an element of order q has a power of order d . Reduction of k/q to lowest terms therefore gives

$$\text{Lsp}(M) = \bigcup_{q \in \mathcal{O}(\Gamma)} \frac{2\pi}{q} \mathbb{N}_{>0}. \quad (\text{A.8})$$

By [83, Corollary 4, p. 152], $\text{sing supp } C_\rho = \{0\} \cup L_\pm(M)$. $\square$

Lemma A.7. *Put*

$$V_\rho(t) = \text{Tr } e^{it\sqrt{\Delta_M + \rho^2}}, \quad W_M(t) = \text{Tr } e^{it\sqrt{\Delta_M}}.$$

Then

$$\text{sing supp } C_\rho = \text{sing supp } V_\rho = \text{sing supp } W_M.$$

Proof. Let μ_j be the Laplace eigenvalues, repeated according to multiplicity, and put $\nu_j = \sqrt{\mu_j + \rho^2}$. Their polynomial counting bound makes V_ρ tempered. The holomorphic series $\sum_j e^{iz\nu_j}$ on $\text{Im } z > 0$ has distributional boundary value V_ρ . By [85, Corollary 2.9],

$$\text{WF}_A(V_\rho) \subset \{(t; \tau) : \tau > 0\}.$$

Thus V_ρ and $\overline{V_\rho}$ have disjoint smooth wavefront sets. Since $2C_\rho = V_\rho + \overline{V_\rho}$ and complex conjugation preserves singular support,

$$\text{sing supp } C_\rho = \text{sing supp } V_\rho. \quad (\text{A.9})$$

Fix a bounded open interval $I \subset \mathbb{R}$ and choose $\theta \in C_c^\infty(\mathbb{R}; \mathbb{R})$ equal to t near $\bar{I}$. Choose a real smooth function b , zero for sufficiently negative ξ , such that

$$b(\xi) = \sqrt{\xi^2 - \rho^2} - \xi = -\frac{\rho^2}{\sqrt{\xi^2 - \rho^2} + \xi}$$

for all sufficiently large positive ξ . Then $\partial_\xi^j b = O(\langle \xi \rangle^{-1-j})$ for every $j \geq 0$. The symbol

$$a(t, \xi) = e^{i\theta(t)b(\xi)}$$

is globally of order zero, $a - 1$ has order -1 , and $|a| = 1$. Hence $A = \text{Op}(a)$ is elliptic. By [69, Lemma 2.14 and Theorem 5.7, pp. 16 and 47], it acts continuously on $S'(\mathbb{R})$. Left quantization gives

$$Ae^{it\nu_j} = e^{it\nu_j + i\theta(t)b(\nu_j)}.$$

On I this is $e^{it\sqrt{\mu_j}}$ for every sufficiently large ν_j . Applying A to the convergent series in $S'(\mathbb{R})$ therefore gives

$$(AV_\rho - W_M)|_I \in C^\infty(I), \quad (\text{A.10})$$

since only finitely many frequencies are excluded.

By [69, Lemma 2.22, pp. 19–20], write $A = P + R$, where $\text{supp } K_P \subset \{|t - u| \leq \varepsilon\}$ for some $\varepsilon > 0$ and R has a symbol of order $-\infty$. The kernel of R belongs to $C^\infty(\mathbb{R}_t; \mathcal{S}(\mathbb{R}_u))$, so R maps tempered distributions to smooth functions. Choose $\psi \in C_c^\infty(\mathbb{R})$ equal to one on a neighborhood of $\bar{I} + [-\varepsilon, \varepsilon]$. Then

$$(AV_\rho - A(\psi V_\rho))|_I = (R((1 - \psi)V_\rho))|_I \in C^\infty(I).$$

The distribution ψV_ρ has compact support. Since A is elliptic, [69, Lemma 6.9, p. 52] gives

$$\text{sing supp}(AV_\rho) \cap I = \text{sing supp } V_\rho \cap I.$$

Together with (A.10), this proves the second equality, since I is arbitrary. $\square$

Corollary A.8. *For every spherical space form M as above,*

$$\text{sing supp } W_M = \text{singsupp}_A W_M = \bigcup_{q \in \mathcal{O}(\Gamma)} \frac{2\pi}{q} \mathbb{Z}. \quad (\text{A.11})$$

For a sphere of radius R , the numerator is $2\pi R$. Two scalar-isospectral spherical space forms have the same ordinary length set.

Proof. The smooth singular-support equality follows from Theorem A.6 and Lemma A.7. The union in (A.11) is $\{0\} \cup \text{Lsp}(M) \cup (-\text{Lsp}(M))$, by (A.8). It contains the analytic singular support by Proposition A.3, and the analytic singular support contains the smooth singular support. This proves the analytic equality. Multiplication of the metric by R^2 changes $W_M(t)$ to $W_M(t/R)$ and every geodesic period T to RT . Equality of the scalar spectra implies equality of the half-wave traces and hence equality of their positive singular supports. $\square$

For odd-dimensional spherical space forms, equality of the element-order sets under scalar isospectrality is [55, Corollary 2.8].

Remark A.9. Let $\Gamma \neq \{I\}$, and let reg be its regular unitary representation. Equation (5.3) at the identity and the decomposition of reg into irreducible summands give, by additivity,

$$W_{\text{reg}} = \sum_{\rho \in \text{Irr}(\Gamma)} (\dim \rho) W_\rho = D_I = W_{\mathbb{S}^n}.$$

The positive periods of the covering sphere are $2\pi\mathbb{N}_{>0}$. Proposition A.3, applied to that sphere, makes this trace real analytic on $\mathbb{R} \setminus 2\pi\mathbb{Z}$. For every $q > 1$ in $\mathcal{O}(\Gamma)$, the quotient period $2\pi/q$ lies outside $2\pi\mathbb{Z}$.

APPENDIX B. COMPACT THREE-GRADINGS AND CENTRAL INTERSECTIONS

B.1. Simple compact three-gradings. Let $\mathfrak{s}$ be a compact simple Lie algebra and let $0 \neq B \in \mathfrak{s}$. Suppose that the eigenvalues of $\text{ad } B$ on $\mathfrak{s}_{\mathbb{C}}$ belong to $\{0, i, -i\}$. Put $\mathfrak{k} = \mathfrak{c}_s(B)$ and $T = \ker(\text{ad } B - i)$. Compact conjugation gives

$$\mathfrak{s}_{\mathbb{C}} = \bar{T} \oplus \mathfrak{k}_{\mathbb{C}} \oplus T.$$

Moreover, $[T, T] = 0$, since $2i$ is not an adjoint eigenvalue. Choose a Cartan subalgebra containing B and positive roots for which $-iB$ is dominant. Then $\mathfrak{p} = \mathfrak{k}_{\mathbb{C}} \oplus T$ is the standard parabolic whose Levi roots vanish on B , and its nilradical is T . The classification of cominuscule parabolics [62, Proposition 2.1, Table 1 and Lemma 2.3] gives $\mathfrak{z}(\mathfrak{k}) = \mathbb{R}B$, the irreducibility of T , and the following table. The third column gives the $[\mathfrak{k}, \mathfrak{k}]$ module; B acts by i .

$\mathfrak{s}$	$[\mathfrak{k}, \mathfrak{k}]$	T	$\dim_{\mathbb{C}} T$
$\mathfrak{su}(p+q)$	$\mathfrak{su}(p) \oplus \mathfrak{su}(q)$	$\mathbb{C}^p \otimes (\mathbb{C}^q)^*$	pq
$\mathfrak{sp}(m)$	$\mathfrak{su}(m)$	$\text{Sym}^2 \mathbb{C}^m$	$m(m+1)/2$
$\mathfrak{so}(2m)$	$\mathfrak{su}(m)$	$\Lambda^2 \mathbb{C}^m$	$m(m-1)/2$
$\mathfrak{so}(d+2)$	$\mathfrak{so}(d)$	$\mathbb{C}^d$	d
$\mathfrak{e}_6$	$\mathfrak{so}(10)$	half-spin	16
$\mathfrak{e}_7$	$\mathfrak{e}_6$	minuscule	27.

(B.1)

Here p, q, m, d are positive integers, and only simple ambient algebras are included. The exceptional modules and their dimensions are given in [37, §31.3]. The low-rank identities $D_2 = A_1 \times A_1$, $D_3 = A_3$ and $B_2 = C_2$ are as in [37, Remark 22.11]. Isomorphisms of graded pairs preserve the normalized element B .

Equip T with the Hermitian form induced by the negative Killing form of $\mathfrak{s}$. The homomorphism $r_T : \mathfrak{k} \rightarrow \mathfrak{u}(T)$ is injective. Indeed, if $\mathfrak{a} = \ker r_T$, then $[\mathfrak{a}_{\mathbb{C}}, T] = [\mathfrak{a}_{\mathbb{C}}, \bar{T}] = 0$ by compact conjugation. Thus $\mathfrak{a}_{\mathbb{C}}$ is an ideal of $\mathfrak{s}_{\mathbb{C}}$ contained in $\mathfrak{k}_{\mathbb{C}}$, and simplicity gives $\mathfrak{a} = 0$.

Lemma B.1. *Let $n \geq 2$ and let V be the defining complex $\mathfrak{su}(n)$ module. Every nontrivial finite-dimensional complex $\mathfrak{su}(n)$ module has dimension at least n . An irreducible module of dimension n is V or V^* . A nontrivial four-dimensional complex $\mathfrak{su}(3)$ module is $V \oplus \mathbb{C}$ or $V^* \oplus \mathbb{C}$.*

Proof. By complete reducibility [37, Theorem 16.4], it suffices to consider irreducible modules. Let $\lambda = \sum_{j=1}^{n-1} a_j \omega_j \neq 0$ be a dominant integral weight. If $a_j > 0$, the Weyl dimension formula and the fundamental representations of type A [37, Proposition 28.1 and §28.3] give

$$\dim L_{\lambda} \geq \dim L_{\omega_j} = \binom{n}{j} \geq n.$$

Equality in the first inequality holds only for $\lambda = \omega_j$, and in the second only for $j = 1, n-1$. For $n = 3$, the same formula is

$$\dim L_{a\omega_1+b\omega_2} = \frac{(a+1)(b+1)(a+b+2)}{2}.$$

It is at least six when $a+b \geq 2$. Complete reducibility gives the last assertion. $\square$

Lemma B.2. *Let $n \geq 2$, let V be the defining complex $\mathfrak{su}(n)$ module, and let $\phi : \mathfrak{su}(n) \rightarrow \mathfrak{k}$ be a homomorphism.*

- (1) *If $T \simeq V$ or V^* as a $\mathfrak{su}(n)$ module, then the graded pair is $\mathfrak{su}(n+1)/\mathfrak{u}(n)$.*
- (2) *Suppose $T = T_+ \oplus T_-$ with $T_+ \simeq V$ and $T_- \simeq V^*$. Let A_T commute with $r_T \circ \phi$, be skew-adjoint for the fixed Hermitian form on T , and act by i on T_+ and by $-i$ on T_- . Then the graded pair is*

$$\mathfrak{su}(2n+1)/\mathfrak{u}(2n) \quad \text{or} \quad \mathfrak{so}(2n+2)/(\mathfrak{so}(2n) \oplus \mathfrak{so}(2)).$$

In the second case there is a unique $A \in \mathfrak{k}$ with $r_T(A) = A_T$. For each of the two graded pairs, the homomorphisms

$$\mathfrak{su}(n) \oplus \mathbb{R}^2 \longrightarrow \mathfrak{s}, \quad (X, a, b) \longmapsto \phi(X) + aA + bB,$$

arising in this way are conjugate by an automorphism of the compact Lie algebra $\mathfrak{s}$ fixing B .

Proof. Since the action on T is nontrivial, simplicity and perfectness of $\mathfrak{su}(n)$ give

$$\ker \phi = 0, \quad \phi(\mathfrak{su}(n)) \subset [\mathfrak{k}, \mathfrak{k}].$$

In the first row of (B.1), a nonzero projection to either defining factor makes its dimension at least n , by Lemma B.1. Order the factors so that $p \leq q$. If $pq = n$, then $(p, q) = (1, n)$. If $pq = 2n$, then $(p, q) = (1, 2n)$ or $(2, n)$. For $n \geq 3$, the second possibility has tangent module $2V$ or $2V^*$ and is excluded. For $n = 2$ and $(p, q) = (2, 2)$, nonzero projections to both factors give $V \otimes V \simeq \text{Sym}^2 V \oplus \mathbb{C}$, which is excluded. If only one projection is nonzero, the tangent module is $2V$, and the pair is the quadric pair under $A_3 = D_3$.

In the symplectic row, the defining m -dimensional module is nontrivial, so $m \geq n$. The equation $m(m+1)/2 = n$ has no solution with $n \geq 2$. The equation $m(m+1)/2 = 2n$ implies $n \leq 3$ and leaves only $(m, n) = (3, 3)$. Its tangent module is $\text{Sym}^2 V$ or $\text{Sym}^2 V^*$, hence is irreducible [37, Example 28.3], and is excluded.

In the alternating row, again $m \geq n$. The equation $m(m-1)/2 = n$ leaves only $(m, n) = (3, 3)$, the projective pair under $D_3 = A_3$. The equation $m(m-1)/2 = 2n$ implies $n \leq 5$ and leaves $(m, n) = (4, 3), (5, 5)$. The second has irreducible tangent module $\Lambda^2 V$ or $\Lambda^2 V^*$ [37, §28.3], and is excluded. For $(m, n) = (4, 3)$, Lemma B.1 identifies the defining module with $V \oplus \mathbb{C}$ or $V^* \oplus \mathbb{C}$; its exterior square is $V \oplus V^*$. The three outer nodes of D_4 are permuted by diagram automorphisms. Their lifts preserve the compact real form [37, §§39.1, 39.4] and carry the normalized grading elements to one another. Thus this pair is the quadric pair with $n = 3$.

In the vector row, $d = n$ is impossible because $\phi(\mathfrak{su}(n)) \subset \mathfrak{so}(n)$ would imply

$$n^2 - 1 \leq \frac{n(n-1)}{2},$$

contrary to $n \geq 2$. The case $d = 2n$ is the quadric pair. For $\mathfrak{e}_6$, the tangent dimensions force $n = 16$ or 8 , whereas $n^2 - 1 > 45 = \dim \mathfrak{so}(10)$. For $\mathfrak{e}_7$, the first case would give $n = 27$ and $n^2 - 1 > 78 = \dim \mathfrak{e}_6$, while $2n = 27$ is impossible.

In the projective pair, r_T identifies $\mathfrak{k}$ with $\mathfrak{u}(T)$, so $A_T = r_T(A)$ for a unique A . Unitary intertwiners on the corresponding T_+ and T_- give a unitary map U intertwining the two tangent representations and their operators A_T . Conjugation by $\text{diag}(U, 1)$ on $\mathfrak{su}(2n+1)$ induces U . This conjugation fixes B .

In the quadric pair, identify $T = \mathbb{C}^{2n}$ with the complexified real defining module. Let $q(x, y) = x^t y$; the Hermitian form is a positive multiple of the standard one. The restrictions of q to T_+ and T_- are zero, since $\text{Sym}^2 V$ and $\text{Sym}^2 V^*$ are nontrivial irreducible modules [37, Example 28.3]. The eigenvalues $i, -i$ therefore give

$$q(A_T x, y) + q(x, A_T y) = 0 \quad (x, y \in T).$$

Thus $A_T^t = A_T^* = -A_T$, so A_T is real and belongs to $\mathfrak{so}(2n)$. It is induced by a unique $A \in \mathfrak{so}(2n) \subset \mathfrak{k}$. Let σ be conjugation relative to the real defining module; it exchanges T_+ and T_- . For two such tangent representations, choose a unitary $\mathfrak{su}(n)$ intertwiner U_+ between their positive A_T eigenspaces. The map equal to U_+ on T_+ and to $\sigma U_+ \sigma$ on T_- is unitary, commutes with σ , and intertwines both $\mathfrak{su}(n)$ and A_T . It is therefore the complexification of a real orthogonal map. Extend this map by the identity on the grading plane. Conjugation gives an automorphism of $\mathfrak{so}(2n+2)$ fixing B , with either sign of the determinant.

In both pairs, injectivity of r_T gives $[A, \phi(\mathfrak{su}(n))] = 0$ and makes each constructed automorphism intertwine the displayed homomorphisms pointwise. $\square$

B.2. Central intersections. For $n \geq 1$, the embeddings of (2.1) satisfy

$$Z(G_n) \cap i_n(P_n) = \{I, -I\}, \quad Z(H_n) \cap j_n(P_n) = \{(1, I), (-1, -I)\}. \quad (\text{B.2})$$

Indeed, $i_n(A, z) = \zeta I_{2n+1}$ implies $A = \zeta I_n = \bar{A}$ and $z = \zeta$, hence $\zeta = \pm 1$. If $j_n(A, z)$ is central, write $c(A) = \varepsilon I_{2n}$ and $r(A \oplus z) = \delta I_{2n+2}$ with $\varepsilon, \delta \in \{1, -1\}$. The first equality gives $A = \varepsilon I_n$; the second gives $A = \delta I_n$ and $z = \delta$. Thus $\varepsilon = \delta$ and $z = \varepsilon$. Conversely, $(A, z) = (\varepsilon I_n, \varepsilon)$, for $\varepsilon = \pm 1$, has the two central images in (B.2).

APPENDIX C. CENTRAL QUOTIENTS AND CHARACTERISTIC CLASSES

C.1. The subgroup lattice and its factor count. For finite abelian C , let $\mathcal{J}(C)$ be its nontrivial cyclic subgroups of prime-power order, and put $j(C) = |\mathcal{J}(C)|$.

Lemma C.1. *The nonzero join-irreducibles of $\text{Sub}(C)$ are exactly $\mathcal{J}(C)$. Choose a generator c_H of each such H . Every $D \leq C$ is generated by the c_H with $H \leq D$.*

Proof. The join-irreducible subgroups of a finite group are its nontrivial cyclic subgroups of prime-power order [1, §1.3, p. 3]. Since $\text{Sub}(C)$ is finite, every $D \leq C$ is the join of its join-irreducibles [78, Lemma 2.8]. Thus $D = \langle c_H : H \in \mathcal{J}(C), H \leq D \rangle$. $\square$

Theorem C.2. *Let E be finite and $1 \neq C \leq Z(E)$. One common subgroup in a product of $r = j(C)$ pairs (G_n, H_n) realizes every marked group E/D , $D \leq C$, with the spectral equalities, curvature, covers and flat-twist spectra of Theorem 10.4. Its all-unitary and all- H groups are E and E/C . Inclusions $D \leq D'$ give the specified maps $E/D \rightarrow E/D'$ on lifted-stabilizer component groups.*

Proof. Apply Theorem 10.4 to the tuple $(c_H)_{H \in \mathcal{J}(C)}$. For $I(D) = \{H \in \mathcal{J}(C) : H \leq D\}$, Lemma C.1 gives $C_{I(D)} = D$, and $D \leq D'$ gives $I(D) \subseteq I(D')$. $\square$

The injection $D \mapsto I(D)$ preserves intersections; its joins are obtained by generating a subgroup from a union and then applying I . Its image need not be closed under union. Taking $C = Z(E)$ realizes all central quotients of E ; for $C = 1$, take one identity entry.

Proposition C.3. *If every subgroup of C is generated by a subtuple of a fixed tuple, then that tuple has length at least $j(C)$. Hence Theorem C.2 has the least number of factors among realizations with this property. Write $C_{(p)} = \prod_{a=1}^{d_p} C_{p^{\lambda_a}}$ for the Sylow p -subgroup of C , with $\lambda_a \geq 1$, and set $S_p(k) = \sum_a \min(k, \lambda_a)$, with $S_p(0) = 0$. Then*

$$j(C) = \sum_{p||C|} \sum_{k=1}^{\max_a \lambda_a} \frac{p^{S_p(k)} - p^{S_p(k-1)}}{p^{k-1}(p-1)}. \quad (\text{C.1})$$

In particular $j((\mathbb{Z}/p)^d) = (p^d - 1)/(p - 1)$ and $j(C_m) = \sum_{p|m} v_p(m)$.

Proof. A subtuple generating a cyclic p -group H must contain a generator of H : otherwise all its entries lie in the unique maximal subgroup of H . A coordinate generates only one subgroup, so there must be at least $j(C)$ coordinates. Apply [93, Corollary 2, equation (9)] to each Sylow subgroup of C and sum over the nontrivial prime-power orders. This gives (C.1); the two specializations follow by substituting $\lambda_a = 1$ and $d_p = 1$, respectively. $\square$

Proposition C.4. *Let $K \leq \prod_{j=1}^r P_{n_j}$ be closed, and suppose its all-unitary quotient has finite fundamental group E . Let $z : K \rightarrow \mathbb{T}^r$ be the restriction of the product of the ambient unitary determinants, $\Gamma = \pi_0 K$, and $C = \ker(E \rightarrow \Gamma)$. There are central generators $c_1, \dots, c_r$ of C such that*

$$\pi_0(\ker z|_{I^c}) = E / \langle c_j : j \in I \rangle.$$

If the vertex boundary maps identify their fundamental groups with these component groups, and every subgroup of C is generated by a subtuple of $(c_1, \dots, c_r)$, then $r \geq j(C)$.

Proof. The homotopy sequence makes $\text{im}(\pi_1(K^0) \rightarrow \mathbb{Z}^r)$ a finite-index subgroup. Thus $z(K^0) = \mathbb{T}^r$. Put $L = \ker z$. The simply connected product of the special unitary groups acts transitively on the all-unitary quotient, with stabilizer L , so $E \simeq \pi_0 L$. The homotopy sequence of $L \rightarrow K \rightarrow \mathbb{T}^r$ gives

$$\mathbb{Z}^r \xrightarrow{\partial} E \longrightarrow \Gamma \longrightarrow 1.$$

For $l \in L$, conjugation by l induces the identity on $\mathbb{T}^r$. Naturality of ∂ gives $l\partial(a)l^{-1} = \partial(a)$ in $\pi_0 L$ for every $a \in \mathbb{Z}^r$. Hence $C = \text{im } \partial$ is central, with generators $c_j = \partial e_j$.

For $I \subseteq [r]$, the sequence $L \rightarrow \ker z|_{I^c} \rightarrow \mathbb{T}^I$ is onto on the right because $z(K^0) = \mathbb{T}^r$. Its boundary homomorphism is $\partial|_{\mathbb{Z}^I}$, and therefore

$$\pi_0(\ker z|_{I^c}) \simeq E / \langle c_j : j \in I \rangle.$$

At this vertex, the boundary identifies this component group with the fundamental group whenever

$$\{0\} \times (C_2)^I \subseteq \text{im}(\pi_1(K^0) \longrightarrow \mathbb{Z}^{I^c} \times (C_2)^I).$$

Indeed, a loop with zero unitary winding represents a class from $\pi_1((\ker z|_{I^c})^0)$, by the homotopy sequence over $\mathbb{T}^{I^c}$; the displayed inclusion kills every remaining orthogonal fundamental-group generator. Proposition C.3 gives the lower bound. $\square$

The markings are the quotient homomorphisms from the fixed group E ; the lower bound applies when every subgroup of C is generated by a subtuple of the specified coordinates.

C.2. Compatible quotient diagrams. Let $\mathcal{D}$ be a nonempty finite diagram of groups over a finite group Γ . Its vertex maps $p_v : E_v \rightarrow \Gamma$ have finitely generated central kernels, and its arrows $f_{vw} : E_v \rightarrow E_w$ satisfy $p_w f_{vw} = p_v$. Set

$$L = \{(e_v) : p_v(e_v) \text{ is independent of } v, f_{vw}(e_v) = e_w \text{ for every arrow}\}. \quad (\text{C.2})$$

Theorem C.5. *There is a central extension $E \twoheadrightarrow \Gamma$ with finitely generated kernel and compatible epimorphisms over Γ $E \twoheadrightarrow E_v$ exactly when every projection $L \rightarrow E_v$ is onto. One may then take $E = L$, and the diagram is realized on the marked component groups of one family in Theorem 10.4, with its spectral equalities, curvature, covers and flat-twist spectra.*

Proof. Compatible epimorphisms give $E \rightarrow L$ whose coordinate images are all E_v , proving necessity. Conversely, surjectivity of one projection makes the common-value map $L \rightarrow \Gamma$ onto. Its kernel $C \leq \prod_v \ker p_v$ is central and finitely generated abelian. Let $D_v = \ker(L \rightarrow E_v) \leq C$. Then $L/D_v \simeq E_v$ with the prescribed marking, and an arrow gives $D_v \leq D_w$ and the required quotient map. Concatenate generating tuples for C and all D_v . The indices of all entries lying in D_v generate D_v and are nested along arrows. Apply Theorem 10.4; for $C = 1$, take one identity entry. Repeated vertices may correspond to the same manifold. For nontrivial finite C , Theorem C.2 gives the sharper full lattice factor count. Ricci curvature is positive precisely at finite fundamental groups. $\square$

Corollary C.6. *If every arrow is onto and the underlying undirected multigraph, retaining each arrow, is a forest, every projection in (C.2) is onto.*

Proof. Fix a vertex v_0 and $e_{v_0} \in E_{v_0}$, and put $\gamma = p_{v_0}(e_{v_0})$. Root its tree at v_0 . Inductively, if w has parent v and e_v is chosen, set $e_w = f_{vw}(e_v)$ when the arrow is $v \rightarrow w$; for an arrow $w \rightarrow v$, choose $e_w \in f_{vw}^{-1}(e_v)$. In both cases $p_w(e_w) = \gamma$. For each other tree, choose a root element above γ and apply the same induction. Every edge joins a vertex to its parent, so all arrow equations hold. The resulting element of L projects to e_{v_0} . $\square$

Example C.7. Take four copies A, B, C, D of C_3 over $\Gamma = 1$, with arrows $A \rightarrow B, A \rightarrow D, C \rightarrow B$ equal to the identity and $C \rightarrow D$ equal to minus the identity. The directed graph is acyclic, but a coherent tuple satisfies $a = b = c$ and $d = a = -c$, hence is zero. None of the limit projections is onto. Separate realizability of the arrows does not give a compatible realization from one source.

C.3. Central characters and projective flatness. Let J be compact and connected with bi-invariant metric Q , and let $K \leq J$ be closed with $T = K^0 \leq Z(K)$. Thus T is a torus and $\Gamma = K/T$ is finite. Put $\mathfrak{m} = \mathfrak{t}^{\perp_Q}$; the canonical principal connection is $\text{pr}_t(g^{-1}dg)$.

Proposition C.8. *An irreducible $\rho : K \rightarrow U(d)$ restricts to $\rho(t) = \chi(t)I_d$ on T . Write $d\chi = i\ell_\chi$ and let $Z_\chi \in \mathfrak{t}$ be its Q -dual. Let β_χ be the invariant real two-form whose value at eK is the expression below for $X, Y \in \mathfrak{m}$. The canonical curvature and vertical Casimir are*

$$F_\rho = i\beta_\chi I_d, \quad \beta_\chi(X, Y) = -\ell_\chi(\text{pr}_t[X, Y]) = -Q([Z_\chi, X], Y), \quad C_\rho = \|\ell_\chi\|_{Q^*}^2 I_d.$$

The projective and endomorphism connections are flat with holonomy factoring through $\pi_1(J/K) \rightarrow \Gamma$. If τ is another irreducible unitary K -representation with the same central character, the connection on $\text{Hom}(E_\rho, E_\tau)$ is flat. Moreover, in the orthonormal wedge convention,

$$\beta_\chi = 0 \iff Z_\chi \in \mathfrak{z}(\mathfrak{j}), \quad \|\beta_\chi\|^2 = -\frac{1}{2}B_j(Z_\chi, Z_\chi).$$

For every ambient product in Theorem 4.3, $\mathfrak{p} \cap \mathfrak{z}(\mathfrak{j}) = 0$; its irreducible canonical connection is flat exactly when $\chi = 1$.

Proof. Schur's lemma gives the scalar T action. Maurer–Cartan gives $\Omega(X, Y) = -\text{pr}_t[X, Y]$, and applying $d\rho = i\ell_\chi I_d$ proves the curvature formula. The orthonormal sum of $-d\rho(u)^2$ gives the Casimir. The projective, endomorphism, and equal-character Hom representations are trivial on T ; their connections are induced from the discrete Γ cover $J/T \rightarrow J/K$. Since $[Z_\chi, \mathfrak{t}] = 0$ and $[Z_\chi, \mathfrak{m}] \subseteq \mathfrak{m}$, the curvature vanishes exactly when $\text{ad } Z_\chi = 0$. For an orthonormal basis e_a of $\mathfrak{m}$,

$$\|\beta_\chi\|^2 = \frac{1}{2} \sum_{a,b} Q([Z_\chi, e_a], e_b)^2 = \frac{1}{2} \|\text{ad } Z_\chi\|_{\text{HS}}^2 = -\frac{1}{2}B(Z_\chi, Z_\chi).$$

The H_n algebra is semisimple. If $\text{diag}(X, \bar{X}, it) = icI$ in the G_n embedding, then $X = icI_n$ and $\bar{X} = icI_n$. Conjugation gives $c = 0$, and then $t = 0$. Thus $\mathfrak{p} \cap \mathfrak{z}(\mathfrak{j}) = 0$ in every ambient product, and a character of connected T has zero differential exactly when it is trivial. $\square$

Corollary C.9. *Let $K \leq \prod_j P_{n_j}$ be closed with $K^0 \leq Z(K)$. For each irreducible unitary K -representation, C_ρ and $\|\beta_\chi\|^2$ are independent of the ambient choices and agree for matched positive factor scales.*

Proof. The isotropy metric is common, and Proposition 4.5 identifies its adjoint actions, hence their Killing forms. Explicitly,

$$B_{\mathfrak{g}_n}|_{\mathfrak{p}_n}((X, it), (Y, is)) = B_{\mathfrak{h}_n}|_{\mathfrak{p}_n}((X, it), (Y, is)) = 4(2n + 1) \operatorname{Re} \operatorname{Tr}(XY) - 4n ts.$$

The common dual vector Z_χ and the preceding proposition give the same curvature norm, while the common isotropy metric gives the same Casimir. Matched scales change the dual vector equally on both sides. $\square$

C.4. Projective multipliers.

Proposition C.10. *Fix a character $\chi : T \rightarrow U(1)$ and a normalized section $s : \Gamma \rightarrow K$. Set $c(\gamma, \delta) = s(\gamma)s(\delta)s(\gamma\delta)^{-1} \in T$, and put $\alpha_\chi = \chi \circ c$. Representations with central character χ correspond to α_χ -projective representations of Γ , by*

$$R_\gamma = \rho(s(\gamma)), \quad R_\gamma R_\delta = \alpha_\chi(\gamma, \delta) R_{\gamma\delta}, \quad \rho(ts(\gamma)) = \chi(t) R_\gamma.$$

Tensor products multiply the multipliers and duals invert them.

Proof. Form the central extension

$$K_\chi = (K \times U(1)) / \{(t, \chi(t)^{-1}) : t \in T\}.$$

The representations of K with central character χ identify with the representations of K_χ of central weight one by $\hat{\rho}([k, \lambda]) = \lambda \rho(k)$. In the coordinates determined by $[s(\gamma), 1]$, multiplication is

$$(\lambda, \gamma)(\mu, \delta) = (\lambda\mu\alpha_\chi(\gamma, \delta), \gamma\delta).$$

Mackey's correspondence [70, Theorem 2.1, p. 270], applied to this extension of the finite group Γ , gives the displayed formulas and preserves intertwiners and irreducibility. His convention is $L_{\gamma\delta} = \sigma(\gamma, \delta)L_\gamma L_\delta$, so $\sigma = \alpha_\chi^{-1}$ [70, §1, p. 267]. Tensor products and duals have the stated multipliers by these formulas. $\square$

Proposition C.11. *Let E be finite, $C \leq Z(E)$, and $i : C \hookrightarrow T$ an embedding in a torus. Set $K = (E \times T) / \{(c, i(c)^{-1}) : c \in C\}$. Let σ be an irreducible unitary representation of E and write $\sigma(c) = \eta(c)I$ for $c \in C$. The character $\eta : C \rightarrow U(1)$ extends along i to a character χ of T . Then $\rho([e, t]) = \sigma(e)\chi(t)$ is irreducible, and its projective E/C multiplier is the push-forward of the extension cocycle by η .*

Proof. The subgroup $i(C)$ is closed in the compact abelian group T . The character-extension theorem [52, Corollary 2.33(ii)] therefore gives $\chi : T \rightarrow U(1)$ with $\chi \circ i = \eta$. The diagonal relation acts by $\eta(c)\chi(i(c))^{-1} = 1$, so ρ descends to K . Its restriction to the embedded subgroup E is σ ; hence ρ is irreducible. A section of $E \rightarrow E/C$ has cocycle c_E , whose image in K is $i(c_E)$. Proposition C.10 gives the multiplier $\eta(c_E)$. $\square$

Remark C.12. The extensions of a fixed η form a coset of $\operatorname{Hom}(T/i(C), U(1))$ in $\operatorname{Hom}(T, U(1))$: their quotients are exactly the characters trivial on $i(C)$. For $\eta \neq 1$ every extension is nontrivial. In the ambient products of Theorem 4.3, the associated canonical connections are therefore nonflat and projectively flat by Proposition C.8. For $\eta = 1$ one may choose $\chi = 1$. The representation then factors through E/C , and its canonical connection is flat.

C.5. Examples and algebraic limits.

Example C.13 (A non-direct-factor kernel). Let E be finite and $C = \langle a \rangle \simeq C_4 \leq Z(E)$. The tuple (a, a^2) has coordinate kernels $1, C, \langle a^2 \rangle, C$ and relation lattice $\{(m, n) : m + 2n \equiv 0 \pmod{4}\}$. It realizes the proper subgroup of order two while retaining E at the all-unitary vertex. Theorem 12.8 separates all four manifolds after padding to distinct odd ranks. For C_8 , all four marked subgroups require three coordinates, since $j(C_8) = 3$; counting vertices alone would give only two.

Proposition C.14. *Let p be an odd prime, $d \geq 2$, $V = \mathbb{F}_p^d$, and $W = \Lambda^2 V$. The multiplication on $V \oplus W$*

$$(v, w)(v', w') = (v + v', w + w' + \tfrac{1}{2}v \wedge v') \quad (\text{C.3})$$

defines a group F of order $p^{d+\binom{d}{2}}$ and exponent p , with $Z(F) = [F, F] = W$. Every finite exponent- p group of nilpotency class at most two and abelianization dimension d is F/U for a subspace $U \leq W$. One family from Theorem C.2 realizes this entire marked quotient diagram with $r = (p^{\binom{d}{2}} - 1)/(p - 1)$ factors. This is the least number of coordinates for which every subgroup of W is generated by a subtuple.

Proof. The alternating bilinear map $(v, v') \mapsto \frac{1}{2}v \wedge v'$ is a 2-cocycle on the additive group V , so (C.3) defines a group. For the commutator convention $[x, y] = x^{-1}y^{-1}xy$,

$$(v, w)^a = (av, aw), \quad [(v, w), (v', w')] = (0, v \wedge v').$$

These identities give the exponent and $[F, F] = W$. If $v \neq 0$, there is v' independent of v , since $d \geq 2$; thus $Z(F) = W$.

Let N be the 2-nilpotent product of d cyclic groups of order p . Its universal property and normal form [71, Definition 1.3, Theorem 1.4 and the following paragraphs, pp. 2–3] give a surjection $N \rightarrow F$ sending its generators to $(e_i, 0)$. The commutators span W , so these images generate F . Both groups have order $p^{d+\binom{d}{2}}$; hence this map is an isomorphism. For a target group G , let H be generated by lifts of an abelianization basis. Then $G = H[G, G]$, and centrality of $[G, G]$ gives $[G, G] = [H, H] \leq H$. Thus $G = H$ is minimally d -generated. By [71, §2, pp. 3–4], it is a quotient of N by a subgroup of $[N, N]$. Under $N \simeq F$, this subgroup is a subspace of W . Theorem C.2 and Proposition C.3 give the simultaneous realization and its factor count. $\square$

For $p = d = 3$, the center has 28 subspaces: 0, thirteen lines, thirteen planes, and W . Thirteen coordinates are necessary and sufficient. The ambient cube has 8192 vertices, all nonhomeomorphic after odd distinct rank padding; the twenty-eight selected markings may include isomorphic abstract quotient groups.

Example C.15 (Nontransitivity). For $D_{16} = \langle r, s : r^8 = s^2 = 1, srs = r^{-1} \rangle$, the central maps $D_{16} \rightarrow D_8 \rightarrow C_2^2$ have kernels $\langle r^4 \rangle$ and $Z(D_8)$, respectively. Their composite kernel $\langle r^2 \rangle$ is not central. Since $Z(D_{16}) = \langle r^4 \rangle$, every central quotient of D_{16} has order sixteen or eight, whereas every quotient of C_2^2 has order at most four. Thus these endpoints have no common central quotient and cannot occur in one common-isotropy homogeneous family by Theorem 11.1, although each adjacent pair can.

C.6. Finite covers and centerless groups.

Proposition C.16. *Scalar-isospectral closed connected manifolds with finite fundamental groups and an isometric universal Riemannian cover have fundamental groups of equal order. In particular, groups E and E/A , with E finite and $A \neq 1$, cannot occur in such a pair.*

Proof. For the common cover X , fiber integration gives $\text{Vol } X = |\pi_1(M)| \text{Vol } M = |\pi_1(N)| \text{Vol } N$. The scalar spectra give $\text{Vol } M = \text{Vol } N > 0$. $\square$

Proposition C.17. *There are strongly isospectral compact normal homogeneous manifolds with positive Ricci curvature and nonisomorphic centerless fundamental groups. Those groups admit no common-isotropy homogeneous realization.*

Proof. Put $V = \mathbb{F}_7^2$ and $E_+ = V \rtimes_{2I} C_3$, $E_- = V \rtimes_{\text{diag}(2,4)} C_3$. Both actions have order three and no nonzero fixed vector. A central element in V must be fixed; one outside V would act nontrivially on V . Hence both centers vanish. The Sylow seven subgroup is unique, since its number divides three and is 1 modulo seven. An isomorphism would conjugate $2I$ to $\text{diag}(2, 4)$ or its square, which is impossible. Each group has one identity, 48 elements of order seven, and 98 of order three: $I + A + A^2 = 0$ makes every element outside V have order three. Their regular embeddings in S_{147} are therefore almost conjugate by the Todd–Komatsu construction [33, Example 7.7]. Embed S_{147} in $J = \text{SU}(148)$

by $P \mapsto \text{diag}(P, \det(P)^{-1})$. The quotients $J/E_{\pm}$ have dimension 21903, the prescribed fundamental groups, and normal metrics with nonnegative sectional and positive Ricci curvature. For every finite-dimensional S_{147} module, the two subgroup character averages agree. Apply this to each eigenspace of an equivariant self-adjoint elliptic operator over J . The invariant multiplicities agree, giving strong isospectrality, including Hodge and canonical tensor operators and cubic operators on equivariant Clifford bundles; spinor bundles require an equivariant spin lift. A common-isotropy realization would give central epimorphisms $E_{\pm} \rightarrow \Gamma$ by Theorem 11.1. Their kernels would be trivial, contradicting $E_+ \neq E_-$. $\square$

C.7. Finite central extensions and parity.

Proposition C.18. *Let E be finite and $C = \prod_{j=1}^r C_{q_j} \leq Z(E)$ a specified cyclic decomposition, with $C_I = \prod_{j \in I} C_{q_j}$. Set $s = \#\{j : 2 \mid q_j\}$, $d_0 = |E| + r$, and $n_j = 2d_0 + 1_{2 \nmid q_j}$. There is a common-isotropy realization of all E/C_I in dimension $\sum_j (2n_j + 1)^2 - r - s$. For $C = 1$, take $r = 1$, $q_1 = 1$.*

Proof. Embed $C = \prod \mu_{q_j} \subset T^r$ and put

$$\bar{K} = (E \times T^r) / \{(c, i(c)^{-1}) : c \in C\}, \quad z_j([e, t]) = t_j^{q_j}.$$

The joint kernel of z is E . Decompose the regular representation $\mathbb{C}[E] = \bigoplus_{\eta} V_{\eta}$ by central characters, and choose $a_{\eta} \in \mathbb{Z}^r$ with $i(c)^{a_{\eta}} = \eta(c)$, taking $a_1 = 0$. Define $R([e, t])|_{V_{\eta}} = t^{a_{\eta}} L_e$ and $\rho = R \oplus z_1 \oplus \cdots \oplus z_r$. The diagonal relations act trivially. If $\rho([e, t]) = 1$, its regular E/C summand gives $e \in C$ and the z_j give $t = i(c')$. Every remaining character gives $\eta(ec') = 1$, hence $ec' = 1$. Thus ρ is faithful of dimension d_0 . For each even q_j , add one circle w_j and set $A_j = \rho \oplus \rho \oplus w_j$ there, and $A_j = \rho \oplus \rho$ for odd q_j , together with z_j . The embedding is faithful. At a vertex I , fix $j \in I$. If q_j is odd, the t_j loop has zero unitary winding and orthogonal winding $q_j e_j = e_j$ in $(C_2)^I$, since each ρ occurs twice. If q_j is even, the w_j loop has these windings. Thus the image of $\pi_1(K^0)$ contains $\{0\} \times (C_2)^I$. The component computation (10.1) and the homotopy sequence give E/C_I . Finally $\dim K = r + s$, which gives the stated dimension. $\square$

Proposition C.19. *Let $C = \langle c \rangle \leq Z(E)$ have order q , with E finite, and set $\bar{K} = (E \times S^1) / \langle (c, \zeta_q^{-1}) \rangle$, $z([e, t]) = t^q$. A faithful $A : \bar{K} \rightarrow U(n)$ for some n can make the H_n boundary map an isomorphism onto E/C , without an auxiliary circle, exactly when q is odd or $c^{q/2} \notin [E, E]$ for even q .*

Proof. The determinant pulls back to $\chi(e)t^d$ and descends exactly when $\chi(c) = \zeta_q^d$. Surjectivity onto $\pi_1(H_n) = C_2$ is $d + q$ odd. For odd q , double a faithful representation to make d even. For even q , odd d forces $\chi(c^{q/2}) = -1$. Conversely, suppose $c^{q/2}$ has nontrivial image in $E/[E, E]$. The nontrivial character of the subgroup it generates extends to $E/[E, E]$ by [52, Corollary 2.33(ii)]. Its pullback $\chi : E \rightarrow U(1)$ satisfies $\chi(c^{q/2}) = -1$. Write $\chi(c) = \zeta_q^d$; then d is odd. Append the descending character $\chi(e)t^d$ to a doubled faithful representation. This realizes the required odd winding. The auxiliary circle in Theorem 10.4 supplies it when this criterion fails. $\square$

Example C.20 (Quaternion quotient in dimension 47). Embed $Q_8 \leq SU(2)$ and take

$$K = ((Q_8 \times S^1) / \langle (-I_2, -1) \rangle) \times S^1_w, \quad A([e, t], w) = \text{diag}(te, w), \quad z = t^2.$$

If $te = I_2$, then $(e, t) = (I_2, 1)$ or $(-I_2, -1)$; the w line then proves faithfulness. The determinant is $t^2 w$, so the w loop supplies the odd orthogonal winding. The quotients of $U(7)$ and $Sp(3) \times SO(8)$ have dimension 47, groups Q_8 , C_2^2 , and the spectral and curvature properties of Theorem 10.4. Since $-I_2 \in [Q_8, Q_8]$, Proposition C.19 excludes the unstabilized circle model.

Example C.21 (Odd Heisenberg quotients). For odd prime p , let $Sv_j = v_{j+1}$ and $Tv_j = \zeta_p^j v_j$ on $j \in \mathbb{Z}/p$. The matrices $\zeta_p^a S^b T^c$ are distinct: the shift determines b , and its nonzero entries determine a, c . They form a group E of order p^3 , with center C_p and quotient C_p^2 , since $TS = \zeta_p ST$. The defining representation σ is faithful and has determinant one: this holds for S, T and $\zeta_p I$. On $K = (E \times S^1) / \langle (\zeta_p I, \zeta_p^{-1}) \rangle$ set $A([e, t]) = t^{p+1} \sigma(e)$ and $z = t^p$. These maps descend. If $z = 1$ and $A = I$,

then $t = \zeta_p^k$, $e = \zeta_p^{-k}I$, precisely the diagonal relation, so (A, z) is faithful. Its orthogonal winding is $p(p+1) + p = p(p+2)$, which is odd. The resulting quotients of $U(2p+1)$ and $Sp(p) \times SO(2p+2)$ have dimension $4p(p+1)$ and groups E, C_p^2 ; for $p = 3$ the dimension is 48. The homomorphism (A, z) is injective; A alone need not be injective.

C.8. Circle metrics and Chern classes.

Proposition C.22. *Let $q \geq 1$. For the circle $(a, b) = (q-1, q)$ of Corollary 11.12, put $D = 2a^2 + b^2$. Its length is $2\pi\sqrt{D}$, and $\text{Vol}(X_q) = \text{Vol}(Y_q) = 16\pi^5/\sqrt{D}$. The covering $SU(2)^3 \rightarrow H_1$ has metric $Q_0 \oplus 2Q_0 \oplus 2Q_0$, where $Q_0 = -\frac{1}{2}\text{Re Tr}_{\mathbb{C}^2}$, and its primitive circle has length $4\pi\sqrt{D}$. For $q = 1$, the metric on $X_1 \cong SU(3)$ is left invariant and, at the identity,*

$$g(X, Y) = Q(X, Y) - Q(X, iE_{33})Q(Y, iE_{33}), \quad Q = -\text{Re Tr}_{\mathbb{C}^3}.$$

Proof. Equation (2.3) gives the circle length. The ambient-volume calculation preceding (6.14) gives $\text{Vol}(G_1) = \text{Vol}(H_1) = 32\pi^6$ for these metrics. Division by $2\pi\sqrt{D}$ gives the quotient volumes. The Spin covering has the metric scales given by (6.7). Its primitive circle maps twice onto K_q by (11.5), so its length is $4\pi\sqrt{D}$. For $q = 1$, orthogonal projection away from the unit vector iE_{33} gives the displayed metric. $\square$

Proposition C.23. *Let $q \geq 1$, and let L_X, L_Y over X_q, Y_q correspond to the weight-one character of K_q . Then*

$$H^2(X_q; \mathbb{Z}) = \mathbb{Z}/q, \quad c_1(L_X) \text{ generates } \mathbb{Z}/q, \quad H^2(Y_q; \mathbb{Z}) = \mathbb{Z}u, \quad c_1(L_Y) = \pm 2u.$$

For every $m \in \mathbb{Z}$, their m th tensor powers have equal canonical rough spectra. The first bundle power is trivial exactly when $q \mid m$; the second is nontrivial for $m \neq 0$. For $q = 1$, both bases are simply connected and L_X is trivial.

Proof. The integral Gysin sequence [47, §4.D, p. 438] gives

$$H^1(U(3); \mathbb{Z}) \rightarrow H^0(X_q; \mathbb{Z}) \xrightarrow{\sim c_1(L_X)} H^2(X_q; \mathbb{Z}) \rightarrow H^2(U(3); \mathbb{Z}).$$

The first map is multiplication by q , since the determinant restricts to the weight- q character of K_q . The diffeomorphism $(S, z) \mapsto S \text{diag}(z, 1, 1)$ from $SU(3) \times S^1$ to $U(3)$ and $\pi_1 SU(3) = \pi_2 SU(3) = 0$ give $H^2(U(3); \mathbb{Z}) = 0$. Thus $c_1(L_X)$ generates $H^2(X_q; \mathbb{Z}) \simeq \mathbb{Z}/q$.

For Y_q , the boundary homomorphism identifies $\pi_2(Y_q)$ with $2\mathbb{Z} \leq \pi_1(K_q) = \mathbb{Z}$, by Corollary 11.12 and its proof. Since Y_q is simply connected, Hurewicz [47, Theorem 4.32] identifies this group with $H_2(Y_q; \mathbb{Z})$. The first Chern class evaluates as the clutching winding, so $H^2(Y_q; \mathbb{Z}) = \mathbb{Z}u$ and $c_1(L_Y) = \pm 2u$. The weight- m character has vertical Casimir m^2/D ; Theorem 4.3 gives the equality of rough spectra. Finally $c_1(L^{\otimes m}) = mc_1(L)$, and complex line bundles on these compact manifolds are classified by c_1 [48, Proposition 3.10]. This gives the assertions about triviality. The case $q = 1$ of Corollary 11.12 gives simple connectivity of both bases. $\square$

For the unitary horizontal vectors $X = (E_{13} - E_{31})/\sqrt{2}$, $Y = i(E_{13} + E_{31})/\sqrt{2}$, one has $[X, Y] = i(E_{11} - E_{33})$ and pairing $a - b = -1$ with the circle generator. The canonical curvature is i/D on this pair. Thus the topologically trivial q th tensor power has nonzero curvature qi/D .

C.9. Integral cohomology in dimension seven.

Proposition C.24. *For $K = P_1$, the simply connected isospectral seven-manifolds $A_7 = U(3)/T^2 \cong N_{1,-1}$ and $B_7 = (Sp(1) \times SO(4))/T^2 \cong Q^{1,1,1}$ satisfy*

$$b_2(A_7) = 1, \quad b_2(B_7) = 2, \quad H^4(A_7; \mathbb{Z}) = 0, \quad H^4(B_7; \mathbb{Z}) = \mathbb{Z}/2.$$

The z character gives a trivial line bundle on A_7 and a primitive nonzero Chern class on B_7 , with equal canonical spectra for every corresponding tensor power.

Proof. The homogeneous identifications and simple connectivity are those at the beginning of Section 12. The kernels of the two winding maps are respectively $\mathbb{Z}(1, 0)$ and $\Lambda = \{(m, l) \in \mathbb{Z}^2 : m + l \in 2\mathbb{Z}\}$. The homotopy sequence and Hurewicz give $b_2(A_7) = 1$ and $b_2(B_7) = 2$.

In the notation of [23, arXiv version, §1], inversion identifies A_7 with $E_{k,l}$ for $k = (1, -1, 0)$ and $l = (0, 0, 0)$. The coordinate sums agree and every freeness condition is $\gcd(1, -1) = 1$. Moreover $|\sigma_2(k) - \sigma_2(l)| = 1$, where σ_2 is the second elementary symmetric function. Hence [23, arXiv version, equation (1.6)] gives $H^4(A_7; \mathbb{Z}) = 0$. The space $B_7 = Q^{1,1,1}$ is the principal circle bundle over $(S^2)^3$ with first Chern class $x + y + z$, where x, y, z are the positive generators of the three factors [10, arXiv version, §2.1]. Its fourth integral cohomology is $\mathbb{Z}/2$ [10, arXiv version, §2.2].

The z character extends to the determinant on $U(3)$ and trivializes the first line bundle by $[g, w] \mapsto (gK, \det(g)w)$. On the second, its Chern functional is $(m, l) \mapsto l$ on Λ ; its value at $(1, 1)$ is one. Theorem 4.3 applies to every tensor power. $\square$

With $T = i \operatorname{diag}(1, -1, 0)$ and $W = iE_{33}$, the metric on $N_{1,-1}$ at the identity coset is $g(X, Y) = Q(X, Y) - Q(X, W)Q(Y, W)$ for $X, Y \in T^\perp \subset \mathfrak{su}(3)$. The remaining diagonal direction $i \operatorname{diag}(1, 1, -2)$ has one third its Q -squared norm. On $B_7 = Q^{1,1,1}$, the normal metric is the quotient of $Q_0 \oplus 2Q_0 \oplus 2Q_0$, by Proposition C.22.

C.10. Central-character Chern and lifting classes.

Proposition C.25. *In Proposition C.8, put $m = |\Gamma|$ and $E_\rho = J \times_K \mathbb{C}^d$. In integral cohomology,*

$$\begin{aligned} m(d^k c_k(E_\rho) - \binom{d}{k} c_1(E_\rho)^k) &= 0 & (1 \leq k \leq d), \\ m(2d c_2(E_\rho) - (d-1)c_1(E_\rho)^2) &= 0 & (d \geq 2), \\ m c_k(\operatorname{End} E_\rho) &= 0 & (k > 0). \end{aligned}$$

Rationally, $c_k(E_\rho) = \binom{d}{k} (c_1(E_\rho)/d)^k$ and $\operatorname{ch}(E_\rho) = d \exp(c_1(E_\rho)/d)$.

Proof. On the degree- m cover $p : J/T \rightarrow J/K$, $p^* E_\rho = L_\chi^{\oplus d}$. For $x = c_1(L_\chi)$ this gives $p^* c_k = \binom{d}{k} x^k$ and $p^* c_1 = dx$, while $p^* \operatorname{End} E_\rho$ is trivial. Each displayed integral expression pulls back to zero. The transfer identity $\operatorname{tr} p^* = m \operatorname{id}$ [47, §3.G, pp. 321–322] gives the integral annihilators. It also makes rational pullback injective, giving the remaining formulas. $\square$

Proposition C.26. *Let $f_M : J/K \rightarrow B\Gamma$ classify $J/T \rightarrow J/K$. For the multiplier $[\alpha_\chi] \in H^2(\Gamma; U(1))$ of Proposition C.10, $f_M^*(\beta[\alpha_\chi]) = 0$ in $H^3(J/K; \mathbb{Z})$, where β is the exponential Bockstein. A nonzero group-cohomology multiplier can satisfy this equality, as in Example C.27.*

Proof. On a good cover, write the K transitions as $k_{ab} = t_{ab} s(\gamma_{ab})$, with γ_{ab} locally constant. Their cocycle identity gives

$$\chi(t_{ab})\chi(t_{bc})\alpha_\chi(\gamma_{ab}, \gamma_{bc}) = \chi(t_{ac}).$$

Thus the pulled-back multiplier is a coboundary in the sheaf of smooth $U(1)$ functions; its exponential connecting class vanishes [75, §4]. The matrices $\chi(t_{ab})R_{\gamma_{ab}}$ are transition functions of E_ρ and satisfy the ordinary matrix cocycle identity. Their scalar factors may vary on the overlaps. A flat lift of this projective connection requires locally constant scalar factors. $\square$

Example C.27. For $Q_8 \rightarrow C_2^2$, the defining representation has central character $-1 \mapsto -1$. A coboundary multiplier would make scalar multiples of the matrices of i, j commute, whereas they still anti-commute. The multiplier is nontrivial, although its exponential Bockstein pulls back to zero on every associated quotient by Proposition C.26.

REFERENCES

- [1] A. Abdollahi, R. Woodroffe and G. Zaimi, *Frankl's conjecture for subgroup lattices*, Electron. J. Combin. 24 (2017), no. 3, Paper P3.25, 9 pp. doi:10.37236/6248; arXiv:1606.00011v2.
- [2] I. Agricola, *Connections on naturally reductive spaces, their Dirac operator and homogeneous models in string theory*, Comm. Math. Phys. 232 (2003), 535–563. doi:10.1007/s00220-002-0743-y; arXiv:math/0202094.

- [3] I. Agricola and J. Henkel, *The Laplace–Beltrami spectrum on naturally reductive homogeneous spaces*, preprint (2025), revised 2026. arXiv:2503.21416v2.
- [4] W. Ambrose and I. M. Singer, *A theorem on holonomy*, Trans. Amer. Math. Soc. **75** (1953), 428–443. doi:10.1090/S0002-9947-1953-0063739-1.
- [5] J. An, J.-K. Yu and J. Yu, *On the dimension datum of a subgroup and its application to isospectral manifolds*, J. Differential Geom. **94** (2013), 59–85. doi:10.4310/jdg/1361889061.
- [6] R. M. Arroyo, A. Pulemotov, and W. Ziller, *The prescribed Ricci curvature problem for naturally reductive metrics on compact Lie groups*, Differential Geom. Appl. **78** (2021), article 101794. arXiv:2001.09441v1.
- [7] O. Baker, *The conjugacy problem for Higman’s group*, Internat. J. Algebra Comput. **30** (2020), no. 6, 1211–1235. doi:10.1142/S0218196720500393; arXiv:1902.06037v1.
- [8] W. Ballmann, *On the construction of isospectral manifolds*, in *Geometric Methods in Inverse Problems and PDE Control*, IMA Vol. Math. Appl., vol. 137, Springer, New York, 2004, 1–14. doi:10.1007/978-1-4684-9375-7_1.
- [9] C. Bär and W. Ballmann, *Guide to elliptic boundary value problems for Dirac-type operators*, in *Arbeitstagung Bonn 2013*, Progress in Mathematics, vol. 319, Birkhäuser/Springer, Cham, 2016, 43–80. doi:10.1007/978-3-319-43648-7_3; arXiv:1307.3021v2.
- [10] N. Benishti, D. Rodríguez-Gómez and J. Sparks, *Baryonic symmetries and M5 branes in the $\text{AdS}_4/\text{CFT}_3$ correspondence*, J. High Energy Phys. **2010** (2010), no. 7, article 024. doi:10.1007/JHEP07(2010)024; arXiv:1004.2045v1.
- [11] D. J. Benson, *Polynomial Invariants of Finite Groups*, London Mathematical Society Lecture Note Series, vol. 190, Cambridge University Press, Cambridge, 1993. doi:10.1017/CBO9780511565809.
- [12] P. Bérard, *Variétés riemanniennes isospectrales non isométriques*, Séminaire Bourbaki, exposé 705 (March 1989), Astérisque **177–178** (1989), 127–154. https://www.numdam.org/item/SB_1988-1989__31__127_0/.
- [13] L. Bérard-Bergery and J.-P. Bourguignon, *Laplacians and Riemannian submersions with totally geodesic fibres*, Illinois J. Math. **26** (1982), no. 2, 181–200. doi:10.1215/ijm/1256046790.
- [14] R. Berndt, *Representations of Linear Groups: An Introduction Based on Examples from Physics and Number Theory*, Vieweg, Wiesbaden, 2007.
- [15] E. Bierstone and P. D. Milman, *Semianalytic and subanalytic sets*, Publ. Math. IHES **67** (1988), 5–42. doi:10.1007/BF02699126.
- [16] R. L. Bishop and B. O’Neill, *Manifolds of negative curvature*, Trans. Amer. Math. Soc. **145** (1969), 1–49. doi:10.1090/S0002-9947-1969-0251664-4.
- [17] I. Biswas, M. Fernández, V. Muñoz and A. Tralle, *On formality of Sasakian manifolds*, J. Topol. **9** (2016), 161–180. doi:10.1112/jtopol/jtv044; arXiv:1402.6861.
- [18] S. Bochner, *Vector fields and Ricci curvature*, Bull. Amer. Math. Soc. **52** (1946), 776–797. doi:10.1090/S0002-9904-1946-08647-4.
- [19] M. Bordoni and V. Marino, *Towers of isospectral manifolds*, Complex Var. Elliptic Equ. **54** (2009), no. 1, 1–7. doi:10.1080/17476930802315157.
- [20] T. Bröcker and T. tom Dieck, *Representations of Compact Lie Groups*, Graduate Texts in Mathematics, vol. 98, Springer-Verlag, New York, 1985. doi:10.1007/978-3-662-12918-0.
- [21] B. Casselman, *Elliptic differential equations—analyticity*, Essays in analysis, University of British Columbia, 1 May 2012. <https://personal.math.ubc.ca/~cass/research/pdf/Analytic.pdf>.
- [22] J. Chazarain, *Formule de Poisson pour les variétés riemanniennes*, Invent. Math. **24** (1974), 65–82. doi:10.1007/BF01418788.
- [23] T. Chinburg, C. Escher and W. Ziller, *Topological properties of Eschenburg spaces and 3-Sasakian manifolds*, Math. Ann. **339** (2007), 3–20. doi:10.1007/s00208-007-0102-6; arXiv:math/0412370v1.
- [24] D. Cianci, *On the Poisson relation for flat and spherical space forms*, preprint (2016), arXiv:1608.02816.
- [25] Y. Colin de Verdière, *Spectrum of the Laplace operator and periodic geodesics: thirty years after*, Ann. Inst. Fourier **57** (2007), no. 7, 2429–2463. doi:10.5802/aif.2339.
- [26] B. Conrad, *Class Functions and Weyl groups*, Math. 210C course notes, Stanford University. <https://math.stanford.edu/~conrad/210CPage/handouts/classfn.pdf>.
- [27] G. Cornelissen and N. Peyerimhoff, *Twisted Isospectrality, Homological Wideness, and Isometry: A Sample of Algebraic Methods in Isospectrality*, SpringerBriefs in Mathematics, Springer, Cham, 2023. doi:10.1007/978-3-031-27704-7; arXiv:2107.00253v1.
- [28] F. D. Cunden, F. Mezzadri and N. O’Connell, *Free fermions and the classical compact groups*, J. Stat. Phys. **171** (2018), 768–801. doi:10.1007/s10955-018-2029-6; arXiv:1705.05932.
- [29] D. A. Dawson and J. Gärtner, *Large deviations from the McKean–Vlasov limit for weakly interacting diffusions*, Stochastics **20** (1987), no. 4, 247–308. doi:10.1080/17442508708833446.
- [30] P. Deligne, P. Griffiths, J. Morgan and D. Sullivan, *Real homotopy theory of Kähler manifolds*, Invent. Math. **29** (1975), 245–274. doi:10.1007/BF01389853.
- [31] A. Derdzinski, *A Myers-type theorem and compact Ricci solitons*, Proc. Amer. Math. Soc. **134** (2006), no. 12, 3645–3648. doi:10.1090/S0002-9939-06-08422-X.

- [32] G. de Rham, *Sur la réductibilité d'un espace de Riemann*, Comment. Math. Helv. **26** (1952), 328–344. doi:10.1007/BF02564308.
- [33] B. de Smit, R. Gornet and C. J. Sutton, *Sunada's method and the covering spectrum*, J. Differential Geom. **86** (2010), 501–537. doi:10.4310/jdg/1303219428; arXiv:0905.0380v2.
- [34] J. J. Duistermaat and V. W. Guillemin, *The spectrum of positive elliptic operators and periodic bicharacteristics*, Invent. Math. **29** (1975), 39–79. doi:10.1007/BF01405172.
- [35] S. Dyatlov, *Lecture notes for 18.155: distributions, elliptic regularity, and applications to PDEs*, MIT, version of 20 October 2026. <https://math.mit.edu/~dyatlov/18.155/155-notes.pdf>.
- [36] N. Ejiri, *A construction of non-flat, compact irreducible Riemannian manifolds which are isospectral but not isometric*, Math. Z. **168** (1979), 207–212. doi:10.1007/BF01214512.
- [37] P. Etingof, *Lie Groups and Lie Algebras*, MIT 18.745 and 18.755 lecture notes. <https://math.mit.edu/~etingof/lnlg.pdf>.
- [38] P. Etingof, *Representations of Lie Groups*, MIT 18.757 lecture notes, Fall 2023, Lecture 14. https://ocw.mit.edu/courses/18-757-representations-of-lie-groups-fall-2023/mit18_757_f23 lec14.pdf.
- [39] B. Farb and S. Weinberger, *Isometries, rigidity and universal covers*, Ann. of Math. (2) **168** (2008), 915–940. doi:10.4007/annals.2008.168.915.
- [40] M. Fernández, A. Fino, A. Kovalev and V. Muñoz, *Nearly parallel G_2 -manifolds: formality and associative submanifolds*, Math. Res. Lett. **31** (2024), 1391–1434. doi:10.4310/MRL.241211042435; arXiv:2208.13046.
- [41] D. H. Fremlin, *Measure Theory. Vol. 4: Topological Measure Spaces*, 2013 edition; updated online text. <https://www1.essex.ac.uk/math/people/fremlin/chap41.pdf>.
- [42] P. B. Gilkey and J. H. Park, *The Bochner Laplacian, Riemannian submersions, heat content asymptotics, and heat equation asymptotics*, Czechoslovak Math. J. **49** (1999), no. 2, 233–240. doi:10.1023/A:1022439916298.
- [43] C. S. Gordon, D. Schueth, and C. J. Sutton, *Spectral isolation of bi-invariant metrics on compact Lie groups*, Ann. Inst. Fourier **60** (2010), no. 5, 1617–1628. doi:10.5802/aif.2567.
- [44] C. S. Gordon and C. J. Sutton, *Spectral isolation of naturally reductive metrics on simple Lie groups*, Math. Z. **266** (2010), no. 4, 979–995. doi:10.1007/s00209-009-0640-6; arXiv:0707.0853.
- [45] A. Grigor'yan, *Heat Kernel and Analysis on Manifolds*, AMS/IP Studies in Advanced Mathematics, vol. 47, American Mathematical Society and International Press, 2009. doi:10.1090/amsip/047.
- [46] G. H. Hardy and M. Riesz, *The General Theory of Dirichlet's Series*, Cambridge Tracts in Mathematics and Mathematical Physics, no. 18, Cambridge University Press, Cambridge, 1915.
- [47] A. Hatcher, *Algebraic Topology*, Cambridge University Press, Cambridge, 2002.
- [48] A. Hatcher, *Vector Bundles & K-Theory*, version 2.2, November 2017. <https://pi.math.cornell.edu/~hatcher/VBKT/VB.pdf>.
- [49] C. S. Gordon and J. P. Rossetti, *Boundary volume and length spectra of Riemannian manifolds: what the middle degree Hodge spectrum doesn't reveal*, Ann. Inst. Fourier **53** (2003), 2297–2314. doi:10.5802/aif.2007.
- [50] R. Gornet, *Riemannian nilmanifolds, the wave trace, and the length spectrum*, Comm. Anal. Geom. **16** (2008), 27–89. doi:10.4310/CAG.2008.v16.n1.a2.
- [51] G. Higman, *A finitely generated infinite simple group*, J. London Math. Soc. **s1-26** (1951), 61–64. doi:10.1112/jlms/s1-26.1.61.
- [52] K. H. Hofmann and S. A. Morris, *The Structure of Compact Groups: A Primer for the Student— A Handbook for the Expert*, third revised and augmented edition, De Gruyter Studies in Mathematics, vol. 25, De Gruyter, Berlin/Boston, 2013.
- [53] S. Hong, *A Lie-algebraic approach to the local index theorem on compact homogeneous spaces*, Adv. Math. **296** (2016), 127–153. doi:10.1016/j.aim.2016.03.038; arXiv:1210.5855v4.
- [54] L. Hörmander, *The Analysis of Linear Partial Differential Operators I: Distribution Theory and Fourier Analysis*, second edition, Springer, Berlin, 1990; Classics in Mathematics reprint, 2003. doi:10.1007/978-3-642-61497-2.
- [55] A. Ikeda, *On the spectrum of a Riemannian manifold of positive constant curvature*, Osaka J. Math. **17** (1980), no. 1, 75–93. doi:10.18910/5171.
- [56] R. Joshi and S. Spallone, *Spinorality of orthogonal representations of reductive groups*, Represent. Theory **24** (2020), 435–469. doi:10.1090/ert/552; arXiv:1901.06232v3.
- [57] T. Kawai, *Construction of local elementary solutions for linear partial differential operators with real analytic coefficients (I): The case with real principal symbols*, Publ. Res. Inst. Math. Sci. **7** (1971/72), 363–397. doi:10.2977/prims/1195193547.
- [58] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, vols. I and II, Interscience Publishers, New York, 1963 and 1969.
- [59] B. Kostant, *A cubic Dirac operator and the emergence of Euler number multiplets of representations for equal rank subgroups*, Duke Math. J. **100** (1999), 447–501. doi:10.1215/S0012-7094-99-10016-0.
- [60] D. Kotschick and S. Terzić, *On formality of generalized symmetric spaces*, Math. Proc. Cambridge Philos. Soc. **134** (2003), no. 3, 491–505. doi:10.1017/S0305004102006540; arXiv:math/0106135v2.
- [61] I. Koval and A. Vig, *Silent orbits and cancellations in the wave trace*, preprint, 2024. arXiv:2408.09238.
- [62] U. Krämer and M. Tucker-Simmons, *On the Dolbeault–Dirac operator of quantized symmetric spaces*, Trans. Lond. Math. Soc. **2** (2015), no. 1, 33–56. doi:10.1112/tlms/tlv002.

- [63] J. M. Lee, *Introduction to Riemannian Manifolds*, second edition, Graduate Texts in Mathematics, vol. 176, Springer, Cham, 2018. doi:10.1007/978-3-319-91755-9.
- [64] Y. Le Jan, *Markov loops, coverings and fields*, Ann. Fac. Sci. Toulouse Math. (6) **26** (2017), 401–416. doi:10.5802/afst.1538.
- [65] Y. Le Jan, *Brownian loops topology*, Potential Anal. **53** (2020), 223–229. doi:10.1007/s11118-019-09765-z.
- [66] Y. Li, Y. Wu, and J. Zhou, *Shortest closed geodesics on three-dimensional isospectral manifolds*, preprint, 2026. arXiv:2609.28380v1.
- [67] M. Ludewig, *Heat kernel asymptotics, path integrals and infinite-dimensional determinants*, J. Geom. Phys. **131** (2018), 66–88. doi:10.1016/j.geomphys.2018.04.012; arXiv:1607.05891v3.
- [68] M. Ludewig, *Strong short-time asymptotics and convolution approximation of the heat kernel*, Ann. Global Anal. Geom. **55** (2019), 371–394. doi:10.1007/s10455-018-9630-4.
- [69] S. Ma, *Lecture notes for pseudodifferential operators and microlocal analysis*, arXiv:2107.12839 (2021).
- [70] G. W. Mackey, *Unitary representations of group extensions. I*, Acta Math. **99** (1958), 265–311. doi:10.1007/BF02392428.
- [71] A. Magidin, *Capable groups of prime exponent and class two*, preprint (2004). arXiv:math/0401423v1.
- [72] A. Malcev, *On isomorphic matrix representations of infinite groups*, Mat. Sbornik N.S. **8(50)** (1940), no. 3, 405–422. <https://www.mathnet.ru/eng/sm6037>.
- [73] A. Mantione and R. Torres, *Geography of 4-manifolds with positive scalar curvature*, Expositiones Math. **39** (2021), no. 4, 566–582. doi:10.1016/j.exmath.2021.05.003; arXiv:2105.12162v1.
- [74] R. J. Miatello and J. P. Rossetti, *Length spectra and p-spectra of compact flat manifolds*, J. Geom. Anal. **13** (2003), no. 4, 631–657. doi:10.1007/BF02921882; arXiv:math/0110325.
- [75] M. K. Murray, *Bundle gerbes*, J. London Math. Soc. (2) **54** (1996), 403–416. doi:10.1112/jlms/54.2.403; arXiv:dg-ga/9407015.
- [76] S. B. Myers, *Riemannian manifolds with positive mean curvature*, Duke Math. J. **8** (1941), no. 2, 401–404. doi:10.1215/S0012-7094-41-00832-3.
- [77] J. C. Nash, *Positive Ricci curvature on fibre bundles*, J. Differential Geom. **14** (1979), no. 2, 241–254. doi:10.4310/jdg/1214434973.
- [78] J. B. Nation, *Notes on Lattice Theory*, University of Hawai‘i course notes. <https://math.hawaii.edu/~jb/lat1-6.pdf>.
- [79] National Institute of Standards and Technology, *Digital Library of Mathematical Functions*, §25.8, equation (25.8.8). <https://dlmf.nist.gov/25.8.E8>.
- [80] J. Neisendorfer and T. Miller, *Formal and coformal spaces*, Illinois J. Math. **22** (1978), 565–580. doi:10.1215/ijm/1256048467.
- [81] B. Nica, *Linear groups—Malcev’s theorem and Selberg’s lemma*, arXiv:1306.2385 (2013).
- [82] J. R. Norris, *A complete differential formalism for stochastic calculus in manifolds*, Séminaire de Probabilités XXVI, Lecture Notes in Math., vol. 1526, Springer, Berlin, 1992, 189–209. doi:10.1007/BFb0084322.
- [83] F. Prüfer, *On the spectrum and the geometry of the spherical space forms. I*, Ann. Global Anal. Geom. **3** (1985), no. 2, 129–154. doi:10.1007/BF01000336.
- [84] D. Schueth, *Constructing isospectral metrics via principal connections*, in *Geometric Analysis and Nonlinear Partial Differential Equations*, Springer, 2003, 69–79. https://doi.org/10.1007/978-3-642-55627-2_4.
- [85] A. Strohmaier, R. Verch, and M. Wollenberg, *Microlocal analysis of quantum fields on curved space-times: Analytic wave front sets and Reeh–Schlieder theorems*, J. Math. Phys. **43** (2002), 5514–5530; doi:10.1063/1.1506381; arXiv:math-ph/0202003.
- [86] A. I. Suciú and H. Wang, *Formality properties of finitely generated groups and Lie algebras*, Forum Math. **31** (2019), 867–905. doi:10.1515/forum-2018-0098; arXiv:1504.08294.
- [87] Z.-W. Sun, *Refining Lagrange’s four-square theorem*, J. Number Theory **175** (2017), 167–190; doi:10.1016/j.jnt.2016.11.008; arXiv:1604.06723.
- [88] C. J. Sutton, *Isospectral simply-connected homogeneous spaces and the spectral rigidity of group actions*, Comment. Math. Helv. **77** (2002), 701–717. doi:10.1007/PL00012438; arXiv:math/0301376.
- [89] C. J. Sutton, *On the Poisson relation for compact Lie groups*, Ann. Global Anal. Geom. **57** (2020), 537–589. doi:10.1007/s10455-020-09712-x; arXiv:1606.07426.
- [90] M. E. Taylor, *Introduction to Lie Groups*, AMS Open Math Notes, OMN:202106.111301, 2021. <https://mtaylor.web.unc.edu/wp-content/uploads/sites/16915/2021/03/liegp.pdf>.
- [91] S. Terzić, *Rational homotopy groups of generalised symmetric spaces*, Math. Z. **243** (2003), 491–523. doi:10.1007/s00209-002-0461-3; arXiv:math/0205318.
- [92] G. Teschl, *Mathematical Methods in Quantum Mechanics: With Applications to Schrödinger Operators*, first edition, Graduate Studies in Mathematics, vol. 99, American Mathematical Society, Providence, RI, 2009. <https://www.mat.univie.ac.at/~gerald/ftp/book-schroe/schroe.pdf>.
- [93] L. Tóth, *On the number of cyclic subgroups of a finite Abelian group*, Bull. Math. Soc. Sci. Math. Roumanie **55(103)** (2012), no. 4, 423–428. <https://ssmr.ro/bulletin/pdf/55-4/Toth.pdf>; arXiv:1203.6201.
- [94] H. Urakawa, *The heat equation on compact Lie group*, Osaka J. Math. **12** (1975), no. 2, 285–297. doi:10.18910/7850.

- [95] R. Vein and P. Dale, *Determinants and Their Applications in Mathematical Physics*, Applied Mathematical Sciences, vol. 134, Springer-Verlag, New York, 1999. doi:10.1007/b98968.
- [96] E. B. Vinberg, *Linear Representations of Groups*, translated by A. Iacob, Birkhäuser, Basel, 1989; uncorrected reprint, Modern Birkhäuser Classics, 2010. doi:10.1007/978-3-0348-0063-1.
- [97] J. Voight, *Quaternion Algebras*, Graduate Texts in Mathematics, vol. 288, Springer, Cham, 2021. doi:10.1007/978-3-030-56694-4.
- [98] J. Wang, Z. Xie and G. Yu, *Approximations of delocalized eta invariants by their finite analogues*, Enseign. Math. **69** (2023), 51–91. doi:10.4171/LEM/1048.
- [99] B. Wilking, *On fundamental groups of manifolds of nonnegative curvature*, Differential Geom. Appl. **13** (2000), 129–165. doi:10.1016/S0926-2245(00)00030-9.
- [100] J. S. Wilson, *Profinite groups with few conjugacy classes of p -elements*, Proc. Amer. Math. Soc. **150** (2022), no. 8, 3297–3305. doi:10.1090/proc/15925; corrected version, arXiv:2204.09936v2.
- [101] J. Yu, *On the dimension datum of a subgroup. II*, arXiv:1803.06210v1 (2018).
- [102] J. Yu, *Some new results on dimension datum*, Bull. Lond. Math. Soc. **54** (2022), no. 3, 929–948. doi:10.1112/blms.12605; arXiv:2008.00378v2.
- [103] S. Zelditch, *The inverse spectral problem*, Surveys in Differential Geometry **9** (2004), 401–467; doi:10.4310/SDG.2004.v9.n1.a12; arXiv:math/0402356.
- [104] W. Ziller, *Lie Groups. Representation Theory and Symmetric Spaces*, lecture notes, University of Pennsylvania, Fall 2010. <https://mathweb.ucsd.edu/~lni/math251B/ziller.pdf>.