

INDEX AND NULLITY OF CRITICAL SPHERICAL CATENOIDS IN HYPERBOLIC THREE-SPACE

IVAN DOROFEEV AND ALEXANDER PIGAZZINI

ABSTRACT. We prove that every critical spherical catenoid in a geodesic ball of hyperbolic three-space has Morse index four and nullity two. This resolves Medvedev's index conjecture throughout the parameter range of the catenoid family. The kernel consists precisely of the normal components of rotations that preserve the center of the ball and tilt the axis of the catenoid. As a consequence, the radius of the supporting ball increases strictly along the family. The proof combines a global boundary inequality obtained from a Wronskian, a positive Dirichlet eigenfunction supplied by the time component of the unit normal, and a Riccati comparison for the angular mode $k = 2$. A ground-state identity then yields positivity of every angular mode $k \geq 2$.

1. INTRODUCTION

The critical catenoid is a basic example in the study of the Morse index of free boundary minimal surfaces. In the Euclidean unit ball in $\mathbb{R}^3$, its index is four, as proved by Devyver [3], Smith and Zhou [11], and Tran [12]; another proof was given by Medvedev [6]. Devyver also proved that every compact orientable free boundary minimal surface in this ball, other than a flat disk, has index at least four [3]. Thus the critical catenoid attains this lower bound.

The analogous index problem in curved space forms concerns critical spherical catenoids. In $\mathbb{H}^3$, these are rotational minimal annuli obtained by free boundary truncation of the spherical catenoids constructed by Mori [8]. The geometry of rotational hypersurfaces in spaces of constant curvature was developed further by do Carmo and Dajczer [1], and the free boundary truncation was considered by Li and Xiong [4]. Lima [5] proved that an embedded free boundary minimal annulus in a geodesic ball of $\mathbb{S}_+^3$ or $\mathbb{H}^3$ that is invariant under the geodesic symmetry about the center of the ball is congruent to a critical rotational annulus; in $\mathbb{H}^3$, these are the critical spherical catenoids. Medvedev [7] proved that the critical spherical catenoid in a geodesic ball of $\mathbb{S}_+^3$ has index four, and that its hyperbolic counterpart has index at least four. He conjectured equality in the hyperbolic case. We prove this equality throughout the parameter range of the critical spherical catenoid family in $\mathbb{H}^3$ and determine the nullity.

The analysis in [10] determines the index and nullity contributions of the angular mode $|k| = 1$ for every parameter. The local result in [9] establishes index four and nullity two when the parameter is sufficiently close to its lower endpoint. The theorem below establishes both equalities for every $a > \frac{1}{2}$.

Theorem 1. *For each $a > \frac{1}{2}$, let Σ_a be the critical spherical catenoid with angular period 2π , parametrized in Section 2, and let $\mathbb{B}(r(a)) \subset \mathbb{H}^3$ be its supporting geodesic ball. Then*

$$\text{ind}(\Sigma_a) = 4, \quad \text{nul}(\Sigma_a) = 2.$$

The Robin kernel is generated by the normal components of the two rotations that tilt the axis of the catenoid while fixing the center of $\mathbb{B}(r(a))$.

2020 *Mathematics Subject Classification.* Primary 53A10; Secondary 53C42, 58J50, 34B24.

Key words and phrases. Free boundary minimal surfaces, hyperbolic space, critical catenoid, Morse index, nullity, Robin eigenvalue problem.

Here the index and nullity refer to the second variation of area for normal variations whose boundary remains on the supporting sphere. In particular, the ambient ball is fixed when taking variations.

The theorem supplies the global hyperbolic counterpart of the Euclidean index-four results and identifies all infinitesimal degeneracies with rotations of the ambient ball. At fixed ambient curvature -1 , both the geometry of the annulus and its supporting radius vary with a . The associated Jacobi operator contains the curvature term $+2$, and the Robin coefficient is $\coth r(a)$; these terms are recorded in Section 3. The proof controls this family of boundary value problems over the entire interval $a > \frac{1}{2}$.

The proof has three main ingredients. A Wronskian identity gives a strict boundary inequality valid for every $a > \frac{1}{2}$. The time component of the unit normal is a positive Dirichlet eigenfunction with eigenvalue two; a codimension-two min–max argument therefore determines the radial mode $k = 0$. Finally, a comparison of logarithmic derivatives, together with an explicit integral inequality, gives a boundary inequality for the mode $k = 2$; a ground-state identity turns it into positivity of that mode, and monotonicity in k^2 completes the spectral count. The geometric and spectral identities used below are included in the proof.

Two consequences are recorded in Section 8. The radius of the supporting ball is a strictly increasing function of a (Corollary 9). The restriction to angular period 2π is essential: for $m \geq 2$, the m -fold cover of Σ_a has index at least $4m$, and its nullity exceeds two exactly when $mk_*(a)$ is an integer, for a threshold frequency $k_*(a) \in (1, 2)$ (Remark 10). This agrees with the non-rotational free boundary minimal annuli of Cerezo [2], which bifurcate from multiple covers of critical spherical catenoids.

2. GEOMETRY OF THE CRITICAL CATENOIDS

We use the hyperboloid model

$$\mathbb{H}^3 = \{X \in \mathbb{R}^{3,1} : \langle X, X \rangle_L = -1, X^0 > 0\}, \quad \langle X, Y \rangle_L = -X^0 Y^0 + \sum_{j=1}^3 X^j Y^j.$$

The center of the balls under consideration is $e_0 = (1, 0, 0, 0)$. Fix $a > \frac{1}{2}$ and define

$$(2.1) \quad K = \sqrt{a^2 - \frac{1}{4}}, \quad A(s)^2 = a \cosh(2s) + \frac{1}{2}, \quad B(s)^2 = a \cosh(2s) - \frac{1}{2},$$

$$(2.2) \quad \varphi(s) = \int_0^s \frac{K}{A(\sigma)^2 B(\sigma)} d\sigma.$$

The positive square roots are understood. The rotational parametrization is

$$(2.3) \quad X(s, \theta) = (A \cosh \varphi, A \sinh \varphi, B \cos \theta, B \sin \theta), \quad (s, \theta) \in \mathbb{R} \times (\mathbb{R}/2\pi\mathbb{Z}).$$

A prime denotes differentiation with respect to s throughout the paper. Set $v = B'$. The identities

$$(2.4) \quad A^2 = 1 + B^2, \quad A' = \frac{Bv}{A}, \quad v = \frac{a \sinh(2s)}{B},$$

$$(2.5) \quad v^2 = 1 + B^2 - \frac{K^2}{B^2}, \quad v' = B + \frac{K^2}{B^3} > 0$$

follow directly from (2.1). In particular, A and B are even, whereas v and φ are odd.

For completeness, introduce the Lorentz-orthonormal frame

$$E_0 = (\cosh \varphi, \sinh \varphi, 0, 0), \quad E_1 = (\sinh \varphi, \cosh \varphi, 0, 0), \quad E_r = (0, 0, \cos \theta, \sin \theta),$$

together with $E_\theta = (0, 0, -\sin \theta, \cos \theta)$. Then $X = AE_0 + BE_r$ and

$$X_s = A'E_0 + A\varphi'E_1 + vE_r, \quad X_\theta = BE_\theta.$$

Using (2.2)–(2.5), we obtain

$$\langle X_s, X_s \rangle_L = \frac{v^2 + K^2/B^2}{A^2} = 1, \quad \langle X_s, X_\theta \rangle_L = 0.$$

Consequently,

$$(2.6) \quad g = ds^2 + B^2 d\theta^2.$$

A unit normal is

$$(2.7) \quad \nu = \frac{K}{A} E_0 - \frac{v}{A} E_1 + \frac{K}{B} E_r.$$

Indeed, (2.4) and (2.5) imply that ν is orthogonal to X, X_s, X_θ and has Lorentzian norm one. A direct computation using (2.2)–(2.5) gives

$$\nu_s = -\frac{K}{B^2} X_s, \quad \nu_\theta = \frac{K}{B^2} X_\theta.$$

Thus, with shape operator $-d\nu$, the principal curvatures are K/B^2 and $-K/B^2$. The surface is minimal and

$$(2.8) \quad |\text{II}|^2 = \frac{2K^2}{B^4}.$$

The parametrization is embedded: the quotient $X^1/X^0 = \tanh \varphi(s)$ is strictly increasing in s , and, for each fixed s , the pair $(X^2, X^3) = B(s)(\cos \theta, \sin \theta)$ determines θ uniquely modulo 2π because $B(s) > 0$.

2.1. Free boundary truncation. Write

$$g_0(s) = A(s) \cosh \varphi(s) = X^0, \quad g_1(s) = A(s) \sinh \varphi(s) = X^1.$$

For the distance ρ from e_0 , one has $X^0 = \cosh \rho$ and

$$\nabla^{\mathbb{H}^3} X^0 = X^0 X - e_0, \quad \langle \nabla^{\mathbb{H}^3} X^0, \nu \rangle_L = \nu^0.$$

It follows that the surface meets a sphere centered at e_0 orthogonally precisely when $\nu^0 = 0$. At a point $s = S > 0$, this condition reads

$$(2.9) \quad \tanh \varphi(S) = \frac{K}{v(S)}.$$

As functions of $S > 0$, the left-hand side increases strictly from zero, whereas the right-hand side decreases strictly from $+\infty$ to zero. Hence (2.9) has exactly one solution $S > 0$.

The critical catenoid is

$$\Sigma_a = X([-S, S] \times (\mathbb{R}/2\pi\mathbb{Z})).$$

It is symmetric with respect to the totally geodesic plane $\{X^1 = 0\}$, which contains the neck circle $s = 0$, and its supporting ball is centered at the point e_0 where this plane meets the axis $\{X^2 = X^3 = 0\}$. Set

$$(2.10) \quad B_0 = B(S), \quad v_0 = v(S), \quad r = \text{arcosh } g_0(S), \quad c = \coth r.$$

Since $g'_0(s) > 0$ for $s > 0$, every interior point of Σ_a lies strictly inside $\mathbb{B}(r)$. At the positive boundary the outward unit conormal is X_s . Orthogonality to the sphere gives

$$X_s(S, \theta) = \frac{\cosh r X(S, \theta) - e_0}{\sinh r}.$$

Taking components yields

$$(2.11) \quad g_0(S) = \cosh r, \quad g'_0(S) = \sinh r, \quad g'_1(S) = c g_1(S),$$

$$(2.12) \quad v_0 = c B_0, \quad g'_0(S) - c g_0(S) = -\frac{1}{\sinh r}.$$

3. THE QUADRATIC FORM AND SEPARATION OF VARIABLES

We use the convention $\Delta = \operatorname{div} \nabla$. For the ambient Ricci tensor, $\operatorname{Ric}_{\mathbb{H}^3}(\nu, \nu) = -2$. If N is the outward unit normal to the supporting sphere, then $\langle \nabla_V^{\mathbb{H}^3} N, V \rangle_L = \coth r$ for every unit vector V tangent to that sphere. Along the free boundary, ν is such a tangent vector. The general free boundary second variation formula [12, Section 2] therefore gives

$$(3.1) \quad Q[u] = \int_{\Sigma_a} (|\nabla u|^2 + (2 - |\mathbf{II}|^2)u^2) dA - c \int_{\partial \Sigma_a} u^2 d\ell, \quad u \in H^1(\Sigma_a).$$

Its associated self-adjoint operator and boundary condition are

$$J = -\Delta - |\mathbf{II}|^2 + 2, \quad \partial_\eta u = cu,$$

where η is the outward unit conormal along $\partial \Sigma_a$. The index is the number of negative eigenvalues, counted with multiplicity; the nullity is the dimension of the kernel. The form domain is $H^1(\Sigma_a)$ and imposes no Robin condition on test functions.

For angular frequency $k \geq 0$, the radial operator is

$$(3.2) \quad J_k u = -\frac{1}{B}(Bu')' + \left(2 + \frac{k^2}{B^2} - \frac{2K^2}{B^4}\right)u$$

in $L^2((-S, S), B ds)$, with boundary conditions

$$(3.3) \quad u'(S) = cu(S), \quad -u'(-S) = cu(-S).$$

After removing the positive angular integration factor, the radial form is

$$(3.4) \quad \begin{aligned} \mathfrak{q}_k[u] &= \int_{-S}^S \left(Bu'^2 + \left(2B + \frac{k^2}{B} - \frac{2K^2}{B^3}\right)u^2 \right) ds \\ &\quad - v_0(u(S)^2 + u(-S)^2), \quad u \in H^1(-S, S). \end{aligned}$$

Each $k \geq 1$ has the two real angular factors $\cos(k\theta)$ and $\sin(k\theta)$; the mode $k = 0$ has one. The coefficients are even in s , so each radial problem splits into even and odd sectors.

For each fixed $a > \frac{1}{2}$, one has $B(s) \geq \sqrt{a - \frac{1}{2}} > 0$ on the compact interval $[-S, S]$. All radial problems are therefore regular Sturm–Liouville problems with positive smooth weight. We use the min–max principle, simplicity of radial eigenvalues, and the oscillation theorem: the j th eigenfunction, with eigenvalues numbered starting at $j = 1$, has exactly $j - 1$ interior zeros. These standard facts are discussed in [13].

4. A GLOBAL BOUNDARY INEQUALITY

For the flat connection D of $\mathbb{R}^{3,1}$, the Gauss formula reads

$$D_U(dX(V)) = dX(\nabla_U^\Sigma V) + \mathbf{II}(U, V)\nu + g(U, V)X.$$

Tracing and using minimality gives $\Delta_\Sigma X = 2X$. The coordinate functions consequently satisfy $\Delta_\Sigma X^j = 2X^j$, so

$$(4.1) \quad g_i'' + \frac{B'}{B}g_i' - 2g_i = 0, \quad J_0 g_i = -|\mathbf{II}|^2 g_i, \quad i = 0, 1.$$

Lemma 2. *For every $a > \frac{1}{2}$, the boundary radius in the rotational coordinates satisfies*

$$B_0^2 > 2K^2.$$

Proof. By (2.5),

$$v'' = v - \frac{3K^2 v}{B^4},$$

and substitution in (3.2) gives $J_0 v = 0$. Moreover, $v(0) = 0$ and $v > 0$ on $(0, S]$. Define

$$W = B(vg_1' - v'g_1).$$

The equations for v and g_1 imply

$$W(0) = 0, \quad W' = B|\Pi|^2 v g_1 > 0 \quad \text{on } (0, S].$$

Using (2.11), we find

$$0 < W(S) = -B_0 g_1(S)(v'(S) - cv_0).$$

On the other hand, (2.5) and $c = v_0/B_0$ give

$$v'(S) - cv_0 = B_0 + \frac{K^2}{B_0^3} - \frac{v_0^2}{B_0} = \frac{2K^2 - B_0^2}{B_0^3}.$$

Since $B_0 g_1(S) > 0$, the asserted inequality follows. $\square$

5. THE MODES ZERO AND ONE

5.1. A Dirichlet eigenfunction.

Lemma 3. *The first Dirichlet eigenvalue of J_0 on $(-S, S)$ is 2. A corresponding positive eigenfunction is*

$$n(s) = \nu^0(s) = \frac{K \cosh \varphi(s) - v(s) \sinh \varphi(s)}{A(s)}.$$

Proof. The Gauss–Weingarten equations in $\mathbb{R}^{3,1}$ imply

$$(5.1) \quad \Delta_\Sigma \nu = -|\Pi|^2 \nu$$

for a minimal surface in $\mathbb{H}^3$. To see the signs explicitly, let $\mathcal{A}$ denote the shape operator. In a geodesic orthonormal frame on the surface, tracing the derivative of $D\nu = -dX \circ \mathcal{A}$ gives

$$\Delta_\Sigma \nu = -dX(\operatorname{div} \mathcal{A}) - |\Pi|^2 \nu - (\operatorname{tr} \mathcal{A})X.$$

The Codazzi equation in constant curvature gives $\operatorname{div} \mathcal{A} = \nabla(\operatorname{tr} \mathcal{A})$, and both trace terms vanish by minimality. This proves (5.1). Taking its time component gives $J_0 n = 2n$.

The function n is even and $n(0) = K/A(0) > 0$. For $0 < s < S$, the uniqueness of the crossing in (2.9) gives $\tanh \varphi(s) < K/v(s)$, so $n(s) > 0$. At $s = \pm S$, the free boundary condition gives $n = 0$. Thus n is a Dirichlet eigenfunction with no interior zero. The oscillation theorem identifies its eigenvalue as the first one. $\square$

Proposition 4. *The radial Robin problem for J_0 has exactly two negative eigenvalues and no zero eigenvalue. One negative eigenfunction is even and the other is odd.*

Proof. Integrating (4.1) by parts in (3.4), and using (2.11)–(2.12), gives

$$(5.2) \quad \mathfrak{q}_0[g_0] = - \int_{-S}^S B|\Pi|^2 g_0^2 ds - 2B_0 \coth r < 0,$$

$$(5.3) \quad \mathfrak{q}_0[g_1] = - \int_{-S}^S B|\Pi|^2 g_1^2 ds < 0.$$

Since g_0 is even and g_1 is odd, their mixed term in the polarized quadratic form is zero. Hence their span is a two-dimensional negative subspace.

Let $\lambda_1 < \lambda_2 < \lambda_3 < \dots$ be the radial Robin eigenvalues. The preceding subspace gives $\lambda_2 < 0$. The Dirichlet form domain $H_0^1(-S, S)$ has codimension two in $H^1(-S, S)$, since its defining conditions are the two endpoint traces. Therefore the span of the first three Robin eigenfunctions contains a nonzero $z \in H_0^1(-S, S)$. Lemma 3 and the min–max principle yield

$$2 \int_{-S}^S Bz^2 ds \leq \mathfrak{q}_0[z] \leq \lambda_3 \int_{-S}^S Bz^2 ds.$$

Thus $\lambda_3 \geq 2$. This proves the exact count and excludes a kernel. The negative tests (5.2) and (5.3) supply one negative direction of each parity, so each parity contributes exactly one negative eigenvalue. $\square$

5.2. The first angular mode.

Proposition 5. *The radial Robin problem for J_1 has one negative eigenvalue and a one-dimensional kernel generated by g'_0 . The negative eigenfunction is even and the kernel function is odd.*

Proof. The identity

$$\left(\frac{B'}{B}\right)' = \frac{2K^2}{B^4} - \frac{1}{B^2}$$

shows, upon differentiating (4.1), that

$$(5.4) \quad J_1 g'_0 = 0, \quad J_1 g'_1 = 0.$$

Set $f = g'_0$. The function f is odd, positive on $(0, S]$, and satisfies $f'(0) = 2g_0(0) > 0$. At $s = S$,

$$f'(S) = 2g_0(S) - c g'_0(S) = 2 \cosh r - c \sinh r = \cosh r = c f(S).$$

Parity gives the other boundary condition in (3.3). Thus f is a Robin eigenfunction with eigenvalue zero and exactly one interior zero. By the oscillation theorem, zero is the second radial eigenvalue, and it is simple. There is exactly one negative eigenvalue.

A first eigenfunction can be chosen strictly positive. Reflection $s \mapsto -s$ preserves the problem, so simplicity makes that eigenfunction even. The kernel generator f is odd. $\square$

6. POSITIVITY OF THE HIGHER ANGULAR MODES

Define the conformal coordinate

$$t = t(s) = \int_0^s \frac{d\sigma}{B(\sigma)}, \quad T = t(S).$$

A dot denotes differentiation with respect to t ; primes continue to denote differentiation with respect to s . We use the same letters for functions composed with $s(t)$. Equation (2.5) gives

$$(6.1) \quad \dot{B} = Bv, \quad \dot{v} = \mathcal{D}(v) := \sqrt{(v^2 - 1)^2 + 4K^2}, \quad v(0) = 0, \quad \mathcal{D}(0) = 2a.$$

Indeed, $\dot{v} = Bv' = B^2 + K^2/B^2 > 0$, and the square of this expression is $(v^2 - 1)^2 + 4K^2$.

In this coordinate the zero-eigenvalue equation is

$$(6.2) \quad \ddot{u} = (2v^2 + k^2 - 2)u,$$

and the full radial form is

$$(6.3) \quad \mathbf{q}_k[u] = \int_{-T}^T (\dot{u}^2 + (2v^2 + k^2 - 2)u^2) dt - v_0(u(T)^2 + u(-T)^2).$$

The norm in the spectral problem is

$$(6.4) \quad \int_{-S}^S B u^2 ds = \int_{-T}^T B^2 u^2 dt.$$

In particular, the conformal change preserves the form and the zero equation; the transformed eigenvalue problem has weight B^2 .

6.1. A boundary logarithmic derivative.

Let u_2 solve

$$(6.5) \quad \ddot{u}_2 = (2v^2 + 2)u_2, \quad u_2(0) = 1, \quad \dot{u}_2(0) = 0.$$

The coefficient is strictly positive. While u_2 is positive, $\ddot{u}_2 > 0$, so u_2 increases from 1 and cannot have a first zero. Hence $u_2 > 0$ on $[0, T]$ and $\dot{u}_2 > 0$ on $(0, T]$. Define

$$p_2 = \frac{\dot{u}_2}{u_2}, \quad \delta_2 = p_2 - v.$$

Lemma 6. *For every $a > \frac{1}{2}$, one has $p_2(T) > v_0$.*

Proof. The Riccati equation for p_2 and (6.1) give

$$(6.6) \quad \dot{\delta}_2 + (p_2 + v)\delta_2 = F_2(v), \quad \delta_2(0) = 0, \quad F_2(v) = v^2 + 2 - \mathcal{D}(v).$$

Since $(v^2 + 2)^2 - \mathcal{D}(v)^2 = 6v^2 + 4 - 4a^2$, we have the exact identity

$$(6.7) \quad F_2(v) = \frac{6v^2 + 4 - 4a^2}{v^2 + 2 + \mathcal{D}(v)}.$$

If $a \leq 1$, then $F_2(v(t)) > 0$ for every $t > 0$. An integrating factor in (6.6) yields $\delta_2(T) > 0$.

Suppose now that $a > 1$. By (6.7), $F_2(v)$ is negative for $v^2 < 2(a^2 - 1)/3$ and positive for $v^2 > 2(a^2 - 1)/3$. Since v is strictly increasing, the function

$$M(t)\delta_2(t) = \int_0^t M(\tau)F_2(v(\tau))d\tau, \quad M(t) = \exp\left(\int_0^t (p_2 + v)d\tau\right) > 0,$$

first decreases and then increases, if the sign change is reached. It follows that

$$(6.8) \quad \delta_2(T) \leq 0 \implies \delta_2(t) < 0 \quad (0 < t < T).$$

If the sign change is not reached before T , the same implication follows from strict decrease throughout the interval.

For a comparison solution, set $w = g'_1$. From the definitions,

$$w = A' \sinh \varphi + A\varphi' \cosh \varphi > 0 \quad (s \geq 0).$$

The function w is even, and $J_1 w = 0$ by (5.4). Consequently, for

$$p_1 = \frac{\dot{w}}{w}, \quad \delta_1 = p_1 - v,$$

we have

$$\dot{p}_1 = 2v^2 - 1 - p_1^2, \quad p_1(0) = 0,$$

and

$$\dot{\delta}_1 + (p_1 + v)\delta_1 = v^2 - 1 - \mathcal{D}(v) < 0, \quad \delta_1(0) = 0.$$

The strict sign holds because $K > 0$, so $\mathcal{D}(v) > |v^2 - 1|$. Thus $p_1 < v$ for $t > 0$.

Assume, for contradiction, that $\delta_2(T) \leq 0$. By (6.8), $p_1 + p_2 < 2v$ on $(0, T)$. The difference $\omega = p_2 - p_1$ satisfies

$$\dot{\omega} + (p_1 + p_2)\omega = 3, \quad \omega(0) = 0.$$

Therefore

$$(6.9) \quad \begin{aligned} \omega(T) &= 3 \int_0^T \exp\left(-\int_\tau^T (p_1 + p_2)dt\right) d\tau \\ &> \frac{3}{B_0^2} \int_0^T B(t)^2 dt = \frac{3}{B_0^2} \int_0^S B(s) ds. \end{aligned}$$

Here we used $\dot{B}/B = v$; the strict inequality holds on every integration interval with $\tau < T$.

At the boundary, (4.1) and $g'_1(S) = (v_0/B_0)g_1(S)$ give

$$p_1(T) = \frac{B_0 g_1''(S)}{g_1'(S)} = \frac{2B_0^2}{v_0} - v_0.$$

Since $p_2(T) \leq v_0$, equation (2.5) yields

$$(6.10) \quad \omega(T) \leq v_0 - p_1(T) = \frac{2(B_0^2 - K^2)}{B_0^2 v_0}.$$

We now prove that (6.9) and (6.10) are incompatible. For $s > 0$, define

$$\mathcal{I}(s) = 3 \int_0^s B(\sigma) d\sigma - \frac{2(B(s)^2 - K^2)}{v(s)}.$$

Let $x = B^2$ and $\kappa = K^2$. Direct differentiation gives

$$\begin{aligned}
 \mathcal{I}' &= -B + \frac{2(B^2 - \kappa)(B + \kappa/B^3)}{v^2} \\
 &= B \left(-1 + \frac{2(x - \kappa)(x^2 + \kappa)}{x(x^2 + x - \kappa)} \right) \\
 (6.11) \quad &= B \frac{(x - 2\kappa)(x^2 - x + \kappa)}{x(x^2 + x - \kappa)}.
 \end{aligned}$$

For the last step, the numerator expands as

$$\begin{aligned}
 &-x(x^2 + x - \kappa) + 2(x - \kappa)(x^2 + \kappa) \\
 &= x^3 - (1 + 2\kappa)x^2 + 3\kappa x - 2\kappa^2 \\
 &= (x - 2\kappa)(x^2 - x + \kappa).
 \end{aligned}$$

The denominator in (6.11) is positive for $s > 0$, because $x^2 + x - \kappa = xv^2 > 0$. Since $a > 1$, we have $\kappa > \frac{3}{4}$, and

$$x^2 - x + \kappa = (x - \frac{1}{2})^2 + \kappa - \frac{1}{4} > 0.$$

Set

$$(6.12) \quad s_* = \frac{1}{2} \operatorname{arcosh}(2a), \quad B(s_*) = \sqrt{2}K, \quad v(s_*) = \sqrt{2}a.$$

Lemma 2 implies $S > s_*$. Equation (6.11) shows that $\mathcal{I}$ is strictly increasing on $(s_*, S]$.

The identity

$$(6.13) \quad \left(\frac{B'}{B} \right)^2 = 1 + \frac{1}{x} - \frac{K^2}{x^2} = \frac{a^2}{K^2} - \frac{(x - 2K^2)^2}{4K^2x^2} \leq \frac{a^2}{K^2}$$

follows by expanding the square and using $a^2 = K^2 + \frac{1}{4}$. As $B' \geq 0$ for $s \geq 0$, integration of $B' \leq (a/K)B$ gives

$$\int_0^{s_*} B \, ds \geq \frac{K}{a} \left(\sqrt{2}K - \sqrt{a - \frac{1}{2}} \right).$$

Using (6.12), we conclude that

$$(6.14) \quad \mathcal{I}(s_*) \geq \frac{K}{a} \left(2\sqrt{2}K - 3\sqrt{a - \frac{1}{2}} \right) > 0.$$

The final strict inequality follows from

$$8K^2 - 9(a - \frac{1}{2}) = (a - \frac{1}{2})(8a - 5) > 0.$$

Thus $\mathcal{I}(S) > 0$. In contrast, (6.9)–(6.10) imply

$$3 \int_0^S B \, ds < \frac{2(B_0^2 - K^2)}{v_0},$$

which says $\mathcal{I}(S) < 0$. This contradiction proves $\delta_2(T) > 0$ also when $a > 1$. $\square$

6.2. A positive quadratic form.

Proposition 7. *Every radial Robin eigenvalue in angular mode $k \geq 2$ is strictly positive.*

Proof. Extend u_2 evenly to $[-T, T]$. The coefficient in (6.5) is even, so the extension is smooth, positive, and solves the same equation. Its logarithmic derivative p_2 is odd. For any $z \in H^1(-T, T)$, writing $z = u_2 h$ and integrating

$$\dot{z}^2 + (2v^2 + 2)z^2 = u_2^2 \dot{h}^2 + \frac{d}{dt}(p_2 z^2)$$

gives

$$(6.15) \quad \begin{aligned} \mathfrak{q}_2[z] &= \int_{-T}^T u_2^2 \left(\frac{d}{dt} \frac{z}{u_2} \right)^2 dt \\ &\quad + (p_2(T) - v_0)(z(T)^2 + z(-T)^2). \end{aligned}$$

The identity first follows for smooth functions and extends to H^1 by density and continuity of endpoint traces. Lemma 6 makes the boundary coefficient strictly positive. If the right-hand side vanishes, then z/u_2 is constant and $z(T) = z(-T) = 0$, which forces $z = 0$. Therefore $\mathfrak{q}_2[z] > 0$ for every nonzero $z \in H^1(-T, T)$.

The minimum of the Rayleigh quotient with the positive weight $B^2 dt$ in (6.4) is attained for this regular problem. Hence its first eigenvalue is strictly positive. Finally,

$$\mathfrak{q}_k[z] = \mathfrak{q}_2[z] + (k^2 - 4) \int_{-T}^T z^2 dt \quad (k \geq 2)$$

proves the assertion for all higher modes, on the full form domain and therefore in both parity sectors. $\square$

7. PROOF OF THE MAIN THEOREM

Proof of Theorem 1. Propositions 4, 5, and 7 give the following contributions. Angular multiplicities are included in the table.

Sector	Index contribution	Nullity contribution
$k = 0$, even	1	0
$k = 0$, odd	1	0
$k = 1$, even	2	0
$k = 1$, odd	0	2
$k \geq 2$, both parities	0	0

The angular Fourier decomposition is orthogonal for both the L^2 inner product and the quadratic form. Its partial sums converge in $H^1(\Sigma_a)$, and hence their boundary traces converge in $L^2(\partial\Sigma_a)$. Thus the decomposition also holds on the full form domain. The span of the negative eigenfunctions in the table has dimension four, and its L^2 -orthogonal complement is nonnegative for Q . A function in the Robin kernel has each Fourier coefficient in the corresponding radial kernel. The mode counts therefore give

$$\text{ind}(\Sigma_a) = 4, \quad \ker J = \text{span}\{g'_0(s) \cos \theta, g'_0(s) \sin \theta\}.$$

In particular, $\text{nul}(\Sigma_a) = 2$.

To identify the kernel geometrically, consider the spatial rotation fields

$$Z_2(X) = (0, -X^2, X^1, 0), \quad Z_3(X) = (0, -X^3, 0, X^1).$$

They preserve the center e_0 and the geodesic ball. Equation (2.7) gives

$$\langle Z_2, \nu \rangle_L = g'_0 \cos \theta, \quad \langle Z_3, \nu \rangle_L = g'_0 \sin \theta.$$

Indeed, $\langle Z_2, \nu \rangle_L = (-B\nu^1 + (K/B)g_1) \cos \theta$ and $\langle Z_3, \nu \rangle_L = (-B\nu^1 + (K/B)g_1) \sin \theta$, while

$$-B\nu^1 + \frac{K}{B}g_1 = \frac{Bv}{A} \cosh \varphi + \frac{K}{AB} \sinh \varphi = g'_0.$$

These rotations tilt the axis of the catenoid. Rotation about the axis itself has zero normal component. The two displayed fields span the entire Robin kernel, as asserted. $\square$

Remark 8. For each fixed $a > \frac{1}{2}$, the theorem gives nondegeneracy modulo the rotations preserving the supporting ball. All spectral comparisons above are made for that fixed parameter; no uniform spectral gap as the parameter approaches an endpoint is required.

8. TWO CONSEQUENCES

Corollary 9. *The supporting radius $r(a)$ depends smoothly on a and satisfies*

$$\frac{dr}{da} > 0 \quad \text{for every } a > \frac{1}{2}.$$

In particular, distinct members of the family $\{\Sigma_a\}_{a>1/2}$ are free boundary in distinct geodesic balls centered at e_0 .

Proof. For $s > 0$, the function $\tanh \varphi - K/v$ has s -derivative $\varphi' \operatorname{sech}^2 \varphi + Kv'/v^2 > 0$. By the implicit function theorem, the solution S of (2.9) depends smoothly on a , and so does $r = \operatorname{arcosh} g_0(S)$, since $g_0(S) > 1$.

All quantities of Section 2 depend smoothly on (s, a) . Let $Y = \partial_a X$, taken at fixed (s, θ) , and set $\psi = \langle Y, \nu \rangle_L$. This is the parametric Jacobi field of [9, Definition 8.1]; its evenness, the value $\psi(0) = 1/(2K)$ and the boundary identity (8.1) below are established in [9, Lemmas 8.2 and 8.3], and we include the short derivations for completeness. Each $X(\cdot, \cdot; a)$ is a minimal immersion of $\mathbb{R} \times (\mathbb{R}/2\pi\mathbb{Z})$, so the normal component of the variation field is a Jacobi field. Since ψ does not depend on θ , this gives $J_0\psi = 0$. The Lorentz reflection $R = \operatorname{diag}(1, -1, 1, 1)$ satisfies $X(-s, \theta) = RX(s, \theta)$ and $\nu(-s, \theta) = R\nu(s, \theta)$ for every a ; hence ψ is even.

At $s = 0$ one has $\varphi = \partial_a \varphi = 0$ and $v = 0$. Since $\partial_a A^2 = \partial_a B^2 = \cosh(2s)$, the vectors Y and ν at $s = 0$ are

$$\left(\frac{1}{2A}, 0, \frac{\cos \theta}{2B}, \frac{\sin \theta}{2B}\right) \quad \text{and} \quad \left(\frac{K}{A}, 0, \frac{K}{B} \cos \theta, \frac{K}{B} \sin \theta\right),$$

with $A = A(0)$ and $B = B(0)$. Using $A(0)^2 - B(0)^2 = 1$ and $A(0)^2 B(0)^2 = K^2$, we obtain

$$\psi(0) = \frac{K}{2} \left(\frac{1}{B(0)^2} - \frac{1}{A(0)^2} \right) = \frac{1}{2K} \neq 0.$$

Now let $s = S$. The boundary relation of Section 2 gives $e_0 = \cosh r X - \sinh r X_s$; in particular $\langle X_s, e_0 \rangle_L = -\sinh r$. Differentiating $\langle X, X \rangle_L = -1$, $\langle \nu, X \rangle_L = 0$ and $\langle \nu, X_s \rangle_L = 0$ in a , and using $\nu_s = -(K/B^2)X_s$, we find

$$\langle Y, X \rangle_L = 0, \quad \langle \partial_a \nu, X \rangle_L = -\psi, \quad \langle \partial_a \nu, X_s \rangle_L = -\psi' - \frac{K}{B^2} \langle Y, X_s \rangle_L.$$

Since $\partial_a g_0 = -\langle Y, e_0 \rangle_L = \sinh r \langle Y, X_s \rangle_L$ at $s = S$ and $g'_0(S) = \sinh r$, differentiating $g_0(S(a)) = \cosh r(a)$ and dividing by $\sinh r$ gives

$$\frac{dS}{da} + \langle Y, X_s \rangle_L = \frac{dr}{da}.$$

Since $\langle \nu_s, e_0 \rangle_L = (K/B_0^2) \sinh r$ at $s = S$, differentiating the free boundary condition $\langle \nu, e_0 \rangle_L|_{s=S(a)} = -\nu^0(S(a)) = 0$, and using the decomposition of e_0 together with the identities above, gives

$$\frac{K}{B_0^2} \sinh r \frac{dS}{da} - \cosh r \psi(S) + \sinh r \left(\psi'(S) + \frac{K}{B_0^2} \langle Y, X_s \rangle_L \right) = 0.$$

Dividing by $\sinh r$ and combining the last two identities, we obtain

$$(8.1) \quad \psi'(S) - c \psi(S) = -\frac{K}{B_0^2} \frac{dr}{da}.$$

Suppose that $dr/da = 0$ for some a . Then (8.1) and the evenness of ψ show that ψ satisfies both conditions in (3.3). Since $J_0\psi = 0$ and $\psi(0) \neq 0$, zero would be an eigenvalue of the radial Robin problem for J_0 , contrary to Proposition 4. Hence dr/da never vanishes and has constant sign. By Lemma 2,

$$\cosh r = A(S) \cosh \varphi(S) \geq \sqrt{1 + B_0^2} > \sqrt{1 + 2K^2},$$

and the right-hand side tends to infinity as $a \rightarrow \infty$. Thus r is unbounded as $a \rightarrow \infty$, which excludes the negative sign. $\square$

Combined with the limit $r(a) \rightarrow 0$ as $a \rightarrow \frac{1}{2}^+$, established in [10], Corollary 9 shows that $a \mapsto r(a)$ is an increasing diffeomorphism of $(\frac{1}{2}, \infty)$ onto $(0, \infty)$. Thus every geodesic ball of $\mathbb{H}^3$ is the supporting ball of an isometric image of exactly one Σ_a . Corollary 9 answers the question of the strict monotonicity of $r(a)$; see [9, Remark 1.3].

Remark 10. The restriction to angular period 2π in Theorem 1 cannot be dropped. The radial form (3.4) is defined for every real $k \geq 0$, and

$$\mathfrak{q}_k[u] - \mathfrak{q}_\ell[u] = (k^2 - \ell^2) \int_{-S}^S \frac{u^2}{B} ds.$$

Since $B(0) \leq B \leq B_0$ on $[-S, S]$, the min-max principle gives, for the radial Robin eigenvalues $\lambda_j(k)$,

$$\frac{k^2 - \ell^2}{B_0^2} \leq \lambda_j(k) - \lambda_j(\ell) \leq \frac{k^2 - \ell^2}{B(0)^2} \quad (0 \leq \ell < k).$$

Hence each $\lambda_j(k)$ is a continuous, strictly increasing function of $k \geq 0$. Propositions 5 and 7 give $\lambda_1(1) < 0 < \lambda_1(2)$, so there is a unique $k_*(a) \in (1, 2)$ with $\lambda_1(k_*(a)) = 0$. For $0 < k < 1$, Propositions 4 and 5 give $\lambda_1(k) < \lambda_2(k) < \lambda_2(1) = 0 < \lambda_3(0) < \lambda_3(k)$; for $1 < k < 2$, one has $\lambda_2(k) > \lambda_2(1) = 0$. On the m -fold cover of Σ_a , of angular period $2\pi m$, the angular frequencies are j/m with integer $j \geq 0$. Each frequency $j/m \in (0, 1)$ therefore contributes four negative directions, so for $m \geq 2$ the index of the cover is at least $4m$. Moreover, the nullity of the cover exceeds two exactly when $mk_*(a)$ is an integer, which again requires $m \geq 2$. This agrees with the construction of Cerezo [2], in which non-rotational free boundary minimal annuli in geodesic balls of $\mathbb{H}^3$ bifurcate from m -fold covers of critical spherical catenoids (called free boundary hyperbolic catenoids in [2]). The bifurcating annuli are invariant under n reflections in planes through the axis, where m and n are coprime and n/m lies in an interval contained in $(\sqrt{2}, \sqrt{3}) \subset (1, 2)$.

Remark 11 (Scope of the argument). Lemmas 2 and 6 rest on identities specific to the rotational family of Section 2, and the separation of variables uses its rotational symmetry. Three ingredients nevertheless have a wider scope.

(i) The identity (5.1) uses neither the explicit parametrization nor the rotational symmetry: for every two-sided minimal hypersurface $\Sigma^n \subset \mathbb{H}^{n+1} \subset \mathbb{R}^{n+1,1}$, the Gauss–Weingarten and Codazzi equations give $\Delta_\Sigma \nu = -|\mathbf{II}|^2 \nu$, hence $J\nu^0 = n\nu^0$ for $J = -\Delta_\Sigma - |\mathbf{II}|^2 + n$. If Σ is free boundary in a geodesic ball centred at e_0 , then $\nu^0 = 0$ on $\partial\Sigma$. What is specific to the present family is the sign property of Lemma 3: ν^0 is positive in the interior, hence it is the first Dirichlet eigenfunction, with eigenvalue 2; this is the input of Proposition 4.

(ii) The last step of Proposition 7 is an instance of the ground-state representation for regular Sturm–Liouville problems. Write the Robin conditions as $\partial_\eta z = \sigma z$ at the endpoints. If a positive solution u of the zero-energy equation satisfies $\partial_\eta u/u > \sigma$ at both endpoints, the substitution $z = uh$ writes the quadratic form as a nonnegative interior square plus boundary terms whose coefficients are positive multiples of $\partial_\eta u/u - \sigma$; the form is therefore positive definite. By symmetry the two conditions reduce here to $p_2(T) > v_0$, which is Lemma 6.

(iii) The proof of Corollary 9 reflects a general transversality principle. Let X_a be a smooth family of free boundary minimal hypersurfaces in the geodesic balls of radius $r(a)$ centred at e_0 , parametrized over a fixed manifold with boundary, and let $\psi = \langle \partial_a X_a, \nu \rangle_L$. Then ψ is a Jacobi field, and differentiating the two free boundary conditions gives

$$\partial_\eta \psi - \coth r \psi = -\mathbf{II}(\eta, \eta) \frac{dr}{da} \quad \text{on } \partial\Sigma,$$

where η is the outward conormal; here one uses that η is a principal direction along $\partial\Sigma$, the supporting sphere being umbilic. For Σ_a one has $\Pi(\eta, \eta) = K/B_0^2$, and this is (8.1). Hence, if dr/da vanished, ψ would lie in the Robin kernel. For Σ_a , ψ is nonzero since $\psi(0) = 1/(2K)$, and Proposition 4 excludes a radial Robin kernel, so $dr/da \neq 0$; the sign is then fixed by the unboundedness of r .

REFERENCES

- [1] M. do Carmo and M. Dajczer, *Rotation hypersurfaces in spaces of constant curvature*, Trans. Amer. Math. Soc. **277** (1983), no. 2, 685–709. <https://doi.org/10.1090/S0002-9947-1983-0694383-X>.
- [2] A. Cerezo, *Free boundary minimal annuli in geodesic balls of $\mathbb{H}^3$* , arXiv:2502.20303 (2025). <https://arxiv.org/abs/2502.20303>.
- [3] B. Devyver, *Index of the critical catenoid*, Geom. Dedicata **199** (2019), 355–371. <https://doi.org/10.1007/s10711-018-0353-2>.
- [4] H. Li and C. Xiong, *A gap theorem for free boundary minimal surfaces in geodesic balls of hyperbolic space and hemisphere*, J. Geom. Anal. **28** (2018), no. 4, 3171–3182. <https://doi.org/10.1007/s12220-017-9953-6>.
- [5] C. Lima, *On uniqueness of free boundary minimal annuli in geodesic balls of $\mathbb{S}_+^3$ and $\mathbb{H}^3$* , arXiv:2503.16763 (2025). <https://arxiv.org/abs/2503.16763>.
- [6] V. Medvedev, *On the index of the critical Möbius band in $\mathbb{B}^4$* , J. Geom. Anal. **33** (2023), no. 3, article no. 93. <https://doi.org/10.1007/s12220-022-01069-w>.
- [7] V. Medvedev, *On free boundary minimal submanifolds in geodesic balls in $\mathbb{H}^n$ and $\mathbb{S}_+^n$* , Math. Z. **310** (2025), no. 1, article no. 10. <https://doi.org/10.1007/s00209-025-03707-9>.
- [8] H. Mori, *Minimal surfaces of revolution in H^3 and their global stability*, Indiana Univ. Math. J. **30** (1981), no. 5, 787–794. <https://doi.org/10.1512/iumj.1981.30.30057>.
- [9] A. Pigazzini, *Analytic local resolution of Medvedev’s Morse index conjecture for the critical spherical catenoid in $\mathbb{H}^3$* , arXiv:2605.13562 (2026). <https://arxiv.org/abs/2605.13562>.
- [10] A. Pigazzini, *Robin nullity in mode $|k| = 1$ and asymptotic radius of the critical spherical catenoid*, arXiv:2605.11244 (2026). <https://arxiv.org/abs/2605.11244>.
- [11] G. Smith and D. Zhou, *The Morse index of the critical catenoid*, Geom. Dedicata **201** (2019), 13–19. <https://doi.org/10.1007/s10711-018-0412-8>.
- [12] H. Tran, *Index characterization for free boundary minimal surfaces*, Commun. Anal. Geom. **28** (2020), no. 1, 189–222. <https://doi.org/10.4310/CAG.2020.v28.n1.a6>. See also <https://arxiv.org/abs/1609.01651v4>.
- [13] A. Zettl, *Sturm–Liouville Theory*, Mathematical Surveys and Monographs, vol. 121, American Mathematical Society, Providence, RI, 2005. <https://doi.org/10.1090/surv/121>.