

DIRAC SPECTRA AND THE TOPOLOGY OF SPACES OF METRICS

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ABSTRACT. Let M be a closed connected spin manifold of dimension $n \geq 3$. We prescribe arbitrary admissible finite Dirac spectra in bounded open symmetric intervals, including multiplicities and nullity. Kernel dimensions are unbounded on each M . Nonzero eigenvalues generically have the least multiplicity allowed by the spinor structures, also at fixed density.

Compatible multiplication perturbations of Dirac operators are approximated in norm resolvent by metric Dirac operators, including first parameter derivatives for compact smooth families. The metrics are arbitrarily C^0 -close, preserve density, and agree with the original metric outside a prescribed neighborhood of the support. In dimensions four and five, such changes in any nonempty open set realize the first admissible nullity above the index minimum.

If invertible metrics exist, ordinary real index difference identifies their components at fixed density with KO_{n+1} ; for $n \geq 5$, each contains a positive-scalar-curvature metric whenever one exists. With density and the metric outside any nonempty open set fixed, the Dirac-minimal space has weak homotopy type $\mathrm{Sp} / \mathrm{Sp}(a)$ for $n \equiv 4 \pmod{8}$, and $\Omega \mathrm{Sp}$ for $n \equiv 5 \pmod{8}$, where $a = |\mathrm{ind}_{\mathbb{H}} D^+|$ and Sp is the stable compact symplectic group. On compact connected spin n -manifolds, invertible boundary metrics admit invertible cylindrical extensions exactly when the relative real index vanishes.

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1. INTRODUCTION

Let D_g be the Dirac operator of a spin manifold with metric g . Charge conjugation and chirality restrict its spectral symmetries and multiplicities; the real index gives a lower bound for its kernel. Relative indices obstruct paths through invertible operators and invertible extensions of boundary metrics. The corresponding existence and classification questions concern the sufficiency of these restrictions for metric Dirac operators [36, 37, 38].

Theorem R realizes every compatible multiplication perturbation $D_{g_b} + V_b$ by metric Dirac operators in norm resolvent. The approximation includes first parameter derivatives, preserves volume density and a prescribed exterior, and permits arbitrarily small C^0 changes of metric. It is relative to parameter sets near which V_b vanishes. Contour integration gives the associated spectral projections and compressed operators. A full potential section then yields a full metric section by the finite-dimensional inverse function theorem. Relative realization also removes multiplication potentials from families of invertible operators and from quaternionic chiral families with index-minimal kernel.

These constructions give the spectral and component theorems below. In dimensions congruent to 4 or 5 modulo 8, a relative boundary construction determines the weak homotopy type of the minimal metric space on the fixed manifold (Theorem T). It retains both the prescribed exterior and the volume density. When the chiral index is nonzero, the comparison retains the finite-dimensional cokernel and the nullhomotopy of its stable class. At index zero the comparison is ordinary real index difference.

All manifolds are smooth, oriented, and spin, with their spin structures fixed. Connected manifolds are nonempty. Clifford multiplication satisfies

$$c(\xi)c(\eta) + c(\eta)c(\xi) = -2g^{-1}(\xi, \eta)I, \quad c(\xi)^* = -c(\xi). \quad (1.1)$$

A metric on a compact manifold with boundary is product near the boundary. When the boundary Dirac operator is invertible, the metric extends over fixed half-infinite product cylinders. Invertibility then refers to the completed operator $D_g : H^1 \rightarrow L^2$. Spaces of smooth metrics and multiplication potentials carry the C^∞ topology. All indices below have ordinary real coefficients; no condition of simultaneous invertibility for flat group-algebra twists is imposed.

1.1. Norm-resolvent approximation. Let X be closed, or let $X = W_\infty$ be obtained from a compact spin manifold by attaching product half-cylinders to its boundary. In the latter case the boundary Dirac operator is invertible, and all changes of metric have compact support. Positive frame comparisons, with their density corrections, identify the operators on

$$\mathcal{D} = H^1(\Sigma X) \subset \mathcal{H} = L^2(\Sigma X). \quad (1.2)$$

A complex-linear endomorphism is *compatible* if it is self-adjoint and satisfies the charge-conjugation and, in even dimension, chirality identities of the Dirac operator. These identities are given in Section 2.

Theorem R (Norm-resolvent realization). *Let $n = \dim X \geq 3$. Let B be a compact smooth parameter manifold, possibly with boundary, and let $B_0 \subset B$ be closed. Suppose*

$$L_b = D_{g_b} + V_b$$

is a smooth family with fixed cylindrical ends, and V_b is a compatible multiplication endomorphism supported in one compact subset of an interior open set U . Assume $V_b = 0$ on a neighborhood of B_0 .

For every compact set

$$\mathcal{K} \subset \{(b, z) \in B \times \mathbb{C} : z \in \rho(L_b)\}$$

and every $\varepsilon > 0$, there is a smooth metric family $\widehat{g}_b$ such that

$$\begin{aligned} \widehat{g}_b &= g_b \text{ on } X \setminus U, & dv_{\widehat{g}_b} &= dv_{g_b}, \\ \widehat{g}_b &= g_b \text{ for } b \text{ near } B_0, & \|\widehat{g} - g\|_{C^1(B; C^0)} &< \varepsilon. \end{aligned} \quad (1.3)$$

In the direct positive spinor comparison, $\mathcal{K}$ belongs to the parameter-resolvent set of $\widehat{D}_b = D_{\widehat{g}_b}$, and

$$\left\| (\widehat{D}_b - z)^{-1} - (L_b - z)^{-1} \right\|_{C_b^1(\mathcal{K}; \mathcal{L}(\mathcal{H}))} < \varepsilon. \quad (1.4)$$

The parameter derivatives in (1.4) are taken on neighborhoods of $\mathcal{K}$ in finitely many parameter charts. No derivative in z is included in this notation. A kernel may lie inside a spectral contour; only the contour must belong to the resolvent set. The metric norm in (1.3) is taken relative to any fixed smooth reference metric and a finite parameter atlas. On cylindrical ends the metrics agree with that reference outside a compact set. The estimate is invariant under replacing these fixed norms by equivalent norms. No spatial C^1 approximation or curvature bound is imposed.

1.2. Generic nonzero eigenvalues. Let M^n be closed and connected, with its spin structure fixed, and write $\mathcal{R}(M)$ for the space of smooth metrics. Put

$$d_n = \begin{cases} 2, & n \equiv 2, 3, 4 \pmod{8}, \\ 1, & \text{otherwise.} \end{cases} \quad (1.5)$$

A nonzero eigenvalue is called *simple* if its complex multiplicity is d_n [56, Definition 6.2.4].

Theorem G (Generic nonzero spectrum). *Let $n \geq 3$. The set*

$$S(M) = \{g \in \mathcal{R}(M) : \dim_{\mathbb{C}} \ker(D_g - \lambda) = d_n \text{ for every } \lambda \in \text{Spec}(D_g) \setminus \{0\}\}$$

is residual and dense in the C^∞ topology. The same conclusion holds in $\mathcal{R}_\mu(M) = \{g : dv_g = \mu\}$ for every positive smooth density μ .

More precisely, for each integer $a \geq 2$, set

$$K_a = [-a, -a^{-1}] \cup [a^{-1}, a], \quad \mathcal{G}_a = \{g : \text{every eigenvalue of } D_g \text{ in } K_a \text{ is simple}\}. \quad (1.6)$$

Each $\mathcal{G}_a$ is C^1 -open and C^∞ -dense, as is its restriction to $\mathcal{R}_\mu(M)$, and $S(M) = \bigcap_{a \geq 2} \mathcal{G}_a$.

The pathwise uniform approximation in Theorem R, relative to the initial parameter, suffices for G. At zero, the generic minimum-kernel theorem [3, 4] gives the simultaneous assertion in Corollary 7.6. Constant scaling and isotopy impose the prescribed density while preserving the nonzero multiplicities.

1.3. Path components. Let M^n be closed and connected, and let μ be a positive smooth density. Set

$$\mathcal{R}_\mu^{\text{inv}}(M) = \{g : dv_g = \mu, \ker D_g = 0\}.$$

For an invertible base metric g_* , let δ_{g_*} denote ordinary real index difference, with the normalization of Section 8.1.

Theorem C (Path components). *Let $n \geq 3$ and $g_* \in \mathcal{R}_\mu^{\text{inv}}(M)$. Then*

$$\delta_{g_*} : \pi_0 \mathcal{R}_\mu^{\text{inv}}(M) \longrightarrow KO_{n+1}$$

is a bijection of pointed sets.

The component counts for $n \equiv 0, 1, \dots, 7 \pmod{8}$ are

$$2, \quad 2, \quad 1, \quad \aleph_0, \quad 1, \quad 1, \quad 1, \quad \aleph_0.$$

The hypothesis of nonemptiness is equivalent to $\alpha(M) = 0$ by [4] and conformal covariance. No exterior is prescribed in Theorem C. Its potential paths may vary on all of M .

Corollary 1.1. *If $n \geq 5$ and M admits a metric of positive scalar curvature, each component of $\mathcal{R}_\mu^{\text{inv}}(M)$ contains such a metric.*

The corollary follows from Theorem C and the integral index-range results of [24, Theorems C, D and 5.3.1]. The density is imposed by constant rescaling and an isotopy, not by a conformal deformation of the positive-scalar-curvature metric. For simply connected spin manifolds of dimension at least five, Corollary 1.1 answers Dahl's question whether every invertible component contains a positive-scalar-curvature metric [37, p. 453]. The homotopy-equivalence question and the formulation involving all flat group-algebra coefficients in [37] are distinct statements.

1.4. Boundary metrics and the real index. Let (W, h) be a compact spin manifold with a prescribed invertible boundary metric. Its relative real Dirac index is denoted by

$$\alpha(W, h) \in KO_n, \quad n = \dim W.$$

It is the index of the operator on the cylindrical completion. Its normalization is fixed in Section 2.

Theorem I (Invertible extensions). *Suppose $n \geq 3$ and W is connected. An invertible product extension of h exists if and only if $\alpha(W, h) = 0$.*

More precisely, if $\alpha(W, h) = 0$, then for every product extension g and every nonempty open $U \subseteq \text{int } W$ there is an invertible extension $\hat{g}$ satisfying

$$\hat{g} = g \text{ on } W \setminus U, \quad dv_{\hat{g}} = dv_g. \quad (1.7)$$

There is also a trace-free, g -self-adjoint endomorphism H , compactly supported in U , such that $D_{g(e^{tH}, \cdot)}$ is invertible for all sufficiently small $t > 0$.

For the groups of Dahl–Große, this gives

$$\theta_n : R_n^{\text{inv}} \xrightarrow{\cong} KO_n, \quad n \geq 0. \quad (1.8)$$

The low degrees are treated in Appendix A. The previously established surjectivity and the definition of R_n^{inv} are those of [38, Section 5].

1.5. Finite spectra. For a closed spin manifold M^n , a finite multiset Ξ of nonzero real numbers is *admissible* if it is symmetric under $\lambda \mapsto -\lambda$ unless $n \equiv 3, 7 \pmod{8}$, and every multiplicity is even when $n \equiv 2, 3, 4 \pmod{8}$. The admissible kernel dimensions are

$n \bmod 8$	$h = \dim_{\mathbb{C}} \ker D \quad (r \in \mathbb{Z}_{\geq 0})$
0	$ \hat{A}(M) + 2r$
1	$\alpha(M) + 2r$
2	$2\alpha(M) + 4r$
3	$2r$
4	$ \hat{A}(M) + 4r$
5	$2r$
6	$2r$
7	r .

(1.9)

Here $\alpha(M) \in \{0, 1\}$ in residues one and two; $\hat{A}(M)$ is the complex chiral index in residues zero and four.

Theorem P (Finite spectral prescription). *Let M^n be closed, connected, and spin, with $n \geq 3$. Let $\Lambda > 0$, let $\Xi \subset (-\Lambda, \Lambda) \setminus \{0\}$ be a finite admissible multiset, and let h satisfy (1.9). There is a smooth metric g for which*

$$\text{Spec}(D_g) \cap ((-\Lambda, \Lambda) \setminus \{0\}) = \Xi, \quad \dim_{\mathbb{C}} \ker D_g = h. \quad (1.10)$$

The equalities in (1.10) include multiplicities. There is no volume, curvature, or conformal-class normalization in Theorem P. In particular, for every such M ,

$$\sup_{g \in \mathcal{R}(M)} \dim_{\mathbb{C}} \ker D_g = \infty$$

(Corollary 19.2). This proves the unbounded-kernel conjecture of Bär [16, Section 8], also stated by Ammann–Ginoux [5, Conjecture 2.2].

1.6. Localized harmonic spinors. For a closed spin four-manifold, put $a = |\text{ind}_{\mathbb{H}} D^+| = \frac{1}{2}|\widehat{A}(M)|$, and denote by A_g the chiral operator of quaternionic index $-a$. A metric is Dirac-minimal precisely when A_g is injective. Its full complex kernel then has dimension $2a$. In dimension five Dirac-minimality means invertibility.

Theorem L (Localized harmonic spinors). *Let M be closed, connected, and spin of dimension four or five. For every metric g , every nonempty open set $U \subset M$, and every $\varepsilon > 0$, there is a smooth metric h such that*

$$h = g \text{ on } M \setminus U, \quad dv_h = dv_g, \quad \|h - g\|_{C_g^0} < \varepsilon. \quad (1.11)$$

In dimension four it satisfies

$$\dim_{\mathbb{H}} \ker A_h = 1, \quad \dim_{\mathbb{H}} \ker A_h^* = a + 1, \quad \dim_{\mathbb{C}} \ker D_h = 2a + 4. \quad (1.12)$$

In dimension five it satisfies

$$\dim_{\mathbb{C}} \ker D_h = 2. \quad (1.13)$$

The metric belongs to a smooth finite-dimensional family h_q , with $h_0 = h$, retaining the exterior and density, for which the derivative of the complete normal matrix at $q = 0$ is an isomorphism. The normal space is $\text{Hom}_{\mathbb{H}}(\mathbb{H}, \mathbb{H}^{a+1})$ in dimension four and $\mathfrak{sp}(1)$ for the matrix of iD in dimension five.

After restriction of its parameter domain, every metric h_q satisfies (1.11). The nullities in (1.12) and (1.13) hold at $q = 0$. If A_g is injective in dimension four, a C^1 -small, density-preserving change makes the metric flat on a ball. When $\ker A_g \neq 0$, and in dimension five, the potential is constructed on the original metric.

1.7. Weak homotopy type of the minimal locus. Fix a metric g_* on a closed connected spin manifold M , put $\mu = dv_{g_*}$, and let $U \subset M$ be nonempty and open. Write

$$\mathcal{R}_{\mu}(U) = \{g : dv_g = \mu, \quad g = g_* \text{ on } M \setminus U\}.$$

Its Dirac-minimal and invertible subspaces are denoted by $\mathcal{R}_{\min, \mu}(U)$ and $\mathcal{R}_{\text{inv}, \mu}(U)$. For $n \equiv 4 \pmod{8}$ put $a = |\text{ind}_{\mathbb{H}} D^+|$; thus $|\widehat{A}(M)| = 2a$. Let $\text{Sp} = \text{colim}_m \text{Sp}(m)$ and $U_{\text{st}} = \text{colim}_m U(m)$.

Theorem T (Weak homotopy type of the minimal locus). *The indicated spaces are nonempty. For every base metric in the corresponding space there are weak homotopy equivalences*

$$\kappa_a : \mathcal{R}_{\min, \mu}(U) \longrightarrow \text{Sp} / \text{Sp}(a), \quad n \equiv 4 \pmod{8}, \quad (1.14)$$

$$\kappa_5 : \mathcal{R}_{\text{inv}, \mu}(U) \longrightarrow \Omega \text{Sp} \simeq \text{Sp} / U_{\text{st}}, \quad n \equiv 5 \pmod{8}. \quad (1.15)$$

The map κ_a is the cokernel comparison to the homotopy fibre of $B \text{Sp}(a) \rightarrow B \text{Sp}$. Its stable nullhomotopy is given by the ambient affine contraction of compatible pairs. If $a = 0$, it is ordinary real index difference. The map κ_5 is ordinary real index difference in the quaternionic skew-adjoint model.

For nonempty open sets $U_0 \subset U_1$, with the same prescribed metric and density, the inclusion of the corresponding minimal spaces is a weak homotopy equivalence.

The quotient in (1.14) is $\text{colim}_r \text{Sp}(r + a) / \text{Sp}(a)$. In particular, $\mathcal{R}_{\min, \mu}(U)$ is $(4a + 2)$ -connected and

$$\pi_{4a+3}(\mathcal{R}_{\min, \mu}(U), g_b) \cong \mathbb{Z} \quad (n \equiv 4 \pmod{8}).$$

In residue five the invertible space is simply connected and

$$\pi_2(\mathcal{R}_{\text{inv}, \mu}(U), g_b) \cong \mathbb{Z}.$$

For $n \equiv 4 \pmod{8}$ with $a = 0$, and for $n \equiv 5 \pmod{8}$, ordinary real index difference induces isomorphisms

$$\pi_k(\mathcal{R}_{\text{inv}, \mu}(U), g_b) \xrightarrow{\cong} KO_{n+k+1}, \quad k \geq 1. \quad (1.16)$$

In dimensions four and five the integer generators are signed counts of kernel lines in transverse fillings. Theorem L places a representing sphere in every prescribed C^0 neighborhood of any metric in $\mathcal{R}_{\mu}(U)$; its initial basepoint is a point of that sphere. At nonzero index the cokernel in (1.14) has rank a and is retained throughout the comparison.

1.8. Relation to earlier results. The spinor comparison and metric variation formulas are due to Bourguignon–Gauduchon [25, Propositions 1 and 5; Theorems 20–21]; the hypersurface formula is given in [20]. Hermann [56] gives the unitary comparison, unique continuation, and analytic perturbation statements in the conventions needed below. A rotating symmetric coframe with a quadratic scalar Dirac term appears in [39, Section 8]. Agricola–Friedrich consider the metric connection with skew-symmetric torsion, whose spin Dirac operator is $D_g + \frac{3}{4}c_g(T)$ [1, Section 2]. Khrabustovskyi–Lotoreichik prove norm-resolvent homogenization for a two-dimensional Dirac operator with position-dependent mass [63]. In Theorem R the approximating operators are Levi-Civita Dirac operators. The approximation preserves support, volume density, and relative parameter conditions, and includes the first parameter derivatives.

Bandara–McIntosh–Rosén prove Riesz-continuity estimates under uniform geometric bounds on the reference metric and a bound on $\|\nabla^g h\|_\infty$ for the compared metric [15, Theorem 3.1 and Remark 3.2]. The elementary oscillatory metrics of Section 4 satisfy $g_N - g = O(N^{-1/2})$ in C^0 , whereas their first spatial derivatives can have order $N^{1/2}$. The cited estimate requires a uniform bound on these derivatives.

In dimension three, Dahl proves generic simplicity and generic simplicity of the nonzero spectrum within each conformal class [35, Theorem 1.2 and Remark 1.3]. The July 2013 correction in the second arXiv version adds the condition $\ker D_g = 0$ in Proposition 3.2. Theorem G proves the nonzero-spectrum assertion of the higher-dimensional conjecture following [35, Theorem 1.2]. Its proof is on fixed spectral windows separated from zero.

Canzani proves conformal genericity under a hypothesis excluding rigid eigenspaces [29, Definition 2.2 and Theorem 2.3]; the nonzero-window argument is [29, Proposition 7.4]. Chatzakou–Helffer give a nonscalar-compression criterion for bounded real-potential perturbations of semibounded self-adjoint operators with compact resolvent and discuss finite windows [31]. Brito–Marrocos study generic simplicity for connection Laplacians and G -simplicity on principal bundles [27]. These operator families differ from the untwisted metric Dirac family. The splitting lemma in Section 7 concerns compatible potentials at every pair $D_g + W$; a conformal direction is added in dimension 2 (mod 8). Its metric consequence follows from R and an analytic path argument. No higher-dimensional conformal-class assertion is made.

For the scalar Laplacian on a closed connected manifold of dimension at least three, Colin de Verdière realizes any prescribed finite list of initial Laplace eigenvalues, counted with multiplicity, with zero simple and all remaining entries positive [32, Theorem 1.2]. He compares interaction matrices over parameter families and obtains the prescribed multiplicities by transversality [32, p. 600]. Here the additional analytic result is the supported relative realization of compatible Dirac perturbations in Theorem R, with prescribed density. The spectral construction also respects the real spinor structures and retains the index-forced zero module through surgery.

Local sections through a prescribed reference for scalar Schrödinger operators and positive Laplace clusters are constructed in [71, Theorems A and 5.3]; at a singular reference they may have only one-half Hölder regularity. Proposition 6.3 transfers a smooth potential section to a smooth metric section by Theorem R; the reference metric may change.

Dahl’s finite-prescription theorem concerns simple nonzero eigenvalues [36, Theorem 3]. Conjecture 9 asks for arbitrary finite spectral data, including the multiplicity of zero, subject to the spinorial symmetries and the index restrictions. Compact-parameter spectral surgery is [36, Theorem 6]. Nowaczyk obtains an eigenvalue of complex multiplicity at least two in dimensions 0, 6, 7 (mod 8), with a prescribed exterior [72, Main Theorem 1]. Theorem P proves Conjecture 9 in dimensions at least three: it prescribes an arbitrary admissible finite multiset and the kernel dimension simultaneously. The Berger eigenvalues are due to Hitchin [60]; his prime-isolation argument is explicitly attributed there to Chowla.

Maier proves generic index-minimality in dimensions two, three, and four [66, Theorems 1.1, 1.2(i), and 1.3(i), p. 409]. Ammann–Dahl–Humbert give a surgery proof in every dimension at least two

[3, Theorem 1.1] and then allow the metric to change in any prescribed nonempty open set [4, Theorem 1.1].

Conjecture A of Bär–Dahl [19] asks whether every closed spin manifold of dimension at least three admits a metric with a nonzero harmonic spinor. In dimensions $3 \pmod{4}$, Bär constructs such a metric for every fixed spin structure [16, Theorem A]; the change may be confined to an arbitrarily small neighborhood of a point [16, Section 4]. Under the same dimensional restriction, fixed twisting bundles and connections are treated in [17, Theorem 4.1]. Crowley–Schick–Steimle prove existence of harmonic spinors in every dimension at least six [33, Theorem 1.4]. Farinelli states the corresponding existence assertion in every dimension at least four [44, Theorem 1.1, p. 3]. The later account [65, Section 1, p. 2] treats general existence in dimensions four and five as open. Theorem L gives an independent localized construction with prescribed density, arbitrarily small C^0 change, the smallest kernel dimension above the index minimum, and an invertible normal derivative.

Dahl–Große define the groups R_n^{inv} and prove surjectivity of $\theta_n : R_n^{\text{inv}} \rightarrow KO_n$ [38, Section 5]. Their Conjecture 5.12 asks for injectivity. Theorem I, Corollary 15.1, and Appendix A prove this conjecture in every degree $n \geq 0$. The case $R_5^{\text{inv}} = 0$ also follows from earlier results: connectedness of the invertible locus on S^4 [2], isotopy-to-concordance [37, Corollary 2.4], and [38, Proposition 5.8]. Together with [38, Theorem 5.3(1)], this also gives existence of an invertible extension for every prescribed invertible boundary metric in dimension five. Theorem I adds a density-preserving modification in any prescribed interior open set and an analytic ray supported there.

Dahl detects distinct invertible components by index difference [37]; Waterstraat proves that there are infinitely many components in dimensions $3 \pmod{4}$ [80]. Theorem C concerns the converse: equal ordinary index differences determine the same component on the fixed manifold. Invertibility of a cylindrical concordance does not imply invertibility of every intrinsic slice operator. A spin-bordism comparison for spaces of invertible metrics is stated in [52, Theorem 1, p. 3]. The later account [65, Remark 1.4] treats this equivalence as an open problem. Combined with [65, Theorem B], the equivalence stated in [52] would imply existence of a metric with a nonzero harmonic spinor on every closed connected spin four-manifold with $\hat{A} = 0$ [65, Remark 1.4]. Surgery maps for spaces of Dirac-minimal metrics are discussed in [2, Remark 1.2(3)], with reference to [74, 52]; the existence of homotopy inverses is stated there as an expectation. The proof of C proceeds directly through compatible multiplication potentials. In the real skew-adjoint case, the zero-current fibre in dimensions at least seventeen is an OT–FKM focal manifold. Its description by a symmetric Clifford system is due to Ferus–Karcher–Münzner [45, Section 4.2(ii), p. 485]; the sphere-bundle description is recorded in [75, Section 1]. Lemma 13.5 applies that description to relative spinor sections.

On a closed Riemannian manifold, fix a compatible Dirac symbol on a finite real Clifford bundle with ample L^2 space, and an invertible pseudo-Dirac operator B . The bounded transform induces a weak homotopy equivalence from the space of smooth pseudo-Dirac paths starting at B and ending at an invertible operator to the corresponding real Fredholm loop-space model [41, Definition 4.2 and Proposition 4.3]. The perturbations in this space are arbitrary compatible pseudodifferential operators of order zero, with fixed principal symbol. Hertl constructs a cubical model of invertible block Dirac operators and factors index difference through positive-scalar-curvature concordances [57, Theorems B and C, p. 8]. Here Sections 8–14 concern multiplication potentials and varying metric symbols. The supported relative fillings of Section 22 give the comparison in T; Theorem R gives its metric form.

The real index conventions are those of [10, 12, 14]. The comparison between path and cylindrical index difference is [41]. The real skew-adjoint case is treated in [26, Theorem 8.1], with the two-plane model also described in [30]. The positive-scalar-curvature corollary requires the integral range statements of [24, Theorems C, D and 5.3.1]; rational surjectivity alone would not suffice. A simply connected closed spin manifold of dimension at least five admits a positive-scalar-curvature metric if and only if $\alpha(M) = 0$ [78, Theorem A]. No scalar-curvature bound is imposed on the approximating metrics.

For L , the four-dimensional point sources first satisfy the cokernel-orthogonality equation. In dimension five, the Green spinor has a nonzero collar current with zero flux. The current identities are related to [21, Section 4]. The nonvanishing exact-form construction is [43, Theorem 18.4.1, p. 171; Corollary 20.3.2, pp. 181–182]; its relative spinor lift is included in Section 21. The three-dimensional eigenspinor construction of divergence-free framings is [64].

Ammann–Dahl prove connectedness of the Dirac-minimal metric space on closed connected spin manifolds of dimensions two and four [2, Theorem 1.1]. Their codimension estimates concern the nonminimal strata [2, Theorem 1.3]. After Theorem 1.4 they state the bound $2(k + 1)$ when $|\hat{A}(M)| = 2k$, with details deferred. For $n = 4$ and $a = 0$, Theorem T with $U = M$ and Lemma 8.1 prove the two-connectedness conjecture of Ammann–Dahl recorded in [65, Remark 1.6]. Theorem T determines the weak homotopy types in every dimension 4 or 5 (mod 8), with fixed exterior and density. The Fredholm interpretation of real K -theory and the equivalence $\Omega \mathrm{Sp} \simeq \mathrm{Sp} / U_{\mathrm{st}}$ are given in [13, 23]. Section 22 identifies the relative rectangular model with $\mathrm{Sp} / \mathrm{Sp}(a)$ and constructs the comparison from supported multiplication potentials. For $a > 0$ the model retains the rank- a cokernel together with a nullhomotopy of its stable class. At index zero it identifies all homotopy groups with ordinary real K -theory, and the inclusions associated to $U_0 \subset U_1$ induce these same identifications.

For $M = S^1 \times S^{11}$ and either spin structure on the circle, Theorem T gives $\pi_4 \mathcal{R}^{\mathrm{inv}}(M) \cong \mathbb{Z}/2$. For the projection $M \rightarrow B\mathbb{Z} = S^1$, the rational higher index target in degree 17 is $\mathbb{Q}$; the theorem of [42, Theorem C, p. 1553] therefore detects an infinite-order class in $\pi_4 \mathcal{R}^+(M)$. Twice this class therefore lifts to an infinite-order element in the fourth homotopy group of the homotopy fibre of $\mathcal{R}^+(M) \hookrightarrow \mathcal{R}^{\mathrm{inv}}(M)$ (Proposition 22.26). This consequence concerns the inclusion of the two spaces; the classification in T concerns the space of Dirac-minimal metrics.

1.9. Proof of the principal results. *Theorem R.* By Lemma 2.2, a compatible potential is a sum of Clifford forms of degrees congruent to three modulo four. Proposition 5.1 replaces a term of degree $\ell \geq 7$ by terms of degrees 3 and $\ell - 4$ through a positive congruence. For each replacement, Lemma 3.2 gives

$$(T_{\mathrm{new}} - z)^{-1} = F(T_{\mathrm{old}} + E - zF^2)^{-1}F.$$

Here F is a positive multiplier with $F^{\pm 1}H^1 = H^1$, $F - I \rightarrow 0$ in $C_b^1(\mathcal{L}(L^2))$, and $E \rightarrow 0$ in $C_b^1(\mathcal{L}(H^1, L^2))$ as the frequency tends to infinity. Descending induction leaves only three-form terms, which Proposition 4.2 removes by changes of metric. All these operations preserve the prescribed support and parameter equalities; the metric changes preserve the density. There are finitely many steps. Choose their frequencies successively so that the j th resolvent error, after conjugation by the preceding comparisons, is less than $2^{-j}\theta$, and the j th metric increment is less than $2^{-j}\eta$. Equation (5.8) bounds the total resolvent error by θ , and (5.9) gives the direct positive spinor comparison. Taking θ and η sufficiently small proves (1.3) and (1.4).

Spectral sections. Contour integration gives the spectral projections and first spectral moments in C^1 . Lemma 6.1 then gives C^1 convergence of the complete compressed matrices in compatible orthonormal frames. For a full potential section with matrix A , the resulting metric matrix $F(A)$ satisfies $\|F - \mathrm{id}\|_{C^1} \rightarrow 0$. Proposition 6.3 applies the inverse function theorem to F and gives a full metric section with the same density and exterior.

Theorem G. Fix a metric g and a window K_a as in (1.6). Lemma 7.1 gives a nonscalar compression on every multiple nonzero eigenspace; in residue two, a conformal variation is also allowed. Lemma 7.3 gives a path of compatible pairs, constant near its initial point $(g, 0)$, with simple terminal spectrum in contours enclosing K_a and separated from zero. Relative realization and Lemma 7.4 give a real-analytic metric path g_t with $g_0 = g$, the same contours, and simple terminal spectrum. Divide the traces of the spectral moments by d_n and apply Newton’s identities. The resulting monic cluster polynomials have the eigenvalues as roots, with their complex multiplicities divided by d_n . The product $\Delta(t)$ of their discriminants is real analytic and satisfies $\Delta(1) \neq 0$. Write $\Delta(t) = t^m \delta(t)$, where $m \geq 0$ and $\delta(0) \neq 0$. Thus $\Delta(t) \neq 0$ for all sufficiently small $t > 0$. Since $g_t \rightarrow g$ in C^∞ , this proves finite-window density (Proposition 7.5). Spectral continuity gives C^1 openness; Baire’s theorem gives G. Constant scaling

and Moser's isotopy impose the prescribed density while preserving multiplicities [69, Theorem 2, Section 4, pp. 292–293].

Theorem C. Let $\mathcal{P}_\mu^{\text{inv}}(M)$ be the space of compatible pairs (g, V) with $dv_g = \mu$ and $D_g + V$ invertible. Proposition 8.2, with $U = M$, identifies its components with those of $\mathcal{R}_\mu^{\text{inv}}(M)$ and preserves ordinary index difference. The pair classification is Theorem 11.6 in residues 3, 7, Theorem 11.11 in residues 2, 4, 5, 6, and Theorems 12.1 and 13.1 in residues 0 and 1. Thus

$$\pi_0 \mathcal{R}_\mu^{\text{inv}}(M) \xrightarrow{\cong} \pi_0 \mathcal{P}_\mu^{\text{inv}}(M) \xrightarrow{\delta_{g^*}} KO_{n+1}$$

is a bijection. For Corollary 1.1, the integral index-range results of [24, Theorems C, D and 5.3.1] give a positive-scalar-curvature representative of every ordinary index-difference class when $n \geq 6$; for $n = 5$ the target is $KO_6 = 0$. Theorem C identifies the invertible components, and Lemma 14.1 imposes the prescribed density. The even- and odd-dimensional index maps are given explicitly in Section 14.

Theorem I. Choose a nonempty open set $U_0 \subseteq U$. If $\alpha(W, h) = 0$, Proposition 9.1 gives a compatible V supported in U_0 with $D_g + V$ invertible. Corollary 5.2 gives an invertible metric $\hat{g}$ with the same density and equal to g outside U_0 . Thus $H = \log(g^{-1}\hat{g})$ satisfies $\text{supp } H \subset \overline{U_0} \subseteq U$ and $\text{tr } H = 0$. The compared operators $L_t : H^1 \rightarrow L^2$ of $g_t = g(e^{tH}, \cdot)$ extend holomorphically near $[0, 1]$ as Fredholm operators. Since L_1 is invertible, $L_z L_1^{-1}$ is a holomorphic Fredholm family on L^2 , equal to I at $z = 1$. The analytic Fredholm theorem [79, Theorem 1.1, p. 2] makes the noninvertible parameters discrete. In particular, D_{g_t} is invertible for all sufficiently small $t > 0$. The converse follows from the vanishing of the index of an invertible operator. Corollary 15.1 and Appendix A give (1.8).

Theorem P. Theorem 16.3 and Corollary 16.4 give full metric sections on sufficiently large balanced modules. The index-minimal kernel on M is supplied by [4, Theorem 1.1]. Lemma 19.1 gives the prescribed nonzero blocks; Proposition 17.3 gives the remaining zero-module equations over a disk bundle. Proposition 18.1, based on [72, Theorem 4.24], transports the complete matrices through spin surgeries of codimension at least three with an arbitrarily small uniform error. Eliminating the invertible complement of the zero block by a Schur complement, and applying the polar comparisons to the nonzero blocks, gives a continuous section close to $(A, X, 0)$. Lemma 17.1 gives a common zero of its equations. At that parameter the nullity is h and the prescribed blocks equal $\lambda_j I$. The unused blocks remain outside $(-\Lambda, \Lambda)$, and preservation of the full band rank excludes further eigenvalues there. This proves (1.10); continuity in the surgery parameter suffices for the intersection argument.

Theorem L. In dimension four, start with zero potential if $\ker A_g \neq 0$. Otherwise apply Lemmas 20.1 and 20.2. Proposition 20.3 then gives a supported full potential section with chiral matrix $q \in \text{Hom}_{\mathbb{H}}(\mathbb{H}, \mathbb{H}^{a+1})$. In dimension five, Propositions 21.2 and 21.3 give a section with matrix q for $i(D_g + V(q))$, where $q \in \mathfrak{sp}(1)$. Proposition 6.3, with arbitrarily small C^0 error, gives the corresponding metric sections. At $q = 0$ the entire zero cluster is the kernel, of the dimensions in (1.12) and (1.13); the normal derivative is the identity. The support and density equalities persist.

Theorem T. The pair comparisons of Proposition 22.24, composed with the inclusions of Proposition 8.2, give

$$\mathcal{R}_{\min, \mu}(U) \longrightarrow \mathcal{P}_{\min, \mu}(U) \longrightarrow \text{Sp} / \text{Sp}(a), \quad n \equiv 4 \pmod{8},$$

$$\mathcal{R}_{\text{inv}, \mu}(U) \longrightarrow \mathcal{P}_{\text{inv}, \mu}(U) \longrightarrow \Omega \text{Sp}, \quad n \equiv 5 \pmod{8}.$$

Both arrows in each composition are weak homotopy equivalences. For the second arrows, the inhomogeneous cylinder reduction (Proposition 22.17) gives the finite matching map. Lemma 22.15 realizes its deformations while retaining the exact boundary family. Rectangular completion retains the cokernel and its stable nullhomotopy; Proposition 22.23 identifies the full relative Cayley class in residue five. The ranks in (22.45) are chosen after the homotopy degree. The relative fillings in Proposition 22.24 give injectivity and surjectivity on every homotopy group, with density and exterior fixed. Corollary 22.25 gives the assertion for $U_0 \subset U_1$. Finally, Propositions 22.4 and 22.5 give the local integer generators and their index normalization.

2. SPINOR STRUCTURES AND FIXED DOMAINS

2.1. Charge conjugation. Let $s_n = 2^{\lfloor n/2 \rfloor}$ be the complex rank of the full spinor bundle. There is a parallel antiunitary map J ; in even dimension there is also the chirality operator Γ . Our conventions are

$n \bmod 8$	J^2	$Jc(\xi) = \epsilon_n c(\xi)J$	$J\Gamma = \tau_n \Gamma J$
0	+1	+1	+1
1	+1	-1	--
2	-1	+1	-1
3	-1	+1	--
4	-1	+1	+1
5	-1	-1	--
6	+1	+1	-1
7	+1	+1	--

(2.1)

The last two sign columns are ϵ_n and τ_n . Since J is parallel, $JD = \epsilon_n DJ$. In even dimension $\Gamma^* = \Gamma$, $\Gamma^2 = I$, and $\Gamma D = -D\Gamma$. The real and quaternionic structures are also recorded in [56, Theorem 2.2.2]; the following model fixes our signs.

On $(\mathbb{C}^2)^{\otimes m}$ set

$$\begin{aligned} c_{2j-1} &= i\sigma_3^{\otimes(j-1)} \otimes \sigma_1 \otimes I^{\otimes(m-j)}, \\ c_{2j} &= i\sigma_3^{\otimes(j-1)} \otimes \sigma_2 \otimes I^{\otimes(m-j)}, \quad 1 \leq j \leq m. \end{aligned} \quad (2.2)$$

Here $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, and $\sigma_3 = \text{diag}(1, -1)$. The odd-indexed c_j are imaginary and the even-indexed c_j are real. If m is odd take $Q = c_2 c_4 \cdots c_{2m}$; if m is even take $Q = c_1 c_3 \cdots c_{2m-1}$. Then $J = Q \circ$ conjugation commutes with every c_j , and

$$J^2 = (-1)^{m(m+1)/2}, \quad \Gamma = i^m c_1 \cdots c_{2m}, \quad J\Gamma = (-1)^m \Gamma J.$$

For dimension $2m + 1$, add $c_{2m+1} = i\Gamma$. Retain J if m is odd and replace it by ΓJ if m is even. These formulas give (2.1). Commutation with the spin action makes them parallel bundle maps.

Definition 2.1. The real vector bundle $\mathcal{A}_n \subset \text{End}_{\mathbb{C}}(\Sigma X)$ consists of the endomorphisms V satisfying

$$V^* = V, \quad JV = \epsilon_n VJ, \quad \Gamma V = -V\Gamma \quad (n \text{ even}). \quad (2.3)$$

For a finite-dimensional Hermitian space E carrying an antiunitary J and, in even dimension, a self-adjoint grading Γ with the prescribed values of J^2 and $J\Gamma = \tau_n \Gamma J$ in (2.1), let $\mathcal{A}_n(E)$ be the space defined by (2.3). We call such a structured space a *zero module*.

A zero module is *balanced* if it admits an invertible element of $\mathcal{A}_n(E)$. For a balanced module, the independent matrix and total complex dimension are as follows, with $r \geq 0$:

$n \bmod 8$	independent matrix	$\dim_{\mathbb{C}} E$
0	$\text{Mat}_r(\mathbb{R})$	$2r$
1	$\text{iso}(2r)$	$2r$
2	$\text{Skew}_{2r}(\mathbb{C})$	$4r$
3	$\text{Herm}_r(\mathbb{H})$	$2r$
4	$\text{Mat}_r(\mathbb{H})$	$4r$
5	$\text{isp}(r)$	$2r$
6	$\text{Sym}_r(\mathbb{C})$	$2r$
7	$\text{Sym}_r(\mathbb{R})$	r .

(2.4)

In even ambient dimension, the independent matrix Z represents

$$D(Z) = \begin{pmatrix} 0 & Z^* \\ Z & 0 \end{pmatrix}. \quad (2.5)$$

In residue five $\mathfrak{isp}(r)$ is taken in the complex $2r$ -dimensional representation. It is not multiplication of quaternionic entries by a chosen quaternionic unit.

In residues zero and four, J preserves the two chiral spaces. The independent block is respectively real or quaternionic; it is invertible precisely when it is square and invertible over that field. In residues two and six, choose chiral coordinates in which

$$J(x, y) = (\varepsilon \bar{y}, \bar{x}), \quad \varepsilon = J^2 \in \{-1, 1\}.$$

Then $JD(Z) = D(Z)J$ is equivalent to $Z^T = \varepsilon Z$. This gives the skew-complex and symmetric-complex rows. In residue one, a real frame for J gives $V = iA$ with A real skew-symmetric. In residues three and seven, V is respectively quaternionic Hermitian or real symmetric. In residue five, $A = -iV$ commutes with J and satisfies $A^* = -A$, so $A \in \mathfrak{sp}(r)$. Real and complex skew-symmetric matrices admit invertible elements exactly in even size. The other displayed matrix spaces admit invertible elements at every size. These facts prove the table, including its balance conditions.

Lemma 2.2. *Clifford multiplication induces a real vector-bundle isomorphism*

$$\bigoplus_{\substack{3 \leq \ell \leq n \\ \ell \equiv 3 \pmod{4}}} \Lambda^\ell T^* X \xrightarrow{c} \mathcal{A}_n. \quad (2.6)$$

Proof. For $n = 0$, the grading is scalar and both sides of (2.6) vanish. Suppose $n \geq 1$. For an increasing multi-index I of length ℓ , set $c_I = c(e^{i_1}) \cdots c(e^{i_\ell})$. Reversal of the factors gives

$$c_I^* = (-1)^{\ell(\ell+1)/2} c_I.$$

Thus an odd monomial is self-adjoint precisely when $\ell \equiv 3 \pmod{4}$. For real covectors, (2.1) gives $Jc_I = \epsilon_n^\ell c_I J = \epsilon_n c_I J$; in even dimension $\Gamma c_I = -c_I \Gamma$. Every summand on the left of (2.6) therefore belongs to $\mathcal{A}_n$.

If $I \neq I'$ have odd cardinality, $c_I^* c_{I'}$ is, up to sign, c_K for a nonempty even multi-index K . For $a \in K$,

$$c(e^a) c_K c(e^a)^{-1} = -c_K.$$

Invariance of trace under conjugation gives $\text{tr } c_K = 0$. Since $\text{tr}(c_I^* c_I) = s_n$, the indicated monomials are linearly independent.

The root-of-unity filter gives

$$\begin{aligned} \sum_{\ell \equiv 3 \pmod{4}} \binom{n}{\ell} &= \frac{1}{4} \sum_{j=0}^3 i^{-3j} (1 + i^j)^n \\ &= 2^{n-2} - 2^{n/2-1} \sin(n\pi/4). \end{aligned} \quad (2.7)$$

For $d = s_n$, the real dimensions of $\mathcal{A}_n$, in residue order $0, \dots, 7$, are

$$\frac{d^2}{4}, \quad \frac{d(d-1)}{2}, \quad \frac{d(d-2)}{4}, \quad \frac{d(d-1)}{2}, \quad \frac{d^2}{4}, \quad \frac{d(d+1)}{2}, \quad \frac{d(d+2)}{4}, \quad \frac{d(d+1)}{2}.$$

These are the dimensions of the matrix spaces derived above. In even positive dimension, multiplication by a unit covector interchanges the chiral fibres, so each has complex dimension $d/2$. The skew-matrix dimension formulas hold in odd as well as even matrix size; balance is not needed here. Substitution of $d = 2^{\lfloor n/2 \rfloor}$ agrees with (2.7) in each residue. The fibre map is therefore bijective. Its coefficients and those of its inverse are smooth in every orthonormal frame, proving the bundle assertion. $\square$

2.2. Domains and local compactness. Let X be closed or have the fixed product ends of Theorem R. In a parameter family, all metrics coincide with the reference metric outside one compact subset, and the potentials have a common compact support. Fix a reference metric g_0 . If $h = g_0(S, \cdot)$, the positive map $S^{-1/2}$ identifies orthonormal frames. Its spin lift is denoted by $\beta_{g_0}^h$. It is determined by

continuation from the identity along $S_t^{-1/2}$, where $S_t = \exp(t \log S)$ and $0 \leq t \leq 1$. This lift depends smoothly on the positive metric matrices and equals the identity at $h = g_0$. The same convention holds for every successive comparison. With $\rho = dv_h/dv_{g_0}$, set

$$U_h = \rho^{-1/2} \beta_{g_0}^h : L^2(\Sigma_{g_0}) \longrightarrow L^2(\Sigma_h). \quad (2.8)$$

It is unitary and intertwines J and Γ ; see [56, Lemmas 2.3.3–2.3.5]. On a compact set of smooth metrics with these fixed ends, U_h and U_h^{-1} have uniform H^1 bounds, and

$$U_h^{-1} D_h U_h : H^1(\Sigma_{g_0}) \longrightarrow L^2(\Sigma_{g_0})$$

depends smoothly on every finite smooth parameter. Indeed, in local frames its first-order coefficients depend smoothly on h , and its zeroth-order coefficients depend smoothly on h and its first spatial derivatives; see [56, Theorem 2.3.6]. Compactness of the varying region gives the asserted operator-norm dependence.

Every metric in question is complete and smooth, with a compact core and fixed product ends. It has positive injectivity radius and bounded Ricci tensor and first covariant derivative. Thus the hypotheses of [15, Theorem 3.1] hold with both compared metrics equal to h : their distance is zero, the comparison is the identity, and $\nabla^h h = 0$. Propositions 3.4 and 3.6 of that paper give essential self-adjointness on C_c^∞ , domain $H^1(\Sigma_h)$ for the closure, and equivalence of its graph norm with the H^1 norm. For bounded self-adjoint V , the operator $D_h + V$ is self-adjoint on the same domain, by [62, Chapter V, Theorem 4.3]; the relative bound of V is zero. In the identification (2.8),

$$\|u\|_{H^1} \leq C(\|(D_h + V)u\|_{L^2} + \|u\|_{L^2}). \quad (2.9)$$

For a fixed compact smooth parameter family, the lower bound on injectivity radius, the relevant geometric bounds, $\|V\|_\infty$, and the H^1 comparison bounds are uniform. Hence C is uniform on that family. The constant may depend on the finite spatial derivative bounds of the family.

A bounded compactly supported multiplication operator is compact from H^1 to L^2 : multiply first by a smooth cutoff equal to one on its support and apply the Rellich–Kondrachov theorem [40, Theorem 15.8] in finitely many bundle charts. It is therefore relatively compact with respect to D_h , by the graph-norm equivalence. Hence it preserves the essential spectrum [62, Chapter IV, Theorem 5.35]. If the boundary operator has gap $(-\lambda_0, \lambda_0)$, its cylindrical extension has no essential spectrum in that interval [38, Proposition 2.1]. The same holds for $D_h + V$. Consequently every compact subinterval contains only finitely many isolated eigenvalues, each of finite multiplicity.

Lemma 2.3. *Let E, F be smooth vector bundles of equal rank on a manifold of positive dimension, and let $A : C^\infty(E) \rightarrow C^\infty(F)$ be a first-order differential operator with smooth coefficients. Suppose its principal symbol at x_0 is invertible at every nonzero real covector. If a distribution T with values in E is supported at x_0 and*

$$AT = \delta_{x_0} v, \quad v \in F_{x_0},$$

then $T = 0$ and $v = 0$.

Proof. Choose coordinates centered at x_0 and local bundle frames. The point-support theorem for distributions, applied to each component, gives [40, Theorem 4.19]

$$T = \sum_{|\alpha| \leq m} t_\alpha \partial^\alpha \delta_0. \quad (2.10)$$

Suppose $T \neq 0$, and let q be the largest order for which some $t_\alpha \neq 0$. Write $A = \sum_j a^j(x) \partial_j + a^0(x)$ and set

$$p_q(\xi) = \sum_{|\alpha|=q} t_\alpha \xi^\alpha, \quad a(0, \xi) = \sum_j a^j(0) \xi_j.$$

The polynomial representing the derivatives of order $q+1$ in AT is $a(0, \xi) p_q(\xi)$. Derivatives of the coefficients contribute only orders at most q . Since AT has order zero, $a(0, \xi) p_q(\xi) = 0$. Invertibility for real $\xi \neq 0$ gives $p_q(\xi) = 0$ there. A polynomial vanishing on this open set is zero, contrary to the definition of q . Hence $T = 0$; the displayed equation then gives $v = 0$. $\square$

Lemma 2.4. *Let $n \geq 3$, $0 < m < n$, and let $b_p(x, y, \xi)$ be a smooth matrix-valued amplitude on a coordinate ball, with p in a compact smooth parameter manifold. Suppose that, in parameter charts,*

$$|\partial_p^\alpha \partial_x^\beta \partial_y^\gamma \partial_\xi^\delta b_p(x, y, \xi)| \leq C_{\alpha\beta\gamma\delta} (1 + |\xi|)^{-m-|\delta|}, \quad |\alpha| \leq 1.$$

The distribution defined by the oscillatory integral

$$K_p(x, y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x-y, \xi \rangle} b_p(x, y, \xi) d\xi$$

is smooth in the spatial variables off the diagonal, is of class C^1 in p there, and satisfies

$$|\partial_p^\alpha K_p(x, y)| \leq C|x - y|^{m-n}, \quad 0 < |x - y| < 1, \quad |\alpha| \leq 1, \quad (2.11)$$

on every smaller coordinate ball. The constant depends on finitely many uniform symbol seminorms. Addition of a smooth kernel with uniform $C_p^1 C_{x,y}^0$ norm preserves this conclusion.

Proof. Apply the inverse-Fourier estimate of [49, Proposition 2.2, p. 55] entrywise to $\partial_p^\alpha b_p$, $|\alpha| \leq 1$, at the Fourier variable $x - y$, with (p, x, y) fixed as auxiliary parameters. In that proposition take the symbol order $r = -m$ and an integer $N > n - m$. Its constants depend only on the listed ξ -seminorms, which are uniform here. Since $-m > -n$, its power bound is precisely (2.11).

For the regularity assertion, use

$$L = \frac{(x - y) \cdot \partial_\xi}{i|x - y|^2}, \quad L e^{i\langle x-y, \xi \rangle} = e^{i\langle x-y, \xi \rangle}.$$

On a compact set off the diagonal, integration by parts with L makes the integral absolutely convergent after any prescribed number of spatial derivatives and at most one parameter derivative. The symbol bounds give a common integrable majorant. Differentiation under the integral proves spatial smoothness and C^1 parameter dependence. This is the amplitude version of the off-diagonal regularity in [61, Theorem 18.1.16]. A smooth remainder is absorbed because $|x - y|^{m-n} \geq 1$ for $0 < |x - y| < 1$. $\square$

2.3. The real index. Write $\alpha = \alpha(M)$ when M is closed, and $\alpha = \alpha(W, h)$ for the cylindrical completion of a compact spin manifold W with fixed invertible boundary metric h . All kernels in the latter case are L^2 kernels of the completed operator. In residues zero and four, respectively,

$$\alpha = \text{ind}_{\mathbb{R}} D^+, \quad \alpha = \text{ind}_{\mathbb{H}} D^+.$$

Here the first index is taken on the real fixed spaces of J ; in the second, J gives the chiral spaces their quaternionic structures. On a closed manifold,

$$\text{ind}_{\mathbb{C}} D^+ = \widehat{A}(M) = \begin{cases} \alpha(M), & n \equiv 0 \pmod{8}, \\ 2\alpha(M), & n \equiv 4 \pmod{8}. \end{cases}$$

The complex index formula is [12, Theorem 5.3]. The coefficient groups are given by [10, Table 2 and Theorem 11.5]; their complexification factors are computed in Section 6 of that paper, immediately before and in formula (6.11).

In residues one and two, respectively,

$$\alpha = \dim_{\mathbb{R}}(\ker D \cap \text{Fix } J) \pmod{2}, \quad \alpha = \dim_{\mathbb{C}} \ker D^+ \pmod{2}.$$

In residue one, iD preserves the real Hilbert space $\text{Fix } J$ and is skew-adjoint there, with domain $H^1 \cap \text{Fix } J$. Its complexification is iD on the complex spinor Hilbert space. In residue two, J exchanges the chiral kernels, so the formula counts one chiral kernel. These identifications are [14, Theorems 3.1 and 3.3]. The groups KO_n vanish in the other four residues.

For cylindrical completions we use the index convention of [38, Section 5.4]. In either case, $D + tV : H^1 \rightarrow L^2$, $0 \leq t \leq 1$, is a continuous Fredholm path for every compactly supported compatible potential V . Fredholmness follows from ellipticity in the closed case [40, Theorem 15.13 and Remark 15.2], and from the essential spectral gap in the cylindrical case. The identities (2.3) hold along this path. Homotopy invariance of the real Fredholm index therefore shows that these perturbations preserve α .

3. RESOLVENTS UNDER POSITIVE CONGRUENCE

Let $\mathcal{D} \subset \mathcal{H}$ be a dense, continuously included complex Hilbert subspace. All unbounded operators in this section have domain $\mathcal{D}$. The parameter b lies in a compact smooth manifold B , possibly with boundary. The C^1 norms include the zeroth-order norm and the first parameter derivatives in a fixed finite atlas, with the usual local extensions at boundary points. For a compact set $\mathcal{K} \subset B \times \mathbb{C}$, the notation $C_b^1(\mathcal{K})$ includes no derivative in z .

Lemma 3.1. *Let $b \mapsto T_b$ be C^1 with values in $\mathcal{L}(\mathcal{D}, \mathcal{H})$, and let $\mathcal{K} \subset B \times \mathbb{C}$ be compact. Suppose $T_b - z$ is bijective for every $(b, z) \in \mathcal{K}$. Then*

$$R_b(z) = (T_b - z)^{-1} : \mathcal{H} \longrightarrow \mathcal{D}$$

is defined on an open neighborhood of $\mathcal{K}$ in $B \times \mathbb{C}$, is C^1 in b , and has a uniform $C_b^1 \mathcal{L}(\mathcal{H}, \mathcal{D})$ bound on $\mathcal{K}$. Moreover,

$$\partial_b R_b = -R_b(\partial_b T_b)R_b. \quad (3.1)$$

Proof. The bounded inverse theorem and smooth inversion on Banach spaces [46, Remark 11.6 and Theorems 11.7, 11.17] apply to $T_b - z : \mathcal{D} \rightarrow \mathcal{H}$. The inverse and its parameter derivatives are continuous, hence bounded on $\mathcal{K}$. Differentiating $(T_b - z)R_b = I$ gives (3.1) in $\mathcal{L}(\mathcal{H}, \mathcal{D})$; compare [62, Chapter I, §4.5, formula (4.28)]. $\square$

Lemma 3.2. *Let T_b and $T_{j,b}$ be self-adjoint on $\mathcal{D}$, with $b \mapsto T_b$ of class C^1 in $\mathcal{L}(\mathcal{D}, \mathcal{H})$. Let $F_{j,b} \in \mathcal{L}(\mathcal{H})$ be positive and invertible, and suppose $F_{j,b}^{\pm 1}$ preserve $\mathcal{D}$ and restrict to bounded operators there for each (j, b) . Assume*

$$F_j T_j F_j = T + E_j \quad (3.2)$$

as operators $\mathcal{D} \rightarrow \mathcal{H}$, where

$$\|F_j - I\|_{C^1(B; \mathcal{L}(\mathcal{H}))} \longrightarrow 0, \quad \|E_j\|_{C^1(B; \mathcal{L}(\mathcal{D}, \mathcal{H}))} \longrightarrow 0. \quad (3.3)$$

For every compact set

$$\mathcal{K} \subset \{(b, z) \in B \times \mathbb{C} : z \in \rho(T_b)\},$$

one has $z \in \rho(T_{j,b})$ on $\mathcal{K}$ for all sufficiently large j , and the resolvents of T_j converge to those of T in $C_b^1(\mathcal{K}; \mathcal{L}(\mathcal{H}))$.

Proof. The domain invariance gives

$$T_j - z = F_j^{-1}(T + E_j - zF_j^2)F_j^{-1} : \mathcal{D} \longrightarrow \mathcal{H}. \quad (3.4)$$

Set $R = (T - z)^{-1}$, $K_j = E_j - z(F_j^2 - I)$, and $B_j = K_j R$. Put

$$\eta_j = \|E_j\|_{C^1(B; \mathcal{L}(\mathcal{D}, \mathcal{H}))} + \|F_j - I\|_{C^1(B; \mathcal{L}(\mathcal{H}))}.$$

Lemma 3.1 and the continuous inclusion $\mathcal{D} \hookrightarrow \mathcal{H}$ give, for all sufficiently large j ,

$$\|B_j\|_{C_b^1(\mathcal{K}; \mathcal{L}(\mathcal{H}))} \leq C\eta_j.$$

Indeed K_j and its parameter derivatives map $\mathcal{D}$ to $\mathcal{H}$, whereas R and its parameter derivatives map $\mathcal{H}$ to $\mathcal{D}$. If $C\eta_j < 1/2$, then

$$S_j = (I + B_j)^{-1}, \quad \|S_j\| \leq 2, \quad \|S_j - I\|_{C_b^1(\mathcal{K}; \mathcal{L}(\mathcal{H}))} \leq C'\eta_j,$$

where $\partial_b S_j = -S_j(\partial_b B_j)S_j$. Since $T + E_j - zF_j^2 = (I + B_j)(T - z)$, inversion of (3.4) gives

$$(T_j - z)^{-1} = F_j R S_j F_j. \quad (3.5)$$

This inverse maps $\mathcal{H}$ into $\mathcal{D}$ for each (j, b) . All factors in (3.5), regarded as operators on $\mathcal{H}$, are C^1 in b . The identity

$$F_j R S_j F_j - R = (F_j - I)R S_j F_j + R(S_j - I)F_j + R(F_j - I)$$

and the product rule give

$$\|(T_j - z)^{-1} - (T - z)^{-1}\|_{C_b^1(\mathcal{K}; \mathcal{L}(\mathcal{H}))} \leq C\eta_j. \quad (3.6)$$

The constant depends on $\mathcal{K}$, T , and the inclusion $\mathcal{D} \hookrightarrow \mathcal{H}$. The restrictions $F_j^{\pm 1}|_{\mathcal{D}}$ are isomorphisms for each (j, b) ; their norms do not enter this constant. $\square$

Remark 3.3. The correction $z(F_j^2 - I)$ is necessary. A positive congruence preserves invertibility at zero but need not preserve a nonzero eigenvalue. For example,

$$\text{diag}(2, 1) \text{diag}(1, -1) \text{diag}(2, 1) = \text{diag}(4, -1).$$

Matrix norms in the next two lemmas are operator norms, and $\text{ad}_H(A) = [H, A]$.

Lemma 3.4. *Let H, A be smooth matrix fields and Y a smooth derivation, possibly depending on b . For each $r > 0$, on $\|H\| \leq r$,*

$$\begin{aligned} e^{-H} Y(e^H) &= YH - \frac{1}{2}[H, YH] + \mathcal{E}_-(H, YH), \\ Y(e^H) e^{-H} &= YH + \frac{1}{2}[H, YH] + \mathcal{E}_+(H, YH), \end{aligned} \quad (3.7)$$

where $\|\mathcal{E}_{\pm}\| \leq C_r \|H\|^2 \|YH\|$. A first parameter derivative satisfies

$$\|\partial_b \mathcal{E}_{\pm}\| \leq C_r (\|H\| \|\partial_b H\| \|YH\| + \|H\|^2 \|\partial_b(YH)\|). \quad (3.8)$$

There is also an expansion

$$e^{-H} A e^H = A - [H, A] + C(H, A),$$

where C satisfies the same bounds with YH replaced by A and $\partial_b(YH)$ by $\partial_b A$.

Proof. The differential formula for the matrix exponential [54, Theorem 5.4, formulas (5.10)–(5.11), p. 115] gives

$$e^{-H} Y(e^H) = \int_0^1 e^{-tH} (YH) e^{tH} dt = \sum_{j=0}^{\infty} \frac{(-1)^j}{(j+1)!} \text{ad}_H^j(YH).$$

The remainder consists of the terms with $j \geq 2$. Since $\|\text{ad}_H\| \leq 2\|H\|$, it is bounded by $\|H\|^2 \|YH\|$ times the convergent series $\sum_{j \geq 2} 2^j r^{j-2} / (j+1)!$. A parameter derivative replaces one of the j factors H , or the factor YH , in each term. The differentiated series has a summable majorant and gives (3.8). The right logarithmic derivative has the same series with positive signs. Finally, $e^{-H} A e^H = e^{-\text{ad}_H} A$ [54, Proposition 3.35, p. 64]; its remainder is estimated in the same way, with $j!$ in place of $(j+1)!$. $\square$

Lemma 3.5. *Let P_b, Q_b be smooth matrix fields on a neighborhood of a compact parameter-coordinate set, and let s be a spatial coordinate independent of b . For $N \geq 1$, put*

$$Z_{b,N} = N^{-1/2} (P_b \cos Ns + Q_b \sin Ns).$$

Fix $k \geq 0$ and smooth vector fields $Y_{1,b}, \dots, Y_{k,b}$, independent of N , satisfying $Y_{v,b}s = 0$. Write $\mathcal{Y}_b = Y_{1,b} \cdots Y_{k,b}$, with $\mathcal{Y}_b = I$ when $k = 0$. For $|\alpha| \leq 1$, uniformly on the compact set,

$$\|\partial_b^\alpha (\mathcal{Y}_b Z_{b,N})\|_\infty \leq C_k N^{-1/2}, \quad (3.9)$$

and

$$\begin{aligned} \|\partial_b^\alpha (\mathcal{Y}_b \partial_s Z_{b,N})\|_\infty &\leq C_k N^{1/2}, \\ \|\partial_b^\alpha (\mathcal{Y}_b ([Z_{b,N}, \partial_s Z_{b,N}] - [P_b, Q_b]))\|_\infty &\leq C_k N^{-1}. \end{aligned} \quad (3.10)$$

The same estimates hold with ∂_s replaced by a smooth family v_b , independent of N , satisfying $v_b s = 1$. Each of

$$\mathcal{E}_{\pm}(Z_{b,N}, v_b Z_{b,N}), \quad C(Z_{b,N}, v_b Z_{b,N})$$

from Lemma 3.4 is $O(N^{-1/2})$ after $\partial_b^\alpha \mathcal{Y}_b$. The constants depend on finitely many jets of the fixed amplitudes and vector fields, but not on N .

Proof. Put $\epsilon = N^{-1/2}$ and $\vartheta = Ns$. Tangency gives

$$\mathcal{Y}_b Z_{b,N} = \epsilon ((\mathcal{Y}_b P_b) \cos \vartheta + (\mathcal{Y}_b Q_b) \sin \vartheta).$$

Parameter differentiation leaves ϑ unchanged and preserves tangency: $(\partial_b Y_{v,b})s = 0$. This proves (3.9), including derivatives of the vector fields. For v_b as in the statement,

$$v_b Z_{b,N} = \epsilon N (-P_b \sin \vartheta + Q_b \cos \vartheta) + \epsilon ((v_b P_b) \cos \vartheta + (v_b Q_b) \sin \vartheta).$$

Tangential derivatives and one parameter derivative give the $O(\epsilon N)$ bound. Moreover,

$$\begin{aligned} [Z_{b,N}, v_b Z_{b,N}] &= [P_b, Q_b] \\ &\quad + \epsilon^2 [P_b \cos \vartheta + Q_b \sin \vartheta, (v_b P_b) \cos \vartheta + (v_b Q_b) \sin \vartheta]. \end{aligned}$$

The prescribed derivatives of the last term are $O(\epsilon^2)$. Taking $v_b = \partial_s$ proves (3.10). Also $[Y_{v,b}, v_b]s = 0$ and $(\partial_b v_b)s = 0$; commuting the single normal derivative with a tangential or parameter derivative therefore adds a tangential derivative.

Set $A_N = v_b Z_{b,N}$ and $D = \partial_b^\alpha \mathcal{Y}_b$. Every subword of D applied to $Z_{b,N}$ is $O(\epsilon)$, and every such subword applied to A_N is $O(\epsilon N)$. The expression $\text{ad}_{Z_{b,N}}^j(A_N)$ is a sum of 2^j products, each with j factors $Z_{b,N}$ and one factor A_N . The product rule gives

$$\|D \text{ad}_{Z_{b,N}}^j(A_N)\|_\infty \leq 2^j (j+1)^{k+1} C_k^{j+1} \epsilon^{j+1} N.$$

For $N \geq 1$, the sum of these bounds with coefficients $1/(j+1)!$, $j \geq 2$, is at most

$$C_k \epsilon^3 N \sum_{j=2}^{\infty} \frac{(2C_k)^j (j+1)^{k+1}}{(j+1)!} \leq C'_k \epsilon.$$

Uniform convergence of the differentiated series justifies termwise differentiation and bounds both $\mathcal{E}_-$ and $\mathcal{E}_+$. Replacing $(j+1)!$ by $j!$ gives the bound for C . $\square$

4. REALIZATION OF THREE-FORM PERTURBATIONS

We construct trace-free tangential logarithmic metric variations of C^0 size $O(N^{-1/2})$, with first normal derivatives $O(N^{1/2})$. Their quadratic commutator produces the prescribed three-form term in the Dirac operator in the limit.

4.1. A local connection formula. Let s be a smooth function without critical points. Put

$$\kappa = |ds|_g, \quad \nu = \frac{\nabla^g s}{\kappa}, \quad v = \frac{\nu}{\kappa}.$$

Locally, the flow of v gives coordinates in which

$$g = a^2 ds^2 + q_s, \quad a = \kappa^{-1}. \quad (4.1)$$

Choose a tangential orthonormal frame $e = (e_1, \dots, e_{n-1})$ and define L by $[v, e_a] = \sum_d e_d L_{da}$. These brackets are tangential because $vs = 1$ and $e_a s = 0$. In flow coordinates, $L = E^{-1} \partial_s E$, where the columns of E are the components of e . Use the column convention

$$(\Omega_0)_{da} = g(\nabla_v e_a, e_d), \quad S_{ad} = g(\nabla_{e_a} v, e_d).$$

The torsion-free identity gives $\Omega_0 = \kappa L + S$. Since Ω_0 is skew-symmetric and S is symmetric,

$$S = -\frac{\kappa}{2}(L + L^T), \quad \Omega_0 = \frac{\kappa}{2}(L - L^T). \quad (4.2)$$

For a real symmetric tangential matrix H , put $b = e^{-H/2}$ and define

$$g_H = a^2 ds^2 + q_s(e^H \cdot, \cdot). \quad (4.3)$$

Here H denotes the endomorphism represented in e by that matrix and extended by zero on $\mathbb{R}v$. Thus the definition is independent of the flow coordinates. The fields ν, v and the function κ are unchanged, and (v, eb) is a g_H -orthonormal frame.

Lemma 4.1. *The tangential normal connection in the frame eb is*

$$\Omega_H = \frac{\kappa}{2}(b^{-1}b' - b'b^{-1}) + \frac{\kappa}{2}(b^{-1}Lb - bL^Tb^{-1}), \quad b' = v(b). \quad (4.4)$$

Proof. The bracket matrix of eb is

$$L_H = b^{-1}Lb + b^{-1}b',$$

since $[v, eb] = e(Lb + b')$. Applying (4.2) to g_H gives $\Omega_H = \frac{\kappa}{2}(L_H - L_H^T)$. Both b and b' are symmetric, which proves (4.4). $\square$

Put $e_0 = v$ and $C = c(v)$. The local spin connection formula [20, equation (4)] gives

$$D = \sum_i c(e_i)e_i + \frac{1}{4} \sum_{i,j,k} \langle \nabla_{e_i} e_j, e_k \rangle c(e_i)c(e_j)c(e_k). \quad (4.5)$$

In particular, the contribution of the tangential normal connection matrix Ω is

$$-\frac{1}{4}C \sum_{a,d} \Omega_{ad}c(e_a)c(e_d). \quad (4.6)$$

The minus sign comes from $\Omega_{ad} = \langle \nabla_v e_d, e_a \rangle$. Define

$$D^\top = \sum_a c(e_a)e_a + \frac{1}{4} \sum_{a,d,f} \langle \nabla_{e_a}^{q_s} e_d, e_f \rangle c(e_a)c(e_d)c(e_f).$$

The spinorial Gauss formula [20, equations (10)–(11)], with its Weingarten matrix equal to $-S$, and the normal connection terms give

$$D = Cv + D^\top + \frac{1}{2} \text{tr}(S)C - \frac{1}{4}C \sum_{a,d} \Omega_{ad}c(e_a)c(e_d) - \frac{1}{2}c(\nabla_v v). \quad (4.7)$$

The last term is the Clifford contraction of the normal-tangential part of ∇_v . In the coordinates (4.1),

$$\nabla_v v = -\text{grad}_{q_s}(\log a), \quad (4.8)$$

as follows from $\Gamma_{ss}^d = -a q_s^{df} \partial_f a$ and $v = a^{-1} \partial_s$.

4.2. Parameter-dependent estimates.

Proposition 4.2. *Let g_b , $b \in B$, be a smooth compact parameter family. Let $\mathcal{T}_b = ds \wedge \beta_b$ be a smooth real three-form family supported in a compact subset of a product coordinate neighborhood in $B \times X$, where s is one of the spatial coordinates and is independent of b . There are smooth metrics $g_{b,N}$, unchanged outside that neighborhood, such that*

$$\begin{aligned} dv_{g_{b,N}} &= dv_{g_b}, & \|g_N - g\|_{C^1(B;C^0)} &\leq CN^{-1/2}, \\ U_{b,N}^{-1} D_{g_{b,N}} U_{b,N} &= D_{g_b} + c_{g_b}(\mathcal{T}_b) + E_{b,N}, \\ \|E_N\|_{C^1(B;L(H^1,L^2))} &\leq CN^{-1/2}. \end{aligned} \quad (4.9)$$

Here $U_{b,N}$ is the direct positive spinor comparison, and C is independent of $N \geq 1$. The metrics may be chosen identical to g_b on any fixed parameter neighborhood whose closure is disjoint from the parameter projection of $\text{supp } \mathcal{T}$.

Proof. For $n \leq 2$, take $g_{b,N} = g_b$. Assume $n \geq 3$ and put $v_b = \nabla^{g_b} s / |ds|_{g_b}^2$. Replace β_b by $\iota_{v_b} \mathcal{T}_b$. This preserves $ds \wedge \beta_b = \mathcal{T}_b$, makes β_b tangential, and gives $\text{supp } \beta = \text{supp } \mathcal{T}$. Orthonormalizing the coordinate fields tangent to s gives a smooth family $e_{1,b}, \dots, e_{n-1,b}$ in the fixed spatial coordinates. Write

$$\beta_b = \frac{1}{2} \sum_{a,d} \beta_{ad,b} e_b^a \wedge e_b^d, \quad K_{ad,b} = 16\beta_{ad,b}.$$

Choose a constant diagonal trace-free real matrix A with distinct entries and set

$$B_{ad,b} = \frac{K_{ad,b}}{A_{aa} - A_{dd}} \quad (a \neq d), \quad B_{aa,b} = 0. \quad (4.10)$$

Then $B_b^T = B_b$, $\text{tr } B_b = 0$, and $[A, B_b] = K_b$.

Choose a smooth cutoff χ equal to one near $\text{supp } \mathcal{T}$, compactly supported in the product neighborhood, and vanishing on the prescribed parameter neighborhood. With $\epsilon = N^{-1/2}$, put

$$H_{b,N} = \epsilon(\chi A \cos Ns + B_b \sin Ns). \quad (4.11)$$

Extend the corresponding tangential endomorphism by zero on the normal line and outside the product neighborhood, and define $g_{b,N}$ by (4.3). Its relative determinant is $e^{\text{tr } H_{b,N}} = 1$. Thus its density agrees with that of g_b , and the density factor in (2.8) is one. In local spin frames lifting the paired orthonormal frames (v_b, e_b) and $(v_b, e_b e^{-H_{b,N}/2})$, the comparison $U_{b,N}$ identifies the Clifford matrices. In particular, it fixes the normal principal term $C_b v_b$.

All estimates below are uniform in b and include one parameter derivative in the fixed spatial coordinates. The fixed amplitudes and background fields have bounded finite jets on the support. Since s is independent of b ,

$$(\partial_b v_b)s = 0, \quad (\partial_b e_{a,b})s = 0.$$

Lemma 3.5 applies to these fields. It bounds H_N and each fixed number of tangential derivatives by $O(\epsilon)$, and $H'_N = v_b H_N$, with the same derivatives, by $O(\epsilon N)$. The bound for H_N and smoothness of the exponential give the metric estimate in (4.9). Suppressing the parameter subscripts, the commutator estimate is

$$\begin{aligned} [H_N, H'_N] &= [\chi A \cos Ns + B \sin Ns, -\chi A \sin Ns + B \cos Ns] + O(N^{-1}) \\ &= K + O(N^{-1}), \end{aligned} \quad (4.12)$$

because $\chi B = B$. The error contains only derivatives of the fixed amplitudes.

Apply Lemmas 3.4 and 3.5 to $-H_N/2$. In (4.4), the logarithmic-derivative remainder is $O(\epsilon^3 N) = O(\epsilon)$, whereas

$$b_N^{-1} L b_N - b_N L^T b_N^{-1} - (L - L^T) = O(\epsilon), \quad b_N = e^{-H_N/2}.$$

These bounds include the first parameter derivatives and any fixed number of tangential derivatives. Therefore

$$\Omega_{H_N} - \Omega_0 = -\frac{\kappa}{8} [H_N, H'_N] + O(N^{-1/2}) = -\frac{\kappa}{8} K + O(N^{-1/2}). \quad (4.13)$$

Equation (4.6) now gives

$$\frac{\kappa}{32} \sum_{a,d} K_{ad} C c(e_a) c(e_d) = \frac{\kappa}{2} \sum_{a,d} \beta_{ad} C c(e_a) c(e_d) = c(ds \wedge \beta). \quad (4.14)$$

We compare the other terms in (4.7) in the same paired frames. The tangential principal coefficients change by $O(\epsilon)$. For the intrinsic connection, write $b = b_N$, $f_a = \sum_i e_i b_{ia}$, and $[e_i, e_j] = \sum_\ell c_{ij}^\ell e_\ell$. Then $[f_a, f_d] = \sum_r \tilde{c}_{ad}^r f_r$, where

$$\tilde{c}_{ad}^r = (b^{-1})_{rt} (b_{ia} b_{jd} c_{ij}^\ell + b_{ia} e_i(b_{\ell d}) - b_{id} e_i(b_{\ell a})),$$

with repeated indices summed. Since $b - I$ and $e_i(b)$ are $O(\epsilon)$, we have $\tilde{c} - c = O(\epsilon)$. The Koszul formula gives

$$2\langle \nabla_{f_a}^{q_s, N} f_d, f_r \rangle_{g_N} = \tilde{c}_{ad}^r - \tilde{c}_{dr}^a + \tilde{c}_{ra}^d.$$

Thus the intrinsic spin connection changes by $O(\epsilon)$, including one parameter derivative. For the trace term, the bracket matrix in the proof of Lemma 4.1 gives the exact identity

$$\text{tr } S_N - \text{tr } S = -\kappa \text{tr}(b_N^{-1} b'_N) = -\kappa v(\log \det b_N) = \frac{\kappa}{2} v(\text{tr } H_N) = 0.$$

Finally, (4.8) gives

$$g_{b,N}(\nabla_{v_b}^{g_{b,N}} v_b, (e_b b_N)_a) = -(e_b b_N)_a (\log a_b).$$

Thus its Clifford contraction changes by $O(\epsilon)$ as well. These are all the terms in (4.7).

Consequently, in each fixed spatial chart and background spin frame, write $E_{b,N} = \sum_j a_{j,b,N} \partial_j + a_{0,b,N}$. All its coefficients have $C_b^1 L_x^\infty$ norm at most $C\epsilon$. The background comparison with the fixed reference spinor bundle in Section 2 is independent of N . In a reference frame it acts by a smooth matrix Q_b , so

$$Q_b^{-1} E_{b,N} Q_b = \sum_j Q_b^{-1} a_{j,b,N} Q_b \partial_j + Q_b^{-1} a_{0,b,N} Q_b + \sum_j Q_b^{-1} a_{j,b,N} (\partial_j Q_b).$$

The required jets of Q_b and Q_b^{-1} are uniformly bounded; hence the same coefficient estimates hold in the fixed reference spaces. On the compact support,

$$\|E_{b,N} u\|_{L^2} \leq C \left(\sum_j \|a_{j,b,N}\|_\infty + \|a_{0,b,N}\|_\infty \right) \|u\|_{H^1}.$$

Apply this also to the parameter derivative and use a fixed finite trivializing cover of the compact support to obtain the last estimate in (4.9). The cutoff construction gives the asserted spatial and parameter equalities. $\square$

Remark 4.3. The three-torus family in [39, Section 8] exhibits the same quadratic connection contribution for an explicit metric depending on one coordinate.

5. REALIZATION OF COMPATIBLE POTENTIALS

5.1. Elementary forms. A smooth family of real differential forms on X , with compact support in $B \times X$, admits a finite decomposition subordinate to product coordinate neighborhoods. Expanding each summand in the spatial coordinate coframe gives terms $ds \wedge \gamma_b$, where s is a spatial coordinate independent of b . Put $v_b = \nabla^{g_b} s / |ds|_{g_b}^2$. Replacing γ_b by $\iota_{v_b}(ds \wedge \gamma_b)$ preserves the term and makes its remaining factor tangential to the level sets of s .

For a term of degree $\ell \geq 7$, choose a smooth tangential orthonormal coframe on the product neighborhood and expand this remaining factor in its wedge basis. Each summand has the form

$$\mathcal{T} = f ds \wedge \alpha \wedge \beta, \tag{5.1}$$

where α is a wedge of three tangential orthonormal covectors and β a wedge of $\ell - 4$ distinct further covectors. The real coefficient f is smooth and compactly supported in the product neighborhood. All cutoffs below have compact support there and equal one near the support of the coefficient they accompany.

Proposition 5.1. *Let $\ell \geq 7$, $\ell \equiv 3 \pmod{4}$, and let $\mathcal{T}$ be as in (5.1). For a fixed compatible potential V_0 , there are positive multipliers F_N and compatible potentials $V_{1,N}$, sums of Clifford forms of degrees 3 and $\ell - 4$, such that*

$$F_N(D_g + V_0 + V_{1,N})F_N = D_g + V_0 + c(\mathcal{T}) + E_N. \tag{5.2}$$

For each fixed compact smooth family $(g_b, V_{0,b}, \mathcal{T}_b)$, there is a constant C independent of $N \geq 1$ such that

$$\|F_N - I\|_{C_b^1 L_x^\infty} \leq CN^{-1/2}, \quad \|E_N\|_{C_b^1 \mathcal{L}(H^1, L^2)} \leq CN^{-1/2}. \tag{5.3}$$

The multipliers equal I and the new potentials vanish outside a common compact interior set. They also satisfy $F_N = I$ and $V_{1,N} = 0$ on every parameter neighborhood where the cutoffs vanish.

Proof. Write $\kappa = |ds|_g$, $v = \nabla^g s / \kappa$, $C = c(v)$, $A = c(\alpha)$, and $B = c(\beta)$. The degrees of A and B are $3 \pmod{4}$, so both are odd self-adjoint Clifford elements. Their covector factors are mutually orthogonal and tangential. Consequently

$$AC = -CA, \quad BC = -CB, \quad AB = (-1)^{3(\ell-4)} BA = -BA.$$

Set $S = CA$ and $T = CB$. These are even self-adjoint endomorphisms, anticommute with C , and commute with J and, when present, Γ . Since $C^2 = -I$, we have $ST = AB$ and $TS = BA$, hence

$$[S, T] = 2AB. \quad (5.4)$$

For a real cutoff χ equal to one near $\text{supp } f$, define

$$\begin{aligned} H_N &= N^{-1/2}(\chi S \cos Ns + fT \sin Ns), & F_N &= e^{H_N}, \\ V_{1,N} &= \kappa \sqrt{N}(-\chi A \sin Ns + fB \cos Ns). \end{aligned} \quad (5.5)$$

Thus F_N is positive and commutes with the spinor structures; $V_{1,N}$ is compatible by Lemma 2.2. The anticommutation of H_N and C gives

$$F_N C F_N = C. \quad (5.6)$$

Let $\dot{H}_N$ denote the part of $\nabla_\nu H_N$ obtained by differentiating the trigonometric factors. Then

$$\dot{H}_N = \kappa \sqrt{N}(-\chi S \sin Ns + fT \cos Ns), \quad V_{1,N} = -C \dot{H}_N.$$

The remaining derivative is

$$G_N := \nabla_\nu H_N - \dot{H}_N = N^{-1/2}(\nabla_\nu(\chi S) \cos Ns + \nabla_\nu(fT) \sin Ns).$$

All estimates in this proof include one parameter derivative in the fixed spatial coordinates. Lemma 3.5, applied to $v = \nu/\kappa$ and to tangential fields, gives

$$H_N = O(N^{-1/2}), \quad G_N = O(N^{-1/2}), \quad \dot{H}_N = O(N^{1/2}).$$

The connection coefficients and κ belong to the fixed smooth family. Their derivatives introduce only fixed factors in these bounds. Since $\chi f = f$, direct expansion gives the exact identity

$$[H_N, \dot{H}_N] = \kappa f[S, T].$$

By Lemma 3.4,

$$\begin{aligned} e^{-H_N} \nabla_\nu(e^{H_N}) - e^{-H_N} \dot{H}_N e^{H_N} &= G_N + \frac{1}{2}[H_N, \dot{H}_N] - \frac{1}{2}[H_N, G_N] \\ &\quad + \mathcal{E}_-(H_N, \nabla_\nu H_N) - \mathcal{C}(H_N, \dot{H}_N). \end{aligned}$$

The last two remainders and their first parameter derivatives are $O(N^{-1/2})$ by (3.8); the factor in each bound is $O(N^{-1})O(N^{1/2})$. Thus the normal zeroth-order terms satisfy

$$\begin{aligned} F_N C \nabla_\nu F_N + F_N V_{1,N} F_N &= C(F_N^{-1} \nabla_\nu F_N - F_N^{-1} \dot{H}_N F_N) \\ &= \frac{1}{2}C[H_N, \dot{H}_N] + O(N^{-1/2}) \\ &= \frac{\kappa}{2}fC[S, T] + O(N^{-1/2}) \\ &= c(f ds \wedge \alpha \wedge \beta) + O(N^{-1/2}). \end{aligned} \quad (5.7)$$

Complete ν to an orthonormal frame $\nu, e_1, \dots, e_{n-1}$ with $e_a s = 0$. Lemmas 3.4 and 3.5 give $\nabla_{e_a} F_N = O(N^{-1/2})$. Moreover,

$$F_N c(e_a) F_N - c(e_a) = O(N^{-1/2}), \quad F_N V_0 F_N - V_0 = O(N^{-1/2}),$$

with constants depending on the fixed $C_b^1 L_x^\infty$ norm of V_0 on the common support. Expanding $F_N D_g F_N$ by the Leibniz rule, its normal principal term is $C \nabla_\nu$ by (5.6). Its remaining first-order error is

$$\sum_a (F_N c(e_a) F_N - c(e_a)) \nabla_{e_a};$$

its remaining zeroth-order error is the remainder in (5.7), together with $\sum_a F_N c(e_a) \nabla_{e_a} F_N$ and $F_N V_0 F_N - V_0$. In fixed background frames all these coefficients have $C_b^1 L_x^\infty$ norm at most $CN^{-1/2}$.

The background comparisons are independent of N , so their spatial and parameter derivatives only change C . On a finite cover of the support,

$$\left\| \sum_j a_j \partial_j u + a_0 u \right\|_{L^2} \leq C \left(\sum_j \|a_j\|_\infty + \|a_0\|_\infty \right) \|u\|_{H^1}.$$

Applying this estimate also to each parameter derivative proves the second bound in (5.3). Smoothness of the exponential and the bound for H_N prove the first.

For every finite N , $F_N^{\pm 1}$ are smooth multipliers equal to I outside a fixed compact set. The Leibniz rule shows that they are bounded isomorphisms on H^1 , as well as on L^2 . Their spatial derivative bounds may depend on N ; Lemma 3.2 requires their domain bounds only for each fixed N . The support and parameter equalities follow from $\text{supp } f \subset \{\chi = 1\}$. $\square$

Proof of Theorem R. Choose a fixed neighborhood of B_0 whose closure is disjoint from the parameter projection of $\text{supp } V$. Cover $\text{supp } V$ by finitely many relatively compact product coordinate neighborhoods with closures in $B \times U$ and disjoint from this neighborhood. All subsequent decompositions and cutoffs may be taken in these product neighborhoods.

By Lemma 2.2, write V_b as a finite sum of real Clifford forms. Decompose the forms of largest degree into terms (5.1). For one such term, take V_0 to be the sum of all other terms and apply Proposition 5.1. Both operators are self-adjoint on H^1 by Section 2: the new multiplication potential is bounded for every finite frequency. Lemma 3.2 therefore replaces the chosen term by degrees 3 and $\ell - 4$, with arbitrarily small C_b^1 resolvent error on $\mathcal{K}$. At each degree only finitely many terms occur, and no replacement creates a term of that degree or a larger one. Descending induction on the degree leaves only three-forms after finitely many steps. At each step all preceding frequencies, the current smooth family, its finite spatial jets, and the preceding spinor comparisons are fixed. Only the frequency of that step is increased.

Decompose the remaining three-form into finitely many terms $ds \wedge \beta_b$ in the fixed spatial coordinates. Proposition 4.2 removes one term by changing the metric. Retain the other terms as literal differential forms. For each retained form ω_b ,

$$U_{b,N}^{-1} c_{g_{b,N}}(\omega_b) U_{b,N} - c_{g_b}(\omega_b) = O_{C_b^1 L_x^\infty}(N^{-1/2}).$$

Indeed, under the positive comparison Clifford multiplication depends smoothly and algebraically on the metric and the form; both the form and the current metric are fixed while this frequency is chosen. Write $\mathcal{T}_b = ds \wedge \beta_b$ for the removed term and $\omega_{\text{rem},b}$ for the sum of the retained forms. Put

$$\tilde{L}_{b,N} = U_{b,N}^{-1} (D_{g_{b,N}} + c_{g_{b,N}}(\omega_{\text{rem},b})) U_{b,N}.$$

Proposition 4.2 and the multiplication estimate give

$$\tilde{L}_{b,N} - (D_{g_b} + c_{g_b}(\mathcal{T}_b + \omega_{\text{rem},b})) = O_{C_b^1 \mathcal{L}(H^1, L^2)}(N^{-1/2}).$$

For each finite N , $U_{b,N}$ is unitary on L^2 and identifies the H^1 domains. Lemma 3.2, with $F_N = I$, gives the required resolvent approximation. Repeating this step removes all the three-form terms.

There are finitely many support enlargements, all inside the chosen product neighborhoods. The resulting metric equals g_b on $X \setminus U$ and near B_0 . Each metric step preserves the volume density by Proposition 4.2; the other steps leave the metric unchanged.

We choose the frequencies to control both the composed comparison and the metric increments. Parameter derivatives are taken in the fixed reference norms of Section 2. For the successive metric families $g^{(0)} = g, \dots, g^{(m)}$, let $U_j : \mathcal{H}_{j-1} \rightarrow \mathcal{H}_j$ be the successive positive comparisons; put $U_j = I$ when the metric is unchanged. Write $R^{(j)}$ for the successive resolvent families and set

$$C_{j-1} = U_{j-1} \cdots U_1, \quad C_0 = I, \quad \Delta_j = U_j^{-1} R^{(j)} U_j - R^{(j-1)}.$$

Then

$$C_m^{-1} R^{(m)} C_m - R^{(0)} = \sum_{j=1}^m C_{j-1}^{-1} \Delta_j C_{j-1}. \quad (5.8)$$

Fix positive tolerances θ and η , to be chosen below. Let M_j bound conjugation by C_{j-1} in $C_b^1(\mathcal{L}(\mathcal{H}))$. This is a finite constant fixed before the j th frequency is chosen. At that step choose, simultaneously,

$$\|\Delta_j\|_{C_b^1(\mathcal{K}; \mathcal{L}(\mathcal{H}_{j-1}))} < \frac{2^{-j}\theta}{1+M_j}, \quad \delta_j := \|g^{(j)} - g^{(j-1)}\|_{C_b^1 C_x^0} < 2^{-j}\eta.$$

For a higher-degree replacement $\delta_j = 0$. For a metric step the two choices follow from Proposition 4.2 and Lemma 3.2. At every step that lemma also keeps $\mathcal{K}$ in the resolvent set. Summation gives composed resolvent error less than θ and $S := \sum_j \delta_j < \eta$.

For the direct comparison (2.8), let V_j compare g with $g^{(j)}$, with $V_0 = I$, and put, in the original spinor bundle,

$$W_j = V_j^{-1} U_j V_{j-1}, \quad W = W_m \cdots W_1.$$

The definitions give $C_m = V_m W$. The pointwise matrix for W_j is a smooth function of $g, g^{(j-1)}, g^{(j)}$ and is I when its last two arguments agree. Choose $s_0 > 0$ so that $S \leq s_0$ keeps all metric values and their first parameter jets in a fixed compact set, with the metric values positive. The mean-value formula, applied also after one parameter derivative, gives

$$\|W_j - I\|_{C_b^1 L_x^\infty} \leq C \delta_j.$$

The constant depends on this compact set and the original family, not on the frequencies or the number of factors. Hence

$$\|W - I\|_{C_b^1 L_x^\infty} \leq C e^{CS} S. \quad (5.9)$$

Indeed, the product norm is bounded by $\prod_j (1 + C \delta_j) \leq e^{CS}$; the telescoping expansion of $W - I$ and the differentiated product each have one factor bounded by $C \delta_j$ in every summand.

Writing $\widehat{R} = R^{(m)}$, set

$$R_{\text{comp}} = C_m^{-1} \widehat{R} C_m, \quad R_{\text{dir}} = V_m^{-1} \widehat{R} V_m.$$

Then $R_{\text{dir}} = W R_{\text{comp}} W^{-1}$. For any C^1 family R of bounded operators,

$$W R W^{-1} - R = (W - I) R W^{-1} + R (W^{-1} - I).$$

The product rule and $d(W^{-1}) = -W^{-1}(dW)W^{-1}$ therefore give, for $S \leq s_0$,

$$\|W R W^{-1} - R\|_{C_b^1 \mathcal{L}(\mathcal{H})} \leq C \|W - I\|_{C_b^1 L_x^\infty} \|R\|_{C_b^1 \mathcal{L}(\mathcal{H})}.$$

Let $R_{\text{tar}} = R^{(0)}$ and $M = 1 + \|R_{\text{tar}}\|_{C_b^1(\mathcal{K}; \mathcal{L}(\mathcal{H}))}$, which is finite by Lemma 3.1. Choose $0 < \eta < \min\{s_0, \varepsilon\}$ so that (5.9) makes the last conjugation error less than $\varepsilon/2$ whenever $S < \eta$ and $\|R\|_{C_b^1(\mathcal{K}; \mathcal{L}(\mathcal{H}))} \leq M$. Choose also $0 < \theta < \min\{1, \varepsilon/2\}$. The preceding frequency choices then give $\|R_{\text{comp}}\|_{C_b^1(\mathcal{K}; \mathcal{L}(\mathcal{H}))} < M$. Its approximation error and the final conjugation error sum to less than ε , proving (1.4). Finally, $\|g^{(m)} - g\|_{C_b^1 C_x^0} \leq S < \varepsilon$ proves the metric estimate in (1.3). $\square$

Corollary 5.2. *In the setting of Theorem R, if $D_g + V$ is invertible and V is compatible and compactly supported in U , there is a metric $\widehat{g}$ with $D_{\widehat{g}}$ invertible, $\widehat{g} = g$ outside U , and $dv_{\widehat{g}} = dv_g$.*

Proof. Take a one-point parameter space and apply Theorem R to $\mathcal{K} = \{0\} \subset \rho(D_g + V)$. $\square$

6. SPECTRAL SUBSPACES AND LOCAL SECTIONS

Let B be a compact smooth parameter manifold, possibly with boundary, and let $L_b : \mathcal{D} \rightarrow \mathcal{H}$ be a smooth family of self-adjoint operators with common domain, endowed with a fixed graph norm. Let γ be an oriented finite union of smooth closed curves contained in $\rho(L_b)$ for every b . Its winding number is one at each eigenvalue of a selected cluster of finite total multiplicity and zero at every other point of $\text{Spec}(L_b)$. Define

$$P_b = \frac{1}{2\pi i} \int_{\gamma} (z - L_b)^{-1} dz, \quad Q_b = \frac{1}{2\pi i} \int_{\gamma} z(z - L_b)^{-1} dz. \quad (6.1)$$

The spectral theorem identifies P_b with the orthogonal projection onto the cluster. The resolvent takes values in $\mathcal{L}(\mathcal{H}, \mathcal{D})$, and $L_b(z - L_b)^{-1} = z(z - L_b)^{-1} - I$ gives $Q_b = L_b P_b$. In particular, Q_b is bounded; see also [62, Chapter III, Theorem 6.17 and equation (6.24)]. For realizations furnished by Theorem R on $B \times \gamma$, contour integration gives

$$\|\hat{P} - P\|_{C_b^1} + \|\hat{Q} - Q\|_{C_b^1} \leq \frac{\text{length}(\gamma)}{2\pi} \left(1 + \sup_{z \in \gamma} |z|\right) \sup_{z \in \gamma} \|(z - \hat{D}_b)^{-1} - (z - L_b)^{-1}\|_{C_b^1 \mathcal{L}(\mathcal{H})}.$$

Thus both moments converge in $C_b^1 \mathcal{L}(\mathcal{H})$. Once $\|\hat{P}_b - P_b\| < 1$, the restriction of P_b to $\text{im } \hat{P}_b$ is injective, so $\hat{P}_b$ has finite rank.

Lemma 6.1. *Let $P_b, \hat{P}_{j,b}$ be C^1 families of finite-rank orthogonal projections and let $Q_b, \hat{Q}_{j,b}$ be bounded spectral moments satisfying*

$$\hat{P}_j \longrightarrow P, \quad \hat{Q}_j \longrightarrow Q \quad \text{in } C_b^1 \mathcal{L}(\mathcal{H}).$$

For $\|\hat{P}_j - P\| < 1$, the polar part of $\hat{P}_j P : \text{im } P \rightarrow \text{im } \hat{P}_j$ is an isometry onto $\text{im } \hat{P}_j$. It intertwines every unitary or antiunitary spinor structure commuting with both projections. Omitting j , its expression in a local C^1 isometric frame $U_b : E \rightarrow \text{im } P_b$ is

$$\hat{U}_b = \hat{P}_b U_b (U_b^* \hat{P}_b U_b)^{-1/2}. \quad (6.2)$$

The corresponding complete matrices satisfy, on compact subsets of the frame domain,

$$\hat{U}_b^* \hat{Q}_b \hat{U}_b - U_b^* Q_b U_b \longrightarrow 0 \quad \text{in } C_b^1 \text{End}(E). \quad (6.3)$$

Proof. Put $q = \|\hat{P} - P\| < 1$ and $G = U^* \hat{P} U$. The close-projection theorem [62, Chapter I, Theorem 6.32], applied on $\text{im } P + \text{im } \hat{P}$, gives equal ranks. For $v \in E$,

$$\langle Gv, v \rangle = \|\hat{P} U v\|^2 \geq (1 - q)^2 \|v\|^2.$$

Hence (6.2) is the polar isometry onto $\text{im } \hat{P}$. Since $G \rightarrow I$ in C^1 , holomorphic matrix functional calculus [62, Chapter I, Section 5.6], on a fixed contour in the domain of $z^{-1/2}$, gives $G^{-1/2} \rightarrow I$ in C^1 . Indeed, the derivative of its resolvent is $(z - G)^{-1}(dG)(z - G)^{-1}$. Thus $\hat{U} \rightarrow U$ in $C^1 \mathcal{L}(E, \mathcal{H})$, and the product rule gives (6.3).

Intrinsically the map is $\hat{P} P (P \hat{P} P|_{\text{im } P})^{-1/2}$. Every preserved structure commutes with its factors: the inverse square root is a uniform limit of polynomials with real coefficients on a positive spectral interval. The formula is equivariant under $U \mapsto UR$ for unitary R , so the local maps define a structure-preserving bundle isometry. $\square$

Definition 6.2. Let E be a zero module. A *full potential section* at zero is a smooth family of compatible multiplication endomorphisms $V(A)$, for A near zero in $\mathcal{A}_n(E)$, such that $D_g + V(A)$ has an isolated finite cluster continuing its entire zero eigenspace at $A = 0$. Its complete matrix is required to equal A in a smooth isometric frame intertwining J and, when present, Γ . A *full metric section* is defined by replacing $D_g + V(A)$ by $D_{g(A)}$.

Proposition 6.3. *Assume $n \geq 3$, and let X, g, U satisfy the geometric hypotheses of Theorem R. Every full potential section whose potentials are supported in one compact subset of U yields a full metric section. The metric section equals g outside U and has volume density dv_g . Its reference metric at zero need not equal g .*

Proof. Equip the real vector space $\mathcal{A}_n(E)$ with a Euclidean norm. After shrinking the section domain, take a closed radius- r ball on which one circle centered at zero separates the full zero cluster from the remaining spectrum. A structure commuting with the operator preserves each eigenspace; a structure

anticommuting with it exchanges the eigenspaces at λ and $-\lambda$. The cluster projections therefore commute with J and Γ , both for the potential family and for its metric realizations.

Apply Theorem R with B equal to this ball and $\mathcal{K} = B \times \gamma$. Lemma 6.1 gives, for a sufficiently accurate realization, a complete metric matrix $F(A) \in \mathcal{A}_n(E)$ satisfying

$$\sup_{\|A\| \leq r} \|F(A) - A\| < r/4, \quad \sup_{\|A\| \leq r} \|DF(A) - I\| < 1/2. \quad (6.4)$$

The matrix F is smooth: the realization and the original frame are smooth, and resolvent inversion and the positive inverse square root in (6.2) preserve smoothness. For $\|Y\| < r/4$, the map $\Phi_Y(A) = Y + A - F(A)$ sends the closed radius- $r/2$ ball into its interior, since $\|\Phi_Y(A)\| \leq \|Y\| + \|A - F(A)\| < r/2$. Its derivative has norm less than $1/2$. The contraction theorem [46, Theorem 10.38] gives a unique fixed point $A(Y)$ in that ball, with $F(A(Y)) = Y$. The derivative DF is invertible there. The inverse function theorem [46, Theorem 11.22 and Corollary 11.35] and uniqueness give smooth dependence on Y and

$$DA(Y) = DF(A(Y))^{-1}, \quad \|DA(Y)\| \leq 2, \quad \|A(Y) - A(Z)\| \leq 2\|Y - Z\|.$$

The last estimate also follows by comparing the two fixed-point equations. Composing the metric family and its structural frame with $Y \mapsto A(Y)$ gives the required section. The equalities in (1.3) are preserved; its reference metric is $\widehat{g}_{A(0)}$. $\square$

Lemma 6.4. *Let $T_t : \mathcal{D} \rightarrow \mathcal{H}$ be a C^1 family in $\mathcal{L}(\mathcal{D}, \mathcal{H})$ of self-adjoint operators with common domain $\mathcal{D} = H^1$, whose graph norms are locally uniformly equivalent. Suppose $\lambda \in \mathbb{R}$ is an isolated eigenvalue of finite multiplicity of T_0 , and put $E = \ker(T_0 - \lambda)$. In the polar frame based on the inclusion of E , the derivative of the complete cluster matrix at zero is $P_E \dot{T}_0|_E$. In particular, for $T_t = D_g + tV$ at a zero eigenspace it is $P_E V|_E$.*

Proof. On a fixed contour around λ , the resolvent identity gives a C^1 family $(z - T_t)^{-1} : \mathcal{H} \rightarrow \mathcal{D}$ for small t , with

$$\frac{d}{dt}(z - T_t)^{-1} = (z - T_t)^{-1} \dot{T}_t (z - T_t)^{-1}.$$

Its contour integral and the polar frame $U_t : E \rightarrow \mathcal{D}$ are therefore C^1 , and U_0 is the inclusion. Differentiating gives

$$\left. \frac{d}{dt} \right|_{t=0} U_t^* T_t U_t = \dot{U}_0^* T_0 U_0 + U_0^* \dot{T}_0 U_0 + U_0^* T_0 \dot{U}_0.$$

Since $T_0 U_0 = \lambda U_0$ and T_0 is self-adjoint, the first and last terms sum to

$$\lambda(\dot{U}_0^* U_0 + U_0^* \dot{U}_0) = \lambda \left. \frac{d}{dt} \right|_{t=0} (U_t^* U_t) = 0.$$

The remaining term represents $P_E \dot{T}_0|_E$. $\square$

Lemma 6.5. *Let S, T be Euclidean, Hermitian, or quaternionic Hermitian bundles over X , with a fixed smooth positive density dv . Let $F \subset C^\infty(S) \cap L^2(S)$ and $E \subset C^\infty(T) \cap L^2(T)$ be finite-dimensional subspaces over the same field $\mathbb{F}$, with their L^2 inner products. Let $\mathcal{V}$ be a smooth real subbundle of $\text{Hom}_{\mathbb{F}}(S, T)$. For a nonempty open set U , define*

$$C(V) = \int_X e_E(x)^* V(x) e_F(x) dv(x), \quad V \in C_c^\infty(U; \mathcal{V}),$$

where e_E, e_F are evaluation. Then

$$\text{im } C = \text{span}_{\mathbb{R}} \{e_E(x)^* Q e_F(x) : x \in U, Q \in \mathcal{V}_x\}. \quad (6.5)$$

Proof. The space on the right of (6.5) is a linear subspace of the finite-dimensional real vector space $\text{Hom}_{\mathbb{F}}(F, E)$, hence closed. Integration puts $C(V)$ in that space. Conversely, extend $Q \in \mathcal{V}_x$ to a smooth section Q_y on a coordinate ball in U . Choose $\chi_\delta \geq 0$ supported in that ball with $\int \chi_\delta dv = 1$ and support tending to x . Then

$$\|C(\chi_\delta Q) - e_E(x)^* Q e_F(x)\| \leq \sup_{y \in \text{supp } \chi_\delta} \|e_E(y)^* Q_y e_F(y) - e_E(x)^* Q e_F(x)\| \longrightarrow 0.$$

The real-linear image of C is finite-dimensional and therefore closed. Every generator on the right belongs to that image, proving the equality over each of the three fields. $\square$

Lemma 6.6. *Let $T_0 = \text{diag}(A, C) : E_1 \oplus E_2 \rightarrow F_1 \oplus F_2$ be bounded between fixed orthogonal Hilbert decompositions. An unbounded-operator domain is equipped with a fixed graph norm. Assume $C : E_2 \rightarrow F_2$ is bijective and $\|C^{-1}\|_{F_2 \rightarrow E_2} \leq c^{-1}$, where $c > 0$. Let $T = T_0 + E$, with $\|E\|_{E_1 \oplus E_2 \rightarrow F_1 \oplus F_2} \leq \delta < c/2$. Then the complementary block is invertible and the Schur complement S satisfies*

$$\|S - A\| \leq \delta + 2\delta^2/c. \quad (6.6)$$

The kernel of T is isomorphic to the kernel of S , and

$$(F_1 \oplus F_2)/\text{im } T \simeq F_1/\text{im } S.$$

One range is closed if and only if the other is closed. In that case these quotients identify with the corresponding Hilbert cokernels.

Suppose linear or antilinear structural maps J_E, J_F preserve the summands and satisfy $J_F T = \epsilon T J_E$, where $\epsilon \in \{1, -1\}$. Then $J_{F_1} S = \epsilon S J_{E_1}$. In the unbounded case the maps must preserve the operator domain. For an operator on an ambient Hilbert space $\mathcal{H} = F_1 \oplus F_2$, suppose both orthogonal projections preserve its domain $\mathcal{D}$ and $E_i = \mathcal{D} \cap F_i$. If $T^\dagger = \epsilon T$, then $S^\dagger = \epsilon S$ on E_1 . Here $\dagger$ is the ambient adjoint, or transpose with respect to a fixed conjugation preserving the splitting and the domain. The adjoint and transpose are taken with respect to the ambient pairings.

Proof. Write $T = \begin{pmatrix} A+E_{11} & E_{12} \\ E_{21} & C+E_{22} \end{pmatrix}$ and $C' = C + E_{22}$. Bounded inverse stability [62, Chapter IV, Theorem 1.16] gives $\|(C')^{-1}\|_{F_2 \rightarrow E_2} \leq (c - \delta)^{-1} \leq 2/c$. Set

$$S = A + E_{11} - E_{12}(C')^{-1}E_{21}.$$

The block bounds give (6.6). The exact factorization is

$$\begin{pmatrix} I & -E_{12}(C')^{-1} \\ 0 & I \end{pmatrix} T \begin{pmatrix} I & 0 \\ -(C')^{-1}E_{21} & I \end{pmatrix} = \begin{pmatrix} S & 0 \\ 0 & C' \end{pmatrix}. \quad (6.7)$$

The bounded invertible triangular factors give the kernel and quotient isomorphisms. The left factor sends $\text{im } T$ onto $\text{im } S \oplus F_2$, proving the assertion about closed ranges.

The structural identity on the complementary block gives

$$J_{E_2}(C')^{-1} = \epsilon(C')^{-1}J_{F_2}.$$

Moving J through the three factors of $E_{12}(C')^{-1}E_{21}$ contributes $\epsilon^3 = \epsilon$. Together with the upper-left block identity, this gives $J_{F_1} S = \epsilon S J_{E_1}$, including charge and grading.

For either ambient adjoint or transpose, write $\dagger$ and assume $T^\dagger = \epsilon T$. Put $K = E_{12}(C')^{-1}$. The ambient pairings and bijectivity of C' give

$$((C')^{-1})^\dagger = \epsilon(C')^{-1}, \quad K^\dagger|_{E_1} = (C')^{-1}E_{21}.$$

In the unbounded case the first identity follows by pairing $C'u$ and $C'v$, with $u, v \in E_2$; the second follows by pairing $u \in E_1$ with $(C')^{-1}y$, $y \in F_2$. The map $K : F_2 \rightarrow F_1$ is bounded in the ambient norms. Thus (6.7) is $LTL^\dagger = \text{diag}(S, C')$, where $L = \begin{pmatrix} I & -K \\ 0 & I \end{pmatrix}$ is bounded and invertible on the ambient space. Its transpose or adjoint $L^\dagger$, and its inverse, preserve the domain. Taking $\dagger$ gives $S^\dagger = \epsilon S$ with that domain. $\square$

7. GENERIC NONZERO EIGENVALUES

Throughout this section M^n is closed and connected, $n \geq 3$, and its spin structure is fixed. Multiplicities are counted according to (1.5). The spaces of compatible pointwise endomorphisms are given in Section 2. On a nonzero eigenspace the scalar field is

$$\mathbb{F}_n = \begin{cases} \mathbb{H}, & n \equiv 2, 3, 4 \pmod{8}, \\ \mathbb{C}, & n \equiv 1, 5 \pmod{8}, \\ \mathbb{R}, & n \equiv 0, 6, 7 \pmod{8}. \end{cases} \quad (7.1)$$

In a real case the eigenspace means the real fixed space of charge conjugation; its real dimension equals the complex dimension of the ordinary eigenspace. In a quaternionic case the ordinary complex dimension is twice its quaternionic dimension. For $n \equiv 1, 5 \pmod{8}$, charge conjugation exchanges the two nonzero eigenspaces of opposite sign and imposes no additional multiplicity on either of them.

7.1. Splitting a nonzero eigenspace.

Lemma 7.1. *Let $L = D_g + W$, where W is a smooth compatible multiplication endomorphism. Suppose $\lambda \neq 0$ and $E = \ker(L - \lambda)$ has $\mathbb{F}_n$ -dimension at least two. For every nonempty open set $U \subset M$ there are $f \in C_c^\infty(U; \mathbb{R})$ and a smooth compatible multiplication endomorphism V supported in U such that*

$$P_E V|_E - \lambda P_E M_f|_E \notin \mathbb{R}I_E. \quad (7.2)$$

If $n \not\equiv 2 \pmod{8}$, one may take $f = 0$.

Proof. Elliptic regularity makes every eigenspinor smooth. The square of $D_g + W - \lambda$ has connection-Laplacian principal part and smooth terms of order at most one. The unique-continuation theorem [56, Theorem 6.2.8] therefore applies. A nonzero eigenspinor is nonzero on an open dense set.

Suppose first that every compatible compression supported in U is scalar. For any local compatible matrix field Q and any real test function χ supported in its trivialization within U , $P_E(\chi Q)|_E$ is scalar. Its off-diagonal entries vanish and its diagonal entries agree. Testing all χ gives the corresponding pointwise identities on U . Quaternionic entries are tested in each of their four real components.

The cases $n \equiv 3, 7 \pmod{8}$. Choose an orthonormal pair $\psi, \phi \in E$ over $\mathbb{F}_n$. At a point $x \in U$ where $u = \psi(x) \neq 0$, the allowed fibre map $Q = uu^*$ has compression to this two-frame of rank at most one. Its first diagonal entry is $|u|^4 > 0$. A scalar two-by-two matrix with that entry has rank two, which is a contradiction.

The case $n \equiv 1 \pmod{8}$. Here $J^2 = I$ and $Jc(\xi) = -c(\xi)J$. In a local J -real frame, compatible potentials have the form iQ , with Q real skew-adjoint. For complex orthonormal $\psi_1, \psi_2 \in E$, the localized identities are

$$\overline{\psi_1} \wedge \psi_2 = 0, \quad \overline{\psi_1} \wedge \psi_1 = \overline{\psi_2} \wedge \psi_2. \quad (7.3)$$

The wedge is complex bilinear. Choose a connected ball in U on which $\psi_1 \neq 0$. The first equation gives $\psi_2 = b\overline{\psi_1}$ there, with b smooth and complex-valued. Substitution into the second gives

$$(1 + |b|^2)\overline{\psi_1} \wedge \psi_1 = 0.$$

Thus ψ_1 is pointwise a complex multiple of a nonzero real spinor, and $\psi_2 = a\psi_1$ for a smooth complex function a . The product rule gives $c(da)\psi_1 = 0$. At a point write $\psi_1 = zw$, $z \neq 0$, $Jw = w$, and $a = a_1 + ia_2$. Then

$$c(da_1)w + i c(da_2)w = 0.$$

The two terms belong respectively to $i\Sigma_{\mathbb{R}}$ and $\Sigma_{\mathbb{R}}$. They vanish separately, and the real Clifford identity implies $da_1 = da_2 = 0$. The function a is constant on the ball. Unique continuation for $\psi_2 - a\psi_1$ contradicts orthonormality.

The case $n \equiv 5 \pmod{8}$. Here $J^2 = -I$, $JL = -LJ$, and compatible potentials have the form $V = -iQ$, where Q is quaternionic-linear and skew-adjoint. For a nonzero fibre vector u ,

$$\{Qu : Q^* = -Q, QJ = JQ\} = u^{\perp_{\mathbb{R}}}, \quad \{Vu : V \text{ compatible}\} = (iu)^{\perp_{\mathbb{R}}}. \quad (7.4)$$

In a quaternionic orthonormal basis with $u = |u|e_1$, write $w/|u| = (a, v)$ for $w \in u^{\perp_{\mathbb{R}}}$. Then

$$\operatorname{Re} a = 0, \quad Q = \begin{pmatrix} a & -v^* \\ v & 0 \end{pmatrix}, \quad Q^* = -Q, \quad Qu = w.$$

This proves the first equality; multiplication by $-i$ gives the second.

Choose complex orthonormal $\psi, \phi \in E$. For every compatible V , scalarity gives $\langle \phi(x), V\psi(x) \rangle_{\mathbb{C}} = 0$. Where $\psi(x) \neq 0$, its real and imaginary parts and (7.4) imply

$$\phi(x), i\phi(x) \in \mathbb{R} i\psi(x).$$

A real line contains no nonzero complex line. Therefore $\phi = 0$ on a nonempty open set, contradicting unique continuation.

The cases $n \equiv 0, 4 \pmod{8}$. Put $\mathbb{F} = \mathbb{R}$ or $\mathbb{H}$, respectively. The operator L is chiral. For $\psi = (u, v) \in \ker(L - \lambda)$, vanishing of either chiral component on an open set implies vanishing of the other there, since $\lambda \neq 0$. Both components of a nonzero eigenspinor are therefore nonzero on open dense sets.

Choose $\mathbb{F}$ -orthonormal $\psi_i = (u_i, v_i)$, $i = 1, 2$. For $A \in \operatorname{Hom}_{\mathbb{F}}(\Sigma^+, \Sigma^-)$, the off-diagonal fibre pairing for the block (2.5) is

$$b_{12}(A) = u_1^* A^* v_2 + v_1^* A u_2.$$

For every $q \in \mathbb{F}$ one has

$$\operatorname{Re}(\bar{q} b_{12}(A)) = \operatorname{Re} \operatorname{tr}(A^*(v_1 q u_2^* + v_2 \bar{q} u_1^*)).$$

The real trace pairing is nondegenerate. The vanishing off-diagonal entries and the equality of the diagonal entries therefore give

$$v_1 u_1^* = v_2 u_2^*, \quad v_1 q u_2^* + v_2 \bar{q} u_1^* = 0 \quad (q \in \mathbb{F}). \quad (7.5)$$

Choose a point of U where $u_1, v_1 \neq 0$. The first equation is equality of two nonzero rank-one maps. Hence

$$u_2 = u_1 a, \quad v_2 = v_1 (\bar{a})^{-1} \quad \text{for some } a \in \mathbb{F} \setminus \{0\}.$$

Substituting $q = a$ into the second equation gives $(|a|^2 + 1)v_1 u_1^* = 0$, a contradiction. This calculation also applies when $n = 4$, with both chiral fibres isomorphic to $\mathbb{H}$.

The case $n \equiv 6 \pmod{8}$. Choose charge coordinates $J(u, v) = (\bar{v}, \bar{u})$. A real orthonormal pair has the form $\psi = (u, \bar{u})$ and $\phi = (a, \bar{a})$. Compatible blocks satisfy $A^T = A$. Their off-diagonal pairing is

$$\langle \psi, V_A \phi \rangle_{\mathbb{R}} = 2 \operatorname{Re}(u^T A a).$$

Both A and iA are allowed. Thus $u \odot a = 0$ as a complex symmetric tensor. If $u_i \neq 0$, the diagonal coefficient gives $a_i = 0$, and the ij coefficient gives $a_j = 0$ for every j . It follows that $\phi = 0$ where $\psi \neq 0$, a contradiction.

The case $n \equiv 2 \pmod{8}$. Suppose every expression in (7.2) is scalar. Set $V = 0$. Since $\lambda \neq 0$, testing all real f gives

$$e_x^* e_x = \rho(x) I_E \quad (x \in U), \quad e_x : E \longrightarrow \Sigma_x. \quad (7.6)$$

Here ρ is smooth and nonnegative. Unique continuation shows that it is positive somewhere. The adjoint is quaternionic; (7.6) also holds in the ordinary complex realization.

Choose quaternionic orthonormal $\psi = (u, v)$ and $\phi = (a, b)$. In charge coordinates

$$J(u, v) = (-\bar{v}, \bar{u}), \quad V_A = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}, \quad A^T = -A.$$

Setting $f = 0$ gives the vanishing pairings with (ψ, ϕ) and $(J\psi, \phi)$. In particular,

$$\langle \psi, V_A \phi \rangle_{\mathbb{C}} = v^* A a + \overline{b^* A u}. \quad (7.7)$$

The equations for A and iA separate the two terms. Repeating with $J\psi$ gives

$$\bar{v} \wedge a = 0, \quad \bar{b} \wedge u = 0, \quad u \wedge a = 0, \quad v \wedge b = 0. \quad (7.8)$$

At a point where $\rho > 0$, if $u, \bar{v}$ are independent, these equations imply $a = b = 0$. Otherwise write $u = \alpha z, \bar{v} = \beta \bar{z}$, with $z \neq 0$ and $(\alpha, \beta) \neq (0, 0)$. The same equations imply $a \in \mathbb{C}z$ and $b \in \mathbb{C}\bar{z}$, also when one coefficient vanishes. The vectors

$$\psi = (\alpha z, \bar{\beta} \bar{z}), \quad J\psi = (-\beta z, \bar{\alpha} \bar{z})$$

span this complex plane: their coefficient determinant is $|\alpha|^2 + |\beta|^2 > 0$. Thus $\phi(x)$ belongs to the quaternionic line of $\psi(x)$. Each alternative contradicts (7.6), which makes their values nonzero and quaternionically orthogonal. This proves (7.2). $\square$

Lemma 7.2. *The endomorphism in (7.2) is the derivative of the complete λ -cluster matrix along an analytic path of compatible pairs based at (g, W) and equal to (g, W) outside U .*

Proof. Set $g_s = e^{2sf} g$, and let β_s be the conformal spin isometry. Define

$$W_s = \beta_s e^{-sf} (W + sV) \beta_s^{-1}, \quad U_s = \beta_s e^{-nsf/2}. \quad (7.9)$$

The map $U_s : L^2(\Sigma_g) \rightarrow L^2(\Sigma_{g_s})$ is unitary. Conformal covariance [35, Section 2] gives

$$U_s^{-1} (D_{g_s} + W_s) U_s = e^{-sf/2} (L + sV) e^{-sf/2}. \quad (7.10)$$

This is an analytic family on the common domain H^1 , with derivative $V - (M_f L + L M_f)/2$ at zero. Since $LP_E = \lambda P_E$,

$$P_E \left(V - \frac{1}{2} (M_f L + L M_f) \right) \Big|_E = P_E V|_E - \lambda P_E M_f|_E.$$

Lemma 6.4 identifies this compression with the derivative of the complete cluster matrix. The comparisons and real scalar factors preserve the charge and chirality identities. The factors are the identity outside U , except for the unchanged term W . $\square$

Lemma 7.3. *Let $\Gamma_1, \dots, \Gamma_s$ be pairwise disjoint positively oriented spectral contours for D_g . Suppose their interiors meet $\mathbb{R}$ in pairwise disjoint bounded intervals whose closures avoid zero. In every sufficiently small smooth neighborhood of $(g, 0)$ there is a smooth path (g_t, V_t) , $0 \leq t \leq 1$, contained in that neighborhood and equal to $(g, 0)$ near $t = 0$, such that every Γ_j remains in the resolvent and every enclosed eigenvalue at $t = 1$ is simple. The variations may be supported in any prescribed nonempty open set.*

Proof. In the fixed spinor comparison (2.8), the operator $D_h + W : H^1 \rightarrow L^2$ depends continuously on h in spatial C^1 and on W in C^0 . The resolvent identity therefore gives a neighborhood on which all contours remain in the resolvent and their ranks are constant. Write

$$m = \sum_{j=1}^s \text{rank}_{\mathbb{C}} P_{\Gamma_j} / d_n.$$

This bounds the number of distinct enclosed eigenvalues. At a multiple enclosed eigenvalue, Lemmas 7.1 and 7.2 give a complete cluster matrix

$$A(s) = \lambda I + sB + O(s^2), \quad B^* = B, \quad B \notin \mathbb{R}I.$$

Choose disjoint intervals around two distinct eigenvalues of B . The matrix $(A(s) - \lambda I)/s$ converges to B ; its spectral projections on those intervals have positive ranks for small $s > 0$. Hence the original cluster separates into at least two distinct groups. Choose s small enough to retain every separation

already obtained, all original contours, and the given neighborhood. If $m = 0$, the constant path satisfies the conclusion. For $m > 0$, let r_0 be the initial number of distinct enclosed eigenvalues. After q splittings that number is at least $r_0 + q$ and at most m . If it is smaller than m , one cluster has structural multiplicity greater than one and admits the preceding deformation. Hence at most $m - r_0$ deformations make every enclosed eigenvalue simple. Each step applies at its current compatible potential W . Smooth changes of the parameter, constant near the endpoints of each subinterval, concatenate the finitely many paths. Their spatial supports have a common compact union in the prescribed open set. $\square$

7.2. Analytic metric paths.

Lemma 7.4. *Let $\tilde{g}_t$, $0 \leq t \leq 1$, be a smooth path of metrics, with $\tilde{g}_0 = g$. Suppose finitely many contours Γ_j belong to $\rho(D_{\tilde{g}_t})$ for every t . There is a path h_t , real analytic in t with smooth spatial coefficients, such that*

$$h_0 = g, \quad h_1 = \tilde{g}_1,$$

and every contour remains in $\rho(D_{h_t})$ with the same rank. Moreover $h_t \rightarrow g$ in C^∞ as $t \rightarrow 0$.

Proof. Put $H(t) = \log(g^{-1}\tilde{g}_t)$. Its Bernstein polynomials and the corresponding metrics are

$$B_N(t) = \sum_{j=0}^N \binom{N}{j} t^j (1-t)^{N-j} H(j/N), \quad h_t^{(N)} = g(e^{B_N(t)}, \cdot). \quad (7.11)$$

Let $M_1 = \sup_t \|H(t)\|_{C^1}$ and let ω_1 be the modulus of continuity of $H : [0, 1] \rightarrow C^1$. For $0 < \delta \leq 1$, splitting the Bernstein sum at $|j/N - t| = \delta$ and using the binomial variance $t(1-t)/N \leq 1/(4N)$ gives

$$\sup_t \|B_N(t) - H(t)\|_{C^1} \leq \omega_1(\delta) + \frac{M_1}{2N\delta^2}. \quad (7.12)$$

Thus $h_t^{(N)} \rightarrow \tilde{g}_t$ uniformly in spatial C^1 , with both endpoints exact. Identify both Dirac families with operators on $L^2(\Sigma_g)$ by (2.8), and write $T_t = D_{\tilde{g}_t}$ and $\Delta_{N,t} = D_{h_t^{(N)}} - T_t$ in this identification. The common domain is H^1 , and the coefficient formula gives $\sup_t \|\Delta_{N,t}\|_{H^1 \rightarrow L^2} \rightarrow 0$.

If at least one contour is present, put $C_2 = \sup_{t,z \in \cup_j \Gamma_j} \|(T_t - z)^{-1}\|_{L^2 \rightarrow L^2}$ and $Z_\Gamma = \max_{z \in \cup_j \Gamma_j} |z|$. Lemma 3.1 and (2.9) give

$$\sup_{t,z \in \cup_j \Gamma_j} \|(T_t - z)^{-1}\|_{L^2 \rightarrow H^1} \leq C_{\text{gr}}(1 + (1 + Z_\Gamma)C_2) =: C_\Gamma < \infty. \quad (7.13)$$

Choose N with $\sup_t \|\Delta_{N,t}\|_{H^1 \rightarrow L^2} < (2C_\Gamma)^{-1}$. Then

$$\|s\Delta_{N,t}(T_t - z)^{-1}\|_{L^2 \rightarrow L^2} < \frac{1}{2} \quad (0 \leq s \leq 1, z \in \cup_j \Gamma_j).$$

Stability of bounded invertibility [62, Chapter IV, Theorem 1.16] keeps every contour in the resolvent of $T_t + s\Delta_{N,t}$. The Riesz projections depend continuously on (t, s) and retain their finite ranks. With no contours, there is no resolvent restriction on N .

For this fixed finite N , $B_N(t)$ is polynomial with smooth spatial coefficients. For real t it is g -self-adjoint, so $h_t^{(N)}$ is a metric for all real t . Its coefficients are analytic in every spatial C^r norm. The direct spinor comparison therefore gives a locally holomorphic Dirac family on the common domain H^1 [56, Lemma A.0.16, pp. 88–90]. Take $h_t = h_t^{(N)}$. Since $B_N(0) = 0$, $h_t \rightarrow g$ in every smooth seminorm as $t \rightarrow 0$. $\square$

Proposition 7.5. *For g and the contours of Lemma 7.3, there is an analytic metric path h_t , $0 \leq t \leq 1$, with $h_0 = g$, such that the contours remain in the resolvent and every enclosed eigenvalue is simple for all sufficiently small $t > 0$.*

Proof. If every contour has rank zero, take $h_t = g$. Otherwise fix a nonempty open set $U \subset M$ and apply Lemma 7.3. Around each distinct terminal eigenvalue choose a small subcontour $\gamma_{j\ell}$, of rank d_n , inside its original contour. Their ranks sum to the original rank. Apply Theorem R with $B = [0, 1]$, $B_0 = \{0\}$, and

$$\mathcal{K} = ([0, 1] \times \bigcup_j \Gamma_j) \cup (\{1\} \times \bigcup_{j,\ell} \gamma_{j\ell}),$$

which is a compact subset of the parameter-resolvent set. The potential vanishes near B_0 , and its spatial supports lie in one compact subset of U , by Lemma 7.3. The realized metric equals g near $t = 0$. For a contour of length $\ell(\gamma)$, the projections satisfy

$$\|\hat{P}_\gamma - P_\gamma\| \leq \frac{\ell(\gamma)}{2\pi} \sup_{z \in \gamma} \|(z - \hat{D})^{-1} - (z - L)^{-1}\|. \quad (7.14)$$

Choose the realization error to make every bound less than $1/2$. The projections are orthogonal, and their ranks agree by [62, Chapter I, Theorem 6.32]. Each terminal subcontour has rank d_n and encloses a real interval disjoint from zero. It therefore encloses exactly one simple eigenvalue of the realized metric operator. No additional eigenvalue occurs in an original contour, since the subcontour ranks exhaust its rank.

Denote the realized smooth path by $\tilde{g}_t$. Lemma 7.4 gives an analytic metric path with the same endpoints and original contour ranks. After slightly extending the real parameter interval, the contours remain in the resolvent. In the fixed comparison with g , put

$$\Pi_j(t) = \frac{1}{2\pi i} \int_{\Gamma_j} (z - D_{h_t})^{-1} dz, \quad Q_j(t) = \frac{1}{2\pi i} \int_{\Gamma_j} z(z - D_{h_t})^{-1} dz. \quad (7.15)$$

These finite-rank operators are locally analytic, and $Q_j(t) = D_{h_t} \Pi_j(t) = \Pi_j(t) Q_j(t)$. Set $m_j = \text{rank}_{\mathbb{C}} \Pi_j / d_n$ and

$$p_{j,k}(t) = d_n^{-1} \text{tr}_{\mathbb{C}}(Q_j(t)^k), \quad 1 \leq k \leq m_j.$$

Local analytic frames of $\text{im } \Pi_j(t)$ show that these traces are analytic. The traces agree on overlapping parameter neighborhoods, since they are independent of the frame. Define

$$e_{j,0} = 1, \quad r e_{j,r} = \sum_{k=1}^r (-1)^{k-1} e_{j,r-k} p_{j,k}, \quad q_j(t, z) = \sum_{r=0}^{m_j} (-1)^r e_{j,r}(t) z^{m_j-r}. \quad (7.16)$$

For real t , this monic polynomial has the enclosed eigenvalues as roots, each repeated according to its $\mathbb{F}_n$ -multiplicity. Indeed, the complex trace repeats every such root d_n times.

Let

$$\Delta(t) = \prod_j \text{disc}_z q_j(t, z). \quad (7.17)$$

Polynomials of degree zero or one contribute the factor one. The function Δ is real analytic on a connected neighborhood of $[0, 1]$ and $\Delta(1) \neq 0$. It is therefore not identically zero near 0. Let $v \geq 0$ be its vanishing order at 0. There are a nonzero real number a and a real analytic function r near 0 such that

$$\Delta(t) = t^v (a + tr(t)). \quad (7.18)$$

Choose $\varepsilon > 0$ so that $|tr(t)| \leq |a|/2$ for $|t| < \varepsilon$. Then $|\Delta(t)| \geq |a|t^v/2 > 0$ for $0 < t < \varepsilon$. Each polynomial $q_j(t, \cdot)$ has distinct roots, so every enclosed eigenvalue has complex multiplicity d_n . $\square$

7.3. Genericity and prescribed volume density.

Proof of Theorem G. Fix an integer $a \geq 2$. For $g \in \mathcal{G}_a$, choose disjoint small real intervals around its eigenvalues in K_a , all avoiding zero. Choose a slightly larger neighborhood of K_a which contains no additional eigenvalue of D_g . Discreteness gives these choices also when an endpoint of K_a is an eigenvalue. Surround each small interval by a contour of rank d_n . Cover the remaining compact

real subset of K_a by finitely many zero-rank contours with closures avoiding the spectrum. All these ranks persist under small spatial C^1 changes of metric by the resolvent identity. Thus $\mathcal{G}_a$ is C^1 -open.

For any metric g , enclose the two components of K_a in larger bounded intervals whose closures avoid zero and whose endpoints avoid $\text{Spec}(D_g)$. Proposition 7.5 gives metrics in $\mathcal{G}_a$ tending to g in C^∞ . The space of smooth metrics is an open subset of the Fréchet space of smooth symmetric two-tensors, hence a Baire space. Since the sets K_a exhaust $\mathbb{R} \setminus \{0\}$,

$$S(M) = \bigcap_{a \geq 2} \mathcal{G}_a$$

is residual and dense. This proves the assertion on $\mathcal{R}(M)$. Under its hypothesis excluding rigid eigenspaces on a dense set of metrics, [29, Section 7.1, Proposition 7.4] gives the corresponding conclusion for fixed windows separated from zero.

Now let $g \in \mathcal{R}_\mu(M)$. Choose $h_j \in S(M)$ with $h_j \rightarrow g$ in C^∞ , and put

$$c_j = \left(\frac{\int_M \mu}{\text{Vol}(M, h_j)} \right)^{1/n}, \quad h'_j = c_j^2 h_j, \quad v_j = dv_{h'_j}. \quad (7.19)$$

Then $c_j \rightarrow 1$, $v_j \rightarrow \mu$, and $\int_M (v_j - \mu) = 0$. With the fixed orientation, regard these densities as top forms. Fix a reference Hodge Laplacian on top forms and its Green operator G_n . Set

$$\alpha_j = d^* G_n (v_j - \mu), \quad d\alpha_j = v_j - \mu.$$

The last equality follows from the Hodge decomposition and the zero integral. Elliptic continuity gives $\alpha_j \rightarrow 0$ in C^∞ . For $0 \leq s \leq 1$, define $X_{j,s}$ by

$$\iota_{X_{j,s}} (\mu + s(v_j - \mu)) = -\alpha_j. \quad (7.20)$$

Moser's theorem applies to the positive volume forms $\mu + s(v_j - \mu)$, whose total volume is independent of s . With the primitive α_j and vector field (7.20), it gives an isotopy $\phi_{j,s}$ with $\phi_{j,0} = \text{Id}$ and $\phi_{j,1}^* v_j = \mu$ [69, Theorem 2, Section 4, pp. 292–293]. Since $\alpha_j \rightarrow 0$ and the forms in (7.20) remain uniformly positive, $X_{j,s} \rightarrow 0$ in C^∞ uniformly in s . Smooth dependence of flows on a compact manifold gives $\phi_{j,1} \rightarrow \text{Id}$ in C^∞ .

Constant rescaling multiplies every Dirac eigenvalue by c_j^{-1} ; it preserves multiplicity. The isotopy $\phi_{j,s}$ lifts to the fixed spin structure, so pullback also preserves multiplicity. Therefore $\phi_{j,1}^* h'_j \in S(M) \cap \mathcal{R}_\mu(M)$ and converges smoothly to g . Each restricted set $\mathcal{G}_a \cap \mathcal{R}_\mu(M)$ is open and dense. For $h = g(e^{H/\cdot}, \cdot)$ with H g -self-adjoint, $dv_h = e^{\text{tr } H/2} \mu$. Logarithmic coordinates therefore identify $\mathcal{R}_\mu(M)$ with the Fréchet space of trace-free g -self-adjoint endomorphism fields. This space is Baire. The countable intersection of the restricted sets proves the assertion. $\square$

Corollary 7.6. *A residual dense set of smooth metrics has index-minimal kernel and simple nonzero spectrum. The same holds among metrics with any prescribed positive smooth density.*

Proof. The Dirac-minimal locus is C^1 -open and dense in every C^r topology, for integers $r \geq 1$, by [4, Theorem 1.1 and Corollary 1.2]. A smooth neighborhood contains a neighborhood determined by finitely many seminorms, so that locus is also C^∞ -dense. Intersect it with the open dense sets $\mathcal{G}_a$ and apply the Baire property. For a prescribed density, apply (7.19)–(7.20) to a sequence in this intersection. Scaling and spin pullback preserve the kernel dimension as well as the nonzero multiplicities. Openness and the Baire property on the density slice give the conclusion. $\square$

8. INVERTIBLE PAIRS AND THE REAL INDEX

All manifolds are smooth. Fix a closed connected spin manifold M^n , $n \geq 3$, with its orientation and spin structure, and a positive smooth density μ . The topology on spaces of metrics and sections is the C^∞ topology. Set

$$\mathcal{R}_\mu(M) = \{g : dv_g = \mu\}, \quad \mathcal{R}_\mu^{\text{inv}}(M) = \{g \in \mathcal{R}_\mu(M) : \ker D_g = 0\}.$$

The Dirac operator is the ordinary untwisted spin operator. Its Clifford multiplication satisfies $c(\xi)c(\eta) + c(\eta)c(\xi) = -2g^{-1}(\xi, \eta)$ and $c(\xi)^* = -c(\xi)$.

A multiplication endomorphism is *compatible* if it is self-adjoint and has the same charge-conjugation and chirality identities as D_g . The independent matrix models are recorded in (2.4). In chiral rows the opposite block is the adjoint of the displayed one. Thus the independent potential is unconstrained apart from the indicated real, complex, quaternionic, transpose, or adjoint condition. These models are the algebraic conventions proved in Section 2. The Clifford-module and real-index conventions agree with [10, 26].

The four nonzero component invariants have local representatives

$n \bmod 8$	local crossing	KO_{n+1}
0	$t : \mathbb{R} \rightarrow \mathbb{R}$	$\mathbb{Z}/2$
1	$t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\mathbb{Z}/2$
3	$tI_{\mathbb{H}}$	$\mathbb{Z}$
7	$tI_{\mathbb{R}}$	$\mathbb{Z}$.

(8.1)

In the other four residues $KO_{n+1} = 0$.

Positive spinor comparison, including the density correction when necessary, identifies the Hilbert spaces and their smooth Sobolev domains. Fixing a reference metric also trivializes the bundle of compatible potentials. Write

$$\begin{aligned} \mathcal{P}_\mu &= \{(g, V) : g \in \mathcal{R}_\mu(M), V \text{ smooth and compatible}\}, \\ \mathcal{P}_\mu^{\text{inv}} &= \{(g, V) \in \mathcal{P}_\mu : D_g + V \text{ invertible}\}. \end{aligned}$$

The full pair space is contractible. Indeed, relative to a reference g_0 of density μ , every metric has the unique form $g_0(e^H \cdot, \cdot)$ with H self-adjoint and trace-free. In these coordinates the pair space is a product of two real Fréchet vector spaces. Multiplication perturbations do not change ellipticity or the ordinary spin index.

For a reference metric g_* with $dv_{g_*} = \mu$ and an open set $U \subset M$, put

$$\begin{aligned} \mathcal{R}_\mu(U) &= \{g \in \mathcal{R}_\mu(M) : g = g_* \text{ on } M \setminus U\}, \\ \mathcal{P}_\mu(U) &= \{(g, V) \in \mathcal{P}_\mu : g \in \mathcal{R}_\mu(U), V = 0 \text{ on } M \setminus U\}. \end{aligned} \tag{8.2}$$

Both spaces are contractible in the logarithmic and linear coordinates above. A superscript *inv* denotes the invertible locus; we also write $\mathcal{R}_{\text{inv}, \mu}(U) = \mathcal{R}_\mu^{\text{inv}}(U)$ and $\mathcal{P}_{\text{inv}, \mu}(U) = \mathcal{P}_\mu^{\text{inv}}(U)$. In dimension $n \equiv 4 \pmod{8}$, choose the chiral block A_g with quaternionic index $-a$, where $a = |\text{ind}_{\mathbb{H}} D_g^+|$. At $a = 0$, retain the chiral grading of Section 8.1. A subscript *min* denotes the locus where the corresponding block of $D_g + V$ is injective. Its adjoint kernel then has quaternionic dimension a . We write these loci as $\mathcal{R}_{\text{min}, \mu}(U)$ and $\mathcal{P}_{\text{min}, \mu}(U)$.

Theorem R applies with $X = M$ and the chosen open set U . In Theorem C, $U = M$. All homotopies below retain the density μ .

8.1. The geometric real index. Write $C_{p,q}$ for the real graded Clifford algebra with p generators of square $+1$ and q generators of square -1 . This is the convention of [26, §2.1]; the same algebra is denoted $\text{Cl}^{q,p}$ in [41, Definition 2.1]. Thus $\text{Cl}_n = C_{0,n}$. The real graded spin bundle is

$$\mathcal{S}_{\text{fr}} = P_{\text{Spin}}(M) \times_{\text{Spin}(n)} \text{Cl}_n.$$

Spin transition functions act on the left. The right Clifford action commutes with them and with the Dirac operator. This is the ordinary real spin family of [60, §4.2, Proposition 4.2].

Let $n = 8k + r$, $0 \leq r \leq 7$, and choose a primitive orthogonal projection e in the even subalgebra of Cl_n . For $r = 0, 4$, choose its simple factor so that

$$\omega_n e = (-1)^{n/4} e, \quad \omega_n = c_1 \cdots c_n.$$

The induced grading then agrees with $\Gamma = i^{n/2} \omega_n$. The graded corner $C_r = e \text{Cl}_n e$ and the reduced bundle $\mathcal{S}_{\text{red}} = \mathcal{S}_{\text{fr}} e$ are given below. The quaternion algebra in this table is purely even. The notation

$E_{\mathbb{F}}^{\pm}$ denotes the chiral spinor bundles with their indicated scalar structure, and $W_{\mathbb{F}}$ denotes the odd spinor bundle with that structure. A subscript $\mathbb{R}$ outside parentheses denotes the underlying real bundle.

r	C_r	$\mathcal{S}_{\text{red}}$	independent block	(KO_n, KO_{n+1})
0	$\mathbb{R}$	$E_{\mathbb{R}}^+ \oplus E_{\mathbb{R}}^-$	$A_{\mathbb{R}}$	$(\mathbb{Z}, \mathbb{Z}/2)$
1	$C_{0,1}$	$W_{\mathbb{R}} \oplus W_{\mathbb{R}}$	$K_{\mathbb{R}}^* = -K_{\mathbb{R}}$	$(\mathbb{Z}/2, \mathbb{Z}/2)$
2	$C_{0,2}$	$(E_{\mathbb{C}}^+ \oplus E_{\mathbb{C}}^-)_{\mathbb{R}}$	$A_{\mathbb{C}}^T = -A_{\mathbb{C}}$	$(\mathbb{Z}/2, 0)$
3	$\mathbb{H} \otimes C_{1,0}$	$W_{\mathbb{H}} \oplus W_{\mathbb{H}}$	$T_{\mathbb{H}}^* = T_{\mathbb{H}}$	$(0, \mathbb{Z})$
4	$\mathbb{H}$	$E_{\mathbb{H}}^+ \oplus E_{\mathbb{H}}^-$	$A_{\mathbb{H}}$	$(\mathbb{Z}, 0)$
5	$\mathbb{H} \otimes C_{0,1}$	$W_{\mathbb{H}} \oplus W_{\mathbb{H}}$	$K_{\mathbb{H}}^* = -K_{\mathbb{H}}$	$(0, 0)$
6	$C_{2,0}$	$(E_{\mathbb{C}}^+ \oplus E_{\mathbb{C}}^-)_{\mathbb{R}}$	$A_{\mathbb{C}}^T = A_{\mathbb{C}}$	$(0, 0)$
7	$C_{1,0}$	$W_{\mathbb{R}} \oplus W_{\mathbb{R}}$	$T_{\mathbb{R}}^* = T_{\mathbb{R}}$	$(0, \mathbb{Z})$.

(8.3)

Here $W_{\mathbb{R}} = \Sigma^J$ in residues one and seven, and J defines the quaternionic structure on $W_{\mathbb{H}}$ in residues three and five. The formal transpose in residues two and six is defined by the charge-conjugate duality between the chiral bundles and the L^2 pairing. On a finite spectral subspace it is the matrix transpose in (2.4). The full reduced operator is $\mathcal{D}(A)$ in the even rows and

$$F_{-}(K) = \begin{pmatrix} 0 & -K \\ K & 0 \end{pmatrix} \quad (r = 1, 5), \quad F_{+}(T) = \begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix} \quad (r = 3, 7).$$

For the Dirac operator, $K = iD$ and $T = D$ on the indicated real or quaternionic space. The grading on the doubled bundles is $\text{diag}(I, -I)$.

The corners follow from the Clifford algebra table and the ungraded identification $\text{Cl}_n^0 \cong \text{Cl}_{n-1}$ [10, Table 1 and Proposition 5.4]. Their actions can also be read directly from (2.1). Write $I_{\mathbb{C}}$ for multiplication by i . In residues two and six, the odd generators are J and $I_{\mathbb{C}}J$: they anticommute and have squares $-I$ in residue two and $+I$ in residue six. In residue four, $I_{\mathbb{C}}$ and J generate the even quaternionic action. In residues three and five this action is diagonal on the doubled bundle; the remaining odd generator is, respectively,

$$U_{+} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad U_{-} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

By the model in Section 2, Clifford multiplication by ω_n on the odd complex spinor bundle is respectively $iI, I, -iI, -I$ in residues 1, 3, 5, 7. On its real doubled model, Clifford multiplication is $F_{-}(ic(\xi))$ in residues 1, 5 and $F_{+}(c(\xi))$ in residues 3, 7. Since ω_n is central, its left and right actions on $\mathcal{S}_{\text{fr}}$ agree. Hence

$$\begin{array}{c|cccc} n \bmod 8 & 1 & 3 & 5 & 7 \\ \hline R_{\omega_n} & -U_{-} & U_{+} & U_{-} & -U_{+}. \end{array}$$

Commutation with these actions, oddness, and self-adjointness give exactly the independent blocks in (8.3).

The projection e is full in Cl_n . Since $a(se) = (as)e$ for every spin transition function a , $\mathcal{S}_{\text{red}}$ is a global graded subbundle. The inverse Morita construction is

$$\mathcal{S}_{\text{red}} \otimes_{C_r} e\text{Cl}_n \longrightarrow \mathcal{S}_{\text{fr}}, \quad x \otimes q \longmapsto xq.$$

For homogeneous matrix units with $E_{11} = e$ and $E_{ij}^* = E_{ji}$, its inverse is $s \mapsto \sum_j sE_{j1} \otimes E_{1j}$. Thus a reduced compatible potential lifts by $x \otimes q \mapsto Vx \otimes q$. The construction preserves grading and adjoints and commutes with spin transition functions and positive spinor comparison. It requires no framing of TM . In particular the doubled bundles in the odd rows retain their residual Clifford action; their two summands are not counted as two independent index modules.

The right Clifford L^2 -module is ample: over a coordinate ball, the fibre is the regular graded Clifford module and each irreducible summand occurs with infinite multiplicity in its space of sections. Let ι be the grading, $E_j = R_{c_j}$, $\omega = \iota E_1 \cdots E_n$, and let F be the bounded transform of the full Clifford-linear

operator. Put $f(t) = t(1 + t^2)^{-1/2}$. In residues three and seven, respectively, the primitive reduction of $\omega F \iota$ is

$$- \text{diag}(f(T), f(T)), \quad + \text{diag}(f(T), f(T)).$$

Indeed, $E_1 \cdots E_n = -R_{\omega_n}$ and $\omega F \iota = (E_1 \cdots E_n)F$ in these residues. After complexification, choose nonzero compactly supported local spinors $a_{\pm}$ on a coordinate patch with $ds \neq 0$ such that $ic(ds)a_{\pm} = \pm |ds|a_{\pm}$. The two symbol eigenspaces are interchanged by Clifford multiplication by a nonzero covector orthogonal to ds . For $T = D + V$,

$$\langle T(e^{iN_s} a_{\pm}), e^{iN_s} a_{\pm} \rangle_{L^2} = \pm N \int_M |ds| |a_{\pm}|^2 dv_g + O(1).$$

Thus T has spectrum unbounded in both directions. Since M is closed, its resolvent is compact, and $f(T)$ has essential spectral values $+1$ and -1 . The full Morita reconstruction repeats these blocks. Hence $\omega F \iota$ is neither essentially positive nor essentially negative, as in [41, Definition 2.6].

These are ample $\text{Cl}^{n,0}$ -Fredholm families in the notation of [41, Definition 2.6]. Their degree is KO^{-n} . The representing theorem and the contractibility of the invertible subspace [41, Theorem 2.7, Lemma 2.8, and Theorem 2.21] assign to each path with invertible endpoints a class in

$$KO^{-n}(I, \partial I) = KO^{-(n+1)}(\text{pt}) = KO_{n+1}.$$

The coefficient groups in (8.3) have the normalization of [10, Table 2 and Theorem 11.5]. The four minimal crossings are those in (8.1): a real chiral line and a real skew two-plane contribute one modulo two; a quaternionic line and a real line crossing from negative to positive contribute $+1$ in residues three and seven. In these two rows the product orientation places the R-factor first, and the chiral operator is $\partial_t + T_t$. Thus complex spectral flow is twice the integer invariant in residue three and equals it in residue seven.

After smooth reparametrization, a coefficient path with invertible endpoints is constant on its endpoint collars and extends constantly to $\mathbb{R}$. Its full Clifford operator is odd, symmetric, and of Dirac type on the closed manifold M , with common domain H^1 . Proposition 3.7 of [41] gives norm-continuity of the bounded transforms, hence a relative family class. For a continuous family of such paths in the smooth coefficient topology, parametrized by a compact CW pair (B, B_0) , suppose the paths are constant on B_0 . Choose a Clifford-linear metric connection constant in the cylinder direction on the endpoint collars and over B_0 . These data satisfy Assumptions 3.11 and 3.13 of [41]; Theorem 3.18 identifies the family and suspension classes. The finite Clifford normalizations are those of [26, Sections 2.3–2.5].

For an invertible base metric g_* , ordinary real index difference is denoted by

$$\delta_{g_*}(g, V) \in KO_{n+1}, \quad \delta_{g_*}(g) = \delta_{g_*}(g, 0).$$

It is the relative real family index of a path from $(g_*, 0)$ to (g, V) . Such a path is a smooth common-domain family of elliptic Fredholm operators with invertible endpoints. In the global coordinates above, affine interpolation between two such paths is a homotopy relative to their endpoints. Their relative family classes therefore agree. The index is additive under concatenation, with the normalizations in (8.1) [26, Sections 2.4, 6.2 and Theorem 8.1].

Lemma 8.1. *The fixed-density inclusion $\mathcal{R}_{\mu}^{\text{inv}}(M) \rightarrow \mathcal{R}^{\text{inv}}(M)$ is a homotopy equivalence. The same assertion holds for compatible pairs, for the chiral minimal loci, and with a prescribed exterior whose density is μ . These equivalences preserve ordinary index difference wherever it is defined.*

Proof. For $g \in \mathcal{R}(M)$ put $f = n^{-1} \log(\mu/dv_g)$, $g_t = e^{2tf} g$, and let $\beta_t : \Sigma_g \rightarrow \Sigma_{g_t}$ be the pointwise conformal spinor isometry. Define

$$V_t = e^{-tf} \beta_t V \beta_t^{-1}.$$

Conformal covariance [35, Section 2] gives

$$(D_{g_t} + V_t)(e^{-(n-1)tf/2} \beta_t \psi) = e^{-(n+1)tf/2} \beta_t (D_g + V) \psi.$$

The multiplication factors are isomorphisms and preserve chirality and charge conjugation. They identify the kernels, including each chiral kernel. Moreover $dv_{g_1} = \mu$, and the path is stationary when $dv_g = \mu$. On a prescribed exterior of density μ one has $f = 0$. Thus these paths are strong deformation retractions of the indicated spaces. Homotopy invariance gives the index assertion. $\square$

8.2. Removal of multiplication potentials.

Proposition 8.2. *The inclusions*

$$\mathcal{R}_{\text{inv},\mu}(U) \longrightarrow \mathcal{P}_{\text{inv},\mu}(U), \quad \mathcal{R}_{\text{min},\mu}(U) \longrightarrow \mathcal{P}_{\text{min},\mu}(U), \quad g \longmapsto (g, 0),$$

are weak homotopy equivalences; the second inclusion concerns the chiral block of index $-a$ in dimension $n \equiv 4 \pmod{8}$. For a compact smooth parameter manifold B and a closed subset $B_0 \subset B$, every smooth family in either pair space whose potential vanishes near B_0 is homotopic to a family of pure metrics, relative to a neighborhood of B_0 . In the invertible locus the homotopy preserves ordinary index difference. The assertions include $U = M$.

Proof. Let B be the parameter manifold. In the direct comparison with g_b , regard all operators below as operators on $L^2(\Sigma_{g_b})$ with domain $H^1(\Sigma_{g_b})$. The estimates are uniform in b , using one fixed reference metric. First suppose that the potentials have support in one compact subset of U . Write $L_b = D_{g_b} + V_b$. Choose a smooth function $s : [0, 1] \rightarrow [0, 1]$ equal to zero and one on the respective endpoint collars, and put

$$A_{b,t} = D_{g_b} + s(t)V_b, \quad C_{b,t} = (1 - s(t))V_b.$$

Thus $A_{b,t} + C_{b,t} = L_b$. Choose $\tau > 2 \sup_b \|V_b\|_\infty$, with $\tau > 0$. Apply Theorem R to the potential $s(t)V_b$, the reference metric g_b , and $\mathcal{K} = B \times [0, 1] \times \{i\tau\}$. The operator $A_{b,t}$ is self-adjoint, so this compact set lies in its resolvent set. The potential has one compact spatial support and vanishes near $B \times \{0\}$ and $B_0 \times [0, 1]$. The theorem gives metrics $\hat{g}_{b,t}$ and compared pure Dirac operators $T_{b,t}$ satisfying

$$\|(T_{b,t} - i\tau)^{-1} - (A_{b,t} - i\tau)^{-1}\| < \eta.$$

The metrics retain the exterior and density, and equal g_b on the initial collar and near $B_0 \times [0, 1]$. If B has boundary, extend the smooth coefficient data to a neighborhood of B , extend s constantly across the endpoint collars, and apply the theorem on a smooth compact neighborhood of $B \times [0, 1]$ with rounded corners. The extension retains the compact spatial support and the vanishing near the prescribed parameter subset. Restriction gives the stated family.

For bounded C and R set $F_C(R) = R(I + CR)^{-1}$ when this is defined. The resolvent formula and its difference identity give

$$F_C((A - i\tau)^{-1}) = (A + C - i\tau)^{-1},$$

$$F_C(S) - F_C(R) = (I + SC)^{-1}(S - R)(I + CR)^{-1}.$$

Here $\|C_{b,t}\| \leq \sup_b \|V_b\|_\infty < \tau/2$. Self-adjointness gives $\|(T_{b,t} - i\tau)^{-1}\|, \|(A_{b,t} - i\tau)^{-1}\| \leq \tau^{-1}$. The Neumann series therefore bounds both inverse factors by two. Consequently

$$\|(T_{b,t} + C_{b,t} - i\tau)^{-1} - (L_b - i\tau)^{-1}\| < 4\eta. \quad (8.4)$$

If L_b is invertible, then $I + i\tau(L_b - i\tau)^{-1}$ has inverse $I - i\tau L_b^{-1}$. Compactness of B and the common-domain inverse theorem give $C_0 = \sup_b \|I - i\tau L_b^{-1}\| < \infty$. For $4\tau\eta C_0 < 1$, the Neumann lemma and (8.4) give invertibility of $I + i\tau(T_{b,t} + C_{b,t} - i\tau)^{-1}$. The identity

$$T_{b,t} + C_{b,t} = [I + i\tau(T_{b,t} + C_{b,t} - i\tau)^{-1}](T_{b,t} + C_{b,t} - i\tau)$$

then gives invertibility on the common domain H^1 .

In the chiral minimal case, the full operator L_b has quaternionic kernel dimension a . There is $\rho > 0$ such that $\text{Spec}(L_b) \cap [-2\rho, 2\rho] \subset \{0\}$ for every b . To see this, choose at each parameter a small circle enclosing only zero. Its quaternionic spectral rank is a and remains a nearby. Every neighboring

kernel also has dimension a , so the entire nearby spectral subspace is the kernel. A finite cover of B then gives the stated ρ . Let $\gamma = \{z : |z| = \rho\}$. For $z \in \gamma$,

$$(L_b - z)^{-1} = (L_b - i\tau)^{-1} [I + (i\tau - z)(L_b - i\tau)^{-1}]^{-1}.$$

The last inverses are uniformly bounded. Equation (8.4), the Neumann lemma, and the same identity for $T_{b,t} + C_{b,t}$ give uniform resolvent convergence on γ as $\eta \rightarrow 0$. The contour projections are therefore within norm less than one. They are finite-rank orthogonal projections commuting with J and have the same complex rank, hence the same quaternionic rank a . The chiral index of $T_{b,t} + C_{b,t}$ is still $-a$. Its total kernel has dimension at least a and is contained in this rank- a projection. Consequently the chosen chiral block is injective throughout the homotopy.

Transport $C_{b,t}$ to the realized spinor bundle by the direct unitary comparison. Compatibility and the pointwise norm are preserved. The homotopy begins at (g_b, V_b) and ends with zero potential. Its density and exterior are fixed, and it is stationary where the potential was zero near the prescribed parameter set.

For the support condition in (8.2), choose $\chi \in C_c^\infty(U)$, $0 \leq \chi \leq 1$, for which

$$\sup_b \|(1 - \chi)V_b\|_\infty < \varepsilon.$$

Such a cutoff exists: the continuous function $x \mapsto \sup_b |V_b(x)|$ vanishes on $M \setminus U$, so its superlevel set at any positive value is compact in U . The interpolation $V_b \mapsto (1 - t + t\chi)V_b$ has the prescribed parameter identities. In the invertible case, choose $\varepsilon \sup_b \|L_b^{-1}\| < 1$. In the chiral case, write A_b for the injective block. With a fixed Hilbert norm on H^1 , let $A_b^\# : L^2 \rightarrow H^1$ be its bounded adjoint. The closed range and injectivity give the left inverse

$$G_b = (A_b^\# A_b)^{-1} A_b^\# : L^2 \rightarrow H^1.$$

It depends continuously on b , so its norm is uniformly bounded. Choose $\varepsilon \sup_b \|G_b\|_{L^2 \rightarrow L^2} < 1$. If $(A_b + E)u = 0$, then $u = -G_b E u$, whence $\|u\|_2 \leq \|G_b\|_{L^2 \rightarrow L^2} \|E\|_\infty \|u\|_2$. Thus $u = 0$. Hence the cutoff interpolation remains in the required locus. It is a continuous homotopy in the smooth topology because χ is fixed. The preceding construction applies to its endpoint.

It remains to pass to continuous maps from spheres and disks. In logarithmic metric and fixed potential coordinates, either locus is an open subset $\mathcal{O}$ of a real Fréchet space: the operator map is continuous into $\mathcal{L}(H^1, L^2)$, and an invertible operator, or an injective Fredholm block, remains so under a sufficiently small perturbation. For a continuous map $h : B \rightarrow \mathcal{O}$ from a compact smooth parameter manifold, choose a continuous seminorm p and $\varepsilon > 0$ such that

$$h(B) + \{v : p(v) < \varepsilon\} \subset \mathcal{O}.$$

Choose a finite cover (U_i) , points $b_i \in U_i$, and a smooth partition of unity (φ_i) with $p(h(b) - h(b_i)) < \varepsilon$ on U_i . Then $h' = \sum_i \varphi_i h(b_i)$ is smooth and $p(h' - h) < \varepsilon$. If h is smooth near a closed set K , choose $\theta = 1$ near K with support in that neighborhood and replace h' by $\theta h + (1 - \theta)h'$. This preserves h near K . The affine homotopy from h to h' remains in $\mathcal{O}$. All these operations preserve the prescribed exterior.

For a based sphere, precompose with a map homotopic to the identity, fixing the basepoint and constant near it. Smooth approximation relative to that neighborhood, followed by the removal above, gives a based pure metric representative. For a pair filling of a metric sphere, first approximate the boundary within the metric locus. Attach its short metric homotopy as an annulus and make the new boundary data constant in the normal collar coordinate. Smooth the resulting pair disk relative to that collar and remove its potential relative to the collar. Attaching the reverse metric annulus gives a pure metric filling of the original sphere. The argument for points and intervals gives the bijection on components, and these relative sphere and disk arguments give the isomorphisms on all homotopy groups. Each deformation is a pair homotopy; homotopy invariance therefore preserves the ordinary relative real family index. $\square$

9. SUPPORTED COMPATIBLE POTENTIALS

9.1. Supported multiplication endomorphisms. Weak unique continuation holds for $D_g + V$ with V smooth. Indeed,

$$(D_g + V)^2 = \nabla^* \nabla + P_1 + P_0,$$

where P_1 and P_0 have orders at most one and zero. Theorem 6.2.8 of [56], due to Aronszajn [7], therefore applies. A solution vanishing on a nonempty open subset of a connected manifold vanishes identically.

Proposition 9.1. *Let $X = W_\infty$ be connected, $n \geq 3$, with fixed invertible cylindrical ends; the case $\partial W = \emptyset$, $X = W$, is allowed. Suppose $\alpha(W, h) = 0$, interpreted as $\alpha(W) = 0$ when the boundary is empty. For every nonempty open $U \subseteq \text{int } W$ there is $V \in C_c^\infty(U; \mathcal{A}_n)$ such that $D_g + V$ is invertible.*

Proof. Choose a closed coordinate ball $K \subseteq U$ with nonempty interior, and put

$$\mathcal{V}_K = \{V \in C^\infty(X; \mathcal{A}_n) : \text{supp } V \subset K\}.$$

The operators $D_g + V$, $V \in \mathcal{V}_K$, are self-adjoint Fredholm operators on H^1 , by Section 2. Their finite nullities form a nonempty subset of $\mathbb{Z}_{\geq 0}$. Choose $V_0 \in \mathcal{V}_K$ attaining its minimum m , and set $L = D_g + V_0$, $F = \ker L$.

A sufficiently small circle about zero encloses no other spectral value of L . For $V \in \mathcal{V}_K$ sufficiently close to V_0 in L^∞ , its Riesz projection has rank m and contains $\ker(D_g + V)$. Minimality gives the reverse dimension inequality. Thus the cluster consists entirely of zero modes and the nullity is constant on this neighborhood in $\mathcal{V}_K$. Its complete matrix is zero. Lemma 6.4 gives, with P the orthogonal projection onto F ,

$$PBP = 0 \quad (B \in C_c^\infty(\text{int } K; \mathcal{A}_n)). \quad (9.1)$$

By Lemma 6.5,

$$e_x^* Q e_x = 0 \quad (x \in \text{int } K, Q \in \mathcal{A}_{n,x}),$$

where $e_x : F \rightarrow \Sigma_x X$ is evaluation. Both identities hold at every potential in the same neighborhood. The real index remains $\alpha(W, h)$ along $D_g + tV_0$.

Suppose $n \equiv 0$ or $4 \pmod{8}$. The chiral kernels have equal dimension over $\mathbb{R}$ or $\mathbb{H}$, respectively. If this dimension is positive, choose nonzero kernel spinors ϕ^+ and ϕ^- of opposite chirality, with $J\phi^\pm = \phi^\pm$ in residue zero. Their nonzero loci are open and dense by unique continuation. At a common nonzero point $x \in \text{int } K$, put $u = \phi^+(x)$, $v = \phi^-(x)$ and $Z = vu^*/|u|^2$. The compatible chiral matrix $D(Z)$ then satisfies

$$\langle v, Zu \rangle = |v|^2 > 0,$$

contrary to the pointwise compression identity.

If $n \equiv 3$ or $7 \pmod{8}$, $Q = I$ is compatible. Hence $e_x^* e_x = 0$ on $\text{int } K$, and unique continuation gives $F = 0$.

If $n \equiv 5 \pmod{8}$, $J^2 = -I$ and $JL = -LJ$. For $0 \neq u \in \Sigma_x X$, the endomorphism

$$Q = uu^* - (Ju)(Ju)^*$$

is self-adjoint and satisfies $JQ = -QJ$. Antiunitarity and $J^2 = -I$ give $\langle u, Ju \rangle = 0$, so $\langle Qu, u \rangle = |u|^4 > 0$. Taking $u = \phi(x)$ for a nonzero $\phi \in F$ contradicts the pointwise compression identity at a point where $\phi(x) \neq 0$.

If $n \equiv 6 \pmod{8}$, charge conjugation exchanges the chiral kernels. For $\phi \in \ker L^+$ and $u = \phi(x)$, pairing ϕ with $J\phi$ in charge coordinates gives

$$u^T Z u = 0 \quad (Z^T = Z).$$

If $u_j \neq 0$, the choice $Z = E_{jj}$ contradicts this identity. Unique continuation and charge conjugation therefore give $F = 0$.

If $n \equiv 1 \pmod{8}$, put $F_{\mathbb{R}} = F \cap \text{Fix } J$. Testing with i times every real skew-symmetric matrix shows that the real evaluation image of $F_{\mathbb{R}}$ has dimension at most one at each point of $\text{int } K$. If $\phi, \psi \in F_{\mathbb{R}}$ are independent, choose a ball in $\text{int } K$ on which $\phi \neq 0$. There $\psi = f\phi$ for a smooth real function f , and

$$0 = L(f\phi) - fL\phi = c(df)\phi.$$

Since $c(\eta)^2 = -|\eta|^2 I$ for real η , one has $df = 0$. Unique continuation extends the resulting constant proportionality to X , a contradiction. Thus $\dim_{\mathbb{R}} F_{\mathbb{R}} \leq 1$. Its parity is the vanishing mod-two index, so $F = 0$.

For $n \equiv 2 \pmod{8}$, the dimension restriction gives $n \geq 10$. The local constancy established above is the hypothesis of Lemma 9.2; hence $\dim_{\mathbb{C}} \ker L^+ \leq 1$. This dimension is even because its mod-two index vanishes. Charge conjugation gives $F = 0$.

In every case L is self-adjoint Fredholm with zero kernel, and therefore has a bounded inverse. $\square$

Lemma 9.2. *Let $X, K, \mathcal{V}_K$ be as above and let $n \equiv 2 \pmod{8}$, $n \geq 10$. Suppose $V_0 \in \mathcal{V}_K$ and put $L = D_g + V_0$. If $\dim_{\mathbb{C}} \ker(D_g + V)$ is constant for V in a neighborhood of V_0 in $\mathcal{V}_K$, then $\dim_{\mathbb{C}} \ker L^+ \leq 1$.*

Proof. Set $F = \ker L$ and $F^+ = \ker L^+$. Local constancy and Lemma 6.4 give (9.1), also after every sufficiently small change of V_0 in $\mathcal{V}_K$. In charge coordinates the independent compatible matrices are complex skew-symmetric. Pairing a positive zero mode with the charge conjugate of another and applying Lemma 6.5 gives

$$v^T(E_{ij} - E_{ji})u = v_i u_j - v_j u_i = 0.$$

Consequently

$$\phi(x) \wedge \psi(x) = 0 \quad (\phi, \psi \in F^+, x \in \text{int } K). \quad (9.2)$$

The identity holds for the positive zero modes of every such nearby operator.

Suppose $\phi, \psi \in F^+$ are independent. On a ball $O \Subset \text{int } K$ where $\phi \neq 0$, write $\psi = f\phi$, with f smooth and complex-valued. If $df = 0$ on O , unique continuation gives constant proportionality on X . Thus $df \neq 0$ somewhere on O . Choose $O' \Subset O$ containing such a point and extend f to a compactly supported smooth function on X , unchanged near O' .

For $B \in C_c^\infty(O'; \mathcal{A}_n)$, let P_t be the zero projection of $L + tB$, and let $U_t : F \rightarrow \text{im } P_t$ be its polar frame based on the inclusion of F . These maps preserve chirality. The identity $P\dot{P}P = 0$ gives $P\dot{U}_0 = 0$. With $\phi_t = U_t\phi$, $\psi_t = U_t\psi$, and

$$R = (L|_{F^\perp})^{-1}(I - P) : L^2 \longrightarrow H^1,$$

one has $\Gamma R = -R\Gamma$. Differentiation in the common domain yields

$$\dot{\phi} = -RB\phi, \quad \dot{\psi} = -RB\psi. \quad (9.3)$$

These derivatives also hold in C_{loc}^∞ . Indeed, the interior elliptic estimates [61, Theorems 18.1.24 and 18.1.29], applied to $L\phi_t = -tB\phi_t$, bound ϕ_t in every local Sobolev norm. The identity $L(\phi_t - \phi) = -tB\phi_t$ and convergence in L^2 then give $\phi_t \rightarrow \phi$ in C_{loc}^∞ . For $q_t = (\phi_t - \phi)/t - \dot{\phi}$ one has

$$Lq_t = -B(\phi_t - \phi), \quad q_t \longrightarrow 0 \quad \text{in } L^2.$$

The same estimates give $q_t \rightarrow 0$ in C_{loc}^∞ . The argument for ψ_t is identical.

Differentiating (9.2) on O' gives $\dot{\psi} - f\dot{\phi} \in \mathbb{C}\phi$. Since $B\psi = fB\phi$ on X , the quotient map $\Pi_x : \Sigma_x^+ \rightarrow \Sigma_x^+/\mathbb{C}\phi(x)$ satisfies

$$\Pi[R, f](B\phi) = 0 \quad \text{on } O'. \quad (9.4)$$

Write $u = \phi(x)$ in the charge coordinates and set $\mathcal{H}_\phi = \{y \in \Sigma^- : u^T y = 0\}$ on O' . For $y \in C_c^\infty(O'; \mathcal{H}_\phi)$, put

$$w = \frac{\bar{u}}{|u|^2}, \quad Z_y = yw^T - wy^T.$$

Then $Z_y^T = -Z_y$ and $Z_y u = y(w^T u) - w(y^T u) = y$. Thus $B = \begin{pmatrix} 0 & Z_y^* \\ Z_y & 0 \end{pmatrix}$ is smooth, compatible, supported in O' , and satisfies $B\phi = y$. Equation (9.4) therefore holds for every compactly supported smooth section of $\mathcal{H}_\phi$.

The operator R is locally pseudodifferential of order -1 . Choose nested interior coordinate balls and a properly supported two-sided elliptic parametrix Q on the larger ball [61, Theorem 18.1.24 and pp. 91–93]. On the product of the smaller balls let S be the difference of the kernels of R and Q . The identities $LR = RL = I - P$ show that $L_x S$ and $L_y^t S$ are smooth there; L_y^t acts on the dual bundle in the second variable. The finite-rank kernel of P is smooth by elliptic regularity. The operator

$$L_x^* L_x + (L_y^t)^* L_y^t$$

has principal symbol $(|\xi|^2 + |\eta|^2)I$ on this product. Elliptic regularity gives $S \in C^\infty$.

The principal symbol of the component $R : \Sigma^- \rightarrow \Sigma^+$ is $ic(\xi)/|\xi|^2$. The composition formula [61, Theorem 18.1.8, p. 71] gives

$$\sigma_{-2}([R, f]) = \frac{c(df)}{|\xi|^2} - 2 \frac{\langle \xi, df \rangle c(\xi)}{|\xi|^4}. \quad (9.5)$$

The pairing here is the complex-bilinear extension of g^{-1} . Composition with the inclusion of $\mathcal{H}_\phi$ and the quotient Π multiplies the principal symbol by the corresponding pointwise maps. Since (9.4) vanishes on every test section,

$$\Pi \left(\frac{c(df)}{|\xi|^2} - 2 \frac{\langle \xi, df \rangle c(\xi)}{|\xi|^4} \right) \Big|_{\mathcal{H}_\phi} = 0. \quad (9.6)$$

At a point where $df \neq 0$, choose a nonzero real covector ξ perpendicular to both $\text{Re } df$ and $\text{Im } df$. Put $r = 2^{n/2-1} = \dim_{\mathbb{C}} \Sigma_x^\pm$. If $g^{-1}(\eta, \eta) \neq 0$, the Clifford relation makes both chiral maps $c(\eta)$ invertible. If $\eta \neq 0$ is null, choose a maximal isotropic subspace containing η . In the exterior model [47, Lemma 20.9 and equation (20.11), pp. 304–305], $c(\eta)$ is a nonzero multiple of exterior multiplication by its first generator. On each parity its image has dimension $2^{n/2-2} = r/2$. Thus every nonzero complex covector satisfies $\text{rank } c(\eta) \geq r/2$ on either chiral space.

Restriction to a hyperplane and quotient by a line each lower the rank of a linear map by at most one. Taking $\eta = df$ and the chosen ξ in (9.6) therefore gives

$$0 = \text{rank}(\Pi c(df)|_{\mathcal{H}_\phi}) \geq r/2 - 2 \geq 8 - 2 > 0,$$

a contradiction. □

10. PARAMETER FAMILIES AND KERNEL INCIDENCE

Throughout this section, M is closed and connected. The spin structure and the positive comparisons of Section 2 are fixed. All Sobolev spaces are taken on the resulting fixed bundles.

Lemma 10.1. *Let $\mathcal{Q}$ be the space of smooth compatible potential fields on $I \times M$, where $I = [0, 1]$, vanishing on fixed endpoint collars, or its subspace defined by $Q_t \psi_t = 0$ for a fixed smooth spinor family ψ_t . The same assertions hold without the variable t . There is a continuous linear map*

$$q : \ell^2 \longrightarrow \mathcal{Q}$$

with dense range, where $\ell^2 = \ell^2(\mathbb{N}; \mathbb{R})$. For every k , this map is bounded with values in C^k ; multiplication by $q(p)$ depends smoothly on p at each fixed Sobolev order.

Let $\mathcal{Z}$ be a second-countable Sobolev Banach manifold, and let F be a smooth Banach bundle section over $\ell^2 \times \mathcal{Z}$. At every zero, suppose that $D_z F$ is Fredholm of index d and that the composition of $D_p F$ with the quotient onto $\text{coker } D_z F$ is surjective. Then $F^{-1}(0)$ is a smooth Banach submanifold, and its projection to ℓ^2 is Fredholm of index d .

Proof. The support and pointwise conditions define a closed subspace of the separable Fréchet space of smooth fields. Choose a dense sequence (Q_j) and increasing C^k seminorms, including time derivatives when present. Choose $0 < a_j \leq 2^{-j}(1 + \|Q_j\|_{C^j})^{-1}$ and set

$$q(p) = \sum_{j \geq 1} a_j p_j Q_j. \quad (10.1)$$

For each k , Cauchy–Schwarz gives

$$\|q(p)\|_{C^k} \leq \left(\sum_{j \geq 1} a_j^2 \|Q_j\|_{C^k}^2 \right)^{1/2} \|p\|_{\ell^2} \leq C_k \|p\|_{\ell^2}.$$

Thus the series converges in every smooth seminorm and preserves the constraints. Its range contains the span of the Q_j , hence is dense. The same estimates hold after time differentiation. The multiplication map $C^k \times H^s \rightarrow H^s$ is bounded for sufficiently large k , which gives the asserted smooth dependence. In particular, the image of a sufficiently small ℓ^2 ball lies in any prescribed C^∞ neighborhood of zero.

At a zero, put $L = D_z F$, $C = D_p F$, and let $[C] : \ell^2 \rightarrow \text{coker } L$ be the induced map. A complement of $\ker L$, together with finitely many parameter directions mapped onto $\text{coker } L$, gives a bounded right inverse for (C, L) . The Banach implicit function theorem [46, Theorem 13.20] gives the zero submanifold. Its tangent projection satisfies

$$\ker D\pi \cong \ker L, \quad \text{im } D\pi = \ker[C], \quad \text{coker } D\pi \cong \text{coker } L.$$

Its Fredholm index is therefore $\text{ind } L = d$. The zero manifold is second countable. The regular-value conclusions of [77, Theorem 1.3 and Corollary 1.5] apply, since its projection is smooth. $\square$

Lemma 10.2. *Let p_t be a smooth coefficient path in logarithmic metric and linear potential coordinates, constant on fixed endpoint collars. Fix finitely many smooth potential directions. The path admits arbitrarily close approximations in any finite collection of spatial and parameter seminorms by smooth curves in finite-dimensional coefficient families analytic in their coefficient vectors. These families contain the fixed directions, and the approximations equal p_t on the prescribed collars.*

Suppose, moreover, that the associated first-order elliptic family L_p is real skew-adjoint. Let ψ_t be a smooth L^2 -unit family with $L_{p_t} \psi_t = 0$. If the incidence differential in the fixed potential directions and the spinor directions is surjective, the approximations may be chosen as incidence curves, arbitrarily close to (p_t, ψ_t) in the prescribed seminorms and equal to it on the collars.

Proof. Choose a smooth scalar function σ , equal to zero and one on the two collars, and put $b_t = (1 - \sigma(t))p_0 + \sigma(t)p_1$. The smooth remainder $p_t - b_t$ vanishes on the collars, with all jets zero at their inner boundaries. Cutting it off inside the complement of the collars changes any prescribed finite collection of seminorms by arbitrarily small amounts. A finite partition of unity in t and local Taylor polynomials then give a finite sum $\sum_j f_j(t)e_j$, with every f_j supported off the collars, which approximates this remainder in those seminorms. Adjoin p_0, p_1 and the fixed potential directions to the span of the e_j .

The resulting family $z \mapsto L(z) : H^s \rightarrow H^{s-1}$ is analytic. Indeed its coefficients are obtained from matrix exponentials, inverses, positive square roots, their first spatial derivatives, and affine potentials. These operations are analytic on the open sets in question, at each sufficiently high finite C^k order.

Write $Q(h) = \sum_{j=1}^q h_j Q_j$ in a basis of the fixed potential span, and let $\iota : \mathbb{R}^q \rightarrow \mathbb{R}^N$ be its coefficient inclusion; thus $L(z + \iota h) = L(z) + Q(h)$. Fix an integer $s \geq 1$ and put

$$\mathcal{E}_t = \mathbb{R}^q \oplus (\psi_t^{\perp \mathbb{R}} \cap H^s), \quad \mathcal{F}_t = \psi_t^{\perp \mathbb{R}} \cap H^{s-1}, \quad P_t = P_{\psi_t^{\perp \mathbb{R}}}.$$

These are smooth real Hilbert bundles over I . Their fibre maps

$$T_t(h, u) = P_t(Q(h)\psi_t + L_{p_t}u) : \mathcal{E}_t \longrightarrow \mathcal{F}_t$$

are surjective. For $s > 1$, an H^1 solution with target $v \in \mathcal{F}_t$ satisfies $L_{p_t} u = v - Q(h)\psi_t + c\psi_t$; elliptic regularity [40, Theorem 15.1 and Remark 15.2] gives $u \in H^s$. A surjective bounded map between Hilbert spaces has a bounded right inverse. In local trivializations of both bundles, $J_{t_0}(T_t J_{t_0})^{-1}$ is a smooth right inverse near t_0 . A finite partition of unity gives a smooth family J_t with $T_t J_t = I$. Put $M_J = \sup_t \|J_t\| < \infty$.

Let $z_0(t)$ be a coefficient approximation, equal to p_t on the collars. For $(h, u) = J_t w$, define

$$\Psi_t(w) = P_t(L_{p_t} + Q(h))(\psi_t + u), \quad \Phi_t(w) = P_t L(z_0(t) + ih)(\psi_t + u).$$

The identities $L_{p_t} \psi_t = 0$ and $T_t J_t = I$ give

$$\Psi_t(w) = w + P_t Q(h)u, \quad \Phi_t(w) - \Psi_t(w) = P_t \Delta_t(\psi_t + u), \quad \Delta_t = L(z_0(t)) - L_{p_t}. \quad (10.2)$$

Set

$$C_P = \sup_t \|P_t\|_{H^{s-1} \rightarrow H^{s-1}}, \quad M_\psi = \sup_t \|\psi_t\|_{H^s}, \quad C_Q = \sup_{|h| \leq 1} \|Q(h)\|_{H^s \rightarrow H^{s-1}}, \quad \delta = \sup_t \|\Delta_t\|_{H^s \rightarrow H^{s-1}}.$$

For $\|w\| \leq r$, differentiation yields

$$\sup_{t, \|w\| \leq r} \|D_w \Phi_t(w) - I\| \leq 2C_P C_Q M_J^2 r + C_P \delta M_J, \quad \sup_t \|\Phi_t(0)\| \leq C_P \delta M_\psi. \quad (10.3)$$

Choose $r > 0$ with $2C_P C_Q M_J^2 r < 1/4$, and then choose the approximation with $C_P \delta M_J < 1/4$ and $C_P \delta M_\psi < r/2$. The map $w \mapsto w - \Phi_t(w)$ takes the closed radius- r ball into itself and has Lipschitz constant less than $1/2$. The contraction theorem [46, Theorem 10.38] gives its unique fixed point w_t . The implicit function theorem [46, Theorem 13.20] gives smooth dependence on t , and

$$\|w_t\| \leq 2 \|\Phi_t(0)\|, \quad \|(h_t, u_t)\| \leq 2 \|J_t\| \|\Phi_t(0)\|. \quad (10.4)$$

The correction is small in every prescribed finite collection of parameter seminorms. To see this, make $\Delta \rightarrow 0$ in $C^k(I, \mathcal{L}(H^s, H^{s-1}))$. In the preceding local trivializations, $\Phi(t, w) = w + N(t, w) + E(t, w)$, where N is quadratic in w and $E(t, w) = P_t \Delta_t(\psi_t + \text{pr}_2 J_t w)$. The fibre inverses of $D_w \Phi$ have norm at most 2. Differentiating $\Phi(t, w_t) = 0$ and proceeding by induction on $j \leq k$ gives

$$\|w\|_{C^k(I, F)} \rightarrow 0, \quad \|(h, u)\|_{C^k(I, \mathbb{R}^q \oplus H^s)} \rightarrow 0.$$

Indeed, after the term $D_w \Phi \partial_t^j w$ is isolated, every remaining term contains either a derivative of Δ or a derivative $\partial_t^\ell w$ with $\ell < j$. Choose s large enough for the prescribed spatial seminorms and apply Sobolev embedding [40, Theorem 12.16]. Since $Q(h)$ lies in a fixed finite span, its coefficient seminorms tend to zero as well.

The projected equation reads $L(z_0 + ih)(\psi_t + u) = c\psi_t$, with $c \in \mathbb{R}$. Skew-adjointness and $u \perp_{\mathbb{R}} \psi_t$ give

$$0 = \text{Re} \langle L(z_0 + ih)(\psi_t + u), \psi_t + u \rangle = c \text{Re} \langle \psi_t, \psi_t + u \rangle = c.$$

Hence the full equation holds. The spinor $\psi_t + u$ is nonzero; normalize it in L^2 , preserving the preceding approximations. On the collars, $\Delta_t = 0$, so uniqueness gives $w_t = 0$. For the corrected spinor φ_t and operator $\tilde{L}_t$,

$$\tilde{L}_t \partial_t^k \varphi_t = - \sum_{j=1}^k \binom{k}{j} (\partial_t^j \tilde{L}_t) \partial_t^{k-j} \varphi_t.$$

Elliptic regularity and the uniform estimates for this compact family give smoothness into every Sobolev space, hence joint spatial and parameter smoothness. $\square$

Lemma 10.3. For $0 \neq u \in \mathbb{H}^r$, $r \geq 1$,

$$\{Qu : Q^* = -Q\} = u^{\perp_{\mathbb{R}}}.$$

For $y \in u^{\perp_{\mathbb{R}}}$, a preimage depends smoothly on (u, y) .

Proof. If $Q^* = -Q$, then $\operatorname{Re}(u^* Qu) = 0$. Conversely, put $s = |u|^2$ and $b = u^* y \in \operatorname{Im} \mathbb{H}$. The matrix

$$Q(u, y) = \frac{yu^* - uy^*}{s} - \frac{ub u^*}{s^2} \quad (10.5)$$

is skew-adjoint. Since $y^* u = -b$, $Q(u, y)u = y + ub/s - ub/s = y$. The formula is smooth for $u \neq 0$. $\square$

Fix a nonempty open set $U \subset M$. In dimension $n \equiv 4 \pmod{8}$, choose the chiral operator of quaternionic index $-a$, $a \geq 0$. In dimension $n \equiv 5 \pmod{8}$, put $K = i(D + V)$, a quaternionic skew-adjoint operator.

Lemma 10.4. *Let P be a compact smooth d -manifold with corners and finitely many embedded boundary faces. Here and below, the faces include P and all nonempty intersections of boundary faces. Let (g_p, V_p) be a smooth family, Dirac-minimal near a closed subset $P_0 \subset P$. For $n \equiv 4, 5 \pmod{8}$, an arbitrarily small smooth compatible potential perturbation, supported in a common compact subset of U and zero near P_0 , makes the incidence of parameters and quaternionic kernel lines a compact smooth manifold with corners of dimension*

$$d - r, \quad r = 4(a + 1) \text{ if } n \equiv 4 \pmod{8}, \quad r = 3 \text{ if } n \equiv 5 \pmod{8}. \quad (10.6)$$

The kernel is the minority chiral kernel in residue four and $\ker K_p$ in residue five. The boundary lies exactly over ∂P ; the incidence over each face F has dimension $\dim F - r$. An incidence with negative indicated dimension is empty.

The perturbation may be zero on any prescribed closed collar contained in an open region where the incidence equation is transverse to zero on every face, including all face intersections. In particular, transverse endpoint families and their face restrictions admit unchanged collars.

Proof. On the L^2 -unit sphere in H^1 , the equation is equivariant for the free right action of $\operatorname{Sp}(1)$. A projective chart at $[\psi]$ has tangent

$$\psi^{\perp \mathbb{H}} \cap H^1 = \{u \in H^1 : \langle \psi, u \rangle_{L^2, \mathbb{H}} = 0\}.$$

Normalization in L^2 is smooth on nonzero H^1 vectors. The target transforms by right multiplication when the unit representative changes; the equation therefore defines a smooth section of an associated Hilbert bundle.

In residue four put $L_p = A_{g_p} + B_p$. Its vertical derivative at a zero is

$$L_p : \psi^{\perp \mathbb{H}} \cap H^1 \longrightarrow L^2(T).$$

The unrestricted operator has quaternionic index $-a$. Removing the kernel line $\psi \mathbb{H}$ from its domain leaves its range unchanged and subtracts one from its index. The vertical real index is therefore $-4(a + 1)$.

Fix a closed ball $K \subset U$ with nonempty interior. Suppose η annihilates the vertical image and every $B\psi$ with B supported in K . The annihilator of the projective tangent is represented by $\psi \mathbb{H}$; hence $L_p^* \eta = \psi q_0$ distributionally, for some $q_0 \in \mathbb{H}$. For every $q \in \mathbb{H}$,

$$\operatorname{Re} \langle \psi q, L_p^* \eta \rangle_{L^2} = \operatorname{Re} \langle L_p(\psi q), \eta \rangle_{L^2} = 0.$$

Taking $q = q_0$ gives $q_0 = 0$. Elliptic regularity makes η smooth. Unique continuation ensures that ψ is nonzero on some subball of the interior of K . On that subball the values $B\psi$ fill the target fibre. Thus η vanishes there, and unique continuation gives $\eta = 0$ on M .

In residue five the equation takes values in $\psi^{\perp \mathbb{R}}$, since K_p is skew-adjoint. Write it locally as $\Pi_\psi K_p \psi$, where Π_ψ is the real orthogonal projection. At a zero the derivative of Π_ψ contributes zero. The vertical operator is

$$K_p : \psi^{\perp \mathbb{H}} \cap H^1 \longrightarrow \psi^{\perp \mathbb{R}} \cap L^2.$$

Its real index is -3 : restriction of the domain subtracts four, and restriction of the target adds one to the index zero of K_p . If $\eta \in \psi^{\perp \mathbb{R}}$ annihilates the vertical image, then $K_p \eta = \psi q_0$ distributionally. For $q \in \mathbb{H}$,

$$\operatorname{Re} \langle \psi q, K_p \eta \rangle_{L^2} = -\operatorname{Re} \langle K_p(\psi q), \eta \rangle_{L^2} = 0.$$

Thus $q_0 = 0$, and η is smooth. If η also annihilates all $Q\psi$ with $Q \in \mathfrak{sp}(\Sigma M)$ supported in K , Lemma 10.3 gives $\eta = f\psi$ wherever $\psi \neq 0$ in the interior of K , with f smooth and real. On a connected subball where $\psi \neq 0$,

$$0 = K_p(f\psi) = ic_{g_p}(df)\psi.$$

Since $c(\xi)^2 = -|\xi|^2 I$ for real covectors, $df = 0$ there. Unique continuation gives $\eta = c\psi$ on M . Its real L^2 orthogonality to ψ gives $c = 0$. The elliptic regularity and unique continuation statements here apply to the chiral operators, their adjoints, and K_p as in Section 9.

For a compact smooth coefficient family, the elliptic estimate [40, Theorem 15.7 and Remark 15.2] gives

$$\|\psi\|_{H^{s+1}} \leq C_s (\|L_p \psi\|_{H^s} + \|\psi\|_2), \quad s \geq 0, \quad (10.7)$$

with K_p in residue five. The constants may be chosen uniformly in p : freeze the coefficients at one parameter, absorb a small $\mathcal{L}(H^{s+1}, H^s)$ difference, and take a finite cover of the parameter space. Normalized kernel spinors consequently have uniform bounds at every Sobolev order. Given a sequence (p_j, ψ_j) of such spinors, pass to $p_j \rightarrow p$. The uniform H^2 bound and the compact embedding $H^2 \hookrightarrow H^1$ [40, Theorem 15.10] give a subsequence converging in H^1 to ψ . Continuity in $\mathcal{L}(H^1, L^2)$ gives

$$\|L_p \psi\|_2 \leq \|L_p(\psi - \psi_j)\|_2 + \|(L_p - L_{p_j})\psi_j\|_2 \rightarrow 0, \quad \|\psi\|_2 = 1.$$

The same argument applies to K_p . Thus the normalized incidence and its quotient by $\operatorname{Sp}(1)$ are compact.

At every zero, the preceding cokernel arguments imply that finitely many smooth potentials supported in K make the vertical differential surjective. A finite cover of the compact incidence provides a finite collection for the whole family. Multiply them by smooth parameter cutoffs equal to one on the respective neighborhoods and zero near P_0 . This is possible because the incidence is disjoint from a neighborhood of P_0 . The combined differential remains surjective near the original incidence. The same holds on every face of P : only spinor and potential directions are required for surjectivity.

Let $y \in \mathbb{R}^N$ be the coefficients of these potentials. For all sufficiently small y , every new zero lies in the preceding neighborhood. Otherwise a sequence $y_j \rightarrow 0$ of zeros outside it would, by (10.7) for the compact family with $|y| \leq \delta$, have a subsequence converging to an original zero. This is impossible. The implicit function theorem now gives a smooth universal zero manifold with corners of dimension $d + N - r$, with the induced faces. Apply Sard's theorem [77, Theorem 1.2] to its projection to $\mathbb{R}^N$ and to the restrictions on the interiors of all its faces. There are finitely many face types and each has a countable atlas; the union of the critical-value sets has measure zero. A common regular value, arbitrarily close to zero, gives the stated incidence and face dimensions. Compactness follows from (10.7). If a dimension is negative, the corresponding regular fibre is empty.

Finally, let C be a prescribed closed collar contained in an open region O of transversality on all faces. Choose open sets

$$C \subset O_1 \subset \overline{O_1} \subset O_2 \subset \overline{O_2} \subset O$$

and a smooth function θ equal to zero on O_1 and one outside O_2 . Multiply the preceding parameter cutoffs by θ . Over $\overline{O_2}$, the differential in the original parameter and spinor variables is surjective, on every face. Compactness of the normalized incidence over this closed set and openness of surjectivity give the same property for all sufficiently small y . Outside O_2 , the potential directions give surjectivity. The common regular-value argument therefore applies with the family unchanged on C . Transversality of an endpoint family and all its face restrictions persists on a collar by the same compactness argument. $\square$

Lemma 10.5. *Let X be path connected and locally path connected, and let $S \subset X$ be nonempty, closed, and path connected. Suppose that every point of S has a product neighborhood*

$$(N, N \cap S) \cong (V \times (-1, 1), V \times \{0\}),$$

where V is path connected. Then $X \setminus S$ has at most two path components.

Proof. The factors V are locally path connected, since the projection $V \times (-1, 1) \rightarrow V$ is open. Restricting the factors and the interval therefore gives a basis of product neighborhoods at S with path-connected factors.

Fix $s_0 \in S$ and such a neighborhood N_0 , with sides N_0^+ and N_0^- . For $x \in X \setminus S$, choose a path $\gamma : [0, 1] \rightarrow X$ from x to s_0 , and put

$$t_0 = \min\{t : \gamma(t) \in S\}.$$

Since S is closed, $t_0 > 0$ and $\gamma([0, t_0)) \subset X \setminus S$. For a product neighborhood N of $\gamma(t_0)$, continuity gives $\eta > 0$ with $\gamma([t_0 - \eta, t_0]) \subset N$. The normal coordinate has constant nonzero sign on $[t_0 - \eta, t_0]$. Thus x lies in the path component containing one side of N .

Join $\gamma(t_0)$ to s_0 by a path in S . A finite subdivision places its successive subpaths in product neighborhoods. At each subdivision point take a smaller product neighborhood inside the overlap of the two consecutive ones. Each of its connected sides lies in one side of either larger neighborhood. They lie in different sides: the smaller neighborhood is open and meets both sides of either larger one. These side correspondences propagate through the finite chain. Since every side is path connected, the initial side and one of N_0^+, N_0^- lie in the same path component of $X \setminus S$. Every point of the complement belongs to one of these at most two components. $\square$

11. INTEGER AND VANISHING INDEX DIFFERENCES

Let M^n be a closed connected spin manifold, $n \geq 3$, with fixed spin structure and positive smooth density μ . The operators are untwisted. The fixed-domain comparisons and the spaces $\mathcal{R}_\mu(M)$ and $\mathcal{P}_\mu$ are those of Section 8, with their C^∞ topologies. The potentials range over all smooth sections of the compatible endomorphism bundle. By Proposition 8.2, with $U = M$, the inclusion $g \mapsto (g, 0)$ induces a bijection on invertible path components and preserves ordinary index difference.

11.1. The two self-adjoint cases. Suppose $n \equiv 3, 7 \pmod{8}$, and put

$$\mathbb{F} = \begin{cases} \mathbb{H}, & n \equiv 3 \pmod{8}, \\ \mathbb{R}, & n \equiv 7 \pmod{8}. \end{cases}$$

In residue three the spinor bundle is a right quaternionic bundle; in residue seven we take the real fixed bundle of charge conjugation. The Dirac operator is self-adjoint and $\mathbb{F}$ -linear. The compatible potentials are the fibrewise self-adjoint $\mathbb{F}$ -linear endomorphisms. In particular, real scalar multiplication is compatible.

Lemma 11.1. *Let $L = D_g + V$, let $\lambda \in \mathbb{R}$, and let $E \subset \ker(L - \lambda)$ have $\mathbb{F}$ -dimension two. If P_E is the orthogonal projection onto E , then*

$$\dim_{\mathbb{R}}\{P_E Q|_E : Q \text{ is a smooth compatible potential}\} \geq 3.$$

Proof. Denote the displayed real vector space by W . It contains I_E . If $\dim_{\mathbb{R}} W \leq 2$, it is spanned by I_E and at most one self-adjoint matrix. The real or quaternionic spectral theorem gives an orthonormal basis ψ, ϕ of E in which every member of W is diagonal. Hence

$$\int_M \langle \psi, Q\phi \rangle_{\mathbb{F}} \mu = 0 \quad \text{for every smooth compatible } Q. \quad (11.1)$$

The square of $L - \lambda$ has the form $\nabla^* \nabla + A_1 + A_0$, with smooth terms of orders at most one and zero. Unique continuation [56, Theorem 6.2.8] implies that the nonzero sets of ψ and ϕ are open and dense.

Choose a point where both spinors are nonzero. The union of their two proper orthogonal subspaces is a proper subset of the fibre. There is therefore a vector v such that $\langle \psi, v \rangle_F$ and $\langle v, \phi \rangle_F$ are nonzero. The self-adjoint rank-one map vv^* satisfies

$$\langle \psi, vv^* \phi \rangle_F = \langle \psi, v \rangle_F \langle v, \phi \rangle_F \neq 0.$$

Extend v smoothly to a ball. Substitution of $Q = fvv^*$ in (11.1), for every real test function f supported in that ball, forces this continuous integrand to vanish there. In the quaternionic case apply the argument to its four real components. This is a contradiction. $\square$

Real analyticity of an operator family means analyticity with values in $\mathcal{L}(\mathcal{D}, \mathcal{H})$, where the common domain is $\mathcal{D} = H^1$.

Lemma 11.2. *Let (g_t, V_t^0) , $0 \leq t \leq 1$, be a real-analytic path in $\mathcal{P}_\mu$, and suppose that $L_0^0 = D_{g_0} + V_0^0$ and $L_1^0 = D_{g_1} + V_1^0$ are invertible. There is a smooth compatible potential W_t , arbitrarily small in $C^1([0, 1]; L^\infty)$ and zero on endpoint collars, such that $L_t = D_{g_t} + V_t^0 + W_t$ has the following properties. The two consecutive ordered eigenvalue labels which bound zero on the initial collar remain in a fixed bounded spectral interval. Every F -eigenvalue in this interval is simple on a compact middle interval; L_t is invertible on the remaining parameter intervals.*

Proof. Choose invertible endpoint collars for L_t^0 and a compact interval $J \Subset (0, 1)$ whose complement lies in those collars. Let χ be smooth, zero on smaller endpoint collars and one near J , with transition regions inside the invertible collars.

Consider first every smooth compatible perturbation satisfying

$$\|W\|_{C^1([0,1];L^\infty)} \leq 1, \quad W = 0 \text{ on the smaller endpoint collars.} \quad (11.2)$$

The uniform elliptic estimate for L_t^0 gives, for an L^2 -unit eigenvector of $L_t^0 + W_t$ with eigenvalue λ ,

$$\|\psi\|_{H^1} \leq C_0(\|L_t^0 \psi\|_2 + \|\psi\|_2) \leq C_1(1 + |\lambda|).$$

The derivatives $\dot{L}_t^0 : H^1 \rightarrow L^2$ are uniformly bounded. Reduction to a finite spectral subspace and the perturbation formulas for Hermitian matrices [62, Chapter II, Theorem 6.8 and equation (6.11)] give locally Lipschitz ordered eigenvalues. At almost every parameter their derivatives are eigenvalues of the compression of $\dot{L}_t^0 + \dot{W}_t$ to the corresponding eigenspace. For a suitable unit eigenvector ψ , therefore,

$$|\dot{\lambda}(t)| \leq \|\dot{L}_t^0 \psi\|_2 + \|\dot{W}_t\|_\infty \leq C_2(1 + |\lambda(t)|). \quad (11.3)$$

The same formulas apply to real symmetric matrices and to the complex representation of quaternionic self-adjoint matrices. The constants depend only on the reference path and the bound (11.2).

At $t = 0$, number the eigenvalues, with multiplicity over F , so that $\lambda_0(0) < 0 < \lambda_1(0)$ are consecutive. Gronwall's inequality gives

$$1 + |\lambda_j(t)| \leq (1 + |\lambda_j(0)|)e^{C_2}, \quad j \in \mathbb{Z}.$$

Thus no label can escape to infinity at a finite parameter. The same estimate backwards in t excludes entry from infinity; local ordered continuation gives a $\mathbb{Z}$ -labeling on $[0, 1]$. Choose Λ strictly larger than the bounds for $j = 0, 1$. It is independent of the potential directions chosen below.

The set of tuples (t, λ, ψ, ϕ) such that $t \in J$, $|\lambda| \leq \Lambda + 1$, and (ψ, ϕ) is an orthonormal F -two-frame in $\ker(L_t^0 - \lambda)$ is compact in $J \times [-\Lambda - 1, \Lambda + 1] \times L^2 \times L^2$. Indeed, the elliptic estimate bounds the spinors in H^1 . Rellich compactness gives strong L^2 and weak H^1 convergence along a subsequence. Continuity of $L_t^0 : H^1 \rightarrow L^2$ gives the limiting eigen-equation, and strong L^2 convergence preserves orthonormality.

If this frame set is empty, take $W = 0$. Otherwise, at each frame Lemma 11.1 gives three compatible potentials with linearly independent compressions. Independence is open because their matrix entries are continuous in the L^2 topology. A finite subcover gives potentials $Q_1, \dots, Q_N$ whose compressions span a space of real dimension at least three on every reference two-plane in this window. Let U_t be

the fixed positive comparison from the reference spinor bundle to $\Sigma_{g_t} M$, and represent each selected direction by Q_j on the reference bundle. Put $Q_j(t) = U_t Q_j U_t^{-1}$. These fields are analytic in t and have constant representatives in the common bundle.

For all sufficiently small $p \in \mathbb{R}^N$, these directions still have compression rank at least three on every two-plane in an eigenspace with $|\lambda| \leq \Lambda$ of

$$L_{t,p} = L_t^0 + \chi(t) \sum_{j=1}^N p_j Q_j(t), \quad t \in J. \quad (11.4)$$

Otherwise a sequence $p_v \rightarrow 0$ and corresponding two-frames would have a subsequence converging to a reference two-frame. Every three-by-three minor of the compressed directions would vanish at the limit, contradicting their selection. The uniform H^1 bound needed here follows from (11.2).

Shrink the coefficient ball so that (11.2) holds, the perturbation is as small as prescribed, and the endpoint collars remain invertible. Since $\chi = 1$ near J , $L_{t,p}$ is real analytic in (t, p) there.

On a compact box in (t, p, λ) , finitely many analytic Fredholm reductions of $L_{t,p} - \lambda$ suffice. At each base point split off its kernel; the complementary block is invertible nearby. Its Schur complement is an analytic self-adjoint $\mathbb{F}$ -matrix with the same nullity as $L_{t,p} - \lambda$. In the complex representation, quaternionic nullity at least two is complex nullity at least four. Thus the repeated-eigenvalue locus is a real-analytic determinantal set in each chart.

By semianalytic stratification [22, Proposition 2.10 and Corollary 2.11], the portions in finitely many compact semianalytic boxes have finite stratifications by real-analytic submanifolds, compatible with all kernel-rank loci. On a stratum S the reduced matrix has constant rank; its kernels and their lifts to H^1 form smooth vector bundles. At a point of S , put $F = \ker(L_{t,p} - \lambda)$. Differentiating the kernel equation along S and projecting onto F gives

$$P_F d(L_{t,p} - \lambda)[\xi]|_F = 0, \quad \xi \in TS. \quad (11.5)$$

The ambient real-linear map on the left has rank at least three, as follows by compressing its potential directions to any two-plane in F . Hence

$$\dim_{\mathbb{R}} S \leq (N + 2) - 3 = N - 1.$$

Sard's theorem [77, Theorem 1.2] implies that the projection of each stratum to $\mathbb{R}^N$ has measure zero. There are finitely many such strata. Choose p , arbitrarily close to zero, outside their projected union and put $W_t = \chi(t) \sum_j p_j Q_j(t)$. All eigenvalues in the chosen window are simple for $t \in J$, and the two distinguished labels remain there by (11.3). $\square$

Remark 11.3. For a fixed metric and any bounded spectral window, an arbitrarily small compatible potential makes every eigenvalue in that window simple over $\mathbb{F}$. In the preceding argument omit t ; then (11.5) gives $\dim_{\mathbb{R}} S \leq (N + 1) - 3 = N - 2$, so the projected exceptional set again has measure zero.

Count spectral flow positively at a passage from negative to positive, with multiplicity over $\mathbb{F}$. For invertible pairs q_0, q_1 , denote it by $\text{sf}_{\mathbb{F}}(q_0, q_1)$; for pure metric endpoints write $\text{sf}_{\mathbb{F}}(g_0, g_1)$. It is independent of the pair path, by contractibility of $\mathcal{P}_{\mu}$ and homotopy invariance of spectral flow [26, Remark 5.2 and Section 6.2].

Lemma 11.4. *If $q_0, q_1 \in \mathcal{P}_{\mu}^{\text{inv}}$ and $\text{sf}_{\mathbb{F}}(q_0, q_1) = 0$, there is a smooth invertible pair path from q_0 to q_1 . For pure metric endpoints its potential may vanish on endpoint collars.*

Proof. Write $q_j = (g_j, V_j)$ and $g_1 = g_0(S, \cdot)$, with S positive and self-adjoint. Equality of the densities gives $\det S = 1$. Put

$$H = \log S, \quad g_t = g_0(e^{tH} \cdot, \cdot), \quad V_t^0 = (1 - t)V_0 + tV_1$$

in the common spinor bundle. Since $\text{tr } H = 0$, this is an analytic path in $\mathcal{P}_{\mu}$. Apply Lemma 11.2 and write $L_t = D_{g_t} + V_t^0 + W_t$. Let λ_0, λ_1 be its consecutive ordered labels which initially bound zero. They are strictly separated on the middle interval.

Let $\nu(t)$ be the greatest ordered label with eigenvalue below zero. On an interval $[t_{i-1}, t_i]$ of a spectral-flow partition, choose $a_i > 0$ outside the spectrum throughout that interval. The greatest label k_i below a_i is constant there, and

$$\dim_{\mathbb{F}} \operatorname{ran} \mathbf{1}_{[0, a_i]}(L_t) = k_i - \nu(t).$$

The spectral-projection formula [80, Section 3, equation (1)], with multiplicities over $\mathbb{F}$ as in Section 8.1, gives

$$\operatorname{sf}_{\mathbb{F}}(L_t) = \sum_i (\nu(t_{i-1}) - \nu(t_i)) = \nu(0) - \nu(1).$$

The homotopy $L_t^0 + sW_t$ fixes both endpoints, so this integer is zero. Therefore $\nu(1) = \nu(0) = 0$. Zero lies in the same numbered gap on the final collar, and the two labels are strictly separated throughout $[0, 1]$.

The intervals $(\lambda_0(t), \lambda_1(t))$ admit a continuous selection equal to zero on smaller endpoint collars: combine zero on the collars with the midpoint in the interior by a partition of unity. Compactness gives a positive uniform distance from the endpoints of these intervals. Smooth approximation relative to the smaller collars gives a smooth real function c such that

$$\lambda_0(t) < c(t) < \lambda_1(t), \quad c(t) = 0 \text{ on the smaller endpoint collars.}$$

Then $L_t - c(t)I$ is invertible. Its potential is $V_t^0 + W_t - c(t)I$, which is compatible and has the prescribed endpoint values. If $V_0 = V_1 = 0$, this potential vanishes on the smaller collars. $\square$

Lemma 11.5. Fix $g_* \in \mathcal{R}_{\mu}^{\operatorname{inv}}(M)$. For every $m \in \mathbb{Z}$ there is $g_m \in \mathcal{R}_{\mu}^{\operatorname{inv}}(M)$ such that $\operatorname{sf}_{\mathbb{F}}(g_*, g_m) = m$.

Proof. Take $g_0 = g_*$. For $m \neq 0$, choose a bounded spectral interval $\mathcal{I}$ whose boundary is disjoint from $\operatorname{Spec} D_{g_*}$ and which contains at least $|m| + 1$ eigenvalues of each sign, counted over $\mathbb{F}$. Such a window exists by the two-sided unboundedness proved in Section 8.1. Remark 11.3 gives a compatible potential Q for which these eigenvalues of $D_{g_*} + Q$ are simple. Choose Q so that

$$\|Q\|_{\infty} < \frac{1}{2} \min \{ \operatorname{dist}(0, \operatorname{Spec} D_{g_*}), \operatorname{dist}(\partial \mathcal{I}, \operatorname{Spec} D_{g_*}) \}.$$

Along $D_{g_*} + sQ$, $0 \leq s \leq 1$, no eigenvalue crosses zero or $\partial \mathcal{I}$. Hence this path is invertible and the numbers of eigenvalues of each sign in $\mathcal{I}$ are unchanged.

If $m > 0$, choose $c > 0$ strictly between the m th and $(m + 1)$ st positive numbers $-\lambda$, where λ ranges over the negative eigenvalues of $D_{g_*} + Q$, in increasing order of $-\lambda$. The path $D_{g_*} + Q + scI$ has exactly m positive crossings. If $m < 0$, choose $c < 0$ so that $-c$ lies strictly between the $|m|$ th and $(|m| + 1)$ st positive eigenvalues; it has exactly $|m|$ negative crossings. After stationary reparametrization, concatenate the two paths. In both cases the endpoint $q_m = (g_*, Q + cI)$ is invertible and

$$\operatorname{sf}_{\mathbb{F}}((g_*, 0), q_m) = m.$$

By Proposition 8.2, with $U = M$, a one-point parameter space and empty relative subset, q_m is joined through invertible pairs to $(g_m, 0)$ for some $g_m \in \mathcal{R}_{\mu}^{\operatorname{inv}}(M)$. Additivity and path independence of spectral flow give $\operatorname{sf}_{\mathbb{F}}(g_*, g_m) = m$. $\square$

Theorem 11.6. If $n \equiv 3, 7 \pmod{8}$ and $g_* \in \mathcal{R}_{\mu}^{\operatorname{inv}}(M)$, then

$$\pi_0 \mathcal{P}_{\mu}^{\operatorname{inv}} \longrightarrow \mathbb{Z}, \quad [q] \longmapsto \operatorname{sf}_{\mathbb{F}}((g_*, 0), q),$$

and

$$\pi_0 \mathcal{R}_{\mu}^{\operatorname{inv}}(M) \longrightarrow \mathbb{Z}, \quad [g] \longmapsto \operatorname{sf}_{\mathbb{F}}(g_*, g),$$

are bijections of pointed sets. With the ordinary real index normalization, both maps are δ_{g_*} .

Proof. Spectral flow vanishes on an invertible path. By additivity, two pairs have equal invariants precisely when their relative spectral flow is zero. Lemma 11.4 then gives an invertible path between them. The pair map is injective. The pair construction in the proof of Lemma 11.5 gives its surjectivity. Proposition 8.2 gives the metric bijection. The normalization in Section 8.1, (8.1) and (8.3), gives

$$\delta_{g_*} = \text{sf}_F, \quad \text{sf}_C = \begin{cases} 2 \text{sf}_H, & n \equiv 3 \pmod{8}, \\ \text{sf}_R, & n \equiv 7 \pmod{8}. \end{cases}$$

□

Remark 11.7. The scalar correction in Lemma 11.4 is spatially constant. Theorem 11.6 concerns the full density slice $\mathcal{R}_\mu(M)$.

11.2. The two complex chiral cases. Suppose $n \equiv 2, 6 \pmod{8}$. Charge conjugation exchanges the chiral bundles. In charge coordinates,

$$J(x, y) = (\varepsilon \bar{y}, \bar{x}), \quad D(Z) = \begin{pmatrix} 0 & Z^* \\ Z & 0 \end{pmatrix}, \quad \varepsilon = \begin{cases} -1, & n \equiv 2 \pmod{8}, \\ +1, & n \equiv 6 \pmod{8}. \end{cases}$$

Compatibility is equivalent to $Z^T = \varepsilon Z$. These independent chiral blocks form a complex vector bundle: they are alternating in residue two and symmetric in residue six. The opposite block is always the adjoint.

Lemma 11.8. *In residues two and six, every nonempty fibre of $\mathcal{P}_\mu^{\text{inv}} \rightarrow \mathcal{R}_\mu(M)$ is smoothly path connected. The paths may be stationary near their endpoints. If both endpoint potentials are supported in a compact set K , the connecting path may be supported in K .*

Proof. Fix g and invertible endpoint potentials with independent blocks Z_0, Z_1 . Put

$$A(z) = D_g^+ + Z_0 + z(Z_1 - Z_0) : H^1(\Sigma^+) \longrightarrow L^2(\Sigma^-).$$

The inverse $A(0)^{-1} : L^2(\Sigma^-) \rightarrow H^1(\Sigma^+)$ is bounded. Therefore

$$A(z)A(0)^{-1} = I + zT, \quad T = (Z_1 - Z_0)A(0)^{-1},$$

is an entire Fredholm family on $L^2(\Sigma^-)$, with T compact by the embedding $H^1 \hookrightarrow L^2$. It is invertible at $z = 0$. The analytic Fredholm theorem [79, Theorem 1.1, p. 2], in one complex variable, implies that its noninvertibility set is discrete. It contains neither zero nor one. Choose a smooth curve from zero to one in its complement, stationary near its endpoints.

Along this curve the operator

$$\begin{pmatrix} 0 & A(z)^* \\ A(z) & 0 \end{pmatrix}$$

is self-adjoint and invertible. It is compatible because the transpose condition on the independent potential block is complex linear. The potential blocks are linear combinations of Z_0, Z_1 and their adjoints, so they remain supported in K . □

Proposition 11.9. *Suppose $n \equiv 2, 6 \pmod{8}$ and $\alpha(M) = 0$. Then $\mathcal{P}_\mu^{\text{inv}}$ and $\mathcal{R}_\mu^{\text{inv}}(M)$ are nonempty and smoothly path connected.*

Proof. By Proposition 9.1, the invertible potential fibre over every metric is nonempty. Let $(g_0, V_0), (g_1, V_1)$ be invertible pairs. Join their metrics by a smooth path g_s in the density slice, stationary on endpoint collars, and identify the potential bundles by positive spinor comparison.

For each s , choose $V^{(s)}$ such that $D_{g_s} + V^{(s)}$ is invertible. Openness of invertibility in $\mathcal{L}(H^1, L^2)$ [62, Chapter IV, Section 1.4, Theorem 1.16] implies that the corresponding operator remains invertible on an interval about s . Near the endpoints take the prescribed V_0, V_1 . A finite subcover and a Lebesgue subdivision $0 = s_0 < \dots < s_k = 1$ give one such potential on each closed interval $[s_{i-1}, s_i]$, with V_0 and

V_1 on the first and last intervals. At each subdivision point, connect the two adjacent potentials in the same invertible fibre by Lemma 11.8.

Concatenate these fibre paths with the metric segments. Reparametrize each of the finitely many pieces to be constant on endpoint collars; the resulting pair path is smooth and has the prescribed endpoints. For pure metric endpoints its potential vanishes on these collars. Apply Proposition 8.2 with $U = M$: first to a point, then to this interval with $B = [0, 1]$ and $B_0 = \{0, 1\}$. It gives nonemptiness and smooth path connectedness of the metric space. $\square$

11.3. The two quaternionic incidence cases. In residue four, charge conjugation preserves chirality, and the independent chiral operator is quaternionic linear. Its potential block is an arbitrary quaternionic homomorphism between the chiral bundles. In residue five, charge conjugation anticommutes with D_g , and

$$K = iD_g$$

is quaternionic linear and skew-adjoint. The correspondence $V \mapsto iV$ identifies compatible potentials with quaternionic skew-adjoint multiplication endomorphisms. Here i is the complex scalar on the spin representation.

If $n = 4$ and $\alpha(M) = 0$, the space $\mathcal{R}_\mu^{\text{inv}}(M)$ is connected by [2, Theorem 1.1] and Lemma 8.1.

Proposition 11.10. *Suppose $n \equiv 4, 5 \pmod{8}$ and $\alpha(M) = 0$. Then $\mathcal{P}_\mu^{\text{inv}}$ and $\mathcal{R}_\mu^{\text{inv}}(M)$ are nonempty and smoothly path connected.*

Proof. Apply Lemma 10.4 to a one-point parameter space. The incidence dimension is -4 in residue four, where the quaternionic chiral index is zero, and -3 in residue five. Thus the incidence is empty. In residue four the chiral block is injective of index zero and hence bijective; in residue five the full kernel is zero. This gives an invertible compatible potential over every metric.

Join two invertible pairs by a smooth path in the contractible space $\mathcal{P}_\mu$, stationary on endpoint collars. Apply Lemma 10.4 with $d = 1$, relative to those collars. The incidence dimension is -3 in residue four and -2 in residue five, so it is empty. The resulting path is invertible and has the prescribed endpoints. For pure metric endpoints its potential vanishes on the collars. Proposition 8.2, with $U = M$, applied first to a point and then to this interval with $B = [0, 1]$ and $B_0 = \{0, 1\}$, gives the metric conclusions. $\square$

Theorem 11.11. *Suppose $n \equiv 2, 4, 5, 6 \pmod{8}$ and $\alpha(M) = 0$. Then $\mathcal{P}_\mu^{\text{inv}}$ and $\mathcal{R}_\mu^{\text{inv}}(M)$ each have one path component. Their pointed component sets are classified by ordinary index difference, with values in $KO_{n+1} = 0$.*

Proof. Apply Propositions 11.9 and 11.10 and the coefficient table (8.3). $\square$

12. REAL CHIRAL OPERATORS

Let M be a closed connected spin manifold of dimension $n \equiv 0 \pmod{8}$, with $n \geq 8$ and $\alpha(M) = 0$, and fix a smooth positive density μ on M . In this section all spinors, bundles, homomorphisms, and dimensions are real. The real half-spin bundles $\Sigma_g^\pm$ have rank

$$r = 2^{n/2-1}.$$

A compatible potential has, in the chiral decomposition, the form

$$V_B = \begin{pmatrix} 0 & B^* \\ B & 0 \end{pmatrix}, \quad B \in C^\infty(M; \text{Hom}_\mathbb{R}(\Sigma_g^+, \Sigma_g^-)).$$

Put $A_{g,B} = D_g^+ + B$. Multiplication by B is compact from H^s to H^{s-1} . Hence the real Fredholm index of $A_{g,B}$ is the real chiral spin index $\alpha(M) = 0$. The full operator $D_g + V_B$ is invertible if and only if $A_{g,B}$ is an isomorphism. For a fixed metric g write

$$\mathcal{P}_g = C^\infty(M; \text{Hom}_\mathbb{R}(\Sigma_g^+, \Sigma_g^-)), \quad \mathcal{P}_g^{\text{inv}} = \{B \in \mathcal{P}_g : A_{g,B} \text{ is invertible}\}.$$

We identify the total space of these chiral potentials, over metrics with density μ , with the compatible pair space $\mathcal{P}_\mu$. Positive spin comparison from Section 2 preserves the real structures, the grading, and the L^2 inner products defined by μ .

All coefficient spaces have the C^∞ topology. Potentials may vary globally on M ; relative conditions below refer to parameter collars.

Theorem 12.1. *For every metric g with density μ , the space $\mathcal{P}_g^{\text{inv}}$ has exactly two path components. The space $\mathcal{P}_\mu^{\text{inv}}$ also has exactly two path components. After the choice of an invertible basepoint, the component invariant is determinant parity. It agrees with the ordinary real index difference in*

$$KO_{n+1} \cong KO_1 \cong \mathbb{Z}/2.$$

Thus the index difference induces a bijection of pointed sets from either component set to $\mathbb{Z}/2$.

12.1. Incidence and smooth Banach parameters. Fix an integer $s > n/2 + 1$. After spin comparison, a smooth compact family of metrics and potentials gives a smooth family of real Fredholm operators

$$A_t : H^s(\Sigma^+) \longrightarrow H^{s-1}(\Sigma^-) \quad (12.1)$$

of index zero between fixed Sobolev spaces. Kernels and adjoint kernels consist of smooth sections. We identify the cokernel with the L^2 adjoint kernel; this identification also represents every linear functional on the finite-dimensional cokernel.

The weak unique continuation statement in Section 9, applied to the full spinors $(\psi, 0)$ and $(0, \eta)$ for $D_g + V_B$, gives weak unique continuation for $A = D_g^+ + B$ and A^* , respectively. Thus the nonzero locus of every nonzero solution is open and dense. For every integer $j \geq 0$, the elliptic estimate is

$$\|u\|_{H^{j+1}} \leq C_j (\|A_t u\|_{H^j} + \|u\|_{L^2}), \quad (12.2)$$

with constants uniform over a fixed smooth compact family; see [40, Theorem 15.7 and Remark 15.2] and the compact-family argument following (10.7).

Lemma 12.2. *Suppose $A\psi = 0$ and $\|\psi\|_{L^2} = 1$. The differential of the equation $(D_g^+ + B)\psi = 0$, with the metric fixed and the spinor constrained to the L^2 unit sphere in H^s , is surjective when arbitrary smooth potential variations are allowed. Its restriction to the spinor variable is Fredholm of index -1 .*

Proof. The differential is

$$(Q, u) \longmapsto Au + Q\psi, \quad \langle u, \psi \rangle_{L^2} = 0. \quad (12.3)$$

Removing the kernel vector ψ from the domain reduces the index by one and leaves the range unchanged. Suppose that $\eta \in \ker A^*$ annihilates every $Q\psi$. If $\eta \neq 0$, unique continuation gives a ball U on which both ψ and η are nonzero. For a nonnegative nonzero smooth function χ compactly supported in U , set

$$Q = \chi \eta \otimes \frac{\psi^*}{|\psi|^2}$$

on U and extend it by zero. Then

$$\langle \eta, Q\psi \rangle_{L^2} = \int_M \chi |\eta|^2 \mu > 0,$$

a contradiction. Thus the potential directions map onto the finite-dimensional cokernel, and finitely many such directions already suffice. The derivative has a bounded right inverse, obtained from these finite-dimensional directions on the cokernel and the inverse of A from a complement of its kernel onto its range. $\square$

For parameter intervals, Lemma 10.1 gives smooth potential directions in a separable Banach parameter space.

12.2. Evaluation at a spinor zero.

Lemma 12.3. *Let $A\psi = 0$, $\|\psi\|_{L^2} = 1$, and $\psi(x) = 0$. The map*

$$(Q, u) \mapsto u(x)$$

from the smooth solutions of

$$Au + Q\psi = 0, \quad \langle u, \psi \rangle_{L^2} = 0$$

onto Σ_x^+ is surjective.

Proof. Let $K = \ker A$ and $C = \ker A^*$, and let G be the generalized inverse with

$$AG = 1 - P_C, \quad GA = 1 - P_K.$$

Elliptic regularity makes G and G^* continuous on smooth sections [40, Theorems 15.1, 15.7, and 15.13; Remark 15.2]; by duality they extend to distributions. Choose an open ball U with closure disjoint from x on which ψ is nonzero. For any $q \in C_c^\infty(U; \Sigma^-)$ orthogonal to C , the choices

$$u = Gq, \quad Q = -q \otimes \frac{\psi^*}{|\psi|^2}$$

on the support of q , extended smoothly by zero, are incidence tangent directions. Indeed, $Au = q$, $Q\psi = -q$, and u is orthogonal to K . Every $u \in K \cap \psi^\perp$ is another incidence tangent direction, with $Q = 0$.

Suppose a covector $\zeta \in (\Sigma_x^+)^*$ annihilates all these evaluation directions. The kernel variations, together with $\psi(x) = 0$, imply

$$P_K(\delta_x \zeta) = 0.$$

Here $\delta_x \zeta$ is the point distribution acting on a spinor u by $\zeta(u(x))$; the fibre metric identifies dual bundles when needed. Define the distribution $\eta = G^*(\delta_x \zeta)$. Then

$$A^* \eta = \delta_x \zeta. \tag{12.4}$$

For every test section q supported in U and orthogonal to C , we have $\langle \eta, q \rangle = \zeta((Gq)(x)) = 0$. The annihilator of this test-section subspace is the span of the restrictions to U of the finitely many elements of C . Explicitly, choose a basis $\phi_1, \dots, \phi_m$ of C and factor the functional $q \mapsto \langle \eta, q \rangle$ through $q \mapsto (\langle \phi_j, q \rangle)_{j=1}^m$. It follows that $\eta|_U = \phi|_U$ for some $\phi \in C$.

Elliptic regularity makes η smooth on $M \setminus \{x\}$. This punctured manifold is connected because $n \geq 2$. The section $\eta - \phi$ satisfies the homogeneous adjoint equation there and vanishes on U . Unique continuation gives $\eta = \phi$ on $M \setminus \{x\}$. Thus $T = \eta - \phi$ is supported at x and

$$A^* T = \delta_x \zeta.$$

The principal symbol of A^* is invertible at nonzero real covectors. Lemma 2.3 gives $T = 0$ and $\zeta = 0$. The annihilator of the evaluation image is therefore zero, and evaluation is surjective. $\square$

Lemma 12.4. *A normalized harmonic spinor with a one-dimensional chiral kernel can be joined, by an arbitrarily short smooth path in the one-dimensional-kernel locus with the metric fixed, to a normalized harmonic spinor transverse to zero. The path can be stationary on its parameter collars. For $n \geq 16$ the final spinor is nowhere zero; for $n = 8$ its zeros are finite.*

Proof. At each zero of the original spinor choose finitely many smooth incidence tangent directions giving the surjectivity of Lemma 12.3. Their surjectivity persists for nearby evaluation points. Compactness of the zero set reduces this to finitely many choices. Add finitely many potential directions which fill the incidence cokernel in Lemma 12.2. In this finite-dimensional potential family, the normalized incidence is, near the original datum, a finite-dimensional smooth manifold: the spinor derivative is Fredholm of index -1 , and the full derivative is split surjective. Its evaluation map, crossed with M , is transverse to zero near the original zero set. After shrinking the incidence

chart, all zeros remain in this neighborhood, because evaluation is continuous in H^s and the original spinor is bounded away from zero on the complementary compact set.

Write a for the coordinates in this incidence chart. For every nonzero multi-index β , differentiation of $A_a \psi_a = 0$ gives

$$A_a \partial_a^\beta \psi_a = - \sum_{0 < \gamma \leq \beta} \binom{\beta}{\gamma} (\partial_a^\gamma A_a) \partial_a^{\beta-\gamma} \psi_a.$$

The coefficients and all their parameter derivatives are smooth. Applying the uniform elliptic estimates to these equations and their difference quotients gives, by induction, smooth dependence on a with values in every Sobolev space. Sobolev embedding then gives joint smoothness in parameters and space. Parametric transversality [59, Chapter 3, §2, Theorem 2.7], applied to this evaluation map on the incidence chart crossed with the closed manifold M , gives arbitrarily near incidence points whose spinors are transverse to zero. A small connected coordinate chart gives a smooth incidence path to one of them.

The kernel dimension is at most one in a sufficiently small coefficient neighborhood. For example, restrict the original operator to the complement of its kernel and cokernel and retain the resulting invertible block under perturbation; the remaining block has a one-dimensional domain. The selected normalized spinor keeps the kernel dimension at least one. The whole small path therefore has kernel dimension exactly one. Reparametrization makes it stationary near its ends. Finally, $r > n$ for $n \geq 16$, whereas $r = n = 8$ in dimension eight. Transversality gives the asserted zero sets. $\square$

12.3. Compatible first derivatives in dimension eight. Let V be an oriented Euclidean space of dimension eight, and let $E^\pm$ be its real half-spin modules. Put

$$c(v)c(w) + c(w)c(v) = -2\langle v, w \rangle, \quad c(v)^* = -c(v).$$

For $T \in \text{Hom}_{\mathbb{R}}(V, E^+)$ define Clifford contraction by

$$C(T) = \sum_{i=1}^8 c(e_i)T(e_i), \quad (12.5)$$

where (e_i) is an orthonormal basis. At a zero of a spinor, the connection term and every multiplication potential vanish. Consequently $D^+ \psi = 0$ at that point is precisely $C(d\psi) = 0$.

Choose a unit $\phi \in E^-$. The map

$$f : V \longrightarrow E^+, \quad f(v) = c(v)\phi$$

is an isometry. Indeed, the symmetric part of the Clifford relation gives $\langle f(v), f(w) \rangle = \langle v, w \rangle$; the remaining bivector is skew-adjoint on a real inner product space. Since the ranks agree, f is onto. Write $T = fB$, with $B = S + K$ its symmetric and skew decomposition. A direct computation gives

$$\begin{aligned} C(fB) &= -(\text{tr } B)\phi + \sum_{i < j} (B_{ji} - B_{ij})c(e_i)c(e_j)\phi \\ &= -(\text{tr } B)\phi + 4\rho(K)\phi, \end{aligned} \quad (12.6)$$

where

$$\rho(K) = \frac{1}{2} \sum_{i < j} K_{ji}c(e_i)c(e_j).$$

The last two terms in (12.6) are orthogonal. Thus the compatible matrix space is

$$L = f^{-1}(\ker C) = \text{Sym}_0(V) \oplus \mathfrak{s}_\phi, \quad \mathfrak{s}_\phi = \{K \in \mathfrak{so}(V) : \rho(K)\phi = 0\}. \quad (12.7)$$

Here $\mathfrak{s}_\phi$ is the infinitesimal stabilizer of ϕ .

Lemma 12.5. *Let $m \geq 3$, and let $E \subset \mathfrak{sl}_m(\mathbb{R})$ be a real linear subspace containing $\text{Sym}_0(\mathbb{R}^m)$. Every invertible $A \in E$ can be joined by a straight segment in $E \cap \text{GL}_m(\mathbb{R})$ to an invertible symmetric trace-free matrix.*

Proof. By the real Schur theorem [50, Theorem 7.4.1], choose orthogonal coordinates putting A in real Schur form. Its diagonal blocks consist of nonzero real eigenvalues λ_j and two-by-two real blocks with nonreal eigenvalues $a_\ell \pm i\beta_\ell$, where $\beta_\ell \neq 0$. In these coordinates choose a symmetric block diagonal matrix B which is b_j on each real block and $r_\ell I_2$ on each nonreal block, subject to

$$\text{sign } b_j = \text{sign } \lambda_j, \quad r_\ell \neq 0, \quad \sum_j b_j + 2 \sum_\ell r_\ell = 0. \quad (12.8)$$

If there are no nonreal blocks, take $b_j = \lambda_j$; tracelessness forces both signs. If there is exactly one nonreal block, $m \geq 3$ ensures a real block is present. Choose the b_j with the indicated signs and with nonzero sum, and set $r_1 = -\frac{1}{2} \sum_j b_j$. If there are at least two nonreal blocks, fix all but two of the r_ℓ , and choose the remaining two to satisfy the trace equation, avoiding their finitely many forbidden zero values. This proves (12.8).

For $0 \leq t \leq 1$, the matrix $(1-t)A + tB$ is still block upper triangular. Its real diagonal blocks cannot vanish, since their endpoint signs agree. The eigenvalues on a nonreal diagonal block are

$$(1-t)(a_\ell \pm i\beta_\ell) + tr_\ell.$$

They are nonreal for $t < 1$ and nonzero at $t = 1$. The segment is therefore invertible throughout. In the original coordinates B is symmetric and trace-free, hence belongs to E ; linearity puts the segment in E . $\square$

Proposition 12.6. *Both sets*

$$L^+ = \{B \in L : \det B > 0\}, \quad L^- = \{B \in L : \det B < 0\}$$

are nonempty and path connected. For either sign and every basepoint in that set, inclusion induces a surjection

$$\pi_1(L^\pm) \longrightarrow \pi_1(GL_8^\pm(\mathbb{R})) \cong \mathbb{Z}/2.$$

The paths and loops can be chosen smooth and stationary at any prescribed concatenation points.

Proof. Choose orthonormal e_1, e_2, e_3 , and define e_4 by

$$c(e_4)\phi = c(e_3)c(e_1)c(e_2)\phi.$$

The isometry f gives $|e_4| = 1$. Its pairing with each of $c(e_j)\phi$, $j = 1, 2, 3$, reduces, by the Clifford relation, to the expectation of a skew bivector against ϕ and is zero. Thus e_1, e_2, e_3, e_4 are orthonormal, and

$$c(e_3)c(e_4)\phi = -c(e_1)c(e_2)\phi.$$

Define K by

$$Ke_1 = e_2, \quad Ke_2 = -e_1, \quad Ke_3 = e_4, \quad Ke_4 = -e_3, \quad K|_{\langle e_1, e_2, e_3, e_4 \rangle^\perp} = 0.$$

Then $\rho(K)\phi = \frac{1}{2}(c(e_1)c(e_2) + c(e_3)c(e_4))\phi = 0$. In an orthonormal extension of this basis, $K = J \oplus J \oplus 0_4$, where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

For each $1 \leq k \leq 7$, the invertible symmetric trace-free matrices with k positive eigenvalues are path connected. Diagonalize by an element of $SO(8)$, changing the sign of one eigenvector if necessary to obtain determinant one, and join this diagonalizing matrix to the identity. The diagonal eigenvalues with prescribed strict signs and sum zero form a convex set. Conjugation followed by interpolation of these eigenvalues gives the asserted paths.

For $1 \leq k \leq 5$, choose $d_5, \dots, d_8 \in \{1, -1\}$ with $k-1$ positive entries and $5-k$ negative entries, and set

$$c_0 = \sum_{j=5}^8 d_j = 2k - 6, \quad a(t) = -\cos(\pi t), \quad b(t) = \sin(\pi t), \quad h(t) = -a(t) - c_0/2.$$

Define

$$B(t) = \text{diag}(a(t), a(t), 5 + h(t), -5 + h(t), d_5, d_6, d_7, d_8) + b(t)K. \quad (12.9)$$

Its trace is zero, so $B(t) \in L$. The determinants of the two nontrivial two-by-two blocks are respectively

$$a(t)^2 + b(t)^2 = 1, \quad h(t)^2 - 25 + b(t)^2 < 0,$$

since $|h(t)| \leq 3$ and $|b(t)| \leq 1$. The four other diagonal entries never vanish. Thus (12.9) is invertible. At both endpoints it is symmetric. At $t = 0$ its first pair is negative and its second pair has one sign of each kind, giving k positive eigenvalues; at $t = 1$ the first pair is positive, giving $k + 2$.

These paths join the signatures in the chains

$$1 \longleftrightarrow 3 \longleftrightarrow 5 \longleftrightarrow 7, \quad 2 \longleftrightarrow 4 \longleftrightarrow 6.$$

The determinant sign of an eight-dimensional symmetric matrix with k positive eigenvalues is $(-1)^k$. Lemma 12.5, applied to L , joins every invertible element of L to the symmetric locus. Hence L^+ and L^- are path connected. The diagonal matrices

$$S_+ = \text{diag}(1, 1, 1, 1, -1, -1, -1, -1), \quad S_- = \text{diag}(7, -1, -1, -1, -1, -1, -1, -1)$$

show that both sets are nonempty.

For the fundamental group, choose one positive and one negative coordinate of S_+ , and let $R(t)$ rotate their plane through angle πt . The path

$$S(t) = R(t)S_+R(t)^{-1}$$

is a loop in L^+ because $R(1)$ commutes with S_+ . It consists of symmetric trace-free orthogonal matrices. Right multiplication by S_+ identifies its target component with $GL_8^+(\mathbb{R})$ and takes it to a full plane rotation through $2\pi t$. This rotation generates $\pi_1 SO(8) \cong \mathbb{Z}/2$ [54, Proposition 13.10 and the following paragraph]. Polar deformation gives the same generator in $\pi_1 GL_8^+(\mathbb{R})$ [54, Theorem 2.17 and Proposition 2.19(1)].

Apply the same conjugation construction to S_- , rotating its positive coordinate and one negative coordinate. Polar retraction takes the resulting loop to $R(t)P_-R(t)^{-1}$, where $P_- = \text{diag}(1, -1, \dots, -1)$. Right multiplication by P_- again gives a full plane rotation. Thus the loop generates the fundamental group of the negative component as well. Paths in $L^\pm$ transport these loops to any chosen basepoint. Reparametrizing finitely many path pieces with constant collars makes all concatenations smooth. $\square$

12.4. Paths with a selected harmonic spinor.

Proposition 12.7. *Suppose (g_0, B_0) and (g_1, B_1) have one-dimensional chiral kernels, and choose normalized real spinors ψ_j spanning those kernels. There is a smooth path (g_t, B_t, ψ_t) , stationary on endpoint collars, with the prescribed endpoints, such that*

$$(D_{g_t}^+ + B_t)\psi_t = 0, \quad \|\psi_t\|_{L^2} = 1$$

for every t . The density is μ throughout. If the two metrics coincide, the metric may be held fixed along the path.

Proof. By Lemma 12.4, join the endpoint pairs by short paths in normalized incidence to pairs whose spinors are transverse to zero. These short paths have one-dimensional kernels and fix their metrics. We construct a path between the perturbed endpoints and then concatenate with these short paths and their reversals. In what follows the endpoints already have transverse zeros. Choose a smooth metric path with density μ , stationary on endpoint collars, and identify its spinor bundles by positive spin comparison. A path of metrics with this density exists in the coordinates of Section 8; take the constant path in the fixed-metric case.

A smooth section homotopy fixed on the endpoint collars is transverse there. Choose finitely many smooth sections supported off smaller collars which span the fibres over the complement of the original collars. The evaluation map of the resulting finite-dimensional family is transverse to

zero for small parameters. Applying [59, Chapter 3, §2, Theorem 2.7] on $M \times (0, 1)$ gives a transverse section homotopy relative to the smaller collars.

The case $n \geq 16$. Here $r > n + 1$. The endpoint spinors are nowhere zero. Choose such a transverse section homotopy on $M \times I$. Since its rank exceeds $\dim(M \times I) = n + 1$, it has no zeros. Denote this family by ψ_t . Define

$$B_t^{\text{can}} = -D_{g_t}^+ \psi_t \otimes \frac{\psi_t^*}{|\psi_t|^2}. \quad (12.10)$$

Then $(D_{g_t}^+ + B_t^{\text{can}})\psi_t = 0$. On an endpoint collar, the difference $B_j - B_t^{\text{can}}$ annihilates the fixed spinor ψ_j . Add these differences with smooth cutoffs supported in the respective collars and equal to one on smaller collars. The equation is preserved and the endpoint potentials become exactly B_j . Division by the smooth positive function $\|\psi_t\|_{L^2}$ normalizes the spinors and leaves the equation unchanged.

The case $n = 8$: the zero curve. The real bundle has rank eight. Choose a transverse section Ψ of the half-spin bundle over $W = M \times I$, agreeing with the endpoint spinors on stationary collars. Then

$$Z = \Psi^{-1}(0)$$

is a compact one-dimensional submanifold, with boundary on $M \times \partial I$. On the endpoint collars it is the product of the finite endpoint zero set with the collar. By the density of Morse functions [59, Chapter 6, §1, Theorem 1.2], choose a small smooth function h on Z , zero on the collars, for which $t|_Z + h$ is Morse. The relative condition follows by cutting off a sufficiently small perturbation in collars where $dt|_Z \neq 0$. For sufficiently small C^1 norm, the maps

$$(x(s), t(s)) \mapsto (x(s), t(s) + uh(s)), \quad 0 \leq u \leq 1,$$

are embeddings and form an isotopy, fixed on the collars. Double W and Z across their product collars, extending this isotopy by the identity on the second copy. The isotopy extension theorem [59, Chapter 8, §1, Theorem 1.3] applies to the doubled closed submanifold. Its vector field can be multiplied by a cutoff equal to one on the support of the prescribed velocity and supported in the interior of the first copy. The resulting extension restricts to an isotopy of W fixed near ∂W . Lift it to the vector bundle by parallel transport. Transporting Ψ preserves transversality and produces the stated zero curve. We retain the notation Ψ, Z for the result.

The total derivative of Ψ induces a normal isomorphism

$$F_0 : NZ \longrightarrow \Sigma^+|_Z. \quad (12.11)$$

At a point $z(s) = (x(s), t(s))$ of Z , write

$$A(s) = d_x \Psi : T_{x(s)} M \longrightarrow \Sigma_{z(s)}^+, \quad b(s) = \partial_t \Psi \in \Sigma_{z(s)}^+.$$

Derivatives of a section at its zeros are independent of the chosen connection. Since Ψ vanishes along Z ,

$$A(s)x'(s) + b(s)t'(s) = 0. \quad (12.12)$$

The required normal isomorphism F_1 must be homotopic to F_0 relative to the collars and must satisfy

$$C(A(s)) = 0 \quad (12.13)$$

along Z . Here Clifford contraction is defined by the actual metric $g_{t(s)}$ at $x(s)$.

Orient NZ by F_0 and the orientation of the real half-spin bundle. The induced orientation on Z is consistent with the orientation of W . All oriented bundles under consideration are trivial along each interval or circle component of Z . Moreover, a spin frame along such a component exists: a principal $\text{Spin}(8)$ bundle over a one-manifold is trivial because the structure group is connected. On intervals the frame can agree with prescribed frames on the endpoint collars. Consequently, along an arc, smooth spin frames identify the compatible vertical jet spaces with the single matrix space L of (12.7).

Regular arcs and folds. Where $t'(s) \neq 0$, the map $T_{x(s)}M \rightarrow N_{z(s)}Z$ induced by inclusion in $T_{z(s)}W$ is an isomorphism. A normal frame is then specified by an invertible vertical derivative $A(s)$, with

$$b(s) = -A(s)x'(s)/t'(s).$$

The compatible choices are exactly $L \cap GL_8(\mathbb{R})$ in the chosen spin frames. Up to a fixed orientation convention, the vertical determinant sign is the normal orientation sign times sign $t'(s)$.

At a critical point of $t|_Z$, the Morse lemma [59, Chapter 6, §1, Theorem 1.1] gives a local parameter with $t(s) = t_0 + s^2$ or $t(s) = t_0 - s^2$. The vector $x'(s)$ is nonzero on a small interval. Set

$$e_1(s) = \frac{x'(s)}{|x'(s)|_{g_{t(s)}}}$$

and extend to a smooth orthonormal vertical frame. Choose a smooth unit spinor $\phi^-(s)$ and set $f_s(v) = c_{g_{t(s)}}(v)\phi^-(s)$. Take seven nonzero real constants $\lambda_2, \dots, \lambda_8$ whose sum is zero, and put

$$\begin{aligned} a(s) &= \frac{t'(s)}{|x'(s)|_{g_{t(s)}}}, & b(s) &= -f_s(e_1(s)), \\ A(s)e_1(s) &= a(s)f_s(e_1(s)), & A(s)e_2(s) &= (\lambda_2 - a(s))f_s(e_2(s)), \\ A(s)e_j(s) &= \lambda_j f_s(e_j(s)) & (3 \leq j \leq 8). \end{aligned} \tag{12.14}$$

The matrix $f_s^{-1}A(s)$ is symmetric and trace-free, so (12.13) holds. Formula (12.12) holds as well. After shrinking the interval, the last seven coefficients of $A(s)$ are nonzero. At the critical point $A(s)$ has rank seven, and $b(s)$ spans the missing image direction. At a regular point $A(s)$ is invertible. Thus $(A(s), b(s))$ is surjective throughout the interval and its kernel is exactly $T_{z(s)}Z$. It therefore defines a normal isomorphism across the fold.

This isomorphism can have either normal orientation. At the critical point its orientation sign is a fixed sign times $\prod_{j=2}^8 \lambda_j$. The choices

$$(6, -1, -1, -1, -1, -1, -1), \quad (5/2, 5/2, -1, -1, -1, -1, -1)$$

have zero sum and opposite product signs. Choose the orientation which agrees with F_0 . The vertical determinant then has the required opposite signs on the two regular branches, because $a(s)$ changes sign. The formula contains the actual $t'(s)$ and therefore applies to both maxima and minima. It prescribes the jets along the curved zero set in orthonormal metric frames.

The relative framing class. Keep F_0 on the endpoint collars; it is already compatible there because the endpoint spinors are harmonic. Choose the fold frames above with its orientation on disjoint small neighborhoods of the finitely many critical points. The remaining pieces of Z are regular arcs. On each, the two prescribed endpoint jets have the same vertical determinant sign: their normal orientation agrees, and t' has constant sign on that arc. Proposition 12.6 joins them through compatible invertible jets. Smooth reparametrizations preserve the prescribed jets on smaller endpoint neighborhoods. This yields a compatible normal isomorphism F_1 of the same orientation as F_0 everywhere.

In trivializations, $F_0^{-1}F_1$ takes values in $GL_8^+(\mathbb{R})$ and equals the identity on the endpoint collars. On an interval, its relative homotopy class is an element of $\pi_1 GL_8^+(\mathbb{R}) = \mathbb{Z}/2$. On a circle, its free homotopy class is determined by the same group, since this fundamental group is abelian and $GL_8^+(\mathbb{R})$ is connected. Every component of Z contains a regular arc, since $t|_Z$ is Morse and is a submersion on the boundary collars. On one such arc insert a based loop of compatible invertible vertical jets. The surjectivity assertion in Proposition 12.6, for the determinant sign on that arc, allows this loop to change the relative normal-frame class by either element of $\mathbb{Z}/2$. The common changes of frame on the arc induce translations and conjugation on the resulting loop in $GL_8(\mathbb{R})$; these induce the identity on its fundamental group $\mathbb{Z}/2$. Choose the loop which cancels the class of $F_0^{-1}F_1$. Do this separately on each component of Z .

It follows that F_0 and the resulting F_1 are homotopic through normal isomorphisms relative to the collars. There are no higher obstructions because Z is one-dimensional. By smoothing in the open bundle of isomorphisms, choose a smooth homotopy F_u , $0 \leq u \leq 1$, fixed on those collars.

Replacement without additional zeros. Take a tubular neighborhood of Z that is a product tube on the endpoint collars, and identify the target bundle with its pullback from Z . In normal coordinates write

$$\Psi(z, y) = F_0(z)y + R(z, y), \quad |R(z, y)| \leq C|y|^2,$$

with C uniform along the compact curve. The family $F_u(z)$ has a positive minimum singular value δ on the compact set $Z \times [0, 1]$. Shrink the tube so that $|R(z, y)| \leq \delta|y|/2$. Choose a smooth function $\chi : [0, \infty) \rightarrow [0, 1]$ such that $\chi(|y|^2)$ is one near the zero section and zero near the tube boundary. Replace Ψ in this tube by

$$\Psi'(z, y) = F_{\chi(|y|^2)}(z)y + R(z, y). \quad (12.15)$$

This is the original section near the tube boundary and on the endpoint collars. For $y \neq 0$ it satisfies

$$|\Psi'(z, y)| \geq \delta|y| - |R(z, y)| \geq \frac{1}{2}\delta|y| > 0. \quad (12.16)$$

Consequently its zero set is exactly Z , its normal derivative is F_1 , and it is transverse to zero. Compatibility gives

$$D_{g_t}^+ \Psi'_t = 0 \quad \text{on } Z. \quad (12.17)$$

Joint division and endpoint values. Put $G = -D_{g_t}^+ \Psi'_t$, a smooth section of Σ^- over W . It vanishes on Z by (12.17). In bundle trivializations near a point of Z , total transversality makes the eight components $z_1, \dots, z_8$ of Ψ' , together with one coordinate σ , a coordinate system on W . On a smaller coordinate neighborhood, Hadamard's formula reads

$$G_i(\sigma, z) = \sum_{j=1}^8 z_j \int_0^1 \frac{\partial G_i}{\partial z_j}(\sigma, \tau z) d\tau. \quad (12.18)$$

The smooth coefficients in this formula define a homomorphism B with $B\Psi' = G$. Away from Z , take $G \otimes \Psi'^*/|\Psi'|^2$. On the endpoint collars retain the prescribed B_0, B_1 , which solve the same equation there. A smooth partition of unity patches these local solutions, since the equation is linear in B . Choose the partition so that only the prescribed endpoint solution occurs on each smaller endpoint collar. We obtain a jointly smooth potential B_t with the exact endpoint values and

$$(D_{g_t}^+ + B_t)\Psi'_t = 0.$$

The division in (12.18) is on $M \times I$. It remains valid at the folds, where the spatial derivative alone has rank seven.

No slice Ψ'_t is the zero section. If a slice were identically zero, its spatial derivative would vanish everywhere, so its total derivative would have rank at most one along that slice, contrary to transversality to a bundle of rank eight. Thus $N(t) = \|\Psi'_t\|_{L^2}$ is a smooth positive function. Replacing Ψ'_t by $N(t)^{-1}\Psi'_t$ preserves the equation, because N depends only on t , and preserves the normalized endpoint spinors. The short initial and final incidence paths now restore the original endpoints. All concatenations can be made smooth and stationary on their collars. $\square$

12.5. Removing the additional kernel.

Lemma 12.8. *Let ψ and ξ be real-linearly independent solutions of $A\xi = 0$ on M . Their values are linearly independent on an open dense subset of M .*

Proof. Pointwise independence is open. Suppose its complement contains a nonempty open set. Unique continuation for ψ gives a ball U in that set on which $\psi \neq 0$. There is a smooth real function

$$f = \frac{\langle \xi, \psi \rangle}{|\psi|^2}$$

on U with $\xi = f\psi$. The Leibniz rule gives

$$0 = A(f\psi) - fA\psi = c(df)\psi.$$

Clifford multiplication by a nonzero real covector is invertible, so $df = 0$ on U . On a smaller connected ball f is constant. Unique continuation for $\xi - f\psi$ then makes the two solutions globally dependent, a contradiction. $\square$

Proposition 12.9. *Let (g_t, B_t, ψ_t) be a smooth family, stationary on endpoint collars, with*

$$A_t \psi_t = 0, \quad \|\psi_t\|_{L^2} = 1, \quad A_t = D_{g_t}^+ + B_t,$$

and with one-dimensional kernels on those collars. There are arbitrarily small jointly smooth potential perturbations Q_t , zero on the collars and satisfying $Q_t \psi_t = 0$ pointwise, for which

$$\ker(A_t + Q_t) = \mathbb{R}\psi_t \quad (0 \leq t \leq 1).$$

For one fixed operator with a selected nonzero harmonic spinor, the same conclusion holds without a time variable or endpoint conditions.

Proof. Apply Lemma 10.1 to the closed space of smooth potential directions with $Q_t \psi_t = 0$ and the prescribed vanishing on collars. Write $Q(p) = q(p)$ for its ℓ^2 parameterization. The hyperplanes $\psi_t^\perp \subset H^s(\Sigma^+)$ form a smooth Hilbert bundle over I : their defining L^2 functionals depend smoothly on t and have smooth normalized representatives. The bundle can be trivialized over the interval. On its sphere bundle

$$S = \{(t, \xi) : \langle \xi, \psi_t \rangle_{L^2} = 0, \|\xi\|_{L^2} = 1\}$$

consider

$$F(p, t, \xi) = (A_t + Q_t(p))\xi. \quad (12.19)$$

At a zero, ψ_t and ξ are two independent kernel vectors. The derivative in the fibre variable is $A_t + Q_t(p)$ on the codimension-two subspace

$$\{u \in H^s : \langle u, \psi_t \rangle_{L^2} = \langle u, \xi \rangle_{L^2} = 0\}.$$

Both removed directions lie in its kernel. The restricted operator therefore has the same range and has Fredholm index -2 .

The derivative in the potential parameters maps onto its cokernel at every zero. Let $\eta \neq 0$ be a smooth adjoint-kernel spinor of $A_t + Q_t(p)$. Lemma 12.8 and unique continuation give a ball U on which ψ_t, ξ are pointwise independent and η is nonzero. On this ball put

$$v = \xi - \frac{\langle \xi, \psi_t \rangle}{|\psi_t|^2} \psi_t.$$

Then $v \neq 0$ and $v \perp \psi_t$ pointwise. Choose a nonnegative nonzero $\chi \in C_c^\infty(U)$ and define at this time

$$Q^0 = \chi \eta \otimes \frac{v^*}{|v|^2}.$$

It satisfies $Q^0 \psi_t = 0$ and $Q^0 \xi = \chi \eta$. For nearby times, $\psi_{t'}$ remains nonzero on the compact support of χ . Extend Q^0 smoothly in the fixed spin comparison, compose it on the right with

$$P_{t'}^\perp = 1 - \frac{\psi_{t'} \otimes \psi_{t'}^*}{|\psi_{t'}|^2},$$

and multiply by a time cutoff equal to one at t . The resulting smooth field is an admissible potential variation, supported in the chosen spatial ball and time interval, and

$$\langle \eta, Q^0 \xi \rangle_{L^2} = \int_M \chi |\eta|^2 \mu > 0.$$

Thus no nonzero cokernel functional annihilates the image of the potential derivative. The dense linear image of the weighted potential directions has the same, surjective image in this finite-dimensional cokernel.

Equation (12.19) is consequently a submersion at all its zeros. There are no zeros on the endpoint collars, since the potential directions vanish there and the kernels are already the selected lines. We may therefore work over the open interior parameter interval. Adding the time variable raises the vertical Fredholm index from -2 to -1 . Lemma 10.1 makes the zero set a separable Banach manifold whose projection to ℓ^2 is Fredholm of index -1 . By [77, Theorem 1.3 and Corollary 1.5], regular parameters are dense and their fibres are empty. For such a parameter no normalized kernel vector orthogonal to ψ_t exists at any time. The selected spinor remains harmonic because every potential variation annihilates it pointwise.

With no time variable the same construction has Fredholm index -2 and again has empty fibres at regular parameters. This proves the fixed-operator assertion as well. The estimates in Lemma 10.1 give arbitrary prescribed C^∞ smallness of the perturbation. $\square$

12.6. The simple-kernel hypersurface and transverse paths. In the rest of the proof, $\mathcal{P}$ denotes either $\mathcal{P}_g$ for a fixed metric or the total space $\mathcal{P}_\mu$. In either case define

$$X = \{p \in \mathcal{P} : \dim \ker A_p \leq 1\}, \quad S = \{p \in \mathcal{P} : \dim \ker A_p = 1\}.$$

Thus $X \setminus S = \mathcal{P}^{\text{inv}}$. The equality follows from the index-zero assumption.

Proposition 12.10. *The set S is nonempty and path connected, both with the metric fixed and in the total density slice.*

Proof. First fix a metric. Parametric transversality [59, Chapter 3, §2, Theorem 2.7], applied to fibrewise perturbations, gives a smooth section of Σ^+ transverse to zero. If $n \geq 16$, this section is nowhere zero because $r > n$. Formula (12.10), with this fixed metric, gives a potential for which the section is harmonic.

If $n = 8$, its zero set is finite. At a zero x , its derivative $T_0 : T_x M \rightarrow \Sigma_x^+$ is an isomorphism. By Proposition 12.6, choose an invertible compatible jet T_1 with the same determinant sign as T_0 . These two isomorphisms belong to the same component of $GL_8(\mathbb{R})$ by polar decomposition and connectedness of $SO(8)$ [54, §1.3.2, Theorem 2.17, and Proposition 2.19(1)]. Join them by a smooth path in that component. In disjoint small coordinate balls around the zeros, apply the replacement (12.15) to this path of derivatives, with the zero-dimensional zero set in place of the curve. The same lower bound (12.16) shows that the section has exactly its original zeros and has derivative T_1 at each. Its Dirac derivative therefore vanishes at every zero. Hadamard division, now in the eight spatial coordinates, gives a smooth potential B for which the section is harmonic. Away from its zeros the rank-one formula gives the local solution, and a partition of unity patches the solutions.

In both dimensions the selected section is not identically zero. Normalize it in L^2 and apply the fixed-operator version of Proposition 12.9. The resulting operator has kernel exactly that real line. Since its index is zero, its cokernel is also a real line. This constructs a point of S over every fixed metric.

Given any two points of S , choose normalized real generators of their kernels. Proposition 12.7 joins them by a family with a selected normalized harmonic spinor and exact endpoint operators. Proposition 12.9 removes all other kernel vectors relative to the endpoint collars. The resulting path lies in S . Both propositions permit the metric to remain fixed when the endpoint metrics coincide. $\square$

Lemma 12.11. *The set X is open in $\mathcal{P}$, and S is closed in X . Near every point of S there is a local product chart in which S is the zero set of one real scalar coordinate and its two local sides consist of invertible operators. A smooth potential direction can be chosen transverse to the wall.*

Proof. At a point A_0 of S , put $K = \ker A_0$ and $C = \ker A_0^*$. With L^2 orthogonal complements in the Sobolev spaces, write

$$H^s(\Sigma^+) = K \oplus E_0, \quad H^{s-1}(\Sigma^-) = C \oplus F_0.$$

The map $d_0 = A_0|_{E_0} : E_0 \rightarrow F_0$ is an isomorphism. For a sufficiently close operator write

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad d : E_0 \rightarrow F_0.$$

The block d remains invertible. Elimination of the off-diagonal blocks reduces invertibility to that of

$$s(A) = a - b d^{-1} c : K \longrightarrow C. \quad (12.20)$$

In particular $\dim \ker A \leq 1$ in this neighborhood. Invertibility is open at the remaining points of X , so X is open. The complement of S in X is the open invertible locus, so S is relatively closed.

At A_0 , both b and c vanish, and the derivative of (12.20) in a potential direction is

$$Ds_{A_0}(Q) = P_C Q|_K.$$

For nonzero $\psi \in K$ and $\eta \in C$, Lemma 12.2 provides a smooth Q with $\langle \eta, Q\psi \rangle_{L^2} \neq 0$. Since K and C are real lines, this is a nonzero derivative of the scalar Schur map.

In a C^k coefficient completion with $k \geq s + 1$, the Schur map is smooth and its derivative in the chosen smooth direction Q is nonzero. The Banach implicit function theorem [46, Theorem 13.20] gives a product chart by solving for the scalar coefficient of Q . This chart restricts to smooth coefficients, since its scalar correction multiplies the fixed smooth field Q . Choose the complementary-coordinate neighborhood convex; its intersection with the smooth coefficients is path connected. The two signs of $s(A)$ are its invertible sides. A sufficiently short segment $A_0 + \varepsilon Q$ therefore meets S only at $\varepsilon = 0$. $\square$

Proposition 12.12. *Any two points of $\mathcal{P}^{\text{inv}}$ can be joined by a smooth coefficient path which is stationary on endpoint collars, has a one-dimensional kernel at finitely many interior times, and is invertible at every other time. Each singular time is a transverse simple crossing. A prescribed smooth metric path, stationary on endpoint collars, may be retained. Consequently X is path connected.*

Proof. Join the endpoints by a smooth path in the full coefficient space, constant on endpoint collars. With the metric fixed this is an affine path in the potentials, followed by a reparametrization. In the total space take a smooth metric path and the spin comparison to interpolate the potentials. Take the unrestricted relative potential directions of Lemma 10.1 and consider

$$F(p, t, \psi) = (A_t + Q_t(p))\psi, \quad \|\psi\|_{L^2} = 1. \quad (12.21)$$

The unit sphere is a smooth Banach submanifold of H^s because the derivative of its squared L^2 norm is nonzero there. Lemma 12.2 proves surjectivity at every zero of (12.21); a spatial potential variation can be extended near the chosen interior time and multiplied by a time cutoff. The vertical spinor derivative has index -1 , and adding time gives index zero. The zero set is therefore a separable Banach manifold whose projection to t^2 is Fredholm of index zero. It is empty on the endpoint collars, where the operators remain the given invertible ones.

Choose an arbitrarily small regular parameter by [77, Theorem 1.3]. The fixed-parameter normalized incidence is a zero-dimensional manifold. If a kernel had dimension $k \geq 2$, its full sphere S^{k-1} of normalized real vectors would lie in this incidence at the same time, contradicting discreteness. Thus every kernel encountered is a real line.

For the chosen smooth compact path, the uniform estimate (12.2) gives a uniform H^{s+1} bound on every normalized kernel spinor. Given a sequence of solutions, pass to a convergent sequence of times and then, by the compact inclusion $H^{s+1} \hookrightarrow H^s$, to a convergent sequence of spinors in H^s . The limit is normalized and solves the limiting equation, by operator-norm continuity in (12.1). The normalized incidence is therefore compact. A compact zero-dimensional manifold is finite. Its elements occur in pairs $\psi, -\psi$, so the number of singular times is finite as well.

At a singular time t_0 , regularity of the parameter means that the fixed-parameter derivative in (t, ψ) is onto. Its spinor part has the range of $A_{t_0} + Q_{t_0}(p)$ and has a one-dimensional cokernel. Consequently the time derivative maps nontrivially to this cokernel. For nonzero kernel and adjoint-kernel generators this is the condition

$$\left\langle \eta, \frac{d}{dt} \Big|_{t=t_0} (A_t + Q_t(p))\psi \right\rangle_{L^2} \neq 0. \quad (12.22)$$

By (12.20), it is precisely transversality to the simple wall.

The asserted paths show that any two invertible points can be joined in X . If an endpoint belongs to S , Lemma 12.11 first joins it to an invertible point by a short normal segment lying in X . Concatenating these segments with an invertible-endpoint path proves that any two points of X can be joined in X . $\square$

12.7. The two components and the real index. The ambient coefficient space $\mathcal{P}$ is contractible and metrizable: $\mathcal{P}_g$ is affine, and $\mathcal{P}_\mu$ is identified with a product of Fréchet vector spaces in Section 8.

The determinant line of the real index-zero family is

$$\lambda_A = \Lambda^{\text{top}} \ker A \otimes (\Lambda^{\text{top}} \text{coker } A)^*. \quad (12.23)$$

The operators $A_p : H^s(\Sigma^+) \rightarrow H^{s-1}(\Sigma^-)$ form a continuous family of bounded Fredholm maps between fixed real Hilbert spaces, as in (12.1). Pullback of the real determinant bundle gives a continuous line bundle with fibres (12.23), including at larger kernels [82, Proposition 5.3 and Corollary 5.4]. Since the base is contractible and metrizable, choose an orientation of this line bundle. At an invertible operator there is the canonical element $1 \otimes 1^*$. Let $\varepsilon(A) \in \{1, -1\}$ be its sign in the chosen orientation. This sign is locally constant on the invertible locus.

Lemma 12.13. *The chosen determinant orientation consistently labels the two local sides of S by opposite signs of ε . In particular it coorients S , and ε changes sign at every transverse simple crossing.*

Proof. The normal line in the Schur chart (12.20) is

$$\text{Hom}_{\mathbb{R}}(K, C) \simeq (\det K \otimes (\det C)^*)^*.$$

The chosen determinant orientation therefore orients this line. A short transverse segment has invertible endpoints, exactly one singular point, real kernel and cokernel of dimension one, and a Schur scalar which changes sign. Proposition 5.9 of [82, 2022 update, Section 5.3] makes the canonical determinant orientations on its two sides opposite. The global determinant orientation makes these signs consistent on overlaps. $\square$

Proposition 12.14. *The invertible locus $\mathcal{P}^{\text{inv}}$ has exactly two path components, distinguished by $\varepsilon \in \{1, -1\}$.*

Proof. The space X is open in the ambient Fréchet space and is therefore locally path connected. Propositions 12.10 and 12.12 and Lemma 12.11 give the hypotheses of Lemma 10.5 for (X, S) . Hence the complement has at most two components. The two sides at any point of S have opposite determinant signs by Lemma 12.13. Local constancy of that sign on invertibles prevents a path between them. Both components are therefore nonempty and the count is exactly two. $\square$

Proposition 12.15. *For invertible endpoints p_0, p_1 , the ordinary real index difference in $KO_{n+1} \cong \mathbb{Z}/2$ is zero if $\varepsilon(A_{p_0}) = \varepsilon(A_{p_1})$ and is the nonzero element otherwise.*

Proof. Choose a path p_t as in Proposition 12.12, and let N be its number of singular parameters. By Section 8.1 and (8.3), its relative real chiral Fredholm class is the ordinary index difference in KO_{n+1} . In each Schur chart the complementary block is invertible, and the remaining real-line map changes sign at the crossing. Each crossing therefore represents the nonzero element of $KO_{n+1} \cong \mathbb{Z}/2$ by (8.1). By additivity the ordinary index difference equals $N \pmod{2}$. Lemma 12.13 gives

$$\varepsilon(A_{p_1}) = (-1)^N \varepsilon(A_{p_0}),$$

which proves the assertion. $\square$

Proof of Theorem 12.1. Proposition 12.14 applies to the fixed-metric potential space and to the total density slice. Choose an invertible basepoint and orient the determinant line so that its canonical element is positive there. The resulting determinant parity takes the values zero and one on the two components, respectively. Proposition 12.15 identifies it with the ordinary real index difference in $KO_{n+1} \cong \mathbb{Z}/2$. This proves the asserted bijections of pointed sets. $\square$

Remark 12.16. Proposition 8.2, with $U = M$, gives a nonempty space $\mathcal{R}_\mu^{\text{inv}}(M)$ and a bijection

$$\pi_0 \mathcal{R}_\mu^{\text{inv}}(M) \longrightarrow \pi_0 \mathcal{P}_\mu^{\text{inv}}$$

which preserves ordinary index difference. Thus, for any invertible base metric g_* with density μ ,

$$\text{inddiff}_{g_*} : \pi_0 \mathcal{R}_\mu^{\text{inv}}(M) \longrightarrow KO_{n+1} \cong \mathbb{Z}/2$$

is a bijection of pointed sets.

13. REAL SKEW-ADJOINT OPERATORS

Throughout this section, M is closed and connected,

$$n \equiv 1 \pmod{8}, \quad n \geq 9, \quad \alpha(M) = 0.$$

All spin structures are fixed. The density is the prescribed density μ . The charge conjugation C satisfies $C^2 = 1$ and $CD_g = -D_g C$. Its fixed bundle $\Sigma_g^{\mathbb{R}}$ has real rank

$$r = 2^{(n-1)/2} = 2\ell, \quad \ell = 2^{(n-3)/2}.$$

The operators and their principal Clifford matrices are

$$K_g = iD_g, \quad Q = iV, \quad L_{g,Q} = K_g + Q, \quad a_g(\xi) = ic_g(\xi). \quad (13.1)$$

We identify tangent and cotangent vectors by g in Clifford expressions. Thus K_g and Q preserve $\Sigma_g^{\mathbb{R}}$, and

$$a_g(\xi)^T = a_g(\xi), \quad a_g(\xi)a_g(\eta) + a_g(\eta)a_g(\xi) = 2g^{-1}(\xi, \eta)I. \quad (13.2)$$

The compatibility identities say exactly that Q is an arbitrary smooth real skew-adjoint endomorphism of $\Sigma_g^{\mathbb{R}}$. Indeed, $C(iV) = iVC$ and $(iV)^* = -iV$; conversely, $V = -iQ$ satisfies the original compatibility identities. Put

$$\mathcal{Q}_g = C^\infty(M; \mathfrak{so}(\Sigma_g^{\mathbb{R}})), \quad \mathcal{Q}_g^{\text{inv}} = \{Q : K_g + Q \text{ is invertible}\}.$$

Multiplication by i identifies this notation with the metric-potential notation $\mathcal{P}_\mu$.

Theorem 13.1. *The spaces $\mathcal{Q}_g^{\text{inv}}$, for every fixed smooth metric g , and $\mathcal{P}_\mu^{\text{inv}}$ are nonempty and have exactly two path components. For either coefficient space $\mathcal{P}$, an invertible base point p_* gives a bijection*

$$\pi_0(\mathcal{P}^{\text{inv}}) \longrightarrow KO_2 \cong \mathbb{Z}/2, \quad [p] \longmapsto \text{sf}_2(L_{p_*}, L_p), \quad (13.3)$$

where the real mod-two spectral flow is computed along any smooth path in $\mathcal{P}$. On $\mathcal{P}_\mu^{\text{inv}}$, with a pure metric base point g_* , this invariant is $\delta_{g_*}(g, V) \in KO_{n+1} \cong KO_2$.

13.1. Fixed domains and the universal incidence. Positive spinor comparison identifies the real spinor bundles and their L^2 spaces. In these identifications every operator in a smooth finite parameter family has domain H^1 . Its coefficients depend smoothly on the metric and its first derivatives; see [56, §2.3] and Section 2. Each $L = L_{g,Q}$ is a real skew-adjoint elliptic operator with compact resolvent. Its kernel consists of smooth sections, and

$$\dim_{\mathbb{R}} \ker L \equiv \alpha(M) = 0 \pmod{2}.$$

For $Q = 0$, this is [14, Theorem 3.1]. The real skew index is kernel parity and is constant under Fredholm homotopies [13, Proposition 5.1, pp. 22–23]. Apply this to the bounded transforms of $K_g + tQ$, which are continuous by Section 8.1.

The logarithmic metric and linear potential coordinates of Section 8 identify $\mathcal{P}_\mu$ with a Fréchet vector space. Both $\mathcal{P}_\mu$ and $\mathcal{Q}_g$ are therefore contractible. In what follows, $\mathcal{P}$ denotes either space. All potential directions are smooth multiplication endomorphisms.

For the local analytic arguments fix an integer $s > n/2 + 1$, and complete the coefficients in C^m , where $m \geq s + 1$. The spinor equation is then a smooth Banach-space equation $H^s \rightarrow H^{s-1}$. The value

of m can be increased to handle any prescribed basic neighborhood in the smooth topology. We always choose finite parameter families with smooth spatial coefficients. Every family of kernel spinors which is smooth in the parameters with values in H^s is jointly smooth in the spatial and parameter variables, by the parameter-dependent elliptic argument at the end of the proof of Lemma 10.2. Unique continuation for $L = i(D_g + V)$ is the one established in Section 9, from [56, Theorem 6.2.8].

Define the real projective incidence

$$\mathcal{I}(\mathcal{P}) = \{(p, [\psi]) : 0 \neq \psi \in \ker L_p\}. \quad (13.4)$$

Locally choose an L^2 -unit representative ψ . The spinor tangent and the equation target are respectively

$$H^s \cap \psi^\perp, \quad H^{s-1} \cap \psi^\perp.$$

The target is the orthogonal complement of the tautological line: skew-adjointness gives $\langle L_p \psi, \psi \rangle_{L^2} = 0$. At a zero the differential in potential and spinor directions is

$$T_\psi(R, u) = Lu + R\psi. \quad (13.5)$$

Lemma 13.2. *The map (13.5) is surjective. It remains surjective if the potential directions are required to have support in any fixed nonempty open set. Finitely many smooth potential directions suffice locally. If F is any subspace of $\ker L$ and*

$$W_F = \{P_F R|_F : R \in \mathcal{Q}_g\} \subset \mathfrak{so}(F),$$

then

$$W_F v = v^\perp \cap F \quad (0 \neq v \in F). \quad (13.6)$$

Proof. Let η annihilate the image of (13.5). The spinor variations imply, in the distributional sense, $L\eta = b\psi$ for a real constant b . Pairing with ψ gives $b = 0$. Elliptic regularity makes η smooth.

Unique continuation gives a ball $U_0 \Subset U$ on which ψ is nowhere zero, where U is the prescribed spatial open set. At a point of this ball the vectors $R\psi$, as R varies through skew endomorphisms, fill $\psi^\perp$. Localization of the potential directions therefore gives $\eta = f\psi$ on the ball, with f real and smooth. Hence

$$0 = L(f\psi) = a_g(df)\psi.$$

The Clifford identity makes $a_g(df)$ invertible wherever $df \neq 0$, so f is constant. Unique continuation gives $\eta = c\psi$ on M ; this represents the zero functional on the target $\psi^\perp$.

The vertical operator $L : \psi^\perp \rightarrow \psi^\perp$ is Fredholm. Its range is closed and its cokernel is finite dimensional. The preceding annihilator argument therefore permits a finite selection of smooth potential directions filling that cokernel. The resulting surjective operator has a bounded right inverse.

For (13.6), an element $\eta \in F \cap v^\perp$ annihilating $W_F v$ satisfies $\langle \eta, Rv \rangle_{L^2} = 0$ for every potential variation. Both v and η lie in $\ker L$, so the same localized argument gives $\eta = cv$, and orthogonality gives $\eta = 0$. Thus (13.6) holds for every $F \subset \ker L$. $\square$

If $k = \dim_{\mathbb{R}} \ker L$, the kernel and cokernel of the vertical projective operator both have dimension $k - 1$. Thus, in a finite parameter family of dimension N for which (13.5) is surjective, the resolved incidence is a manifold of dimension N . Above a constant- k -kernel stratum its fibre is $\mathbb{R}P^{k-1}$.

Lemma 13.3. *In local unit representatives, incidence evaluation is transverse to the zero section and to every smooth fibrewise radial submanifold of the real spinor bundle. For an antipodally invariant radial target and a compact family of incidence spinors, these transversality statements can be obtained in finitely many smooth potential directions. Consequently such a radial target of codimension greater than $n + d$ can be avoided by a generic perturbation of an incidence family over a compact smooth d -manifold, possibly with boundary, relative to a closed parameter subset on a neighborhood of which the prescribed family already avoids it.*

Proof. It suffices to consider a fixed point x , since surjectivity onto the vertical normal directions implies transversality of evaluation on the product with M . Let w be a conormal at $\psi(x)$. Radiality gives

$$\langle w, \psi(x) \rangle = 0. \quad (13.7)$$

At the zero spinor this identity holds for every w . Suppose that $u \mapsto \langle w, u(x) \rangle$ vanishes on the tangent space of incidence. By the bounded right inverse in Lemma 13.2, this functional factors through (13.5). There are consequently a distribution η and $b \in \mathbb{R}$ such that

$$L\eta = \delta_x w + b\psi, \quad \langle \eta, R\psi \rangle = 0 \quad \text{for all smooth skew } R. \quad (13.8)$$

Pairing the first equation with ψ , together with (13.7) gives $b = 0$.

Away from x , the distribution η is smooth. On a nonzero ball for ψ in $M \setminus \{x\}$, the second equation in (13.8) gives $\eta = f\psi$. The Clifford identity again gives $df = 0$. Since $M \setminus \{x\}$ is connected, unique continuation gives $\eta = c\psi$ on that punctured manifold. The distribution $T = \eta - c\psi$ is therefore supported at x and satisfies $LT = \delta_x w$.

The symbol $a_g(x, \xi)$ is invertible for every nonzero real ξ . Lemma 2.3 gives $T = 0$ and $w = 0$. Thus the conormal annihilator of the evaluation derivative is zero. Applying this to every $w \perp \psi(x)$ gives

$$\{u(x) : (R, u) \in \ker T_\psi\} + \mathbb{R}\psi(x) = \Sigma_{g,x}^{\mathbb{R}}. \quad (13.9)$$

Smooth tangent pairs suffice for this identity. To see this, choose a right inverse J of T_ψ whose coefficient component lies in the span of finitely many smooth potential directions, as in Lemma 13.2. It sends a smooth target to a smooth pair: its coefficient component is smooth, and the equation for its spinor component is elliptic with smooth right-hand side. Approximate an arbitrary tangent pair v by smooth ambient pairs v_i . Then $v_i - JT_\psi v_i$ are smooth tangent pairs converging to v . Their evaluations are dense in the finite-dimensional image, so finitely many smooth tangent pairs realize (13.9).

Let B be the compact parameter manifold of an incidence map and write the given map as $(p_b, [\psi_b])$. At each (b, x) , select both potential directions filling the equation cokernel and the coefficient components of smooth incidence tangent pairs needed for (13.9). A finite cover of $B \times M$ selects finitely many smooth reference-bundle potential directions $R_1, \dots, R_N$. Consider, near the original section, the finite-dimensional incidence

$$\mathcal{J} = \left\{ (b, z, [\psi]) : \left(L_{p_b} + \sum_{j=1}^N z_j R_j \right) \psi = 0 \right\}, \quad \sigma(b) = (b, 0, [\psi_b]).$$

The derivative in $(z, [\psi])$ is surjective and Fredholm of index N . The Banach implicit function theorem therefore makes $\mathcal{J}$ a smooth manifold of dimension $N + d$ near $\sigma(B)$. The projection $\pi_B : \mathcal{J} \rightarrow B$ is a submersion. Indeed, for any prescribed base tangent $\dot{b}$, the term $(\partial_b L)(\dot{b})\psi$ in the differentiated equation can be cancelled by solving $T_\psi(R, u) = -(\partial_b L)(\dot{b})\psi$ with the selected potential directions. Its fibres have dimension N . Equation (13.9), now with tangent pairs in $\ker d\pi_B$, persists in a neighborhood of $\sigma(B) \times M$.

Choose finitely many smooth sections $v_1, \dots, v_a$ spanning $\sigma^* \ker d\pi_B$, and a fibrewise local addition Θ_b from a neighborhood of zero in that bundle into the fibre of $\mathcal{J}$ at b . Such sections are obtained from finite local frames and a partition of unity; a vertical Riemannian exponential gives the local addition. Let χ vanish on a neighborhood of the prescribed closed subset and be positive elsewhere, with its zero set contained in the region on which the original family already avoids the target. For small $\varepsilon \in \mathbb{R}^a$, define actual sections of π_B by

$$\sigma_\varepsilon(b) = \Theta_b \left(\chi(b) \sum_{j=1}^a \varepsilon_j v_j(b) \right).$$

Where $\chi > 0$, parameter derivatives span the fibre tangent space, and (13.9) gives transversality to every radial target. Where $\chi = 0$, the section is unchanged and already avoids the prescribed target.

Parametric transversality [59, Chapter 3, §2, Theorem 2.7], applied separately on the interior and the boundary of B , gives a perturbed evaluation map transverse to that target on both. The inverse image has dimension at most $n + d$ minus its codimension, and is empty under the stated inequality. The construction is expressed in local unit representatives; antipodal invariance makes these assertions agree on their overlaps.

The perturbed operators have their original smooth metric coefficients and potentials $Q_b + \sum_j z_j(b, \epsilon) R_j$. Parameter differentiation and elliptic regularity make the local unit spinor representatives jointly smooth in the spatial and parameter variables. For the cone $\{\psi \neq 0 : |j(\psi)| = |\psi|^2\}$, first remove spinor zeros; the resulting compact image lies in an annulus on which that cone is closed. This also permits all perturbations to be taken within a fixed compact neighborhood of the relevant evaluations. $\square$

13.2. Currents and connectedness of the full incidence. For a real spinor ψ , define its current and squared length by

$$g(j(\psi), X) = \langle a_g(X^b)\psi, \psi \rangle, \quad \rho(\psi) = |\psi|^2. \quad (13.10)$$

In an orthonormal frame, the vectors $a_i\psi$ are mutually orthogonal and have length $|\psi|$: for $i \neq j$, $a_i a_j$ is skew-symmetric. Bessel's inequality gives

$$|j(\psi)| \leq \rho(\psi), \quad \operatorname{div}_g j(\psi) = 2\langle K_g \psi, \psi \rangle. \quad (13.11)$$

Indeed, in a normal orthonormal frame at the point in question, compatibility of the spin connection with a_g gives

$$\operatorname{div}_g j(\psi) = 2 \sum_i \langle a_g(e^i) \nabla_{e_i} \psi, \psi \rangle = 2\langle K_g \psi, \psi \rangle.$$

Thus an incidence spinor has divergence-free current. Conversely, if ψ is nowhere zero and its current is divergence free, put $y = -K_g \psi$. Then $y \perp \psi$ pointwise, and the formula

$$Q(\psi) = \frac{y\psi^T - \psi y^T}{|\psi|^2} \quad (13.12)$$

defines a smooth skew potential satisfying $(K_g + Q(\psi))\psi = 0$.

Lemma 13.4. *Suppose $n \geq 17$. For every incidence point and every neighborhood of it, there is an incidence path in that neighborhood from the given point to one whose spinor is nowhere zero and satisfies*

$$|j(\psi)| < |\psi|^2$$

at every point. Such a spinor can be deformed through nowhere-zero spinors with divergence-free current to a zero-current spinor.

Proof. Let $\psi \neq 0$ satisfy $|j(\psi)| = |\psi|^2$. Equality in that Bessel inequality is equivalent to

$$a_g(v)\psi = \psi, \quad v = \frac{j(\psi)}{|\psi|^2}, \quad |v| = 1.$$

Conversely, this eigenvector equation implies $\langle j(\psi), v \rangle = |\psi|^2$, and hence equality and the displayed formula for v . The positive and negative eigenspaces of $a_g(v)$ have the same dimension ℓ , since Clifford multiplication by a unit covector perpendicular to v interchanges them. The cone $\{\psi \neq 0 : |j(\psi)| = |\psi|^2\}$ is therefore the image of the nonzero positive eigenbundle over S^{n-1} . This parametrization is an embedding: its inverse is $\psi \mapsto (j(\psi)/|\psi|^2, \psi)$; its differential is injective because $a_g(\dot{v})\psi = 0$ forces $\dot{v} = 0$. Its codimension in the real spinor fibre is

$$2\ell - (\ell + n - 1) = \ell - n + 1 > n + 1. \quad (13.13)$$

The real rank 2ℓ also exceeds $n + 1$. Lemma 13.3, first for zeros and then for this cone in a compact annulus, gives the required nearby incidence point. A path in a sufficiently small incidence chart joins it to the original point.

For a nowhere-zero spinor ψ_0 with strict inequality, fix $j_0 = j(\psi_0)$ and solve the pointwise ordinary differential equation

$$\frac{\partial \psi_s}{\partial s} = -\frac{a_g(j_0^\dagger) \psi_s}{2|\psi_s|^2}, \quad \psi_{s=0} = \psi_0. \quad (13.14)$$

The Clifford anticommutator gives

$$\frac{\partial}{\partial s} \langle a_i \psi_s, \psi_s \rangle = -(j_0)_i, \quad j(\psi_s) = (1-s)j_0.$$

Writing $\rho_s = |\psi_s|^2$, we obtain

$$\frac{d\rho_s}{ds} = -\frac{(1-s)|j_0|^2}{\rho_s}, \quad \rho_s^2 = \rho_0^2 - (2s-s^2)|j_0|^2.$$

On the compact manifold, the strict current inequality gives

$$\rho_s^2 \geq \min_M(\rho_0^2 - |j_0|^2) > 0 \quad (0 \leq s \leq 1).$$

The solution therefore extends smoothly through $s = 1$, with smooth spatial dependence. The currents remain divergence free, and $j(\psi_1) = 0$. Formula (13.12) gives incidence potentials throughout the deformation. $\square$

Lemma 13.5. *For $n \geq 17$, the unit zero-current fibre*

$$Z = \{\psi \in \mathbb{R}^{2\ell} : |\psi| = 1, j(\psi) = 0\}$$

is $(\ell - n - 1)$ -connected, and hence is n -connected. The associated zero-current spinor bundle admits a smooth section over M . Any two prescribed unit zero-current sections extend to a smooth section over $M \times I$, also when the metric varies along a prescribed smooth path stationary on endpoint collars.

Proof. Choose one Clifford generator in the form

$$a_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad a_i = \begin{pmatrix} 0 & E_i \\ E_i^T & 0 \end{pmatrix} \quad (1 \leq i \leq n-1).$$

Each E_i is orthogonal, and the vectors $E_1 v, \dots, E_{n-1} v$ are orthogonal for $v \neq 0$. For $\psi = (u, v)$, the unit zero-current equations are

$$|u| = |v| = \frac{1}{\sqrt{2}}, \quad \langle u, E_i v \rangle = 0 \quad (1 \leq i \leq n-1).$$

These equations define the OT-FKM focal manifold M_+ for $m = n-1$ and $m_2 = \ell - n > 0$ [45, Section 4.2]. The sphere-bundle description [75, Section 1] is

$$S^{\ell-n} \longrightarrow Z \longrightarrow S^{\ell-1}, \quad (u, v) \longmapsto \sqrt{2}v. \quad (13.15)$$

The homotopy exact sequence [55, Theorem 4.41] gives $\pi_q(Z) = 0$ for $q \leq \ell - n - 1$. Here $n \equiv 1 \pmod{8}$, $n \geq 17$, and $\ell = 2^{(n-3)/2}$, so $\ell - n - 1 \geq n$. The intrinsic current equations make this smooth compact fibre $\text{Spin}(n)$ -invariant, so it defines an associated bundle.

For M and $(M \times I, M \times \partial I)$, the section-extension obstructions have coefficients $\pi_{q-1}(Z)$ with $q \leq n+1$. These groups vanish. Relative obstruction theory [55, §4.3, pp. 416–419] therefore gives a section in the first case and an extension of the two prescribed sections in the second, also for the bundle along a varying metric. Extend the endpoint sections constantly on their stationary collars. Relative smooth approximation [81, Theorem 11] gives smooth sections with the same collar values. Positive amplitudes can be joined independently. $\square$

Dimension nine requires a different deformation. Choose the nine Clifford generators in the same block form, now with $\ell = 8$. The eight vectors $E_i v$ fill $\mathbb{R}^8$ for $v \neq 0$. Therefore

$$(|u|^2 - |v|^2)^2 + 4 \sum_{i=1}^8 \langle u, E_i v \rangle^2 = (|u|^2 + |v|^2)^2.$$

Thus $|j(\psi)| = |\psi|^2$ for every spinor. On unit spinors, the current is the $\text{Spin}(9)$ -equivariant Hopf map

$$S^7 \longrightarrow S^{15} \longrightarrow S^8. \quad (13.16)$$

The fibre over $v \in S^8$ is the unit sphere in the eight-dimensional positive eigenspace of $a_g(v)$. This is the smooth $\text{Spin}(9)$ -equivariant octonionic Hopf fibration [73, Section 4].

Lemma 13.6. *Let $n = 9$. Given two nowhere-zero incidence spinors, and a smooth metric path between their metrics, stationary on endpoint collars, there is a smooth nowhere-zero spinor path with divergence-free current and the prescribed endpoint spinors. The path can be stationary on endpoint collars. For each fixed metric there also exists a nowhere-zero spinor with divergence-free current.*

Proof. The unit-spinor fibre S^{15} is 14-connected, and $\dim(M \times I) = 10$. Relative obstruction theory [55, §4.3, pp. 416–419] and relative smooth approximation [81, Theorem 11] give a smooth unit-spinor path along the metric path, equal to the prescribed unit spinors on the endpoint collars. Join their positive lengths as well, and denote the resulting nowhere-zero spinor path by ψ_t^0 . Write

$$\alpha_t^0 = \iota_{j(\psi_t^0)} dv_{g_t}.$$

These eight-forms are nonzero; their endpoint values are closed by (13.11). Choose a smooth function $\chi : I \rightarrow [0, 1]$, equal to zero and one on the two collars, and put

$$\gamma_t = (1 - \chi(t))\alpha_0^0 + \chi(t)\alpha_1^0.$$

Every γ_t is closed in the spatial variables, and $\gamma_t = \alpha_t^0$ on the collars.

Extend these data constantly to $t \leq 0$ and $t \geq 1$. On the base $M \times \mathbb{R}$, consider the bundle $\Lambda^7 T^*M$ pulled back from M . Its first-jet relation is

$$\gamma_t + d_x \beta \neq 0. \quad (13.17)$$

Here d_x differentiates only in the spatial variables. Ampleness follows on every principal affine slice of the full first-jet bundle over $M \times \mathbb{R}$. If a principal covector is $\xi = (\xi_x, \xi_t)$, a rank-one derivative change $\xi \otimes \eta$ changes $d_x \beta$ by $\xi_x \wedge \eta$. For $\xi_x \neq 0$, this linear map has rank $\binom{8}{7} = 8$. The forbidden set in the slice is thus empty or an affine subspace of codimension eight. Its complement is connected and has the whole slice as convex hull: after quotienting by the affine subspace this is the corresponding assertion for $\mathbb{R}^8 \setminus \{0\}$. For $\xi_x = 0$, the relation is constant on the slice; a slice through an admissible jet is entirely admissible. Thus (13.17) is open and ample, including pure-time principal covectors.

Spatial alternation

$$T^*M \otimes \Lambda^7 T^*M \longrightarrow \Lambda^8 T^*M$$

has a smooth right inverse; for example, in an orthonormal frame one may take $\omega \mapsto \frac{1}{8} \sum_i e^i \otimes \iota_{e_i} \omega$. Take the underlying section $\beta = 0$, give its formal spatial derivative this right inverse applied to $\alpha_t^0 - \gamma_t$, and give it formal time derivative zero. This is a formal solution of (13.17); it is the holonomic zero jet on the collars. Choose $\varepsilon > 0$ so that the formal solution is holonomic on a neighborhood of $M \times ((-\infty, \varepsilon] \cup [1 - \varepsilon, \infty))$. The relative open-ample h-principle [43, Theorem 18.4.1 and Remark 1, p. 171], in the formulation [67, Theorem 3.5], gives a homotopy through the relation to a holonomic solution, fixed on that closed subset. Restricting to $M \times I$ gives a homotopy through nonzero eight-forms from α_t^0 to

$$\alpha_t = \gamma_t + d_x \beta_t.$$

Each final form is spatially closed. After a stationary reparametrization in the homotopy variable, relative smooth approximation [81, Theorem 11], in the open bundle of nonzero eight-forms, makes this homotopy smooth in all variables, with its initial and final forms and its time collars fixed.

Contraction with dv_{g_t} identifies this form homotopy with a homotopy of nonzero vectors. Normalize those vectors and lift their directions through (13.16), starting at $\psi_t^0/|\psi_t^0|$. A smooth connection on this compact-fibre bundle gives a smooth horizontal lift. On the endpoint collars the base homotopy is constant, so the lift is constant there. Finally multiply the lifted unit spinor by the positive square root of the length of the prescribed current vector. The resulting spinors have current corresponding to α_t , hence have divergence-free current, and their endpoint values are exactly the prescribed ones.

For existence at a fixed metric, first choose a nowhere-zero spinor; the same obstruction argument on the nine-dimensional base gives one. By [43, Corollary 20.3.2, pp. 181–182], with $n = 9$ and $p = 8$, its current eight-form is homotopic through nonzero eight-forms to a nonzero exact eight-form. The Hopf lift and the length rescaling give the required spinor. $\square$

Proposition 13.7. *For $\mathcal{P} = \mathcal{Q}_g$ and for $\mathcal{P} = \mathcal{P}_\mu$, the full projective incidence $\mathcal{I}(\mathcal{P})$ is nonempty and path connected. The connecting paths may be chosen smooth and stationary on endpoint collars.*

Proof. For $n \geq 17$, Lemma 13.5 gives a global smooth unit zero-current spinor at any fixed metric. For $n = 9$, apply the existence assertion in Lemma 13.6. In both cases (13.12) produces an incidence point. Its kernel has even dimension, and therefore has dimension at least two.

To join two arbitrary incidence points, first apply Lemma 13.3 to reach nowhere-zero spinors by incidence paths. For the resulting endpoint metrics g_0, g_1 , put $H = \log(g_0^{-1}g_1)$ and choose a smooth $\chi : I \rightarrow [0, 1]$ which is zero near 0 and one near 1. The path

$$g_t = g_0(e^{\chi(t)H}, \cdot)$$

is stationary on these collars. If $dv_{g_0} = dv_{g_1}$, then $\text{tr } H = 0$, so $dv_{g_t} = dv_{g_0}$. For a fixed-metric family take $H = 0$. When $n \geq 17$, apply Lemma 13.4 to make both spinors satisfy the strict current inequality, and then contract their currents to zero at their respective metrics. Join the resulting zero-current spinors along this metric path by Lemma 13.5. In dimension nine apply Lemma 13.6 to the same path. All scalar parameter changes can be made constant on endpoint collars, so the constructed paths concatenate smoothly.

It remains to retain the prescribed potentials. On a collar where $g_t = g_0$ and $\psi_t = \psi_0$, the difference

$$B_0 = Q_0 - Q(\psi_0)$$

annihilates ψ_0 . Add $\theta_0(t)B_0$, where $\theta_0 = 1$ near the endpoint and is supported in this stationary collar. This preserves the equation and restores Q_0 exactly. Do the same at the other endpoint and at each concatenation where necessary. Thus the construction gives a path in the required incidence space. For the fixed-metric assertion all metric paths in the construction are constant. $\square$

13.3. Compression on arbitrary kernel subspaces. We identify $\Lambda^2 F$ with $\mathfrak{so}(F)$ by

$$u \wedge v \longleftrightarrow uv^T - vu^T, \quad \langle A, B \rangle = -\frac{1}{2} \text{tr}(AB).$$

For a subspace $F \subset \ker L$, let

$$e_x : F \longrightarrow \Sigma_{g,x}^{\mathbb{R}}, \quad U_x = \text{im}(e_x^*) \subset F$$

be evaluation and its adjoint image.

Lemma 13.8. *For every subspace $F \subset \ker L$,*

$$W_F = \text{span}_{x \in M} \Lambda^2 U_x. \quad (13.18)$$

In particular W_F is spanned by decomposable skew matrices. For every even-dimensional such F , with $k = \dim_{\mathbb{R}} F$,

$$\dim W_F \geq \begin{cases} 1, & k = 2, \\ 4, & k = 4, \\ k + 1, & k \geq 6. \end{cases} \quad (13.19)$$

Proof. Lemma 6.5 gives

$$W_F = \text{span}_{x \in M} e_x^* \mathfrak{so}(\Sigma_{g,x}^{\mathbb{R}}) e_x = \text{span}_{x \in M} \Lambda^2 U_x.$$

The pointwise identity follows by writing a skew matrix as a sum of rank-two matrices. This proves (13.18).

When $k \geq 2$, evaluation cannot have rank at most one on any nonempty open ball. Otherwise choose a nonzero $\psi \in F$ and a smaller ball on which it is nowhere zero. Every $\phi \in F$ then has the form $f\psi$ with f real and smooth. The common equation gives $a_g(df)\psi = 0$, so f is constant, and unique continuation makes ϕ a constant multiple of ψ globally. This contradicts $k \geq 2$. In particular W_F contains a nonzero decomposable matrix.

Equation (13.6) gives $\dim W_F \geq k - 1$. If $k = 2$, this is the assertion. If $k = 4$ and $\dim W_F = 3$, the map

$$W_F \longrightarrow v^\perp \cap F, \quad A \longmapsto Av$$

would be an isomorphism for every $v \neq 0$. Every nonzero $A \in W_F$ would then have zero kernel. This contradicts the nonzero decomposable matrix, whose rank is two. Hence $\dim W_F \geq 4$.

For even $k \geq 6$, the decomposable span and $\dim W_F \geq k - 1$ give two independent decomposable matrices $A, B \in W_F$. Their common kernel has dimension at least $k - 4 > 0$; choose $0 \neq v$ in it. The surjective map $W_F \rightarrow v^\perp \cap F$ then has a kernel containing the independent matrices A, B . Therefore

$$\dim W_F \geq (k - 1) + 2 = k + 1.$$

□

13.4. Compression on a four-dimensional real kernel. In the prepared finite analytic families of Lemma 13.12, the first-order estimate gives ambient codimension at least four for smooth strata with four-dimensional real kernel. Their projective lifts have fibre $\mathbb{R}P^3$, leaving only the lower bound one for their codimension in the resolved incidence. The additional directions and second-order argument of Lemma 13.13 improve the ambient and lifted bounds to five and two, respectively.

Lemma 13.9. *Suppose $F = \ker L$, $\dim_{\mathbb{R}} F = 4$, and $\dim W_F = 4$. Put $A = W_F^\perp \subset \Lambda^2 F$. There are symplectic forms $\alpha, \beta \in A$ and a complex structure $J = \alpha^{-1}\beta$ on F such that*

$$\alpha \wedge \beta = 0, \quad \alpha \wedge \alpha = \beta \wedge \beta \neq 0, \quad J^2 = -I.$$

On a dense open subset of M , U_x is a J -complex line. There are arbitrarily close distinct points x, y in that subset for which the real pairing $\alpha|_{U_x \times U_y}$ is nondegenerate.

Proof. The plane A has no nonzero decomposable member. Indeed, if $v \wedge w \in A$, after taking $w \perp v$ the orthogonality condition would say $\langle w, Bv \rangle = 0$ for every $B \in W_F$. Equation (13.6) would force $w = 0$.

On $\Lambda^2 \mathbb{R}^4$, the wedge quadratic form has signature $(3, 3)$, and its null cone is the set of decomposable forms. Its restriction to the two-plane A has no nonzero null vector and is therefore definite. Choose an orthogonal basis for this restriction, and rescale its members to have equal nonzero square. Denote the resulting forms by α, β .

In a Darboux basis for α , write

$$\alpha = e^{12} + e^{34}, \quad \beta = b(e^{12} - e^{34}) + ce^{13} + de^{14} + ee^{23} + fe^{24}.$$

The condition on the wedge squares gives $-b^2 - cf + de = 1$. Regarding the two forms as skew matrices, multiplication gives

$$(\alpha^{-1}\beta)^2 = (b^2 + cf - de)I = -I.$$

Moreover $J^T \alpha = \alpha J = \beta$. It follows that $\Omega = \alpha - i\beta$ is a nondegenerate complex bilinear alternating form for this complex structure.

By (13.18), each U_x is isotropic for both α and β . It has dimension at most two, since α is symplectic. If $\dim U_x = 2$, then $U_x = U_x^{\perp \alpha}$, and $\beta(U_x, U_x) = \alpha(JU_x, U_x) = 0$ gives $JU_x = U_x$. The rank-two locus

is open. It is dense because a ball on which evaluation had rank at most one would contradict the argument in Lemma 13.8.

The complex line U_x cannot be constant on an open patch: a nonzero vector of F orthogonal to that fixed real plane would define a harmonic spinor vanishing on the patch. Unique continuation would make it zero. There are consequently arbitrarily close x, y in a rank-two patch with distinct complex lines U_x, U_y . In complex dimension two, a nondegenerate complex alternating form pairs distinct complex lines nontrivially. If its value in complex bases is $a + ib \neq 0$, the matrix of its real part on the corresponding real planes is

$$\begin{pmatrix} a & -b \\ -b & -a \end{pmatrix}, \quad \det = -(a^2 + b^2) \neq 0.$$

Since $\operatorname{Re} \Omega = \alpha$, this is the required pairing. $\square$

Lemma 13.10. *Under the hypotheses of Lemma 13.9, let P be the orthogonal projection onto F , and let G be the inverse of L on $F^\perp$, extended by zero on F . For the Schur map at L , there are smooth skew potentials R, S , supported in disjoint small balls, such that*

$$d\Phi(R) = d\Phi(S) = 0, \quad \langle \alpha, d^2\Phi(R, S) \rangle \neq 0. \quad (13.20)$$

Proof. Take the fixed decomposition $F \oplus F^\perp$, with H^1 as the domain on the second summand. Lemma 6.6 gives a real skew Schur matrix Φ on F , with the same kernel dimension as the full operator, on a coefficient neighborhood of L . Its derivatives are

$$d\Phi(R) = PRP, \quad d^2\Phi(R, S) = -P(RGS + SGR)P. \quad (13.21)$$

Indeed, both off-diagonal factors in that Schur formula vanish at $T = 0$; differentiating their product gives the two quadratic terms in (13.21).

The identities $LG = GL = I - P$, with P having a smooth kernel, imply $G \in \Psi^{-1}$, by the parametrix argument in the proof of Lemma 9.2. Its principal inverse symbol is, up to the Fourier factor, $a_g(\xi)/|\xi|_g^2$. In geodesic normal coordinates centred at the source point y , the Fourier transform of the frozen homogeneous symbol is the Euclidean Dirac kernel. Hence

$$G(x, y) = c_n d(x, y)^{1-n} a_g(v_{x,y}) \tau_{y,x} + O(d(x, y)^{2-n}), \quad c_n \neq 0. \quad (13.22)$$

Here $v_{x,y}$ is the terminal unit covector of the short geodesic from y to x , and $\tau_{y,x}$ is spin parallel transport along it. The sign convention is absorbed into c_n . The symbol after subtraction of its principal homogeneous part has order at most -2 . Lemma 2.4, with $m = 2$, bounds its kernel by $Cd(x, y)^{2-n}$. Freezing the principal coefficients at y changes the order -1 symbol by $O(d(x, y))$; the same lemma, with $m = 1$, gives the same kernel bound for this difference. The normal coordinates, parallel transport and coefficients depend smoothly on y in a smaller relatively compact patch, so these bounds are uniform there. The leading matrix in (13.22) is invertible, with inverse norm $|c_n|^{-1}d(x, y)^{n-1}$. The relative error is $O(d(x, y))$. A Neumann series therefore shows that $G(x, y)$ is invertible for all sufficiently close distinct x, y in this patch.

Choose such x, y as in Lemma 13.9. Put $E_z = \operatorname{im}(e_z) \subset \Sigma_{g,z}^{\mathbb{R}}$ for $z = x, y$. These images have rank two on small disjoint neighborhoods of the selected points. Since $G(x, y)$ is invertible, the map

$$E_y^\perp \longrightarrow E_x^\perp, \quad q \longmapsto (I - P_{E_x})G(x, y)q$$

has rank at least $r - 4 > 0$: the domain $E_y^\perp$ has codimension two, the matrix $G(x, y)$ is invertible, and projection onto $E_x^\perp$ has kernel of dimension two. Choose $p \in E_x^\perp, q \in E_y^\perp$ with

$$c = p^T G(x, y)q \neq 0.$$

The nondegenerate pairing of U_x, U_y gives $a \in E_x, b \in E_y$ for which

$$r_0 = e_x^* a, \quad s_0 = e_y^* b, \quad r_0^T \alpha s_0 \neq 0.$$

Extend a, p smoothly as sections of $E_z, E_z^\perp$ on the first neighborhood, and b, q on the second. The pointwise endomorphisms

$$R_z^0 = p_z a_z^T - a_z p_z^T, \quad S_z^0 = q_z b_z^T - b_z q_z^T$$

map the respective evaluation images into their orthogonal complements. Multiply them by nonnegative normalized smooth bumps $\rho_\varepsilon, \sigma_\varepsilon$ shrinking to x, y , with integrals one, and extend by zero. For the resulting potentials $R_\varepsilon, S_\varepsilon$,

$$e_z^* R_\varepsilon(z) e_z = e_z^* S_\varepsilon(z) e_z = 0 \quad \text{for every } z.$$

Hence $PR_\varepsilon P = PS_\varepsilon P = 0$ exactly.

The two supports are disjoint. Put $\mathcal{K}(z, w) = e_z^* R_z^0 G(z, w) S_w^0 e_w$ on their product. For the unit-mass nonnegative cutoffs,

$$\left\| \iint \rho_\varepsilon(z) \sigma_\varepsilon(w) \mathcal{K}(z, w) d\mu(z) d\mu(w) - \mathcal{K}(x, y) \right\| \leq \sup_{(z, w) \in \text{supp } \rho_\varepsilon \times \text{supp } \sigma_\varepsilon} \|\mathcal{K}(z, w) - \mathcal{K}(x, y)\|.$$

The right side tends to zero by continuity off the diagonal. Thus

$$PR_\varepsilon GS_\varepsilon P \longrightarrow e_x^* R_x^0 G(x, y) S_y^0 e_y = -c r_0 s_0^T. \quad (13.23)$$

The sign follows from $e_x^* R_x^0 = -r_0 p^T$ and $S_y^0 e_y = q s_0^T$. The transpose of the first mixed term is

$$(PR_\varepsilon GS_\varepsilon P)^T = -PS_\varepsilon GR_\varepsilon P,$$

because all three operators are skew-adjoint. Consequently (13.21) and (13.23) give

$$\langle \alpha, d^2 \Phi(R_\varepsilon, S_\varepsilon) \rangle \longrightarrow c r_0^T \alpha s_0 \neq 0.$$

Choose sufficiently small positive ε , and fix $R = R_\varepsilon, S = S_\varepsilon$. This proves (13.20). $\square$

13.5. Finite analytic families and avoidance. All finite coefficient families below are analytic in their coefficient vector. Their parameter curves are smooth and constant on endpoint collars.

Lemma 13.11. *Let $p_t, 0 \leq t \leq 1$, be a smooth coefficient path in $\mathcal{P}$, stationary on endpoint collars. There are finitely many smooth potential directions and a neighborhood of its image with the following properties. The selected potential directions give line-incidence surjectivity at every kernel line in this neighborhood. On every even-dimensional subspace F of every kernel there, their compression has rank at least four if $\dim F = 4$, and at least $\dim F + 1$ if $\dim F \geq 6$.*

The path can be approximated, with its endpoint collars fixed, by a smooth curve in a finite-dimensional family analytic in its coefficient vector, containing all selected potential directions. If $(p_t, [\psi_t])$ is an incidence path, its approximation can be corrected to an incidence path in that finite family, with the exact prescribed endpoints.

Proof. Uniform elliptic estimates along the compact path give

$$\|\psi\|_{H^1} \leq C \|\psi\|_{L^2} \quad (\psi \in \ker L_{p_t}).$$

They also give uniform estimates in every higher Sobolev norm. Kernel dimensions on a neighborhood of the path have a finite upper bound K : around each parameter choose a spectral contour enclosing zero and finitely many eigenvalues, preserve its finite rank on a neighborhood, and take a finite cover.

For $k \leq K$, consider all L^2 -orthonormal k -frames in kernels over the path. This set, with the parameter t included, is compact in the product L^2 topology. Indeed, after taking a subsequence $t_j \rightarrow t$, the uniform H^1 estimates and Rellich compactness give strong L^2 convergence of every frame vector and weak H^1 convergence. The equation passes to the limit in distributions, and the limit vectors remain orthonormal. They therefore form a frame in $\ker L_{p_t}$.

For each orthonormal pair v, w in a kernel, there is a potential R with $\langle w, Rv \rangle_{L^2} \neq 0$, by (13.6). A finite cover of the compact set of such pairs selects finitely many potential directions detecting

every pair. Their compression satisfies (13.6) at every kernel line: otherwise a unit vector in the orthogonal complement of its image would give an undetected orthonormal pair. For every even k , apply Lemma 13.8 to the subspace spanned by each k -frame. At that frame select a nonzero minor of the required size in the compression matrix of finitely many potential directions. Another finite cover selects potential directions for all such frames, and there are only finitely many $k \leq K$.

These choices persist near the compact coefficient path. If persistence failed in every neighborhood, there would be coefficients converging to the path and normalized failing kernel frames. Uniform H^1 estimates give a convergent subsequence as before. The limiting frame belongs to a subspace of a reference kernel, possibly a larger kernel. Its compressed entries are limits of the corresponding entries, since the spinors converge strongly in L^2 and the selected multiplication potential directions are bounded. Failure would contradict one of the chosen nonzero minors, or one of the chosen pairings.

For an incidence path, its tautological real line bundle over I is trivial. Choose a smooth L^2 -unit lift ψ_t . This lift is stationary on every collar where the projective incidence path is stationary. Lemma 10.2, with the selected potential directions, now gives the approximation and, in the incidence case, its relative correction in the same finite analytic family. The surjectivity required there is the one proved above. All selected compression inequalities persist on the chosen coefficient neighborhood. $\square$

Fix z_0 and put $F_0 = \ker L(z_0)$. In the decomposition $\mathcal{H} = F_0 \oplus F_0^\perp$, the complementary block is an isomorphism $H^1 \cap F_0^\perp \rightarrow L^2 \cap F_0^\perp$ and remains so near z_0 . Its inverse and its skew Schur matrix $\Phi(z)$ are real analytic, and $\ker L(z)$ is the graph of $\ker \Phi(z)$. Kernel dimensions are therefore specified by the finitely many analytic minors of Φ . By [22, Proposition 2.10 and Corollary 2.11], these rank loci admit a locally finite compatible real analytic stratification. Each constant-rank stratum has a smooth kernel bundle, whose projectivization is a smooth submanifold of the resolved incidence. A countable collection of these charts covers the second countable coefficient neighborhood. For each coefficient stratum and each lifted stratum, transversality is residual by [59, Chapter 3, §2, Theorem 2.1]. The countable intersection is residual as well. For interval paths whose endpoint collars avoid these strata, perturbations are supported off the collars.

Lemma 13.12. *Let $z \mapsto L(z)$, $z \in \mathcal{U} \subset \mathbb{R}^N$, be a finite analytic family prepared as in Lemma 13.11. A smooth constant- k -kernel stratum has codimension at least four for $k = 4$, and at least $k + 1$ for even $k \geq 6$. In the N -dimensional resolved incidence, its projective lift has codimension at least two when $k \geq 6$.*

Proof. At a point of the stratum, differentiate a smooth local basis $\psi_1, \dots, \psi_k$ of its kernel along a tangent vector v . Pairing

$$L \dot{\psi}_i + \dot{L}(v) \psi_i = 0$$

with ψ_j gives

$$\langle \psi_j, \dot{L}(v) \psi_i \rangle_{L^2} = 0 \quad (1 \leq i, j \leq k).$$

Thus the stratum tangent is contained in the kernel of the compressed derivative. Its rank is at least the rank of the selected potential compressions, which has the bounds in Lemma 13.11. This gives the ambient codimension assertions. The projective lift has fibre $\mathbb{R}P^{k-1}$; its dimension is therefore at most

$$N - (k + 1) + (k - 1) = N - 2.$$

The ambient resolved incidence has dimension N , by Lemma 13.2. $\square$

Lemma 13.13. *At a four-dimensional kernel there are a C^m -open coefficient neighborhood and finitely many smooth potential directions such that, in every finite analytic family containing those directions, each smooth constant-four-kernel stratum in that neighborhood has ambient codimension at least five.*

Proof. If the full potential compression has rank at least five, retain five independent compression directions. Their independence persists, and the tangent argument of Lemma 13.12 proves the assertion.

Suppose that the full potential compression has rank four. Retain four potential directions $B_1, \dots, B_4$ whose first compressions are independent, and also the potential directions R, S from Lemma 13.10. Take a fixed rank-four Schur chart on a C^m -open coefficient neighborhood of this operator, before restriction to any finite family. At its center the Schur target is $\mathfrak{so}(F)$, and α annihilates the potential derivative image. For a nearby coefficient z , put

$$C_i(z) = d\Phi_z(B_i), \quad E(z) = \text{span}\{C_1(z), \dots, C_4(z)\}.$$

The four $C_i(z)$ remain independent. Let $G_{ij}(z) = \langle C_i(z), C_j(z) \rangle$. This Gram matrix is invertible. For $V = R, S$, put

$$d_j^V(z) = \langle C_j(z), d\Phi_z(V) \rangle, \quad b^V(z) = G(z)^{-1} d^V(z).$$

Subtract these components:

$$\begin{aligned} \widehat{R}(z) &= R - \sum_{i=1}^4 b_i^R(z) B_i, & \sum_i b_i^R(z) C_i(z) &= P_{E(z)} d\Phi_z(R), \\ \widehat{S}(z) &= S - \sum_{i=1}^4 b_i^S(z) B_i, & \sum_i b_i^S(z) C_i(z) &= P_{E(z)} d\Phi_z(S). \end{aligned}$$

Identify the Schur targets in this chart with the fixed target at its center, and put $\widehat{\alpha}(z) = P_{E(z)^\perp} \alpha$. The inverse Gram matrix and these orthogonal projections depend smoothly on z . At the center, $b^R = b^S = 0$ and $\widehat{\alpha} = \alpha$. After shrinking the neighborhood,

$$\left\langle \widehat{\alpha}(z), d^2\Phi_z(\widehat{R}(z), \widehat{S}(z)) \right\rangle \neq 0. \quad (13.24)$$

When taking the derivative in this formula, the two displayed potential vectors are held fixed at the chosen z .

Now restrict to the selected finite analytic family, including all of its existing metric directions. At a four-kernel point its derivative $d\Phi_z$ has rank at least four. If the rank is at least five, the first-order codimension estimate applies. If the rank is four, its image is exactly $E(z)$. Only in this case do the adjusted vectors satisfy

$$d\Phi_z(\widehat{R}) = d\Phi_z(\widehat{S}) = 0, \quad \widehat{\alpha} \perp \text{im } d\Phi_z.$$

The annihilator here includes the derivatives in all metric directions of the selected family.

Assume a smooth constant-four-kernel stratum had codimension four at this point. In this rank-four chart the stratum lies in $\Phi^{-1}(0)$. Its tangent is contained in $\ker d\Phi_z$; equality of dimensions makes the two spaces equal. Both adjusted potential vectors are tangent to the stratum. For a local parametrization f of the stratum, choose tangent vectors u, v with $df(u) = \widehat{R}$, $df(v) = \widehat{S}$. Differentiating $\Phi \circ f = 0$ twice gives

$$d^2\Phi_z(\widehat{R}, \widehat{S}) + d\Phi_z(d^2f(u, v)) = 0.$$

Pairing with $\widehat{\alpha}$ annihilates the second term and contradicts (13.24). The codimension is therefore at least five. □

Proposition 13.14. *The simple incidence*

$$\mathcal{I}_2(\mathcal{P}) = \{(p, [\psi]) \in \mathcal{I}(\mathcal{P}) : \dim_{\mathbb{R}} \ker L_p = 2\}$$

is nonempty and path connected. A connecting path between prescribed simple incidence points can be chosen smooth and stationary on endpoint collars.

Proof. First let two simple incidence points be given. Proposition 13.7 gives a smooth incidence path between them. Apply Lemma 13.11, with exact endpoint collars, to put a nearby corrected incidence path into a finite analytic family with all the first-order potential directions.

In each finite spectral chart, partition the coefficient domain by the rank of the real-analytic reduced matrix. Choose a locally finite smooth stratification compatible with these rank loci, as in Lemma 13.12. For each stratum with even kernel dimension $k \geq 6$, Lemma 13.12 bounds the dimension of its projective lift by $N - 2$. A smooth map of an interval that is transverse to each of these lifted strata misses all of them. Relative finite-dimensional transversality gives an arbitrarily close such path, fixed on the simple endpoint collars. Thus the path can first be arranged to have kernel dimension at most four everywhere.

Let C_4 be the subset of its coefficient image where the kernel has dimension four. Upper semicontinuity of kernel dimension, and the absence of larger kernels, make C_4 closed in the compact path image. It is compact.

At every point of C_4 , apply Lemma 13.13. A finite subcover selects finitely many additional potential directions. Enlarge the existing finite coefficient span by these directions. The previously constructed path is still a curve in this enlarged analytic family, with the new coefficients zero. By upper semicontinuity, the entire compact path has an open ambient neighborhood O in which every kernel has dimension at most four. Let O_4 be the union of the finitely many neighborhoods on which the selected four-kernel estimates hold. Replace O by the open set

$$O \cap (\{p : \dim_{\mathbb{R}} \ker L_p < 4\} \cup O_4).$$

It still contains the whole path, and each of its four-kernel points lies in O_4 . Restrict the enlarged coefficient family to this neighborhood. Line-incidence surjectivity is preserved by adding potential directions.

Every four-kernel stratum now has codimension at least five, by Lemma 13.13. Its projective fibre has dimension three, so its lift has codimension at least two in the new resolved incidence manifold. A second relative perturbation of the interval misses all these lifted strata. The resulting incidence path has kernel dimension exactly two.

For nonemptiness, start with any incidence point from Proposition 13.7. In a finite transverse incidence chart, prepared with the first-order potential directions on all relevant kernel subspaces, the projective lifts of the strata with $k \geq 6$ have codimension at least two. They cannot fill an open neighborhood. Choose a nearby incidence point with kernel at most four. A two-dimensional kernel gives the required incidence point. If it has dimension four, add the potential directions of Lemma 13.13 and take a small neighborhood with no larger kernel. The lifted four-kernel strata again have codimension at least two and cannot fill that incidence neighborhood. A point outside them has a two-dimensional kernel. $\square$

13.6. The simple-kernel hypersurface and the ordinary real index. Write

$$Z_4 = \{p \in \mathcal{P} : \dim_{\mathbb{R}} \ker L_p \geq 4\}, \quad X = \mathcal{P} \setminus Z_4, \quad S = \{p \in X : \dim_{\mathbb{R}} \ker L_p = 2\}.$$

Lemma 13.15. *The space X is path connected. The subset S is a nonempty path-connected embedded hypersurface, closed in X . The complement $X \setminus S = \mathcal{P}^{\text{inv}}$ has at most two path components.*

Proof. The set X is open in the Fréchet coefficient space, by upper semicontinuity of kernel dimension. It is therefore locally path connected. Join two points of X by a smooth coefficient path by the global linear coordinates of $\mathcal{P}$. Make the path stationary on endpoint collars. Lemma 13.11 puts a nearby path, with these collars fixed, in a finite analytic family with the first-order compression bounds. Every stratum with kernel dimension at least four has ambient codimension at least four, by Lemma 13.12. Relative transversality of the coefficient path itself therefore avoids Z_4 . This proves path connectedness of X .

At $p \in S$, the Schur target on $F = \ker L_p \cong \mathbb{R}^2$ is one-dimensional:

$$\Phi(q) = f(q)J_0, \quad J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Equation (13.6) gives a smooth potential direction R with $df_p(R) \neq 0$. The scalar Banach implicit-function construction in Lemma 12.11, applied to f , gives a product chart on the smooth coefficient

space. Its wall is $f = 0$ and its invertible sides are $f > 0$ and $f < 0$, since the complementary block remains invertible. Both sides are locally path connected.

The set S is closed in X because noninvertibility is closed and X has no larger kernels. It is nonempty and path connected by Proposition 13.14: the projection from $\mathcal{I}_2(\mathcal{P})$ onto S is surjective.

The local wall factor in Lemma 12.11 is convex. Lemma 10.5 therefore applies to (X, S) and gives at most two path components of $X \setminus S$. $\square$

For a smooth common-domain path of real skew Fredholm operators with invertible endpoints, let

$$\text{sf}_2(L_t) \in KO_2 \cong \mathbb{Z}/2$$

be the unbounded real spectral flow. Its construction in [26, §6.2] is defined for gap-continuous paths and satisfies homotopy invariance, concatenation, and vanishing on invertible paths. It agrees with the $\mathbb{Z}/2$ -valued spectral flow of [30]; see [26, Theorem 5.5(2)]. The normalization in [26, §2.4.4, equation (2.20)] is

$$[(\mathbb{R}^2, J_0)] \mapsto \dim_{\mathbb{C}}(\mathbb{R}^2, J_0) \pmod{2} = 1 \in KO_2. \quad (13.25)$$

Thus a transverse real skew two-plane crossing has value one.

In the fixed comparisons, these paths and homotopies are continuous in $\mathcal{L}(H^1, L^2)$, and their graph norms are uniformly equivalent to the H^1 norm on compact parameter sets, by Section 2. They are therefore gap continuous [26, §6.1, equation (6.1)].

Proof of Theorem 13.1. Choose a point of the nonempty simple wall S . In the scalar Schur chart of Lemma 13.15, a small transverse potential curve has two invertible endpoints on the opposite local sides. Equivalently, take the rank-two spectral projection near this point and a smooth orthogonal identification of its real range. The cluster matrix is $b(t)J_0$, where $b(0) = 0$ and $b'(0) \neq 0$; the complementary operator remains invertible. Its spectral flow is therefore the two-plane value (13.25), namely one.

If an invertible path joined these two endpoints, its concatenation with the transverse curve would be a loop with mod-two spectral flow one. The loop contracts in the global linear coefficient coordinates of $\mathcal{P}$, with its invertible base point fixed. Homotopy invariance gives spectral flow zero, a contradiction. Thus the two local sides belong to different path components. Lemma 13.15 gives the opposite bound. There are exactly two components, and both are nonempty.

For an invertible base point p_* , two coefficient paths from p_* to p differ by a loop in the contractible coefficient space. Their spectral flows are equal. This defines (13.3). The invariant is constant on every invertible path, and the preceding local crossing shows that both values occur. Since there are exactly two components, the map is a bijection.

For $n \equiv 1 \pmod{8}$, the real skew family in (8.3) is precisely $L = i(D_g + V)$. Section 8.1 identifies its real spectral flow with the ordinary relative family index in $KO_{n+1} \cong KO_2$, with the two-plane normalization (8.1). For a path from $(g_*, 0)$ to (g, V) , its value is therefore $\delta_{g_*}(g, V)$, as asserted. $\square$

By Theorem R and Proposition 8.2, with $U = M$, the inclusion identifies the components of $\mathcal{P}_\mu^{\text{inv}}$ with those of $\mathcal{R}_\mu^{\text{inv}}(M)$ and preserves this ordinary index difference. Theorem 13.1 therefore proves the $n \equiv 1 \pmod{8}$ case of Theorem C, with two components.

14. COMPONENTS AND POSITIVE SCALAR CURVATURE

Proof of Theorem C. Since D_{g_*} is invertible, $\alpha(M) = 0$. For $n \equiv 3, 7 \pmod{8}$, Theorem 11.6 gives the bijection for invertible pairs. For $n \equiv 2, 4, 5, 6 \pmod{8}$ it is Theorem 11.11; for $n \equiv 0, 1 \pmod{8}$ it is given by Theorems 12.1 and 13.1, respectively. These cases exhaust (2.4). Proposition 8.2, with $U = M$, identifies the pair components with the metric components of the same density and preserves index difference. The component counts follow from KO_{n+1} . $\square$

Lemma 14.1. *Let M be closed, connected, and oriented, and let μ be a positive smooth density. Every positive-scalar-curvature metric h can be joined through positive-scalar-curvature metrics to a metric h_μ*

with $dv_{h_\mu} = \mu$. If M is spin with its spin structure fixed, this path preserves its ordinary invertible-Dirac component.

Proof. Set

$$a = \left(\frac{\int_M \mu}{\text{Vol}_h(M)} \right)^{2/n}, \quad h' = ah.$$

Positive constant scaling joins h to h' through metrics of positive scalar curvature. The positive top forms $dv_{h'}$ and μ have equal integrals. Apply Moser's theorem [69, Theorem 2, Section 4, pp. 292–293] to $(1-t)\mu + t dv_{h'}$. It gives an isotopy ϕ_t , starting at the identity, with $\phi_1^* dv_{h'} = \mu$. The path $\phi_t^* h'$ retains positive scalar curvature and ends at the required h_μ . In the spin case the isotopy has the spin lift continued from the identity; Dirac naturality and constant scaling preserve invertibility throughout both paths. $\square$

Proof of Corollary 1.1. Choose $g_+ \in \mathcal{R}^+(M)$. For even $n \geq 6$, [24, Theorem D], in homotopy degree zero, gives the surjectivity of

$$\delta_{g_+} : \pi_0 \mathcal{R}^+(M) \longrightarrow KO_{n+1}.$$

For $n = 2m + 1 \geq 7$, [24, Theorem C] gives a map

$$\pi_2 MT\text{Spin}(2m) \longrightarrow \pi_0 \mathcal{R}^+(M)$$

whose composite with δ_{g_+} is the Thom homomorphism to KO_{2m+2} . By [24, Theorem 5.3.1], with $d = 2m$ and $k = 2 \leq 2m + 1$, this homomorphism is onto $ko_{2m+2} \cong KO_{n+1}$. For $n = 5$ the target is $KO_6 = 0$.

Given $g \in \mathcal{R}^{\text{inv}}(M)$, choose $h \in \mathcal{R}^+(M)$ with

$$\delta_{g_+}(h) = \delta_{g_+}(g).$$

The ordinary index differences in the two spaces agree. Theorem C and Lemma 8.1 identify the invertible components, so g and h lie in the same one. If g has density μ , apply Lemma 14.1 to h . It yields a positive-scalar-curvature metric h_μ of density μ in that same invertible component. Apply the deformation retraction of Lemma 8.1 to a connecting path from g to h_μ . It fixes both endpoints and produces a path in $\mathcal{R}_\mu^{\text{inv}}(M)$, which proves the fixed-density assertion.

If M is simply connected and $n \geq 5$, then $\alpha(M) = 0$ implies $\mathcal{R}^+(M) \neq \emptyset$ by [78, Theorem A]. If $\alpha(M) \neq 0$, both $\mathcal{R}_\mu^+(M)$ and $\mathcal{R}_\mu^{\text{inv}}(M)$ are empty. $\square$

15. INVERTIBLE BOUNDARY EXTENSIONS

15.1. Metric cancellation and the relative group.

Proof of Theorem I. An invertible Dirac operator has zero real index. Conversely, suppose that $\alpha(W, h) = 0$, and choose a nonempty open set $U_0 \Subset U$. Proposition 9.1 gives a compatible potential V supported in U_0 for which $D_g + V$ is invertible. Corollary 5.2, with the same open set, gives an invertible metric $\hat{g}$ with $dv_{\hat{g}} = dv_g$ and $\hat{g} = g$ on $W \setminus U_0$.

Put $H = \log(g^{-1}\hat{g})$. Then H is smooth and g -self-adjoint, and

$$\text{supp } H \subset \overline{U_0} \Subset U, \quad \text{tr } H = \log \det(g^{-1}\hat{g}) = 0.$$

Thus $g_t = g(e^{tH}, \cdot)$ has density dv_g and agrees with g outside U_0 . With reference metric g in (2.8), put

$$L_t = U_{g_t}^{-1} D_{g_t} U_{g_t} : H^1 \longrightarrow L^2.$$

The frame map is $e^{-tH/2}$. The local comparison formula [56, Theorem 2.3.6], whose coefficients depend on this map, its inverse and their first spatial derivatives, gives a holomorphic extension $L_z : H^1 \rightarrow L^2$ near $[0, 1]$. Here holomorphy is in operator norm, since the varying coefficients

have compact support. Each L_t is Fredholm by ellipticity and the fixed invertible ends. Openness of Fredholmness gives a connected complex neighborhood Ω of $[0, 1]$ on which every L_z is Fredholm. The operator L_1 is invertible and $L_1^{-1} : L^2 \rightarrow H^1$ is bounded. Hence

$$A(z) = L_z L_1^{-1} \in \mathcal{L}(L^2), \quad A(1) = I,$$

is a holomorphic Fredholm family on Ω . By the analytic Fredholm theorem [79, Theorem 1.1, p. 2], its noninvertible parameters form a discrete subset of Ω . Since $0 \in \Omega$, there is an $\varepsilon > 0$ such that D_{g_t} is invertible for $0 < t < \varepsilon$. Moreover, $g_t \rightarrow g$ in C^∞ . $\square$

Corollary 15.1. *For $n \geq 3$, $\theta_n : R_n^{\text{inv}} \rightarrow KO_n$ is an isomorphism.*

Proof. Let $[M, h] \in \ker \theta_n$. The empty representative is zero. Otherwise, attach spin one-handles to $M \times [0, 1]$ along interior discs in distinct components of $M \times \{1\}$, with framings for which the spin structure extends. The upper boundary is connected after finitely many attachments; denote it by N . The side boundary remains $\partial M \times [0, 1]$, with metric $h + dt^2$. If $\partial M \neq \emptyset$, its cylindrical completion has Dirac square, on product spinors,

$$-\partial_t^2 + D_h^2 \geq \lambda_h^2 > 0, \quad \lambda_h = \text{dist}(0, \text{spec } D_h).$$

If $\partial M = \emptyset$, the side boundary is empty. The handle trace and this side boundary therefore give $[M, h] = [N, h]$ by the defining relation [38, Section 5.2]. By [38, Section 5.4], $\alpha(N, h) = \theta_n([N, h]) = 0$. Theorem I gives an invertible product extension over N , so $[N, h] = 0$ by [38, Corollary 5.6]. Thus θ_n is injective. Its surjectivity is proved in [38, Section 5.4]. $\square$

16. SPECTRAL MATRICES ON BERGER PRODUCTS

16.1. One middle-weight projector. Let $k \geq 1$ be an integer, and put $j = k - \frac{1}{2}$. Let V_j be the irreducible unitary $SU(2)$ -module of complex dimension $2k$, with action ρ . Its orthonormal weight basis is denoted by v_m , $-j \leq m \leq j$. Let P_m be orthogonal projection onto $\mathbb{C}v_m$.

Lemma 16.1. *For $k \geq 1$ and every nonempty open $O \subset SU(2)$,*

$$\text{span}_{\mathbb{R}}\{\rho(x)P_{1/2}\rho(x)^{-1} : x \in O\} = \text{Herm}_{\mathbb{C}}(V_j). \quad (16.1)$$

The same equality holds with $P_{-1/2}$.

Proof. The Clebsch–Gordan decomposition [47, Exercise 11.11, p. 151] is

$$\text{End}_{\mathbb{C}}(V_j) = \bigoplus_{\ell=0}^{2k-1} V_\ell.$$

Each summand occurs once and has a one-dimensional weight-zero subspace. Conjugate transpose preserves each summand, so that subspace contains a nonzero Hermitian vector T . Write $Tv_m = q_m v_m$, with $q_m \in \mathbb{R}$. In $V_\ell = \text{Sym}^{2\ell} \mathbb{C}^2$, the weight-zero monomial is $z^\ell w^\ell$. The Weyl transformation $(z, w) \mapsto (w, -z)$ gives

$$q_{-m} = (-1)^\ell q_m. \quad (16.2)$$

Let J_1, J_2, J_3 be the angular-momentum matrices, normalized by $[J_1, J_2] = iJ_3$ and cyclic permutations, with $J_3 v_m = m v_m$. The Casimir identity on V_ℓ [8, Section 2.1, equations (2.4)–(2.5)] is $\sum_a [J_a, [J_a, T]] = \ell(\ell+1)T$. Its diagonal entries are

$$\begin{aligned} \ell(\ell+1)q_m &= (j-m)(j+m+1)(q_m - q_{m+1}) \\ &\quad + (j+m)(j-m+1)(q_m - q_{m-1}). \end{aligned} \quad (16.3)$$

Terms with indices outside $[-j, j]$ are omitted. If $\text{tr}(P_{1/2}T) = 0$, then $q_{1/2} = q_{-1/2} = 0$ by (16.2). For $j > 1/2$, substitute successively $m = 1/2, 3/2, \dots, j-1$ in (16.3). The forward coefficient is positive, and every positive entry vanishes. Reflection gives every negative entry. For $j = 1/2$ the first two entries are all the entries. Either case contradicts $T \neq 0$.

Thus $P_{1/2}$ has nonzero projection onto every irreducible summand. Since the decomposition is multiplicity free, [34, Section 3, equation (13)] implies that its orbit has complex span $\text{End}_{\mathbb{C}}(V_j)$. Every orbit element is Hermitian, so its real span is $\text{Herm}_{\mathbb{C}}(V_j)$. If a Hermitian endomorphism annihilates the orbit for $x \in O$, its pairing with that orbit is a real-analytic function on connected $SU(2)$ vanishing on O . It therefore vanishes everywhere, and the endomorphism is zero. The Weyl element w exchanges $P_{1/2}$ and $P_{-1/2}$; applying the preceding equality to Ow gives the assertion for $P_{-1/2}$. $\square$

16.2. The Berger kernel and its evaluation. Choose left-invariant fields X_1, X_2, X_3 with $[X_1, X_2] = 2X_3$ and cyclic permutations. Let ω_i be the dual coframe. For $t > 0$, put

$$g_t = t^2 \omega_1^2 + \omega_2^2 + \omega_3^2. \quad (16.4)$$

In the left-invariant spin frame take $c_1 = i\sigma_3$, $c_2 = i\sigma_1$, and $c_3 = i\sigma_2$, so that $c_1 c_2 c_3 = I$. Hitchin's formulas [60, Propositions 3.1–3.2], in these conventions, give

$$D_t = t^{-1} c_1 X_1 + c_2 X_2 + c_3 X_3 + (t/2 + t^{-1})I, \quad (16.5)$$

and, besides the positive eigenvalues $t/2 + p/t$ with $p \geq 1$,

$$\frac{t}{2} \pm \sqrt{4pq + \frac{(p-q)^2}{t^2}}, \quad p, q \geq 1. \quad (16.6)$$

In the Peter–Weyl summand $V_j^* \otimes V_j \otimes \mathbb{C}^2$, the first factor is the multiplicity space. Take $d\rho(X_1) = -2iJ_3$, $d\rho(X_2) = -2iJ_1$, and $d\rho(X_3) = -2iJ_2$. On the pair

$$v_m \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_{m+1} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad -j \leq m < j,$$

the operator matrix is

$$(t/2 + t^{-1})I + \begin{pmatrix} 2m/t & 2\sqrt{(j-m)(j+m+1)} \\ 2\sqrt{(j-m)(j+m+1)} & -2(m+1)/t \end{pmatrix}. \quad (16.7)$$

This block corresponds to $p = j + m + 1$ and $q = j - m$ in (16.6).

Lemma 16.2. *If k is prime, $k \equiv 3 \pmod{4}$, and $t = 4k$, then*

$$\ker D_t \cong V_{k-1/2}^*, \quad \dim_{\mathbb{C}} \ker D_t = 2k.$$

In the fixed spin frame, evaluation followed by projection onto either spinor component has adjoint square equal to a positive constant times the orbit of $P_{1/2}$ in the contragredient representation.

Proof. For $t = 4k$ with k prime and $k \equiv 3 \pmod{4}$, the only zero in (16.6) has $p = q = k$; this is the observation of Chowla recorded in [60, p. 32, Remarks (1)(a)].

Thus $j = k - 1/2$ and $m = -1/2$. The block (16.7) is $\begin{pmatrix} t/2 & 2k \\ 2k & t/2 \end{pmatrix}$, whose zero eigenvector at $t = 4k$ is $(1, -1)/\sqrt{2}$. For $\ell \in V_j^*$ the corresponding spinor is

$$\Psi_{\ell}(x) = \sqrt{\frac{k}{\text{Vol}(g_t)}} \begin{pmatrix} \ell(\rho(x)v_{-1/2}) \\ -\ell(\rho(x)v_{1/2}) \end{pmatrix}. \quad (16.8)$$

For Haar measure dx of total mass one, Schur orthogonality [34, Section 3, equation (9)] gives

$$\int_{SU(2)} \rho(x) P_m \rho(x)^* dx = \frac{I}{2k}.$$

Since $dv_{g_t} = \text{Vol}(g_t) dx$,

$$\int_{SU(2)} |\ell(\rho(x)v_m)|^2 dv_{g_t} = \frac{\text{Vol}(g_t)}{2k} \|\ell\|^2.$$

Thus (16.8) is an L^2 isometry on V_j^* . Its first-row adjoint square is

$$\frac{k}{\text{Vol}(g_t)} \bar{\rho}(x) P_{-1/2} \bar{\rho}(x)^* \quad (16.9)$$

in dual coordinates. The second row has $P_{1/2}$. The contragredient representation is unitarily equivalent to V_j , and Lemma 16.1 applies to both. $\square$

16.3. Flat factors and compatible compression. In the rest of this section, k is prime and $k \equiv 3 \pmod{4}$. Let $n \geq 3$, and put $d = n - 3$. Give $T^d = \mathbb{R}^d / (2\pi\mathbb{Z})^d$ its flat metric and its spin structure with constant spinors, and set

$$X_{n,k} = S_{4k}^3 \times T^{n-3}. \quad (16.10)$$

Hitchin gives the product formula for harmonic spinors [60, p. 12, Remark (4)]. In the product spinor identification, with an extra factor $\mathbb{C}^2$ when d is odd,

$$D_{X_{n,k}}^2 = D_{S_{4k}^3}^2 \otimes I + I \otimes D_{T^d}^2, \quad (16.11)$$

where identity on the extra factor is understood. The torus kernel consists of parallel spinors. Set

$$W_n = \begin{cases} \ker D_{T^d}, & d \text{ even}, \\ \ker D_{T^d} \otimes \mathbb{C}^2, & d \text{ odd}. \end{cases}$$

For $d = 0$, take $T^0 = \{\text{pt}\}$, $\text{Vol}(T^0) = 1$, $D_{T^0} = 0$, and $W_3 = \mathbb{C}$. The product formula and Lemma 16.2 give

$$E_{n,k} = \ker D_{X_{n,k}} \cong V_{k-1/2}^* \otimes W_n, \quad \dim_{\mathbb{C}} W_n = 2^{\lfloor n/2 \rfloor - 1}. \quad (16.12)$$

Evaluation identifies the parallel torus spinors with the full torus spinor fibre. If the parallel sections have their L^2 norm, multiplication of evaluation by $\text{Vol}(T^d)^{1/2}$ makes this identification unitary. Thus in the product spin frame the spinor fibre is $\mathbb{C}^2 \otimes W_n$. If Q is projection onto the first coordinate line in $\mathbb{C}^2$, then $Q \otimes T$ is a Hermitian fibre endomorphism for every $T \in \text{Herm}_{\mathbb{C}}(W_n)$. The product conventions are also given in [36, Section 3].

Theorem 16.3. *For every nonempty open $U \subset X_{n,k}$, the map*

$$C_c^\infty(U; \mathcal{A}_n) \longrightarrow \mathcal{A}_n(E_{n,k}), \quad V \longmapsto P_E V|_E \quad (16.13)$$

is onto. It has a finite-dimensional linear right inverse, and it yields a full potential section supported in U .

Proof. Put $E = E_{n,k}$ and $j = k - 1/2$. For unrestricted complex Hermitian multiplication endomorphisms, Lemma 6.5 identifies the compression image with the real span of $e(x, y)^* Q' e(x, y)$, where e is evaluation and Q' ranges over the Hermitian fibre endomorphisms. Choose a nonempty product box $U_3 \times U_T \subseteq U$. Equation (16.9) and the torus normalization give, for $x \in U_3$, $y \in U_T$, and $T \in \text{Herm}_{\mathbb{C}}(W_n)$,

$$e(x, y)^* (Q \otimes T) e(x, y) = \frac{k}{\text{Vol}(g_{4k}) \text{Vol}(T^d)} \bar{\rho}(x) P_{-1/2} \bar{\rho}(x)^* \otimes T.$$

By Lemma 16.1, these matrices span

$$\text{Herm}_{\mathbb{C}}(V_j^*) \otimes_{\mathbb{R}} \text{Herm}_{\mathbb{C}}(W_n) \cong \text{Herm}_{\mathbb{C}}(E).$$

Thus unrestricted compression is onto.

On Hermitian endomorphisms put $\mathcal{J}(A) = JAJ^{-1}$ and, in even dimension, $\mathcal{G}(A) = \Gamma A \Gamma$. They are commuting real-linear involutions: $J^2 = \pm I$, $\Gamma^2 = I$, and $J\Gamma = \pm \Gamma J$. The projection onto (2.3) is

$$\Pi = \frac{I + \epsilon_n \mathcal{J}}{2} \quad (n \text{ odd}), \quad \Pi = \frac{I + \epsilon_n \mathcal{J}}{2} \frac{I - \mathcal{G}}{2} \quad (n \text{ even}).$$

Since P_E commutes with J and, in even dimension, with Γ , compression intertwines these projections on the bundle and on E . For $A \in \mathcal{A}_n(E)$, an unrestricted preimage V therefore gives a compatible preimage ΠV with the same support.

Choose compatible preimages $V_1, \dots, V_N$ of a real basis of $\mathcal{A}_n(E)$. Their supports lie in one compact subset of U , and they define a linear right inverse s of compression. Choose a circle centered at zero isolating E . For A near zero, the enclosed spectral projection of $D_{X_{n,k}} + s(A)$ commutes with J and, in even dimension, with Γ . The polar frame of Lemma 6.1 therefore preserves these structures. Let $F(A) \in \mathcal{A}_n(E)$ be the complete cluster matrix in that frame. Then $F(0) = 0$ and $dF_0 = I$ by Lemma 6.4. The inverse function theorem [46, Theorem 11.22 and Corollary 11.35] gives the full potential section $A \mapsto s(F^{-1}(A))$ near zero. After shrinking its domain, the complementary spectrum remains separated from zero. $\square$

Corollary 16.4. *In every dimension $n \geq 3$, the manifold $S^3 \times T^{n-3}$ admits full metric sections of arbitrarily large balanced zero-module size. The metric variations may have fixed pointwise density and support in any prescribed nonempty open set, relative to the resulting zero-centered metric. Every smaller balanced module occurs after fixing an invertible complementary block.*

Proof. Apply Proposition 6.3 to Theorem 16.3. There are arbitrarily large primes $3 \pmod{4}$ [6, Theorem 7.1, p. 147]. The module sizes in (16.12) therefore tend to infinity.

For even n , the grading on the extra $\mathbb{C}^2$ factor gives equal chiral ranks. In residues one and two, the skew sizes are respectively

$$k 2^{\lfloor n/2 \rfloor}, \quad k 2^{\lfloor n/2 \rfloor - 1},$$

both are even, since the least dimensions in these residues are nine and ten. The other cases of (2.4) impose no further restriction. Hence $E_{n,k}$ is balanced.

For a smaller balanced module choose a structural orthogonal decomposition $E_{n,k} = E' \oplus E''$. Its complement is balanced: in the chiral cases subtract equal ranks, and in the skew cases subtract even sizes. Fix a sufficiently small invertible $B \in \mathcal{A}_n(E'')$ and restrict the full metric section to matrices $A \oplus B$, with $A \in \mathcal{A}_n(E')$. Its value at $0 \oplus B$ has zero module E' . Choose a spectral radius smaller than $\text{dist}(0, \text{spec } B)$ and the original exterior gap, and restrict to $\|A\|$ below that radius. The central cluster is then exactly the retained block. The density and the metric outside U remain fixed relative to this new central metric. $\square$

17. KERNEL DIMENSIONS AND THE REAL INDEX

The parameter manifolds below are compact. Vector bundles carry fixed smooth Euclidean, Hermitian, or quaternionic Hermitian metrics. Matrix bundles have the associated Hilbert–Schmidt inner products, and direct sums have the product inner product. Write $|\cdot|_{\text{fib}}$ for these fibre norms. The disk bundle $D_\eta(N) = \{(b, A) : |A|_{\text{fib}} \leq \eta\}$, with $\eta > 0$, is a smooth manifold with boundary. The norm of a section is its supremum fibre norm.

Operator norms between orthogonal summands are denoted by $\|\cdot\|_{\text{op}}$. For each residual matrix T , with components $(T_1, \dots, T_s)$, finite rank gives a constant $C_{\text{fib}} \geq 1$, uniform over the base, such that

$$|(T_1, \dots, T_s)|_{\text{fib}} \leq C_{\text{fib}} \|T\|_{\text{op}}.$$

For an individual matrix, the Hilbert–Schmidt norm dominates the operator norm.

Lemma 17.1. *Let B be a compact smooth manifold without boundary, and let $\eta > 0$. Let $N, E \rightarrow B$ be real vector bundles such that $\text{rank } E = \dim B$. Suppose E admits a smooth section with exactly one transverse zero. On $D_\eta(N)$ let*

$$s_0(b, A) = (A, 0)$$

be the section of the pullback of $N \oplus E$. Every continuous section s satisfying $\|s - s_0\| < \eta/2$ has a zero in the interior of the disk bundle.

Proof. For $\pi : N \rightarrow B$, let $\mathcal{E} = \pi^*(N \oplus E)$ over the full total space N . Its rank equals $\dim D_\eta(N)$. If σ is the section in the hypothesis, then

$$s_\varepsilon(b, A) = (A, \varepsilon \sigma(b)), \quad \varepsilon > 0,$$

has exactly one transverse zero. Its derivative normal to the zero section is $\text{diag}(I, \varepsilon d\sigma)$.

Suppose that s has no zero. Compactness permits a smooth approximation s' with no zero and with $\|s' - s_0\| < \eta/2$. In the homotopy

$$H_t = (1 - t)s_\varepsilon + ts',$$

the N -component on $\partial D_\eta(N)$ has norm at least $\eta - \|s' - s_0\| > \eta/2$. Round the corners of $D_\eta(N) \times [0, 1]$ in a collar on which H has no zero. The boundary map into the full total space of $\mathcal{E}$ is transverse to its closed zero section and meets it exactly once. Its mod-two intersection number is zero by the Boundary Theorem [53, Chapter II, Section 4, p. 80], a contradiction. Finally, $|s| \geq \eta - \|s - s_0\| > \eta/2$ on the boundary. $\square$

17.1. Integer indices. Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{H}$, of real dimension $d_{\mathbb{F}}$. Suppose the fixed index-minimal module has chiral parts $\mathbb{F}^a$ and 0 , with $a \geq 0$, in domain and range order. If the nonzero minimal chiral part lies in the range, take adjoints throughout. Fix $c > 0$ and $0 < \eta < c/4$. To add $r \geq 1$ zero modes in both chiralities, set

$$B = \text{Gr}_r(\mathbb{F}^{a+r}), \quad L \rightarrow B \tag{17.1}$$

with its tautological bundle. The auxiliary domain and range are both the fixed space $\mathbb{F}^{a+r}$. For $A \in \text{End}_{\mathbb{F}}(L)$ define

$$Z_{L,A} = cP_{L^\perp} + AP_L, \quad |A|_{\text{fib}} \leq \eta < c/4. \tag{17.2}$$

Adjoin the fixed domain summand $\mathbb{F}^a$. The complementary block maps $L^\perp$ isomorphically to $L^\perp$. Its elimination leaves a section of

$$\text{Hom}_{\mathbb{F}}(L \oplus \mathbb{F}^a, L) = \text{End}_{\mathbb{F}}(L) \oplus \text{Hom}_{\mathbb{F}}(\mathbb{F}^a, L) \tag{17.3}$$

close to $(A, 0)$.

Proposition 17.2. *Every sufficiently small continuous family of $\mathbb{F}$ -linear perturbations of (17.2), with the fixed domain summand adjoined, has a parameter at which the full chiral kernel and cokernel have dimensions $r + a$ and r over $\mathbb{F}$.*

Proof. Put $L_0 = (\text{im } B_0)^\perp$, where $B_0 : \mathbb{F}^a \rightarrow \mathbb{F}^{a+r}$ is injective. The bundle $E = \text{Hom}_{\mathbb{F}}(\mathbb{F}^a, L)$ has real rank $d_{\mathbb{F}}ar = \dim_{\mathbb{R}} B$. Its section $s(L) = P_L B_0$ vanishes exactly at L_0 , because $P_L B_0 = 0$ is equivalent to $L \subset (\text{im } B_0)^\perp$, and both spaces have dimension r .

Write $L = \text{graph}(X)$, with $X : L_0 \rightarrow \text{im } B_0$, and let $j_X = \begin{pmatrix} I \\ X \end{pmatrix} (I + X^* X)^{-1/2}$. This is an isometric frame of L . In this frame,

$$j_X^* B_0 = (I + X^* X)^{-1/2} X^* B_0.$$

Its derivative at zero is $X \mapsto X^* B_0$. Since $B_0 : \mathbb{F}^a \rightarrow \text{im } B_0$ is an isomorphism, this is a real-linear isomorphism onto $\text{Hom}_{\mathbb{F}}(\mathbb{F}^a, L_0)$. Thus s has one transverse zero.

Let δ bound the full matrix perturbation in operator norm. Choose

$$\delta < c/2, \quad C_{\text{fib}} \left(\delta + \frac{2\delta^2}{c} \right) < \eta/2.$$

Lemma 6.6 then gives a section of (17.3) whose distance from $(A, 0)$ is less than $\eta/2$ in the chosen smooth fibre norm. Lemma 17.1, with $N = \text{End}_{\mathbb{F}}(L)$ and the bundle E above, gives a zero of this section. At that point its domain has dimension $r + a$ and its range dimension r . The complementary map is invertible, so these are exactly the chiral kernel and cokernel dimensions. This includes $a = 0$: then B is a point and $E = 0$. $\square$

In residue zero $a = |\widehat{A}(M)|$, and the resulting complex nullity is $|\widehat{A}(M)| + 2r$. In residue four $a = |\widehat{A}(M)|/2$, and it is $|\widehat{A}(M)| + 4r$.

17.2. The nonzero index in residue one. Fix $r \geq 1$, $c > 0$, and $0 < \eta < c/4$. The fixed index-minimal module is $\mathbb{R}$. Let $B = \mathbb{C}P^r$ parameterize the complex hyperplanes $L \subset \mathbb{C}^{r+1}$. Regard $\mathbb{C}^{r+1}$ as a real space of dimension $2r + 2$. On the real two-plane $L^\perp$ place cJ_0 , where J_0 is multiplication by i ; on $L_{\mathbb{R}}$ place $A \in \mathfrak{so}(L_{\mathbb{R}})$ with $|A|_{\text{fib}} \leq \eta < c/4$. After adjoining this real line, the skew Schur complement belongs to

$$\mathfrak{so}(L_{\mathbb{R}} \oplus \mathbb{R}) = \mathfrak{so}(L_{\mathbb{R}}) \oplus L_{\mathbb{R}}. \quad (17.4)$$

The last bundle has rank $2r = \dim_{\mathbb{R}} \mathbb{C}P^r$. For a fixed nonzero $v \in \mathbb{C}^{r+1}$, the section $L \mapsto P_L v$ has the single transverse zero $L = v^\perp$. For $L_0 = v^\perp$ and $X \in \text{Hom}_{\mathbb{C}}(L_0, \mathbb{C}v)$, the graph formula gives the derivative $X \mapsto X^*v$. Its inverse sends $w \in L_0$ to $X_w(u) = v\langle w, u \rangle / |v|^2$, with the Hermitian product linear in u . It is therefore a real-linear isomorphism.

The complementary block has inverse norm $1/c$. For a structural perturbation satisfying the bounds in the proof of Proposition 17.2, Lemmas 6.6 and 17.1, with $N = \mathfrak{so}(L_{\mathbb{R}})$ and $E = L_{\mathbb{R}}$, give a parameter where the residual section vanishes. At this parameter the residual real skew matrix is zero on a space of dimension $2r + 1$. Multiplication by i returns the self-adjoint form. The complex nullity is exactly $2r + 1$.

17.3. The nonzero index in residue two. Fix $r \geq 1$, $c > 0$, and $0 < \eta < c/4$. The fixed index-minimal module has chiral parts $\mathbb{C}$ and $\mathbb{C}$, exchanged by charge conjugation. Let $B = \mathbb{H}P^r$ parameterize quaternionic hyperplanes $L \subset \mathbb{H}^{r+1}$. As a complex space L has dimension $2r$. Let K be the standard antiunitary of the ambient space $\mathbb{H}^{r+1}$, so that $K^2 = -I$. On the complementary quaternionic line take $c\omega$, where

$$\omega(u, v) = \langle Ku, v \rangle_{\mathbb{C}}.$$

Since K is antiunitary and $K^2 = -I$, this form is complex bilinear, skew, and has singular values one. Its restriction to $L^\perp$ is therefore invertible with inverse norm $1/c$ after multiplication by c . On L place a complex skew form A with $|A|_{\text{fib}} \leq \eta < c/4$.

After adjoining the fixed complex line and eliminating the complementary complex two-plane, the residual bundle is

$$\Lambda_{\mathbb{C}}^2(L \oplus \mathbb{C})^* = \Lambda_{\mathbb{C}}^2 L^* \oplus L^*. \quad (17.5)$$

Its last summand has real rank $4r = \dim_{\mathbb{R}} \mathbb{H}P^r$. Fix a nonzero complex linear functional $\ell(w) = \langle v, w \rangle_{\mathbb{C}}$ on $\mathbb{H}^{r+1}$. The section $L \mapsto \ell|_L$ vanishes exactly at

$$L_0 = \text{span}_{\mathbb{C}}\{v, Kv\}^\perp.$$

Indeed quaternionic invariance of L makes orthogonality to v equivalent to orthogonality to both v and Kv . In graph coordinates its derivative is

$$X \mapsto \ell \circ X, \quad X \in \text{Hom}_{\mathbb{H}}(L_0, \mathbb{H}v).$$

If $\ell \circ X = 0$, then for every $w \in L_0$ both $\ell(Xw)$ and $\ell(KXw) = \ell(XKw)$ vanish. In the complex basis (v, Kv) of $\mathbb{H}v$, these two equalities make both components of Xw zero. Thus $X = 0$. Both spaces have real dimension $4r$, so this derivative is an isomorphism.

For a structural perturbation satisfying the bounds in the proof of Proposition 17.2, Lemma 6.6 gives a residual section within $\eta/2$ of $(A, 0)$. Lemma 17.1, with $N = \Lambda_{\mathbb{C}}^2 L^*$ and $E = L^*$, gives a zero Schur complement of complex size $2r + 1$. Both chiral kernels have dimension $2r + 1$, and the total complex nullity is $4r + 2$.

17.4. Minimal kernel dimensions. If the fixed index-minimal module is zero, let E_0 be a balanced zero module of the prescribed dimension in (2.4). On the ball in $\mathcal{A}_n(E_0)$ take the reference matrix A . Lemma 17.1, with B a point, $N = \mathcal{A}_n(E_0)$, and $E = 0$, gives a zero after every sufficiently small perturbation. A fixed invertible complement may be eliminated by Lemma 6.6.

If the prescribed nullity is the index minimum, no auxiliary zero block is needed. After elimination of the invertible complement, the residual matrix space is respectively $\text{Hom}_{\mathbb{F}}(\mathbb{F}^a, 0)$ or its adjoint,

$\mathfrak{so}(1)$, $\text{Skew}_1(\mathbb{C})$, or the zero space. Each is $\{0\}$. The kernel therefore has exactly the prescribed dimension. This includes dimension zero.

Proposition 17.3. *For each admissible nullity in (1.9), an auxiliary balanced matrix family can be chosen on a compact disk bundle so that, after adjoining a fixed index-minimal module, every sufficiently small structural perturbation has a parameter with exactly that nullity. Finitely many additional independent matrix equations can be imposed simultaneously if their reference map is the identity on additional trivial disk factors.*

Proof. The preceding constructions give every nullity in (1.9). For additional equations, replace N in Lemma 17.1 by $N \oplus \underline{V}$, where V is the finite-dimensional real space in which these equations take values. On the disk bundle for the product fibre metric, the reference section is $(A, X, 0)$. Lemma 17.1 gives a common zero after every sufficiently small perturbation. The complementary inverse bounds and the fibre norm comparison constant are fixed before the perturbation size is chosen. $\square$

18. SPECTRAL SUBSPACES UNDER SURGERY

Compact-family spectral surgery is given in [72, Theorem 4.24], extending [19] and [36, Theorem 6, third arXiv version]. Let $b \mapsto g_b$ be a compact C^2 -continuous family of smooth metrics on a closed spin manifold, and fix a finite spectral band whose endpoints avoid every spectrum. A fixed spin surgery of codimension at least three admits a C^2 -continuous family of smooth metrics for which the eigenvalues in that band, counted with multiplicity, are uniformly arbitrarily close to the original ones. The band ranks are equal, and the metrics agree outside any prescribed neighborhood of the surgery sphere. The smooth compact-parameter connected-sum construction in odd dimensions is also given in [16, Section 7, Remark 4].

An eigenbundle isomorphism under surgery is proved in [72, Theorem 4.26 and Remark 4.27].

Proposition 18.1. *Let B be compact, let $b \mapsto g_b$ be a C^2 -continuous family of smooth metrics on a closed spin manifold N , and fix a spin surgery of codimension $q \geq 3$. Write $D_b = D_{g_b}$. Let $\alpha, \beta : B \rightarrow \mathbb{R}$ be continuous, with $\alpha_b < \beta_b$ and $\alpha_b, \beta_b \notin \text{Spec}(D_b)$. Suppose that*

$$E_b = \mathbf{1}_{(\alpha_b, \beta_b)}(D_b) L^2(\Sigma_{g_b})$$

has fixed rank. For every $\eta > 0$ and every neighborhood V of the surgery sphere, there is a C^2 -continuous family $\tilde{g}_b$ of smooth metrics on the surgered manifold, equal to g_b outside V , for which $\tilde{E}_b = \mathbf{1}_{(\alpha_b, \beta_b)}(\tilde{D}_b) L^2(\Sigma_{\tilde{g}_b})$ has the same rank as E_b . There is a continuous bundle isometry $W : E \rightarrow \tilde{E}$ such that

$$\sup_{b \in B} \|W_b^* \tilde{D}_b W_b - D_b|_{E_b}\| < \eta.$$

If E is invariant under the spinor structures, W intertwines their restrictions. If $\alpha_b > 0$ and a spinor structure exchanges the bands (α_b, β_b) and $(-\beta_b, -\alpha_b)$, the isometry on their direct sum may also preserve the two summands.

Proof. For rank zero, the spectral surgery theorem gives the assertion. Suppose the rank is positive. Locally index the eigenvalues in the selected interval. The spectral bundles are continuous by [72, Theorems 3.3–3.4]. Norm-resolvent continuity and compactness give

$$d := \inf_{b \in B} \text{dist}(\{\alpha_b, \beta_b\}, \text{Spec}(D_b)) > 0.$$

All constants below are independent of b and of the surgery radius.

The old cluster has a uniform L^2 -to- L^∞ bound. For $D_b \psi = \lambda \psi$ in the band, the square formula and Kato's inequality give, for $u = (|\psi|^2 + \varepsilon^2)^{1/2}$,

$$\Delta_b u \leq C_0 u.$$

The constant C_0 is uniform because $|\lambda|$ and the scalar curvatures of the compact C^2 -family are bounded. With the nonnegative Laplacian convention, this is

$$\partial_i(a_b^{ij}\partial_j u) + d_b u \geq 0, \quad a_b^{ij} = \sqrt{\det g_b} g_b^{ij}, \quad d_b = C_0 \sqrt{\det g_b}.$$

On a fixed finite atlas these coefficients have common ellipticity and boundedness constants. The local boundedness estimate [48, Theorem 8.17, p. 194], with $p = 2$ and zero right-hand side, applies on nested coordinate balls. Uniform equivalence of the coordinate and metric measures and $\varepsilon \rightarrow 0$ give

$$\|\psi\|_\infty \leq C_1 \|\psi\|_2. \quad (18.1)$$

Expansion in an orthonormal eigenbasis gives the same inequality on E_b , with the additional factor $\sqrt{\text{rank } E}$.

Fix tubular neighborhoods T_r of the surgery sphere with respect to a background metric. Choose $0 \leq \chi_\rho \leq 1$ equal to zero on T_ρ and to one outside $T_{2\rho}$, with $|d\chi_\rho|_{g_b} \leq C/\rho$. Take ρ small enough that $T_{2\rho} \subset V$. Theorem 4.24 of [72], with any sufficiently small spectral error $\varepsilon < d/3$, gives a C^2 -continuous family $\tilde{g}_b$ equal to g_b outside $T_{\rho/2}$ and with the same band ranks. On the common exterior, identify the spinor bundles by the spin structure of the surgery. Set

$$A_b = D_b|_{E_b}, \quad C_{\rho,b} : E_b \longrightarrow L^2(\Sigma_{\tilde{g}_b}), \quad C_{\rho,b}\psi = \chi_\rho\psi,$$

extending by zero across the modified region. This map is continuous in b , has norm at most one, and takes values in $D(\tilde{D}_b)$. The volume of $T_{2\rho}$ is $O(\rho^q)$ uniformly. Hence

$$\begin{aligned} C_{\rho,b}^* C_{\rho,b} &= I + O(\rho^q), \\ R_b &:= \tilde{D}_b C_{\rho,b} - C_{\rho,b} A_b = d\chi_\rho \cdot |_{E_b}, \quad \|R_b\| \leq C\rho^{q/2-1}. \end{aligned} \quad (18.2)$$

Let P_b be the new band projection. Every eigenvalue of $\tilde{D}_b$ on $\text{im}(I - P_b)$ lies outside (α_b, β_b) , at distance at least d from the spectrum of A_b . For an orthonormal eigenbasis e_j of A_b , with eigenvalues λ_j , the spectral theorem gives

$$\|(I - P_b)C_{\rho,b}e_j\| \leq d^{-1} \|(\tilde{D}_b - \lambda_j)(I - P_b)C_{\rho,b}e_j\| \leq d^{-1} \|R_b e_j\|.$$

Consequently $\|(I - P_b)C_{\rho,b}\| \leq \sqrt{\text{rank } E} d^{-1} \|R_b\|$. Put

$$G_b = C_{\rho,b}^* P_b C_{\rho,b}, \quad W_b = P_b C_{\rho,b} G_b^{-1/2}.$$

Since

$$G_b = C_{\rho,b}^* C_{\rho,b} - ((I - P_b)C_{\rho,b})^* ((I - P_b)C_{\rho,b}),$$

we have $\|G_b - I\| \leq C(\rho^q + \rho^{q-2})$. For small ρ , $\frac{1}{2}I \leq G_b \leq I$. Thus $W_b^* W_b = I$, and equality of ranks makes W_b onto $\tilde{E}_b$. Continuity of the factors gives continuity of W . The identity

$$W_b^* \tilde{D}_b W_b = G_b^{1/2} A_b G_b^{-1/2} + G_b^{-1/2} C_{\rho,b}^* P_b R_b G_b^{-1/2} \quad (18.3)$$

gives

$$\|W_b^* \tilde{D}_b W_b - A_b\| \leq C(\|G_b - I\| + \|R_b\|) \leq C\rho^{q/2-1}.$$

Since $q \geq 3$, decreasing ρ gives the prescribed η .

Suppose a spinor structure preserves E_b . If it reverses the sign of D_b , each eigenvalue λ of A_b has $-\lambda$ in the same band; both have distance at least d from its endpoints. The spectral error is less than d , so the new band is likewise invariant. The common exterior spin identification and the real cutoff intertwine the structures. The projections therefore intertwine them, and so do $G_b^{-1/2}$ and W_b , by the real functional calculus argument in Lemma 6.1.

For the two nonzero bands, apply the preceding construction to $E_{b,+}$ and write

$$C_{b,+} = \chi_\rho|_{E_{b,+}}, \quad G_{b,+} = C_{b,+}^* P_{b,+} C_{b,+}, \quad W_{b,+} = P_{b,+} C_{b,+} G_{b,+}^{-1/2}. \quad (18.4)$$

Let $S_b : E_{b,+} \rightarrow E_{b,-}$ be a unitary or antiunitary sign-reversing structure, with new counterpart $\tilde{S}_b$. Set

$$W_{b,-} = \tilde{S}_b W_{b,+} S_b^{-1}. \quad (18.5)$$

For $A_{b,\pm} = D_b|_{E_{b,\pm}}$ and $\tilde{A}_{b,\pm} = \tilde{D}_b|_{\tilde{E}_{b,\pm}}$,

$$W_{b,-}^* \tilde{A}_{b,-} W_{b,-} - A_{b,-} = -S_b (W_{b,+}^* \tilde{A}_{b,+} W_{b,+} - A_{b,+}) S_b^{-1}.$$

Thus the two errors have the same norm. The positive construction intertwines every sign-preserving structure. If T_b preserves sign, apply this identity to $S_b^{-1} T_b S_b$ on $E_{b,+}$; it gives $W_{b,-} T_b = \tilde{T}_b W_{b,-}$ on $E_{b,-}$. For a sign-reversing $T_b = J_b$ or Γ_b , the product $S_b^{-1} T_b$ preserves sign; hence $W_{b,-} T_b = \tilde{T}_b W_{b,+}$ on $E_{b,+}$. The corresponding identity on $E_{b,-}$ follows from $T_b^2 = \tilde{T}_b^2 = \pm I$. Therefore $W_{b,+} \oplus W_{b,-}$ preserves both summands and intertwines the spinor structures. $\square$

18.1. Spin bordisms. The model $X_n = S^3 \times T^{n-3}$ bounds $D^4 \times T^{n-3}$ as a spin manifold. For $n \geq 5$, the spin-bordism surgery theorem [51, proof of Theorem B, pp. 431–432], applied to the simply connected target S^n , gives a finite sequence of spin surgeries of codimension at least three from X_n to S^n ; see also [36, proof of Proposition 7, p. 7, third arXiv version]. Disconnected intermediate levels are permitted in the compact-parameter spectral theorem.

For $n = 4$, surgery on the circle in $S^1 \times S^3$ gives S^4 . Indeed the complement of $S^1 \times \text{int } D^3$ is $S^1 \times D^3$, and the surgery union is

$$(S^1 \times D^3) \cup (D^2 \times S^2) = \partial(D^2 \times D^3).$$

Rounding corners identifies the right side with S^4 . The two framings differ by the generator of $\pi_1 SO(3)$. The framing change is induced by a rotation of D^3 depending on the circle coordinate; this map extends over $S^1 \times D^3$, so both framings give this smooth manifold. The nontrivial loop in $SO(3)$ lifts to a path in $\text{Spin}(3)$ with opposite endpoints. Thus the two framings induce the two spin structures on the circle. The bounding one extends over the attached handle [19, Section 1]. This is the four-dimensional spin surgery in [36, Section 3].

For $n = 3$, the auxiliary manifold is S^3 . Connected-sum surgery with the prescribed manifold has codimension $n \geq 3$ and gives $M \# S^n \cong M$ with its original spin structure. The attachments are independent of the parameters. For a sequence of m surgeries, let A_j be the successive compressed operators and W_j the isometries of Proposition 18.1. If $W = W_m \cdots W_1$, then

$$\|W^* A_m W - A_0\| \leq \sum_{j=1}^m \|W_j^* A_j W_j - A_{j-1}\|.$$

Since m is finite, the total error can be made arbitrarily small.

19. FINITE SPECTRAL PRESCRIPTION

19.1. Nonzero matrix blocks. By (2.1), the multiplicity field for a nonzero eigenvalue is

$$\mathbb{F}_n = \begin{cases} \mathbb{H}, & n \equiv 2, 3, 4 \pmod{8}, \\ \mathbb{C}, & n \equiv 1, 5 \pmod{8}, \\ \mathbb{R}, & n \equiv 0, 6, 7 \pmod{8}. \end{cases} \quad (19.1)$$

Let $H = \lambda I + X$, where $\lambda \neq 0$ and $X \in \text{Herm}_m(\mathbb{F}_n)$. In the symmetric spectral classes, take $\lambda > 0$ and H positive definite. In residues three and seven either sign is allowed.

Lemma 19.1. *Every such matrix occurs as a block in a balanced zero module. The inclusion of the selected subspace can be chosen independently of H and compatibly with its multiplicity field. If charge conjugation or chirality changes the spectral sign, its image subspace carries the block determined by that identity.*

Proof. For $n \equiv 3, 7 \pmod{8}$, take H itself. For $n \equiv 0, 4 \pmod{8}$, put $Z = H$. The $\mathbb{F}_n$ -linear isometries $f_{\pm}(u) = (u, \pm u)/\sqrt{2}$ satisfy

$$D(H)f_{\pm} = \pm f_{\pm}H.$$

The same formula applies in residue six, since a real symmetric H is complex symmetric. Here $J(x, y) = (\bar{y}, \bar{x})$ induces conjugation on the positive subspace: $Jf_+(u) = f_+(\bar{u})$.

In residue two, represent the quaternionic Hermitian matrix H on $\mathbb{C}^{2m}$ and put

$$J_0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad Z = HJ_0.$$

The identities $H^* = H$ and $HJ_0 = J_0\bar{H}$ give $Z^T = -J_0\bar{H} = -HJ_0 = -Z$. For $f_+(u) = (-J_0u, u)/\sqrt{2}$ and $f_-(u) = (J_0u, u)/\sqrt{2}$,

$$D(Z)f_+(u) = (-J_0Hu, Hu)/\sqrt{2} = f_+(Hu),$$

$$D(Z)f_-(u) = (-J_0Hu, -Hu)/\sqrt{2} = -f_-(Hu).$$

The charge conjugation $J(x, y) = (-\bar{y}, \bar{x})$ satisfies $Jf_+(u) = f_+(-J_0\bar{u})$. Thus the positive subspace has its quaternionic structure, commuting with H .

In residue five, take $T = \text{diag}(H, -\bar{H})$ and $J(u, v) = (-\bar{v}, \bar{u})$. Then $T^* = T$, $J^2 = -I$, and

$$JT(u, v) = (H\bar{v}, \bar{H}\bar{u}) = -TJ(u, v).$$

The two coordinate inclusions give H and $-\bar{H}$.

In residue one write $H = A + iB$, with $A^T = A$, $B^T = -B$, and set

$$R(H) = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}, \quad J_0 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

Then $R(H)^T = R(H)$ and $R(H)J_0 = J_0R(H)$, so $J_0R(H)$ is real skew. On the complexification, the isometries $f_+(u) = (u, -iu)/\sqrt{2}$ and $f_-(u) = (u, iu)/\sqrt{2}$ satisfy

$$-iJ_0R(H)f_+ = f_+H, \quad -iJ_0R(H)f_- = -f_-\bar{H}.$$

Every displayed inclusion is independent of H . The underlying modules satisfy the balance conditions of (2.4). $\square$

19.2. Separation of spectral intervals.

Proof of Theorem P. Let λ_j be the distinct prescribed nonzero values, and let m_j be their multiplicities over $\mathbb{F}_n$. In the symmetric classes list only the positive values. Put

$$V = \bigoplus_j \text{Herm}_{m_j}(\mathbb{F}_n), \quad H_j = \lambda_j I + X_j, \quad X = (X_j)_j \in V. \quad (19.2)$$

Choose disjoint closed intervals about these values and a symmetric closed interval about zero, all contained in $(-\Lambda, \Lambda)$. Include the opposite intervals in the symmetric classes. For each interval take the positively oriented circle having it as a diameter; write γ_j and γ_0 for the corresponding contours.

Fix $M_0 > \Lambda + 3$. By [4, Theorem 1.1], the prescribed spin manifold admits a metric $g_{\min}$ whose complex nullity is

$$a(M) = \begin{cases} |\widehat{A}(M)|, & n \equiv 0, 4 \pmod{8}, \\ \alpha(M), & n \equiv 1 \pmod{8}, \\ 2\alpha(M), & n \equiv 2 \pmod{8}, \\ 0, & n \equiv 3, 5, 6, 7 \pmod{8}. \end{cases} \quad (19.3)$$

Constant scaling multiplies the Dirac operator by the inverse length factor. Rescale $g_{\min}$ so that its nonzero eigenvalues have absolute value greater than $3M_0$, and put $K_{\min} = \ker D_{g_{\min}}$.

For $h > a(M)$, take the zero family of Proposition 17.3, with $c = M_0$. Write B for its base, N for its normal bundle, and E for the remaining residual bundle. If $a(M) = 0$, then B is a point and $E = 0$. For $h = a(M)$, take B to be a point and $N = E = 0$. In residues zero and four, when the nonzero minimal chiral summand lies in the range, take the adjoint chiral matrices throughout, as in Section 17. This interchanges kernel and cokernel and preserves their sum.

On the disk bundle

$$Y = D_\eta(N \oplus \underline{V})$$

combine the auxiliary zero family with the nonzero blocks of Lemma 19.1. Each summand is balanced. Choose a strictly larger balanced module from Corollary 16.4 containing their structural orthogonal direct sum. Its complement is balanced by (2.4); on that complement fix an allowed matrix whose singular values are M_0 . Choose $\eta > 0$ sufficiently small that every variable spectral block lies in the interior of its chosen interval and $\eta < M_0/4$. The resulting auxiliary matrices B_y depend smoothly on $y \in Y$ and satisfy $\|B_y\| \leq M_0$.

Let $A \mapsto g(A)$ be the full metric section on this module given by Corollary 16.4. On a sufficiently small closed matrix ball its complementary spectrum has distance at least $d_{\text{aux}} > 0$ from zero. Choose a fixed $\epsilon > 0$ such that ϵB_y lies in that ball for every y and $d_{\text{aux}}/\epsilon > 3M_0$. Set

$$g_y^{\text{aux}} = \epsilon^2 g(\epsilon B_y). \quad (19.4)$$

Under the constant spinor comparison, $D_{\epsilon^2 g} = \epsilon^{-1} D_g$. Thus the selected matrix of (19.4) is B_y , and every eigenvalue outside this cluster has absolute value greater than $3M_0$. On the disjoint union with $(M, g_{\min})$, the complete matrix in $(-2M_0, 2M_0)$ is

$$T_y = B_y \oplus 0_{K_{\min}}, \quad \|T_y\| \leq M_0.$$

The projection inside γ_0 has complex rank h : its range is the residual module of Section 17, including $K_{\min}$. For $h = a(M)$ it is precisely $K_{\min}$. This includes $h = 0$.

For all the chosen contours, including the negative contours in the symmetric classes, set

$$d_\ell = \inf_{y \in Y, z \in \gamma_\ell} \text{dist}(z, \text{Spec}(T_y)) > 0, \quad \kappa_\ell = \frac{|\gamma_\ell|}{\pi d_\ell^2}.$$

The spectra lie in the interiors of the selected intervals or at $\pm M_0$; compactness of Y gives the strict lower bounds.

Apply the finite spin-surgery sequence of Section 18 to the auxiliary manifold, and then form its connected sum with $(M, g_{\min})$. The parameter space Y is compact, the initial metric family is smooth, and the endpoints $\pm 2M_0$ avoid its spectrum. Proposition 18.1 and the finite composition estimate identify the entire resulting band with the reference module. For any sufficiently small $\delta > 0$, the resulting matrix $\tilde{T}_y$ satisfies

$$\sup_{y \in Y} \|\tilde{T}_y - T_y\| < \delta.$$

These matrices are continuous and retain all the spinor identities. The metrics are smooth and depend continuously on y in C^2 .

The reference residual module and its orthogonal complement form smooth bundles over B . The complementary block has singular values bounded below by a constant $\mu > 0$: its variable nonzero blocks lie in intervals avoiding zero, and its remaining singular values equal M_0 . Take the chiral Schur complement in even dimensions, and the Hermitian Schur complement in residues three and seven. In residues one and five, take the real or quaternionic skew-adjoint Schur complement of $-i\tilde{T}_y$. In these two residues the normal variable and the residual matrix are both expressed in skew-adjoint coordinates by $T \mapsto -iT$.

The resulting section S_0 takes values in $N \oplus E$. For nonzero index the identifications are (17.3), (17.4), and (17.5), with the adjoint in the opposite integer-index case. At zero index, let E_0 be the selected residual zero module; the target is $\mathcal{A}_n(E_0)$ in the indicated matrix coordinates. For $h = a(M)$ the

residual space is zero, by the minimal-kernel construction of Section 17. In every case the reference section is $S_{\text{ref}} = (A, 0)$, with absent summands omitted. Lemma 6.6 applies to these orthogonal splittings; for $\delta < \mu/2$, it preserves the structural identities and gives

$$\|S_0 - S_{\text{ref}}\|_{\text{op}} \leq \delta + \frac{2\delta^2}{\mu} \leq 2\delta.$$

Let $P_\ell, \tilde{P}_\ell$ be the Riesz projections of $T_y, \tilde{T}_y$ for y_ℓ . For $\delta < d_\ell/2$, the resolvent identity gives

$$\|(z - \tilde{T}_y)^{-1} - (z - T_y)^{-1}\| \leq \frac{2\delta}{d_\ell^2}, \quad z \in \gamma_\ell.$$

Integration yields

$$q_\ell := \|\tilde{P}_\ell - P_\ell\| \leq \kappa_\ell \delta.$$

Put $\delta_{\text{polar}} = \min_\ell (2\kappa_\ell)^{-1}$. For $\delta < \delta_{\text{polar}}$, the projections have the same ranks by [62, Chapter I, Theorem 6.32].

For an independent nonzero block, let U_j be its fixed isometric inclusion from Lemma 19.1. The polar formula (6.2) gives

$$G_j = U_j^* \tilde{P}_j U_j, \quad \tilde{U}_j = \tilde{P}_j U_j G_j^{-1/2}.$$

These expressions are continuous in y . Since $\|G_j - I\| \leq q_j < 1/2$ and $G_j \leq I$, the spectral theorem gives

$$\|G_j^{-1/2}\| \leq \sqrt{2}, \quad \|G_j^{-1/2} - I\| \leq 2q_j, \quad \|\tilde{U}_j - U_j\| \leq 4q_j.$$

Thus $\tilde{H}_j = \tilde{U}_j^* \tilde{T}_y \tilde{U}_j$ satisfies

$$\|\tilde{H}_j - H_j\| \leq \delta + 2M_0 \|\tilde{U}_j - U_j\| \leq C_j \delta, \quad C_j = 1 + 8M_0 \kappa_j. \quad (19.5)$$

The polar isometry intertwines the sign-preserving spinor structures, as in the proof of Lemma 6.1. Consequently $\tilde{H}_j \in \text{Herm}_{m_j}(\mathbb{F}_n)$; the sign-reversing structures determine the negative partners.

Fix a constant $C_{\text{fib}} \geq 1$ comparing the product fibre norm on $N \oplus \underline{V} \oplus E$ with

$$\left(\|S\|_{\text{op}}^2 + \sum_j \|Y_j\|_{\text{op}}^2 \right)^{1/2}, \quad S \in N \oplus E, \quad Y_j \in \text{Herm}_{m_j}(\mathbb{F}_n).$$

It is uniform because these bundles have finite rank over the compact base. Put $K_0 = (4 + \sum_j C_j^2)^{1/2}$. After the reference family and these constants have been fixed, choose the total surgery error so that

$$0 < \delta < \min \left\{ \frac{\mu}{2}, \frac{1}{2} \min_\ell d_\ell, \frac{\eta}{2C_{\text{fib}}K_0}, \frac{M_0 - \Lambda}{2}, \delta_{\text{polar}} \right\}.$$

The continuous section with components S_0 and $(\tilde{H}_j - \lambda_j I)_j$, ordered in $N \oplus \underline{V} \oplus E$, differs from $(A, X, 0)$ by less than $C_{\text{fib}}K_0\delta < \eta/2$. The constructions of Section 17 give a compact smooth base B without boundary, $\text{rank } E = \dim B$, and a section of E with one transverse zero. Lemma 17.1, applied with $N \oplus \underline{V}$, therefore gives a parameter with $S_0 = 0$ and

$$\tilde{H}_j = \lambda_j I \quad \text{for every } j. \quad (19.6)$$

For $h = a(M)$ this is the same lemma with B a point and $N = E = 0$. If also $V = 0$, the parameter space is a point.

At this parameter, Lemma 6.6 identifies the kernel and cokernel with those of the vanishing residual matrix. Their dimensions are those of Section 17, and the full complex kernel has dimension h . Every kernel vector belongs to the band and to the projection inside γ_0 . That projection has rank h , so its range is exactly the kernel.

Each nonzero projection retains its reference rank. Equation (19.6) makes its compressed operator the prescribed scalar matrix. A block of size m_j contributes complex multiplicity $2m_j$ over $\mathbb{H}$ and m_j over $\mathbb{R}$ or $\mathbb{C}$. In the symmetric classes the negative partners have the same multiplicities. Thus these subspaces give exactly Ξ .

The remaining reference eigenvalues equal M_0 or $-M_0$. For the full finite self-adjoint matrices, the ordered-eigenvalue inequality [62, Chapter II, Theorem 6.10, p. 126] gives $|\widehat{\nu}_k - \nu_k| < \delta$. Hence every entry corresponding to an unused reference block has absolute value greater than $M_0 - \delta > \Lambda$. The number of these entries, together with the ranks of the zero projection and all the prescribed nonzero projections, equals the rank of T_γ . Proposition 18.1 preserves that entire band rank, so these eigenvalues exhaust the spectrum in $(-2M_0, 2M_0)$. There is no further eigenvalue in $(-\Lambda, \Lambda)$. The final spin manifold is $M \# S^n \cong M$ with its prescribed spin structure. This proves (1.10). $\square$

Corollary 19.2. *For every closed connected spin manifold of dimension at least three,*

$$\sup_g \dim_{\mathbb{C}} \ker D_g = \infty.$$

Proof. Each row of (1.9) contains arbitrarily large integers. Apply Theorem P with an empty nonzero multiset. $\square$

20. LOCAL HARMONIC SPINORS IN DIMENSION FOUR

Let M^4 be closed, connected, and spin, and fix a nonempty open set $U \subset M$. Interchange the chiralities, if necessary, so that

$$A_g : H^1(S) \longrightarrow L^2(T), \quad \text{ind}_{\mathbb{H}} A_g = -a, \quad a \geq 0.$$

Both S and T have quaternionic rank one. By Lemma 2.2 and (2.5), the real-linear bundle map

$$\Lambda^3 T^*M \longrightarrow \text{Hom}_{\mathbb{H}}(S, T), \quad \tau \longmapsto c_g(\tau)|_S \quad (20.1)$$

is an isomorphism. A section B of the target corresponds to the compatible self-adjoint potential $\begin{pmatrix} 0 & B^* \\ B & 0 \end{pmatrix}$.

20.1. A flat ball with prescribed density.

Lemma 20.1. *For every $p \in U$, the metric g can be approximated in C^1 by smooth metrics g_1 which are flat on a neighborhood of p , equal g outside U , and satisfy $dv_{g_1} = dv_g$. If A_g is injective, g_1 can be chosen so that A_{g_1} is injective.*

Proof. Take normal coordinates x at p , with $dv_g = J(x) dx$. Here $J = 1 + O(|x|^2)$ and $dJ = O(|x|)$. On a smaller coordinate box define

$$y_1 = \int_0^{x_1} J(t, x_2, x_3, x_4) dt, \quad y_i = x_i \quad (i > 1).$$

The Jacobian determinant is J , and

$$y - x = O(|x|^3), \quad Dy - I = O(|x|^2), \quad D^2y = O(|x|).$$

Thus $dv_g = dy$, and the matrix $G(y)$ of g satisfies

$$\det G = 1, \quad G - I = O(|y|^2), \quad dG = O(|y|).$$

Choose χ_r equal to zero on $|y| \leq r/3$ and to one on $|y| \geq 2r/3$, with $|d\chi_r| \leq C/r$. Replace G on this box by

$$G_r = \exp(\chi_r \log G).$$

It equals G near the boundary of the box and I near p . Since $\text{tr} \log G = \log \det G = 0$, its determinant is one. Differentiation of the exponential integral formula in Lemma 3.4 gives

$$\|G_r - G\|_{C^0} \leq Cr^2, \quad \|G_r - G\|_{C^1} \leq Cr.$$

These tensors extend by g to all of M .

Viewed as a complex elliptic operator on the closed manifold M , $A_g : H^1(S) \rightarrow L^2(T)$ has closed range and finite-dimensional kernel by [61, Theorem 19.5.1, pp. 225–226]; the fixed density identifies sections with half-density-valued sections. Under the positive spinor comparisons (2.8),

$$\|A_{g_r} - A_g\|_{H^1 \rightarrow L^2} \leq C' \|g_r - g\|_{C^1} \rightarrow 0.$$

If A_g is injective, the perturbation theorem [61, Theorem 19.1.5, pp. 182–183] implies that A_{g_r} is injective for all sufficiently small r . $\square$

20.2. Cokernel-orthogonal point sources.

Lemma 20.2. *Suppose A_{g_1} is injective and g_1 is flat on a ball contained in U . There is a smooth $B \in C_c^\infty(U; \text{Hom}_{\mathbb{H}}(S, T))$ such that $\ker(A_{g_1} + B) \neq 0$.*

Proof. Write $A = A_{g_1}$ and $E = \ker A^*$, of quaternionic dimension a . Select $m > a$ distinct points p_i in the flat ball. The quaternionic-linear map

$$\bigoplus_{i=1}^m T_{p_i} \rightarrow E, \quad (v_i) \mapsto \sum_i \text{ev}_{p_i}^* v_i$$

has a nonzero kernel. Choose a nonzero vector there and discard its zero entries. The distribution $\eta = \sum_i \delta_{p_i} v_i$ is nonzero and annihilates E .

Since A is injective, A^*A is an invertible elliptic operator. Set

$$R = (A^*A)^{-1}A^*.$$

The elliptic calculus [61, Theorems 18.1.8 and 18.1.24, pp. 71 and 87; bundle calculus, pp. 91–93] gives R of order -1 . The identity $AR = I - P_E$ extends to distributions, since P_E has a smooth kernel. Hence

$$\psi = R\eta, \quad A\psi = \eta. \quad (20.2)$$

After a constant quaternionic unitary change of the target frame if necessary, A has one of the following expressions in flat chiral frames:

$$\mathcal{D} = \partial_0 + \mathbf{i}\partial_1 + \mathbf{j}\partial_2 + \mathbf{k}\partial_3, \quad \overline{\mathcal{D}} = \partial_0 - \mathbf{i}\partial_1 - \mathbf{j}\partial_2 - \mathbf{k}\partial_3.$$

Suppose first that $A = \mathcal{D}$ in these frames. The fundamental solution of $\Delta_{\mathbb{R}^4} = \sum_{j=0}^3 \partial_j^2$ is $\Phi_0(x) = -1/(4\pi^2|x|^2)$ [48, Section 2.4, equation (2.12), p. 17]. Since $\mathcal{D}\overline{\mathcal{D}} = \Delta_{\mathbb{R}^4}$,

$$E_0(x) = \overline{\mathcal{D}}\Phi_0(x) = \frac{\bar{x}}{2\pi^2|x|^4}, \quad \mathcal{D}E_0 = \delta_0.$$

Elliptic regularity gives $\psi = E_0(x - p_i)v_i + h_i$, where h_i is smooth and $\mathcal{D}h_i = 0$. Choose disjoint balls about the p_i , contained in the flat ball, such that

$$|h_i(x)| < \frac{|v_i|}{4\pi^2|x - p_i|^3} \quad (x \neq p_i).$$

Put

$$u(x) = x_0 + x_1\mathbf{i} + x_2\mathbf{j} - x_3\mathbf{k}, \quad \mathcal{D}u = 1 - 1 - 1 + 1 = 0. \quad (20.3)$$

For $\theta \in S^3$ and $s \in [0, 1]$, define

$$Q_s(\theta) = \theta_0 + (\cos(\pi s)\theta_1 - \sin(\pi s)\theta_2)\mathbf{i} \\ + (\sin(\pi s)\theta_1 + \cos(\pi s)\theta_2)\mathbf{j} - \theta_3\mathbf{k}.$$

Then $|Q_s(\theta)| = 1$, $Q_0(\theta) = u(\theta)$, and $Q_1(\theta) = \bar{\theta}$. In one of the chosen balls write $x - p_i = \rho\theta$ and take $0 < r_1 < r_2 < r_3$ smaller than its radius. Choose smooth $\chi, \zeta : [0, \infty) \rightarrow [0, 1]$ with

$$\chi = 0 \ (\rho \leq r_1), \quad \chi = 1 \ (\rho \geq r_2), \quad \zeta = 0 \ (\rho \leq r_2), \quad \zeta = 1 \ (\rho \geq r_3).$$

For $\rho > 0$, set

$$b(\rho) = (1 - \chi(\rho))\rho + \frac{\chi(\rho)}{2\pi^2\rho^3}, \quad \phi_i(p_i + \rho\theta) = b(\rho)Q_{\chi(\rho)}(\theta)v_i + \zeta(\rho)h_i(p_i + \rho\theta).$$

For $\rho \leq r_1$ this is $u(x - p_i)v_i$, so it extends smoothly across p_i . For $\rho \geq r_3$ it equals ψ . If $\zeta = 0$, then $|\phi_i| = b(\rho)|v_i| > 0$. If $\zeta \neq 0$, then $\chi = 1$ and

$$|\phi_i| \geq |E_0(x - p_i)v_i| - |h_i(x)| > \frac{1}{2}|E_0(x - p_i)v_i| > 0.$$

For $A = \bar{\mathcal{D}}$, the fundamental solution is $x/(2\pi^2|x|^4)$. Replace $Q_s(\theta)$ by $Q_s(\bar{\theta})$; the inner map is $u(\bar{x})$ and $\bar{\mathcal{D}}(u(\bar{x})) = 1 - 1 - 1 + 1 = 0$.

Set $\phi = \phi_i$ in these balls and $\phi = \psi$ outside them. This defines a smooth section. It satisfies $A\phi = 0$ near the centers, near the ball boundaries, and outside the balls. On each punctured ball define

$$B = -\frac{(A\phi)\phi^*}{|\phi|^2}. \quad (20.4)$$

Here $\phi^*(w) = \langle \phi, w \rangle_{\mathbb{H}}$. Thus B is quaternionic-linear and $B\phi = -A\phi$. It vanishes near the centers and ball boundaries, so extension by zero gives $B \in C_c^\infty(U; \text{Hom}_{\mathbb{H}}(S, T))$ and $(A + B)\phi = 0$. Since ϕ_i is nonzero on a punctured ball, $\phi \neq 0$. $\square$

20.3. A chiral kernel of quaternionic dimension one.

Proposition 20.3. *Let $L = A_g + B_0$ have nonzero kernel, and let B_0 be smooth and supported in U . There is a smooth supported perturbation B_1 such that*

$$\dim_{\mathbb{H}} \ker(A_g + B_1) = 1, \quad \dim_{\mathbb{H}} \ker(A_g + B_1)^* = a + 1.$$

At this operator, finitely many supported multiplication variations induce an isomorphism onto the normal space $\text{Hom}_{\mathbb{H}}(\mathbb{H}, \mathbb{H}^{a+1})$ of its zero chiral matrix.

Proof. The multiplication operator $B_0 : H^1(S) \rightarrow L^2(T)$ is compact, so $\text{ind}_{\mathbb{H}} L = -a$ [61, Corollary 19.1.8, p. 183]. Write $F = \ker L$, $E = \ker L^*$, and $\dim_{\mathbb{H}} F = k$; then $\dim_{\mathbb{H}} E = k + a$. Unique continuation for L and L^* follows from [7, Remark 3, p. 248], applied to L^*L and LL^* : their principal symbols are $|\xi|_g^2 I$, and their coefficients are smooth. Compression defines a real-linear map

$$C : C_c^\infty(U; \text{Hom}_{\mathbb{H}}(S, T)) \longrightarrow \text{Hom}_{\mathbb{H}}(F, E), \quad C(B) = P_E B|_F. \quad (20.5)$$

By Lemma 6.5, for every $x \in U$ and every $q \in \mathbb{H}$ its image contains uqv^* , where $v^* : F \rightarrow S_x$ and $u^* : E \rightarrow T_x$ are evaluation in local quaternionic unit frames. If $k = 1$, retain $B_1 = B_0$; the normal-surjectivity argument below, with $\ell = F$ and $C = E$, gives the required potential directions. Assume henceforth $k \geq 2$. Choose $k - 1$ points for which the vectors v_j and u_j are independent in their respective spaces. For the inductive choice, fix the previous spans $V \subsetneq F$ and $W \subsetneq E$. The set of points at which the source evaluation line is not contained in V is open and dense in U . If its complement contained a nonempty open set, every element of $V^\perp \setminus \{0\}$ would vanish there, in contradiction to unique continuation. The target condition is open and dense because a cokernel spinor vanishing on an open set vanishes identically by unique continuation for L^* . Their intersection contains a point distinct from the preceding ones.

The matrix

$$M_0 = \sum_{j=1}^{k-1} u_j v_j^* \quad (20.6)$$

belongs to $\text{im } C$ and has rank $k - 1$. Indeed the two column maps are injective, and the adjoint of the source column map is surjective. Put $\ell = \ker M_0$ and $C = (\text{im } M_0)^\perp$; their quaternionic dimensions are 1 and $a + 1$.

The real-linear map

$$B \longmapsto P_C C(B)|_\ell \quad \text{into } \text{Hom}_{\mathbb{H}}(\ell, C) \quad (20.7)$$

is onto. To prove this, choose a unit generator ψ of ℓ . A real annihilator is represented by $e \in C$ and would satisfy

$$\int_M \operatorname{Re}\langle e(x), B(x)\psi(x) \rangle dv_g = 0 \quad \text{for every supported } B.$$

On a ball in U on which $\psi \neq 0$, multiplication by an arbitrary quaternionic-linear map sends $\psi(x)$ to every vector of T_x . Consequently e vanishes on that ball. Since $e \in \ker L^*$, unique continuation gives $e = 0$.

Choose B_* with $C(B_*) = M_0$ and potential directions $V_1, \dots, V_d$ whose images under (20.7) form a real basis, where $d = 4(a+1)$. Set $B(q) = B_* + \sum q_\nu V_\nu$. Give $H^1(S)$ the graph inner product of L . Since $LF = 0$, F and $F^\perp \cap H^1(S)$ are orthogonal for this inner product, where $F^\perp$ denotes the L^2 orthogonal complement. The restriction

$$L_c = L|_{F^\perp \cap H^1(S)} : F^\perp \cap H^1(S) \longrightarrow E^\perp \subset L^2(T)$$

is bijective with bounded inverse. Write $B(q) = (B_{ij}(q))$ in these domain and range decompositions. For small (t, q) , Lemma 6.6 gives the finite Schur matrix

$$S(t, q) = tC(B(q)) - t^2 B_{12}(q)(L_c + tB_{22}(q))^{-1} B_{21}(q). \quad (20.8)$$

The complementary inverse is smooth, so $S(t, q)/t$ extends smoothly across $t = 0$.

Split $F = \ell \oplus \ell^\perp$ and $E = C \oplus \operatorname{im} M_0$. At $(t, q) = (0, 0)$ the lower-right block of S/t is invertible. Eliminate it by Lemma 6.6. The remaining column $s(t, q) \in \operatorname{Hom}_{\mathbb{H}}(\ell, C)$ satisfies

$$s(0, 0) = 0, \quad d_q s(0, 0)[q] = P_C C\left(\sum q_\nu V_\nu\right)|_\ell.$$

This derivative is an isomorphism. The implicit function theorem [46, Theorem 13.20, p. 71] gives a smooth $q(t)$, with $q(0) = 0$ and $s(t, q(t)) = 0$. Fix a sufficiently small $t \neq 0$. Both eliminated operators remain invertible. The kernel and cokernel of $L + tB(q(t))$ therefore have exactly the dimensions of ℓ and C . This gives $B_1 = B_0 + tB(q(t))$.

At the resulting operator the argument for (20.7) applies to its actual kernel and cokernel. Apply Lemma 6.4 to the self-adjoint family

$$D(A_g + B) = \begin{pmatrix} 0 & (A_g + B)^* \\ A_g + B & 0 \end{pmatrix}.$$

Its full zero space at $B = B_1$ is the sum of the actual kernel and cokernel. A symmetric spectral circle and the polar frame of Lemma 6.1 preserve their chiral quaternionic structures. The derivative of the chiral block is $\dot{B} \mapsto P_{\ker(A_g+B_1)^*} \dot{B}|_{\ker(A_g+B_1)}$. Restrict to d directions on which this derivative is an isomorphism. The inverse function theorem [46, Theorem 11.22 and Corollary 11.35, pp. 61–63] then gives a full potential section with exact column $q \mapsto q$ near zero in $\mathbb{H}^{a+1}$. $\square$

Proof of the four-dimensional assertion of Theorem L. If $\ker A_g \neq 0$, put $g' = g$ and apply Proposition 20.3 with $B_0 = 0$. Otherwise, Lemma 20.1 gives a metric g' with

$$\|g' - g\|_{C_g^0} < \varepsilon/2, \quad dv_{g'} = dv_g, \quad g' = g \text{ on } M \setminus U,$$

to which Lemma 20.2 and Proposition 20.3 apply.

In either case there is a full potential section with exact column $q \mapsto q$, of chiral quaternionic ranks 1 and $a+1$. Restrict its parameter domain to a small closed ball on which one symmetric circle separates the full zero cluster from the remaining spectrum. The finitely many potential directions have one common compact support in U . Apply Proposition 6.3; in its application of Theorem R require also

$$\sup_q \|h_q - g'\|_{C_g^0} < \varepsilon/2.$$

The resulting metric section has exact column $q \mapsto q$ and satisfies $dv_{h_q} = dv_g$ and $h_q = g$ on $M \setminus U$. The triangle inequality gives $\|h_q - g\|_{C_g^0} < \varepsilon$. At $q = 0$, the whole kernel is the selected cluster, so

$$\dim_{\mathbb{H}} \ker A_{h_0} = 1, \quad \dim_{\mathbb{H}} \ker A_{h_0}^* = a+1, \quad \dim_{\mathbb{C}} \ker D_{h_0} = 2a+4.$$

The derivative of its normal column is the identity. This proves (1.11) and (1.12). $\square$

21. LOCAL HARMONIC SPINORS IN DIMENSION FIVE

Fix a closed connected spin five-manifold (M, g) . Its spinor bundle is a right quaternionic bundle of rank two. By (2.1), its charge conjugation satisfies $J^2 = -I$ and $JD_g = -D_gJ$. Put

$$K = iD_g, \quad \gamma(X) = ic_g(X).$$

Then K is quaternionic-linear and skew-adjoint; $\gamma(X)$ is quaternionic Hermitian, with $\gamma(X)^2 = |X|^2I$.

21.1. The spinor current.

Lemma 21.1. *For a smooth spinor field ϕ , define the real vector field j_ϕ by*

$$g(j_\phi, X) = \operatorname{Re}\langle \phi, \gamma(X)\phi \rangle. \quad (21.1)$$

It satisfies

$$|j_\phi| = |\phi|^2, \quad \operatorname{div} j_\phi = 2 \operatorname{Re}\langle \phi, iD_g\phi \rangle. \quad (21.2)$$

At a point x with $\phi(x) \neq 0$, the equation $(D_g\phi)(x) + c_g(\tau_x)\phi(x) = 0$ has a real three-form solution $\tau_x \in \Lambda^3 T_x^*M$ if and only if $\operatorname{div} j_\phi(x) = 0$. On an open set where ϕ is nowhere zero and the divergence vanishes, τ can be chosen smoothly.

Proof. The matrices $\gamma(e_j)$ span the traceless quaternionic Hermitian endomorphisms of $\mathbb{H}^2$ and satisfy

$$\gamma(e_j)\gamma(e_k) + \gamma(e_k)\gamma(e_j) = 2\delta_{jk}I.$$

For the inner product $(S, T) = \frac{1}{2} \operatorname{Re} \operatorname{tr}_{\mathbb{H}}(ST)$ they are orthonormal. In a spin frame the map in (21.1) is, up to a fixed orthogonal identification,

$$(u, v) \mapsto (2u\bar{v}, |u|^2 - |v|^2) \in \mathbb{H} \oplus \mathbb{R}.$$

Its squared norm is $4|u|^2|v|^2 + (|u|^2 - |v|^2)^2 = (|u|^2 + |v|^2)^2$. At a point choose a normal orthonormal frame. Since $\nabla\gamma = 0$ and the $\gamma(e_j)$ are Hermitian, differentiation gives

$$\operatorname{div} j_\phi = 2 \sum_j \operatorname{Re}\langle \phi, \gamma(e_j)\nabla_{e_j}\phi \rangle = 2 \operatorname{Re}\langle \phi, iD_g\phi \rangle.$$

For a unit spinor η , let α_η be the one-form of [21, Definition 4.8 and Lemma 4.11]. With $j_1 = -i$, its normalization is $j_\eta^b = -\alpha_\eta$.

By Lemma 2.2 and (2.4), $\tau \mapsto ic_g(\tau)$ identifies $\Lambda^3 T^*M$ with $\mathfrak{sp}(\Sigma M)$. By Lemma 10.3, a nonzero $u \in \mathbb{H}^2$ admits a skew-adjoint solution of $Qu = y$ precisely when $\operatorname{Re}(u^*y) = 0$; the displayed solution in that lemma is smooth in (u, y) and vanishes when $y = 0$. Apply it with $u = \phi$ and $y = -iD_g\phi$. The compatibility condition is the second equality in (21.2). $\square$

21.2. A nonzero spinor with divergence-free current.

Proposition 21.2. *For every nonempty open $U \subset M$, there is $\tau \in C_c^\infty(U; \Lambda^3 T^*M)$ such that $\ker(D_g + c_g(\tau)) \neq 0$.*

Proof. If D_g has a kernel, take $\tau = 0$. Otherwise choose $p \in U$ and $0 \neq v \in \Sigma_p M$. The distribution $\psi = D_g^{-1}(\delta_p v)$ is smooth and harmonic away from p . In normal coordinates centered at p and the Bourguignon–Gauduchon trivialization, Hermann’s expansion [56, Corollary 5.3.4 and Theorem 5.3.10, pp. 55 and 58–60], with $n = 5$ and $\lambda = 0$, gives

$$\psi(x) = -\frac{1}{\omega_4} \frac{c_p(x)v}{|x|^5} + O(|x|^{-3}), \quad \omega_4 = \operatorname{vol}(S^4). \quad (21.3)$$

Since $|c_p(x)v| = |x| |v|$, there is $r_0 > 0$ such that

$$|\psi(x)| \geq \frac{|v|}{2\omega_4} |x|^{-4} > 0 \quad (0 < |x| < r_0).$$

Choose a closed coordinate ball $B \subseteq U$ about p , with a boundary collar on which $\psi \neq 0$. On the compact exterior $M \setminus \text{int } B$ the current is divergence-free. Stokes' theorem gives

$$\int_{\partial B} \iota_{j_\psi} dv_g = 0. \quad (21.4)$$

Trivialize the quaternionic spinor bundle on B . Since $\pi_4(S^7) = 0$ [55, Corollary 4.9, p. 349], the normalized collar spinor extends continuously to B , with its prescribed values on a smaller collar. Relative smooth approximation [81, Theorem 11, p. 99] gives a smooth such extension. Extend $\log |\psi|$ smoothly from that collar to B and exponentiate. The resulting smooth nowhere-zero spinor ϕ_0 equals ψ near ∂B . Put

$$\alpha_0 = \iota_{j_{\phi_0}} dv_g \in \Omega^4(B).$$

This form is nowhere zero and is closed near ∂B . On a collar $C \simeq S^4 \times [0, 1]$ its de Rham class is determined by the integral over ∂B . This integral vanishes by (21.4). Hence $\alpha_0 = d\beta_\partial$ on C for a smooth three-form β_∂ . Extend a cutoff multiple of β_∂ by zero to obtain $\beta_0 \in \Omega^3(B)$ with $d\beta_0 = \alpha_0$ on a smaller collar.

The Levi-Civita connection identifies a first jet of a three-form with its value and covariant derivative. For $L \in T^*B \otimes \Lambda^3 T^*B$ set

$$\mathcal{A}(L) = \sum_{j=1}^5 e^j \wedge L(e_j), \quad S(\alpha)(X) = \frac{1}{4} \iota_X \alpha.$$

Then $\mathcal{A}(\nabla\beta) = d\beta$ and $\mathcal{A}S = I$. Thus the value β_0 and formal derivative

$$L_0 = \nabla\beta_0 + S(\alpha_0 - d\beta_0) \quad (21.5)$$

define a section of $J^1(\Lambda^3 T^*B)$ with $\mathcal{A}(L_0) = \alpha_0 \neq 0$. It equals $j^1\beta_0$ on the fixed collar.

The first-order relation $\mathcal{A}(L) \neq 0$ is open and ample. Indeed, in a principal affine subspace with covector $\xi \neq 0$, the derivative varies by $\xi \otimes \eta$ and its alternating part by $\xi \wedge \eta$. The latter map has rank $\binom{4}{3} = 4$. The excluded set is therefore empty or an affine subspace of codimension four. Its complement is connected and its convex hull is the whole principal subspace; see also [43, Theorem 20.3.1 and Corollary 20.3.2, pp. 181–182].

Apply the relative form of the open-ample theorem [43, Theorem 18.4.1 and Remark 1, p. 171] on $\text{int } B$, fixing an outer annulus that is relatively closed in $\text{int } B$ and has a neighborhood on which (21.5) is holonomic. It gives a continuous homotopy of formal solutions from (β_0, L_0) to $j^1 b_1$, where $b_1 \in C^1(B; \Lambda^3 T^*B)$, fixed on that annulus. Its alternating parts form a continuous homotopy through nonzero four-forms from α_0 to db_1 . Relative C^1 approximation [59, Chapter 2, Theorem 2.5, p. 48], in the coordinate trivialization over $\text{int } B$ and extended by the fixed boundary values, gives a smooth three-form β_1 , equal to b_1 on a smaller collar, such that

$$\|d\beta_1 - db_1\|_{C^0(B)} < \frac{1}{2} \min_B |db_1|.$$

Consequently $(1-s)db_1 + s d\beta_1 \neq 0$ for $0 \leq s \leq 1$. Append this segment and reparameterize the resulting homotopy to be constant near its endpoints. Apply relative smooth approximation [81, Theorem 11, p. 99] to the bundle of nonzero four-forms over $B \times [0, 1]$, relative to smaller closed endpoint collars and a smaller closed spatial collar. The section is smooth on neighborhoods of these fixed sets. The resulting smooth homotopy is fixed on the spatial collar and satisfies

$$\alpha_t \neq 0, \quad \alpha_{t=0} = \alpha_0, \quad \alpha_{t=1} = d\beta_1. \quad (21.6)$$

Write $\alpha_t = \iota_{Y_t} dv_g$. The normalized currents $Y_t/|Y_t|$ define a homotopy in the unit tangent sphere bundle. On B the map $\phi \mapsto j_\phi$ on unit spheres is the quaternionic Hopf bundle $S^7 \rightarrow S^4$ [55, Example 4.46, p. 378]. Lift the homotopy, starting at $\phi_0/|\phi_0|$, by horizontal lifting for a smooth bundle connection. Its time derivative is zero on the fixed collar, so the lift stays fixed there. If $\hat{\phi}_1$

is the terminal unit lift, put $\phi = \sqrt{|Y_1|} \hat{\phi}_1$. Since $j_t \phi = t^2 j_\phi$ for $t > 0$, $j_\phi = Y_1$. On the fixed collar, $|Y_1| = |j_\psi| = |\psi|^2$ and $\hat{\phi}_1 = \psi/|\psi|$. Thus ϕ is nowhere zero on B , equals ψ on the collar, and satisfies $d(\iota_{j_\phi} dv_g) = d^2 \beta_1 = 0$.

Extend ϕ by ψ outside B . Lemma 21.1, with $y = -iD_g \phi$, gives $Q = ic_g(\tau)$ on B and $(iD_g + Q)\phi = 0$. On the boundary collar $D_g \phi = 0$, so the solution in Lemma 10.3 gives $Q = 0$. Extension by zero yields $\tau \in C_c^\infty(U; \Lambda^3 T^*M)$ and $(D_g + c_g(\tau))\phi = 0$ on M . Since ϕ is nowhere zero on B , it is a nonzero element of this kernel. $\square$

21.3. Exact quaternionic nullity.

Proposition 21.3. *The potential in Proposition 21.2 can be chosen so that $\dim_{\mathbb{C}} \ker(D_g + c_g(\tau)) = 2$. At this potential three supported variations, after a smooth change of parameters, give the complete zero-cluster matrix $q \mapsto q \in \mathfrak{sp}(1)$ for $i(D_g + c_g(\tau))$.*

Proof. Put $L_0 = D_g + c_g(\tau_0)$, $K = iL_0$, and $F = \ker K$, with $\dim_{\mathbb{H}} F = k \geq 1$. These operators have domain H^1 , and $K^* = -K$. Their kernels consist of smooth sections. Unique continuation holds for every $D_g + c_g(\tau)$ with τ smooth: its square has scalar principal symbol $|\xi|_g^2 I$, so [7, Remark 3, p. 248] applies.

For

$$C : C_c^\infty(U; \mathfrak{sp}(\Sigma M)) \longrightarrow \mathfrak{sp}(F), \quad Q \longmapsto P_F Q|_F,$$

Lemma 6.5 gives

$$vqv^* \in \text{im } C, \quad v = G_x^* u, \quad x \in U, \quad u \in \Sigma_x M, \quad q \in \text{Im } \mathbb{H},$$

where $G_x : F \rightarrow \Sigma_x M$ is evaluation. Indeed, uqu^* is skew-adjoint. The vectors $G_x^* u$ span F : an element orthogonal to all of them vanishes on U , hence on M .

If $k = 1$, put $\ell = F$. If $k \geq 2$, choose $k - 1$ quaternionically independent vectors of this form and put

$$M_0 = \sum_{j=1}^{k-1} v_j i v_j^*, \quad \ell = \ker M_0. \quad (21.7)$$

If $W : \mathbb{H}^{k-1} \rightarrow F$ has columns v_j , then $M_0 = W \text{diag}(i, \dots, i) W^*$. The map W is injective and W^* is surjective. Thus M_0 has rank $k - 1$, $\dim_{\mathbb{H}} \ell = 1$, and $M_0|_{\ell^\perp} : \ell^\perp \rightarrow \ell^\perp$ is invertible.

In either case let ψ be an L^2 -unit generator of ℓ . Unique continuation gives a ball $O \Subset U$ on which $\psi \neq 0$. Choose a real $\rho \in C_c^\infty(O)$ with $\int_M \rho dv_g = 1$. For $q \in \text{Im } \mathbb{H}$, set

$$Q_q(x) = \rho(x) \frac{\psi(x) q \psi(x)^*}{|\psi(x)|^4} \quad (x \in O), \quad Q_q = 0 \quad \text{on } M \setminus O. \quad (21.8)$$

These are smooth skew-adjoint multipliers, and

$$\langle \psi, Q_q \psi \rangle_{L^2, \mathbb{H}} = \int_M \rho q dv_g = q.$$

Hence $P_\ell C(Q_q)|_\ell = q$ in the frame ψ .

Suppose $k \geq 2$. Choose $Q^\circ \in C_c^\infty(U; \mathfrak{sp}(\Sigma M))$ with $C(Q^\circ) = M_0$, and put $Q(q) = Q^\circ + Q_q$. Equip H^1 with the graph norm

$$\|u\|_K^2 = \|u\|_2^2 + \|Ku\|_2^2.$$

By (2.9) this norm is equivalent to the H^1 norm, and $H^1 = F \oplus (F^\perp \cap H^1)$ is an orthogonal decomposition for it. The complementary operator

$$K_c : F^\perp \cap H^1 \longrightarrow F^\perp$$

is bijective and has a bounded inverse. Write $Q_{ij}(q)$ for the blocks of $Q(q)$ relative to $F \oplus F^\perp$. For small (t, q) , Lemma 6.6, with these fixed graph domains, gives the skew-adjoint Schur matrix

$$S(t, q) = tC(Q(q)) - t^2 Q_{12}(q)(K_c + tQ_{22}(q))^{-1} Q_{21}(q).$$

The complementary inverse is smooth. Thus $S(t, q)/t$ extends smoothly to $t = 0$, with value $C(Q(q))$, and takes values in $\mathfrak{sp}(F)$.

At $(t, q) = (0, 0)$ this matrix is M_0 . Apply Lemma 6.6 in the orthogonal decomposition $F = \ell \oplus \ell^\perp$. Its residual matrix $s(t, q) \in \mathfrak{sp}(\ell)$ satisfies

$$s(0, 0) = 0, \quad D_q s(0, 0) \dot{q} = P_\ell C(Q \dot{q})|_\ell = \dot{q}.$$

The derivative of the Schur product is zero at $(0, 0)$, since both off-diagonal blocks of M_0 vanish. The implicit function theorem [46, Theorem 13.20, p. 71] gives a smooth $q(t)$ with $q(0) = 0$ and $s(t, q(t)) = 0$. For a sufficiently small fixed $t \neq 0$, both complementary operators remain invertible. The two kernel isomorphisms of Lemma 6.6 give

$$\ker(K + tQ(q(t))) \simeq \ell.$$

By Lemma 2.2, this operator is $K_1 = i(D_g + c_g(\tau_1))$ for a smooth three-form τ_1 compactly supported in U . If $k = 1$, take $\tau_1 = \tau_0$ and $K_1 = K$. In both cases $E = \ker K_1$ is a quaternionic line.

Repeat (21.8) with an L^2 -unit generator of E , and denote the resulting multipliers by R_η , $\eta \in \text{Im } \mathbb{H}$. Consider the self-adjoint family

$$T_\eta = -i(K_1 + R_\eta) = D_g + c_g(\tau_1) - iR_\eta$$

on the common domain H^1 . A small symmetric spectral circle separates the entire zero eigenspace E of T_0 from the remaining spectrum. It remains in the resolvent for small η . Since $JT_\eta = -T_\eta J$, the cluster projection commutes with J . Lemma 6.1 gives a smooth quaternionic isometric frame U_η based on E . Put

$$f(\eta) = iU_\eta^* T_\eta U_\eta \in \mathfrak{sp}(E).$$

Lemma 6.4, applied to T_η , gives

$$f(0) = 0, \quad Df(0)\dot{\eta} = iP_E(-iR_{\dot{\eta}})|_E = P_ER_{\dot{\eta}}|_E = \dot{\eta}.$$

The inverse function theorem [46, Theorem 11.22 and Corollary 11.35, pp. 61–63] gives a smooth change of parameters for which the complete matrix of $i(D_g + c_g(\tau(q)))$ is q . The corresponding self-adjoint matrix is $-iq \in \mathcal{A}_5(E)$. $\square$

Proof of the five-dimensional assertion of Theorem L. Propositions 21.2 and 21.3 give a full potential section on the fixed metric g , with self-adjoint matrix $-iq$, where $q \in \mathfrak{sp}(1)$. Restrict its parameter domain to a small closed ball on which one symmetric circle separates the full zero cluster from the remaining spectrum. The finitely many potential directions have one common compact support in U .

Apply Proposition 6.3, with the self-adjoint coordinate $A = -iq$. In its application of Theorem R require also

$$\sup_q \|h_q - g\|_{C_g^0} < \varepsilon.$$

The resulting smooth metric family satisfies

$$h_q = g \text{ on } M \setminus U, \quad dv_{h_q} = dv_g,$$

and its complete matrix for iD_{h_q} is q . At $q = 0$ the whole kernel is the selected quaternionic line, so $\dim_{\mathbb{C}} \ker D_{h_0} = 2$. The derivative of its normal matrix is the identity. $\square$

22. SPACES OF DIRAC-MINIMAL METRICS

Let M , g_* , μ , and the nonempty open set U be as in (8.2). All spaces carry the smooth topology. In the first three subsections put

$$(\mathcal{Z}, r) = \begin{cases} (\mathcal{R}_{\min, \mu}(U), 4a + 4), & n = 4, \\ (\mathcal{R}_{\text{inv}, \mu}(U), 3), & n = 5. \end{cases}$$

In dimension four the chosen chiral block has quaternionic index $-a$.

22.1. Relative incidence families.

Lemma 22.1. *The space $\mathcal{Z}$ is nonempty. Every smooth map $S^j \rightarrow \mathcal{Z}$, $0 \leq j \leq r - 2$, extends to a smooth disk in $\mathcal{Z}$.*

Proof. Logarithmic metric coordinates extend the sphere to a smooth family in $\mathcal{R}_\mu(U)$ over D^{j+1} , constant in the normal coordinate near its boundary. Since $j + 1 < r$, Lemma 10.4 gives supported potentials, zero on that collar, for which every operator is minimal. Proposition 8.2, relative to the collar, gives the required metric disk. The same argument with a one-point parameter space proves nonemptiness. For continuous sphere maps, smooth approximation in the open minimal locus and the resulting short metric homotopy give continuous fillings with the original boundary, as in Proposition 8.2. $\square$

22.2. Oriented incidence.

Lemma 22.2. *Let the family in Lemma 10.4 be transverse, with oriented parameter manifold of dimension r or $r + 1$. Every incident kernel is a quaternionic line, and the incidence manifold has a canonical orientation compatible with its boundary.*

Proof. If a kernel had quaternionic dimension at least two, its projective space would contain $\mathbb{H}P^1$. This is a smooth four-dimensional submanifold in the incidence fibre over that parameter. A transverse incidence manifold has dimension at most one, which is impossible.

In dimension four, at a simple kernel the minority line ℓ and its cokernel C have quaternionic ranks 1 and $a + 1$. Block elimination gives the normal space

$$\mathrm{Hom}_{\mathbb{H}}(\ell, C).$$

Choose orthonormal quaternionic frames. They identify it with the oriented real space $\mathbb{H}^{a+1}$. Frame changes belong to $\mathrm{Sp}(1) \times \mathrm{Sp}(a + 1)$, a connected group, and consequently preserve this orientation. No choice of a global right-quaternionic structure on this Hom bundle is required. The derivative in the parameters is onto the normal space, since the incidence is transverse. Its kernel, the tangent of the incidence manifold, is oriented by the parameter orientation and this normal orientation.

This is also the orientation of the real determinant line of the quaternionic vertical operator on a projective slice; changing its unit representative through $\mathrm{Sp}(1)$ preserves it.

In dimension five the normal space is $\mathfrak{sp}(\ell)$. Quaternionic unit frames identify it with $\mathrm{Im} \mathbb{H}$. Their changes act by the adjoint representation of $\mathrm{Sp}(1)$, with image in $SO(3)$, so this real three-space is canonically oriented. Let $\mathcal{I}$ denote the incidence manifold in the oriented parameter manifold P , and let N be its canonically oriented normal bundle. We orient $\mathcal{I}$ by the convention

$$\mathrm{or}(TP) = \mathrm{or}(T\mathcal{I}) \otimes \mathrm{or}(N).$$

On a product boundary collar the outward normal to ∂P is also the outward normal to $\partial \mathcal{I}$ within $\mathcal{I}$. With outward-normal-first boundary orientations, cancellation of this common first factor gives

$$\mathrm{or}(T\partial P) = \mathrm{or}(T\partial \mathcal{I}) \otimes \mathrm{or}(N).$$

There is no change of order past N , so the same convention applies to its even rank $4a + 4$ in dimension four and odd rank 3 in dimension five. This proves compatibility with boundary restriction. $\square$

22.3. Local integer classes.

Proposition 22.3. *For every basepoint g_b in the indicated space there are homomorphisms*

$$\begin{aligned} \lambda_4 &: \pi_{4a+3}(\mathcal{R}_{\min, \mu}(U), g_b) \longrightarrow \mathbb{Z}, \\ \lambda_5 &: \pi_2(\mathcal{R}_{\mathrm{inv}, \mu}(U), g_b) \longrightarrow \mathbb{Z}. \end{aligned} \tag{22.1}$$

They are defined by the signed number of kernel lines in a generic potential-perturbed disk filling.

Proof. For a smooth based map $f : S^{r-1} \rightarrow \mathcal{Z}$, choose a smooth filling g_p , $p \in D^r$, in $\mathcal{R}_\mu(U)$, equal to the boundary family on a collar. Lemma 10.4 gives a generic supported potential perturbation zero there. Its incidence is a compact oriented zero-manifold. Define $\lambda(f)$ to be its signed cardinality.

For two fillings write $g_p^{(i)} = g_*(e^{H_p^{(i)}}, \cdot)$, $i = 0, 1$. On the common boundary collar their logarithms agree. The formula

$$H_{p,t} = (1-t)H_p^{(0)} + tH_p^{(1)}$$

retains self-adjointness, zero trace, and spatial support. Interpolate the potentials in the fixed positive spinor identifications. Compatibility is real-linear, so this gives a homotopy relative to the boundary collar. Reparameterize it to be constant near its ends. Orient $D^r \times [0, 1]$ by $dt \wedge \text{or}(D^r)$. Apply the relative part of Lemma 10.4, retaining the two transverse endpoint families and the side collar. The resulting incidence is a compact oriented one-manifold. Its boundary consists of the final zero-manifold and the negative of the initial one. Their signed counts agree.

For a homotopy of the boundary sphere in the minimal locus, extend the given side family across $D^r \times I$ in logarithmic metric coordinates. Keep it fixed on a side collar, where there is no minority kernel. Relative transversality produces a compact oriented one-dimensional incidence whose boundary is the difference of the two endpoint incidences. Their signed cardinalities are equal.

For addition in π_{r-1} , take two disk fillings whose boundary maps are constant near their basepoints. Join the disks by a parameter one-handle on which the family is constantly g_b . There are no incidences on this handle. Smoothing the corners gives a disk whose boundary represents the sum of the two sphere maps. Its signed count is the sum of the two counts. Thus λ is a homomorphism. Smooth approximation as in Lemma 22.1 extends the definition to continuous representatives. $\square$

Proposition 22.4. *The homomorphisms in (22.1) are split-surjective. Every C^0 neighborhood of every metric in $\mathcal{R}_\mu(U)$ contains a sphere with signed count one, based at one of its own points.*

Proof. Theorem L gives a transverse metric disk arbitrarily C^0 -close to the prescribed metric, with the same density and exterior. In dimension four its entire small chiral matrix has quaternionic ranks 1 and $a+1$ and is an exact normal column, after local reparameterization. On the boundary this column is nonzero, so the minority kernel vanishes. In dimension five its iD matrix is an exact element of $\mathfrak{sp}(1)$ and is nonzero on the boundary. The complementary spectral gaps exclude all other zero modes. Orient the disk by its normal coordinates. Its single incidence has sign $+1$.

The signed count of this disk computes the homomorphism of Proposition 22.3. Indeed a sufficiently small generic supported perturbation preserves its cluster ranks and exterior gap. Its incidence is the zero set of the resulting column or $\mathfrak{sp}(1)$ matrix. The sum of its signs is the Brouwer degree, unchanged from that of the identity because its boundary stays away from zero. Thus the boundary sphere has $\lambda = 1$.

Choose a boundary metric as basepoint. Lemma 22.1 gives a path to any prescribed basepoint; basepoint transport preserves the signed count, although this path may leave the chosen neighborhood. If x is the transported class, then $\lambda(x) = 1$, and $m \mapsto mx$ splits λ ; here $r-1 \geq 2$. $\square$

Proposition 22.5. *At index zero, the normal disks of Theorem L in dimensions four and five represent primitive ordinary real index-difference classes.*

Proof. For index zero, the normal disk classes have parameter dimensions $r = 4$ when $n = 4$ and $r = 3$ when $n = 5$. Hence

$$x_n \in KO^{-n}(D^r, S^{r-1}) \cong KO^{-(n+r)}(\text{pt}) = KO^{-8}(\text{pt}).$$

The complexifications of the quaternionic chiral and skew-adjoint models in (8.3) retain the Morita reduction:

$$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong M_2(\mathbb{C}), \quad M_2(\mathbb{C})e \cong \mathbb{C}^2, \quad e = \text{diag}(1, 0).$$

For a quaternionic multiplicity space V , the primitive corner $(V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C})e$ is its complex space of dimension $2 \dim_{\mathbb{H}} V$. Thus each quaternionic block gives one defining complex block.

In dimension four the normal chiral map is $A(q) = L_q : \mathbb{H} \rightarrow \mathbb{H}$, for $q \in D^4$. On S^3 its complex clutching map is the defining representation $\mathrm{Sp}(1) = \mathrm{SU}(2) \rightarrow \mathrm{U}(2)$. The associated complex rank-two Hopf bundle has second Chern number ± 1 . Its reduced complex class is therefore primitive.

In dimension five put $K(v) = L_v = iD(v)$ for $v = ai + bj + ck \in \mathrm{Im} \mathbb{H}$. Here $D(v) = -i\rho(v)$, where ρ is the defining complex representation. In the right-complex basis $(1, j)$,

$$D(v) = \begin{pmatrix} a & -c + ib \\ -c - ib & -a \end{pmatrix}, \quad D(v)^2 = |v|^2 I.$$

Its positive eigenline on S^2 is a Hopf line. To fix the Bott normalization, let S be an irreducible real Clifford module for four generators $c_1, \dots, c_4$ of square -1 , and put $\gamma = c_1 c_2 c_3 c_4$. In the Morita presentation with grading $\iota = \mathrm{diag}(\gamma, -\gamma)$, write

$$F_K = \begin{pmatrix} 0 & -K \\ K & 0 \end{pmatrix}, \quad e_5 = \begin{pmatrix} 0 & -\gamma \\ \gamma & 0 \end{pmatrix}.$$

Here γ acts on the Clifford factor and K on the quaternionic multiplicity factor. Thus $e_5^2 = -I$; e_5 anticommutes with the first four generators and with ι , and commutes with F_K . Ebert's Bott map [41, formula (2.23)] becomes

$$F_K + s(e_5 \iota) = \begin{pmatrix} 0 & sI - K \\ sI + K & 0 \end{pmatrix}.$$

Its chiral map is $sI + K(v) = L_{s+v}$, exactly the four-dimensional clutching family. The two local classes thus have primitive complex images in total degree eight.

Let $\beta \in K^{-2}(\mathrm{pt})$ be the complex Bott generator and $\lambda \in KO^{-8}(\mathrm{pt})$ the real Bott generator. The complexification formula is $c(\lambda) = \beta^4$ [10, (6.11) and Theorem 11.5]. Thus $c : KO^{-8}(\mathrm{pt}) \rightarrow K^{-8}(\mathrm{pt})$ is an isomorphism, and both x_4 and x_5 are primitive generators up to sign. The factor two in complexification of the degree-four coefficient generator does not occur in this total degree. These are the relative operator classes defining ordinary index difference. For the cylinder definition, use [41, Assumption 3.11 and Theorem 3.18]: the manifold is closed, the parameter pair is finite CW, and the metric paths give odd Clifford-linear pseudo-Dirac families, constant and invertible on the end collars. Based representatives are constant at the basepoint, so the relative hypothesis also holds. $\square$

22.4. The comparison maps. Consider the supported pair spaces of (8.2). Throughout this subsection $n \equiv 4$ or $5 \pmod{8}$. The support condition means equality with the prescribed data on $M \setminus U$. All additional potentials have support in one compact subset of U , chosen for the compact parameter family. In residue four choose the chiral source so that $\mathrm{ind}_{\mathbb{H}} A = -a \leq 0$, and put $Y_a(U) = \mathcal{P}_{\min, \mu}(U)$. In residue five put $Y_5(U) = \mathcal{P}_{\mathrm{inv}, \mu}(U)$. Fix a base pair b in the relevant locus. Nonemptiness will also follow from Proposition 22.24.

Here and below $\mathrm{Sp} = \mathrm{colim}_m \mathrm{Sp}(m)$ and $\mathrm{U}_{\mathrm{st}} = \mathrm{colim}_m \mathrm{U}(m)$. The quotient $\mathrm{Sp} / \mathrm{Sp}(a)$ means $\mathrm{colim}_r \mathrm{Sp}(r+a) / \mathrm{Sp}(a)$. All comparisons with these direct limits are comparisons of weak homotopy types.

Identify the spinor bundles by the positive comparison based at g_b , including its density correction. Write a pair as (H, V) , where $g = g_b(e^H \cdot, \cdot)$ and $\mathrm{tr} H = 0$. The contraction from b to (H, V) is

$$c_t(H, V) = (tH, (1-t)V_b + tV), \quad 0 \leq t \leq 1. \quad (22.2)$$

It fixes b and preserves the exterior condition. With no density condition the same formula applies without $\mathrm{tr} H = 0$.

For residue four let $\mathcal{F}_{-a}$ be the norm-topological space of bounded quaternionic Fredholm maps between the fixed, infinite dimensional Hilbert spaces $H^1(\Sigma^+)$ and $L^2(\Sigma^-)$, of index $-a$. Let $\mathcal{I}_a$ be its injective locus. The operator $A_{g,V} : H^1 \rightarrow L^2$ depends continuously on the pair. The Fredholm models give

$$\mathcal{I}_a \simeq B\mathrm{Sp}(a), \quad \mathcal{F}_{-a} \simeq B\mathrm{Sp}, \quad \mathrm{hofib}(\mathcal{I}_a \longrightarrow \mathcal{F}_{-a}) \simeq \mathrm{Sp} / \mathrm{Sp}(a). \quad (22.3)$$

Take negative reduced index for the middle identification. Thus the restriction to $\mathcal{J}_a$ sends a cokernel C to $[C] - a$ and is the usual stabilization $B\mathrm{Sp}(a) \rightarrow B\mathrm{Sp}$. With ordinary reduced index its value would be $a - [C]$.

Polar decomposition retracts $\mathcal{J}_a$ onto isometric embeddings with an a -dimensional orthogonal complement. The map to this complement has contractible Hilbert-isometry fibres, and hence gives the first identification. The quaternionic Fredholm model and contractibility of these Hilbert groups are the real Clifford Fredholm model of [13, Theorem A(k), §4, and §5, pp. 22–23]; the proof of Proposition 4.1 there explicitly includes Hilbert spaces over $\mathbb{H}$. Finally the fibration $\mathrm{Sp} / \mathrm{Sp}(a) \rightarrow B\mathrm{Sp}(a) \rightarrow B\mathrm{Sp}$ gives the last identification. Consequently

$$\kappa_a : Y_a(U) \longrightarrow \mathrm{Sp} / \mathrm{Sp}(a) \quad (22.4)$$

is represented by $A_{g,V} \in \mathcal{J}_a$, together with the reverse of the ambient path $t \mapsto A_{c_t(g,V)}$ from A_b to $A_{g,V}$. It records both the cokernel and this nullhomotopy of its stable class. For $a = 0$ this is ordinary quaternionic index difference, with its original sign. Indeed ordinary index difference is represented by the ambient path from the basepoint to the endpoint. Reversing that path in the homotopy-fibre convention negates the suspended relative class; passing from reduced index to negative reduced index negates it once more. The two changes cancel. Explicitly, the quaternionic scalar disk $A(q) = L_q : \mathbb{H} \rightarrow \mathbb{H}$, $q \in D^4$, has boundary image $q \mapsto L_q$ in $\mathrm{Sp}(1)$. This is the defining clutching map of Proposition 22.5, rather than its inverse. Orient the local Bott disk so that this ordinary relative class is the chosen positive generator; this fixes the same sign for the comparison. For $a > 0$ no cokernel framing is chosen: the map retains the cokernel and its stable nullhomotopy.

In residue five put $A = i(D_g + V)$ in the quaternionic skew-adjoint model of Section 2, and put

$$u(A) = (A + I)(A - I)^{-1} = (D_g + V - i)(D_g + V + i)^{-1}. \quad (22.5)$$

This is a quaternionic unitary which differs from I by a compact operator. It is norm-continuous by the fixed-domain resolvent identity. Let Sp_c be the group of all such unitaries, and let $\mathcal{V} \subset \mathrm{Sp}_c$ be the open subset avoiding -1 . The inclusion $\mathrm{Sp} \rightarrow \mathrm{Sp}_c$ is a weak equivalence by [70, Theorem II.14(i), pp. 109–110], applied over $\mathbb{H}$ with $p = \infty$. Here the compact operators carry the operator norm and the finite matrix groups the direct limit topology. The polar deformation retractions of the finite matrix groups and of the invertible compact perturbations of I commute with the inclusions.

The principal logarithm identifies $\mathcal{V}$ with $\{Z : Z^* = -Z, Z \text{ compact}, \|Z\| < \pi\}$. For $y \in Y_5(U)$ concatenate the ambient path $u(A_{c_t(y)})$ with

$$s \mapsto \exp((1-s)\log u(A_y) + s\log u(A_b)).$$

This is a loop based at $u(A_b)$, constant when $y = b$. It defines

$$\kappa_5 : Y_5(U) \longrightarrow \Omega \mathrm{Sp}_c \simeq \Omega \mathrm{Sp} \simeq \mathrm{Sp} / \mathrm{U}_{\mathrm{st}}. \quad (22.6)$$

The last equivalence is Bott periodicity [23, Theorems I–II, (1.3) and (1.6)]. The Fredholm interpretation is ordinary real index difference, in degree KO_{n+k+1} on π_k ; the Cayley convention is also the one in Proposition 22.23. In particular it fixes its sign.

Lemma 22.6. *Let a continuous pair family over a disk be minimal on its boundary. Suppose domain and target isomorphisms, defined over the disk, reduce its chiral operator to $F : \mathbb{H}^r \rightarrow \mathbb{H}^{r+a}$ and an invertible block. Its boundary class under (22.4) is represented by $F(F^*F)^{-1/2} \in \mathrm{Sp}(r+a) / \mathrm{Sp}(a)$, with its cokernel bundle and stable trivialization.*

Proof. The cokernel of the reduced map is identified with that of the original map. On the boundary its stable trivialization is

$$\mathbb{H}^r \oplus \mathrm{coker} F \longrightarrow \mathbb{H}^{r+a}, \quad (v, w) \mapsto F(F^*F)^{-1/2}v + w.$$

This is the Stiefel model of the homotopy fibre in (22.3). The pair disk and the ambient contraction supply homotopic nullhomotopies, since the ambient pair space is contractible. All finite coordinate

changes extend over the disk and transform its ambient nullhomotopy. The infinite invertible block gives no additional class: its Hilbert trivializations are homotopic through invertible trivializations by the contractibility used in (22.3). Thus the reduction is a reduction of the pair $(\mathcal{F}_{-a}, \mathcal{I}_a)$ and retains the displayed trivialization. $\square$

22.5. Boundary domains and supported perturbations. All Sobolev spaces in the boundary constructions are quaternionic. In dimension $n \equiv 4 \pmod{8}$ the operator is $A = D^+ + Q$, with $Q : \Sigma^+ \rightarrow \Sigma^-$ quaternionic linear. In dimension $n \equiv 5 \pmod{8}$ it is $A = i(D + V)$; thus A and its multiplication term are quaternionic skew-adjoint. Metrics and coefficients are fixed on every boundary collar under consideration. Positive spinor comparison therefore gives fixed Sobolev domains.

On a compact connected manifold W with a product boundary collar, write $K = \ker B$ for the kernel of the tangential operator. The *low APS domain* allows the strictly negative spectral subspace of B , and the *full APS domain* additionally allows K . Changing the normal-Clifford identification reverses both displayed spectral signs. These domains are elliptic, their adjoints are the domains prescribed by Green's formula, and allowing K increases the quaternionic index by $\dim_{\mathbb{H}} K$ [18, Theorems 4.6 and 4.9, Example 4.21, Corollary 6.2]. Their realizations and self-adjoint doubles have smooth kernels and satisfy the elliptic Sobolev estimates.

In the skew-adjoint case write

$$A = \sigma(\partial_t + B), \quad \sigma^* = \sigma, \quad \sigma^2 = I, \quad \sigma B = -B\sigma. \quad (22.7)$$

Suppose that $\sigma|_K$ has signature (N, N) over $\mathbb{H}$. The Green annihilator of the negative spectral subspace is that subspace plus K . Consequently

$$A_{\text{low}}^* = -A_{\text{full}}, \quad \text{ind}_{\mathbb{H}} A_{\text{low}} = -N, \quad \text{ind}_{\mathbb{H}} A_{\text{full}} = N. \quad (22.8)$$

Indeed the two indices are opposite and differ by $2N$.

Fix a compact smooth parameter manifold P of dimension d , possibly with boundary, and a nonempty open set $O \Subset \text{int } W$. Consider smooth coefficient families with the fixed collar and APS domain above. Perturbations in the next lemma have common compact support in O .

Lemma 22.7. *Arbitrarily small compatible perturbations give the following conclusions, relative to any closed parameter set near which the conclusion holds. For a quaternionic chiral family with a fixed elliptic APS domain and index $-r$, $r \geq 0$, injectivity can be attained if $d < 4(r + 1)$. For index r , surjectivity can be attained under the same inequality. Thus, if $\dim_{\mathbb{H}} K = m$ and the low index is $-r$, low injectivity and full surjectivity can be attained simultaneously when*

$$d < 4(r + 1), \quad d < 4(m - r + 1), \quad 0 \leq r \leq m. \quad (22.9)$$

For the skew family (22.7), low injectivity can be attained if $d < 4N + 3$; full surjectivity then follows.

The same assertions apply to closed families: the bounds are $d < 4(r + 1)$ for chiral index $-r$ and $d < 3$ for quaternionic skew-adjoint operators.

Proof. Consider unit L^2 kernel vectors modulo the free right action of $\text{Sp}(1)$. A local slice at ψ has tangent $\psi^{\perp_{\mathbb{H}}} \cap \text{dom } A$. In the chiral case the vertical derivative is the restriction of A to this slice, of real Fredholm index $-4(r + 1)$. If a real cokernel vector ϕ annihilates it and every supported variation $Q\psi$, then $\phi \in \text{dom } A^*$ and $A^*\phi = \psi q$ for some $q \in \mathbb{H}$. Pairing with ψq gives $q = 0$. Elliptic boundary regularity makes ϕ smooth. Where $\psi \neq 0$, the vectors $Q(x)\psi(x)$ fill the target fibre. Hence ϕ vanishes on a nonempty open subset of O and is zero by unique continuation. Unique continuation here applies to A^*A and AA^* , whose principal part is a connection Laplacian and whose remaining terms have order at most one; see [56, Theorem 6.2.8].

In the skew case the equation takes values in the real hyperplane $\psi^{\perp_{\mathbb{R}}}$, since its boundary trace is null for the Green form. The vertical real index is therefore $4 \text{ind}_{\mathbb{H}} A_{\text{low}} - 4 + 1 = -4N - 3$. An annihilator $\phi \in \psi^{\perp_{\mathbb{R}}}$ again satisfies $A_{\text{low}}^*\phi = \psi q$, and pairing with all $\psi q'$ gives $q = 0$. For a nonzero vector $v \in \mathbb{H}^s$, the vectors Qv with $Q^* = -Q$ form $v^{\perp_{\mathbb{R}}}$: they are the tangent vectors to the transitive $\text{Sp}(s)$ -action on the unit sphere. Thus $\phi = f\psi$ on the nonzero locus of ψ in O , with f real. On a

connected subball there, the two equations give $ic(df)\psi = 0$, whence $df = 0$. Unique continuation gives $\phi = c\psi$ on W ; real orthogonality gives $c = 0$.

The fixed-domain boundary estimates give a uniform H^2 bound for normalized kernel vectors. Rellich compactness in H^1 therefore allows the finite-control construction of Lemma 10.4. The universal derivative is surjective by the calculations above. Its incidence projection to the control space has index $d - 4(r + 1)$ or $d - 4N - 3$. A common regular value for all boundary faces has empty fibre when this number is negative.

For the relative assertion, take nested neighborhoods of the closed parameter set on which the required property holds. Choose the parameter cutoff zero on the smaller neighborhood and one outside the larger one. Small perturbations preserve the property on the transition region. The same finite-control argument applies elsewhere. Apply the chiral argument to the adjoint for surjectivity, and use the sum of the two control spaces for the simultaneous assertion. On a closed manifold the skew vertical index is -3 . $\square$

22.6. Auxiliary spin domains with large boundary kernel.

Lemma 22.8. *In every dimension $n \equiv 4$ or $5 \pmod{8}$ there are separating hypersurfaces $\Sigma \subset S^n$, product metrics near Σ , and compatible pairs on the two complementary domains W_0, W_1 with the following properties.*

If $n \equiv 4 \pmod{8}$, then $\dim_{\mathbb{H}} \ker B = 2r$, where r is arbitrarily large; both low APS operators are injective of index $-r$, and both full APS operators are surjective of index r . If $n \equiv 5 \pmod{8}$, then $\ker B$ has σ -signature (N, N) , with N arbitrarily large; both low APS operators are injective of index $-N$, and both full APS operators are surjective of index N . In either case B has a positive spectral gap off its kernel. A designated interior ball in W_1 can be Euclidean with zero potential.

Proof. Put $m = n - 3$. The torus T^m embeds as an oriented hypersurface in $\mathbb{R}^{m+1}$. To see this inductively, take the boundary of a rank-two tubular neighborhood of $T^{m-1} \subset \mathbb{R}^m \subset \mathbb{R}^{m+1}$. Its inclusion in $\mathbb{R}^{m+3}$ has a framed rank-three normal bundle. A tubular neighborhood in a ball of S^n is therefore $T^m \times D^3$.

The induced torus spin structure can be made periodic on every circle. Reframing by $a : T^m \rightarrow SO(3)$ changes it by the obstruction to lifting a to $\text{Spin}(3)$. Rotations in a fixed two-plane with angle $\sum_j \ell_j \theta_j$, $\ell_j \in \{0, 1\}$, give every class in $H^1(T^m; \mathbb{Z}/2)$. The chosen framing gives a product identification of the tube.

Choose an integer $b \geq 1$, put $g = 4b - 1$, and embed a genus- g handlebody H_g in $\text{int } D^3$. Set

$$W_0 = T^m \times H_g, \quad \Sigma = T^m \times \partial H_g, \quad W_1 = S^n \setminus \text{int } W_0.$$

Both domains are connected. The surface spin structure induced from H_g is bounding. To prescribe its metric, take a hyperelliptic double cover $\pi : C \rightarrow \mathbb{P}^1$ of genus g , put $H = \pi^* \mathcal{O}(1)$, and take $\kappa = H^{2b-1}$. The double-cover identities

$$K_C = H^{g-1}, \quad \pi_* \mathcal{O}_C = \mathcal{O} \oplus \mathcal{O}(-g-1)$$

give $\kappa^2 = K_C$ and $h^0(C, \kappa) = 2b$. This spin structure is bounding by [9, Proposition 4.1]. The mapping class group is transitive on the even surface spin structures [28, Lemma 2.15]; hence it can be identified with the already specified spin structure on ∂H_g . Pull back a metric in this conformal class. Serre duality and $\kappa^2 = K_C$ give complex dimension $2b$ for both chiral Dirac kernels. This boundary metric extends over H_g and can be product near its boundary; the identifying surface diffeomorphism need not extend over H_g .

Give T^m its flat periodic metric and W_0 the product metric. If m is odd, the boundary product operator has symmetric spectrum: on a nonzero surface eigenpair its eigenvalues are $\pm \sqrt{\mu^2 + \lambda^2}$, and on the surface kernel the two chiralities give μ and $-\mu$ with equal multiplicities. Thus $\eta(B) = 0$ and

$$\dim_{\mathbb{H}} \ker B = 2b \cdot 2^{(n-4)/2} = 2r. \quad (22.10)$$

All positive-degree Pontryagin forms of W_0 vanish, since its curvature comes from the three-dimensional factor. The APS formula [11, Theorem 4.2] therefore gives

$$\operatorname{ind}_{\mathbb{C}} D_{W_0, \text{low}}^+ = \int_{W_0} \hat{A} - \frac{\eta(B) + \dim_{\mathbb{C}} \ker B}{2} = -2r.$$

The quaternionic index is $-r$. Extending the product collar metric over W_1 , the integral of $\hat{A}$ on W_1 is also zero, because its sum with the integral on W_0 is $\hat{A}(S^n) = 0$. The opposite tangential operator is $-B$, so the same calculation applies to W_1 . Allowing K gives index r .

If m is even, each torus chiral kernel has complex dimension $2^{m/2-1}$. The even product formula gives

$$\dim_{\mathbb{H}} \ker B^+ = \dim_{\mathbb{H}} \ker B^- = 2^{m/2-1}(2b) = N. \quad (22.11)$$

Here the boundary has dimension 4 (mod 8), so each chiral kernel is quaternionic. Its chirality is σ up to an overall sign; (22.8) gives indices $-N$ and N .

Make the metric Euclidean on a ball in the interior of W_1 . Lemma 22.7, with parameter dimension zero and controls disjoint from that ball and the boundary, imposes all the injectivity and surjectivity conditions. The fixed elliptic operator B has discrete spectrum, so its nonzero spectrum is separated from zero. $\square$

22.7. Connected sums with a fixed APS boundary condition. For an elliptic realization A_D put

$$\mathcal{D}_D = \begin{pmatrix} 0 & A_D^* \\ A_D & 0 \end{pmatrix}, \quad \operatorname{dom} \mathcal{D}_D = \operatorname{dom} A_D \oplus \operatorname{dom} A_D^*. \quad (22.12)$$

It is self-adjoint with compact resolvent. Every cutoff constant near the old boundary preserves this domain.

Let $D_{1,p}, D_{2,p}$ be a compact continuous family of smooth first-order self-adjoint elliptic boundary problems in dimension $n \geq 3$, with fixed boundary domains and collars. On common Euclidean interior balls of radius R , assume that both operators have one fixed model: a spin Dirac operator of finite multiplicity, or a fixed sum of positive and negative Dirac copies. The connected-sum realization $D_{\rho,p}$ retains the old boundary domains. Write $E_I(D)$ for the spectral subspace of D over I .

Proposition 22.9. *Fix $\Lambda, \epsilon > 0$. Suppose that $D_{1,p} \oplus D_{2,p}$ has at most k complex eigenvalues in $(-\Lambda, \Lambda)$, counted with multiplicity, for every p . There is $\rho_0 > 0$, independent of p , such that for $0 < \rho < \rho_0$ and $|\lambda| < \Lambda - 2\epsilon$,*

$$\dim_{\mathbb{C}} E_{\{\lambda\}}(D_{1,p} \oplus D_{2,p}) \leq \dim_{\mathbb{C}} E_{[\lambda-\epsilon, \lambda+\epsilon]}(D_{\rho,p}), \quad (22.13)$$

$$\dim_{\mathbb{C}} E_{[\lambda-\epsilon, \lambda+\epsilon]}(D_{\rho,p}) \leq \dim_{\mathbb{C}} E_{[\lambda-2\epsilon, \lambda+2\epsilon]}(D_{1,p} \oplus D_{2,p}). \quad (22.14)$$

The realization is continuous in (p, ρ) for $\rho > 0$, and smooth for smooth coefficient families. If the operators and cutoffs are quaternionic-linear, the inequalities also hold in quaternionic dimension.

Proof. Take the rotationally symmetric neck of [17, proof of Theorem 2.1]. Its four radii are

$$t_2 = \rho, \quad t_1 = \rho^4/2, \quad t_{-1} = t_1/2, \quad t_{-2} = 2^{-9}\rho^{16}.$$

The metric is unchanged outside the two balls of radius t_{-2} . The neck profile f equals $|t|$ away from its middle, has absolute derivative at most one, and is positive. Choose $0 \leq \chi_i \leq 1$ equal to zero for $r \leq t_{-1}$ and one for $r \geq t_1$, with $|d\chi_i| \leq 4/t_1$. It is one near every old boundary.

The estimates used in [17, proof of Theorem 2.1, pp. 9–12] are local. Its Proposition 3.2 and two corollaries apply on the Euclidean annuli; Proposition 3.3 applies on the warped neck. For sufficiently small ρ , depending only on Λ , they give, for every original λ -eigenvector u ,

$$\|\chi_i u\|^2 \geq \frac{1}{2}\|u\|^2, \quad \|[D_i, \chi_i]u\|^2 \leq 2^{11}\rho\|u\|^2. \quad (22.15)$$

For a sum u of at most $k + 1$ orthogonal connected-sum eigenvectors with eigenvalues in $(-\Lambda, \Lambda)$, they give

$$\|u\|_{\{t|t<t_1\}} \leq \rho(k+1)\|u\|, \quad \|[D_i, \chi_i]u\| \leq 2^{11/2}\rho^{1/2}(k+1)\|u\|. \quad (22.16)$$

These estimates concern only the Euclidean annuli and the round neck. Their hypotheses hold because $n \geq 3$, the absolute spherical Dirac eigenvalues are at least one, and $\Lambda\rho + \frac{1}{2}\|f'\|_\infty \leq 1$. They apply separately to the positive and negative Dirac copies. The estimates are supported in the interior gluing region.

Cutoff transplantation of the original λ -eigenspace gives orthogonal images from the two pieces, with squared residual quotient at most $2^{12}\rho$. For $2^{12}\rho < \epsilon^2$, projection onto the glued spectral interval of radius ϵ is injective. This proves (22.13).

For (22.14), let $l \leq k$ be the original spectral rank in $[\lambda - 2\epsilon, \lambda + 2\epsilon]$. On any span V of $l + 1$ glued eigenvectors in $[\lambda - \epsilon, \lambda + \epsilon]$, the cutoff map $Ju = (\chi_1 u, \chi_2 u)$ satisfies, for small ρ ,

$$\|Ju\|^2 \geq \frac{1}{2}\|u\|^2, \quad \|((D_1 \oplus D_2) - \lambda)Ju\| \leq \frac{5}{4}\epsilon\|u\|.$$

Indeed (22.16) applies since $l + 1 \leq k + 1$. The squared residual quotient is at most $25\epsilon^2/8 < 4\epsilon^2$. Projection of JV onto the original spectral subspace of radius 2ϵ would be injective, contrary to its dimension l .

Both projection arguments use the form $u \mapsto \|(D - \lambda)u\|^2$ on $\text{dom } D$. The cutoffs are one near the old boundary and hence preserve every imposed boundary condition. This is the only change from the closed case in [17]. One may take $0 < \rho_0 < R$ sufficiently smaller than

$$\min\{2^{-4}, (100\Lambda^2)^{-1}, (2\Lambda)^{-1}, (2(k+1))^{-1}, 2^{-17}\epsilon^2/(k+1)^2\}.$$

It depends on no exterior coefficient. Fixed neck formulas give the parameter assertion. Quaternionic cutoffs and spectral projections allow the same dimension argument over $\mathbb{H}$. $\square$

Let X have a fixed collar, a Euclidean zero-potential interior ball, and an injective low APS operator of quaternionic index $-r$. Let A_p be a compact closed chiral family of index $-a$, with $\ker A_p = 0$ and common Euclidean zero-potential gluing balls. In the skew case take a closed invertible family with the same local gluing data, and put $a = 0$.

Corollary 22.10. *For all sufficiently small common radii $\rho > 0$, the low APS operator on $X \# M_p$ is injective of index $-(r + a)$. Its double has a uniform positive gap off its kernel, independent of p and ρ .*

Proof. The disconnected double has $r + a$ quaternionic zero modes. Choose $\delta > 0$ so that it has no other spectrum in $[-4\delta, 4\delta]$, uniformly in p ; compactness of the injective family supplies this choice. The upper inequality of Proposition 22.9 bounds the glued spectral rank in $[-\delta, \delta]$ by $r + a$. The APS boundary correction is unchanged. The connected-sum spin bordism is relative to this boundary; the APS formula and bordism invariance therefore give $\text{ind}_{\mathbb{H}} A_{X \# M_p, \text{low}} = \text{ind}_{\mathbb{H}} A_{X, \text{low}} + \text{ind}_{\mathbb{H}} A_p = -(r + a)$. If x, y are its kernel and cokernel dimensions, then $x + y \leq r + a$ and $x - y = -(r + a)$. Hence $x = 0$, $y = r + a$, and the entire indicated spectral interval consists of these forced zero modes. In the skew case the low index remains $-r$ by (22.8). Its double is locally unitarily equivalent to $D \oplus (-D)$, so the same spectral estimate applies. $\square$

22.8. Low APS coordinates and their uniform bounds. Fix $X = H^{1/2}(\Sigma)$, with its equivalent spectral norm, and write $X = X_{\text{allow}} \oplus K \oplus X_{\text{forb}}$. Boundary trace is denoted by γ , and P is the projection onto K . Assume that A_{low} is injective. Put $C = \ker A_{\text{low}}^*$, and let $G : L^2 \rightarrow \text{dom } A_{\text{low}}$ be its generalized inverse: $GA_{\text{low}} = I$ and $A_{\text{low}}G = I - P_C$.

Lemma 22.11. *There are bounded maps*

$$w : K \oplus X_{\text{forb}} \rightarrow H^1(W), \quad \beta : K \rightarrow C, \quad \eta : X_{\text{forb}} \rightarrow C,$$

$$H : K \oplus X_{\text{forb}} \rightarrow X_{\text{allow}}, \quad J : L^2 \rightarrow X_{\text{allow}}$$

such that $Au = f$, $P\gamma u = k$, and $Q_{\text{forb}}\gamma u = b$ if and only if

$$u = Gf + w(k, b), \quad \beta k + \eta b = P_C f. \quad (22.17)$$

Its allowed trace is $Jf + H(k, b)$. The unconstrained domain coordinates are

$$H^1(W) \longrightarrow C^\perp \oplus K \oplus X_{\text{forb}}, \quad u \longmapsto ((I - P_C)Au, P\gamma u, Q_{\text{forb}}\gamma u), \quad (22.18)$$

a bounded isomorphism with inverse $(h, k, b) \mapsto Gh + w(k, b)$. In these coordinates A is $(h, k, b) \mapsto (h, \beta k + \eta b)$.

All maps vary continuously over compact fixed-domain families, and smoothly in smooth parameters. For a family with the same collar coefficients and a common positive nonzero spectral gap δ for the low double, the maps β, η, H, J are uniformly bounded in the stated boundary norms.

Proof. Choose a bounded trace lift $\mathcal{L} : K \oplus X_{\text{forb}} \rightarrow H^1(W)$ supported in the fixed collar and satisfying

$$P\gamma \mathcal{L}(k, b) = k, \quad Q_{\text{forb}}\gamma \mathcal{L}(k, b) = b.$$

Set

$$\begin{aligned} w(k, b) &= \mathcal{L}(k, b) - GA\mathcal{L}(k, b), & \beta k &= P_C A\mathcal{L}(k, 0), \\ \eta b &= P_C A\mathcal{L}(0, b), & H(k, b) &= Q_{\text{allow}}\gamma w(k, b), & J &= Q_{\text{allow}}\gamma G. \end{aligned}$$

Then $Aw = \beta k + \eta b$, and w has the prescribed traces. Subtracting $\mathcal{L}(k, b)$ from an arbitrary section puts it in the low domain. The identity $GA_{\text{low}} = I$ proves (22.18) and (22.17). It also proves uniqueness. The constant index and injectivity make C a vector bundle; orthogonal projection and inversion on its complement give the parameter assertions.

For the last assertion, $\|G\|_{L^2 \rightarrow L^2} \leq \delta^{-1}$, while $A\mathcal{L}$ has a uniform norm because it is supported in the unchanged collar. This bounds β and η . If $v = Gf$, multiply v by a fixed cutoff ζ supported in that collar and equal to one near Σ . Regard ζv as a section on the original fixed cap. It has its low APS boundary condition, and

$$A(\zeta v) = \zeta(I - P_C)f + [A, \zeta]v.$$

The fixed-cap elliptic estimate and trace theorem give

$$\|\gamma Gf\|_{H^{1/2}} \leq C(\|f\|_{L^2} + \|Gf\|_{L^2}) \leq C(1 + \delta^{-1})\|f\|_{L^2}. \quad (22.19)$$

Applying this also to $f = A\mathcal{L}(k, b)$ bounds H and J . The constant depends only on the fixed collar and δ . $\square$

The domain extension gives the exact sequence

$$0 \longrightarrow \ker A_{\text{full}} \xrightarrow{P\gamma} K \xrightarrow{\beta} C \longrightarrow \text{coker } A_{\text{full}} \longrightarrow 0. \quad (22.20)$$

Indeed the full domain in (22.18) has $b = 0$. If A_{full} is surjective, put $E = \ker \beta$ and $e = -\beta^\dagger \eta$. The harmonic trace condition is then

$$P_{E^\perp}(k - eb) = 0, \quad \gamma_{\text{allow}} u = H(k, b). \quad (22.21)$$

For skew operators E is a maximal null subspace of (K, σ) : Green's identity makes it null and (22.8) gives its dimension N .

Lemma 22.12. *In Corollary 22.10, let $C_X = \ker A_{X, \text{low}}^*$ and $C_{p, \rho} = \ker A_{X \# M_p, \text{low}}^*$. There are continuous isometric embeddings $j_{p, \rho} : C_X \rightarrow C_{p, \rho}$ such that, uniformly in p as $\rho \rightarrow 0$,*

$$j_{p, \rho}^* \beta_{p, \rho} \longrightarrow \beta_X, \quad j_{p, \rho}^* \eta_{p, \rho} \longrightarrow \eta_X \quad (22.22)$$

in the indicated operator norms. Their boundary traces converge in $H^{1/2}$ to those of the original cap modes.

Proof. Transplant $\phi \in C_X$ by the cutoff of Proposition 22.9 and set

$$J_{p,\rho}^0 \phi = \chi_\rho \phi - G_{p,\rho}^* A_{p,\rho}^* (\chi_\rho \phi).$$

The L^2 adjoint G^* maps into $\text{dom } A_{\text{low}}^*$ and satisfies $A_{\text{low}}^* G^* = I$, by $GA_{\text{low}} = I$. Hence $J_{p,\rho}^0 \phi \in C_{p,\rho}$. The finite-dimensional smooth space C_X has a bounded pointwise evaluation norm near the gluing point. Thus

$$\|A^*(\chi_\rho \phi)\|_{L^2} \leq Ct_1^{(n-2)/2} \|\phi\|_{L^2} = o(1) \|\phi\|_{L^2}.$$

The uniform inverse bound makes the correction $o(1)$ in L^2 . It is harmonic on the fixed boundary collar; the adjoint version of (22.19) makes its boundary trace $o(1)$ in $H^{1/2}$. The cutoff also loses $o(1)$ of the norm of ϕ . Therefore $(J^0)^* J^0 = I + o(1)$, and $j = J^0((J^0)^* J^0)^{-1/2}$ is isometric and has the same boundary-trace convergence. On the fixed collar $\mathcal{C}$, $\|j\phi - \phi\|_{L^2(\mathcal{C})} = o(1) \|\phi\|_{L^2}$ uniformly for $\phi \in C_X$, since $\chi_\rho = 1$ there and the polar factor is $I + o(1)$. The section $\mathcal{AL}$ is supported on this collar, so

$$\langle j\phi, \beta_{p,\rho} k \rangle = \langle j\phi, \mathcal{AL}(k, 0) \rangle = \langle \phi, A_X \mathcal{L}(k, 0) \rangle + o(1) \|\phi\| \|k\|.$$

The same argument with $\mathcal{L}(0, b)$ proves the second convergence, uniformly for bounded $b \in X_{\text{forb}}$. $\square$

22.9. Boundary evolution operators. The boundary is connected. Fix a nonempty open subset O ; every multiplier below has compact spatial support in O . A matrix on a subspace is extended by zero on its orthogonal complement.

Lemma 22.13. (i) *Let E be a quaternionic subspace of a finite spectral sum of a self-adjoint quaternionic Dirac-type operator B on a connected closed manifold. Then*

$$\text{Lie}_{\mathbb{R}}\{P_E V P_E : V \in C_c^\infty(O, \text{End}_{\mathbb{H}} \Sigma)\} = \text{End}_{\mathbb{H}}(E).$$

(ii) *Let $K = \ker B$. Suppose a quaternionic self-adjoint involution σ on K and a smooth field of quaternionic self-adjoint involutions σ_x on the spinor fibres satisfy $e_x \sigma = \sigma_x e_x$, where e_x is evaluation. Then*

$$\text{Lie}_{\mathbb{R}}\{P_K V P_K : V^* \sigma_x + \sigma_x V = 0\} = \mathfrak{g}_\sigma := \{X \in \text{End}_{\mathbb{H}} K : X^* \sigma + \sigma X = 0\}.$$

For signature (N, N) this is $\mathfrak{sp}(N, N)$.

Proof. For a zero-dimensional controlled space both Lie algebras are zero; assume henceforth that its dimension is positive. A section in a finite spectral sum cannot vanish on an open set unless it is zero. Indeed, $\prod_{\mu \neq \lambda} (B - \mu)$ isolates its λ -component. The operator $(B - \lambda)^2$ has scalar Laplace principal part and smooth terms of order at most one. The system version of unique continuation in [7, pp. 235–236 and Remark 3, p. 248] therefore applies to each component. In particular, the nonzero locus of every nonzero section under consideration is dense and open.

Put $R_x = \text{im } e_x^*$. Normalized multiplier bumps concentrating at $x \in O$ give $e_x^* V_x e_x$ as a limit of compressions. The space of compressions is a linear subspace of a finite-dimensional real vector space and is closed. These pointwise matrices therefore belong to it. In (i) they give all of $\text{End}_{\mathbb{H}}(R_x)$.

Let $\mathfrak{g}$ be the generated algebra. It is closed under adjoints. If a quaternionic subspace W is invariant, so is $W^\perp$, and $\int v^* V u = 0$ for $u \in W$, $v \in W^\perp$ and every allowed V . For nonzero u, v , choose a nonnegative bump χ in their common nonzero locus in O and set $V = \chi v u^*$. Then

$$\int v^* V u = \int \chi |u|^2 |v|^2 > 0.$$

Thus the action is quaternionically irreducible.

Choose W of maximal dimension with $\text{End}_{\mathbb{H}}(W) \subset \mathfrak{g}$; some nonzero R_x supplies one. If $W \neq E$, irreducibility gives $B_0 \in \mathfrak{g}$ with $X = (I - P_W) B_0 P_W \neq 0$. Since $P_W \in \mathfrak{g}$,

$$X = \frac{1}{2} (\text{ad}_{P_W}^2 B_0 - \text{ad}_{P_W} B_0) \in \mathfrak{g}.$$

Choose $v \in W$ with $w = Xv \neq 0$. For $u \in W$ of norm one and $q \in \mathbb{H}$,

$$[X, v q u^*] = w q u^*, \quad [w q u^*, u w^*] = w q w^* - |w|^2 u q u^*.$$

Together with adjoints, these matrices give $\text{End}_{\mathbb{H}}(W \oplus w\mathbb{H})$, contradicting maximality. This proves (i).

For (ii), R_x is σ -invariant, and $G_x = e_x^* e_x|_{R_x}$ is positive invertible and commutes with σ . For $T \in \mathfrak{g}_{\sigma}(R_x)$, the matrix $e_x G_x^{-1} T G_x^{-1} e_x^*$ on e_x , extended by zero, is σ_x -skew and compresses to T . Such fibre matrices extend locally to allowed smooth multipliers, by applying $V \mapsto (V - \sigma_x V^* \sigma_x)/2$ to any extension. The bump argument thus puts a nonzero full σ -skew block in $\mathfrak{g}$. This algebra is closed under ordinary adjoints and conjugation by σ .

It is again quaternionically irreducible. Otherwise nonzero $u \in W$, $v \in W^{\perp}$ satisfy $v^* V u = 0$ pointwise for every allowed fibre matrix. Write $V = \sigma_x Q$, $Q^* = -Q$. For a unit vector u , the vectors Qu fill its real tangent hyperplane: if $\text{Re}(u^* w) = 0$, then

$$Q = wu^* - uw^* - u(u^* w)u^*, \quad Q^* = -Q, \quad Qu = w.$$

Hence $(\sigma_x v)\mathbb{H}$ lies in the real line $\mathbb{R}u$ and $v = 0$ where $u \neq 0$. This contradicts the common dense nonzero loci.

Choose a maximal σ -invariant W with $\mathfrak{g}_{\sigma}(W) \subset \mathfrak{g}$. If it is proper, choose $B_0 \in \mathfrak{g}$ with a nonzero block $W \rightarrow W^{\perp}$ and subtract its W block. Choose a unit σ -eigenvector $e \in W$ on which that block is nonzero; write $\sigma e = \varepsilon e$ and $J_e(p) = e p e^*$ for imaginary p . Since $J_e(i)^2 = -P_e$ and $J_e(i)B_0 J_e(i) = 0$,

$$C = -[J_e(i), [J_e(i), B_0]] = P_e B_0 + B_0 P_e \in \mathfrak{g}.$$

For some $\eta \in \{1, -1\}$, the matrix $(C + \varepsilon \eta \sigma C \sigma)/2$ is nonzero on e . Dividing by the norm of its value there gives

$$B(1) = z e^* - \varepsilon \eta e z^*, \quad |z| = 1, \quad \sigma z = \eta z, \quad z \perp W.$$

Put $B(q) = z q e^* - \varepsilon \eta e \bar{q} z^*$. Direct multiplication gives, for imaginary p, s ,

$$\begin{aligned} [J_e(p), B(q)] &= B(-qp), \\ [B(s), B(1)] &= 2\varepsilon \eta (J_e(s) - J_z(s)). \end{aligned}$$

The first identity gives every $B(q)$; the second gives $\mathfrak{sp}(1)$ on $z\mathbb{H}$. For a σ -eigenunit $f \in W \cap e^{\perp}$, with sign ε_f ,

$$[B(1), e q f^* - \varepsilon \varepsilon_f f \bar{q} e^*] = z q f^* - \eta \varepsilon_f f \bar{q} z^*.$$

These are the remaining cross blocks of $\mathfrak{g}_{\sigma}(W \oplus z\mathbb{H})$, again a contradiction. $\square$

For compact continuous smooth-coefficient families of these data with fixed band ranks, finitely many continuous families of allowed supported multipliers and Lie words suffice. Choose spanning words locally, extend their multipliers using parameter cutoffs, and take a finite cover. Their spanning map L_p has the continuous right inverse $L_p^*(L_p L_p^*)^{-1}$. This is a bundle formula and requires no global eigenframe. In the applications below B is fixed. The independent normal-Clifford block of a chiral potential is an arbitrary quaternionic multiplier, with adjoint opposite block; in the skew model it is σ -skew. Thus these are allowed controls.

For the endpoint statements, write K for the controlled space and $m = \dim_{\mathbb{H}} K$. Fix a time interval compactly contained in $(0, 1)$.

Lemma 22.14. *For $G = \text{GL}_m(\mathbb{H})$ in case (i), or $G = \text{Sp}(N, N)$ in case (ii), the endpoint map of $u' = -P_K V(t) P_K u$ has a local section at I . Its values are smooth controls with the fixed spatial and time supports; it is continuous in C^{∞} and sends I to zero.*

Proof. Choose Lie monomials Y_j of lengths ℓ_j in finitely many compressed generators $-P_K V P_K$, spanning the real Lie algebra. Recursive group commutators give analytic exponential words

$$W_j(s) = I + s^{\ell_j} Y_j + O(s^{\ell_j+1}).$$

Use $W_j(z^{1/\ell_j})$ for $z \geq 0$ and $W_j((-z)^{1/\ell_j})^{-1}$ for $z \leq 0$. The result is C^1 at zero, with expansion and derivative

$$I + z Y_j + O(|z|^{1+1/\ell_j}), \quad Y_j + O(|z|^{1/\ell_j}).$$

Realize the two words on two fixed lists of disjoint time slots, using amplitudes $(\max(z, 0))^{1/\ell_j}$ and $(\max(-z, 0))^{1/\ell_j}$. A smooth bump of integral one realizes each exponential exactly. Reversing the slots and signs realizes the inverse word. Every amplitude vanishes at zero.

The concatenated endpoint has derivative $L : (z_j) \mapsto \sum_j z_j Y_j$ at zero. Composing with a linear right inverse of L and a chart at I gives derivative I . The C^1 inverse function theorem [46, Theorem 11.22, p. 61] gives the section. Its controls are finite sums of fixed bumps with continuous amplitudes. If $\ell = \max_j \ell_j$, every fixed C^∞ seminorm is bounded by $C\|T - I\|^{1/\ell}$ near I . For compact families the preceding bundle right inverses and the parameter version of the same inverse function theorem give one uniform section neighborhood. $\square$

Lemma 22.15. *Let P be compact, $A \subset P$ closed, and V_p^0 a continuous C^∞ family of controls with endpoint H_p , zero on a reserved final interval. If a continuous G -homotopy from H_p to T_p is fixed on A , there are continuous C^∞ controls with endpoints T_p which equal V_p^0 for every $p \in A$.*

Proof. For the given homotopy $T_{p,s}$ put $Q_{p,s} = T_{p,s}H_p^{-1}$. Then $Q_{p,0} = I$ and $Q_{p,s} = I$ on A . Compactness gives a subdivision $0 = s_0 < \dots < s_l = 1$ such that

$$\Delta_{p,j} = Q_{p,s_j} Q_{p,s_{j-1}}^{-1}$$

lies in the neighborhood of Lemma 22.14 for every p, j . Place the section controls for these increments in successive fixed subintervals of the reserved interval. Their endpoint product is $\Delta_{p,l} \dots \Delta_{p,1} = Q_{p,1}$, so the full endpoint is $Q_{p,1}H_p = T_p$. On A every increment is I and every added control is zero. Thus equality there holds for the controls themselves. $\square$

Both groups G are connected. Applying the lemma to a path from I therefore realizes every endpoint by a finite smooth control. For a compact control family with $s_{\min}(F_p) \geq \delta > 0$, where $F_p = (\beta_0, \beta_1 T_p)$, Proposition 22.17 gives, for all sufficiently large L ,

$$\|M_{L,p} - F_p\| < \delta/2, \quad \|M_{L,p}k\| \geq (\delta/2)\|k\|.$$

A final interval can be reserved by time reparametrization. Choose a smooth nondecreasing $\phi : [0, 1] \rightarrow [0, 1]$ with $\phi(0) = 0$ and $\phi = 1$ on the desired final interval, and put $\phi_s(t) = (1-s)t + s\phi(t)$. Replacing $V(t)$ by $\phi'_s(t)V(\phi_s(t))$ preserves its endpoint. For controls zero near the original endpoints this is a compact continuous family of smooth controls, with a reserved interval at $s = 1$. If F_p is injective, the preceding estimate keeps this annulus injective for one sufficiently long cylinder. Its final boundary control is then held fixed.

22.10. The inhomogeneous problem on a long cylinder. Let B be the fixed self-adjoint quaternionic tangential operator. Put $K = \ker B$, $P = 1_{\{0\}}(B)$, $Q = I - P$, $Q_\pm = 1_{\mathbb{R}_\pm}(B)$, $X = H^{1/2}(\Sigma)$, and $X_\pm = Q_\pm X$, using the equivalent spectral norm on X . Assume $|B| \geq \gamma_0 > 0$ on $K^\perp$. Let $V_z(t, x)$ be a compact continuous C^∞ family of quaternionic multipliers and define

$$U'_z + PV_zP U_z = 0, \quad U_z(0) = I, \quad T_z = U_z(1), \quad \epsilon = L^{-1}.$$

For smooth gluing require $V_z = 0$ near $t = 0, 1$. The cylinder operator in normal-Clifford coordinates is then $\partial_s + B + L^{-1}V_z(s/L, x)$ on $[0, L] \times \Sigma$. Pulling back by increasing diffeomorphisms which are translations in fixed end collars places all lengths on one fixed manifold. The pulled-back coefficients vary smoothly with $L \geq 1$ and join the unchanged exterior coefficients. In the skew case impose $V_z^* \sigma + \sigma V_z = 0$.

Parameter derivatives below keep ϵ fixed and are taken in smooth finite-dimensional control parameters. For each prescribed finite order assume uniform bounds on the corresponding coefficient derivatives.

Lemma 22.16. *For sufficiently small $\epsilon > 0$, the map*

$$\begin{aligned} C_{\epsilon,z} : H^1([0, 1] \times \Sigma) &\longrightarrow L^2([0, 1] \times \Sigma) \oplus K \oplus X_+ \oplus X_-, \\ w &\longmapsto ((\partial_t + \epsilon^{-1}B + V_z)w, Pw(0), Q_+w(0), Q_-w(1)) \end{aligned} \tag{22.23}$$

is a bounded isomorphism. For data (f, k, a, d) its remaining traces $k' = Pw(1)$, $b = Q_-w(0)$, $c = Q_+w(1)$ satisfy

$$k' = T_z k + E_{\epsilon, z}(k, a, d) + F_{\epsilon, z} f, \quad (22.24)$$

$$(b, c) = B_{\epsilon, z}(k, a, d) + J_{\epsilon, z} f, \quad (22.25)$$

with

$$\|E_{\epsilon, z}\| + \|B_{\epsilon, z}\| \leq C\epsilon, \quad \|F_{\epsilon, z}\| \leq C, \quad \|J_{\epsilon, z}\| \leq C\epsilon^{1/2}.$$

For $f = 0$, put $u = Pw$, $v = Qw$. Then

$$\|u - U_z k\|_{C_t^0 K} \leq C\epsilon(\|k\| + \|a\|_X + \|d\|_X), \quad (22.26)$$

$$\|v\|_{C_t^0 X} \leq C(\|a\|_X + \|d\|_X + \epsilon\|k\|), \quad \|v\|_{L_t^1 X} \leq C\epsilon(\|k\| + \|a\|_X + \|d\|_X). \quad (22.27)$$

The constants are uniform. The homogeneous estimates hold after the specified control derivatives.

Endpoint vanishing is unnecessary for this analytic lemma. After affine reparametrization it applies on every fixed subinterval of positive length, with that interval's fundamental matrix.

Proof. For $V = 0$ the incoming high solution and the zero-incoming Green operator are

$$\begin{aligned} v_{\text{in}}(t) &= e^{-tB_+/\epsilon} a + e^{-(1-t)|B_-|/\epsilon} d, \\ (G_\epsilon h)(t) &= \int_0^t e^{-(t-s)B_+/\epsilon} Q_+ h(s) ds - \int_t^1 e^{-(s-t)|B_-|/\epsilon} Q_- h(s) ds. \end{aligned}$$

The incoming term has $C_t^0 X$ norm at most $\|a\|_X + \|d\|_X$, $L_t^1 X$ norm at most $\epsilon\gamma_0^{-1}(\|a\|_X + \|d\|_X)$, and outgoing norm at most $e^{-\gamma_0/\epsilon}(\|a\|_X + \|d\|_X)$. The exponential kernels give

$$\|G_\epsilon\|_{C_t^0 X \rightarrow C_t^0 X} + \|G_\epsilon\|_{L_t^1 X \rightarrow L_t^1 X} \leq C\epsilon, \quad \|G_\epsilon\|_{L_t^2 L^2 \rightarrow L_t^2 L^2} \leq \epsilon/\gamma_0. \quad (22.28)$$

For a positive eigenvalue λ and $v(0) = 0$,

$$\int_0^1 |v' + \lambda v/\epsilon|^2 = \int_0^1 (|v'|^2 + \lambda^2 |v|^2/\epsilon^2) + \lambda |v(1)|^2/\epsilon.$$

For $\lambda < 0$, $v(1) = 0$ gives the nonnegative boundary term $|\lambda| |v(0)|^2/\epsilon$. Cauchy-Schwarz in either integral also gives $|\lambda| |v(t)|^2 \leq (\epsilon/2) \|h_\lambda\|_{L^2}^2$. Summing over the eigenmodes proves

$$\|\partial_t G_\epsilon h\|_{L^2} + \epsilon^{-1} \|B G_\epsilon h\|_{L^2} \leq C \|h\|_{L^2}, \quad \|G_\epsilon h\|_{C_t^0 X} \leq C\epsilon^{1/2} \|h\|_{L^2}. \quad (22.29)$$

Spectral truncation gives the stated C^0 representative; the elliptic graph norm of B gives the spatial H^1 norm. The same mode calculation shows that the incoming semigroups belong to H^1 exactly for data in $X_\pm$.

Let $R_z : L_t^1 K \rightarrow C_t^0 K$ solve $u' + P V_z P u = g$, $u(0) = 0$. The coupled equations are

$$u = U_z k + R_z(Pf - P V_z v), \quad v = v_{\text{in}} + G_\epsilon Q f - G_\epsilon Q V_z(u + v).$$

Eliminating u gives

$$(I + \mathcal{K}_{\epsilon, z})v = v_{\text{in}} + G_\epsilon Q f - G_\epsilon Q V_z(U_z k + R_z P f), \quad \mathcal{K}_{\epsilon, z} = G_\epsilon Q V_z(I - R_z P V_z).$$

Multiplication by V_z is uniformly bounded on X and L^2 ; K is finite dimensional. Thus $\|\mathcal{K}_{\epsilon, z}\| \leq C\epsilon$ on $C_t^0 X$, $L_t^1 X$, and $L_t^2 L^2$. Its Neumann inverse gives the homogeneous solution in the first space and the source solution in the last, with uniqueness in $L_t^2 L^2$.

For $f = 0$, the $L_t^1 X$ equation and the low equation give

$$\|v\|_{L_t^1 X} \leq C\epsilon(\|a\|_X + \|d\|_X + \|u\|_{C_t^0 K}), \quad \|u\|_{C_t^0 K} \leq C\|k\| + C\|v\|_{L_t^1 X}.$$

Absorption and then the $C_t^0 X$ equation prove (22.26)–(22.27). At the outgoing endpoints the high integrals are bounded by $C\epsilon\|u + v\|_{C_t^0 X}$, in addition to the incoming exponential remainder. This proves the bound for $B_{\epsilon, z}$; evaluating $-R_z P V_z v$ at 1 proves the bound for $E_{\epsilon, z}$.

For source data alone the L^2 inverse gives $\|v\|_{L^2} \leq C\epsilon\|f\|_{L^2}$ and $\|u\|_{C^0_K} \leq C\|f\|_{L^2}$. Apply (22.29) to $Q(f - V_z(u + v))$. This gives $v \in H^1$ and the outgoing $C\epsilon^{1/2}\|f\|_{L^2}$ bound; the low equation gives $u \in H^1$ and the bound for $F_{\epsilon,z}$. The homogeneous solution is also in H^1 , since its incoming term is in H^1 and its remaining high source is in L^2 . These formulas give a bounded inverse into H^1 for each fixed $\epsilon > 0$.

At fixed ϵ , differentiate the finite fundamental matrices and $(I + \mathcal{K}_{\epsilon,z})^{-1}$. Every derivative of $\mathcal{K}_{\epsilon,z}$ retains the factor G_ϵ ; the inverse derivative identity $\partial S^{-1} = -S^{-1}(\partial S)S^{-1}$ and induction preserve the homogeneous $O(\epsilon)$ bounds. All remaining factors have the specified uniform derivative bounds. For an interval of length $h > 0$, affine rescaling replaces ϵ by ϵ/h and V by hV , proving the final observation. $\square$

Join compact pieces W_0, W_1 by this cylinder. In the common boundary identification their low APS domains allow X_+ and X_- , respectively. Assume these realizations are injective. Let $C_i, \beta_i, \eta_i, H_i, J_i$ denote the maps and spaces of Lemma 22.11, with uniformly bounded boundary maps over the parameter family.

Proposition 22.17. *For all sufficiently large finite L , the glued quaternionic operator $H^1 \rightarrow L^2$ has an exact bounded reduction to identity operators and a finite map*

$$M_{L,p,z} : K \longrightarrow C_{0,p} \oplus C_{1,p}, \quad M_{L,p,z} = (\beta_{0,p}, \beta_{1,p} T_z) + O(L^{-1}). \quad (22.30)$$

The error is uniform, including the thin families of Corollary 22.10, and has the same control derivatives. The index is $\dim_{\mathbb{H}} K - \dim_{\mathbb{H}} C_0 - \dim_{\mathbb{H}} C_1$.

This is a reduction of the Fredholm pair with its injective locus, and identifies the actual cokernel bundles there. On any compact subset where the limiting finite map is injective, the straight homotopy to that map remains injective for sufficiently large L . In the skew case the assertion concerns the quaternionic Fredholm pair; Proposition 22.23 computes the skew class.

Proof. In the skew case multiply the cylinder equation by σ , a bounded target isomorphism. On the direct sum of the three H^1 spaces include the transmission equations:

$$\mathcal{A}(u_0, w, u_1) = (A_0 u_0, A_C w, A_1 u_1, \gamma u_0 - w(0), \gamma u_1 - w(L)).$$

The trace-difference map onto $X \oplus X$ has a bounded right inverse supported in fixed collars. Splitting by it, and subtracting its contribution to the first three target coordinates, identifies $\mathcal{A}$ with the closed operator plus $I_{X \oplus X}$.

Apply (22.18) and (22.23). The change from s to $t = s/L$ is a bounded domain and source isomorphism for each finite L . The domain coordinates are

$$(h_0, k_0, b), \quad (f, k, a, d), \quad (h_1, k_1, c), \quad h_i \in C_i^\perp.$$

The source target coordinates are (h_0, f, h_1) . Subtracting their contributions leaves these identity blocks and the homogeneous transmission block

$$\begin{aligned} k_0 - k &= 0, & b - B_{\epsilon,z}^-(k, a, d) &= 0, & c - B_{\epsilon,z}^+(k, a, d) &= 0, \\ a - H_0(k_0, b) &= 0, & d - H_1(k_1, c) &= 0, & k_1 - T_z k - E_{\epsilon,z}(k, a, d) &= 0, \end{aligned}$$

with finite residuals $\beta_0 k_0 + \eta_0 b$ and $\beta_1 k_1 + \eta_1 c$. The right sides of these equations may be arbitrary target vectors; the elimination is on the full operator.

Suppress the parameters, put $h = (a, d)$, and write

$$B_\epsilon(k, h) = B_K k + B_H h, \quad E_\epsilon(k, h) = E_K k + E_H h,$$

where all four maps are $O(\epsilon)$. After eliminating k_0, b, c , write the allowed-trace equations as

$$h = \mathcal{H}_K(k, k_1) + \mathcal{H}_B(B_K k + B_H h).$$

Both $\mathcal{H}_K$ and $\mathcal{H}_B$ are uniformly bounded. Consequently

$$h = D_0 k + D_1 k_1, \quad (D_0, D_1) = (I - \mathcal{H}_B B_H)^{-1} (\mathcal{H}_K + \mathcal{H}_B B_K \text{pr}_1)$$

has uniformly bounded coefficients. The last low equation gives

$$k_1 = S_\epsilon k, \quad S_\epsilon = (I - E_H D_1)^{-1} (T_z + E_K + E_H D_0) = T_z + O(\epsilon).$$

Thus $h = (D_0 + D_1 S_\epsilon)k$ and $(b, c) = B_K k + B_H(D_0 + D_1 S_\epsilon)k = O(\epsilon)k$. Substitution into the two residuals defines M_L and proves (22.30).

Each elimination is a bounded invertible row or column operation, retaining an identity block for every eliminated variable. The reduction is therefore exact, including its cokernel, and varies continuously in the parameters. Triangular factors contract through invertible factors; the small blocks contract linearly to I . The remaining Hilbert coordinate changes contribute no class by quaternionic Kuiper contractibility [13, §4, proof of Proposition 4.1], as in Lemma 22.6.

If the limit has least singular value $\delta > 0$ on the indicated compact subset, choose $\|M_L - F\| < \delta/2$ there. Every point of the straight homotopy then has least singular value at least $\delta/2$. Differentiating the displayed Schur complements at fixed L proves the control derivative assertion. Exterior-parameter derivatives obey the same bounds whenever the corresponding boundary maps and their derivatives are uniformly bounded. $\square$

22.11. Insertion inside a prescribed ball. Fix a nonempty open subset U of a closed connected spin manifold M , and a compact continuous smooth-coefficient family a_p of compatible pairs. In dimension 4 (mod 8), assume its chiral operator is injective of index $-a$. In dimension 5 (mod 8), assume its skew operator is invertible. Choose the caps of Lemma 22.8 and a smooth cylinder control V_* whose endpoint satisfies

$$F_* = (\beta_0, \beta_1 T_*) : K \longrightarrow C_0 \oplus C_1 \quad \text{is an isomorphism.} \quad (22.31)$$

Such an endpoint exists: in the chiral case send $\ker \beta_0$ to a complement of $\ker \beta_1$; in the skew case send the maximal null plane E_0 to a null complement of E_1 . The preceding control lemmas realize these endpoints.

Proposition 22.18. *There are $L_0 < \infty$ and $\rho > 0$ for which a_p is homotopic, in the same injective or invertible locus and with support in a ball in U , to the family*

$$W_0 \cup ([0, L_0] \times \Sigma, V_*) \cup (W_1 \#_\rho M_p).$$

The collars and B are common. The latter exterior has injective low APS operator of index $-(r + a)$ in the chiral case and $-N$ in the skew case. For this same ρ , every $L \geq L_0$ remains in the required locus. Injectivity, and invertibility in the skew case, persist uniformly for $L \geq L_0$ under sufficiently small perturbations of the exterior coefficients and rescaled controls, with the collars fixed. If the perturbations also fix the data outside the chosen ball, the homotopy retains that support.

Proof. For the compact injective family there is C with $\|u\|_{H^1} \leq C\|A_p u\|_{L^2}$. In normal coordinates, $g_{ij} = \delta_{ij} + O(|x|^2)$ and $\partial g_{ij} = O(|x|)$, uniformly in p . A cutoff on a ball of radius s therefore changes the Dirac operator by $O(s) : H^1 \rightarrow L^2$, throughout its interpolation to the Euclidean metric. Since $n \geq 3$,

$$\|1_{B_s} Q u\|_{L^2} \leq C s \|Q\|_\infty \|u\|_{H^1}.$$

Removing the potential there has the same small norm. If the total change has norm below $1/(2C)$, the modified operator is bounded below by $1/(2C)$. This preserves chiral injectivity, and skew invertibility follows from index zero.

The Euclidean charts can be made common. In a reference chart at the fixed center x_0 , let G_p be the metric matrix and use the normal embedding $\phi_p(x) = \exp_{x_0}^{g_p}(G_p^{-1/2} x)$. Its linear part is positive definite. On a sufficiently small common ball, $\phi_{p,t}(x) = (1-t)x + t\phi_p(x)$ is an isotopy of embeddings: its derivative is a convex combination of uniformly positive matrices plus a uniformly $O(|x|)$ term.

The supported isotopy extension theorem [59, Chapter 8, §1, Theorem 1.4] extends it on a smaller ball. Parameter continuity follows by using the velocity field

$$Y_{p,t}(\phi_{p,t}(x)) = \chi(x)\partial_t\phi_{p,t}(x)$$

with one cutoff χ equal to one on that smaller ball and extending it by zero. Its flows are supported in a fixed ball in U and depend continuously on p . Pullback, with the spin lift determined by the isotopy, preserves kernels and makes the Euclidean zero-potential gluing ball common.

By Corollary 22.10, the low inverse gap and the boundary-coordinate bounds on $W_1 \#_\rho M_p$ are uniform for $0 < \rho < \rho_1$. Lemma 22.12 gives isometries $j_{p,\rho} : C_1 \rightarrow C_{1,p,\rho}$ with

$$(I \oplus j_{p,\rho}^*)(\beta_0, \beta_{1,p,\rho} T_*) \longrightarrow F_*$$

uniformly in p . Put $\delta = s_{\min}(F_*) > 0$. Since $\|I \oplus j_{p,\rho}^*\| = 1$, the unprojected rectangular map has least singular value at least $\delta/2$ after decreasing ρ_1 . Its target rank is $\dim_{\mathbb{H}} K + a$ in the chiral case. The uniform C/L error in Proposition 22.17 gives one L_0 with $C/L_0 < \delta/4$, valid for every $0 < \rho < \rho_1$ and $L \geq L_0$. The glued operator is injective; in the skew case its index is zero and it is invertible. The strict inequalities persist for small perturbations of the stated data. At L_0 , interpolation in coefficient coordinates joins such a perturbation to the displayed endpoint within this neighborhood. It fixes the exterior of the chosen ball when the perturbation does.

The two original caps joined by the length- L_0 cylinder similarly give an invertible pair b on S^n . Its designated gluing ball is Euclidean with zero potential. Choose a rotationally symmetric, zero-potential pair b_{rd} conformal to the round sphere and Euclidean on that ball. Conformal covariance makes it invertible. Interpolate the two pairs relative to the ball. The closed case of Lemma 22.7, with parameter dimension one and controls in its complement, gives an invertible path b_t , fixed near its endpoints and on a smaller gluing ball.

Apply Proposition 22.9 to the compact family (a_p, b_t) . In the chiral case the disconnected doubles have exactly a quaternionic zero modes and a common nonzero gap. The upper spectral bound and the index $-a$ imply injectivity after gluing. In the skew case the same bound excludes zero. Choose one positive ρ below this pathwise threshold and ρ_1 . This produces a homotopy from $a_p \#_\rho b_{\text{rd}}$ to $a_p \#_\rho b$. The already chosen L_0 works at this radius for every subsequent $L \geq L_0$.

The round attachment is joined to the original family by conformal covariance. Its replacement disk is radial, $ds^2 + f(s)^2 g_{S^{n-1}}$, Euclidean near the outer boundary, with $f > 0$ away from the center and smooth odd expansion $f(s) = s + O(s^3)$ there. Solve

$$\frac{dr}{r} = \frac{ds}{f(s)}, \quad r = f(s) \text{ on the outer Euclidean collar}, \quad u = \log \frac{f(s)}{r(s)}.$$

Then r is a radial diffeomorphism and the disk metric is $e^{2u}(dr^2 + r^2 g_{S^{n-1}})$. The odd expansion makes u smooth at the center; $u = 0$ on the outer collar. This identifies $M \# S^n$ with M relative to the exterior. The potential vanishes on the replacement disk, so interpolating u to zero preserves kernels and fixes the exterior pair. Combining this interpolation, the sphere path, and the initial preparation proves the proposition. All radii used are positive. $\square$

For relative constructions retain the preparation homotopy as a parameter annulus, whose outer boundary is the original family. For a prescribed density apply the supported conformal normalization of Lemma 8.1, including its rescaling of the potential.

22.12. Rectangular completion and cancellation. Quaternionic stabilization [70, Theorem II.12 and its proof, pp. 108–109] gives

$$\pi_i \text{Sp}(q) \longrightarrow \pi_i \text{Sp} \quad \text{an isomorphism for } i \leq 4q + 1, \quad \text{onto for } i = 4q + 2. \quad (22.32)$$

Comparing the fibrations with common fibre $\text{Sp}(a)$ shows that $\text{Sp}(r + a)/\text{Sp}(a) \rightarrow \text{Sp}/\text{Sp}(a)$ is $(4(r + a) + 2)$ -connected.

Let $a \geq 0$, $r \geq \max(1, a)$, $0 \leq k \leq 4r + 1$, and $d = k + 1$. Let $E_0, E_1 \subset D^d \times \mathbb{H}^{2r}$ have ranks $r, r - a$. Choose separate source and target trivializations over the disk, carrying E_0 and E_1 to fixed coordinate subspaces. In these coordinates put

$$G = \mathrm{GL}_{2r}(\mathbb{H}), \quad \mathcal{U}_a = \{T \in G : qT|_{E_0} \text{ is injective}\},$$

where q is quotient by E_1 . The trivializations extend over the disk and therefore preserve the relative extension problem.

Lemma 22.19. (a) Let $T_0 : S^k \rightarrow \mathcal{U}_a$ have a specified extension $H : D^d \rightarrow G$. If the stabilized Stiefel class of $qT_0|_{E_0}$ vanishes, there is an extension $T : D^d \rightarrow \mathcal{U}_a$ which is G -homotopic to H relative to S^k .

(b) Every based class in $\pi_k(\mathrm{Sp} / \mathrm{Sp}(a))$, $k \geq 1$, is represented, for sufficiently large r , by the first quotient columns of a based map to $\mathcal{U}_a$ that is based nullhomotopic in G .

Proof. For (a), polar normalization retracts the injective quotient columns onto the Stiefel space. The stated connectivity extends the boundary columns to a continuous family $F_p : \mathbb{H}^r \rightarrow \mathbb{H}^{r+a}$, $p \in D^d$, with $\ker F_p = 0$. Extend their E_1 components X over the disk and set $L = iF + X$, where i includes $E_1^\perp$. The columns of L are independent. Their orthogonal complement has rank r and is trivial over the disk; choose a frame W . Every completion is uniquely

$$[L, LC + WQ] = [L, W] \begin{pmatrix} I & C \\ 0 & Q \end{pmatrix}, \quad C \in \mathrm{Mat}_r(\mathbb{H}), \quad Q \in \mathrm{GL}_r(\mathbb{H}). \quad (22.33)$$

This frame is chosen only over the disk, after extending F .

On S^k , the given T_0 determines C, Q . Both T_0 and $[L, W]$ extend to G . Their quotient is the triangular matrix in (22.33), so contracting C shows that $[Q]$ maps to zero in $\pi_k G$. For $k \geq 1$, base these maps by constant translations. The stable range makes $\pi_k \mathrm{GL}_r(\mathbb{H}) \rightarrow \pi_k G$ injective; hence Q extends. The matrix C extends radially. This gives an extension T in $\mathcal{U}_a$.

Its difference from H is the boundary-identity map TH^{-1} , representing a class in $\pi_d G$. Right multiplication by $\mathrm{diag}(I_r, v)$, where $v : (D^d, S^k) \rightarrow (\mathrm{GL}_r(\mathbb{H}), I)$, preserves L . The block inclusion is onto on π_d since $d \leq 4r + 2$. Choose v to cancel $[TH^{-1}]$. Indeed the new difference is

$$(T \mathrm{diag}(I_r, v) T^{-1})(TH^{-1}).$$

Contracting the disk in the conjugating factor T reduces its action to constant conjugation, while $v = I$ on the boundary. Constant conjugation acts trivially on homotopy groups because G is connected. The new difference is nullhomotopic relative to the boundary, giving the required homotopy. For $k = 0$, the injection and completion spaces are connected and $\pi_1 G = 0$; the same construction gives the required path and relative homotopy.

For (b), represent the stable class by a based $F : S^k \rightarrow V_r(\mathbb{H}^{r+a}) = \mathrm{Sp}(r+a) / \mathrm{Sp}(a)$ in a stage large enough for the preceding stable ranges. Use $\mathbb{H}^{2r} = \mathbb{H}^{r+a} \oplus \mathbb{H}^{r-a}$, $E_1 = 0 \oplus \mathbb{H}^{r-a}$, and set $L = (F, 0)$. The rank- r complement of $\mathrm{im} L$ is stably trivial. It is trivial since $\pi_k B \mathrm{Sp}(r) \rightarrow \pi_k B \mathrm{Sp}$ is injective for $k \leq 4r + 2$, by (22.32). Choose its frame W with the prescribed basepoint value and complete to $R = [L, W]$, based at I . Surjectivity of $\pi_k \mathrm{GL}_r(\mathbb{H}) \rightarrow \pi_k G$ supplies a based v for which $R \mathrm{diag}(I_r, v)$ is nullhomotopic. Its quotient columns remain F .

For a different reference $T_* \in \mathcal{U}_a$, use $R = T T_*^{-1}$ and the transverse planes $T_* E_0, E_1$. Choose a basis of $T_* E_0 \oplus C_a \oplus E_1$, with C_a an a -dimensional complement. The construction based at $R = I$ then gives the assertion based at $T = T_*$. $\square$

22.13. Quaternionic isotropic subspaces. Let $K = \mathbb{H}^N \oplus \mathbb{H}^N$ have form $\sigma = \mathrm{diag}(I, -I)$, and let E_0, E_1 be maximal null planes. Put

$$G_N = \mathrm{Sp}(N, N), \quad \mathcal{U}_N = \{T \in G_N : T E_0 \cap E_1 = 0\}.$$

Lemma 22.20. The inclusion $\mathcal{U}_N \rightarrow G_N$ is equivalent up to homotopy to the diagonal $\mathrm{Sp}(N) \rightarrow \mathrm{Sp}(N)^2$. Consequently

$$\mathrm{hofib}(\mathcal{U}_N \rightarrow G_N) \simeq \Omega \mathrm{Sp}(N), \quad \pi_d(G_N, \mathcal{U}_N) \cong \pi_d \mathrm{Sp}(N) \quad (d \geq 2).$$

The absolute class (α, β) maps to $\beta - \alpha$; stabilization is an isomorphism for $d \leq 4N + 1$.

Let $T_0 : S^k \rightarrow \mathcal{U}_N$ extend to $H : D^{k+1} \rightarrow G_N$. If $[H]$ vanishes stably as a relative class and $k + 1 \leq 4N + 1$, there is an extension $T : D^{k+1} \rightarrow \mathcal{U}_N$ which is G_N -homotopic to H relative to the boundary.

Proof. A maximal null plane is the graph of a quaternionic unitary from the positive to the negative summand: its positive projection is injective, hence an isomorphism, and nullity says $L^*L = I$. Thus the null Grassmannian is $\text{Sp}(N)$. Writing $E_1 = \text{graph } R$, its transverse chart consists of L with $R^{-1}L - I$ invertible. The mutually inverse maps

$$C = (R^{-1}L + I)(R^{-1}L - I)^{-1}, \quad R^{-1}L = (C - I)^{-1}(C + I)$$

identify it with the vector space $C^* = -C$.

Choose $T_* \in \mathcal{U}_N$ and replace T by TT_*^{-1} ; the source plane is then $\bar{E}_0 = T_*E_0$, transverse to E_1 . Choose dual null bases for $\bar{E}_0, E_1$. The stabilizer of $\bar{E}_0$ consists of

$$\begin{pmatrix} A & AZ \\ 0 & (A^*)^{-1} \end{pmatrix}, \quad A \in \text{GL}_N(\mathbb{H}), \quad Z^* = -Z.$$

Contract Z to zero and use the polar retraction of A . Its compact subgroup is the diagonal $\text{Sp}(N)$ in the corresponding positive and negative coordinates. The orbit map to the null Grassmannian is a principal stabilizer bundle. Over the contractible transverse chart its fibre inclusion is a homotopy equivalence; this identifies $\mathcal{U}_N \rightarrow G_N$ with the stabilizer inclusion.

For the polar decomposition $T = uP$ in G_N , $\sigma P \sigma = P^{-1}$ and u commutes with σ . The path uP^{1-s} retracts G_N onto $\text{Sp}(N)^2$ and fixes the compact stabilizer. The quotient of these compact groups is

$$(a, b) \mapsto b a^{-1} \in \text{Sp}(N).$$

Its fibre is the diagonal, so its fibre sequence gives the stated homotopy fibre and relative groups. Multiplication and inversion give $(\alpha, \beta) \mapsto \beta - \alpha$ on these groups. The stabilization range is (22.32).

For $k \geq 1$, the diagonal map on π_k is injective. Since T_0 extends to G_N , it therefore extends to $T_1 : D^{k+1} \rightarrow \mathcal{U}_N$. Write $[T_1 H^{-1}] = (\alpha, \beta)$ in $\pi_{k+1} \text{Sp}(N)^2$. Its relative image is zero stably; the stated range gives $\alpha = \beta$ already at this finite stage. Right-multiply T_1 by a boundary-identity map in the stabilizer of E_0 representing the negative diagonal class. This preserves $T_1 E_0$ and cancels $[T_1 H^{-1}]$, by the conjugation argument in Lemma 22.19. It gives the required relative homotopy. For $k = 0$, both spaces are connected and G_N is simply connected, which gives the assertion directly. For $N = 0$, both G_N and $\mathcal{U}_N$ are a point. $\square$

22.14. Quaternionic skew operators and finite boundary relations. In dimension $n \equiv 5 \pmod{8}$, put $A = i(D + V)$. The charge conjugation identities make A quaternionic-linear and skew-adjoint. All Hilbert spaces, adjoints, and dimensions in this subsection are quaternionic. Put

$$\text{Cay}(A) = (A + I)(A - I)^{-1}.$$

Thus $\text{Cay}(i(D + V)) = (D + V - i)(D + V + i)^{-1}$. Let Sp_c denote the norm-topological group of quaternionic unitaries I -compact, and put

$$\text{Sp}_c^\circ = \mathcal{V} = \{U \in \text{Sp}_c : U + I \text{ is invertible}\}.$$

The map $U \mapsto (U - I)(U + I)^{-1}$ identifies this open set with the real vector space of compact quaternionic skew-adjoint operators. For a compact CW pair (P, P_0) , a norm-resolvent-continuous family of compact-resolvent skew-adjoint operators, invertible on P_0 , therefore represents

$$K \text{Sp}^{-1}(P, P_0) = K O^{-5}(P, P_0) \quad \text{by a map to } (\text{Sp}_c, \text{Sp}_c^\circ). \quad (22.34)$$

This is the quaternionic form of the real Fredholm model [13, Theorem A(k) and Sections 2, 4–5]. More explicitly, $\text{Cl}_{0,4} = \text{Mat}_2(\mathbb{H})$ has volume involution ω ; on its simple module tensored with a quaternionic multiplicity space, $\omega \otimes A$ is skew-adjoint and anticommutes with the four Clifford

generators. Conversely every such operator has this form. The degree-five real Morita identification therefore has a single quaternionic multiplicity factor. The displayed Cayley convention fixes its sign. One can compare it directly with the exponential representative of the bounded transform $F = A(1 - A^2)^{-1/2}$. On the imaginary spectral parameter ix , the two representatives are $-\exp(2i \arctan x)$ and $-\exp(\pi ix / \sqrt{1 + x^2})$. For $0 \leq t \leq 1$ put

$$\phi_t(x) = (1 - t)2 \arctan x + t\pi x(1 + x^2)^{-1/2}.$$

This odd function lies in $(-\pi, \pi)$, has the sign of x , and tends to $\pm\pi$ at the two ends, uniformly in t . Thus $-e^{i\phi_t(x)} - 1$ belongs to $C_0(\mathbb{R})$ and varies continuously there in the uniform norm. Functional calculus gives a norm-continuous homotopy of $I +$ compact quaternionic unitaries whose -1 eigenspace is $\ker A$. It preserves the relative invertible locus.

Let S be a closed densely defined injective skew-symmetric operator with closed range and finite-dimensional $C = \ker(-S^*)$. Suppose that S has an invertible skew-adjoint extension A_0 with compact inverse R . Write $K : C \hookrightarrow \mathcal{H}$ for the isometric inclusion and $S_{\max} = -S^*$. For a maximal-null subspace $\mathcal{B} \subset C \oplus C$ for the form

$$h((x, y), (x', y')) = \langle y, x' \rangle + \langle x, y' \rangle,$$

define the extension $A_{\mathcal{B}}$ by

$$u = Rf + Kx, \quad f = S_{\max}u, \quad (x, K^*f) \in \mathcal{B}.$$

Regard $\mathcal{B}$ as a linear relation, with $\ker \mathcal{B} = \{x : (x, 0) \in \mathcal{B}\}$ and Cayley transform $(y - x) \mapsto y + x$. This convention includes vertical relations.

Lemma 22.21. *The operator $A_{\mathcal{B}}$ is skew-adjoint with compact resolvent. Its Cayley transform is homotopic to $\text{Cay}(\mathcal{B}) \oplus I_{C^\perp}$ through $\text{Sp}_{\mathbb{C}}$, with constant -1 eigenspace $K \ker \mathcal{B}$. The homotopy is continuous in $K, R, \mathcal{B}$ and hence preserves invertibility on every relative parameter set.*

Proof. Every $u \in \text{dom } S_{\max}$ has the stated unique decomposition, since $S_{\max}(u - Rf) = 0$. Set $\Gamma_0 u = x$ and $\Gamma_1 u = K^*f$. Skewness of R gives

$$\langle S_{\max}u, v \rangle + \langle u, S_{\max}v \rangle = \langle \Gamma_1 u, \Gamma_0 v \rangle + \langle \Gamma_0 u, \Gamma_1 v \rangle. \quad (22.35)$$

The boundary map is onto, since $u = RKy + Kx$ has boundary values (x, y) . Its kernel is $\text{dom } S$, since the radical of the left side is that domain, by $S_{\max}^* = -S$. The annihilator of a maximal-null $\mathcal{B}$ is $\mathcal{B}$ itself. Formula (22.35) thus proves equality of the extension and minus its adjoint, including their domains. For a graph-bounded sequence $u = Rf + Kx$, both f and $x = K^*(u - Rf)$ are bounded. Compactness of R and finite dimension of C give a convergent subsequence in $\mathcal{H}$. Thus the maximal domain embeds compactly, which proves compactness of the extension resolvent.

The defining domain formula gives the equality of relations

$$A_{\mathcal{B}}^{-1} = R + K\mathcal{B}^{-1}K^*. \quad (22.36)$$

The domain on the right is $\{f : K^*f \in \text{ran } \mathcal{B}\}$, and its multivalued part is $K \ker \mathcal{B}$. Put

$$Q_0 = K\mathcal{B}^{-1}K^*, \quad Q_t = tR + Q_0, \quad E_t = (I - Q_t)^{-1}, \quad V_t = 2E_t - I.$$

For a maximal skew relation on a finite-dimensional space,

$$\|u \mp q\|^2 = \|u\|^2 + \|q\|^2 \quad ((u, q) \in Q).$$

The maps $(u, q) \mapsto u \mp q$ are therefore injective; equality of dimensions makes them isomorphisms. Applied on C , this shows that E_0 depends continuously on $\mathcal{B}$. On $C^\perp$ it is I , so $E_0 - I$ has finite rank. For $q_0 \in Q_0 u$ put $y = u - q_0$ and $x = y - tRu$. Then $u = E_0 y$ and $x = (I - tRE_0)y$, so

$$E_t = E_0(I - tRE_0)^{-1} \quad (22.37)$$

once the last inverse exists. If $y = tRE_0y$ and $u = E_0y$, then $y \in u - Q_0u$ and hence $u \in Q_tu$. Skewness gives $u = 0$, then $y = 0$. The Fredholm alternative applies to $I - tRE_0$ and proves its invertibility. The same argument for $I + Q_t$ shows that V_t is onto. The identity $\|u + q\| = \|u - q\|$ shows it is unitary. Formula (22.37) proves norm continuity and $V_t - I$ compact.

The kernel of E_t is the multivalued part of Q_t , namely $K \ker \mathcal{B}$, so this is exactly $\ker(V_t + I)$ for every t . Finally (22.36) gives

$$V_1 = \text{Cay}(A_{\mathcal{B}}), \quad V_0 = \text{Cay}(\mathcal{B}) \oplus I_{C^\perp}.$$

All formulas are invariant under unitary changes of frame of C ; they therefore apply to finite kernel bundles as well. $\square$

22.15. A fixed midpoint restriction. Let P be a compact CW complex, and join two compact caps by a product cylinder of length L . Assume the coefficients are continuous in the smooth topology, the tangential operator B and its product collar are fixed, and the low APS realizations on both caps are injective. In the present symmetry class their full APS realizations are surjective by (22.8). On the cylinder write

$$A_{L,p} = \sigma(\partial_s + B + L^{-1}V_p(s/L)), \quad \sigma^* = \sigma, \quad \sigma^2 = I, \quad B^* = B, \quad B\sigma = -\sigma B, \quad V_p^*\sigma + \sigma V_p = 0. \quad (22.38)$$

The controls form one compact smooth family, supported away from the outer ends. Put $K_0 = \ker B$, let P_K be its orthogonal projection, and assume that $\sigma|_{K_0}$ has signature (N, N) . Let $E_0, E_1 \subset K_0$ be the cap low Cauchy spaces. They are maximal-null N -planes. Let T_- and T_+ be the transfers of $u' + P_K V_p P_K u = 0$ on $[0, 1/2]$ and $[1/2, 1]$, and put $T = T_+ T_-$. Since $P_K \sigma = \sigma P_K$, their generators satisfy $Z^* \sigma + \sigma Z = 0$; hence $T_\pm^* \sigma T_\pm = \sigma$.

At the midpoint define

$$\text{dom } S_{L,p} = \{u \in H^1(M_L) : P_K \gamma_{L/2} u = 0\}, \quad S_{L,p} u = A_{L,p} u. \quad (22.39)$$

Write γ_l, γ_r for the traces from the two sides of the cut and put

$$\mathcal{Q} = K_{0,l} \oplus K_{0,r}, \quad h_{\mathcal{Q}} = \sigma \oplus (-\sigma), \quad \Delta = \{(x, x) : x \in K_0\}.$$

Proposition 22.22. *For all sufficiently large L , uniformly on P , the restriction S_L is injective and Fredholm of quaternionic index $-2N$. Its maximal domain consists of the piecewise H^1 sections whose high traces agree at the cut; its boundary quotient is $(\mathcal{Q}, h_{\mathcal{Q}})$, and the closed extension corresponds to Δ .*

The kernels $C_{L,p} = \ker(-S_{L,p}^*)$ form a continuous rank- $2N$ bundle. Their low traces are maximal-null planes $\Lambda_{L,p}$ and satisfy

$$\Lambda_{L,p} = (T_- E_{0,p}) \oplus (T_+^{-1} E_{1,p}) + O(L^{-1}), \quad (22.40)$$

where the error is in the norm of orthogonal projections.

Proof. The empty-boundary case of [18, Theorems 4.9 and 4.11], applied to the formally self-adjoint Dirac-type operator $-iA$, gives the closed domain H^1 and equivalence of its graph and H^1 norms. Rellich's theorem gives compact resolvent. The APS domains on the compact caps are elliptic domains in that source. The fixed collar and low injectivity give the $H^{1/2}$ trace maps of Lemma 22.11. The compact control family satisfies the coefficient bounds of Lemma 22.16 on the full cylinder and each half. The half-interval estimates use only the coefficient bounds and the length of the interval; they do not require the control to vanish at the midpoint.

The trace in (22.39) is onto K_0 , by a smooth collar lift. Its kernel is dense in L^2 , since it contains smooth sections supported away from the cut. It is graph-closed of codimension $2N$ in the closed domain. Hence $S_{L,p}$ is Fredholm of index $-2N$. To prove injectivity, let u be a closed zero mode, and let k be its left low value. The two cap high-trace equations and Lemma 22.16 give

$$\|a\|_{H^{1/2}} + \|d\|_{H^{1/2}} \leq C\|k\|, \quad P_K u(L/2) = T_- k + O(L^{-1})k.$$

Indeed, the cap equations and the cylinder traces give

$$\|a\|_{H^{1/2}} + \|d\|_{H^{1/2}} \leq C\|k\| + \frac{C}{L}(\|a\|_{H^{1/2}} + \|d\|_{H^{1/2}}).$$

Absorb the last term and apply the low estimate at $s = L/2$. Put $C_T = \max(1, \sup_{p \in P} \|T_-^{-1}\|)$. A zero midpoint trace implies $\|k\| \leq C_T CL^{-1}\|k\|$. Thus $k = 0$ for large L , then $a = d = 0$. Cylinder uniqueness and low injectivity on both caps give $u = 0$.

Suppress L, p , and write $\tau = P_K \gamma_{L/2}$ on the closed H^1 domain. If $u \in \text{dom}(-S^*)$ and $f = -S^*u$, the continuous functional

$$v \mapsto \langle f, v \rangle + \langle u, Av \rangle, \quad v \in H^1(M_L),$$

vanishes on $\ker \tau$ and therefore factors through K_0 . Choose $w \in K_0$ representing it as $\langle w, \tau v \rangle$. There is a smooth piecewise section j with only a low jump, $\gamma_l j - \gamma_r j = \sigma w$, supported near the cut. Green's formula gives

$$\langle A_{pw} j, v \rangle + \langle j, Av \rangle = \langle w, \tau v \rangle.$$

Subtracting this identity shows that $u - j \in \text{dom } A^* = H^1$. Thus u is piecewise H^1 and its high traces agree. Conversely, Green's formula for any such section has only the low jump term, which vanishes on $\ker \tau$; hence it belongs to $\text{dom}(-S^*)$. This also proves $-S^*u = A_{pw}u$.

Independent low lifts show that the two low traces map this maximal domain onto $\mathcal{Q}$, with kernel $\text{dom } S$. Green's formula is exactly $h_{\mathcal{Q}}$. Since S is injective with closed range, $-S^*$ is onto and has kernel dimension $2N$. The maximal domain is a fixed piecewise H^1 Banach space; its graph norm is equivalent to that norm by the closed graph theorem. On this domain the surjective maps $S_{\max, p}$ are continuous. For a right inverse R_* at p_* , the formula $R_p = R_*(S_{\max, p} R_*)^{-1}$ gives local continuous right inverses. The projections $I - R_p S_{\max, p}$ therefore define a continuous kernel bundle in piecewise H^1 . Its low trace is injective: a zero low trace would give a vector in $\ker S$. Green's formula makes its image null; its dimension $2N = \frac{1}{2} \dim \mathcal{Q}$ makes it maximal null. This also proves continuity of Λ_L .

For (22.40), retain independent midpoint low values. Write k for the left end value and ℓ for the value on the right side of the midpoint. Write (a_0, d_-, a_+, d_1) for the four incoming high data of the two half-cylinders. Their outgoing data satisfy

$$(b_0, c_m) = O(L^{-1})(k, a_0, d_-), \quad (b_m, c_1) = O(L^{-1})(\ell, a_+, d_1),$$

and the right final low value is $k_1 = T_+ \ell + O(L^{-1})(\ell, a_+, d_1)$. The four high equations are

$$a_0 = H_0(k, b_0), \quad d_- = b_m, \quad a_+ = c_m, \quad d_1 = H_1(k_1, c_1).$$

Here H_i are the cap maps of Lemma 22.11. After substituting the cylinder formulas, this is an equation $h = G(k, \ell) + \mathcal{E}h$, with $\|\mathcal{E}\| \leq C/L$. It has a unique bounded solution, continuous in the parameters. The remaining cap equations are therefore

$$F_L(k, \ell) = 0, \quad F_L = (P_{E_0^\perp} k, P_{E_1^\perp} T_+ \ell) + O(L^{-1})(k, \ell). \quad (22.41)$$

Indeed the exact cap constraints are $P_{E_0^\perp}(k - e_0 b_0) = 0$ and $P_{E_1^\perp}(k_1 - e_1 c_1) = 0$.

The leading map F_∞ has the uniformly bounded right inverse $(u, v) \mapsto (u, T_+^{-1}v)$. Thus F_L is onto, with right inverse $R_L = R_\infty(F_L R_\infty)^{-1}$. The map $(I - R_L F_L)|_{\ker F_\infty}$ is $I + O(L^{-1})$ and identifies $\ker F_\infty$ with $\ker F_L$. The two midpoint traces of the reconstructed section are

$$(T_- k + O(L^{-1})(k, \ell), \ell).$$

Thus the map from $\ker F_\infty = E_0 \oplus T_+^{-1}E_1$ to Λ_L differs from $(k, \ell) \mapsto (T_- k, \ell)$ by $O(L^{-1})$ and has a uniform lower bound. For an injective map Z , the projection onto its image is $Z(Z^* Z)^{-1}Z^*$. Applying this formula proves (22.40). $\square$

22.16. The relative Cayley class. Fix a positive–negative polarization of $\mathcal{Q}$ and write maximal-null planes as graphs of unitaries. For a domain plane Δ and a Cauchy plane Λ put

$$C(\Delta, \Lambda) = -U_{\Delta}^{-1} U_{\Lambda}. \quad (22.42)$$

This is a unitary, and $C + I$ is invertible precisely when $\Delta \cap \Lambda = 0$.

Proposition 22.23. *Under the midpoint hypotheses, let $P_0 \subset P$ be a closed subcomplex on which $TE_0 \cap E_1 = 0$. For sufficiently large L , A_L is invertible on P_0 , and its class in (22.34) is represented by (22.42) for the pair*

$$(\Delta, (TE_0) \oplus E_1) \quad \text{in } (K_0 \oplus K_0, \sigma \oplus (-\sigma)). \quad (22.43)$$

For fixed E_0, E_1 , the induced map from the transmission pair $(G_N, \mathcal{U}_N)$ has coefficient one: on $\text{Sp}(N)^2$ it is $[b] - [a]$. It induces an isomorphism on π_d , with target KO_{5+d} , for $1 \leq d \leq 4N + 1$.

Proof. For the invertible reference extension in Lemma 22.21, fix a positive–negative polarization of $\mathcal{Q}$ and write $\Lambda_L = \text{graph } U$. The plane $\text{graph}(-U)$ is maximal null and transverse to Λ_L . Its extension A_0 is skew-adjoint, by the exact boundary quotient, and has zero kernel. The piecewise H^1 domain embeds compactly into L^2 , so this extension is Fredholm with compact resolvent; skew-adjointness makes it onto. Its inverse R is compact. These inverses are continuous. Let J be a fixed right inverse of the finite trace map γ and let P_p be the orthogonal projection onto the reference extension plane. Near p_* , the map

$$W_p = I + J(P_p - P_{p_*})\gamma$$

is invertible on the maximal domain and carries the domain at p_* onto the domain at p : its trace on the former plane is P_p . The maps $A_{0,p} W_p$ are continuous bounded bijections from the fixed domain, with its piecewise H^1 norm, to L^2 . Thus $R_p = W_p(A_{0,p} W_p)^{-1}$ is continuous, also as a map to the maximal domain. Compactness of P gives uniform inverse bounds. The low trace followed by positive projection is an isomorphism $C_{L,p} \rightarrow \mathcal{Q}_+$. If Z_p is its inverse, then $K_p = Z_p(Z_p^* Z_p)^{-1/2}$ is a continuous isometric embedding of the fixed space $\mathcal{Q}_+$ onto $C_{L,p}$. Thus Lemma 22.21 applies globally, also when the closed nullity jumps. Let $V_p^f \in \text{Sp}(\mathcal{Q}_+)$ be its finite Cayley representative transported from $C_{L,p}$ by K_p . To place it in a fixed summand, stabilize once and use

$$K_{p,\theta} = (\cos \theta K_p, \sin \theta I) : \mathcal{Q}_+ \longrightarrow \mathcal{H} \oplus \mathcal{Q}_+, \quad I + K_{p,\theta}(V_p^f - I)K_{p,\theta}^*, \quad 0 \leq \theta \leq \pi/2.$$

This is a unitary homotopy from the representative on $C_{L,p}$ to the same matrix on the fixed second summand. Its spectrum away from 1 is unchanged, so it preserves the relative condition.

In that lemma's coordinates put $p = (x+y)/\sqrt{2}$, $q = (x-y)/\sqrt{2}$. The Green form is $\langle p, p \rangle' - \langle q, q \rangle'$, and the Cauchy plane $y = 0$ has graph I . A relation $\mathcal{B}$ has graph unitary

$$U_{\mathcal{B}} = (I - \mathcal{B})(I + \mathcal{B})^{-1}, \quad \text{Cay}(\mathcal{B}) = -U_{\mathcal{B}}^{-1}.$$

These formulas mean the maps $x + y \mapsto x - y$ on the relation, so also hold for vertical relations. The Green isometry $\text{diag}(I, U_{\Lambda}^{-1})$ changes the two graph unitaries to $U_{\Lambda}^{-1} U_{\Delta}$ and I . The graph-pair matrix in these coordinates is therefore $-(U_{\Lambda}^{-1} U_{\Delta})^{-1} = -U_{\Delta}^{-1} U_{\Lambda}$, as in (22.42). Its -1 eigenspace is isomorphic to $\Delta \cap \Lambda$.

The relative class is independent of the boundary coordinates. A simultaneous Green isometry g preserves intersections. With $J = \text{diag}(I, -I)$, its polar decomposition is $g = kH$, where $kJ = Jk$ and $JHJ = H^{-1}$. Thus kH^t is a path of Green isometries from k to g . Apply it simultaneously to both planes; their transversality is unchanged. For $k = \text{diag}(a, d)$ the representative changes to aCa^{-1} . This is stably homotopic to C through conjugations: if R_{θ} is the ordinary real block rotation, then

$$\text{diag}(a, I)R_{\theta} \text{diag}(I, a^{-1})R_{\theta}^{-1}, \quad 0 \leq \theta \leq \pi/2,$$

runs from $\text{diag}(a, a^{-1})$ to I . Conjugating $C \oplus I$ along this path preserves the entire spectrum. This proves invariance of the full relative class, including for parameter-dependent Green coordinates.

The two coordinate systems above consequently give the same relative class, so (22.42) represents the class of V_p^f .

Proposition 22.22 compares actual maximal-null planes. If U_L, U_∞ are their graph unitaries, then

$$U_s = U_\infty \exp(s \log(U_\infty^{-1} U_L)), \quad 0 \leq s \leq 1,$$

is defined by the principal logarithm for large L and remains $O(L^{-1})$ from U_∞ . Compactness of P_0 gives a positive least singular value for $U_\Delta - U_\infty$ there. Increasing L makes the whole path transverse to Δ on P_0 . Lemma 22.21 and (22.42) identify the actual family class with this limiting pair. Applying T_+ to both boundary factors fixes Δ and sends the limiting Cauchy plane to $(TE_0) \oplus E_1$. The coordinate-invariance argument just given proves (22.43).

This also covers independent end gauges. If $E_i(p) = g_i(p)E_i^*$, then the simultaneous Green isometry $\text{diag}(g_1^{-1}, g_1^{-1})$ fixes Δ and sends (22.43) to

$$(\Delta, (\tilde{T}E_0^* \oplus E_1^*)), \quad \tilde{T} = g_1^{-1} T g_0.$$

Thus the complete relative class is unchanged when the end planes and the transmission map are transformed together.

For the normalization, choose $T_* \in \mathcal{U}_N$. Replacing T by $T T_*^{-1}$ and E_0 by $T_* E_0$ makes I the basepoint. A fixed block-unitary change then makes $E_0 = \text{graph } I$. Choose $K_0 = \mathbb{H}_+^N \oplus \mathbb{H}_-^N$ with $\sigma = \text{diag}(I, -I)$, and write $TE_0 = \text{graph } L$ and $E_1 = \text{graph } R$. Give the doubled boundary space positive coordinates (x_l^+, x_r^-) and negative coordinates (x_l^-, x_r^+) . Direct substitution gives

$$U_\Delta = \text{diag}(L, R^{-1}), \quad U_\Delta = J_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad C = -J_0 \text{diag}(L, R^{-1}). \quad (22.44)$$

The condition $C + I$ invertible is exactly $L - R$ invertible. For $T = \text{diag}(a, b)$ one has $L = b a^{-1}$, so the stable *absolute* homotopy map of (22.44) is $[b] - [a]$; R and $-J_0$ are constant. Stable multiplication and block sum give the same operation, as follows from the same block rotation used above.

This absolute calculation determines the relative map without removing $-J_0$ by a boundary-changing homotopy. Indeed, the exact graph-pair map sends $\mathcal{U}_N$ into Sp_c° . By Lemma 22.20, the map $\pi_{d-1}(\mathcal{U}_N) \rightarrow \pi_{d-1}(G_N)$ is the injective diagonal map. For $d \geq 2$ the long exact sequence therefore identifies $\pi_d(G_N, \mathcal{U}_N)$ with the quotient of $\pi_d(G_N)$ by the diagonal image. Since Sp_c° is contractible, the corresponding target relative group is $\pi_d(\text{Sp}_c)$. Naturality gives exactly the quotient homomorphism $[b] - [a]$. For $d = 1$ both groups vanish. The stabilization range in Lemma 22.20 gives the isomorphism in this range. Finally (22.34) on (D^d, S^{d-1}) gives KO_{5+d} ; for $d = k + 1$ this is KO_{n+k+1} . $\square$

22.17. Homotopy groups of compatible pairs.

Proposition 22.24. *Let M be closed, connected and spin, and let $U \subset M$ be nonempty and open. The supported spaces $Y_4(U)$ in residue four and $Y_5(U)$ in residue five are nonempty. The maps (22.4) and (22.6) are weak homotopy equivalences.*

Proof. Work first without a density constraint. Lemma 8.1 normalizes families relative to their already normalized parameters. The closed case of Lemma 22.7, with perturbations in one ball in U , proves nonemptiness. Applied to a coefficient path relative to its endpoints, it proves path connectedness: the relevant codimensions are $4(a + 1) > 1$ and $3 > 1$. The targets are connected. It remains to prove bijectivity on π_k for $k \geq 1$.

Families with a product cylinder. Put $d = k + 1$. Given $f : S^k \rightarrow Y$, make it constant near its basepoint and approximate it smoothly in logarithmic metric and linear multiplier coordinates, relative to that neighborhood. The compact family of injective operators $H^1 \rightarrow L^2$ has a common positive lower bound. A sufficiently close approximation and its interpolation therefore stay in Y . Retain that interpolation as an outer annulus, so that the eventual filling has exactly the original continuous boundary.

Choose two disjoint balls compactly contained in U . Use the first for the insertion and the second, in the connected exterior, for supported genericity. By the unbounded ranks of Lemma 22.8, choose

$$\begin{cases} r \geq \max(1, a), & d < 4(r - a + 1), & k \leq 4r + 1, & n \equiv 4 \pmod{8}, \\ d < 4N + 3, & d \leq 4N + 1, & & n \equiv 5 \pmod{8}. \end{cases} \quad (22.45)$$

Proposition 22.18 gives a homotopy in Y to a family containing a product cylinder. Retain it as a second annulus. Its tangential operator, product collars, and cylinder control are fixed, its neck radius is positive, and its limiting matching map is uniformly injective. The control has a final interval on which it is zero. Every subsequent lengthening remains in Y .

In residue four the fixed cap has low and full indices $-r, r$; the exterior has indices $-r - a, r - a$ and is low-injective. Make its full boundary family surjective by a sufficiently small relative perturbation from Lemma 22.7. The low gap and the limiting injection remain positive throughout its interpolation. First choose a cylinder length valid for this compact interpolation and reach it by lengthening; then perform the perturbation at that length. Retain the resulting annuli. In residue five low injectivity already implies full surjectivity by (22.8).

Extend the exterior coefficients over D^d , constant on a boundary parameter collar and fixed on the spatial collar and $M \setminus U$. Indeed, in one spinor comparison, convex interpolation of positive metric tensors and affine interpolation of multipliers, with a radial cutoff, give an extension constant near the centre. Smooth it relative to the boundary collar. Apply Lemma 22.7 inside the second ball, with perturbations zero near S^k . In residue four its indices are $-r - a$ for the low problem and $-(r - a)$ for the adjoint full problem; the first line of (22.45) gives both avoidance inequalities. In residue five use its second line. The resulting disk is low-injective and full-surjective, so its cokernel and low Cauchy bundles extend over the disk.

Injectivity in residue four. Write $K = \mathbb{H}^{2r}$. The exact sequence (22.20) gives surjections $\beta_i : K \rightarrow C_i$, with

$$\dim_{\mathbb{H}} C_0 = r, \quad \dim_{\mathbb{H}} C_1 = r + a, \quad E_i = \ker \beta_i, \quad \dim_{\mathbb{H}} E_0 = r, \quad \dim_{\mathbb{H}} E_1 = r - a.$$

Choose independent gauges $g_i : D^d \rightarrow \mathrm{GL}_{\mathbb{H}}(K)$ with $g_i \bar{E}_i = E_i(p)$ and choose quotient frames over the disk. Such choices exist because the subspace and quotient bundles are trivial on the contractible disk. Put $\tilde{T} = g_1^{-1} T g_0$. Elimination of the first surjective block in $(\beta_0, \beta_1 T)$ leaves $F = q \tilde{T}|_{\bar{E}_0} : \mathbb{H}^r \rightarrow \mathbb{H}^{r+a}$, where q is the quotient by $\bar{E}_1$ in the chosen frame.

Extend the actual boundary control over the disk in its linear control space, retaining the final zero interval, and call its endpoint H ; put $\tilde{H} = g_1^{-1} H g_0$. The resulting coefficient disk computes the comparison of its boundary: any two coefficient fillings are homotopic relative to their common boundary in the contractible ambient coefficient space. Proposition 22.17 and Lemma 22.6 identify that comparison with the stable Stiefel class of F , including the cokernel and its stable trivialization. The retained annuli identify it with $\kappa_a[f]$, with the induced basepoint transport.

If $\kappa_a[f] = 0$, Lemma 22.19(a) gives a transverse $\tilde{T}$, equal to $\tilde{H}$ on S^k and G -homotopic to it relative to S^k . Set $T = g_1 \tilde{T} g_0^{-1}$ and transform the homotopy by the same formula. It joins H to T relative to S^k . Lemma 22.15 realizes T while retaining the exact boundary control.

Injectivity in residue five. Now K has signature (N, N) and E_0, E_1 are maximal-null N -plane bundles. Choose independent gauges $g_i : D^d \rightarrow \mathrm{Sp}(N, N)$ with $g_i \bar{E}_i = E_i(p)$. They exist because the orbit maps to the null Grassmannian are locally trivial and their pullbacks to a disk have sections. Extend the actual boundary control over the disk, retaining its final zero interval, and denote its endpoint and boundary endpoint by H and T_0 . Put

$$\tilde{H} = g_1^{-1} H g_0, \quad \tilde{T}_0 = g_1^{-1} T_0 g_0.$$

The simultaneous Green isometry $\mathrm{diag}(g_1^{-1}, g_1^{-1})$ fixes Δ and changes the limiting Cauchy plane to $(\tilde{H} \bar{E}_0) \oplus \bar{E}_1$. Thus Proposition 22.23, applied to this ambient disk, identifies $\kappa_5[f]$ with the stable relative class of

$$(\tilde{H}, \tilde{T}_0) : (D^d, S^k) \longrightarrow (G_N, \mathcal{U}_N),$$

with coefficient $\beta - \alpha$. If this class is zero, Lemma 22.20 gives a transverse extension $\tilde{T}$ homotopic to $\tilde{H}$ relative to the exact boundary. Set $T = g_1 \tilde{T} g_0^{-1}$ and undo the homotopy by the same formula. Then $T = T_0$ on S^k , and Lemma 22.15 retains the original boundary control.

In either residue the entire constructed disk of limiting matching maps has a minimum singular value $\delta > 0$. Choose its final finite length so that

$$\sup_{p \in D^d} \|M_{L,p} - (\beta_{0,p}, \beta_{1,p} T_p)\| < \delta/2.$$

Proposition 22.17 gives a disk in Y . Its boundary is the boundary family with its prescribed control at this length. Attach the lengthening, exterior-perturbation, insertion, and approximation annuli. The outer boundary is the original f , with its actual coefficients. This proves injectivity.

Surjectivity in residue four. Fix a base pair containing this cylinder and $\xi \in \pi_k(\text{Sp} / \text{Sp}(a))$, choosing the rank sufficiently large first. Lemma 22.19(b) gives a transverse endpoint $T = RT_*$ with quotient columns representing ξ , and a based homotopy $R_s : S^k \rightarrow G$ from I to R . Apply Lemma 22.15 to the reference control, with parameter space $S^k \times [0, 1]$ and endpoint homotopy

$$T_{(x,s),t} = R_{st}(x)T_*, \quad 0 \leq t \leq 1.$$

One subdivision works for this compact family. On $s = 0$ and the basepoint line every added control is zero. The coefficient family therefore descends to the cone D^{k+1} and is based on its boundary. A common sufficiently long cylinder gives a minimal boundary sphere. Its ambient disk, the Fredholm reduction, and Lemma 22.6 show that its comparison is ξ .

Surjectivity in residue five. Choose N in the stable range and a representative

$$g : (D^{k+1}, S^k) \longrightarrow (\text{Sp}(N), I)$$

of the prescribed class in $\pi_{k+1} \text{Sp}$. By homotopy extension, make $g = I$ on a radial segment from the centre to the boundary basepoint: this segment contracts to its boundary endpoint relative to that endpoint. Put

$$H(p) = \text{diag}(I, g(p))T_*, \quad Q_{p,s} = H(sp)T_*^{-1}.$$

The paths $Q_{p,s}$ start at I and are constant on the chosen segment. Lift their increments by the local section of Lemma 22.14, using one uniform subdivision and placing the added controls after the reference control. They vanish on that segment. On S^k their endpoints are I , while the controls themselves may vary as loops. A common sufficiently long cylinder therefore gives an invertible boundary sphere based at the reference pair. Proposition 22.23 identifies its disk class with $[g]$, since the varying compact block is (I, g) . These spheres may be based at a longer reference cylinder. Transport along its lengthening and insertion paths gives surjectivity at the original base pair; path connectedness gives it at every basepoint. All constructed controls are continuous in the smooth spatial topology: their amplitudes are continuous and their spatial and time functions belong to a fixed finite span. Finally normalize every disk and annulus by Lemma 8.1. It fixes the original normalized boundary. The contraction (22.2) on the unrestricted coefficient space restricts to the same contraction on the density slice, so its comparison maps restrict as well. The normalized homotopies and fillings consequently prove the required isomorphisms with density fixed. $\square$

22.18. Naturality of localization.

Corollary 22.25. *Let $U_0 \subset U_1$ be nonempty open subsets, with the same density and prescribed exterior metric. The inclusion of the supported minimal pair spaces in residue four, or the supported invertible pair spaces in residue five, is a weak homotopy equivalence. The same holds for the corresponding metric spaces, by Proposition 8.2.*

Proof. Choose the base pair in the smaller space. Take its positive spinor comparison on both spaces and the single contraction (22.2) on the larger coefficient space. Its restriction is the contraction for the smaller space. The operator, cokernel, ambient path, and, in residue five, the logarithmic return

path therefore agree under inclusion. The two comparison maps commute with inclusion, before passing to their stable models. Proposition 22.24 and two-out-of-three give the pair assertion. Choose a minimal metric basepoint in the smaller space and apply the supported relative equivalences of Proposition 8.2 in the same square to obtain the metric assertion. $\square$

22.19. Homotopy groups of metric spaces.

Proof of Theorem T. Compose the pair comparisons of Proposition 22.24 with the inclusions of Proposition 8.2. Their homotopy fibres and Cayley conventions are those of (22.4) and (22.6). Both inclusions are weak homotopy equivalences, with density and exterior fixed. This proves (1.14) and (1.15). Corollary 22.25 proves the assertion about $U_0 \subset U_1$.

The fibration $\mathrm{Sp}(a) \rightarrow \mathrm{Sp}(a+1) \rightarrow S^{4a+3}$ and (22.32) show that

$$S^{4a+3} = \mathrm{Sp}(a+1)/\mathrm{Sp}(a) \longrightarrow \mathrm{Sp}/\mathrm{Sp}(a)$$

induces isomorphisms on homotopy groups through degree $4a+5$. Thus the target is $(4a+2)$ -connected and its next homotopy group is $\mathbb{Z}$. Similarly $\mathrm{Sp}(1) = S^3 \rightarrow \mathrm{Sp}$ induces isomorphisms through degree five. Hence $\Omega \mathrm{Sp}$ is simply connected with $\pi_2 = \mathbb{Z}$. The Fredholm identifications (22.3) and (22.34) identify the comparison homomorphisms at index zero with KO_{n+k+1} , giving (1.16).

In dimensions four and five, the homomorphisms of Proposition 22.3 are therefore surjections between infinite cyclic groups, by Proposition 22.4. They are isomorphisms. That proposition also gives the stated C^0 localization of their generators, while Proposition 22.5 fixes their integral index normalization at index zero. $\square$

22.20. Positive scalar curvature and the ordinary Dirac operator. For a positive-scalar-curvature metric g_0 on a closed spin manifold, let

$$\mathcal{F}_{g_0}(M) = \mathrm{hofib}_{g_0}(\mathcal{R}^+(M) \longrightarrow \mathcal{R}^{\mathrm{inv}}(M)).$$

The inclusion is defined by the Lichnerowicz formula. Both spaces have the smooth topology; no density or exterior is prescribed here.

Proposition 22.26. *Let $M = S^1 \times S^{11}$, with either spin structure on the circle, and let $g_0 \in \mathcal{R}^+(M)$. Then $\pi_4(\mathcal{F}_{g_0}(M))$ contains an element of infinite order.*

Proof. Projection $M \rightarrow S^1 = B\mathbb{Z}$ induces an isomorphism on fundamental groups, hence a split surjection. The discrete group $\mathbb{Z}$ is second countable, torsion-free, and has rational homological dimension one. Its proper isometric translation action on $\mathbb{R}$ gives the Haagerup property. Thus [58, Theorem 1.1], with scalar complex coefficients, and [76, Corollary 2.13] give the real Baum–Connes isomorphism. Thus

$$KO_{17}(C_{\mathbb{R},r}^*\mathbb{Z}) \cong KO_{17}(S^1) \cong KO_{17}(\mathrm{pt}) \oplus KO_{16}(\mathrm{pt}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}.$$

The middle equality is the suspension splitting of S_+^1 . Theorem C of [42] applies with $2n = 12 \geq 6$, $r = 4$, and $q = 1$: the fundamental-group map is split surjective, g_0 has positive scalar curvature, and $4 \geq 1 - 12 - 1$. Its index-difference image spans the rational target, which is one-dimensional. Hence there is an element

$$x \in \pi_4(\mathcal{R}^+(M), g_0)$$

whose higher real index difference has nonzero projection to the infinite cyclic summand. Thus x and $2x$ have infinite order.

The manifold admits a product metric of positive scalar curvature, so its ordinary spin index vanishes. Theorem T and Lemma 8.1 give

$$\pi_4(\mathcal{R}^{\mathrm{inv}}(M), g_0) \cong \pi_4(\mathrm{Sp}) = KO_{17}(\mathrm{pt}) = \mathbb{Z}/2.$$

Hence the image of $2x$ under the inclusion is zero. The homotopy exact sequence of its fibre gives an element $y \in \pi_4(\mathcal{F}_{g_0}(M))$ mapping to $2x$. If a positive integer killed y , it would kill $2x$. Therefore y has infinite order. $\square$

APPENDIX A. THE OBSTRUCTION GROUPS IN DEGREES AT MOST TWO

Let $0 \leq n \leq 2$. The homomorphism

$$j_n : \Omega_n^{\text{Spin}} \longrightarrow R_n^{\text{inv}}, \quad [X] \longmapsto [X, -],$$

satisfies $\theta_n j_n = \alpha_n$, where α_n is the real spin index [38, Sections 5.2 and 5.4]. The low-dimensional spin bordism and index computations give

$$\alpha_n : \Omega_n^{\text{Spin}} \xrightarrow{\cong} KO_n, \quad KO_0 = \mathbb{Z}, \quad KO_1 = KO_2 = \mathbb{Z}/2;$$

see [68, p. 201], [14, Theorem 3.1, p. 147], and [9, Section 4, pp. 57–58].

Every degree-zero representative is closed. In degree one the boundary Dirac operator is zero, so an invertible boundary is empty. Thus j_0 and j_1 are surjective.

Let $[W, h] \in R_2^{\text{inv}}$. Each boundary circle has zero real index and therefore has bounding spin structure. Let g_{tor} be the metric on $\mathbb{R}^2$ in [52, Appendix A, p. 29]. Its Dirac operator is invertible, and its cylindrical cross-section has length 2π . For a boundary circle of length L , truncate $(\mathbb{R}^2, (L/2\pi)^2 g_{\text{tor}})$ in its product end. The resulting spin disk has invertible cylindrical completion. An arclength spin isometry identifies its boundary with the oppositely oriented boundary circle of W .

Let C be the disjoint union of these disks, with $\partial C = -\partial W$, and put $\widehat{W} = W \cup_{\partial W} C$. If $\partial W = \emptyset$, take $C = \emptyset$. Since $C = \widehat{W} \setminus \text{int } W$, [38, Lemma 5.5, p. 31], applied with $M = \widehat{W}$ and $N = W$, gives

$$[W, h] = [\widehat{W}, -] = j_2([\widehat{W}]).$$

Hence j_2 is surjective. Since $\theta_n j_n = \alpha_n$ is an isomorphism and j_n is surjective, θ_n is an isomorphism for $0 \leq n \leq 2$.

REFERENCES

- [1] I. Agricola and T. Friedrich, *The Casimir operator of a metric connection with skew-symmetric torsion*, J. Geom. Phys. **50** (2004), 188–204. doi:10.1016/j.geomphys.2003.11.001.
- [2] B. Ammann and M. Dahl, *The space of Dirac-minimal metrics is connected in dimensions 2 and 4*, SIGMA Symmetry Integrability Geom. Methods Appl. **21** (2025), Paper No. 102, 18 pp. doi:10.3842/SIGMA.2025.102.
- [3] B. Ammann, M. Dahl, and E. Humbert, *Surgery and harmonic spinors*, Adv. Math. **220** (2009), 523–539. arXiv:math/0606224.
- [4] B. Ammann, M. Dahl, and E. Humbert, *Harmonic spinors and local deformations of the metric*, Math. Res. Lett. **18** (2011), 927–936. doi:10.4310/MRL.2011.v18.n5.a10. Theorem locators refer to arXiv:0903.4544v3.
- [5] B. Ammann and N. Ginoux, *Dirac-harmonic maps from index theory*, Calc. Var. Partial Differential Equations **47** (2013), 739–762. doi:10.1007/s00526-012-0534-z. The conjecture is cited from arXiv:1110.1309v2.
- [6] T. M. Apostol, *Introduction to Analytic Number Theory*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1976. doi:10.1007/978-1-4757-5579-4.
- [7] N. Aronszajn, *A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order*, J. Math. Pures Appl. (9) **36** (1957), 235–249.
- [8] T. Aschieri, B. Ruba, and J. P. Solovej, *SU(2)-equivariant quantum channels: semiclassical analysis*, Comm. Math. Phys. **405** (2024), Article No. 298. doi:10.1007/s00220-024-05178-1.
- [9] M. F. Atiyah, *Riemann surfaces and spin structures*, Ann. Sci. École Norm. Sup. (4) **4** (1971), 47–62. DOI: 10.24033/asens.1205.
- [10] M. F. Atiyah, R. Bott, and A. Shapiro, *Clifford modules*, Topology **3**, Suppl. 1 (1964), 3–38. doi:10.1016/0040-9383(64)90003-5.
- [11] M. F. Atiyah, V. K. Patodi, and I. M. Singer, *Spectral asymmetry and Riemannian geometry. I*, Math. Proc. Cambridge Philos. Soc. **77** (1975), 43–69. doi:10.1017/S0305004100049410.
- [12] M. F. Atiyah and I. M. Singer, *The index of elliptic operators: III*, Ann. of Math. (2) **87** (1968), 546–604. doi:10.2307/1970717.
- [13] M. F. Atiyah and I. M. Singer, *Index theory for skew-adjoint Fredholm operators*, Publ. Math. IHÉS **37** (1969), 5–26. doi:10.1007/BF02684885.
- [14] M. F. Atiyah and I. M. Singer, *The index of elliptic operators: V*, Ann. of Math. (2) **93** (1971), 139–149. doi:10.2307/1970757.
- [15] L. Bandara, A. McIntosh, and A. Rosén, *Riesz continuity of the Atiyah–Singer Dirac operator under perturbations of the metric*, Math. Ann. **370** (2018), 863–915. doi:10.1007/s00208-017-1610-7. Theorem locators refer to arXiv:1603.03647v2.
- [16] C. Bär, *Metrics with harmonic spinors*, Geom. Funct. Anal. **6** (1996), 899–942. doi:10.1007/BF02246994.
- [17] C. Bär, *Harmonic spinors for twisted Dirac operators*, Math. Ann. **309** (1997), 225–246. doi:10.1007/s002080050111. Theorem locators refer to arXiv:dg-ga/9706016.

- [18] C. Bär and W. Ballmann, *Guide to elliptic boundary value problems for Dirac-type operators*, in *Arbeitstagung Bonn 2013*, Progr. Math. **319**, Birkhäuser, Cham, 2016, pp. 43–80. doi:10.1007/978-3-319-43648-7_3. Theorem locators refer to arXiv:1307.3021v2.
- [19] C. Bär and M. Dahl, *Surgery and the spectrum of the Dirac operator*, J. Reine Angew. Math. **552** (2002), 53–76. arXiv:math/0201195.
- [20] C. Bär, P. Gauduchon, and A. Moroianu, *Generalized cylinders in semi-Riemannian and spin geometry*, Math. Z. **249** (2005), 545–580. doi:10.1007/s00209-004-0718-0. Equation numbers refer to arXiv:math/0303095v1.
- [21] G. Bazzoni, L. Martín-Merchán, and V. Muñoz, *Spin-harmonic structures and nilmanifolds*, Comm. Anal. Geom. **32** (2024), 153–201. arXiv:1904.01462v2.
- [22] E. Bierstone and P. D. Milman, *Semianalytic and subanalytic sets*, Publ. Math. IHÉS **67** (1988), 5–42. doi:10.1007/BF02699126.
- [23] R. Bott, *The stable homotopy of the classical groups*, Ann. of Math. (2) **70** (1959), 313–337. doi:10.2307/1970106.
- [24] B. Botvinnik, J. Ebert, and O. Randal-Williams, *Infinite loop spaces and positive scalar curvature*, Invent. Math. **209** (2017), 749–835. doi:10.1007/s00222-017-0719-3. Theorem locators refer to arXiv:1411.7408v3.
- [25] J.-P. Bourguignon and P. Gauduchon, *Spineurs, opérateurs de Dirac et variations de métriques*, Comm. Math. Phys. **144** (1992), 581–599. doi:10.1007/BF02099184.
- [26] C. Bourne, A. L. Carey, M. Lesch, and A. Rennie, *The KO-valued spectral flow for skew-adjoint Fredholm operators*, J. Topol. Anal. **14** (2022), no. 2, 505–556. doi:10.1142/S1793525320500557. Section and theorem locators refer to arXiv:1907.04981v2 (6 April 2020).
- [27] G. C. Brito and M. A. M. Marrocos, *On the generic simplicity of the spectrum for connection Laplacian and G-simplicity on principal bundles*, preprint, 13 February 2026. arXiv:2602.12884v1.
- [28] A. Calderon and N. Salter, *Higher spin mapping class groups and strata of Abelian differentials over Teichmüller space*, Adv. Math. **389** (2021), 107926. doi:10.1016/j.aim.2021.107926.
- [29] Y. Canzani, *On the multiplicity of eigenvalues of conformally covariant operators*, Ann. Inst. Fourier **64** (2014), 947–970. doi:10.5802/aif.2870.
- [30] A. L. Carey, J. Phillips, and H. Schulz-Baldes, *Spectral flow for skew-adjoint Fredholm operators*, J. Spectr. Theory **9** (2019), 137–170. doi:10.4171/JST/243.
- [31] M. Chatzakou and B. Helffer, *Generic simplicity for self-adjoint operators under bounded potential perturbations*, preprint, version 2 (16 July 2026). arXiv:2605.31368v2.
- [32] Y. Colin de Verdière, *Construction de laplaciens dont une partie finie du spectre est donnée*, Ann. Sci. École Norm. Sup. (4) **20** (1987), 599–615. doi:10.24033/asens.1546.
- [33] D. Crowley, T. Schick, and W. Steimle, *Harmonic spinors and metrics of positive curvature via the Gromoll filtration and Toda brackets*, J. Topol. **11** (2018), 1077–1099. doi:10.1112/topo.12081.
- [34] G. M. D’Ariano, P. Perinotti, and M. F. Sacchi, *Informationally complete measurements and group representation*, J. Opt. B: Quantum Semiclass. Opt. **6** (2004), S487–S491. doi:10.1088/1464-4266/6/6/005.
- [35] M. Dahl, *Dirac eigenvalues for generic metrics on three-manifolds*, Ann. Global Anal. Geom. **24** (2003), 95–100. doi:10.1023/A:1024231524848. The correction and theorem locators refer to arXiv:math/0204232v2 (3 July 2013).
- [36] M. Dahl, *Prescribing eigenvalues of the Dirac operator*, Manuscripta Math. **118** (2005), 191–199. arXiv:math/0311172v3. doi:10.1007/s00229-005-0583-0.
- [37] M. Dahl, *On the space of metrics with invertible Dirac operator*, Comment. Math. Helv. **83** (2008), 451–469. doi:10.4171/CMH/132.
- [38] M. Dahl and N. Große, *Invertible Dirac operators and handle attachments on manifolds with boundary*, J. Topol. Anal. **6** (2014), 339–382. arXiv:1203.3637v2.
- [39] R. J. Downes, M. Levitin, and D. Vassiliev, *Spectral asymmetry of the massless Dirac operator on a 3-torus*, J. Math. Phys. **54** (2013), 111503. arXiv:1306.5689.
- [40] S. Dyatlov, *Lecture notes for 18.155: distributions, elliptic regularity, and applications to PDEs*, Massachusetts Institute of Technology, version of 20 October 2026. Lecture notes.
- [41] J. Ebert, *The two definitions of the index difference*, Trans. Amer. Math. Soc. **369** (2017), 7469–7507. doi:10.1090/tran/7133. Theorem locators refer to arXiv:1308.4998v3.
- [42] J. Ebert and O. Randal-Williams, *Infinite loop spaces and positive scalar curvature in the presence of a fundamental group*, Geom. Topol. **23** (2019), 1549–1610. doi:10.2140/gt.2019.23.1549.
- [43] Y. Eliashberg and N. Mishachev, *Introduction to the h-Principle*, Graduate Studies in Mathematics 48, American Mathematical Society, Providence, RI, 2002. doi:10.1090/gsm/048.
- [44] S. Farinelli, *Riemannian metrics and harmonic sections of spinor bundles*, preprint, 2012, version 10 (2024). arXiv:1204.3248v10. doi:10.48550/arXiv.1204.3248.
- [45] D. Ferus, H. Karcher, and H.-F. Münzner, *Cliffordalgebren und neue isoparametrische Hyperflächen*, Math. Z. **177** (1981), 479–502. doi:10.1007/BF01219082. English translation by T. E. Cecil, arXiv:1112.2780.
- [46] D. S. Freed, *Multivariable analysis*, lecture notes, version of 2 February 2021. Author notes.
- [47] W. Fulton and J. Harris, *Representation Theory: A First Course*, Graduate Texts in Mathematics 129, Springer-Verlag, New York, 1991. doi:10.1007/978-1-4612-0979-9.

- [48] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Classics in Mathematics, Springer, Berlin, 2001, reprint of the 1998 edition. doi:10.1007/978-3-642-61798-0.
- [49] H. Gimpelrein and G. Grubb, *Heat kernel estimates for pseudodifferential operators, fractional Laplacians and Dirichlet-to-Neumann operators*, J. Evol. Equ. **14** (2014), 49–83. doi:10.1007/s00028-013-0206-2.
- [50] G. H. Golub and C. F. Van Loan, *Matrix Computations*, third edition, Johns Hopkins University Press, Baltimore, 1996.
- [51] M. Gromov and H. B. Lawson, Jr., *The classification of simply connected manifolds of positive scalar curvature*, Ann. of Math. (2) **111** (1980), 423–434. doi:10.2307/1971103.
- [52] N. Große and N. Pederzani, *Homotopy equivalence of spaces of metrics with invertible Dirac operator*, preprint, 2019, version 2 (2022). arXiv:1910.09167v2. doi:10.48550/arXiv.1910.09167.
- [53] V. Guillemin and A. Pollack, *Differential Topology*, Prentice-Hall, Englewood Cliffs, NJ, 1974.
- [54] B. C. Hall, *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*, second edition, Graduate Texts in Mathematics 222, Springer, Cham, 2015. doi:10.1007/978-3-319-13467-3.
- [55] A. Hatcher, *Algebraic Topology*, Cambridge University Press, Cambridge, 2002. Author text.
- [56] A. Hermann, *Dirac eigenspinors for generic metrics*, doctoral dissertation, Universität Regensburg, 2012. arXiv:1201.5771v3.
- [57] T. Hertl, *Concordances in positive scalar curvature and index theory*, doctoral dissertation, Georg-August-Universität Göttingen, 2022; arXiv:2303.07844v1 (2023). doi:10.48550/arXiv.2303.07844.
- [58] N. Higson and G. Kasparov, *E-theory and KK-theory for groups which act properly and isometrically on Hilbert space*, Invent. Math. **144** (2001), 23–74. doi:10.1007/s002220000118.
- [59] M. W. Hirsch, *Differential Topology*, Graduate Texts in Mathematics 33, Springer, New York, 1976. doi:10.1007/978-1-4684-9449-5.
- [60] N. Hitchin, *Harmonic spinors*, Adv. Math. **14** (1974), 1–55. doi:10.1016/0001-8708(74)90021-8.
- [61] L. Hörmander, *The Analysis of Linear Partial Differential Operators III: Pseudo-Differential Operators*, Classics in Mathematics, Springer, Berlin, 2007, reprint of the 1994 edition. doi:10.1007/978-3-540-49938-1.
- [62] T. Kato, *Perturbation Theory for Linear Operators*, second edition, corrected printing, Grundlehren der mathematischen Wissenschaften 132, Springer-Verlag, Berlin, 1980; reprinted in Classics in Mathematics, Springer-Verlag, Berlin, 1995. doi:10.1007/978-3-642-66282-9.
- [63] A. Khrabustovskyi and V. Lotoreichik, *Homogenization of the Dirac operator with position-dependent mass*, Rev. Mat. Complut. **39** (2026), 381–405. doi:10.1007/s13163-025-00543-7.
- [64] F. Lin, *Divergence-free framings of three-manifolds via eigenspinors*, Doc. Math. **31** (2026), 445–451. doi:10.4171/DM/989.
- [65] S. Lockman, *A non-trivial index difference on surfaces of genus at least 3*, preprint, version 2 (7 August 2026). arXiv:2606.12283v2. doi:10.48550/arXiv.2606.12283.
- [66] S. Maier, *Generic metrics and connections on Spin- and Spin^c-manifolds*, Comm. Math. Phys. **188** (1997), 407–437. doi:10.1007/s002200050171.
- [67] J. Martínez-Aguinaga and Á. del Pino, *Convex integration with avoidance and hyperbolic (4, 6) distributions*, 2021, arXiv:2112.14632v1. Theorem 3.5 is the open-ample theorem.
- [68] J. Milnor, *Spin structures on manifolds*, Enseign. Math. (2) **9** (1963), 198–203. doi:10.5169/seals-38784.
- [69] J. Moser, *On the volume elements on a manifold*, Trans. Amer. Math. Soc. **120** (1965), 286–294. doi:10.1090/S0002-9947-1965-0182927-5.
- [70] K.-H. Neeb, *Classical Hilbert–Lie groups, their extensions and their homotopy groups*, Banach Center Publ. **55** (2002), 87–151. doi:10.4064/bc55-0-6.
- [71] A. Nifa, *Local realization of spectral clusters*, preprint, version 2, 2026. doi:10.5281/zenodo.23224622.
- [72] N. Nowaczyk, *Existence of Dirac eigenvalues of higher multiplicity*, Math. Z. **284** (2016), 285–307. Theorem locators refer to arXiv:1504.00961v1. doi:10.1007/s00209-016-1657-2.
- [73] L. Ornea, M. Parton, P. Piccinni, and V. Vuletescu, *Spin(9) geometry of the octonionic Hopf fibration*, Transform. Groups **18** (2013), no. 3, 845–864. doi:10.1007/s00031-013-9233-x. See also arXiv:1208.0899v2, Section 4.
- [74] N. Pederzani, *On the spin cobordism invariance of the homotopy type of the space $\mathcal{R}^{\text{inv}}(M)$* , doctoral dissertation, Universität Leipzig, 2018. urn:nbn:de:bsz:15-qucosa2-213917.
- [75] C. Qian, Z. Tang, and W. Yan, *Clifford systems, harmonic maps and metrics with non-negative curvature*, Pacific J. Math. **320** (2022), no. 2, 391–424. doi:10.2140/pjm.2022.320.391. Section locators refer to arXiv:2202.09078v2 (23 August 2022).
- [76] T. Schick, *Real versus complex K-theory using Kasparov’s bivariant KK-theory*, Algebr. Geom. Topol. **4** (2004), 333–346. doi:10.2140/agt.2004.4.333.
- [77] S. Smale, *An infinite dimensional version of Sard’s theorem*, Amer. J. Math. **87** (1965), 861–866.
- [78] S. Stolz, *Simply connected manifolds of positive scalar curvature*, Ann. of Math. (2) **136** (1992), 511–540. doi:10.2307/2946598.
- [79] M. E. Taylor, *Multidimensional analytic Fredholm theory*, author notes, Theorem 1.1, p. 2.
- [80] N. Waterstraat, *A remark on the space of metrics having non-trivial harmonic spinors*, J. Fixed Point Theory Appl. **13** (2013), 143–149. arXiv:1206.0499v2.
- [81] C. Wockel, *A generalization of Steenrod’s approximation theorem*, Arch. Math. (Brno) **45** (2009), no. 2, 95–104. DML-CZ.

- [82] A. Zinger, *The determinant line bundle for Fredholm operators: construction, properties, and classification*, Math. Scand. **118** (2016), 203–268. The orientation discussion in Section 5.3 refers to the post-publication update of 30 May 2022, arXiv:1304.6368v4.

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