

SHARP SCHRÖDINGER INEQUALITIES FOR EMBEDDED HYPERSURFACES

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ABSTRACT. Let $n \geq 2$, let h denote the second fundamental form, and put $\Delta = -\operatorname{div} \nabla$. Every closed connected smooth embedded minimal hypersurface $M^n \subset \mathbb{S}^{n+1}$ which is not totally geodesic satisfies $\Delta + |h|^2 \geq n$ in quadratic-form sense on $H^1(M)$. In particular, $\int_M |h|^2 dv_g \geq n \operatorname{Vol}(M)$, as conjectured by Perdomo, with equality precisely for minimal Clifford hypersurfaces.

Let $Q > 0$ be self-adjoint on $\mathbb{R}^{n+1}$, and put $a = \lambda_{\min}(Q)$ and $F(x) = \frac{1}{2}\langle Qx, x \rangle$. On a hypersurface with unit normal ν , write $d\mu_F = e^{-F} dv_g$, $\Delta_F = \Delta + \langle \nabla F, \nabla \rangle$ and $U = |h|^2 + \langle Q\nu, \nu \rangle$. Every connected complete smooth properly embedded hypersurface $M^n \subset \mathbb{R}^{n+1}$ without boundary, stationary for the weighted area $e^{-F} dv_g$ and not a hyperplane, satisfies $\Delta_F + U \geq 2a$ in quadratic-form sense on $C_c^\infty(M)$ in $L^2(M, d\mu_F)$. Its weighted volume is finite, and $\int_M U d\mu_F \geq 2a\mu_F(M)$. Equality in this average inequality holds precisely for the cylinders $\{x + y : x \in E, |x| = \sqrt{p/a}, y \in E^\perp\}$, where $1 \leq p \leq n$, E is a linear subspace of $\ker(Q - a\operatorname{Id})$ and $\dim E = p + 1$. For $Q = \frac{1}{2}\operatorname{Id}$, this gives the Gaussian average-curvature inequalities conjectured by Li and Zhao.

In the spherical case, Morse index $n + 3$ characterizes the minimal Clifford hypersurfaces. Every smooth entire graph stationary for the quadratic density in a Q -eigendirection with eigenvalue $\beta \leq 2a$ is a hyperplane through the origin.

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1. INTRODUCTION

Let $X : M^n \hookrightarrow \mathbb{S}^{n+1}$ be a closed connected embedded minimal hypersurface, $n \geq 2$, with induced metric g and second fundamental form h . Put $S = |h|^2$ and $\Delta = -\operatorname{div} \nabla$. Simons' identity [39, Theorem 5.3.1] gives

$$\int_M S(S - n) dv_g = \int_M |\nabla h|^2 dv_g. \quad (1.1)$$

Perdomo's conjecture [35, Conjecture B] asks for $\int_M S dv_g \geq n \operatorname{Vol}(M)$ when M is not totally geodesic, with equality precisely for minimal Clifford hypersurfaces; see also [25, Conjecture 1.5]. The quadratic-form inequality below implies this average inequality.

1.1. Minimal hypersurfaces of the sphere.

Theorem 1.1. *Let $n \geq 2$, and let $X : M^n \hookrightarrow \mathbb{S}^{n+1}$ be a closed connected smooth minimal embedding which is not totally geodesic. Then*

$$\int_M (|\nabla f|^2 + S f^2) dv_g \geq n \int_M f^2 dv_g \quad (f \in H^1(M)). \quad (1.2)$$

In particular,

$$\int_M S dv_g \geq n \operatorname{Vol}(M). \quad (1.3)$$

Equality in (1.3) holds if and only if, after an ambient isometry,

$$X(M) = \mathbb{S}^p \left(\sqrt{p/n} \right) \times \mathbb{S}^{n-p} \left(\sqrt{(n-p)/n} \right), \quad 1 \leq p \leq n-1. \quad (1.4)$$

Ge–Li [25, Theorem 1.4] obtained the lower bound $\int_M S dv_g \geq \delta(n) \operatorname{Vol}(M)$ for some $\delta(n) > 0$ depending only on n . Wang–Wang [42, Theorem 1.2] obtained the coefficient n under invariance by reflection in $n+2$ mutually perpendicular great hyperspheres. The classification under the pointwise identity $S \equiv n$ is due to Chern–do Carmo–Kobayashi [16, Main theorem, p. 60] and Lawson [29].

For $n = 2$, the hypersurface M in Theorem 1.1 has genus $g \geq 1$ [39, §5.4, p. 96], and Gauss–Bonnet gives

$$\int_M S dv_g = 2 \operatorname{Vol}(M) + 8\pi(g-1).$$

Thus the average inequality holds in this dimension, and its equality case is Brendle's classification of embedded minimal tori [7]. The embedding hypothesis in Theorem 1.1 is essential. In dimensions $n \geq 3$, closed connected oriented minimal immersions with exactly two principal curvatures and non-Clifford image satisfy $\int_M S dv_g < n \operatorname{Vol}(M)$ [36].

1.2. Quadratic Gaussian densities. Let Q be a positive-definite self-adjoint endomorphism of $\mathbb{R}^{n+1}$, and set

$$a = \lambda_{\min}(Q), \quad b = \lambda_{\max}(Q), \quad F(z) = \frac{1}{2} \langle Qz, z \rangle. \quad (1.5)$$

For a Euclidean hypersurface with unit normal ν , put

$$h(v, z) = \langle \bar{D}_\nu \nu, dX(z) \rangle, \quad H = \operatorname{tr}_g h, \quad S = |h|^2. \quad (1.6)$$

Here $\bar{D}$ is the Euclidean connection. Stationarity for the density e^{-F} is equivalent to

$$H = \langle QX, \nu \rangle; \quad (1.7)$$

see [14, Proposition 2]. Define

$$\begin{aligned} d\mu_F &= e^{-F \circ X} dv_g, & \Delta_F &= -\operatorname{div} \nabla + \langle (QX)^T, \nabla \rangle, & \mathcal{L}_F &= -\Delta_F, \\ q_\nu &= \langle Q\nu, \nu \rangle, & U &= S + q_\nu. \end{aligned} \quad (1.8)$$

Theorem 1.2. *Let $n \geq 2$, and let $X : M^n \hookrightarrow \mathbb{R}^{n+1}$ be a connected complete smooth proper embedding without boundary satisfying (1.7). Assume that $X(M)$ is not a hyperplane. Then*

$$\int_M (|\nabla f|^2 + U f^2) d\mu_F \geq 2a \int_M f^2 d\mu_F \quad (f \in C_c^\infty(M)). \quad (1.9)$$

The weighted volume is finite, and

$$\int_M U d\mu_F \geq 2a \mu_F(M). \quad (1.10)$$

Equality in (1.10) holds if and only if there are an integer $1 \leq p \leq n$ and a linear subspace $E \subset \ker(Q - a\text{Id})$, $\dim E = p + 1$, such that

$$X(M) = \{x + y : x \in E, |x| = \sqrt{p/a}, y \in E^\perp\}. \quad (1.11)$$

In the compact equality case, $p = n$ and $Q = a\text{Id}$.

The integral in (1.10) is allowed to be infinite. The Friedrichs realization of $\Delta_F + U$ has spectrum in $[2a, \infty)$. Since $q_v \leq b$, one also has $\Delta_F + S \geq 2a - b$ in quadratic-form sense. The second variation of weighted area has quadratic form $\int_M (|\nabla f|^2 - U f^2) d\mu_F$ [14, 27]. For the quadratic weight, the ambient Bakry–Émery tensor is Q and is parallel. The Simons and normal-component equations are therefore specializations of [15, arXiv v1, (23), Theorem 3 and Proposition 2], as stated in Lemma 5.1.

Write F for $F \circ X$, and put $\mathcal{D} = \text{div } \nabla$ and $C_0 = \text{tr } Q - a$. The function F is nonnegative and proper, and

$$\mathcal{D}F \leq C_0, \quad \mathcal{D}F - \frac{1}{2}|\nabla F|^2 + aF \leq C_0.$$

Thus $\int_M e^{-F/2} dv_g < \infty$: for noncompact M this follows from Cheng–Vieira–Zhou [17, Theorem 1.1(ii), arXiv version] with $\alpha = \frac{1}{2}$ and $\beta = a$, and for compact M it is immediate. Lemma 6.1 then gives $\mu_F(M) < \infty$ and $F \in L^1(M, d\mu_F)$.

For $Q = \frac{1}{2}\text{Id}$, equation (1.7) is the self-shrinker equation. Write $d\gamma = e^{-|X|^2/4} dv_g$ and $\Delta_\gamma = \Delta_F$ at this scale. Theorem 1.2 becomes

$$\Delta_\gamma + S \geq \frac{1}{2}, \quad \int_M S d\gamma \geq \frac{1}{2} \gamma(M). \quad (1.12)$$

Equality in the average inequality characterizes, up to an orthogonal transformation, the cylinders $S^p(\sqrt{2p}) \times \mathbb{R}^{n-p}$, $1 \leq p \leq n$. Cheng–Zhou [18, Theorem 1.3] prove that properness, polynomial volume growth and finite Gaussian volume are equivalent for complete immersed self-shrinkers. The entropy identity in Lemma 7.5, from [19, (7.27), p. 788], therefore converts (1.12) into the inequalities of Li–Zhao [31, Conjectures 1.6–1.7], for $n \geq 2$.

Cao–Li [10, Theorem 1.1] and Ding–Xin [21, Proposition 3.1] classify complete self-shrinkers with $S \leq \frac{1}{2}$, under polynomial volume growth or properness, respectively. For complete properly embedded self-shrinkers, Li–Zhao [32, arXiv v3, Theorems 1.2–1.3] obtain a hyperplane or a generalized round cylinder under $S < \frac{3}{4}$, and under $S \leq \frac{3}{4}$ when $n = 2$. Their scale and that of Cao–Li are related to the present one by

$$\widehat{X} = X/\sqrt{2}, \quad \widehat{g} = \frac{1}{2}g, \quad \widehat{S} = 2S, \quad e^{-|\widehat{X}|^2/2} = e^{-|X|^2/4}. \quad (1.13)$$

In particular, Li–Zhao’s upper bounds $\widehat{S} < 3/2$ and $\widehat{S} \leq 3/2$ give the stated bounds on S . For the scalar drift Laplacian, Ding–Xin [20, Theorem 1.3] and Brendle–Tsiamis [9, Theorem 1.1] give the weighted Poincaré inequality with coefficient $1/4$ for mean-zero functions, in the compact and properly embedded cases, respectively.

1.3. Graphical and spectral consequences.

Theorem 1.3. *Let $n \geq 2$, let Q satisfy (1.5), and let $X : M^n \hookrightarrow \mathbb{R}^{n+1}$ be connected, complete, smooth, properly embedded, without boundary, and satisfy (1.7). Suppose that a nonzero constant vector v satisfies*

$$Qv = \beta v, \quad \beta \leq 2a, \quad \langle v, v \rangle > 0 \quad \text{on } M. \quad (1.14)$$

Then $X(M)$ is a hyperplane through the origin. In particular, every entire smooth graph in such an eigendirection is a hyperplane.

For $Q = a\text{Id}$, dilation by $\sqrt{2a}$ reduces the entire-graph conclusion to Wang's theorem [41], and the positive-angle conclusion to Ding–Xin–Yang [22, arXiv v1, Theorem 3.1]. Hurtado–Palmer–Rosales [26, Theorem 4.26] treat entire graphs for product densities $e^{\eta(x)+\mu(t)}$, with a radial decay hypothesis and $\text{Hess } \eta(\xi, \xi) \leq c \leq \mu''(t)$ for every unit horizontal vector ξ . For $Q = \text{diag}(A, \beta)$, $\eta(x) = -\langle Ax, x \rangle/2$ and $\mu(t) = -\beta t^2/2$, the radial bound is $\langle \nabla(\eta + \mu), \nabla|(x, t)| \rangle \leq -a|(x, t)|$ away from the origin, and the Hessian condition is $-\lambda_{\min}(A) \leq c \leq -\beta$. Thus that theorem gives the entire-graph conclusion when $\beta = a$. Theorem 1.3 also includes the range $a < \beta \leq 2a$.

Perdomo's estimate [35, Theorem 3.1], together with Theorem 1.1, gives the embedded index- $(n+3)$ classification in Corollary 7.3. The implication from the average inequality to this classification is recorded in [35, 25]. For $n = 2$, the classification is Urbano's theorem [40]. Chen–Wang [13, Corollary 1.3] obtain the lower bound $n+4$ for the index under the additional condition $\lambda_1(\Delta) < n$.

On a compact hypersurface in Theorem 1.1 or Theorem 1.2, let λ_j be the eigenvalues of Δ or Δ_F , respectively, and ζ_j those of $\Delta + S$ or $\Delta_F + U$, in nondecreasing order and with multiplicity. Proposition 7.2 gives

$$\zeta_j \geq c + \left(\frac{\min_M w}{\max_M w} \right)^2 \lambda_j \quad (j \geq 0), \quad c = \begin{cases} n & \text{in the sphere,} \\ 2a & \text{in Euclidean space,} \end{cases}$$

where w is the positive tangent-ball width defined below. The comparison follows from the quadratic-form min–max principle [28, Chapter VII, §6.4, Remark 6.2].

1.4. Tangent-ball curvatures and weak inequalities. Tangent-ball curvatures and their flow inequalities appear in [2, 5, 3]. Theorem 2 of [5] gives unnormalized viscosity inequalities; positive speed is an additional hypothesis of its quotient Corollary 3. Two-point functions also occur in Brendle's theorem [7] and in the constant-mean-curvature theorem of Andrews–Li [6]. Proposition 3 of [6, arXiv v3] is the dimension-two antecedent of the finite-contact calculation in Proposition 4.1; its mean curvature equals $H/2$ in the trace convention (1.6). The reflected tangent-space identity and the cubic identity at a diagonal contact occur in [4, Lemmas 2.2 and 2.4]. Brendle [8, Propositions 3, 5 and Corollary 6] proves semiconvexity and a distributional inequality with the terms $2 \sum_i \phi_i^2 / (\phi - \kappa_i)$ on the strict-contact region of a mean-convex flow.

For a hypersurface in Theorem 1.1 or Theorem 1.2, let k, ℓ be the two tangent-ball curvature functions of (3.2), corresponding to the two orientations. In Euclidean space each signed supremum is replaced by its maximum with zero; in the sphere the signed suprema are retained. Lemma 3.3 gives $w := k + \ell > 0$. The functions k, ℓ are locally semiconvex by Lemma 3.2. Their distributional Hessians have positive semidefinite singular parts [24, Theorems 6.8–6.9]; see also [33, Section 2 and Appendix]. The contact inequalities of Propositions 4.1 and 5.2, together with Lemma 2.5, give

$$\mathcal{L}_F w + V w \, d\mu_F \geq 2w^{-1} |\nabla w|^2 \, d\mu_F \quad (1.15)$$

as measures, with the convention (2.2). Here $F = 0$ and $V = S - n$ in the spherical case, and $V = U - 2a$ in the quadratic Gaussian case. The diagonal-contact lemma supplies the zero numerators at zero gap denominators, with arbitrary multiplicities of the principal curvatures.

Put $u = w^{-1}$. Proposition 2.4 gives the nonnegative locally finite measure $\mathfrak{m} = (\Delta_F + V)u$ and the identity

$$\int_M (|\nabla f|^2 + V f^2) \, d\mu_F = \int_M w^{-2} |\nabla(wf)|^2 \, d\mu_F + \int_M w f^2 \, d\mathfrak{m} \quad (f \in C_c^\infty(M)).$$

For positive smooth solutions, this is the ground-state transformation [37, Lemma 2.4]. For a closed connected embedded non-totally-umbilic hypersurface of constant trace mean curvature H in $\mathbb{S}^{n+1}$, Theorem 4.2 gives the same identity with $F = 0$ and $V = S - n - H(k - \ell)$. Reversing the normal interchanges k, ℓ and changes H to $-H$, so this potential is unchanged.

2. POSITIVE SUPERSOLUTIONS AND QUADRATIC FORMS

Let (M, g) be a smooth n -dimensional Riemannian manifold without boundary, let $F \in C^\infty(M)$, and put

$$d\mu_F = e^{-F} dv_g, \quad \mathcal{L}_F = \operatorname{div} \nabla - \langle \nabla F, \nabla \rangle, \quad \Delta_F = -\mathcal{L}_F. \quad (2.1)$$

All function spaces are real. The space $H^1(M, d\mu_F)$ consists of functions whose value and weak gradient are square integrable with respect to $d\mu_F$. On a closed manifold it coincides with $H^1(M)$, with an equivalent norm. The notation $K \Subset M$ means that K is a compact subset of M . A C^2 function ϕ is an upper test for a continuous function z at p if $\phi(p) = z(p)$ and $\phi \geq z$ on a neighbourhood of p .

The unweighted case is $F = 0$. For locally Lipschitz z , define

$$\langle \mathcal{L}_F z, \eta \rangle_F = - \int_M \langle \nabla z, \nabla \eta \rangle d\mu_F, \quad \eta \in C_c^\infty(M). \quad (2.2)$$

When $\mathcal{L}_F z$ is a measure, it includes the weight: a smooth density a represents the measure $a d\mu_F$.

Lemma 2.1. *Let $K \Subset M$, and let $z \geq 0$ be Lipschitz with support in K . There are nonnegative $z_j \in C_c^\infty(M)$, all supported in one compact set, such that $z_j \rightarrow z$ uniformly and in $H^1(M, d\mu_F)$.*

Proof. Choose finitely many relatively compact coordinate balls U_i and nonnegative $\rho_i \in C_c^\infty(U_i)$ with $\sum_i \rho_i = 1$ near K . Extend each $\rho_i z$ by zero in its coordinate space and convolve with a nonnegative standard mollifier. For sufficiently small radii, the convolutions have supports in fixed compact subsets of the respective charts. Their pullbacks therefore sum to a nonnegative smooth function with a common compact support. By [24, Theorem 4.1(ii), (vi)], these sums converge uniformly and in the coordinate H^1 norms to $\sum_i \rho_i z = z$. On the fixed compact sets, the smooth weight and metric are bounded above and below by positive constants. The coordinate and weighted H^1 norms are therefore equivalent. $\square$

Consequently, if z is locally Lipschitz and $\mathcal{L}_F z$ is a locally finite signed Radon measure, then (2.2) remains valid for compactly supported Lipschitz tests. Indeed, apply Lemma 2.1 to the positive and negative parts of a test: uniform convergence passes the measure integral, and H^1 convergence passes the gradient integral. An inequality between such measures may therefore be tested against any nonnegative compactly supported Lipschitz function.

A function z is locally semiconvex if, on every sufficiently small coordinate ball, $z + C|x|^2/2$ is convex for some $C \geq 0$. Such a function is locally Lipschitz and has an Alexandrov Hessian almost everywhere [24, Theorems 6.7(i) and 6.9].

Lemma 2.2. *If z is locally semiconvex, its coordinate Hessian is a locally finite symmetric matrix measure. Its singular part is positive semidefinite, and its absolutely continuous density is the Alexandrov Hessian almost everywhere. The singular part of $\mathcal{L}_F z$, relative to $d\mu_F$, is nonnegative.*

Proof. The Euclidean assertions follow from [24, Theorems 6.8–6.9], applied locally to $z + C|x|^2/2$. Subtracting the smooth quadratic changes only the absolutely continuous part. Write

$$\mathcal{H} = (\partial_{ij} z) = A dx + \mathcal{H}^s, \quad \omega = e^{-F} \sqrt{\det g}.$$

The coordinate expression of (2.2) is

$$\mathcal{L}_F z = \omega g^{ij} \mathcal{H}_{ij} + \omega (-g^{ij} \Gamma_{ij}^k \partial_k z - g^{ij} (\partial_i F) (\partial_j z)) dx. \quad (2.3)$$

The parenthesized function belongs to L_{loc}^∞ . Since $\mathcal{H}^s$ is positive semidefinite, write $\mathcal{H}^s = C d\sigma$, where $\sigma = \operatorname{tr} \mathcal{H}^s \geq 0$ and $C \geq 0$ σ -almost everywhere. Thus

$$(\mathcal{L}_F z)^s = \omega \operatorname{tr}(g^{-1/2} C g^{-1/2}) d\sigma \geq 0.$$

$\square$

In almost-everywhere inequalities, $\mathcal{L}_F z$ denotes the density of its absolutely continuous part with respect to $d\mu_F$.

Lemma 2.3. *Let z be locally semiconvex and let $G \in L^1_{\text{loc}}(M, d\mu_F)$. If $\mathcal{L}_F z \geq G$ almost everywhere, then $\mathcal{L}_F z \geq G d\mu_F$ as measures. At an Alexandrov point, every non-strict upper-test inequality whose expression is continuous in the Hessian, with value and gradient fixed, holds for the Alexandrov Hessian. Hence it holds for the absolutely continuous density of the coordinate Hessian almost everywhere.*

Proof. The first assertion follows from the nonnegative singular part in Lemma 2.2. At an Alexandrov point, take coordinates with origin at that point and write

$$z(x) = z(0) + p \cdot x + \frac{1}{2} x^T B x + o(|x|^2). \quad (2.4)$$

For every $\varepsilon > 0$, $\phi_\varepsilon(x) = z(0) + p \cdot x + \frac{1}{2} x^T B x + \varepsilon |x|^2$ is an upper test on a sufficiently small ball. Its value and gradient at the origin are independent of ε , and $D^2 \phi_\varepsilon(0) = B + 2\varepsilon \text{Id} \rightarrow B$. Apply the upper-test inequality to ϕ_ε and let $\varepsilon \downarrow 0$. $\square$

For the ground-state transformation with a positive solution, see [37, Lemma 2.4]. The following proposition gives the identity with a nonnegative measure $(\Delta_F + V)u$.

Proposition 2.4. *Let $w > 0$ be locally Lipschitz, suppose that $\mathcal{L}_F w$ is a locally finite signed Radon measure, and let $V \in L^\infty_{\text{loc}}(M)$. Assume*

$$\mathcal{L}_F w + V w d\mu_F \geq 2w^{-1} |\nabla w|^2 d\mu_F. \quad (2.5)$$

Then $u = w^{-1}$ is locally Lipschitz and

$$\mathfrak{m} = (\Delta_F + V)u \quad (2.6)$$

is a nonnegative locally finite Radon measure. For every $f \in C_c^\infty(M)$,

$$\mathcal{Q}_V(f) := \int_M (|\nabla f|^2 + V f^2) d\mu_F = \int_M w^{-2} |\nabla(wf)|^2 d\mu_F + \int_M w f^2 d\mathfrak{m}. \quad (2.7)$$

Proof. On every compact set, w has a positive lower bound. Hence $u = w^{-1}$ is locally Lipschitz. The Sobolev chain rule [24, Theorem 4.4(ii)] gives $\nabla u = -w^{-2} \nabla w$. By (2.5), the measure

$$\mathfrak{n} = \mathcal{L}_F w + (V w - 2w^{-1} |\nabla w|^2) d\mu_F$$

is nonnegative and locally finite. For $\eta \in C_c^\infty(M)$, the function η/w^2 is a compactly supported Lipschitz test. Hence

$$\begin{aligned} \int_M \frac{\eta}{w^2} d\mathfrak{n} &= - \int_M \langle \nabla w, \nabla(\eta/w^2) \rangle d\mu_F + \int_M \left(\frac{V\eta}{w} - \frac{2\eta |\nabla w|^2}{w^3} \right) d\mu_F \\ &= \int_M (\langle \nabla u, \nabla \eta \rangle + V u \eta) d\mu_F. \end{aligned} \quad (2.8)$$

Thus $\mathfrak{m} = w^{-2} \mathfrak{n}$, which proves (2.6).

For $f \in C_c^\infty(M)$, the function f^2/u is also a compactly supported Lipschitz test. The product rule and (2.6) give

$$\begin{aligned} \mathcal{Q}_V(f) - \int_M u^2 |\nabla(f/u)|^2 d\mu_F &= \int_M (\langle \nabla u, \nabla(f^2/u) \rangle + V f^2) d\mu_F \\ &= \int_M \frac{f^2}{u} d\mathfrak{m}. \end{aligned}$$

Substitute $u = w^{-1}$ to obtain (2.7). $\square$

Lemma 2.5. *Let $V \in L^\infty_{\text{loc}}(M)$, let k, ℓ be locally semiconvex, let $w = k + \ell > 0$, and let $\kappa_1, \dots, \kappa_n$ be the eigenvalues of a smooth symmetric endomorphism of TM . At each point where k and ℓ are differentiable, choose an orthonormal eigenbasis $e_1, \dots, e_n$ and put $k_i = dk(e_i)$ and $\ell_i = d\ell(e_i)$. No smooth choice of eigenbasis is assumed. Suppose $k \geq \kappa_i$ and $\ell \geq -\kappa_i$, and assume almost everywhere that*

$$\mathcal{L}_F k + V k \geq 2 \sum_i \frac{k_i^2}{k - \kappa_i}, \quad \mathcal{L}_F \ell + V \ell \geq 2 \sum_i \frac{\ell_i^2}{\ell + \kappa_i}. \quad (2.9)$$

A zero denominator is required to have zero numerator, and that summand is set equal to zero. Then (2.5) holds.

Proof. For $A, B > 0$,

$$\frac{r^2}{A} + \frac{s^2}{B} - \frac{(r+s)^2}{A+B} = \frac{(Br - As)^2}{AB(A+B)} \geq 0. \quad (2.10)$$

If $A = 0$ and $r = 0$, the inequality $r^2/A + s^2/B \geq (r+s)^2/(A+B)$ is an equality under the stated convention. The case $B = 0, s = 0$ is identical. Apply this inequality with $A = k - \kappa_i, B = \ell + \kappa_i, r = k_i$ and $s = \ell_i$. Since $A + B = w > 0$, the two denominators do not vanish simultaneously. Summing over i gives (2.5) almost everywhere. Since $2w^{-1}|\nabla w|^2 - Vw \in L^\infty_{\text{loc}}(M)$, Lemma 2.3, applied to w , gives the measure inequality. $\square$

Lemma 2.6. *Under the hypotheses of Proposition 2.4, suppose that $V \in L^1(M, d\mu_F)$ and that there are $\chi_j \in C_c^\infty(M)$ with $0 \leq \chi_j \uparrow 1$, equal to one eventually on every compact set, such that $\int_M |\nabla \chi_j|^2 d\mu_F \rightarrow 0$. Then*

$$\int_M V d\mu_F = \int_M |\nabla \log w|^2 d\mu_F + \int_M w d\mathbf{m}. \quad (2.11)$$

Both terms on the right are finite. On a closed manifold, the identity follows directly by taking $f = 1$ in (2.7).

Proof. Apply (2.7) to χ_j :

$$\int_M (|\nabla \chi_j|^2 + V \chi_j^2) d\mu_F = \int_M |\nabla \chi_j + \chi_j \nabla \log w|^2 d\mu_F + \int_M w \chi_j^2 d\mathbf{m}. \quad (2.12)$$

By dominated convergence and the energy hypothesis, the left side tends to $q := \int_M V d\mu_F < \infty$. For each point, $\chi_j = 1$ on a neighbourhood of that point for all sufficiently large j . Fatou's lemma for the first term and monotone convergence for the second give

$$\int_M |\nabla \log w|^2 d\mu_F + \int_M w d\mathbf{m} \leq q.$$

Thus both integrals are finite. In particular,

$$\left| 2 \int_M \chi_j \langle \nabla \chi_j, \nabla \log w \rangle d\mu_F \right| \leq 2 \|\nabla \chi_j\|_{L^2(d\mu_F)} \|\nabla \log w\|_{L^2(d\mu_F)} \rightarrow 0. \quad (2.13)$$

Expand the square in (2.12). The gradient energy and the mixed term tend to zero, while monotone convergence applies to $\chi_j^2 |\nabla \log w|^2$ and $w \chi_j^2$. Taking limits gives (2.11). $\square$

Lemma 2.7. *On a closed manifold, multiplication by a Lipschitz function w is bounded on $H^1(M, d\mu_F)$. If $\min_M w > 0$, this multiplication operator is invertible. On a connected manifold, a locally Lipschitz function with zero weak gradient is constant.*

Proof. For smooth f , the Sobolev product rule [24, Theorem 4.4(i)] gives $\nabla(wf) = w\nabla f + f\nabla w$. Hence

$$\|wf\|_{L^2} \leq \|w\|_\infty \|f\|_{L^2}, \quad \|\nabla(wf)\|_{L^2} \leq \|w\|_\infty \|\nabla f\|_{L^2} + \|\nabla w\|_\infty \|f\|_{L^2}, \quad (2.14)$$

where the L^2 norms are taken with respect to $d\mu_F$. Smooth approximation [24, Theorem 4.2], in finitely many charts, extends these estimates to $H^1(M, d\mu_F)$. If $\min_M w > 0$, the chain rule [24, Theorem 4.4(ii)] gives $\nabla(w^{-1}) = -w^{-2}\nabla w$, so the same estimates apply to w^{-1} . Finally, Morrey's inequality [24, Theorem 4.10(i)], with any exponent greater than n , makes a locally Lipschitz function with zero weak gradient constant on each coordinate ball. Continuity and connectedness give constancy on M . $\square$

3. TANGENT-BALL CURVATURES

Let $X : M^n \hookrightarrow N$ be a smooth connected embedded hypersurface without boundary, with $n \geq 2$. Suppose either that $N = \mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ and M is closed, or that $N = \mathbb{R}^{n+1}$ and X is proper.

Lemma 3.1. *The complement $N \setminus X(M)$ has exactly two connected components. There is a smooth global unit normal pointing from either specified component to the other.*

Proof. The image $X(M)$ is a closed subset of N , by compactness or properness. Since N is simply connected, the separation theorem [23, Section 3.2] gives two components with common boundary $X(M)$. The local unit normals directed toward either chosen component agree on overlaps and define a smooth global unit normal. $\square$

Fix such a normal ν . Let $\bar{D}$ be the flat connection of the containing Euclidean space, and put

$$h(\nu, z) = \langle \bar{D}_\nu \nu, dX(z) \rangle = g(B\nu, z), \quad H = \text{tr } B, \quad S = |h|^2.$$

Identify tangent vectors with their images under dX . For $x \neq y$, define

$$d(x, y) = X(y) - X(x), \quad A(x, y) = \frac{1}{2}|d(x, y)|^2, \quad c(x, y) = -\frac{\langle \nu(x), d(x, y) \rangle}{A(x, y)}. \quad (3.1)$$

In the sphere, $A = 1 - \langle X(x), X(y) \rangle$. Define

$$\begin{aligned} \mathbb{S}^{n+1} : \quad k &= \sup_{y \neq x} c(x, y), & \ell &= \sup_{y \neq x} [-c(x, y)], \\ \mathbb{R}^{n+1} : \quad k &= \max\{0, \sup_{y \neq x} c(x, y)\}, & \ell &= \max\{0, \sup_{y \neq x} [-c(x, y)]\}. \end{aligned} \quad (3.2)$$

Put $w = k + \ell$. Reversing ν interchanges k and ℓ and leaves w unchanged. The spherical envelopes are not truncated at zero.

Lemma 3.2. *The functions k, ℓ are finite, locally Lipschitz, and locally semiconvex. If $\kappa_1, \dots, \kappa_n$ are the principal curvatures, then*

$$k \geq \max_i \kappa_i, \quad \ell \geq -\min_i \kappa_i. \quad (3.3)$$

In the spherical case, $k(p) > \max_i \kappa_i(p)$ implies that the supremum is attained at a point distinct from p . In the Euclidean case, the same holds provided also that $k(p) > 0$.

Proof. For $|v| = 1$, the Gauss formula along $\gamma(t) = \exp_x(tv)$ gives

$$A(x, \gamma(t)) = \frac{1}{2}t^2 + O(t^3), \quad -\langle \nu(x), X(\gamma(t)) - X(x) \rangle = \frac{1}{2}h_x(v, v)t^2 + O(t^3). \quad (3.4)$$

Thus $c(x, \gamma(t)) \rightarrow h_x(v, v)$, which gives (3.3).

Fix $p \in M$. Choose a relatively compact convex normal neighbourhood of p with a smooth orthonormal frame. Shrink $\varepsilon > 0$ so that, for $d_M(p, x) \leq \varepsilon/4$, the map $\exp_x$ is defined and injective on $\{|u| \leq \varepsilon\}$, with image in that neighbourhood. The radius may also be chosen so that every pair with $d_M(p, x) \leq \varepsilon/4$ and $d_M(p, y) < \varepsilon/2$ is joined there by its unique short geodesic. In the chosen frame, let v_x correspond to $v \in \mathbb{S}^{n-1}$. Consider

$$(x, v, t) \mapsto c(x, \exp_x(tv_x)), \quad 0 < t \leq \varepsilon. \quad (3.5)$$

Both numerator and denominator vanish to second order at $t = 0$. For any smooth P with $P(x, v, 0) = \partial_t P(x, v, 0) = 0$,

$$\frac{P(x, v, t)}{t^2} = \int_0^1 (1-s) \partial_t^2 P(x, v, st) ds.$$

The divided denominator equals $1/2$ at $t = 0$. Consequently, (3.5) extends smoothly to $t = 0$, with value $h_x(v_x, v_x)$. On the compact parameter set, this family and its first two x derivatives are uniformly bounded.

Take the fixed far set $\{y : d_M(p, y) \geq \varepsilon/2\}$. Every remaining competitor occurs in (3.5), since its distance from x is less than $3\varepsilon/4$. Properness makes the intersection of the far set with the inverse

image of any closed bounded ambient set compact. On this intersection, $d_M(x, y) \geq \varepsilon/4$; injectivity and compactness therefore give a positive lower bound for $A(x, y)$. The quotient and its first two x derivatives are uniformly bounded there.

In the Euclidean case, put $P(x, y) = \langle v(x), X(y) - X(x) \rangle$. As $|X(y)| \rightarrow \infty$, uniformly in the fixed compact set of x ,

$$|P| + |\partial_x P| + |\partial_x^2 P| \leq C|X(y)|, \quad A \geq c_1|X(y)|^2, \quad |\partial_x A| + |\partial_x^2 A| \leq C|X(y)|.$$

Differentiating $c = -P/A$ gives

$$|c(x, y)| + |\partial_x c(x, y)| + |\partial_x^2 c(x, y)| \leq C|X(y)|^{-1}. \quad (3.6)$$

The near and far families thus have common bounds for their values and first two coordinate derivatives. Every member is a quotient or a directional limit of quotients, and every competitor occurs in one of the families. Their supremum is the untruncated envelope. In Euclidean space, adjoin the constant zero function. On a smaller convex coordinate ball, all members f_α have a common Lipschitz bound and satisfy $D^2 f_\alpha \geq -C \text{Id}$. The supremum is finite and Lipschitz, and

$$\sup_\alpha f_\alpha(x) + \frac{C}{2}|x|^2 = \sup_\alpha \left(f_\alpha(x) + \frac{C}{2}|x|^2 \right)$$

is convex. This proves the assertions for k . The same derivative bounds hold for $-c$, and hence for ℓ .

For attainment, take a maximizing sequence $y_j \neq p$ in either strict case of the statement. After passage to a subsequence, properness gives one of three alternatives: $y_j \rightarrow q \neq p$, $y_j \rightarrow p$, or, in Euclidean space, $|X(y_j)| \rightarrow \infty$. The second has upper limit at most $\max_i \kappa_i(p)$ by the smooth diagonal extension; the third has limit zero by (3.6). The strict hypotheses exclude both. In the first case, continuity gives $c(p, q) = k(p)$. $\square$

Lemma 3.3. *In the spherical case, $w > 0$ everywhere unless the image is a round totally umbilic hypersphere. In the Euclidean case, $w > 0$ everywhere unless the image is an affine hyperplane.*

Proof. In the sphere, $w(p)$ is the supremum minus the infimum of $c(p, \cdot)$. If $w(p) = 0$, there is a $t \in \mathbb{R}$ such that $c(p, y) = t$ for every $y \neq p$. Thus

$$\langle v(p) - tX(p), X(y) \rangle = -t \quad (y \in M). \quad (3.7)$$

Since $|v(p) - tX(p)|^2 = 1 + t^2$, this affine section of $\mathbb{S}^{n+1}$ is a round n -sphere. The immersion into that section is a local diffeomorphism. Its image is open and, by compactness, closed, so it is the entire section.

In Euclidean space, $k, \ell \geq 0$. If $w(p) = 0$, then $c(p, y) = 0$ for every $y \neq p$, and $X(M) \subset X(p) + v(p)^\perp$. The image is open in this hyperplane by the inverse function theorem and closed by properness; hence it is the entire hyperplane. On a compact nonexceptional hypersurface, continuity gives $\min_M w > 0$. $\square$

Denote the two components of $N \setminus X(M)$ by Ω_- and Ω_+ , with v pointing from Ω_- to Ω_+ .

Lemma 3.4. *Let $B \subset N \setminus X(M)$ be connected and meet Ω_- . At $Y = X(q) \in \partial B$, suppose that B is locally $\{G < 0\}$, where G is smooth, $G(Y) = 0$, $\nabla^N G(Y) \neq 0$, and $\nabla^N G(Y) \perp T_q M$. Then $\nabla^N G(Y)/|\nabla^N G(Y)| = v(q)$.*

Proof. Connectedness gives $B \subset \Omega_-$. Put $V = \nabla^N G(Y)/|\nabla^N G(Y)|$. For small $t > 0$,

$$G(\exp_Y(-tV)) = -t|\nabla^N G(Y)| + O(t^2) < 0,$$

so $\exp_Y(-tV) \in \Omega_-$. Since $V \in \{v(q), -v(q)\}$ and $\exp_Y(tV(q)) \in \Omega_+$, one has $V = v(q)$. $\square$

3.1. Covariant identities. Put $c_0 = 1$ in the sphere and $c_0 = 0$ in Euclidean space. The Gauss and Weingarten formulas are

$$\bar{D}_v dX(z) = dX(\nabla_v z) - h(v, z)v - c_0 g(v, z)X, \quad \bar{D}_v v = dX(Bv). \quad (3.8)$$

With the convention $R(v, z)t = \nabla_v \nabla_z t - \nabla_z \nabla_v t - \nabla_{[v, z]}t$, the Gauss and Codazzi equations read

$$\begin{aligned} R(v, z)t &= c_0(g(z, t)v - g(v, t)z) + h(z, t)Bv - h(v, t)Bz, \\ (\nabla_v h)(z, t) &= (\nabla_z h)(v, t). \end{aligned} \quad (3.9)$$

In particular, ∇h is fully symmetric. Write $\mathcal{D} = \operatorname{div} \nabla$, also denoting by $\mathcal{D}$ the trace of the covariant Hessian of a tensor. Simons' identity in a space of constant curvature [38, (1.20)] is

$$\mathcal{D}h = \nabla^2 H + (nc_0 - S)h + Hh^2 - c_0 Hg, \quad (3.10)$$

where $(h^2)(v, z) = g(B^2 v, z)$ and $H = \operatorname{tr} h$. The mean curvature in [38] is H/n ; its formula (1.20) is expressed in terms of $\sum_i h_{ii}$ and does not assume that H is constant.

3.2. Diagonal contacts.

Lemma 3.5. *Let k be the envelope for v , let ϕ be a C^2 upper test at p , and suppose $k(p) = \kappa_{\max}(p)$. Put $E = \ker(k(p)\operatorname{Id} - B_p)$. Then*

$$d\phi|_E = 0. \quad (3.11)$$

Let Z be a smooth tangent field and put $L_Z = \mathcal{D} - \nabla_Z$. For any orthonormal eigenbasis with $e_1 \in E$, write $\phi_i = d\phi(e_i)$ and $h_{ij;r} = (\nabla_{e_r} h)(e_i, e_j)$. At p ,

$$L_Z \phi \geq (L_Z h)_{11} + 2 \sum_{j=1}^n \sum_{\kappa_i < k} \frac{h_{1i;j}^2}{k - \kappa_i} \geq (L_Z h)_{11} + 2 \sum_{\kappa_i < k} \frac{\phi_i^2}{k - \kappa_i}. \quad (3.12)$$

Proof. Fix a unit vector $v \in E$. The directional limits give

$$\kappa_{\max}(p) \leq \sup_{y \neq p} c(p, y) \leq k(p) = \kappa_{\max}(p).$$

For the reversed normal, the least principal curvature is therefore $-k(p) = \inf_{y \neq p} [-c(p, y)]$. The local cubic calculation in [4, proof of Lemma 2.4], applied to $-v$ and v , gives

$$(\nabla_v h)(v, v) = 0. \quad (3.13)$$

The C^2 tensor $T = \phi g - h$ is positive semidefinite near p , and $T_p(v, \cdot) = 0$. For a local unit extension V of v , the scalar $T(V, V)$ has a minimum equal to zero at p . Differentiation gives

$$d\phi(z) = (\nabla_z h)(v, v) \quad (z \in T_p M). \quad (3.14)$$

Taking the derivative in the direction v and applying (3.13) gives $d\phi(v) = 0$. Since v is arbitrary, (3.11) follows.

For $v = e_1$, equation (3.14) gives $(\nabla_Z T)_{11} = 0$. The local null-eigenvector inequality [1, proof of Theorem 3.2] therefore gives, for every real array Γ_j^i ,

$$0 \leq (L_Z T)_{11} - 4 \sum_{j,i} \Gamma_j^i (\nabla_j T)_{1i} + 2 \sum_{j,i,r} \Gamma_j^i \Gamma_j^r T_{ir}.$$

At p , $T_{ir} = (k - \kappa_i)\delta_{ir}$. Choose

$$\Gamma_j^i = \begin{cases} -h_{1i;j}/(k - \kappa_i), & \kappa_i < k, \\ 0, & \kappa_i = k. \end{cases}$$

For $\kappa_i < k$, one has $(\nabla_j T)_{1i} = -h_{1i;j}$. Substitution gives the first inequality in (3.12). Retaining the terms with $j = 1$ and applying Codazzi and (3.14), $h_{1i;1} = h_{11;i} = \phi_i$, gives the second. $\square$

4. THE SPHERICAL INEQUALITY

Let $M^n \hookrightarrow \mathbb{S}^{n+1}$ be closed, connected, and of constant trace mean curvature H . Assume that M is not totally umbilic. Let k, ℓ be the spherical envelopes in (3.2), and put $w = k + \ell$. By Lemma 3.3, $\min_M w > 0$.

Since H is constant, (3.8) and (3.10) give

$$\mathcal{D}X = -nX - Hv, \quad \mathcal{D}v = -Sv - HX, \quad \mathcal{D}h = (n - S)h + H(h^2 - g). \quad (4.1)$$

Proposition 4.1. *Let ϕ be a C^2 upper test for k at p . In an orthonormal principal frame at p , put $\phi_i = d\phi(e_i)$. Then*

$$\mathcal{D}\phi + (S - n)k - H(k^2 - 1) \geq 2 \sum_{\kappa_i < k} \frac{\phi_i^2}{k - \kappa_i}. \quad (4.2)$$

Moreover, $\phi_i = 0$ whenever $\kappa_i = k$.

Proof. If $k(p) = \kappa_{\max}(p)$, Lemma 3.5 with $Z = 0$ and (4.1) give both assertions.

Suppose that $k(p) > \kappa_{\max}(p)$. By Lemma 3.2, there is $q \neq p$ with $c(p, q) = k(p)$. Write

$$X = X(p), \quad Y = X(q), \quad A = 1 - \langle X, Y \rangle > 0, \quad d = Y - X, \quad b_i = \langle e_i, Y \rangle. \quad (4.3)$$

In the following calculation, $k = k(p)$ and $\kappa_i = \kappa_i(p)$. The function

$$Z(x, y) = \phi(x)(1 - \langle X(x), X(y) \rangle) + \langle v(x), X(y) \rangle \quad (4.4)$$

is nonnegative for x near p and every $y \in M$, and vanishes at (p, q) . Its value and its first derivatives in x give

$$\langle v(p), Y \rangle = -kA, \quad A\phi_i = (k - \kappa_i)b_i. \quad (4.5)$$

On $\mathbb{S}^{n+1}$ put

$$G(z) = k(1 - \langle X, z \rangle) + \langle v(p), z \rangle, \quad \mathcal{B} = \{G < 0\}.$$

The affine normal of G has length $\sqrt{1 + k^2}$, and $-k/\sqrt{1 + k^2} \in (-1, 1)$. Thus $\mathcal{B}$ is a nonempty connected spherical cap. The cap is disjoint from $X(M)$. Since $\nabla^S G(X) = v(p)$, it meets Ω_- near X ; connectedness gives $\mathcal{B} \subset \Omega_-$. At Y ,

$$\nabla^S G(Y) = v(p) + kd =: W, \quad |W|^2 = 1 - 2k^2A + 2k^2A = 1. \quad (4.6)$$

Since $G \circ X$ has a minimum at q , one has $W \perp T_q M$. Moreover, $G(Y) = 0$ and $W \neq 0$ give $Y \in \partial \mathcal{B}$. Lemma 3.4 gives $v(q) = W$.

By [4, Lemma 2.2], the reflected vectors

$$\tilde{e}_i = e_i - \frac{b_i}{A}d \quad (4.7)$$

form an orthonormal basis of $T_q M$. Put $v_i = (e_i, \tilde{e}_i) \in T_{(p,q)}(M \times M)$. In the spatial Hessian formula of [3, Section 2, p. 335], take ambient curvature 1 and $F(h) = \text{tr } h$. Then $\dot{F}^{ij} = \delta_{ij}$, $F(p) = F(q) = H$, and $\nabla F = 0$; the concavity inequality in that formula is an equality because F is linear. Since $|d|^2 = 2A$ and $\langle v(p), v(q) \rangle = 1 - k^2A$, it gives

$$\sum_i \text{Hess}_{M \times M} c(v_i, v_i) = (n - S)k + H(k^2 - 1) + \frac{2}{A^2} \sum_i (k - \kappa_i)b_i^2. \quad (4.8)$$

The function $(x, y) \mapsto \phi(x) - c(x, y)$ has a local minimum equal to zero at (p, q) . Consequently, (4.5) and (4.8) give

$$\mathcal{D}\phi \geq \sum_i \text{Hess}_{M \times M} c(v_i, v_i) = (n - S)k + H(k^2 - 1) + 2 \sum_i \frac{\phi_i^2}{k - \kappa_i}. \quad (4.9)$$

All the denominators are positive in this case. $\square$

Theorem 4.2. *For a closed connected embedded non-totally-umbilic CMC hypersurface of $\mathbb{S}^{n+1}$, put*

$$w = k + \ell, \quad V = S - n - H(k - \ell), \quad u = w^{-1}. \quad (4.10)$$

Then $(\Delta + V)u = m$ is a nonnegative finite measure and

$$\int_M (|\nabla f|^2 + V f^2) dv_g = \int_M w^{-2} |\nabla(wf)|^2 dv_g + \int_M w f^2 dm \quad (4.11)$$

for every $f \in C^\infty(M)$. In particular, $\Delta + V \geq 0$ on $H^1(M)$. The function V and the measure m are unchanged by reversing the normal.

Proof. At every common Alexandrov point of k and ℓ , Lemma 2.3, Proposition 4.1, and normal reversal give

$$\begin{aligned} \mathcal{D}k + (S - n)k - H(k^2 - 1) &\geq 2 \sum_i \frac{k_i^2}{k - \kappa_i}, \\ \mathcal{D}\ell + (S - n)\ell + H(\ell^2 - 1) &\geq 2 \sum_i \frac{\ell_i^2}{\ell + \kappa_i}. \end{aligned} \quad (4.12)$$

A zero denominator has zero numerator by Lemma 3.5, and its summand is zero. In the quadratic upper tests from Lemma 2.3, the value and gradient at the contact point are fixed. Thus these conventions persist as the Hessians converge to the Alexandrov Hessians.

The common Alexandrov points have full measure. Since

$$(S - n)(k + \ell) - H(k^2 - \ell^2) = Vw, \quad (4.13)$$

addition and (2.10) yield

$$\mathcal{D}w + Vw \geq 2w^{-1} |\nabla w|^2 \quad \text{almost everywhere.}$$

The functions k, ℓ are locally semiconvex by Lemma 3.2. Moreover, $2w^{-1} |\nabla w|^2 - Vw \in L^\infty(M)$, since M is compact and $\min_M w > 0$. Lemma 2.3 therefore gives the measure inequality (2.5). Proposition 2.4, with $F = 0$, gives m and (4.11). Its nonnegative right side implies $\Delta + V \geq 0$ on $C^\infty(M)$. Since $V \in L^\infty(M)$, the quadratic form is continuous on $H^1(M)$; smooth density [24, Theorem 4.2] gives the asserted inequality there.

Normal reversal sends (H, k, ℓ) to $(-H, \ell, k)$, leaving w, V, u , and m unchanged. $\square$

Proof of Theorem 1.1. A minimal totally umbilic hypersurface has $h = (H/n)g = 0$. Hence Theorem 4.2 applies under the hypotheses of Theorem 1.1. With $H = 0$, it gives $V = S - n$ and (1.2). Taking $f = 1$ in (4.11) gives

$$\int_M (S - n) dv_g = \int_M |\nabla \log w|^2 dv_g + \int_M w dm \geq 0. \quad (4.14)$$

If equality holds, then $\nabla w = 0$ almost everywhere and $m = 0$, since $\min_M w > 0$. By Lemma 2.7, w is constant. Thus

$$0 = m = (S - n)w^{-1} dv_g,$$

and continuity gives $S = n$. Proposition 4.4 identifies the image with (1.4). Conversely, Lemma 4.5 gives $S = n$ on every minimal Clifford hypersurface. $\square$

Lemma 4.3. *Let $\mathcal{E}$ be a smooth rank- r subbundle of $M \times \mathbb{R}^N$, with M connected. Suppose that the Euclidean derivative of every local section of $\mathcal{E}$ belongs to $\mathcal{E}$. Then $\mathcal{E}_p$ is a fixed linear subspace of $\mathbb{R}^N$, independent of p .*

Proof. The Euclidean connection restricts to a connection on $\mathcal{E}$. Its parallel transport along any smooth path is a linear isomorphism of the endpoint fibres [34, Section 3.3.2]. Its parallel sections have zero Euclidean derivative and are therefore constant vectors in $\mathbb{R}^N$. Thus the endpoint fibres are equal as subspaces of $\mathbb{R}^N$. Connectedness gives the conclusion. $\square$

Let B be a parallel self-adjoint endomorphism of TM , where (M, g) is any connected Riemannian manifold. For a smooth curve γ , let P_t denote parallel transport from $\gamma(0)$ to $\gamma(t)$. Then

$$\frac{d}{dt}(P_t^{-1}B_{\gamma(t)}P_t) = P_t^{-1}(\nabla_{\dot{\gamma}}B)P_t = 0. \quad (4.15)$$

Thus the eigenvalues and their multiplicities are constant. The spectral projections $\prod_{\mu \neq \lambda} \frac{B - \mu \text{Id}}{\lambda - \mu}$ are smooth and parallel, so their images are smooth parallel eigendistributions.

Proposition 4.4 (Chern–do Carmo–Kobayashi and Lawson). *Let $M^n \hookrightarrow \mathbb{S}^{n+1}$ be closed, connected, and minimal. If $S \equiv n$, its image is congruent to a product (1.4).*

Proof. This is the codimension-one case of [16, Main theorem, p. 60]. $\square$

Lemma 4.5. *Let $p, q \geq 1$ be integers with $p + q = n$, and let $r, t > 0$ satisfy $r^2 + t^2 = 1$. On $\mathbb{S}^p(r) \times \mathbb{S}^q(t) \subset \mathbb{S}^{n+1}$, write $X = (rx, ty)$ with $x \in \mathbb{S}^p(1)$ and $y \in \mathbb{S}^q(1)$, and choose $v = (tx, -ry)$. Then*

$$k = t/r, \quad \ell = r/t, \quad S - n - H(k - \ell) = 0. \quad (4.16)$$

In the minimal case $r^2 = p/n$, $S = n$, and equality holds in (1.3).

Proof. The principal curvatures are $\alpha = t/r$ and $\beta = -r/t$, with multiplicities p and q . For a distinct second point (rx', ty') , put $A_1 = 1 - \langle x, x' \rangle$ and $A_2 = 1 - \langle y, y' \rangle$. Then

$$c = \frac{rt(A_1 - A_2)}{r^2A_1 + t^2A_2}. \quad (4.17)$$

This is a weighted average of α and β , with nonnegative weights r^2A_1 and t^2A_2 . Each endpoint is attained by fixing one factor and varying the other. Hence $k = \alpha$ and $\ell = -\beta$. Since $\alpha\beta = -1$,

$$H(k - \ell) = (p\alpha + q\beta)(\alpha + \beta) = p\alpha^2 + q\beta^2 - n = S - n.$$

Finally, $H = 0$ is equivalent to $pt^2 = qr^2$, which gives $r^2 = p/n$ and $S = n$. $\square$

5. THE QUADRATIC GAUSSIAN INEQUALITY

Let $Q > 0$ be self-adjoint, and retain (1.5)–(1.8). Let $X : M^n \hookrightarrow \mathbb{R}^{n+1}$ be complete, connected, properly embedded, and without boundary. Choose the normal from Lemma 3.1, and assume $H = \langle QX, \nu \rangle$.

Let X_t be a smooth variation equal to X outside a fixed compact set, with $X_0 = X$ and $\dot{X}_0 = f\nu$. Put $g_t = X_t^* g_{\text{Eucl}}$ and $J_t = dv_{g_t}/dv_g$. The relative weighted area

$$\mathcal{A}(t) = \int_M (e^{-F \circ X_t} J_t - e^{-F \circ X}) dv_g$$

is finite because the integrand has fixed compact support. The first-variation formula [14, Proposition 2], with mean curvature vector $-H\nu$, gives

$$\mathcal{A}'(0) = \int_M (H - \langle QX, \nu \rangle) f d\mu_F. \quad (5.1)$$

Thus stationarity is equivalent to (1.7).

Lemma 5.1 (Cheng–Mejia–Zhou). *Put $Z = (QX)^T$ and $Q_T = P_T Q|_{TM}$. Identifying self-adjoint endomorphisms with their symmetric bilinear forms,*

$$\mathcal{L}_F h = hQ_T + Q_T h - Uh, \quad (5.2)$$

$$\frac{1}{2} \mathcal{L}_F S = |\nabla h|^2 + 2 \text{tr}(h^2 Q_T) - US, \quad (5.3)$$

$$\mathcal{L}_F \nu = Q\nu - U\nu. \quad (5.4)$$

In (5.4), $\mathcal{L}_F$ acts on the ambient components of ν .

Proof. The ambient curvature vanishes, $\overline{\text{Ric}}_F = Q$, and $\bar{\nabla}Q = 0$. The tensor equation [15, arXiv v1, (23)] therefore gives (5.2), and the scalar equation [15, arXiv v1, Theorem 3] gives (5.3). For every constant ambient vector v , [15, arXiv v1, Proposition 2] gives

$$\mathcal{L}_F \langle v, v \rangle = \langle Qv, v \rangle - U \langle v, v \rangle.$$

Since Q is self-adjoint, this is (5.4). $\square$

Let k, ℓ be the nonnegative Euclidean envelopes in (3.2). Assume that $X(M)$ is not a hyperplane. Then $w = k + \ell > 0$ by Lemma 3.3.

Proposition 5.2. *Let ϕ be a C^2 upper test for k at p . In an orthonormal principal frame at p , put $\phi_i = d\phi(e_i)$. Then*

$$\mathcal{L}_F \phi + (U - 2a)k \geq 2 \sum_{\kappa_i < k} \frac{\phi_i^2}{k - \kappa_i}. \quad (5.5)$$

If $k(p) = \kappa_i(p)$, then $\phi_i = 0$. If $k(p) = 0$, then $\nabla\phi(p) = 0$.

Proof. If $k(p) = 0$, then $\phi \geq k \geq 0$ near p and $\phi(p) = 0$. Hence $\nabla\phi(p) = 0$ and $\mathcal{D}\phi(p) \geq 0$, which proves (5.5).

Suppose that $k(p) = \kappa_{\max}(p) > 0$. For a unit vector e_1 in the maximal eigenspace, (5.2) gives

$$(\mathcal{L}_F h)_{11} = (2\langle Qe_1, e_1 \rangle - U)k. \quad (5.6)$$

Lemma 3.5, with $Z = (QX)^T$, gives the zero components and

$$\mathcal{L}_F \phi + (U - 2a)k \geq 2k(\langle Qe_1, e_1 \rangle - a) + 2 \sum_{\kappa_i < k} \frac{\phi_i^2}{k - \kappa_i}. \quad (5.7)$$

Since $k > 0$ and $Q \geq a\text{Id}$, this implies (5.5).

Suppose now that $k(p) > \max\{0, \kappa_{\max}(p)\}$. By Lemma 3.2, there is $q \neq p$ with $c(p, q) = k(p)$. Write

$$X = X(p), \quad Y = X(q), \quad d = Y - X, \quad A = |d|^2/2, \quad b_i = \langle e_i, d \rangle, \quad H_x = H(p), \quad H_y = H(q). \quad (5.8)$$

Here $k = k(p)$, $\kappa_i = \kappa_i(p)$, and $H_i = dH_p(e_i)$. The function

$$\mathcal{Z}(x, y) = \frac{\phi(x)}{2} |X(y) - X(x)|^2 + \langle v(x), X(y) - X(x) \rangle \quad (5.9)$$

is nonnegative for x near p and every $y \in M$, and vanishes at (p, q) . Its value and first derivatives in x give

$$\langle v(p), d \rangle = -kA, \quad A\phi_i = (k - \kappa_i)b_i. \quad (5.10)$$

Its derivatives in y give $W := v(p) + kd \perp T_q M$, and (5.10) gives $|W| = 1$.

The open ball $\mathcal{B}$ of centre $X - v(p)/k$ and radius $1/k$ is disjoint from $X(M)$. It meets the negative-normal component near X , so its connectedness and Lemma 3.1 give $\mathcal{B} \subset \Omega_-$. At $Y \in \partial\mathcal{B}$, the gradient of $z \mapsto k|z - X|^2/2 + \langle v(p), z - X \rangle$ is W . Lemma 3.4 gives

$$v(q) = v(p) + kd. \quad (5.11)$$

By [4, Lemma 2.2], the reflected vectors

$$\tilde{e}_i = e_i - \frac{b_i}{A}d \quad (5.12)$$

form an orthonormal basis of $T_q M$. Put $v_i = (e_i, \tilde{e}_i)$.

The function $\Psi(x, y) = \phi(x) - c(x, y)$ has a minimum zero at (p, q) , and $\text{Hess } \mathcal{Z} = A \text{Hess } \Psi$ there. The spatial derivative formulas [5, (8)–(10), p. 29], with unit chord $-d/|d|$, give

$$T := \sum_i \text{Hess}_{M \times M} \mathcal{Z}(v_i, v_i) \geq 0, \quad (5.13)$$

$$T = A(\mathcal{D}\phi + Sk - H_x k^2) - 2 \sum_i \phi_i b_i + \sum_i H_i b_i + H_x - H_y.$$

Put $Z_i = \langle QX, e_i \rangle$. Differentiating (1.7) at p gives

$$H_i = \langle Qe_i, v(p) \rangle + \kappa_i Z_i. \quad (5.14)$$

Stationarity and (5.11) give

$$H_y - H_x = \langle Qd, v(p) \rangle + k\langle QX, d \rangle + k\langle Qd, d \rangle, \quad (5.15)$$

$$\sum_i H_i b_i = \langle Qd, v(p) \rangle + kAq_v + \sum_i \kappa_i Z_i b_i. \quad (5.16)$$

Indeed, (5.10) gives

$$d = \sum_i b_i e_i - kAv(p), \quad \langle QX, d \rangle = \sum_i Z_i b_i - kAH_x. \quad (5.17)$$

Subtracting (5.15) from (5.16) and applying (5.10) yields

$$\sum_i H_i b_i + H_x - H_y = -A \sum_i Z_i \phi_i + k^2 AH_x + kAq_v - k\langle Qd, d \rangle. \quad (5.18)$$

Substitution in (5.13), with $e = d/|d|$, gives

$$\frac{T}{A} = \mathcal{L}_F \phi + (U - 2\langle Qe, e \rangle)k - 2 \sum_i \frac{\phi_i^2}{k - \kappa_i}. \quad (5.19)$$

Hence

$$\mathcal{L}_F \phi + (U - 2a)k = \frac{T}{A} + 2k(\langle Qe, e \rangle - a) + 2 \sum_i \frac{\phi_i^2}{k - \kappa_i}. \quad (5.20)$$

Here $T \geq 0$, $A > 0$, $k > 0$, and $\langle Qe, e \rangle \geq a$. All denominators are positive. This proves (5.5). $\square$

Proposition 5.3. *For every hypersurface in this section whose image is not a hyperplane, put $V = U - 2a$ and $u = w^{-1}$. Then*

$$\mathcal{L}_F w + (U - 2a)w \geq 2w^{-1}|\nabla w|^2 \quad (5.21)$$

distributionally. The distribution $\mathfrak{m} = (\Delta_F + U - 2a)u$ is a nonnegative locally finite measure, and

$$\int_M (|\nabla f|^2 + (U - 2a)f^2) d\mu_F = \int_M w^{-2}|\nabla(wf)|^2 d\mu_F + \int_M wf^2 d\mathfrak{m} \quad (5.22)$$

for every $f \in C_c^\infty(M)$.

Proof. The functions k, ℓ are locally semiconvex by Lemma 3.2. Normal reversal preserves S, q_v, U and $\mathcal{L}_F$, and interchanges k, ℓ . At common Alexandrov points, Lemma 2.3 and Proposition 5.2 give (2.9) with $V = U - 2a$. A zero denominator has zero numerator by that proposition. Lemma 2.5 gives (5.21) as a measure inequality. For every compact $K \Subset M$, continuity gives $\min_K w > 0$. Proposition 2.4 therefore gives $\mathfrak{m}$ and (5.22). $\square$

Quadratic-form assertion of Theorem 1.2. Equation (5.22) gives (1.9). On the complexification of $L^2(M, d\mu_F)$, let $T_0 = \Delta_F + U$ with domain $C_c^\infty(M; \mathbb{C})$. This domain is dense, and weighted integration by parts makes T_0 symmetric, with

$$q(f) := \langle T_0 f, f \rangle_{L^2(d\mu_F)} = \int_M (|\nabla f|^2 + U|f|^2) d\mu_F \geq 2a\|f\|_{L^2(d\mu_F)}^2. \quad (5.23)$$

The inequality follows by applying (1.9) to the real and imaginary parts of f . By [28, Chapter VI, Corollary 1.28], q is closable. Its closure has lower bound $2a$ and defines the Friedrichs extension T of T_0 [28, Chapter VI, Theorem 2.6 and §2.3]. Thus $\sigma(T) \subset [2a, \infty)$. Finally, $q_v \leq b$ gives, for real $f \in C_c^\infty(M)$,

$$\int_M (|\nabla f|^2 + Sf^2) d\mu_F \geq (2a - b) \int_M f^2 d\mu_F. \quad (5.24)$$

$\square$

6. WEIGHTED EXHAUSTION AND RIGIDITY

Let $Q > 0$ be self-adjoint, let $a = \lambda_{\min}(Q)$, and let $F(z) = \frac{1}{2}\langle Qz, z \rangle$. Let $X : M^n \hookrightarrow \mathbb{R}^{n+1}$, $n \geq 2$, be a connected complete smooth proper embedding without boundary satisfying (1.7).

Lemma 6.1. *The weighted volume of M is finite, and $F \circ X \in L^1(M, d\mu_F)$.*

Proof. Write F for $F \circ X$. Since $F \geq a|X|^2/2$ and X is proper, F is nonnegative and proper. The identities $\nabla F = (QX)^T$ and $\text{Hess } F(v, z) = \langle Qv, z \rangle - Hh(v, z)$ give

$$\mathcal{D}F = \text{tr } Q - q_v - H^2, \quad |\nabla F|^2 = |QX|^2 - H^2, \quad (6.1)$$

$$\mathcal{L}_F F = \text{tr } Q - q_v - |QX|^2 \leq C_0 - 2aF, \quad C_0 = \text{tr } Q - a. \quad (6.2)$$

Here $q_v \geq a$ and $Q^2 \geq aQ$. In particular,

$$\mathcal{D}F - \frac{1}{2}|\nabla F|^2 + aF = \text{tr } Q - q_v - \frac{1}{2}H^2 - \frac{1}{2}|QX|^2 + aF \leq C_0, \quad \mathcal{D}F \leq C_0. \quad (6.3)$$

If M is noncompact, Cheng–Vieira–Zhou [17, Theorem 1.1(ii), arXiv version], with $\alpha = \frac{1}{2}$, $\beta = a$ and upper bound C_0 , gives $\int_M e^{-F/2} dv_g < \infty$. The same integral is finite when M is compact. The inequalities

$$e^{-F} \leq e^{-F/2}, \quad Fe^{-F} \leq \frac{2}{e} e^{-F/2}$$

give both conclusions. $\square$

Choose a smooth nonincreasing $\chi : [0, \infty) \rightarrow [0, 1]$, equal to one on $[0, 1]$ and zero on $[2, \infty)$. For $R > 0$, set

$$\chi_R = \chi(|X|/R), \quad |\nabla \chi_R| \leq \|\chi'\|_\infty / R. \quad (6.4)$$

The functions χ_R are smooth also at $X = 0$, where they are locally constant. They have compact support, increase to one, and are eventually one on every compact set. Lemma 6.1 gives

$$\int_M |\nabla \chi_R|^2 d\mu_F \leq R^{-2} \|\chi'\|_\infty^2 \mu_F(M) \rightarrow 0. \quad (6.5)$$

Average inequality in Theorem 1.2. Suppose $X(M)$ is not a hyperplane, and let w, m be as in Proposition 5.3. If $\int_M S d\mu_F = \infty$, then (1.10) follows from $a \leq q_v \leq b$ and $\mu_F(M) < \infty$. Otherwise $V = U - 2a \in L^1(M, d\mu_F)$. Lemma 2.6, applied with the cutoffs (6.4), gives

$$\int_M (U - 2a) d\mu_F = \int_M |\nabla \log w|^2 d\mu_F + \int_M w dm \geq 0. \quad (6.6)$$

$\square$

Lemma 6.2. *Suppose $X(M)$ is not a hyperplane. Equality in (1.10) implies $U = 2a$ and $\nabla h = 0$.*

Proof. Equality implies $S \in L^1(M, d\mu_F)$. Equation (6.6) therefore gives $\nabla \log w = 0$ almost everywhere and $\int_M w dm = 0$. By Lemma 2.7 and connectedness, $w = c > 0$ is constant. Hence

$$0 = m = c^{-1}(U - 2a) d\mu_F.$$

Since U is smooth, $U = 2a$ everywhere. Thus

$$0 \leq S = 2a - q_v \leq a. \quad (6.7)$$

Equation (5.3) becomes

$$\frac{1}{2}\mathcal{L}_F S = B, \quad B = |\nabla h|^2 + 2 \text{tr}(h^2(Q_T - a\text{Id})) \geq |\nabla h|^2. \quad (6.8)$$

Indeed, an orthonormal eigenbasis (e_i) of Q_T , with eigenvalues $q_i \geq a$, gives

$$\text{tr}(h^2(Q_T - a\text{Id})) = \sum_i (q_i - a) |Be_i|^2 \geq 0. \quad (6.9)$$

By (6.4), (6.5) and [26, Theorem 2.2, (iv) $\Rightarrow$ (i)], for the weight e^{-F} every smooth function bounded above and satisfying $\mathcal{L}_F v \geq 0$ is constant. Applied to S , this gives $B = 0$ in (6.8), hence $\nabla h = 0$. $\square$

Proposition 6.3. *A connected complete smooth properly embedded Euclidean hypersurface without boundary with parallel shape operator is either an affine hyperplane or an entire product $\mathbb{S}^p(r, c) \times E_0$, where $1 \leq p \leq n$, $r > 0$, E_0 is a linear subspace of dimension $n - p$, and the sphere lies in $E_0^\perp$. In the nonhyperplane case, (1.7) and $U = 2a$ imply (1.11).*

Proof. Equation (4.15) gives constant principal curvatures. The Euclidean classification [11, §2, p. 3] makes the image an open subset of an affine hyperplane, a hypersphere or a spherical cylinder. Properness makes the image closed in the ambient space, hence in that model. Each model is connected, so the image is the whole model.

In the nonhyperplane case, write $X(M) = \mathbb{S}^p(r, c) \times E_0$, with $E_1 = E_0^\perp$ and $c \in E_1$. With the outward spherical normal ω , stationarity reads

$$\frac{p}{r} = \langle Q(c + r\omega + z), \omega \rangle \quad (\omega \in E_1, |\omega| = 1, z \in E_0). \quad (6.10)$$

Varying z gives $\langle Qz, \omega \rangle = 0$ for all $z \in E_0$ and all unit $\omega \in E_1$. Thus $Q(E_0) \subset E_0$; self-adjointness gives $Q(E_1) \subset E_1$. The equations at ω and $-\omega$ are

$$p/r = \langle Qc, \omega \rangle + r\langle Q\omega, \omega \rangle, \quad p/r = -\langle Qc, \omega \rangle + r\langle Q\omega, \omega \rangle.$$

Subtracting gives $\langle Qc, \omega \rangle = 0$ on the unit sphere of E_1 . Since $Qc \in E_1$, one has $Qc = 0$, and positive definiteness gives $c = 0$. Adding gives $\langle Q\omega, \omega \rangle = p/r^2$. Polarization yields $Q|_{E_1} = (p/r^2)\text{Id}$. On the cylinder $S = q_\nu = p/r^2$. Therefore $U = 2a$ gives $r^2 = p/a$ and $E_1 \subset \ker(Q - a\text{Id})$, as in (1.11). $\square$

Equality in Theorem 1.2. Equality gives $\nabla h = 0$ and $U = 2a$ by Lemma 6.2. Since $X(M)$ is not a hyperplane, Proposition 6.3 gives (1.11). Conversely, let E and p be as in (1.11), and put $r = \sqrt{p/a}$. The product is connected, complete and properly embedded. Its outward principal curvatures are $1/r$ with multiplicity p and zero with multiplicity $n - p$. The subspaces $E, E^\perp$ are Q -invariant, and $Q|_E = a\text{Id}$. Hence $H = p/r = ar = \langle QX, \nu \rangle$ and $S = q_\nu = a$. Lemma 6.1 gives finite weighted volume, so the product attains equality. If it is compact, $p = n$ and $E = \mathbb{R}^{n+1}$, whence $Q = a\text{Id}$. $\square$

If $h = 0$, Proposition 6.3 gives an affine hyperplane $X(M) = x_0 + \nu^\perp$ with constant unit normal ν . Stationarity gives

$$0 = \langle Q(x_0 + y), \nu \rangle \quad (y \in \nu^\perp). \quad (6.11)$$

Varying y gives $Q\nu = \beta\nu$ with $\beta > 0$; at $y = 0$ the equation gives $\beta\langle x_0, \nu \rangle = 0$. Thus $X(M) = \nu^\perp$. On this hyperplane $k = \ell = 0$.

7. SPECTRAL AND GRAPHICAL CONSEQUENCES

7.1. Graphs in eigendirections. For $Q = a\text{Id}$, the dilation $X \mapsto \sqrt{2a}X$ reduces Theorem 1.3 to the hemisphere theorem of Ding–Xin–Yang [22, arXiv v1, Theorem 3.1].

Proof of Theorem 1.3. Rescale v so that $|v| = 1$ and put $\theta = \langle \nu, v \rangle > 0$. Equation (5.4) and $Qv = \beta v$ give

$$\mathcal{L}_F \theta = (\beta - U)\theta. \quad (7.1)$$

Suppose that $X(M)$ is not a hyperplane. The ground-state identity (2.7), with θ^{-1} in place of w , $V = \beta - U$ and $\mathfrak{m} = 0$, gives

$$\int_M \theta^2 |\nabla(f/\theta)|^2 d\mu_F = \int_M (|\nabla f|^2 + (\beta - U)f^2) d\mu_F \quad (f \in C_c^\infty(M)). \quad (7.2)$$

By Theorem 1.2,

$$0 \leq \int_M \theta^2 |\nabla(f/\theta)|^2 d\mu_F \leq 2 \int_M |\nabla f|^2 d\mu_F + (\beta - 2a) \int_M f^2 d\mu_F. \quad (7.3)$$

Take $f = \chi_R$ from (6.5). If $\beta < 2a$, the right side tends to $(\beta - 2a)\mu_F(M) < 0$, a contradiction. If $\beta = 2a$, then, for every compact $K \subset M$ and all sufficiently large R ,

$$\int_K |\nabla \log \theta|^2 d\mu_F \leq 2 \int_M |\nabla \chi_R|^2 d\mu_F \longrightarrow 0.$$

Thus θ is constant by connectedness, and (7.1) gives $U = 2a$. Since $\mu_F(M) < \infty$, equality holds in (1.10). Theorem 1.2 then gives (1.11); in particular, $v(x) \in \ker(Q - a\text{Id})$ for every $x \in M$. Since $v \in \ker(Q - 2a\text{Id})$ and $a > 0$, self-adjointness of Q gives $\langle v, v \rangle = 0$, a contradiction. Hence $X(M)$ is a hyperplane, and (6.11) places it through the origin. $\square$

Corollary 7.1. *Let v be a unit eigenvector of Q with eigenvalue $\beta \leq 2a$. Every smooth entire graph*

$$X(y) = y + f(y)v, \quad y \in v^\perp,$$

satisfying (1.7) is a hyperplane through the origin.

Proof. The graph is diffeomorphic to $v^\perp$ and is properly embedded by [30, Proposition 5.7]. Since the ambient Euclidean metric is complete, its induced metric is complete by [30, Problem 13-18(b)]. Choose the unit normal

$$v = \frac{v - \nabla f}{\sqrt{1 + |\nabla f|^2}}, \quad \langle v, v \rangle = \frac{1}{\sqrt{1 + |\nabla f|^2}} > 0.$$

Theorem 1.3 gives the conclusion. $\square$

7.2. Compact spectral comparison. On a closed connected manifold of positive dimension with smooth density $d\mu_F = e^{-F} dv_g$, let

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$$

be the eigenvalues of Δ_F , counted with multiplicity. The weight has positive upper and lower bounds, so the weighted and unweighted Sobolev norms are equivalent and $H^1(M) \hookrightarrow L^2(M, d\mu_F)$ is compact. The form min-max principle is given in [28, Chapter VII, §6.4, Remark 6.2].

Proposition 7.2. *Let (M, g) be a closed connected smooth Riemannian manifold of positive dimension, let $F \in C^\infty(M)$, and put $d\mu_F = e^{-F} dv_g$ and $\Delta_F = -\text{div } \nabla + \langle \nabla F, \nabla \rangle$. Let $c \in \mathbb{R}$, let $P \in L^\infty(M; \mathbb{R})$, and suppose that a positive Lipschitz function w satisfies*

$$\int_M (|\nabla f|^2 + (P - c)f^2) d\mu_F \geq \int_M w^{-2} |\nabla(wf)|^2 d\mu_F \quad (f \in H^1(M)). \quad (7.4)$$

Let ζ_j be the eigenvalues of $\Delta_F + P$, counted with multiplicity in nondecreasing order. Then

$$\zeta_j \geq c + \vartheta \lambda_j \quad (j \geq 0), \quad \vartheta = \left(\frac{\min_M w}{\max_M w} \right)^2. \quad (7.5)$$

Proof. Put $u = w^{-1}$. By Lemma 2.7, multiplication by u is an automorphism of $H^1(M)$. For $f = uv \neq 0$, (7.4) gives

$$\frac{\int_M (|\nabla f|^2 + (P - c)f^2) d\mu_F}{\int_M f^2 d\mu_F} \geq \frac{\int_M u^2 |\nabla v|^2 d\mu_F}{\int_M u^2 v^2 d\mu_F} \geq \vartheta \frac{\int_M |\nabla v|^2 d\mu_F}{\int_M v^2 d\mu_F}. \quad (7.6)$$

The map $E \mapsto u^{-1}E$ is a bijection on the set of $(j + 1)$ -dimensional subspaces of $H^1(M)$. Taking the supremum over $0 \neq f \in E$ and then the infimum over these subspaces gives (7.5) by the form min-max principle. $\square$

On the closed hypersurfaces of Theorems 1.1 and 1.2, (4.11) with $H = 0$ and (5.22), respectively, imply (7.4) for smooth f . Smooth density and Lemma 2.7 extend these inequalities to $H^1(M)$. Taking $(F, P, c) = (0, S, n)$ in the spherical case and $(P, c) = (U, 2a)$ in the quadratic case gives

$$\zeta_j(\Delta + S) \geq n + \vartheta \lambda_j(\Delta), \quad \zeta_j(\Delta_F + U) \geq 2a + \vartheta \lambda_j(\Delta_F). \quad (7.7)$$

On a minimal Clifford hypersurface, w is constant and $S = n$ by Lemma 4.5. On the compact equality model (1.11), the outward normal gives $k = \sqrt{a/n}$, $\ell = 0$, and $U = 2a$. Thus $\vartheta = 1$ and equality holds for every j in the respective comparisons.

7.3. The embedded low-index classification. For a closed two-sided minimal hypersurface of $\mathbb{S}^{n+1}$, its Morse index is the number of negative eigenvalues, counted with multiplicity, of

$$J = \Delta - S - n. \quad (7.8)$$

Perdomo's theorem [35, Theorem 3.1, p. 564], with ambient sphere dimension $N = n + 1$, gives

$$\text{ind}(M) = n + 3 \implies \int_M S dv_g \leq n \text{Vol}(M) \quad (7.9)$$

for every closed oriented minimally immersed $M^n \rightarrow \mathbb{S}^{n+1}$.

Corollary 7.3. *Let $n \geq 2$. A closed connected embedded minimal hypersurface of $\mathbb{S}^{n+1}$ has Morse index $n + 3$ if and only if it is a minimal Clifford hypersurface.*

Proof. By Lemma 3.1, M is two-sided and oriented. The equator has index 1 [35, p. 560]. Suppose that $\text{ind}(M) = n + 3$. Then M is not totally geodesic, and (7.9) and Theorem 1.1 give

$$n \text{Vol}(M) \leq \int_M S dv_g \leq n \text{Vol}(M).$$

The equality case of Theorem 1.1 gives the conclusion. Conversely, every minimal Clifford hypersurface has index $n + 3$ [35, p. 560]. $\square$

7.4. Self-shrinkers and Gaussian curvature. Set $Q = \frac{1}{2}\text{Id}$, $d\gamma = e^{-|X|^2/4} dv_g$, and $\Delta_\gamma = -\mathcal{D} + \frac{1}{2}\langle X^\top, \nabla \rangle$. Stationarity becomes $H = \frac{1}{2}\langle X, \nu \rangle$.

Corollary 7.4. *Let $M^n \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a connected complete properly embedded self-shrinker without boundary, other than a hyperplane. Then*

$$\int_M (|\nabla f|^2 + S f^2) d\gamma \geq \frac{1}{2} \int_M f^2 d\gamma \quad (f \in C_c^\infty(M)), \quad (7.10)$$

and

$$\int_M S d\gamma \geq \frac{1}{2} \gamma(M). \quad (7.11)$$

Equality in (7.11) holds precisely for $\mathbb{S}^p(\sqrt{2p}) \times \mathbb{R}^{n-p}$, $1 \leq p \leq n$, up to an orthogonal transformation.

Proof. Here $a = q_\nu = 1/2$ and $U = S + 1/2$. Substitution in (1.9) and (1.10) gives the two inequalities. Since $\ker(Q - a\text{Id}) = \mathbb{R}^{n+1}$, (1.11) gives the stated cylinders. Among them, precisely the model with $p = n$ is compact. $\square$

For complete immersed self-shrinkers, properness, polynomial volume growth and finite Gaussian area are equivalent by Cheng–Zhou [18, Theorem 1.3].

Lemma 7.5 (Colding–Minicozzi). *Let $X : M^n \rightarrow \mathbb{R}^{n+1}$ be a smooth complete self-shrinking immersion without boundary, with $n \geq 2$ and $H = \langle X, \nu \rangle/2$ in every local unit normal. Suppose that*

$$\text{Vol}_g(X^{-1}(B_R(0))) \leq C(1 + R)^d \quad (R > 0)$$

for some $C > 0$ and $d \geq 0$. Define

$$F_{y,t}(M) = (4\pi t)^{-n/2} \int_M e^{-|X-y|^2/(4t)} dv_g, \quad \Lambda(M) = \sup_{y \in \mathbb{R}^{n+1}, t > 0} F_{y,t}(M). \quad (7.12)$$

Then

$$\Lambda(M) = F_{0,1}(M) = (4\pi)^{-n/2} \gamma(M). \quad (7.13)$$

Proof. Polynomial volume growth gives $(1 + |X|)^m e^{-\varepsilon|X|^2} \in L^1(M, dv_g)$ for every $m \geq 0$ and $\varepsilon > 0$. Fix $y \in \mathbb{R}^{n+1}$ and $c > -1$. Put $t_s = 1 + cs^2$ and $G(s) = F_{s y, t_s}(M)$ for $0 \leq s \leq 1$. Then $t_s \geq \min\{1, 1 + c\} > 0$. The identity of Colding–Minicozzi [19, p. 788, (7.27)] gives

$$G'(s) = -\frac{s}{2t_s^2} (4\pi t_s)^{-n/2} \int_M |csX^\perp + y^\perp|^2 e^{-|X-sy|^2/(4t_s)} dv_g \leq 0. \quad (7.14)$$

Here $cs(X - sy)^\perp + t_s y^\perp = csX^\perp + y^\perp$. Thus $F_{y, 1+c}(M) \leq F_{0,1}(M)$. Since every $t > 0$ is $1 + c$ for some $c > -1$, taking the supremum gives $\Lambda(M) \leq F_{0,1}(M)$. The reverse inequality follows by taking $y = 0$, $t = 1$. $\square$

Consequently, (7.11) is equivalent to

$$\int_M S e^{-|X|^2/4} dv_g \geq \frac{1}{2} (4\pi)^{n/2} \Lambda(M), \quad (7.15)$$

with equality precisely for the cylinders of Corollary 7.4, and for $\mathbb{S}^n(\sqrt{2n})$ in the closed case. These are Conjectures 1.6–1.7 of Li–Zhao [31], in dimensions $n \geq 2$.

7.5. The operator on a punctured cone. Let $L^n \subset \mathbb{S}^{n+1}$ be a smooth minimal hypersurface. On $C = (0, \infty) \times L$, with $X_C(r, p) = rp \in \mathbb{R}^{n+2}$, one has

$$g_C = dr^2 + r^2 g_L, \quad d\gamma_C = e^{-r^2/4} r^n dr dv_L, \quad S_C = r^{-2} S_L.$$

Thus

$$\Delta_{\gamma, C} = -\partial_r^2 - \frac{n}{r} \partial_r + r^{-2} \Delta_L + \frac{r}{2} \partial_r.$$

For $v \in C^\infty(L)$,

$$(\Delta_{\gamma, C} + S_C - \frac{1}{2})(rv) = r^{-1}(\Delta_L + S_L - n)v \quad (r > 0). \quad (7.16)$$

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